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LECTURE I. PHYSICAL-REGION ANALYTICITY PROPERTIES OF MANY-PARTICLE AMPLITUDES

Contents

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ABSTRACT

The first lecture summarizes results obtained during the past few years on the analytic structure of many-particle amplitudes in the physical region. The results are derived mainly from the cluster decomposition and macrocausality requirements. The second lecture describes the macrocausality requirement.

I. INTRODUCTION

Recent work[1] on the multiperipheral model has focused attention on the properties of many-particle amplitudes in the physical region. In this lecture I shall summarize the principal results obtained during the past few years concerning the analytic properties of many-particle amplitudes in the physical region. Only the assumptions and conclusions are described. References are given to the proofs.

The results have been derived from S-matrix principles, and one main aim is to clearly describe these principles as they apply to this work.

The principal conclusions are a description of the analytic structure of the complete surface of physical-region singularities, and a formula for the discontinuity around an arbitrary physical-region singularity surface. This formula is similar to the one proposed by Cutkosky, but there are important differences. These are discussed.

Another important result is a fundamental theorem that describes the physical-region analytic structure of integrals of the general type generated by the unitarity equations. The locations of the singularity surfaces of all of these functions are specified, and the general rule for continuing these functions around their singularity surfaces is given.

Finally, two interesting expressions for S are given. The first is an infinite-series expansion that resembles the Feynman expansion except that (1) each line of the diagram corresponds to a <u>physical</u> particle (and the integrations are accordingly on-mass-shell), and (2) each vertex of the diagram corresponds to minus the physical-region connected part of S^{-1}. The second expression is similar, except that now (1) the sum is only over "structure diagrams", and (2) the propagator corresponding to each line is the exact physical-region S matrix. Because the sum in this second expression is over structure diagrams, only a finite number of terms contribute in any bounded portion of the physical region.

The n-particle amplitude is defined only on the set defined by the conservation-law and mass constraints. The appropriate notion of analyticity is thus the notion of analyticity on an "algebraic variety." This is a standard mathematical concept, and a very useful one. Because it is still unfamiliar to many physicists, I shall begin by describing it.

II. ANALYTICITY ON THE MASS SHELL

A. The Mass Shell

The n-particle amplitude is a function of n four-vectors p_i. However, it is defined only on the surface defined by the mass-shell constraints

$$p_i^2 - m_i^2 = 0 \qquad (\text{each } i) \quad , \tag{2.1}$$

and the conservation laws

$$\sum_{in} p_i - \sum_{out} p_i = 0 \quad . \tag{2.2}$$

The set of complex four-vector p_i that satisfy (2.1) and (2.2) is called the (complex) mass shell $\mathcal{M}_c$.

B. Algebraic Varieties

A set defined by the vanishing of a set of polynomials is called an algebraic variety by mathematicians. Since the left-hand sides of (2.1) and (2.2) are polynomials in p_i, the mass shell $\mathcal{M}_c$ is an algebraic variety in the space of the n complex four-vectors p_i.

An n-particle amplitude is sometimes expressed in terms of functions of scalar invariants. For the cases $n > 5$ this again leads to an algebraic variety. For if one wishes to have a "basic set" of scalar invariants such that all others are expressed unambiguously in terms of the basic ones, then the basic set must include more than just $3n - 10$ elements, even though the dimension of the mass shell in the space of invariants is $3n - 10$. In fact, the mass shell in invariant space is a $(3n - 10)$ dimensional algebraic variety in an $(n^2 - 3n)/2$ dimensional space of basic invariants. Asribekov[2] has shown how to choose convenient basic sets of $(n^2 - 3n)/2$ scalar invariants. Recent works[3] on the n-particle Veneziano formula make use of these sets.

Toller[4] has expressed the n-particle amplitudes in terms of parameters of the Lorentz group acting on the external particles. In this space the mass shell is again an algebraic variety.

Thus from many points of view the mass shell is an algebraic variety. To define analyticity on an algebraic variety we first introduce the notion of an analytic submanifold.

C. Analytic Submanifolds

Let $\mathcal{M}_c^0$ be the points of $\mathcal{M}_c$ where all of the n four-vectors p_i are parallel. The set $\mathcal{M}_c - \mathcal{M}_c^0 \equiv W_c$ is called the "restricted" (complex) mass shell. It is an analytic submanifold in the space of the n four-vectors p_i. (All the masses m_i are here assumed to be strictly positive.)

An analytic submanifold is a set that is locally analytically equivalent to a flat space. Specifically, an analytic submanifold of an n-dimensional space of points $Z = (z_1, \cdots z_n)$ is a set $\mathcal{S}$ such that each point p of $\mathcal{S}$ has a (full) neighborhood U^p and set $G^p(Z) \equiv \left(g_1^p(Z), \cdots g_n^p(Z)\right)$ of n functions all analytic in U^p such that $J^p \equiv \partial G^p/\partial Z$ is nonzero at each point of U^p and such that $\mathcal{S}$ coincides inside U^p with the set defined by $g_1^p(Z) = g_2^p(Z) = \cdots g_m^p(Z) = 0$, where $m = m^p$ is some positive integer $\leqslant n$. Thus $G^p(Z)$ maps $\mathcal{S} \cap U^p$ analytically into the flat space $g_1{}^p = \cdots = g_m{}^p = 0$. The condition $J^p \equiv \partial G^p/\partial Z \neq 0$ ensures that the mapping is nonsingular.[5] That is, the inverse mapping $Z^p(G)$ is well-defined and analytic on the image $\tilde{U}^p$ of U^p.

The number m^p is the <u>codimension</u> of $\mathcal{S}$ at p, and $n - m^p$ is the <u>dimension</u> of $\mathcal{S}$ at p. The coordinates $(g_{m+1}{}^p, \cdots g_n{}^p)$ are called <u>local coordinates</u> of $\mathcal{S}$ at p. They are adjusted so that $Z^p(0) = Z_p$.

D. Analyticity on Analytic Submanifolds

If a function is defined near p only on points of an analytic submanifold $\mathcal{S}$ (of dimension $<n$) then the usual definition of analyticity breaks down, because the function is not defined on a full neighborhood of p. However, there is a completely natural generalization:[6] a function F defined near p only on an analytic submanifold $\mathcal{S}$ that contains p is said to be analytic at p if $F^p(G) \equiv F[Z^p(G)]$ is an analytic function at $G = 0$ of the local coordinates $g_{m+1}{}^p, \cdots g_n{}^p$ of $\mathcal{S}$ at p.

The fact that the mapping $G^p(Z)$ is nonsingular ensures that this definition of analyticity is independent of the particular set of local coordinates used to define analyticity at p.

This definition of analyticity involves only the values of F on the manifold $\mathcal{S}$. If there were a function $\tilde{F}(Z)$ defined in a full neighborhood of p of $\mathcal{S}$, and analytic at p, then the restriction of this function to $\mathcal{S}$ would clearly be analytic at p, since then $F^p(G) = \tilde{F}(Z^p(G))$ would be analytic in all the g_i^p at $G = 0$, and hence also in the local coordinates $g_{m+1}^p, \cdots g_n^p$.

Conversely, if F defined on $\mathcal{S}$ near p is analytic at p, then there exists an analytic function $\tilde{F}(Z)$ defined on a full neighborhood of p that is analytic at p and that coincides with F on $\mathcal{S}$ near p. For instance, one can trivially extend $F^p(G)$ to a function $\tilde{F}^p(G)$ defined and analytic in a full neighborhood of $G = 0$. Then the function $\tilde{F}(Z)$ defined by $\tilde{F}(Z) \equiv \tilde{F}^p(G^p(Z))$ has the required properties. Thus one can locally extend a function F defined only on $\mathcal{S}$ and analytic at p to function $\tilde{F}(Z)$ analytic at p. This extension is highly nonunique. And generally these various local extensions will not fit together to give a global extension.

The above remarks show, however, that the definition of analyticity given above is equivalent to the following one: F defined on $\mathcal{S}$ is analytic at p of $\mathcal{S}$ if there is an $\tilde{F}(Z)$ that is analytic in Z at p (in the usual sense), and that coincides with F on $\mathcal{S}$ in some finite neighborhood of p. (This does not imply, however, that there is a single global function $\tilde{F}(Z)$ defined on a full neighborhood of $\mathcal{S}$.)

E. Analyticity on the Mass Shell

At points of the restricted mass shell W_c the above definition is applicable. For the definition of analyticity on the mass shell $\mathcal{M}_c$ we give two possible candidates:

A function F defined only on $\mathcal{M}_c$ is said to be analytic in the "weak sense" at p of $\mathcal{M}_c$ if for some neighborhood $U_c^p \subset \mathcal{M}_c$ of p the function F is continuous on U_c^p and analytic on $U_c^p \cap W_c$.

A function F defined on only $\mathcal{M}_c$ is said to be analytic in the "strong sense" at p of $\mathcal{M}_c$ if F is the restriction to $\mathcal{M}_c$ of a function $\tilde{F}(Z)$ defined and analytic in a full neighborhood of p.

These two definitions are not obviously equivalent. But Hepp[7] has noted that in fact they are equivalent, due to a theorem by Oka.

Thus we may use either of these definitions.

F. Fundamental Properties

The definition of analyticity on $\mathcal{M}_c$ given above is the natural one. But is it useful? Do the usual consequences of analyticity carry over? Mathematicians have given a lot of attention to this problem, and the answer is yes. The fact that one does not have a global set of basic coordinates, but must generally use different sets of local coordinates at different points, does not disrupt things very much. In particular, the following properties hold:

1. Analytic Continuation. Analytic continuation on the variety is defined in essentially the usual way. This continuation is unique: it does not depend on the choices of local coordinates.

2. Cauchy-Poincaré Theorem. A contour integral over a smooth (m-n) real-dimensional contour in an m-n complex-dimensional analytic submanifold can be continuously distorted through a domain of

analyticity in this submanifold without changing its value, provided the boundary remains fixed. The contour has $(m-n)$ real dimensions, whereas the manifold has $2(m-n)$ real dimensions. The contour integral is defined by integrating in the local variables, but with the appropriately signed Jacobian $\partial Z/\partial G = J^{-1}$. For details see Ref. 8.

3. "Mandelstam" Representation

Let $\{S_i\}$ be a finite set of (nonconstant) polynomials in the (space $\underset{\sim}{C}^n$ of) n variables z_i. Let C_i be a curve in the S_i plane. Let $\mathcal{C}_i$ be the inverse image of C_i. (It will be a set of codimension one in $\underset{\sim}{C}^n$ Suppose V is an ℓ dimensional algebraic variety in $\underset{\sim}{C}^n$. And suppose F is defined only on V and is analytic at all points of $D \equiv V - \bigcup \mathcal{C}_i$. Let R_c be a large rectangular box in $\underset{\sim}{C}^n$. Then F on $D \cap R_c$ can be expressed as

$$F(Z) = \sum_{\lambda} F_{\lambda}(Z) + B(Z) \quad ,$$

where

$$F_{\lambda}(Z) = \int_{R_c} \Delta_{\lambda}(S') \, K(Z, S') \prod_{j=1}^{\ell} \frac{dS'_{i(\lambda,j)} \, (2\pi i)^{-1}}{S'_{i(\lambda,j)} - S_{i(\lambda,j)}(Z)}$$

and $B(Z)$ is a similar contribution from the boundary of R_c. The function $\Delta_{\lambda}(S')$ is the ℓ-fold multiple discontinuity at the intersection of the ℓ cuts $\mathcal{C}_{i(\lambda,j)}$, and $K(Z,S')$ is a known kernel.

This is the generalization of the Mandelstam representation to a function defined only on an algebraic variety. For more details see Ref. 9.

These properties show that the natural definition of analyticity introduced above is also a useful definition. For other useful properties see Ref. 10.

III. CLUSTER DECOMPOSITION AND LANDAU SURFACES

A. Cluster Decomposition

The n-particle amplitude is assumed to contain terms corresponding to different disjoint subsets of the n particles interacting only among themselves. This decomposition into terms corresponding to different clusters of particles interacting among themselves is called the cluster decomposition.

The term of an n-particle amplitude corresponding to all n particles interacting with each other is called the connected part of that amplitude. This same function is assumed to represent the interaction of this cluster of particles also when they occur in a larger reaction. A derivation of this cluster decomposition from physical requirements is given in Ref. 11.

B. Bubble Diagram Functions

The unitarity equation is

$$SS^{\dagger} = I \ , \tag{3.1}$$

or equivalently

$$S^{\dagger} = S^{-1} \ . \tag{3.2}$$

Insertion of the cluster decompositions of S and S^{-1} into equations like

$$SS^{-1} = I \tag{3.3}$$

and

$$S = SS^{-1}S \quad , \tag{3.4}$$

etc., decomposes the expressions on the two sides into sums of terms. These terms are conveniently represented by bubble diagrams. For example, a term in the right side of (3.4) for the case of four incoming and four outgoing particles is represented by the bubble diagram in Fig. 1.

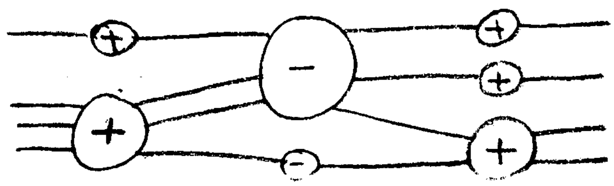

Fig. 1. A bubble diagram B representing a term F^B on the left side of (3.4), for the case of four incoming and four outgoing particles. The plus bubbles represent the cluster terms of S and the minus bubbles represent the cluster terms of S^{-1}. The lines represent mass-shell particles and there is a mass-shell integration over all internal lines.

Precise rules have been given[12] for converting each bubble diagram B into a well-defined function F^B. Equations that arise from the cluster properties of S and S^{-1} then take the form

$$\sum_{B \in \mathcal{C}} F^B = 0 \quad , \tag{3.5}$$

where $\mathcal{C}$ is a class of bubble diagrams.

C. Landau Singularities

Insertion of the cluster decompositions of S and S^{-1} into equations like (3.3), (3.4), and their generalizations, gives nontrivial equations of the form

$$S_c^{\pm} = \sum_{B \in \mathcal{C}} F^B \quad , \tag{3.6}$$

where $S_c^{\pm}$ is the connected part of $S^{\pm 1}$. These expressions for $S_c^{\pm}$ can be inserted in place of the corresponding factor on the right to give still other equations of this same form.

The conservation law of energy-momentum (2.2) is satisfied by the n-particle amplitude, and by its connected part.[13] Because of this constraint the various individual terms in the integral equations (3.6) for a given connected part will vanish below certain corresponding thresholds. But they will not in general vanish just above this threshold. Thus the individual terms in these expressions for the connected part must generally have singularities at these thresholds. These explicit singularities are known to be confined to Landau surfaces.[14]

It is generally the case that the explicit singularities of certain of the terms on the right of (3.6) cannot be balanced by the explicit singularities of the other terms on the right. This means that some of the $S_c^{\pm}$ must themselves have singularities, provided they are not all zero.

Maximal analyticity asserts that the functions $S_c^{\pm}$ have no physical-region singularities except those required by the equations (3.6). The statement of maximal analyticity given in Ref. 14 says more: it requires the singularities of $S_c^{\pm}$ to be confined to the union of the surfaces of the singularities explicitly appearing in the various terms of (3.6), and hence to the union of all Landau surfaces. Certain "$i\epsilon$ rules" were also required. These two conditions on the allowed singularities follow from causality requirements.

IV. CONSEQUENCES OF MACROCAUSALITY[18]

A. Positive-α Rule

The macrocausality condition itself will be discussed in the second lecture. One of its consequences is that the physical-region singularities of $S_c^{\pm}$ (divided by the conservation law δ) are confined to the positive-α branches of the Landau surfaces. These surfaces will be described in Section V.

This positive-α rule was obtained originally by Landau in his study of singularities of Feynman diagrams. In that context the positive-α rule arose from the $i\epsilon$ resolution of the singularity of the Feynman propagator function. That arose in turn from a causality requirement. The physical origin of the positive-α rule is thus the same for us as for Landau.

B. $i\epsilon$ Rule

Macrocausality also gives the important $i\epsilon$ rule. This rule assert that each connected part (divided by the conservation law delta function) can be represented as the limit of a unique analytic function. It moreover specify the set of allowed direction from which the limit is to be taken. This direction is, roughly speaking, the intersection of the "upper-half planes" associated with the various singularity surfaces that pass through the point. The precise definition is given later.

It should be noted that the various individual terms of the equations (3.6) that according to maximal analyticity generate the singularities of $S_c^{\pm}$ do not generally satisfy this $i\epsilon$ requirement. In particular, they generally vanish on one side of their threshold singularities but not on the other. Thus the functions on the two sides are certainly not parts of a single analytic function. The $i\epsilon$ rule therefore demands strong connections between the singularities of the various terms of (3.6).

V. THE ANALYTIC STRUCTURE OF L^+

A. The Landau Equations

The integrations in the definition of a bubble diagram function F^B are effectively constrained by the various mass-shell and conservation-law delta functions. A product of delta functions signifies that one should transform to a set of variables that contains the arguments of all the delta functions as independent variables, and then omit the corresponding integrations. The condition for such a set of variables to exist is that the corresponding "Landau equations" not be soluble.

Landau equations are associated with Landau diagrams. A Landau diagram D is a diagram that is topologically equivalent to a spacetime diagram representing a multiple-scattering process. Thus the vertices of D, which correspond to the individual scattering processes, are connected by lines corresponding to particles moving forward in time. This condition imposes a certain partial ordering requirement on the vertices: it must be possible to draw all the lines of D directed from right to left.

A real four-vector variable p_j is assigned to each line of D, and a real scalar variable α_j is assigned to each internal line of D. Then the Landau equations corresponding to D are the mass-shell constraints,

$$p_j^2 = m_j^2, \quad p_j^0 > 0 \quad , \tag{5.1}$$

for each line of D, the energy-momentum conservation law,

$$\sum_{\text{into } V} p_j = \sum_{\text{out of } V} p_j \quad , \tag{5.2}$$

for each vertex V of D, and the loop equation

$$\sum_{\text{loop}} \pm \alpha_j p_j = 0 \quad , \tag{5.3}$$

for each closed internal loop of D. The sign in (5.3) is plus if the line is directed along the loop, and minus otherwise.

The vector $\alpha_j p_j$ can be regarded as the spacetime distance between the creation and annihilation of particle j. The loop equation says that the sum of these vectors around any closed loop of the diagram must take one back to the starting point. The condition that α_j be positive is the condition that the particle moves forward in time.[19]

The Landau surface L(D) is the set of p_j corresponding to external lines of D such that the Landau equations for D have a solution with some $\alpha_j \neq 0$. The set $L^+(D)$ is the subset of L(D) such that the Landau equations can be satisfied subject to the additional conditions $\alpha_j \geqslant 0$.

B. The Basic Diagrams D_β

Suppose two vertices of some D are connected by a set consisting of more than one line. One can insert on these lines trivial extra vertices corresponding to forward scattering, without changing the Landau surface $L^+(D)$. Thus each point of $L^+(D)$ belongs to the $L^+(\hat{D})$ of an infinite number of different $\hat{D}$.

It is convenient to deal with diagrams that do not have these trivial extra vertices. Let a trivial part of a diagram D be a part consisting of internal lines that is connected to the rest of the diagram only at its initial and final vertex, but which has also other vertices. A diagram D having no trivial parts is called a basic diagram, and is denoted by D_β. Any D can be generated by inserting trivial parts into some D_β.

It is easy to see that

$$L^+ \equiv \bigcup L^+(D) = \bigcup L^+(D_\beta) \quad . \tag{5.4}$$

That is, every positive-α point lies on the $L^+(D_\beta)$ of some basic diagram D_β.

Only a finite number of D_β give $L^+(D_\beta)$ entering any bounded portion of the physical region,[20] but an infinite number of D have this property.

C. The Basic Surfaces $L_0^+(D_\beta)$

Let $\mathcal{M}_0$ be the points of $\mathcal{M}$ such that at least two initial p_j are parallel or at least two final p_j are parallel. Let $L_0^+(D_\beta)$ be the points of $L^+(D_\beta) - \mathcal{M}_0$ such that the Landau equations for $L^+(D_\beta)$ have no solution with some $\alpha_j = 0$. Then one can show[21] that

$$L^+ = \bigcup L_0^+(D_\beta) + \mathcal{M}_0 \quad . \tag{5.5}$$

The basic surfaces $L_0^+(D_\beta)$ are codimension-one analytic submanifolds of $\mathcal{M} - \mathcal{M}_0$.[21] That is, any point p on $L_0^+(D_\beta)$ has a neighborhood $U^p \subset \mathcal{M} - \mathcal{M}_0$ such that $L_0^+(D_\beta)$ coincides in U^p with the set $f = 0$, where f is an analytic function (of the local coordinates of $\mathcal{M}$) in U^p and satisfies $\nabla f \neq 0$ throughout U^p. Thus $L^+ - \mathcal{M}_0$ is the union of these simple surfaces, each having a well-defined normal, whose direction changes gradually as p moves on it.

This property of L^+ is important in the formulation of the $i\epsilon$ rules. Let the sign of f be chosen so that ∇f points in the direction that $L_0^+(D_\beta)$ would move if the internal masses m_i were formally increased by a common scale factor. Then the "upper half plane" corresponding to $L_0^+(D_\beta)$ at point p is defined by $\operatorname{Im} G(p') \cdot \nabla f(p) > 0$, where p' is a variable point near p, and $G(p')$ is the set of local coordinates of p'.

If only one basic surface $L_0^+(D_\beta)$ passes through p of $L^+ - \mathcal{M}_0$ then the $i\epsilon$ rule says that S_c^+/δ near p can be represented as the limit of a function analytic in a domain in the upper-half plane corresponding to $L_0^+(D_\beta)$ at p.

If there are several $L_0^+(D_\beta)$ that contain p, then the analyticity domain lies in the intersection of the corresponding upper-half planes, provided this intersection is nonempty.

There are, however, points p of $L^+ - \mathcal{M}_0$ such that this intersection is empty.

These points would give serious trouble, were it not for the independence property described below. For if the contours of integrations in the functions F^B were trapped between such surfaces then the functions F^B would be nowhere analytic, and the whole S-matrix analysis of singularities would break down.

The vital independence property, which allows one to get around this problem, asserts that S has a decomposition

$$S = \sum S[D_\beta] \quad , \tag{5.6}$$

where $S[D_\beta]$ has singularities only on $L^+(D_\beta)$. It can be shown[21] that for any single D_β the $i\epsilon$ rules corresponding to the basic surfaces of D_β^+ and all its contractions are mutually compatible: for any p of $L^+(D_\beta)$ these various basic surfaces L_0^+ at p have upper-half planes that have a nonempty intersection. The $i\epsilon$ rules are applied to the individual terms of (5.6).

The basis of the independence property is discussed at the end of the next section.

VI. THE FUNDAMENTAL THEOREM[16,22]

A. Assumptions

1. Positive-α Rule. The singularities of the connected parts of S/δ and S^{-1}/δ are confined to L^+ (δ is the conservation law delta function).

2. Independence Property. In any bounded portion R of the physical region the connected part of S [and similarly of S^{-1}] decomposes into a sum of terms $\overset{\pm}{S}[D_\beta]$, one for each basic diagram D_β such that $L^+(D_\beta)$ enters R, and the singularities of $\overset{\pm}{S}[D_\beta]/\delta$ in $R - \mathcal{M}_0$ are confined to $L^+(D_\beta)$.

3. $i\epsilon$ Rule. For any p in $L^+(D_\beta) - \mathcal{M}_0$ let $I_p^+[D_\beta]$ [resp. $I_p^-[D_\beta]$] be the (necessarily nonempty) intersection of the upper-half [resp. lower-half] planes corresponding to the basic surfaces L_0^+ through p that are associated with D_β and its contractions. Then the function $S^{\pm}[D_\beta]/\delta$ can be represented over $\mathcal{R} - \mathcal{M}_0$ as the limit $\operatorname{Im} p' \to 0$ of a function that is analytic (and unique) in the set

$$\{p' : \operatorname{Re} p' \equiv p \in R - \mathcal{M}_0,\ p' \in C_p \cap \delta_p\} \quad ,$$

where δ_p is a (sufficiently small) neighborhood of p. For p on $\mathcal{M} - L^+(D_\beta) - \mathcal{M}_0$ the set C_p is the whole space. For p on $L^+(D_\beta) - \mathcal{M}_0$ the set C_p is an open convex cone lying in $I_p^{\pm}[D_\beta]$. This cone can be made arbitrarily close to $I_p^{\pm}[D_\beta]$ in δ_p by taking δ_p sufficiently small. This representation is discussed in Ref. 22.

4. Boundedness. For any unit-normed individual particle wave functions ψ_i one has

$$|S[\psi_i]| \leqslant C ,$$

where "norm" is the L^2 norm of ψ. [This bound follows directly from the probability interpretation of S, the constant C being unity. This boundedness property is used to show that contributions to F^B from small neighborhoods of points corresponding to $\mathcal{M}_0$ go to zero like the volume of the neighborhoods.]

B. Consequences

Let B be any connected bubble diagram. Then F^B satisfies the following properties:

1. Generalized Positive-α Rule. The singularities of F^B/δ are confined to the union of the Landau surfaces $L^{\pm}(D_B)$. A D_B is a diagram constructed by inserting some Landau diagram D_b into each bubble b of B. The surface $L^{\pm}(D_B)$ is the Landau surface corresponding to D_B, with the additional stipulation that the Landau parameter α_j corresponding to a line j that comes from inside a bubble b of sign σ_b must satisfy $\sigma_b \alpha_j \geqslant 0$. The original lines of B, which lie outside all the bubbles b, have no such constraint.

2. Generalized Independence Property. Let B be the connected bubble diagram. In any bounded portion R of $\mathcal{M}$ the function F^B decomposes into a finite number of terms, one for each way a set of basic diagrams can be introduced into the set of bubbles b of B,

subject to the condition that these basic diagrams D_β give surfaces $L^+(D_\beta)$ that enter the regions allowed by energy conservation. The singularities of the term $F^B[D_B]$ corresponding to the diagram D_B are confined to $L^{\pm}(D_B)$.

3. Generalized $i\epsilon$ Rule. For any p of $L^{\pm}(D_B) - \mathcal{M}_0$ let $\{\alpha_i(p), p_i(p)\}$ be the internal parameters corresponding to a solution at p of the Landau equations corresponding $L^{\pm}(D_B)$. The corresponding "upper-half plane" is defined by $\text{Im}\ \sigma(p', p) > 0$, where

$$\sigma(p', p) \equiv \sum \alpha_i(p)[p_i(p') - p_i(p)] \quad .$$

Here p' is a variable point of $\mathcal{M}_c$ and $\{p_i(p')\}$ is any set of values of the internal p_i of D_B that satisfy the conservation law constraints corresponding to the external values p'. [The function $\sigma(p', p)$ will not depend on the particular way the $p_i(p')$ are chosen.]

Let $I_p[D_B]$ be the intersection of the upper-half planes corresponding to all solutions $\{\alpha_i(p), p_i(p)\}$ of the Landau equations of $L^{\pm}(D_B)$ at p. Let $\mathcal{M}_0[D_B]$ be the subset of points p of $L^{\pm}(D_B) - \mathcal{M}_0$ such that $I_p[D_B]$ is empty. Then for any bounded region R, F^B can be represented over $R - \mathcal{M}_0 - \mathcal{M}_0[D_B]$ as the limit $\text{Im}\ p' \to 0$ of a function that is analytic (and unique) in the set

$$\{p' : \text{Re}\ p' \equiv p \in R - \mathcal{M}_0 - \mathcal{M}_0[D_B],\ p' \in C_p \cap \delta_p\}$$

defined just as before, with D_B in place of D_β.

4. Generalized Boundedness. For any unit-normed individual particle wave functions ψ_i one has[13]

$$F^B[\psi_i] \leqslant C \quad .$$

C. The Justification of the Independence Property

The $i\epsilon$ rule derived from macrocausality in Ref. 18 applies specifically to points that lie on only one surface $L^+(D_\beta)$. The independence property allows this rule to be applied additively at points lying on several such surfaces.

This independence property, or at least the implied additivity of physical-region singularities associated with different D_β, is crucial to the S-matrix approach. It can be justified in several different ways.

One way to justify the independence property is to invoke (for the first time) maximal analyticity. The point is that the independence property is regenerated by (3.6), in the sense that the fundamental theorem ensures that the right-hand side of any equation (3.6) can be split into a well-defined set of terms $F(D_\beta)$, one for each D_β, such that the positive-α singularities of $F(D_\beta)$ are confined to $L^+(D_\beta)$: positive-α singularities corresponding to different D_β are separated into different terms.

There is a tacit assumption at this point that the non-positive-α singularities in (3.6) exactly cancel out, and thus do not affect the physical-region analytic structure. This cancellation assumption, though an extra technical assumption, is highly plausible, and it can very likely be proved.[23]

Another way to phrase the argument is to say that if one assumes the independence property, then one can derive from the cluster properties of S and S^{-1} explicit formulas for the discontinuities. These formulas imply that the singularities associated with different basic diagrams are independent. Therefore singularities that violate the independence property are not required by the cluster properties of S and S^{-1}. Hence maximal analyticity says they are not present.

The global decomposition demanded by the independence property is exhibited in Section VIII.

VII. THE GENERAL PHYSICAL-REGION DISCONTINUITY FORMULA[23]

The formula for the discontinuity of S around $L_0^+(D_\beta)$ is obtained by replacing each vertex V of D_β by the entire S matrix corresponding to V, and replacing the set of lines α joining each pair of vertices of D_β by S_α^{-1}, where $S_\alpha = P_\alpha S P_\alpha$ and $S_\alpha^{-1} S_\alpha = P_\alpha$. Here P_α is zero when acting on a set of lines of mass less than M_α, and is unity otherwise. The mass of a set of lines is the sum of the rest masses corresponding to those lines, and M_α is the mass of the set of lines α of D_β.

The action of S_α^{-1} on S is to effectively shift it onto another sheet. Thus the S_α^{-1} can be eliminated from the discontinuity formulas in favor of functions S on other sheets. The utility of this is not apparent, because one will ultimately want to eliminate these extra functions in favor of the physical ones, which is done by means of S_α^{-1}.

Notice also that the single S_α^{-1} generates the unphysical sheets of the S's corresponding to all the different choices of the other (free) variables of S: each S_α^{-1} generates the unphysical sheets of an infinite number of different functions S. Thus the operators S_α^{-1} provide the compact and convenient way to express the information about these sheets.

Finally the S_α^{-1} have simple $i\epsilon$ properties,[23] and the fundamental theorem can be readily extended to functions containing them. This provides the tool for treating singularities that lie at real points on the various unphysical sheets. One effectively expresses the functions on these sheets in terms of bubble diagram functions, to which the theorem applies.

VIII. TWO FEYNMAN-LIKE ON-MASS-SHELL FORMULAS FOR S

A. An Infinite-Series Expansion for S

The equation $SS^{-1} = I$ has the formal iterative solution

$$S = \sum (R^-)^n \quad , \tag{8.1}$$

where $R^{\pm} = \pm(S^{\pm 1} - I)$. If one converts this to bubble diagram functions one obtains[23]

$$S = \sum F^{B^-} \quad , \tag{8.2}$$

where the sum is over all topologically distinct bubble diagrams B^- all bubbles of which are nontrivial minus bubbles. (Trivial bubbles have exactly one incoming and exactly one outgoing line.) The nontrivial minus bubbles now represent the connected part of R^-.

B. A Finite Expression for S

There is one term in (8.2) for each Landau diagram D. If one groups together the terms corresponding to each basic diagram D_β one gets

$$S = \sum S[D_\beta] \quad . \tag{8.3}$$

The term $S[D_\beta]$ is obtained by replacing each vertex of D_β by the corresponding minus bubble and each set of lines α of D_β by $S^{(\alpha)}$, where $S^{(\alpha)}$ is the sum of terms of S of (8.2) such that the mass of the lines at all scattering stages is less than or equal to M_α, the mass of the lines of α. Thus $S^{(\alpha)}$ contains all the terms of S that contribute just above the normal threshold at M_α, but has no higher normal threshold singularities.

If one now combines all terms corresponding to D_β's that have the same structure s (i.e., that differ only by the values of the M_α), then the various $S^{(\alpha)}$ connecting the two vertices corresponding to α add to give S, which acts between the bubbles that represent these two vertices.

A typical term is shown in Fig. 2

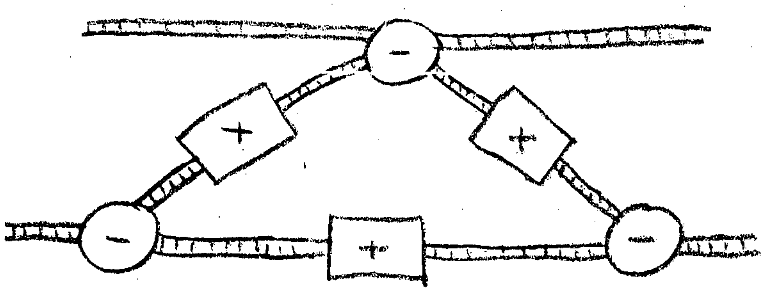

Fig. 2. The diagram representing a contribution to S corresponding to a triangle structure s. The shaded strips represent arbitrary sets of lines.

Only a finite number of different structures will contribute in any bounded portion of the physical region.

Lecture II. MACROCAUSALITY

Contents

I. THE PHYSICAL IDEAS

A. Classical Limit

Macrocausality requires that classical ideas and estimates become valid in classical limits. In ordinary quantum theory the passage to classical limits is ensured by the Schroedinger equation. In S-matrix theory there is no Schroedinger equation, and the correspondence to classical theory must be introduced in another way.

B. Finite-Range Interactions

Macrocausality also embodies the requirement that interactions be of finite range. Massless particles and their effects are thus ignored.

C. Causality in Classical Physics

What is "causality"? Philosophers have debated this at length. An example of what I mean is provided by a rock thrown at a window: the window breaks when the rock hits it, and not before. Another illustration is a set of moving billiard balls: they are deflected when they hit each other, and not before. The common phrasing is that the deflections are "caused" by the impacts. But the essential phenomenological fact is the existence of a relationship between changes of velocity, on the one hand, and spacetime separations on the other.

A quantitative description of the billiard ball situation is provided by Newton's laws of motion, as embodied by the laws of conservation of energy and momentum. Each ball is conceived to carry conserved energy-momentum, which is transmitted between them at impact. The size of the region of impact corresponds to the range of the interactions.

A notion of causality is contained in this picture: the relationship of things at an earlier time to things at a later time is understood in terms of the idea of particles that move from one region of impact to another. Their motions are restricted by the requirement that the conserved energy-momentum carried by the particles is transmitted between them only during their impacts.

This notion of causality based on Newton's laws of motion is the primitive causality notion in physics. It has predictive power, and this power would become exact in certain idealized limits.

This latter fact suggests a more restrictive notion of causality, namely that things at one instant of time exactly determine things at all later times.

Attempts to implement this second notion of causality lead to difficulties. One must introduce new "things", besides just the particles, to represent the exact situation at the instants of time during the interaction. These new things have no clear status in phenomena, because all attempts to measure them are disrupted by the interactions associated with the process of measurement. This physical ambiguity has attendant mathematical ambiguities, and it is not at all clear that phenomena actually have the structure ascribed to it by this notion of causality.

The S-matrix viewpoint is that one should not try to implement this restrictive notion of causality. Instead, one should try to determine the mathematical structure of correlations between measurements within the broader class of theories not restricted by requiring the existence of some preconceived type of representation of systems at instants of time. Also, one should focus attention on measurements for which a decoupling can be made between a measured system and the measuring devices.

Although in S-matrix theory the notion of causality based on instants of time is not enforced, one must not lose the macroscopic classical structure that underlies the primitive causality notion. Macrocausality is the requirement that this classical structure emerge in the appropriate macroscopic limit, and that moreover classical estimates become valid in this limit.

II. THE MATHEMATICAL IDEAS

A. Classical Limit

Consider first a freely moving particle of mass m. Let its momentum-space wave function be

$$\varphi_{\gamma\tau}(p) = \chi(p)\, e^{-\gamma\tau(\vec{p}-\vec{P})^2}\, \delta(p^2 - m^2)\, \theta(p^0) \quad , \tag{2.1}$$

where P is a mass-shell four-vector ($P^2 = m^2$), and $\chi(p)$ is an infinitely differentiable function of compact support. Suppose further that $\chi(p)$ is analytic in some neighborhood $N(P)$ of P, and is bounded by unity both in $N(P)$ and at real p. The class of all such $\varphi_{\gamma\tau}(p)$ will be called $\Omega\big(P, N(P)\big)$.

Consider next the translated function

$$\varphi_{\gamma\tau}^{u\tau}(p) = \varphi_{\gamma\tau}(p)\, e^{ip\cdot u\tau} \quad . \tag{2.2}$$

Its Fourier transform is

$$\tilde{\varphi}_{\gamma\tau}^{u\tau}(x) = \int \varphi_{\gamma\tau}^{u\tau}(p)\, e^{-ipx}\, dp \quad . \tag{2.3}$$

We shall be interested in the limit $\tau \to \infty$. Consider therefore the "scaled coordinate"

$$x' = x/\tau \quad . \tag{2.4}$$

The physical displacement $u\tau$ in x space is a constant displacement u in x' space.

The function $\tilde{\varphi}_{\gamma\tau}^{u\tau}(x'\tau)$ has an important property specified in the following theorem.

Theorem A. Let $\Gamma(P, u)$ be the line in x' space that contains u and has direction P. Let R be any closed bounded set in x' space that does not intersect $\Gamma(P, u)$. Let $\Omega \equiv \Omega(P, N(P))$ be some fixed class. Then there are three constants

$$C > 0, \quad \alpha > 0, \quad \text{and} \quad \gamma > 0$$

such that

$$|\tilde{\varphi}_{\gamma\tau}^{u\tau}(x'\tau)| < C\, e^{-\alpha\gamma\tau} \tag{2.5}$$

for all $\tau \geqslant 0$, all x' in R, all positive $\gamma \leqslant \gamma_0$, and all $\varphi_{\gamma\tau}$ in Ω.

The content of this theorem is this: the function $\tilde{\varphi}_{\gamma\tau}^{u\tau}(x'\tau)$, considered as a function of τ and x', collapses exponentially to the line $\Gamma(P, u)$ as $\tau \to \infty$.

The limit $\tau \to \infty$ corresponds to a classical limit. The momentum spread Δp goes to zero like $\tau^{-\frac{1}{2}}$, and the coordinate spread $\Delta x'$ also goes to zero like $\tau^{-\frac{1}{2}}$. The trajectory region in x'-space collapses to a classical trajectory, i.e., to the line $\Gamma(P, u)$.

In S-matrix theory the only role of Planck's constant is to fix the scale of space and time. That is, the x that occurs in $\exp ipx$ should be written as $x_{ph}/\hbar$, where x_{ph} is the physical spacetime coordinate and $\hbar$ is Planck's constant in the units used to measure x_{ph} and p. If one writes $x = x_{ph}/\hbar = x'\tau$, one sees that holding the physical coordinates fixed and taking the limit $\hbar \to 0$ is the same as holding x' fixed and taking $\tau \to \infty$. Looked at the other way, holding x' fixed and taking $\tau \to \infty$ expands the physics into the macroscopic domain.

Consider next a scattering process involving n particles. It is described by a corresponding transition amplitude

$$A = S[\varphi_1^*, \cdots \varphi_n] \quad . \tag{2.6}$$

Let each of these wave functions φ_i be replaced by a wave function of the form (2.2). Then A becomes [suppressing the unimportant dependence on the functions $\chi(p)$]

$$A^\tau \equiv A^\tau(P_1, P_2, \cdots P_n; u_1, u_2, \cdots u_n) \quad .$$

The limit $\tau \to \infty$, considered in x' space, gives a classical limit in which the ith (initial or final) particle has the spacetime trajectory $\Gamma(P_i, u_i)$.

B. Finite-Range Interactions

Any finite interval in the actual physical space x reduces to a point in $x' = x/\tau$ space, in the limit $\tau \to \infty$. Thus any finite-range interaction becomes a zero-range interaction in x' space.

Consider the classical multiple-scattering processes in x' space that would be allowed if only zero range interactions were allowed. They are represented by multiple-scattering diagrams D with point interaction vertices. For certain values of the set of arguments (P, u) of $A^{\tau}(P_1,\cdots P_n; u_1,\cdots u_n)$ there exists a multiple-scattering diagram D that would fit onto the corresponding external lines $\Gamma(P_i, u_i)$. The points (P, u) corresponding to these sets of values (of the 8n components) are called causal. The points (P, u) that are compatible with no multiple-scattering diagram are called noncausal. Most points (P, u) are noncausal, since generally the lines $\Gamma(P_i, u_i)$ will not intersect at all, and there will therefore be no possible initial vertex of D.

In these multiple-scattering diagrams D we allow internal lines corresponding to all possible physical particles. These lines are required to begin and end at point vertices, at which energy-momentum is conserved. The external lines are fixed by (P, u).

C. Macrocausality in S-Matrix Theory

Macrocausality requires that classical ideas and estimates become valid in the limit $\tau \to 0$. To make the classical estimates one regards $|A^{\tau}|^2$ as the probability function for a statistical ensemble of classical scattering experiments. The coordinate-space and momentum-space wave functions of the initial particles define the classical momentum-space and coordinate-space distributions of the statistical ensembles of initial particles, and the wave functions for the final particles represent the corresponding detection efficiencies.[11] The estimates are made by summing the properly weighted contributions of all the possible classical scattering processes. These scattering processes are required to proceed via finite-range interactions: each particle is conceived to carry conserved energy-momentum, which can be transmitted between them by interactions whose effects are exponentially damped under spacetime dilation.

III. THE MACROCAUSALITY CONDITION

The general macroscopic causality requirement has the following specific consequence: Let $P \equiv (P_1, \cdots P_n)$ and $u \equiv (u_1, \cdots u_n)$ represent the arguments of A^τ. Fix P and let $\mathcal{Q}(P)$ be the set of all u such that (P, u) is noncausal. Let $\hat{\mathcal{Q}}$ be some closed, bounded subset of $\mathcal{Q}(P)$. Let the classes $\Omega_i \equiv \Omega(P_i, N_i(P_i))$ be fixed. Then there are three constants $C > 0$, $\alpha > 0$, and $\gamma > 0$ such that

$$|A^\tau| < C\, e^{-\alpha\gamma\tau} \tag{3.1}$$

for all $\tau \geqslant 0$, all $0 < \gamma < \gamma_0$, all positive $\gamma < \gamma_0$, and all $\varphi_{\gamma\tau}(p_i)$ in Ω_i.

To get this condition one first uses the bound $|A^\tau| \leqslant 1$, which follows from the probability interpretation, to reduce the problem to that of the asymptotic behavior.

For any u in $\hat{\mathcal{Q}}$ it is, by definition, impossible to find a classical process corresponding to P. Thus every one of the processes contributing to $|A^\tau|^2$ must fail in some way to satisfy all the classical conditions.

If one allows the energy-momentum to be transmitted over any finite distance Δ in x' space by any finite-range interaction, then the contribution from this process must contain a factor $e^{-a\Delta\tau}$, where $a > 0$ is some positive constant. Thus for some sufficiently small α the contribution from such a term would satisfy the required bound.

There might be a process D for which the momenta of the initial and final particles of the statistical ensembles are not exactly those given by P. But then there would be a corresponding factor $e^{-\gamma\tau\sum(\Delta p_i)^2}$ coming from the wave functions $\varphi_{\tau\gamma}(p)$.

Another possibility is that the space position of some interaction is not on the corresponding line $\Gamma(P_i, u_i)$, which defines the center trajectory of the statistical ensemble corresponding to particle i. In this case the required factor comes from Theorem A.

For most processes all three of these things will happen. But continuity properties ensure that the minimum of the corresponding constants α in (3.1) is some strictly positive $\alpha > 0$. The bound (3.1) with this α holds for all processes, and hence for their weighted average.

The macrocausality condition used in Ref. 18 includes, in addition to (3.1), an analogous condition for $\gamma = 0$. The fall-off in this case is no longer exponential, but merely rapid (i.e., faster than any inverse power of τ). This rapid fall-off comes from a rapid fall-off property of $\tilde{\varphi}_{o\tau}^{u\tau}(x'\tau)$, analogous to (2.5), that arises from the infinite differentiability of $\chi(p)$.

Notice that the macrocausality condition refers only to noncausal situations; it gives a bound in cases where no classical process can occur. Since it says nothing about cases where a classical process can occur, it is considerably weaker than the requirement stated in the preceding chapter.

IV. CONCLUSIONS

As discussed in the first lecture, the macrocausality condition implies the positive-α rule and the $i\epsilon$ rule.[18] And conversely, these analyticity properties imply the macrocausality condition.[18]

The macrocausality condition refers only to noncausal points; it says nothing about situations in which classical processes are possible. Correspondingly, the positive-α and $i\epsilon$ rule do not assert that any singularity actually exists; they merely allow certain singularities to exist.

On the other hand, the cluster properties of S and S^{-1} require the functions $S^{\pm}/\delta$ to have singularities.

The main result described in the first lecture is essentially this: if one permits only those singularities of $S^{\pm}/\delta$ allowed by macrocausality, then the equations that follow from the cluster decomposition of $SS^{-1} = I$, and the similar identities, require all of the allowed singularities to actually be present. That is, discontinuity formulas are derived for all allowed singularities, and these generally give a nonzero discontinuity.

It is interesting that, although the macrocausality requirement is invoked only for noncausal points, the derived discontinuity formulas are in complete harmony with macrocausality requirements at causal points. That is, the discontinuity formulas are such as to give a spacetime structure to macroscopic phenomena that is in complete accord with the multiple-scattering picture of classical physics.[11]

Thus from the cluster properties plus the macrocausality condition associated with noncausal points we have deduced the complete general physical-region analytic structure, and also the multiple-scattering structure of macroscopic spacetime phenomena.

The reciprocal manner in which the various ideas lead to each other permits the assumptions to be stated in various ways. In the above discussion the cluster properties and macrocausality for noncausal points were taken as the starting points. The cluster properties themselves can be derived directly from very weak macroscopic spatial cluster properties.[11] Thus the basic assumptions are simply macroscopic spacetime fall-off properties that embody the requirement that interactions have finite range.

A slightly different way of stating the assumption is to say that all physical-region singularities are associated with particle processes: S-matrix theory is essentially a particle theory (as opposed to a field theory) based on analyticity, and the basic assumption is that the only singularities are those associated with particles. This requirement means specifically that all physical-region singularities are positive-α singularities that obey the plus $i\epsilon$ rule. The justification for this specific interpretation of the general statement (that all singularities are associated with particle processes) is placed on a sound basis by the collective results described in these lectures.

Once the allowed singularities are restricted to those associated with particle processes, then the cluster properties actually <u>require</u> these singularities to be present. Thus we find that the only singularities actually present are those absolutely required by the cluster properties. And this property is by itself in a sense sufficient. For if the cluster properties can be satisfied with just these singularities, then these alone can be required by the cluster properties.

The formulation of the analyticity assumption of S-matrix theory as the statement that the only singularities are those actually required by the cluster properties (together with other principles such as unitarity, etc.) has a certain philosophic appeal. But from a technical standpoint it raises difficulties. For it presupposes that the solution is unique. But the tremendous complexity of the entire problem makes it unlikely that man will ever succeed in actually proving uniqueness in the full global sense. This leaves one in eternal doubt as to the exact content of the basic assumption.

The practical problem is never to prove uniqueness in the full global sense. Rather it is to understand the mathematical structure of observed phenomena. One starts with certain known aspects of the phenomena, and attempts to correlate these aspects with others, some of which may be unknown. The basic assumption in actual practice is that all singularities are associated with particle processes. This form of the assumption, which is given a well-defined meaning in the physical region by the analysis of macrocausality requirements, can be extended in a natural way to nonphysical-region singularities associated with unstable particles.

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