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INCIDENT RATIONAL CURVES

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ABSTRACT. We study families of rational curves on an algebraic variety satisfying incidence conditions. We prove an analogue of bend-and-break: that is, we show that under suitable conditions, such a family must contain reducibles. In the case of curves in $\mathbb{P}^n$ incident to certain complete intersections, we prove the family is irreducible.

Since the seminal work of Mori and Miyaoka [4] and [3], rational curves on algebraic varieties, especially Fano manifolds, have been much studied. In particular Harris and his school (see for instance [2], [8], [9] and references therein) have studied the case of rational curves on general Fano hypersurfaces, with particular attention to the question of dimension and irreducibility of the family of curves of given degree.

Our interest here is in families of rational curves on a given variety X that are incident to a fixed subvariety Y, i.e. meet Y in a specified number of (unspecified) points. This on the one hand generalizes bend-and-break, which is the case where Y consists of 2 points, and on the other hand is related to rational curves on hypersurfaces, thanks to the fact (see [7]) that a hypersurface X_d of degree d in $\mathbb{P}^n$ admits a 'nice' degeneration (with double points only and smooth total space) to the union of a hypersurface of degree $d-1$ with the blowup of $\mathbb{P}^{n-1}$ in a complete intersection subvariety Y of type $(d-1, d)$, and rational curves on X_d are thereby related to rational curves in $\mathbb{P}^{n-1}$ meeting Y in a specified number of points.

Here in §1 we present two kinds of results of bend-and-break type (arbitrary ambient space). In the first, we make some disjointness conditions on the incident subvarieties, for example (see Theorem 1) a pair of disjoint subvarieties Y_1, Y_2 meeting the curves in question in 1 (resp. 2) points. In the second result (see Theorem 5) we assume given an 'overfilling' family, i.e. one having at least ∞^1 members through a point of the ambient space, together with a subvariety, possibly reducible, meeting the curves in 2 points.

In §2 we specialize to the case of curves of given degree e in $\mathbb{P}^n$, $n \geq 3$, that are a times incident to a fixed general complete intersection of type (c, d) with $a \leq e$ and

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$c+d\leq n$. We prove in this case that the family is irreducible and its general member is well behaved.

In this paper we will work over $\mathbb{C}$ (though the results probably hold over an arbitrary algebraically closed field, at least if resolution of singularities is known through dimension k).

1. INCIDENTAL BEND-AND-BREAK

Notations and conventions. The following set-up will be in effect throughout this section.

(i) X is an irreducible projective variety of dimension $n\geq 3$;
(ii) $\pi:\mathcal{C}\to B$ is a proper flat family over an irreducible projecive base variety of dimension $k\geq n-1$, with fibres $C_b=\pi^{-1}(b)$, so that for general b, C_b is a nonsingular rational curve;
(iii) $f:\mathcal{C}\to X$ is a surjective morphism that has degree 1 on a general fibre of π.

A family as in (iii), i.e. such that f is surjective, is said to be *filling*. If in addition $\dim(B)\geq n$, so that through a point $x\in X$ there are at least ∞^1 curves $f(C_b)$, it is said to be *overfilling*.

Theorem 1. *Under notations and conventions as above, assume moreover there are subvarieties $Y_1,Y_2\subset X$ of respective codimension at least 1 (resp. at least 2) with $Y_1\cap Y_2=\emptyset$, such that for general $b\in B$, $f(C_b)$ meets Y_1 (resp. Y_2) in at least 1 (resp. at least 2) points.*

Then the family $\mathcal{C}/B$ has a reducible fibre C_b.

Proof. Assume for contradiction all fibres C_b are irreducible. With no loss of generality we may assume $\dim(B)=n-1$. After suitable base-change we may assume B is smooth. Actually the argument below will use only a general curve-section of B, so it's enough to assume B normal. Let L be a very ample line bundle on B and set $H=f^*(\mathcal{O}_X(1))$ where $\mathcal{O}_X(1)$ is a very ample line bundle on X. Then I claim that after a further base-change we may assume that

$$\mathcal{C}=\mathbb{P}(E)$$

where E is a rank-2 vector bundle on B: indeed, if base-change enough so that π admits a section $D\subset\mathcal{C}$, we can take $E=\pi_*(\mathcal{O}(D))$. Subsequently, after a further base-change, we may assume that $\wedge^2E$ is divisible by 2 in the Picard group, hence, after a suitable twist, we may assume $\wedge^2E=\mathcal{O}_B$ and in particular, as divisors,

$$c_1(E)\equiv_{\text{num}}0. \quad (1)$$

We will henceforth take c_1 to have values in the Neron-Severi group, so $c_1(E) = 0$. By assumption, we have 'multisections' $S'_i \subset \mathcal{C}$, i.e. possibly reducible subvarieties $S'_i \subset \mathcal{C}, i = 1, 2$, generically finite of degree at least 1 (resp. at least 2) over B such that

$$f(S'_i) \subset Y_i, i = 1, 2.$$

Base-changing via $S'_i \to B$, the pullback of S'_i splits of a section. Then after a further base-change, we may assume we have 3 distinct *sections* S_1, S_2, S_3 such that

$$f(S_1) \subset Y_1, f(S_2), f(S_3) \subset Y_2.$$

Each section S_i corresponds to an exact sequence

$$0 \to P_i \to E \to Q_i \to 0,$$

where $c_1(P_i) = -c_1(Q_i)$ thanks to $c_1(E) = 0$. Since $Y_1 \cap Y_2 = \emptyset$, it follows that

$$S_1 \cap S_2 = S_1 \cap S_3 = \emptyset \tag{2}$$

and hence

$$c_1(P_1) = c_1(Q_2) = c_1(Q_3). \tag{3}$$

For $i = 2, 3$ set $Z_i = f(S_i), m_i = \dim(Z_i) \leq n - 2$. Note that each of S_2, S_3 collapses under f, i.e. while S_i has codimension 1 in $\mathcal{C}$, Z_i has codimension 2 or more in $f(\mathcal{C}) = X$. Identifying S_i with B, let

$$f_i : B \to Z_i$$

be the resulting map, and let F_i be a general fibre of f_i which has codimension m_i. Note that

$$H^{m_i}.S_i \sim \deg(Z_i)F_i. \tag{4}$$

Now, we have

$$H^{m_i+1}S_i\pi^*(L)^{k-m_i-2} = 0, \tag{5}$$

while, by surjectivity of f,

$$H^{m_i+2}\pi^*(L)^{k-m_i-2} > 0. \tag{6}$$

Therefore the Hodge index theorem implies that

$$H^{m_i}S_i^2\pi^*(L)^{k-m_i-2} < 0. \tag{7}$$

Now as S_i is a section, we have

$$\mathcal{O}_{S_i}(S_i) = 2Q_i.$$

In view of (4), (7) means

$$\deg(Y_i)F_i.c_1(P_i)L^{k-m_i-2} > 0, \tag{8}$$

so we may assume

$$F_i.c_1(P_i).L^{k-m_i-2} > 0, i = 2, 3. \tag{9}$$

Now, since the sections S_i are pairwise distinct, the natural map $P_i \to Q_j$ must be nontrivial, hence injective, for all $i \neq j$, hence $c_1(P_2)$ has negative degree on a general curve section of F_3. Thus

$$c_1(P_2).F_3.L^{k-m_i-2} < 0. \tag{10}$$

But this obviously contradicts (3). □

Remark 2. The situation of Theorem 1 is not a priori amenable to the usual bend-and-break because there is not necessarily a curve $f(C_b)$, much less a 1-parameter family of such, through given points $y_1 \in Y_1, y_2 \in Y_2$.

Remark 3. The last part of the proof above an be shortened somewhat by restricting to a 1-parameter subfamily going through a fixed point of Y_2, which allows us to assume that f contracts S_3 to a point while B is 1-dimensional. Then the disjointness condition 2 implies that S_2 and S_3 are numerically equivalent. This, together with the fact that S_2 and S_3 are distinct and S_3 is contracted, easily yields a contradiction.

The hypotheses of Theorem 1 afford tweaking in various ways, for example the following.

Theorem 4. *Under Notation and Conventions as in the Introduction, assume given subvarieties* $Y_1, Y_2, Y_3 \subset X$ *meeting a general* $f(C_b)$ *such that*

(i) each Y_i *has has codimension 3 or more;*

(ii) $\dim(Y_2 \cap Y_3) + \dim(Y_1) \leq n - 3$*;*

(iii) $Y_1 \cap Y_2 \cap Y_3 = \emptyset$*;*

(iv) the subfamily of B *consisting of curves* $f(C_b)$ *that are contained in* $Y_2 \cap Y_3$ *is of codimension* > 2.

Then there is a reducible fibre C_b*.*

Proof. We may assume each Y_i corresponds to a section S_i of $\mathcal{C}/B$, which in turn corresponds to an exact sequence

$$0 \to P_i \to E \to Q_i \to 0, i = 1, 2, 3.$$

If $S_2 \cap S_3 = \emptyset$ then $c_1(Q_2) = c_1(P_3)$ and we easily get a contradiction as above because both P_2 and P_3 inject into Q_1.

Now suppose $S_2 \cap S_3 \neq \emptyset$. If $\pi(S_2 \cap S_3)$ has dimension $< n-2$, it yields an $(n-3)$-dimensional family entirely contained in $Y_2 \cap Y_3$, against our hypotheses. Hence $S_2 \cap S_3$ projects to an $(n-2)$-dimensional subfamily $B' \subset B$ and the restricted family $\mathcal{C}'/B'$ has desjoint sections corresponding to S_1 and S_2 which get contracted to Y_1 and $Y_2 \cap Y_3$ respectively. By Assumption (ii) this family has ∞^1 members through a pair of fixed points on Y_1 and $Y_2 \cap Y_3$, so standard bend-and-break applies.

□

Next we give a bend-and-break type result for overfilling families.

Theorem 5. *Under Notations and Conventions as in the Introduction, assume further*
(i) there is a subvariety $Y \subset X$ of codimension 2 or more such that a general $f(C_b)$ meets Y in 2 or more points;
(ii) $\dim(B) \geq n$.
Then there is a reducible fibre C_b.

Proof. We begin as in the proof of Theorem 1, arguing for contradiction and base-changing so that $\mathcal{C} = \mathbb{P}(E)$ with $c_1(E) = 0$ numerically, and so that we have 2 sections S_1, S_2 collapsing to Y. Let $P_i \subset E$ be the line subbundle corresponding to S_i as before and let F_0 be a component of a general fibre of f over X. Note $\dim(F_0) = \dim(B) + 1 - n \geq 1$; replacing B by a suitable subfamily we may assume $\dim(F_0) = 1$.

We next aim to show that the subfamily family of curves going through a fixed point of X is disjoint from that where the sections S_1 and S_2 meet (i.e., informally speaking, where the curves are 'tangent' to Y). To this end, I claim now that $\pi^*(P_i).F_0 = 0, i = 1, 2$. To see this let F_i be a component of a general fibre of $f|_{S_i}$. Thus $\dim(F_i) \geq 2$. Then using Hodge index as above we see that for any ample line bundle A on B,

$$\pi^*(A)^{\dim(F_i)}\pi^*(P_i).F_i > 0.$$

Since F_i lies on S_i which projects isomorphically to B, this implies that $\pi^*(P_i)|_{F_i}$ is $\mathbb{Q}$-effective. Since F_i is filled up by curves algebraically equivalent to F_0, it follows that

$$\pi^*(P_i).F_0 \geq 0, i = 1, 2.$$

Now as S_1 and S_2 are distinct, the composite of the injection $P_1 \to E$ and the projection $E \to Q_2$ yields an injection $P_1 \to Q_2$ and thus $Q_2 - P_1$ is effective. Hence, as F_0 is a general fibre, we have

$$0 \leq \pi^*(P_1).F_0 \leq \pi^*(Q_2).F_0 \leq 0.$$

Thus $P_i.F_0 = 0, i = 1, 2$, as claimed. Since H^n is a positive multiple of F_0, we have

$$\pi^*(P_i).H^n = 0, i = 1, 2. \tag{11}$$

Now recall the injection of invertible sheaves

$$P_1 \to Q_2.$$

Its zero locus, which is numerically $Q_1 + Q_2 = -(P_1 + P_2)$, is just the locus of points in the base B over which the sections $S_1 = \mathbb{P}(Q_1)$ and $S_2 = \mathbb{P}(Q_2)$ intersect, i.e.

$$Q_1 + Q_2 \equiv_{\text{num}} \pi_*(S_1 \cap S_2).$$

Since we know

$$H^n.(\pi^*(Q_i + Q_j)) = 0,$$

we conclude that

$$f(\pi^{-1}\pi(S_1 \cap S_2) \subsetneq X.$$

This means exactly that the locus of curves going through a general point of X is disjoint from that where S_1 and S_2 intersect.

Now we can easily conclude. Let $x \in X$ be general, and let $B_1 \to B$ be a component of the the normalization of $\pi(f^{-1}(x))$, which we may assume is a smooth curve. Let $\mathcal{C}_1/B_1$ be the pullback $\mathbb{P}^1$ bundle. Then $\mathcal{C}_1$ is endowed with 3 pairwise disjoint sections, namely S_1', S_2' corresponding to S_1, S_2 (disjoint because $x \notin f(\pi^{-1}\pi(S_1 + S_2)))$, plus a section T contracting to x. But this is evidently impossible: writing the corresponding rank-2 bundle on B_1 as $A_1 \oplus A_2$ corresponding to S_1', S_2', the subbundle corresponding to T is isomorphic on the one hand to A_1, on the other hand to A_2, hence $A_1 \simeq A_2$ and $\mathcal{C}_1$ is a product bundle, which has no contractible sections. Contradiction.

□

Example 6. See [7] for context and motivation. Let Y be a smooth codimension-c subvariety of $\mathbb{P}^n, c \geq 2$. Any component B of the family of rational curves meeting Y in a points is at least $(e+1)(n+1) - 4 - a(c-1)$-dimensional. Assume $2 \leq a \leq e$ and that a general curve in B is smooth. For a general curve C in the family, the normal bundle to C in $\mathbb{P}^n$ is a quotient of a sum of copies of $\mathcal{O}(e)$, hence it is a direct sum of line bundles of degrees $\geq e$. Therefore since $a \leq e$, the secant bundle N_C^s is semipositive, hence the family is filling. Hence it follows from Theorem 5 that B will parametrize some reducible curves.

It is shown in [7] that when Y is a $(d-1, d)$ complete intersection, there is only one component B as above, i.e. the family is irreducible, which also implies the existence of reducibles in this case. Another irreducibility result is given in the next section.

2. IRREDUCIBILITY

For integers e, a, n and a smooth subvariety $Y \subset \mathbb{P}^n$, we denote by $M_e(a, Y)$ the projective scheme parametrizing triples $(f, C, \mathfrak{a})$ where (f, C) is in the Kontsevich space of stable maps $f : C \to \mathbb{P}^n$ with C of genus 0 and no marked points, and $\mathfrak{a}$ is a length-a subscheme $\mathfrak{a} \subset f^{-1}(Y)$ (see [1]). Our purpose is to prove

Theorem 7. *Assume Y is a general complete intersection of type (c, d) with $c + d \leq n, n \geq 3$. Then for all $a \leq e$, $M_e(a, Y)$ is irreducible of dimension $(e+1)(n+1) - 4 - a$ and for its general point C is $\mathbb{P}^1$, f has degree 1. and $f^{-1}(Y) = \mathfrak{a}$. Moreover if $a \leq e - 1$ or $c + d \leq n - 3$, f is an embedding.*

Proof. The idea is to degenerate Y to

$$Y_0 = (H_1 \cup ... \cup H_c) \cap (H_{c+1} \cup ... \cup H_{c+d})$$

where $H_0, ..., H_n$ form a basis for the hyperplanes in $\mathbb{P}^n$. We will prove first that the assertions of the Theorem, except for the irreducibility, which is false, hold for Y_0 in place of Y.

Write

$$\mathfrak{a} = \sum_p \mathfrak{a}_p \tag{12}$$

where $p \in C$ are distinct and $\mathfrak{a}_p$ is supported at p and has length a_p with $\sum a_p = a$. We begin by analyzing the case where C is irreducible, i.e. $C = \mathbb{P}^1$. In that case f amounts to an $(n+1)$ tuple of e-forms:

$$f = [\phi_0, ..., \phi_n], \phi_i \in H^0(\mathcal{O}_C(e))$$

defined up to a constant factor and up to reparametrization. Here $\phi_i = f^*(H_i)$. Because any component of $M_e(a, Y_0)$ has codimension at most a in the space of all maps, while it is $e + 1 > a$ conditions for any ϕ_i to vanish, so we may assume all $\phi_i \neq 0$, i.e. $f(C)$ is not contained in any coordinate hyperplane H_i. Also, the conditions on f to appear below will involve ϕ_i only for $i > 0$, so the general $(n+1)$-tuple satisfying them will have no common zero, making the corresponding rational map a morphism.

To each p appearing in (12) we associate index-sets

$$I(p) \subset [1, c], J(p) \subset [c+1, c+d]$$

with

$$f(p) \in (\bigcup_{i \in I(p)} H_i) \cap (\bigcup_{j \in J(p)} H_j).$$

Then

$$a_p \leq \min(|I(p)|, |J(p)|). \tag{13}$$

The condition (13) means that

$$\sum_{i \in I(p)} \mathrm{ord}_p(\phi_i) \geq a_p, \sum_{i \in J(p)} \mathrm{ord}_p(\phi_i) \geq a_p, \tag{14}$$

which amounts to $2a_p$ conditions on f: namely, if L_p denotes a linear form with zero set p, that

$$L_p^{a_p} | \prod_{i \in I(p)} \phi_i, L_p^{a_p} | \prod_{i \in J(p)} \phi_i,$$

and for different points p these conditions are linearly independent. In fact these conditions define a union of linear spaces each of which is of the form

$$\{(\phi_.) : \mathrm{ord}_p(\phi_i) \geq b_{p,i}, \forall i \in I(p) \cup J(p)\},$$

where the $(b_{p,.})$ is sequence of nonnegative integers satisfying

$$\sum_{i \in I(p)} b_{p,i} = a_p, \sum_{i \in J(p)} b_{p,i} = a_p.$$

Thus it is $2a$ conditions to map a given subscheme $\mathfrak{a}$ to Y_0 and $2a - r, r = |\mathrm{supp}(\mathfrak{a})|$ conditions on f to map some unspecified subscheme of type (a_p) (i.e. isomorphic to $\mathfrak{a}$ as above) to Y_0. Since $a \geq r$ with equality iff $\mathfrak{a}$ is reduced, it follows that $f(C)$ is transverse to Y. Also, an easy dimension count shows that f cannot have degree > 1 to its image. Moreover, via multiplication by $\mathfrak{a}$, $\mathcal{O}_C(e - a) \to \mathcal{O}_C(e)$, the linear system corresponding to f contains $n + 1$ unrestricted sections of $\mathcal{O}_C(e - a)$, which is very ample if $a < e$. Finally if $c + d \leq n - 3$ the system contains 4 or more unresticted sections of $\mathcal{O}(e)$, namely $\phi_0, \phi_{c_d+1}, ...\phi_n$, so again it is very ample. Thus, we have shown the Theorem holds for the part of $M_e(a, Y_0)$ corresponding to irreducible curves.

Next we analyze the case where C has nodes. Having a node is already 1 condition on f so it suffices to proves that having a length-a subscheme map into Y_0 is at least a further conditions. The map f may be viewed as a projection of a rational normal tree in $\mathbb{P}^e$, (connected) union of rational normal curves in their respective spans. The foregoing analysis goes through unchanged for points p that are smooth on C, so suppose p is a node, with local branch coordinates x, y. The structure of $\mathfrak{a}_p$ is well understood (see [5]) and in any case $\mathfrak{a}_p$ contains a subscheme $\mathfrak{a}'_p = Z(x^\alpha, y^\beta)$ with $\alpha, \beta > 0$ and $\alpha + \beta \geq a_p$. Analyzing as above, it is at least $2(\alpha + \beta - 1)$ conditions on f to map $\mathfrak{a}'_p$, hence $\mathfrak{a}_p$, into Y_0, and this is $> \mathfrak{a}_p$ unless $\alpha = \beta = 1$. In the latter case, if $\mathfrak{a}_p = \mathfrak{a}'_p$ then $\mathfrak{a}_p = Z(x, y)$ has length 1 while the number of conditions is 2. Finally, assume $\mathfrak{a}'_p \neq \mathfrak{a}_p$. This means $\mathfrak{a}_p$ is a tangent vector, i.e. a length-2 locally principal subscheme of the form $\mathfrak{a}_p = Z(x + ty), t \neq 0$. Note we may assume $I(p), J(p)$ are singletons, or else f must map a node of C to a proper substratum of Y_0 (of dimension $n - 3$ or less), which is at least 3 conditions. Then the condition on f to map $\mathfrak{a}_p$ into Y_0 is first that it must map p, a node of C, to the top stratum of Y_0 (2 conditions), and then the (2-dimensional) Zariski tangent space to $f(C)$ at $f(p)$ must be non-transverse to Y_0, which is 3 conditions in total. This completes the proof of the Theorem, except for the irreducibility, for Y_0 and hence for Y.

Note that the components of $M_e(a, Y_0)$ are of the form $M_e(a., i., j.)$ where for each k, $1 \leq i_k \leq c < j_k \leq d$ and $\sum a_k = a$, and the general curve in $M_e(a., i., j.)$ has a_k points on the top stratum that is open dense in $H_{i_k} \cap H_{j_k}$ for each k. Thus $M_e(a, Y_0)$ is highly reducible. Anyhow for such a curve C, the normal bundle N (strictly speaking, the normal bundle to the map f) is a quotient of a sum of line bundles of degree e hence is itself a sum of line bundles of degree e or more. Consequently, thanks to the condition $a \leq e$, the 'secant bundle' N^s, which parametrizes deformations preserving the incidence to Y (cf. [6]), is semipositive.

Note that each $M_e(a., i., j.)$ contains curves of the form $C' \cup_x L$, where C' is general in $M_{e-1}(a'., i'., j'.)$ where $(a'., i'., j'.)$ is obtained from $(a., i., j.)$ by replacing a single a_k by $a_k - 1$ (and omitting (a_k, i_k, j_k) if $a_k = 1$), and L is a line joining a general point $x = f(p)$ on $f(C)$ with a general point of $H_{i_k} \cap H_{j_k}$. Indeed such a curve is clearly unobstructed as secant thanks to the fact that $N^s_L(-x)$ and $N^s_{C'}(-p)$ are both sums of line bundles of degree -1 or more, hence have vanishing H^1.

With that said, the irreducibility of $M_e(a, Y)$ follows easily by induction on e, the case $e = 1$ being trivial thanks to Y itself being irreducible (this is where we use $n \geq 3$): let B be an irreducible component of $M_e(a, Y), a \leq e$, and consider its limit B_0 in $M_e(a, Y_0)$, which is a sum of components $M_e(a., i., j.)$. Because each of these contains an unobstructed curve of type $C'_x \cup L$ as above, B contains a similar curve of the form $C''_x \cup L$ with $C'' \in M_{e-1}(a-1, Y)$ and since the latter family may be assumed irreducible, B is unique so $M_e(a, Y)$ is irreducible. □

Remark 8. The low-degree hypothesis on Y does not seem necessary for irreducibility; on the other hand absent some upper bound on a like $a \leq e$, $M_e(a, Y)$ may contain components parametrizing curves having a component contained in Y so irreducibility may fail. Another obvious question is as to the Kodaira dimension of $M_e(a, Y)$: probably maximal for large e, a but it's not clear.

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