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The decoupling of defective linear dynamical systems in free motion

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ABSTRACT

It was demonstrated in two earlier papers that there exists a real, linear, time-varying transformation that decouples any non-defective linear dynamical system in free vibration in the configuration space. As an extension of this work, the present paper represents the first systematic effort to decouple defective systems. It is shown that the decoupling of defective systems is a rather delicate procedure that depends on the multiplicities of the system eigenvalues. While any defective system can be decoupled with the eigenvalues kept invariant, the geometric multiplicities of these eigenvalues may not be preserved. Several numerical examples are provided to illustrate the theoretical developments.

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1. Introduction

Coordinate coupling in linear dynamical systems under viscous damping has long been viewed as an undesired phenomenon with respect to system analysis in both practice and theoretical pursuits. Consequently, the decoupling of dynamical systems is a subject with a long history that attracts much attention from researchers even to this day. The problem of decoupling a linear dynamical system is concerned with developing a transformation that simultaneously diagonalizes the coefficient matrices that define the system in terms of its distribution of inertia and viscoelasticity. In free vibration, the equation of motion of an n -degree-of-freedom linear dynamical system subject to viscous damping has the matrix–vector representation

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{0}, \quad (1)$$

where the real and order n coefficient matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are positive definite (i.e., rigid body modes have been removed) and correspond to the system inertia, damping, and elasticity, respectively, and the real n -dimensional column vector $\mathbf{x}(t)$ denotes the generalized coordinates. System (1) is often coupled since the mass, damping, and stiffness matrices are generally not diagonal. Coupling is not an inherent property of a system but rather depends on the choice of generalized coordinates. This paper is concerned with devising a general methodology for decoupling by which any system (1) is transformed into

$$\ddot{\mathbf{p}}(t) + \mathbf{D}\dot{\mathbf{p}}(t) + \mathbf{\Omega}\mathbf{p}(t) = \mathbf{0} \quad (2)$$

for which the coefficient matrices $\mathbf{D}$ and $\mathbf{\Omega}$ are real, diagonal, of order n , and represent the damping and elasticity properties, respectively, of the decoupled system, and the real n -long column vector $\mathbf{p}(t)$ denotes the decoupled (or modal) coordinates.

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It is well known that a class of systems of the form (1) can be decoupled by congruence transformation in the n -dimensional configuration space using the eigenvectors of the undamped system. Associated with the undamped form of system (1) is the generalized eigenvalue problem (e.g., see [1])

$$\lambda \mathbf{M} \mathbf{u} = \mathbf{K} \mathbf{u} \quad (3)$$

that, because the mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$ are real and positive definite, generates n real and positive eigenvalues λ_k ($k=1,2,\dots,n$) with corresponding real eigenvectors $\mathbf{u}_k$ that are orthogonal with respect to $\mathbf{M}$ and $\mathbf{K}$. It is customary to normalize the eigenvectors in accordance with $\mathbf{u}_j \cdot (\mathbf{M} \mathbf{u}_k) = \delta_{jk}$, where $\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b}$ for any vectors $\mathbf{a}$ and $\mathbf{b}$, and δ_{jk} is the Kronecker delta. Upon normalization and arranging the eigenvectors in a modal matrix $\mathbf{U}$, one may define a linear time-invariant coordinate transformation

$$\mathbf{x}(t) = \mathbf{U} \mathbf{p}(t), \quad \mathbf{U} = [\mathbf{u}_1 \mid \dots \mid \mathbf{u}_n], \quad (4)$$

such that application of transformation (4) converts system (1) into the form (2) with

$$\mathbf{D} = \mathbf{U}^T \mathbf{C} \mathbf{U}, \quad \mathbf{\Omega} = \mathbf{U}^T \mathbf{K} \mathbf{U} = \bigoplus_{k=1}^n \lambda_k. \quad (5)$$

If the matrix $\mathbf{D}$ is diagonal upon congruence transformation, then system (1) has been decoupled by a procedure referred to as classical modal analysis. Consequently, a system (1) for which $\mathbf{D}$ is diagonal is said to be classically damped. Since coordinate transformation (4) is real, the modal response $\mathbf{p}(t)$ can be identified with a displacement, and hence classical modal analysis is amenable to physical interpretation. A necessary and sufficient condition for classical damping [2] relates the mass matrix $\mathbf{M}$, damping matrix $\mathbf{C}$, and stiffness matrix $\mathbf{K}$ of system (1) according to

$$\mathbf{C} \mathbf{M}^{-1} \mathbf{K} = \mathbf{K} \mathbf{M}^{-1} \mathbf{C}. \quad (6)$$

Practically speaking, condition (6) implies a fairly uniform distribution of energy dissipation in a system [3,4], but there is no reason for this to be true of any given system. As a result, system (1) may not be decoupled by classical modal analysis in general. Consequently, much attention has been given to the development of decoupling approximation schemes [5–16] for analyzing non-classically damped systems, along with the so-called coupling or non-proportionality indices [17–28] that quantify their degree of coordinate coupling.

Of course, one may also consider decoupling in the $2n$ -dimensional state space. In view of the inadequacy of classical modal analysis, a procedure known as complex modal analysis was developed to decouple non-classically damped systems of the form (1) in the state space via complex congruence transformation (e.g., see [29,30]). However, complex modal analysis is limited by the requirement that system (1) be non-defective (i.e., every eigenvalue of system (1) has an eigenvector), and there is a clear numerical disadvantage to analysis in the $2n$ -dimensional state space over the n -dimensional configuration space. Moreover, unlike classical modal analysis, complex modal analysis provides little in the way of physical insight since the complex congruence transformation involved generally makes it impossible to identify the $2n$ state variables with displacements and velocities. Alternative methods for decoupling include the recently proposed structure-preserving transformations by Garvey and others [31–34], but the case of defective systems (that is, those systems with eigenvalues which do not have corresponding eigenvectors) is not considered. In fact, it is quite common in the literature to avoid the issue of systems with defective eigenvalues, a trend likely motivated by the practical reason that it is rare to obtain exactly repeated eigenvalues. However, recent efforts have revealed that defective systems may be more common in practice than originally thought [35,36], and hence greater attention to the analysis of defective systems seems justified.

According to [37,38], all linear dynamical systems cannot be decoupled, or even partially decoupled, in the configuration space via linear time-invariant transformations. Consequently, if system (1) is non-classically damped and is to be decoupled in the configuration space (if even possible), then the decoupling transformation must at least be time-varying, or perhaps even nonlinear. This decoupling transformation would of course reduce to the familiar linear time-invariant transformation (4) in the event that system (1) is classically damped. In two preceding papers [39,40], it was shown that there does indeed exist a real, linear, time-shifting transformation that decouples a non-classically damped homogeneous system (1) in the configuration space. Decoupling of system (1) in the configuration space is achieved by a process, referred to as phase synchronization, in which the components of the non-classically damped modes of vibration are synchronized to yield classically damped modes. When cast in the state space, the decoupling transformation is real, linear, but time-invariant. Furthermore, it was demonstrated that this decoupling procedure is a direct generalization of classical modal analysis. Only non-defective systems are emphasized in these two papers. While defective systems with complex eigenvalues are mentioned, the case of defective real eigenvalues is not discussed.

The goal of this paper is to extend the decoupling methodology presented in [39,40] for free vibration to include systems of the form (1) possessing defective real or complex eigenvalues. We also clarify the decoupling procedure under real eigenvalues and offer a broader perspective of the decoupling operations. The organization of this paper is as follows. In Section 2, preliminary information on the decoupling of non-defective systems is presented to aid in subsequent developments. It is demonstrated in Section 3 how, through analysis in the state space, the homogeneous system (1) may be completely decoupled when it possesses defective eigenvalues, real or complex, if one does not insist on preserving geometric multiplicities. The generalized decoupling transformation developed in this section is shown to be a direct

generalization of classical modal analysis. Several examples are provided in Section 4 to illustrate the decoupling process. Lastly, we summarize the major results of this paper in Section 5.

2. Preliminaries

Prior to tackling defective systems, it is instructive to review the methodology detailed in [39,40] by which any non-defective system (1) is decoupled. Additional details on decoupling are provided in [41]. The preliminary information presented here will prove helpful in generalizing this decoupling procedure to treat defective systems.

2.1. The quadratic eigenvalue problem

Assume that the homogeneous equation of motion (1) has as its solution $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$. The scalar parameter λ , termed an eigenvalue, and the associated n -dimensional column vector $\mathbf{v}$, referred to as an eigenvector, are solutions of the quadratic eigenvalue problem (e.g., see [42–44])

$$(\mathbf{M}\lambda^2 + \mathbf{C}\lambda + \mathbf{K})\mathbf{v} = \mathbf{0}. \quad (7)$$

Solution to Eq. (7) yields $2n$ eigenvalues and eigenvectors that are generally complex. When system (1) is non-defective and simple, the $2n$ eigenvalues λ_j ($j=1,2,\dots,2n$) are distinct and may be divided into two categories: $2c$ complex and $2r=2(n-c)$ real eigenvalues. Since the system matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are real, the $2c$ complex eigenvalues and their associated eigenvectors necessarily form c complex conjugate pairs. The $2r$ real eigenvalues may be formed into r real quadratic conjugate pairs [40]. Recall that two real numbers are referred to as real quadratic conjugates if they are the roots of the same quadratic equation. While a complex number has a unique complex conjugate, the same is not true of real quadratic conjugation. In fact, there are $(2r)!/(2^r r!)$ different ways to pair the $2r$ real eigenvalues, but the mathematical structure created by real quadratic conjugation proves to be quite useful for decoupling: non-defective systems with complex eigenvalues and those with real eigenvalues may both be decoupled via phase synchronization by essentially the same algorithm, differing only in whether certain parameters are real, imaginary, or complex. As an example, it is possible to express all $2n$ eigenvalues, real or complex, in the complex notation $\lambda_j = \alpha_j + i\omega_j$, where ω_j is real for complex eigenvalues and imaginary for real eigenvalues. We refer the reader to [40] for the finer details of real quadratic conjugation. Some of the eigenvalues of a non-defective system (1) may be repeated so long as those eigenvalues that are repeated possess a full set of corresponding linearly independent eigenvectors. In this case, system (1) is said to be semi-simple.

2.2. Decoupling of non-defective systems

Let system (1) be simple so that its $2c$ complex and $2r$ real eigenvalues are distinct. By pairing eigensolutions via complex and real quadratic conjugation, the response $\mathbf{x}(t)$ of system (1) has the equivalent representations (e.g., see [43])

$$\mathbf{x}(t) = \sum_{k=1}^n (c_k \mathbf{v}_k e^{\lambda_k t} + \hat{c}_k \hat{\mathbf{v}}_k e^{\hat{\lambda}_k t}) = \mathbf{V} e^{\mathbf{\Lambda} t} \mathbf{c} + \hat{\mathbf{V}} e^{\hat{\mathbf{\Lambda}} t} \hat{\mathbf{c}}, \quad (8)$$

where $\mathbf{\Lambda}$ is an order n diagonal matrix of the $n=c+r$ eigenvalues λ_k ($k=1,2,\dots,n$) (that constitute the c complex conjugate and r real quadratic conjugate pairs), $\mathbf{V}$ is an order n matrix whose columns are the corresponding eigenvectors $\mathbf{v}_k$, $\mathbf{c}$ is an n -long vector of coefficients c_k determined by applying the initial conditions $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$, and the ornamenting hat denotes either complex or real quadratic conjugation, whichever is appropriate:

$$\mathbf{\Lambda} = \bigoplus_{k=1}^n \lambda_k, \quad \mathbf{V} = [\mathbf{v}_1 \mid \dots \mid \mathbf{v}_n], \quad \mathbf{c} = [c_1 \ \dots \ c_n]^T. \quad (9)$$

The arrangement of eigenvalues in $\mathbf{\Lambda}$ and its conjugate $\hat{\mathbf{\Lambda}}$ depends on the specified eigenvalue pairing scheme (that is, complex conjugate pairs for complex eigenvalues, and one of the $(2r)!/(2^r r!)$ pairings for distinct real eigenvalues). It is convenient to normalize the eigenvectors $\mathbf{v}_k$ and their conjugates $\hat{\mathbf{v}}_k$ in accordance with

$$2\lambda_k \mathbf{v}_k \cdot (\mathbf{M}\mathbf{v}_k) + \mathbf{v}_k \cdot (\mathbf{C}\mathbf{v}_k) = 2i\omega_k, \quad (10)$$

$$2\hat{\lambda}_k \hat{\mathbf{v}}_k \cdot (\mathbf{M}\hat{\mathbf{v}}_k) + \hat{\mathbf{v}}_k \cdot (\mathbf{C}\hat{\mathbf{v}}_k) = -2i\omega_k, \quad (11)$$

which reduces to normalization with respect to the mass matrix $\mathbf{M}$ in the event that system (1) is either undamped or classically damped [45]. It was demonstrated in [39,40] that phase synchronization decouples any non-defective system (1) into the form (2) with coefficient matrices $\mathbf{D}$ and $\mathbf{\Omega}$ given by

$$\mathbf{D} = -(\mathbf{\Lambda} + \hat{\mathbf{\Lambda}}), \quad \mathbf{\Omega} = \mathbf{\Lambda} \hat{\mathbf{\Lambda}}. \quad (12)$$

It is implied by Eq. (12) that system (2) consists of n decoupled degrees of freedom $p_k(t)$ that are each governed by the equation of motion

$$\ddot{p}_k(t) - (\lambda_k + \hat{\lambda}_k)\dot{p}_k(t) + \lambda_k\hat{\lambda}_k p_k(t) = 0 \quad (13)$$

for which c of these subsystems are underdamped and the remaining r are overdamped by pairing eigenvalues as complex conjugates and real quadratic conjugates, respectively. It can be verified that the solution to Eq. (13) is

$$p_k(t) = c_k e^{\lambda_k t} + \hat{c}_k e^{\hat{\lambda}_k t}, \quad (14)$$

where it is important to note that phase synchronization has not disturbed the coefficients c_k , which appear earlier in Eq. (8). It was illustrated in [39,40] that when Eq. (14) is rewritten in terms of sinusoidal functions, the parameter ω_k corresponds to a damped frequency of vibration. Since ω_k is imaginary for real quadratic conjugate eigenvalues, pure exponential decay may be thought of as imaginary vibration. To establish a transformation that recovers the response $\mathbf{x}(t)$ from the decoupled solution $\mathbf{p}(t)$, begin by writing Eq. (8) and its derivative as a state equation:

$$\begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{V} & \hat{\mathbf{V}} \\ \mathbf{V}\Lambda & \hat{\mathbf{V}}\hat{\Lambda} \end{bmatrix} \begin{bmatrix} e^{\Lambda t} \mathbf{c} \\ e^{\hat{\Lambda} t} \hat{\mathbf{c}} \end{bmatrix}. \quad (15)$$

Based on the structure of the decoupled degree of freedom $p_k(t)$ in Eq. (14), we may express the decoupled response in matrix–vector form as $\mathbf{p}(t) = e^{\Lambda t} \mathbf{c} + e^{\hat{\Lambda} t} \hat{\mathbf{c}}$. Casting $\mathbf{p}(t)$ and its derivative in the form of a state equation

$$\begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \Lambda & \hat{\Lambda} \end{bmatrix} \begin{bmatrix} e^{\Lambda t} \mathbf{c} \\ e^{\hat{\Lambda} t} \hat{\mathbf{c}} \end{bmatrix} \quad (16)$$

for which $\mathbf{I}$ denotes the identity matrix of order n , unless otherwise indicated by a subscript. It follows from Eqs. (15) and (16) that the response $\mathbf{x}(t)$ of non-defective system (1) and its decoupled response $\mathbf{p}(t)$ obtained via phase synchronization are related in the state space by

$$\begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{V} & \hat{\mathbf{V}} \\ \mathbf{V}\Lambda & \hat{\mathbf{V}}\hat{\Lambda} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \Lambda & \hat{\Lambda} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix}. \quad (17)$$

The initial conditions $\mathbf{p}(0)$ and $\dot{\mathbf{p}}(0)$ of the decoupled system are then connected to the system initial conditions $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$ by the state equation

$$\begin{bmatrix} \mathbf{p}(0) \\ \dot{\mathbf{p}}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \Lambda & \hat{\Lambda} \end{bmatrix} \begin{bmatrix} \mathbf{V} & \hat{\mathbf{V}} \\ \mathbf{V}\Lambda & \hat{\mathbf{V}}\hat{\Lambda} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{x}(0) \\ \dot{\mathbf{x}}(0) \end{bmatrix}. \quad (18)$$

While some of the matrices in Eqs. (17) and (18) contain complex elements, the overall transformation is real. It is interesting to note that, because the $2n$ non-defective eigenvalues and associated eigenvectors are paired as n complex and/or real quadratic conjugates, the order $2n$ transformation matrices of Eq. (17) are partitioned into blocks of size n . Consequently, it is possible to extract an analytical expression for the response $\mathbf{x}(t)$ only from transformation (17), making it unnecessary to evaluate the larger state equation. Isolating the upper half of Eq. (17) yields

$$\mathbf{x}(t) = \mathbf{T}_1 \mathbf{p}(t) + \mathbf{T}_2 \dot{\mathbf{p}}(t), \quad (19)$$

where the transformation matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ are given by

$$\mathbf{T}_1 = (\mathbf{V}\hat{\Lambda} - \hat{\mathbf{V}}\Lambda)(\hat{\Lambda} - \Lambda)^{-1}, \quad \mathbf{T}_2 = (\hat{\mathbf{V}} - \mathbf{V})(\hat{\Lambda} - \Lambda)^{-1}. \quad (20)$$

Thus, after solving the decoupled system (2), defined by the coefficient matrices (12), with initial conditions (18), the response $\mathbf{x}(t)$ of system (1) may be recovered directly via transformation (19). All parameters required for decoupling are obtained through solution of the quadratic eigenvalue problem (7). The decoupling transformation is such that the eigenvalues of system (1) and their multiplicities are preserved (i.e., the decoupling transformation is strictly isospectral). The numerical advantage of solving Eq. (19) over transformation (17) is obvious. In the event that some of the eigenvalues of system (1) are repeated but have a full set of associated eigenvectors (i.e., when system (1) is semi-simple), Eq. (19) remains a valid transformation so long as repeated real eigenvalues are not paired as real quadratic conjugates, which would violate the strictly isospectral nature of the decoupling transformation. To summarize, if the coupled n -degree-of-freedom system (1) is non-defective, it may be decoupled by phase synchronization into a set of n single-degree-of-freedom oscillators, c of which are underdamped and $r = n - c$ of which are overdamped. Due to the non-uniqueness of real quadratic conjugation, there exists $(2r)!/(2^r r!)$ possible ways to construct the coefficient matrices $\mathbf{D}$ and $\mathbf{\Omega}$ that define the decoupled system (2).

2.3. Reduction to classical modal analysis

Should system (1) be classically damped and the eigenvectors have been normalized according to Eqs. (10) and (11), then the eigenvectors $\mathbf{v}_k = \hat{\mathbf{v}}_k$ correspond to the classical normal modes $\mathbf{u}_k$ of the undamped system. As a result, $\mathbf{V} = \hat{\mathbf{V}} = \mathbf{U}$,

and hence the transformation matrices $\mathbf{T}_1 = \mathbf{U}$ and $\mathbf{T}_2 = \mathbf{O}$, where $\mathbf{O}$ denotes the zero matrix of order n . Consequently, Eq. (19) simplifies to modal transformation (4), implying that phase synchronization reduces to classical modal analysis. This reduction still holds even if the eigenvectors are not normalized in accordance with Eqs. (10) and (11), albeit with $\mathbf{V} \neq \mathbf{U}$, since eigenvectors are unique up to a multiplicative constant. Lastly, it should be noted that the pairing scheme for the real eigenvalues of system (1) that results in $\mathbf{V} = \hat{\mathbf{V}} = \mathbf{U}$ may not coincide with the pairing scheme used for decoupling, so the reduction of phase synchronization to classical modal analysis for real eigenvalues is not as clean as it is for complex eigenvalues, for which there exists a unique and natural pairing.

3. Decoupling of defective systems

In the event that system (1) is defective, solution of the quadratic eigenvalue problem (7) reveals that some of the eigenvalues are repeated and do not have associated with them a full complement of linearly independent eigenvectors. Such eigenvalues are termed defective. An example of a defective eigenvalue is one that is repeated more than n times. Let λ_k denote the k th eigenvalue of system (1), and suppose it is defective. The number of times the defective eigenvalue λ_k is repeated is referred to as its algebraic multiplicity m_k . Considering the quadratic matrix pencil $\mathbf{Q}(\lambda) = \mathbf{M}\lambda^2 + \mathbf{C}\lambda + \mathbf{K}$, the geometric multiplicity ρ_k of the defective eigenvalue λ_k is given by the dimension of the null space of $\mathbf{Q}(\lambda_k)$, and hence it is equivalent to the number of linearly independent eigenvectors associated with λ_k . For obvious reasons, the geometric multiplicity cannot exceed either the algebraic multiplicity m_k or the system order n , whichever is smaller: $\rho_k \leq \min(m_k, n)$. The associated ρ_k eigenvectors $\mathbf{v}_j^k$ ($j = 1, 2, \dots, \rho_k$) are supplemented with $m_k - \rho_k$ generalized eigenvectors to form a complete set of vectors for λ_k , where the final eigenvector $\mathbf{v}_{\rho_k}^k$ and the generalized eigenvectors constitute a Jordan chain of length $m_k - \rho_k + 1$ characterized by the following recursive scheme (see [44]):

$$\begin{aligned} \mathbf{0} &= \mathbf{Q}(\lambda_k) \mathbf{v}_{\rho_k}^k, \\ \mathbf{0} &= \mathbf{Q}(\lambda_k) \mathbf{v}_{\rho_k+1}^k + \mathbf{Q}'(\lambda_k) \mathbf{v}_{\rho_k}^k, \\ \mathbf{0} &= \mathbf{Q}(\lambda_k) \mathbf{v}_{\rho_k+2}^k + \mathbf{Q}'(\lambda_k) \mathbf{v}_{\rho_k+1}^k + \frac{1}{2} \mathbf{Q}''(\lambda_k) \mathbf{v}_{\rho_k}^k, \\ &\vdots \\ \mathbf{0} &= \mathbf{Q}(\lambda_k) \mathbf{v}_{m_k}^k + \mathbf{Q}'(\lambda_k) \mathbf{v}_{m_k-1}^k + \frac{1}{2} \mathbf{Q}''(\lambda_k) \mathbf{v}_{m_k-2}^k. \end{aligned} \quad (21)$$

In the above sequence, $\mathbf{Q}'(\lambda_k)$ and $\mathbf{Q}''(\lambda_k)$ denote the first and second derivatives, respectively, of $\mathbf{Q}(\lambda)$ with respect to λ and evaluated at the defective eigenvalue λ_k :

$$\mathbf{Q}'(\lambda_k) = \left. \frac{d\mathbf{Q}(\lambda)}{d\lambda} \right|_{\lambda=\lambda_k} = 2\mathbf{M}\lambda_k + \mathbf{C}, \quad (22)$$

$$\mathbf{Q}''(\lambda_k) = \left. \frac{d^2\mathbf{Q}(\lambda)}{d\lambda^2} \right|_{\lambda=\lambda_k} = 2\mathbf{M}. \quad (23)$$

We begin our treatment of defective systems in free motion by formulating a generalized procedure for decoupling system (1) into the form (2) and retrieving the response $\mathbf{x}(t)$ in the state space from the decoupled response $\mathbf{p}(t)$. We then demonstrate how this procedure is applied to defective systems with complex eigenvalues, followed by application to the case of defective real eigenvalues. While the decoupling of systems in free motion with defective complex eigenvalues was briefly touched upon in [39], we provide an alternative and more in-depth analysis in this paper.

3.1. A generalized state space representation for decoupling

Regardless of the nature of its eigenspectrum, the response $\mathbf{x}(t)$ of system (1) has the general form (e.g., see [42,43])

$$\mathbf{x}(t) = \mathbf{V}_x \mathbf{e}^{\mathbf{J}_x t} \mathbf{c} \quad (24)$$

for which $\mathbf{J}_x$ is an order $2n$ Jordan matrix with the eigenvalues on the diagonal, $\mathbf{V}_x$ is an $n \times 2n$ matrix of the associated eigenvectors and generalized eigenvectors, and $\mathbf{c}$ is a $2n$ -dimensional column vector of coefficients determined from the initial conditions $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$. Writing Eq. (24) and its derivative in the form of a state equation,

$$\begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{V}_x \\ \mathbf{V}_x \mathbf{J}_x \end{bmatrix} \mathbf{e}^{\mathbf{J}_x t} \mathbf{c} = \mathbf{S}_x \mathbf{e}^{\mathbf{J}_x t} \mathbf{c}, \quad (25)$$

where the order $2n$ matrix $\mathbf{S}_x$ is invertible since $\mathbf{J}_x$ and $\mathbf{V}_x$ constitute a Jordan pair.

Defective or not, we assume that system (1) may be decoupled into the form (2) such that the diagonal coefficient matrices $\mathbf{D}$ and $\mathbf{\Omega}$ are always given by Eq. (12), where the arrangement of eigenvalues in the diagonal matrix $\mathbf{\Lambda}$ and its conjugate $\hat{\mathbf{\Lambda}}$ is dependent on the choice of eigensolution pairing. It has been illustrated in the literature that it is possible to decouple a non-defective system (1) while preserving both the algebraic and geometric multiplicities of the associated eigenvalues (i.e., the decoupling transformation is strictly isospectral). Phase synchronization [39,40] in the n -dimensional configuration space is an example of how decoupling is achieved in such a way, as is the method of structure-preserving

transformations in the $2n$ -dimensional state space proposed by Garvey and others [31–34]. If one insists that geometric multiplicities be preserved, then a defective system (1) may only be partially decoupled [46]. Our main goal is to demonstrate how defective systems can be decoupled, at any cost, by not requiring geometric multiplicities to be preserved.

Since phase synchronization for a non-defective system (1) does not disturb the eigensolution coefficients, we shall construct the decoupling transformation for a defective system (1) so that it also preserves these coefficients. Consequently, the modal response $\mathbf{p}(t)$ may be cast in a matrix–vector form analogous to Eq. (24):

$$\mathbf{p}(t) = \mathbf{V}_p e^{\mathbf{J}_p t} \mathbf{c}. \quad (26)$$

Here, $\mathbf{J}_p$ is an order $2n$ Jordan matrix consisting of the system eigenvalues, but it may not necessarily be the same as $\mathbf{J}_x$. The matrix $\mathbf{J}_p$ has the same diagonal elements (i.e., the eigenvalues) as $\mathbf{J}_x$, but the off-diagonal elements are generally different since we do not impose preservation of geometric multiplicities. The matrix $\mathbf{V}_p$ is an $n \times 2n$ matrix of vectors that essentially pairs the eigensolutions according to a specified pairing scheme. How $\mathbf{J}_p$ and $\mathbf{V}_p$ are constructed will be clarified in the subsequent developments. Expressing Eq. (26) and its derivative as a state equation,

$$\begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{V}_p \\ \mathbf{V}_p \mathbf{J}_p \end{bmatrix} e^{\mathbf{J}_p t} \mathbf{c} = \mathbf{S}_p e^{\mathbf{J}_p t} \mathbf{c}, \quad (27)$$

where the order $2n$ matrix $\mathbf{S}_p$ will always be invertible so long as eigenvalues are paired to preserve the second-order structure of the equation of motion (1). Combining Eqs. (25) and (27), the response $\mathbf{x}(t)$ of system (1) is recovered from the modal response $\mathbf{p}(t)$ in the state space according to

$$\begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{V}_x \\ \mathbf{V}_x \mathbf{J}_x \end{bmatrix} e^{\mathbf{J}_x t} e^{-\mathbf{J}_p t} \begin{bmatrix} \mathbf{V}_p \\ \mathbf{V}_p \mathbf{J}_p \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} = \mathbf{S}_x e^{\mathbf{J}_x t} e^{-\mathbf{J}_p t} \mathbf{S}_p^{-1} \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} \quad (28)$$

and thus the modal initial conditions $\mathbf{p}(0)$ and $\dot{\mathbf{p}}(0)$ are connected to the system initial conditions $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$ by the state equation

$$\begin{bmatrix} \mathbf{p}(0) \\ \dot{\mathbf{p}}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{V}_p \\ \mathbf{V}_p \mathbf{J}_p \end{bmatrix} \begin{bmatrix} \mathbf{V}_x \\ \mathbf{V}_x \mathbf{J}_x \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{x}(0) \\ \dot{\mathbf{x}}(0) \end{bmatrix}. \quad (29)$$

As one would expect, transformation (28) reduces to classical modal analysis in the event that system (1) is classically damped. Because classical modal analysis constitutes a strictly isospectral transformation, the Jordan matrices $\mathbf{J}_x$ and $\mathbf{J}_p$ are identical, simplifying the matrix exponential product $e^{\mathbf{J}_x t} e^{-\mathbf{J}_p t}$ to an order $2n$ identity matrix, which leaves the transformation $\mathbf{S}_x \mathbf{S}_p^{-1}$. If the system eigenvectors are normalized in accordance with Eqs. (10) and (11), then there exists a particular pairing of the non-defective real eigenvalues such that $\mathbf{S}_p \mathbf{S}_p^{-1} = \mathbf{U} \oplus \mathbf{U}$, and thus the upper half of the state equation (28) yields the classical modal transformation $\mathbf{x}(t) = \mathbf{U} \mathbf{p}(t)$. Multiplicative constants appear here and there if normalizations (10) and (11) are not used. Moreover, the coefficient matrices $\mathbf{D}$ and $\mathbf{\Omega}$ defined by Eq. (12) are the same as those given by classical modal analysis.

A generalized procedure for decoupling system (1) into the form (2) and recovering the response $\mathbf{x}(t)$ from the decoupled solution $\mathbf{p}(t)$ via state space transformation is summarized as follows. (i) Solve the quadratic eigenvalue problem (7) to determine the system eigenvalues and eigenvectors. If system (1) is defective, additional analysis is required to obtain generalized eigenvectors as needed. (ii) Based on the nature of the system's eigenspectrum, specify an appropriate eigenvalue pairing scheme, and arrange the eigenvalues accordingly in the diagonal matrix $\mathbf{\Lambda}$ and its conjugate $\hat{\mathbf{\Lambda}}$. (iii) Construct the matrices $\mathbf{J}_x$ and $\mathbf{V}_x$ from the system eigenvalues, eigenvectors, and generalized eigenvectors. (iv) Form the matrices $\mathbf{J}_p$ and $\mathbf{V}_p$ such that the solution $\mathbf{p}(t)$ of the decoupled system (2) satisfies the specified pairing scheme. (v) Obtain the initial conditions $\mathbf{p}(0)$ and $\dot{\mathbf{p}}(0)$ for the decoupled system via transformation (29). (vi) Construct the coefficient matrices $\mathbf{D}$ and $\mathbf{\Omega}$ in accordance with Eq. (12), and then determine the response $\mathbf{p}(t)$ of the decoupled system. (vii) Retrieve the response $\mathbf{x}(t)$ from the decoupled solution $\mathbf{p}(t)$ using transformation (28).

The heart of the decoupling procedure described here lies in steps (ii) through (iv). To illustrate, we shall demonstrate how the above procedure relates to the decoupling of a non-defective system (1) with mixed eigenvalues ($2c$ complex and the remaining $2r$ real) as presented in Section 2. Because the eigenvalues are generally semi-simple, they may be paired as complex and real quadratic conjugates to yield c underdamped and r overdamped (not necessarily distinct) decoupled degrees of freedom, respectively. Consequently, the first $n=c+r$ eigenvalues and eigenvectors (that constitute the n conjugate pairs) may be arranged in the matrices $\mathbf{\Lambda}$ and $\mathbf{V}$, respectively, in accordance with Eq. (9). Comparing the summation representation of the response $\mathbf{x}(t)$ given in Eq. (8) to the general form (24), it is clear that $\mathbf{J}_x = \mathbf{\Lambda} \oplus \hat{\mathbf{\Lambda}}$ and $\mathbf{V}_x = [\mathbf{V} | \hat{\mathbf{V}}]$. The decoupled solutions $p_k(t)$ given by Eq. (14) imply that $\mathbf{J}_p = \mathbf{J}_x$ and $\mathbf{V}_p = [\mathbf{I} | \mathbf{I}]$ in Eq. (26). As a result, Eqs. (28) and (29) reduce to the familiar transformations (17) and (18), respectively, for a non-defective system (1).

There currently do not exist criteria for eigenvalue pairing when system (1) is defective. Put another way, it is not clear how to form $\mathbf{\Lambda}$ and its conjugate $\hat{\mathbf{\Lambda}}$ when the eigenvalues are defective in order to construct the coefficient matrices $\mathbf{D}$ and $\mathbf{\Omega}$ that define the decoupled system (2). This paper presents a solution to this problem. We begin by considering systems of the type (1) whose defective eigenvalues are all of unit geometric multiplicity. We place this restriction on the

geometric multiplicities only to simplify the presentation of new decoupling methods for the reader's convenience. Decoupling systems with defective complex eigenvalues is more straightforward than for those with defective real eigenvalues, so we first detail how the generalized decoupling procedure is applied to the defective complex eigenvalue case. When the eigenvalues are real and defective, how one proceeds with decoupling depends on whether the eigenvalues' algebraic multiplicities are even or odd. The more straightforward case of even algebraic multiplicities is considered first, followed by an investigation of two scenarios involving odd algebraic multiplicities. We end by briefly demonstrating how the decoupling methodologies developed for the unit geometric multiplicity case may be generalized when the geometric multiplicities are not restricted to unity.

3.2. State transformation of the equation of motion

Before we apply the generalized decoupling procedure outlined in the previous section, let us first further examine the nature of the state space transformation (28). Suppose we cast system (1) and its decoupled form (2), respectively, in the non-symmetric state space realizations (e.g., see [47])

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \ddot{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} = \mathbf{A} \begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix}, \quad (30)$$

$$\begin{bmatrix} \dot{\mathbf{p}}(t) \\ \ddot{\mathbf{p}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{\Omega} & -\mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} = \mathbf{B} \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix}. \quad (31)$$

Since the eigenvalues of system (1) and their algebraic multiplicities are preserved during decoupling, the non-symmetric order $2n$ matrices $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues (i.e., they are isospectral), but the corresponding geometric multiplicities may be different (that is, $\mathbf{A}$ and $\mathbf{B}$ are not strictly isospectral), which is reflected in the associated Jordan matrices $\mathbf{J}_x$ and $\mathbf{J}_p$, respectively. In the general case of a defective system (1), what is the real transformation that converts its state space representation (30) into Eq. (31), from which the decoupled system (2) is extracted? In other words, how are the matrices $\mathbf{A}$ and $\mathbf{B}$ related? To answer this question, begin by writing the general state space transformation (28) in the more compact form

$$\begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} = \mathbf{S}_x e^{\mathbf{J}_x t} e^{-\mathbf{J}_p t} \mathbf{S}_p^{-1} \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} = \mathbf{S}(t) \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix}. \quad (32)$$

Substitute the state vector (32) and its derivative into Eq. (30) and then premultiply the resulting equation by $\mathbf{S}^{-1}(t)$ to obtain

$$\begin{bmatrix} \dot{\mathbf{p}}(t) \\ \ddot{\mathbf{p}}(t) \end{bmatrix} = (\mathbf{S}^{-1}(t)\mathbf{A}\mathbf{S}(t) - \mathbf{S}^{-1}(t)\dot{\mathbf{S}}(t)) \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix}. \quad (33)$$

Comparing Eqs. (31) and (33) reveals that the constant matrices $\mathbf{A}$ and $\mathbf{B}$ are related by the time-varying transformation

$$\mathbf{B} = \mathbf{S}^{-1}(t)\mathbf{A}\mathbf{S}(t) - \mathbf{S}^{-1}(t)\dot{\mathbf{S}}(t). \quad (34)$$

Time variation in Eq. (34) is a direct result of not requiring that geometric multiplicities be preserved during decoupling. In other words, transformation (34) is time-varying when the Jordan matrices $\mathbf{J}_x$ and $\mathbf{J}_p$ are not identical. In the event that $\mathbf{J}_x = \mathbf{J}_p$, the matrix $\mathbf{S}(t) = \mathbf{S}$ is independent of time t , and hence Eq. (34) reduces to the similarity transformation $\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$. One situation for which this reduction occurs is when system (1) is non-defective, in which case the constant matrix $\mathbf{S}$ is given by the order $2n$ transformation in state equation (17).

3.3. Restriction on the geometric multiplicity

As mentioned previously, in order to streamline the introduction of new details, we shall assume that every defective eigenvalue λ_k of system (1) has unit geometric multiplicity (i.e., $\rho_k = 1$), and hence the length of its associated Jordan chain is given by the algebraic multiplicity m_k . In the case of defective complex conjugate eigenvalues, relaxation of this constraint results in a trivial modification of the decoupling procedure. For real defective eigenvalues, however, the situation is more delicate and requires some discussion.

3.4. Defective oscillatory motion (complex eigenvalues)

Suppose that solution of the quadratic eigenvalue problem (7) reveals that the $2n$ eigenvalues of system (1) are complex and defective. As the system matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are real, these eigenvalues necessarily form n_c complex conjugate pairs, where $2n_c < 2n$. Let m_k ($k = 1, 2, \dots, n_c$) be the algebraic multiplicity of the defective complex eigenvalue λ_k . Then m_k is also the algebraic multiplicity of $\bar{\lambda}_k$ so that $m_1 + \dots + m_{n_c} = n$. Under the assumption of unit geometric multiplicity, each defective eigenvalue λ_k has a single (complex) eigenvector $\mathbf{v}_k^c$ associated with it. Let $\mathbf{J}_k$ be a Jordan block of order m_k formed

from the defective eigenvalue λ_k :

$$\mathbf{J}_k = \begin{bmatrix} \lambda_k & 1 & 0 & \cdots & 0 \\ 0 & \lambda_k & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_k & 1 \\ 0 & \cdots & 0 & 0 & \lambda_k \end{bmatrix}. \quad (35)$$

In addition, construct an $n \times m_k$ matrix $\mathbf{V}_k$ whose columns contain the single eigenvector $\mathbf{v}_1^k$ and the m_k-1 generalized eigenvectors that constitute the m_k -long Jordan chain associated with λ_k , and take $\mathbf{c}_k$ to be an m_k -dimensional vector consisting of the complex coefficients c_j^k ($j = 1, 2, \dots, m_k$) for the corresponding eigensolutions:

$$\mathbf{V}_k = [\mathbf{v}_1^k \mid \cdots \mid \mathbf{v}_{m_k}^k], \quad \mathbf{c}_k = [c_1^k \cdots c_{m_k}^k]^T. \quad (36)$$

In terms of the Jordan pairs $(\mathbf{J}_k, \mathbf{V}_k)$ and their complex conjugates, the response $\mathbf{x}(t)$ may be expressed as

$$\mathbf{x}(t) = \sum_{k=1}^{n_c} \mathbf{V}_k e^{\mathbf{J}_k t} \mathbf{c}_k + \bar{\mathbf{V}}_k e^{\bar{\mathbf{J}}_k t} \bar{\mathbf{c}}_k. \quad (37)$$

If system (1) is non-defective and it possesses a complex eigenvalue λ_k repeated m_k times, then associated with λ_k are m_k independent underdamped oscillators that execute oscillatory free motion, governed by the equation of motion (13), with identical decay rate α_k and damped frequency ω_k . The initial conditions, however, are generally different because the eigensolution coefficients c_k are typically not the same. In progressing to defective oscillatory systems, each defective complex eigenvalue λ_k with algebraic multiplicity m_k yields m_k identical and independent underdamped systems subject to different sets of initial conditions upon decoupling. Specifically, every decoupled coordinate $p_j^k(t)$ ($j = 1, 2, \dots, m_k$) corresponding to the defective eigenvalue λ_k satisfies the same equation of motion

$$\ddot{p}_j^k(t) - (\lambda_k + \bar{\lambda}_k) \dot{p}_j^k(t) + \lambda_k \bar{\lambda}_k p_j^k(t) = 0 \quad (38)$$

for which the solution $p_j^k(t)$ is of the form

$$p_j^k(t) = c_j^k e^{\lambda_k t} + \bar{c}_j^k e^{\bar{\lambda}_k t}. \quad (39)$$

The decoupled equation of motion (38) implies that the n_c eigenvalues (that constitute the n_c complex conjugate pairs) may be arranged in the diagonal matrix $\mathbf{\Lambda}$ according to

$$\mathbf{\Lambda} = \bigoplus_{k=1}^{n_c} \mathbf{\Lambda}_k, \quad \mathbf{\Lambda}_k = \lambda_k \mathbf{I}_{m_k}. \quad (40)$$

The (complex) conjugate of $\mathbf{\Lambda}$ immediately follows. Given Eq. (40) and its conjugate, we may then construct the coefficient matrices $\mathbf{D}$ and $\mathbf{\Omega}$ in accordance with Eq. (12) to establish the decoupled system (2). Eq. (40) permits us to write the Jordan block (35) in the form $\mathbf{J}_k = \mathbf{\Lambda}_k + \mathbf{N}_k$, where $\mathbf{N}_k$ is an order m_k nilpotent matrix of 1's on the superdiagonal. Comparing Eqs. (24) and (37) reveals that the matrices $\mathbf{J}_x = \mathbf{J} \oplus \bar{\mathbf{J}}$ and $\mathbf{V}_x = [\mathbf{V} \mid \bar{\mathbf{V}}]$, where the order n block diagonal matrix $\mathbf{J}$ and the $n \times n$ matrix $\mathbf{V}$ of eigenvectors and generalized eigenvectors are given by

$$\mathbf{J} = \bigoplus_{k=1}^{n_c} \mathbf{J}_k = \mathbf{\Lambda} + \mathbf{N}, \quad \mathbf{V} = [\mathbf{V}_1 \cdots \mathbf{V}_{n_c}]. \quad (41)$$

Based on the arrangement of eigenvalues in $\mathbf{\Lambda}$, the modal response vector $\mathbf{p}(t)$ has the conformable structure

$$\mathbf{p}(t) = [\mathbf{p}_1^T(t) \cdots \mathbf{p}_{n_c}^T(t)]^T, \quad \mathbf{p}_k(t) = [p_1^k(t) \cdots p_{m_k}^k(t)]^T \quad (42)$$

from which we deduce that $\mathbf{J}_p = \mathbf{\Lambda} \oplus \bar{\mathbf{\Lambda}}$ and $\mathbf{V}_p = [\mathbf{V} \mid \bar{\mathbf{V}}]$ in order to satisfy the response (39) for each decoupled coordinate $p_j^k(t)$. In addition, it can be shown, by exploiting the commutativity of $\mathbf{\Lambda}_k = \lambda_k \mathbf{I}_{m_k}$ and $\mathbf{N}_k$ in multiplication, that $\mathbf{e}^{\mathbf{J}_x t} \mathbf{e}^{-\mathbf{J}_p t} = \mathbf{e}^{\mathbf{N}t} \oplus \mathbf{e}^{\bar{\mathbf{N}}t}$. Consequently, the state space formulation (28) for defective oscillatory motion has the equivalent time-dependent representations

$$\begin{aligned} \begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} &= \begin{bmatrix} \mathbf{V} & \bar{\mathbf{V}} \\ \mathbf{VJ} & \bar{\mathbf{VJ}} \end{bmatrix} \begin{bmatrix} \mathbf{e}^{\mathbf{N}t} & \mathbf{0} \\ \mathbf{0} & \mathbf{e}^{\bar{\mathbf{N}}t} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda} & \bar{\mathbf{\Lambda}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{V} & \bar{\mathbf{V}} \\ \mathbf{VJ} & \bar{\mathbf{VJ}} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda} & \bar{\mathbf{\Lambda}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{e}^{\mathbf{N}t} & \mathbf{0} \\ \mathbf{0} & \mathbf{e}^{\bar{\mathbf{N}}t} \end{bmatrix} \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix}. \end{aligned} \quad (43)$$

The modal and system initial conditions are related by the state equation

$$\begin{bmatrix} \mathbf{p}(0) \\ \dot{\mathbf{p}}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda} & \bar{\mathbf{\Lambda}} \end{bmatrix} \begin{bmatrix} \mathbf{V} & \bar{\mathbf{V}} \\ \mathbf{VJ} & \bar{\mathbf{VJ}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{x}(0) \\ \dot{\mathbf{x}}(0) \end{bmatrix}. \quad (44)$$

As with non-defective systems, it is possible to isolate an order n transformation from the state space representation (43) to directly give the response $\mathbf{x}(t)$:

$$\mathbf{x}(t) = \mathbf{T}_1 e^{\mathbf{N}t} \mathbf{p}(t) + \mathbf{T}_2 e^{\mathbf{N}t} \dot{\mathbf{p}}(t), \quad (45)$$

where the transformation matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ are still as defined in Eq. (20), but conjugation is strictly complex. Upon solution of the decoupled system with initial conditions (44), the response $\mathbf{x}(t)$ of system (1) may be obtained directly from transformation (45). In addition to the quadratic eigenvalue problem (7), it becomes necessary to evaluate the recursive scheme (21) to obtain all parameters required for retrieving the system response $\mathbf{x}(t)$ from Eq. (45). In the event that system (1) is non-defective, the nilpotent matrix $\mathbf{N} = \mathbf{O}$, and thus the Jordan matrix $\mathbf{J} = \mathbf{\Lambda}$. As a result, Eqs. (43)–(45) reduce to transformations (17)–(19), respectively, associated with non-defective eigenvalues, as expected. In summary, if the coupled n -degree-of-freedom system (1) is defective with n_c pairs of complex conjugate eigenvalues of respective algebraic multiplicity m_k , then it may be decoupled into n underdamped degrees of freedom corresponding to n_c collections of m_k identical, underdamped single-degree-of-freedom oscillators subject to different initial conditions.

The partitioning of the order $2n$ transformation matrices in state equation (43) into blocks of size n makes it convenient to look further into the mechanics of transformation (34) and how it relates the state matrices $\mathbf{A}$ and $\mathbf{B}$ of the non-symmetric state space realizations (30) and (31), respectively. Based on Eq. (43), the matrix $\mathbf{S}(t)$ and its derivative are given by, respectively,

$$\mathbf{S}(t) = \begin{bmatrix} \mathbf{V} & \bar{\mathbf{V}} \\ \mathbf{VJ} & \bar{\mathbf{VJ}} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda} & \bar{\mathbf{\Lambda}} \end{bmatrix}^{-1} \begin{bmatrix} e^{\mathbf{N}t} & \mathbf{O} \\ \mathbf{O} & e^{\mathbf{N}t} \end{bmatrix} = \mathbf{S}_0 \begin{bmatrix} e^{\mathbf{N}t} & \mathbf{O} \\ \mathbf{O} & e^{\mathbf{N}t} \end{bmatrix}, \quad (46)$$

$$\dot{\mathbf{S}}(t) = \mathbf{S}_0 \begin{bmatrix} \mathbf{N} & \mathbf{O} \\ \mathbf{O} & \mathbf{N} \end{bmatrix} \begin{bmatrix} e^{\mathbf{N}t} & \mathbf{O} \\ \mathbf{O} & e^{\mathbf{N}t} \end{bmatrix}. \quad (47)$$

Inserting Eqs. (46) and (47) into transformation (34) yields

$$\mathbf{B} = \begin{bmatrix} e^{-\mathbf{N}t} & \mathbf{O} \\ \mathbf{O} & e^{-\mathbf{N}t} \end{bmatrix} \left(\mathbf{S}_0^{-1} \mathbf{A} \mathbf{S}_0 - \begin{bmatrix} \mathbf{N} & \mathbf{O} \\ \mathbf{O} & \mathbf{N} \end{bmatrix} \right) \begin{bmatrix} e^{\mathbf{N}t} & \mathbf{O} \\ \mathbf{O} & e^{\mathbf{N}t} \end{bmatrix}, \quad (48)$$

where it can be verified that

$$\mathbf{S}_0^{-1} \mathbf{A} \mathbf{S}_0 = \begin{bmatrix} \mathbf{N} & \mathbf{I} \\ -\mathbf{\Omega} & \mathbf{N} - \mathbf{D} \end{bmatrix}. \quad (49)$$

Finally, by exploiting the commutativity in blockwise multiplication of the diagonal coefficient matrices $\mathbf{D}$ and $\mathbf{\Omega}$ with the matrix exponential $e^{\mathbf{N}t}$, it is straightforward to show that the resulting matrix $\mathbf{B}$ is indeed the order $2n$ state matrix in Eq. (31).

3.5. Defective non-oscillatory motion (real eigenvalues)

Suppose that solution of the quadratic eigenvalue problem (7) for system (1) yields real eigenvalues λ_k that are repeated and defective. By Theorem 4.2 in [42], every eigenvector $\mathbf{v}_j^k$ associated with a defective real eigenvalue λ_k satisfies

$$2\mathbf{v}_j^k \cdot (\mathbf{M}\mathbf{v}_j^k)\lambda_k + \mathbf{v}_j^k \cdot (\mathbf{C}\mathbf{v}_j^k) = 0. \quad (50)$$

However, the quadratic eigenvalue problem (7) implies that

$$\lambda_k = -\frac{\mathbf{v}_j^k \cdot (\mathbf{C}\mathbf{v}_j^k)}{2\mathbf{v}_j^k \cdot (\mathbf{M}\mathbf{v}_j^k)} \pm i \frac{\sqrt{4(\mathbf{v}_j^k \cdot (\mathbf{M}\mathbf{v}_j^k))(\mathbf{v}_j^k \cdot (\mathbf{K}\mathbf{v}_j^k)) - (\mathbf{v}_j^k \cdot (\mathbf{C}\mathbf{v}_j^k))^2}}{2\mathbf{v}_j^k \cdot (\mathbf{M}\mathbf{v}_j^k)}. \quad (51)$$

Eqs. (50) and (51) are simultaneously satisfied when the discriminant of Eq. (51) vanishes, resulting in

$$\lambda_k = -\sqrt{\frac{\mathbf{v}_j^k \cdot (\mathbf{K}\mathbf{v}_j^k)}{\mathbf{v}_j^k \cdot (\mathbf{M}\mathbf{v}_j^k)}} \pm i0 = \alpha_k \pm i0. \quad (52)$$

Thus, defective real eigenvalues are such that $\omega_k = 0$, implying that these eigenvalues correspond to the boundary between real vibration (complex eigenvalues) and imaginary vibration (non-defective real eigenvalues). Consequently, we associate defective real eigenvalues with critical damping. Critical damping in multi-degree-of-freedom systems, classically damped or not, has been discussed extensively in the literature [48–53]. The common thread in these works is that every defective real eigenvalue λ_k giving rise to a critically damped degree of freedom has algebraic multiplicity $m_k = 2$. However, there is generally no reason why the algebraic multiplicities of the defective real eigenvalues should be limited in this way for a non-classically damped system. We place no such restriction on the algebraic multiplicities in this paper. By a thorough examination, we shall demonstrate that decoupling of systems with defective real eigenvalues is a delicate procedure that depends on the multiplicities of these eigenvalues.

3.5.1. Even algebraic multiplicity

Consider the case in which system (1) possesses $n_r < 2n$ defective real eigenvalues λ_k ($k = 1, 2, \dots, n_r$), each of even algebraic multiplicity m_k and unit geometric multiplicity, such that $m_1 + \dots + m_{n_r} = 2n$. Construct an order m_k Jordan block $\mathbf{J}_k$, an $n \times m_k$ matrix $\mathbf{V}_k$ of the single eigenvector $\mathbf{v}_k^k$ and the $m_k - 1$ generalized eigenvectors, and an m_k -long vector $\mathbf{c}_k$ of eigensolution coefficients according to Eqs. (35) and (36), respectively. In this case, the response $\mathbf{x}(t)$ may be expressed as

$$\mathbf{x}(t) = \sum_{k=1}^{n_r} \mathbf{V}_k \mathbf{e}^{\mathbf{J}_k t} \mathbf{c}_k. \quad (53)$$

Recall that a critically damped single-degree-of-freedom oscillator in free vibration with natural frequency ω possesses a repeated and defective eigenvalue $\lambda = -\omega$ and is governed by the equation of motion (e.g., see [54])

$$\ddot{x}(t) + 2\omega\dot{x}(t) + \omega^2 x(t) = 0, \quad (54)$$

which has as its solution

$$x(t) = (c_1 + c_2 t) e^{-\omega t}, \quad (55)$$

where the real coefficients c_1 and c_2 are determined by the initial conditions $x(0)$ and $\dot{x}(0)$. Since defective real eigenvalues in multi-degree-of-freedom systems also correspond to critical damping, we seek decoupled degrees of freedom governed by an equation of motion of the form (54). Specifically, much like the case of defective complex conjugate eigenvalues, associated with each defective real eigenvalue λ_k of even algebraic multiplicity m_k are $\bar{m}_k = m_k/2$ identical, critically damped single-degree-of-freedom systems described by

$$\ddot{p}_j^k(t) - 2\lambda_k \dot{p}_j^k(t) + \lambda_k^2 p_j^k(t) = 0 \quad (56)$$

that are subject to different initial conditions $p_j^k(0)$ and $\dot{p}_j^k(0)$ ($j = 1, 2, \dots, \bar{m}_k$). It follows from the decoupled equation of motion (56) that the n_r eigenvalues may be arranged in $\mathbf{\Lambda}$ as

$$\mathbf{\Lambda} = \bigoplus_{k=1}^{n_r} \mathbf{\Lambda}_k, \quad \mathbf{\Lambda}_k = \lambda_k \mathbf{I}_{\bar{m}_k}, \quad (57)$$

where $\mathbf{\Lambda}$ is its own conjugate since the eigenvalues are identical in the case of critical damping. The decoupled system (2) is subsequently established by using Eq. (57) to form the coefficient matrices $\mathbf{D}$ and $\mathbf{\Omega}$ according to Eq. (12). Next, from Eq. (53), we deduce that

$$\mathbf{J}_x = \bigoplus_{k=1}^{n_r} \mathbf{J}_k, \quad \mathbf{V}_x = [\mathbf{V}_1 \quad \dots \quad \mathbf{V}_{n_r}]. \quad (58)$$

As an extension of the single-degree-of-freedom response (55), the $\bar{m}_k$ decoupled solutions $p_j^k(t)$ associated with the equation of motion (56) for the defective eigenvalue λ_k may be arranged in a column vector as

$$\mathbf{p}_k(t) = \begin{bmatrix} p_1^k(t) \\ p_2^k(t) \\ \vdots \\ p_{\bar{m}_k}^k(t) \end{bmatrix} = \begin{bmatrix} (c_1^k + c_2^k t) e^{\lambda_k t} \\ (c_3^k + c_4^k t) e^{\lambda_k t} \\ \vdots \\ (c_{m_k-1}^k + c_{m_k}^k t) e^{\lambda_k t} \end{bmatrix}. \quad (59)$$

Casting Eq. (59) in matrix–vector form,

$$\mathbf{p}_k(t) = \mathbf{E}_k e^{\mathbf{\Gamma}_k t} \mathbf{c}_k \quad (60)$$

for which the order m_k Jordan matrix $\mathbf{\Gamma}_k$ and the $\bar{m}_k \times m_k$ matrix $\mathbf{E}_k$ have the block diagonal structures

$$\mathbf{\Gamma}_k = \bigoplus_{j=1}^{\bar{m}_k} \begin{bmatrix} \lambda_k & 1 \\ 0 & \lambda_k \end{bmatrix}, \quad \mathbf{E}_k = \bigoplus_{j=1}^{\bar{m}_k} [1 \mid 0]. \quad (61)$$

Finally, with the modal response $\mathbf{p}(t)$ given by Eq. (42) (with $n_c = n_r$) so as to be conformable to $\mathbf{\Lambda}$, it follows that

$$\mathbf{J}_p = \bigoplus_{k=1}^{n_r} \mathbf{\Gamma}_k, \quad \mathbf{V}_p = \bigoplus_{k=1}^{n_r} \mathbf{E}_k. \quad (62)$$

The response $\mathbf{x}(t)$ of system (1) may then be obtained via Eq. (28) upon solution of the decoupled system with modal initial conditions given by Eq. (29). Unlike the cases in which system (1) is non-defective or has defective complex eigenvalues only, the order $2n$ transformation matrices $\mathbf{S}_x$ and $\mathbf{S}_p$ generally do not simplify into neatly partitioned matrices with order n blocks when the eigenvalues are real and defective, and the same is true of the matrix exponential product $\mathbf{e}^{\mathbf{J}_x t} \mathbf{e}^{-\mathbf{J}_p t}$. Consequently, it is generally not possible to extract an order n transformation from the state equation (28) in a concise form, such as in Eqs. (19) and (45), that recovers the response $\mathbf{x}(t)$ directly. However, in principle, one may isolate such a transformation from Eq. (28) using a computer algebra system upon calculating $\mathbf{S}_x$ and $\mathbf{S}_p$. In the special case when every defective real eigenvalue λ_k has algebraic multiplicity $m_k = 2$, the Jordan matrices $\mathbf{J}_p$ and $\mathbf{J}_x$ are identical, and hence both algebraic and geometric multiplicities are preserved during decoupling. As a result, transformation (28) is no longer

time-dependent because the matrix exponential product $\mathbf{e}^{\mathbf{J}_k t} \mathbf{e}^{-\mathbf{J}_p t}$ simplifies to an order $2n$ identity matrix. Moreover, when system (1) and its decoupled form (2) are cast in the state space according to Eqs. (30) and (31), respectively, the relationship (34) between the state matrices $\mathbf{A}$ and $\mathbf{B}$ reduces to a similarity transformation. To summarize, when the coupled n -degree-of-freedom system (1) has n_r defective real eigenvalues λ_k with even algebraic multiplicities m_k , it may be decoupled into n degrees of freedom that are critically damped. Associated with each of the n_r defective eigenvalues are $m_k/2$ identical, critically damped systems subject to different initial conditions.

3.5.2. Odd algebraic multiplicity with an unpaired distinct eigenvalue

Let the n -degree-of-freedom system (1) have all real eigenvalues, of which $2r+1$ are distinct. By the method of phase synchronization, $2r$ of these eigenvalues are paired as real quadratic conjugates to generate r decoupled systems that are overdamped. What then is to be done with the unpaired distinct eigenvalue? We shall refer to this eigenvalue as a free eigenvalue and denote it by λ_f . Since $2r+1$ is odd, then $2(n-r)-1$ is also odd. Assuming each defective eigenvalue has unit geometric multiplicity, it is implied that, even if some of the defective eigenvalues are of even algebraic multiplicity, at least one eigenvalue will necessarily have odd algebraic multiplicity. In order to decouple a system of this nature, we will need to introduce an additional, and rather peculiar, criterion for the pairing of eigenvalues. To streamline our presentation, we consider the case when $r=0$ so that system (1) has only one distinct eigenvalue, which is necessarily a free eigenvalue λ_f . In addition, we assume that the remaining defective real eigenvalues are identical, i.e., there is a single defective real eigenvalue λ_1 of algebraic multiplicity $m_1 = 2n-1$. The corresponding response $\mathbf{x}(t)$ is given by

$$\mathbf{x}(t) = \mathbf{V}_1 \mathbf{e}^{\mathbf{J}_1 t} \mathbf{c}_1 + \mathbf{v}_f e^{\lambda_f t} c_f, \quad (63)$$

where the order m_1 Jordan block $\mathbf{J}_1$, the $n \times m_1$ matrix $\mathbf{V}_1$ of vectors, and the m_1 -dimensional vector $\mathbf{c}_1$ of eigensolution coefficients are as detailed in Eqs. (35) and (36), respectively. In Eq. (63), $\mathbf{v}_f$ and c_f denote the eigenvector and eigensolution coefficient, respectively, associated with the free eigenvalue λ_f . Let $\hat{m}_1 = m_1 - 1 = 2(n-1)$. Similar to the case of even algebraic multiplicity, the single defective real eigenvalue λ_1 of odd algebraic multiplicity $m_1 = 2n-1$ has associated with it $\bar{m}_1 = \hat{m}_1/2 = n-1$ decoupled degrees of freedom $p_j^1(t)$ ($j = 1, 2, \dots, \bar{m}_1$) that are critically damped and governed by the equation of motion (56) with $k=1$. Since system (1) is to be decoupled into n independent subsystems, what is the remaining degree of freedom and its corresponding equation of motion? The free eigenvalue λ_f remains unpaired, and because the defective eigenvalue λ_1 is the only other eigenvalue, we must conclude that it is necessary to pair λ_1 and λ_f in order to completely decouple system (1). Treating the eigenvalues λ_1 and λ_f as real quadratic conjugates, this unusual pairing implies that the remaining degree of freedom $p^*(t)$ is effectively overdamped and governed by the equation of motion

$$\ddot{p}^*(t) - (\lambda_1 + \lambda_f) \dot{p}^*(t) + \lambda_1 \lambda_f p^*(t) = 0. \quad (64)$$

Consequently, by Eqs. (56) (with $k=1$) and (64), the eigenvalues may be arranged in $\mathbf{\Lambda}$ and its conjugate $\hat{\mathbf{\Lambda}}$ according to

$$\mathbf{\Lambda} = \lambda_1 \mathbf{I}_{\bar{m}_1} \oplus \lambda_1, \quad \hat{\mathbf{\Lambda}} = \lambda_1 \mathbf{I}_{\bar{m}_1} \oplus \lambda_f \quad (65)$$

from which the coefficient matrices $\mathbf{D}$ and $\mathbf{\Omega}$ defining the decoupled system (2) may be determined via Eq. (12). A comparison of Eqs. (24) and (63) reveals that

$$\mathbf{J}_x = \mathbf{J}_1 \oplus \lambda_f, \quad \mathbf{V}_x = [\mathbf{V}_1 \mid \mathbf{v}_f]. \quad (66)$$

With the assistance of Eqs. (59)–(61), it is straightforward to show that the responses $p_j^1(t)$ of the $\bar{m}_1$ critically damped degrees of freedom have the equivalent representations

$$\mathbf{p}_1(t) = \begin{bmatrix} p_1^1(t) \\ p_2^1(t) \\ \vdots \\ p_{\bar{m}_1}^1(t) \end{bmatrix} = \begin{bmatrix} (c_1^1 + c_2^1 t) e^{\lambda_1 t} \\ (c_3^1 + c_4^1 t) e^{\lambda_1 t} \\ \vdots \\ (c_{m_1-1}^1 + c_{m_1}^1 t) e^{\lambda_1 t} \end{bmatrix} = \mathbf{E}_1 \mathbf{e}^{\mathbf{\Gamma}_1 t} \mathbf{c}_1 \quad (67)$$

for which the matrices $\mathbf{\Gamma}_1$ and $\mathbf{E}_1$ are

$$\mathbf{\Gamma}_1 = \bigoplus_{j=1}^{\bar{m}_1} \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}, \quad \mathbf{E}_1 = \bigoplus_{j=1}^{\bar{m}_1} [1 \mid 0]. \quad (68)$$

The response $p^*(t)$ of the overdamped degree of freedom corresponding to the equation of motion (64) is given by

$$p^*(t) = c_f e^{\lambda_f t} + c_{m_1}^1 e^{\lambda_1 t}. \quad (69)$$

It is important to note that the eigensolution coefficient associated with the defective eigenvalue λ_1 is the coefficient $c_{m_1}^1$ that remains after having established $\bar{m}_1$ critically damped systems. This ensures that the initial conditions for $p^*(t)$ are independent of those for the critically damped solutions $p_j^1(t)$. Structuring the modal response vector as

$$\mathbf{p}(t) = [\mathbf{p}_1^T(t) \quad p^*(t)]^T \quad (70)$$

to be conformable to Λ , it is implied by Eqs. (67) and (69) that

$$\mathbf{J}_p = \Gamma_1 \oplus \lambda_1 \oplus \lambda_f, \quad \mathbf{V}_p = \mathbf{E}_1 \oplus [1 \mid 1]. \quad (71)$$

We are now in a position to solve the decoupled system and then retrieve the response $\mathbf{x}(t)$ of system (1) via transformation (28). In summary, if the coupled n -degree-of-freedom system (1) possesses a free eigenvalue λ_f and a single defective real eigenvalue λ_1 with odd algebraic multiplicity $m_1 = 2n-1$ (i.e., $r=0$), then the system may be decoupled into $(m_1-1)/2 = n-1$ identical, independent critically damped oscillators subject to different initial conditions and an overdamped degree of freedom formed by pairing the defective eigenvalue λ_1 with λ_f . In the general case when $r > 0$, the $2r$ distinct real eigenvalues of system (1) are arranged into r real quadratic conjugate pairs, and thus the matrices Λ , $\hat{\Lambda}$, $\mathbf{J}_x$, $\mathbf{V}_x$, $\mathbf{J}_p$, and $\mathbf{V}_p$ are constructed blockwise by a direct sum of Eqs. (65), (66) and (71) with their respective non-defective counterparts. Lastly, since the pairing of non-defective real eigenvalues as real quadratic conjugates is not unique, it should be no surprise that the free eigenvalue λ_f is also not unique, and hence the coefficient matrices $\mathbf{D}$ and $\mathbf{\Omega}$ of the decoupled system may take on a variety of forms.

Is it possible to decouple a system of this type by leaving the free eigenvalue λ_f unpaired and have Eq. (28) as a valid transformation to determine the system's response $\mathbf{x}(t)$? A short answer to this question is “no” because the second-order structure inherent in transformation (28), based on eigenvalue pairing, would otherwise be violated. To elaborate upon this, replace the matrix $[1 \mid 1]$ in $\mathbf{V}_p$ from Eq. (71) with $[0 \mid 1]$ so that the free eigenvalue λ_f is no longer paired with the defective real eigenvalue λ_1 . It can be verified that the order $2n$ transformation matrix $\mathbf{S}_p$ is singular when $\mathbf{V}_p$ has this structure, and hence the response $\mathbf{x}(t)$ of system (1) may not be recovered from Eq. (28). Indeed, since it is implied from Eq. (69) that the decoupled solution $p^*(t)$ would correspond to a massless first-order system in this case, system (1) is decoupled not into the form (2), but rather

$$\mathbf{A}_2 \ddot{\mathbf{p}}(t) + \mathbf{A}_1 \dot{\mathbf{p}}(t) + \mathbf{A}_0 \mathbf{p}(t) = \mathbf{0}, \quad (72)$$

$$\mathbf{A}_2 = \mathbf{I}_{\bar{m}_1} \oplus \mathbf{0}, \quad \mathbf{A}_1 = -(2\lambda_1 \mathbf{I}_{\bar{m}_1} \oplus \lambda_f), \quad \mathbf{A}_0 = (\lambda_1)^2 \mathbf{I}_{\bar{m}_1} \oplus \lambda_f^2. \quad (73)$$

From Eq. (73), the leading coefficient matrix $\mathbf{A}_2$ is singular, and hence the decoupled system (72) has an infinite eigenvalue (e.g., see [44, p. 255]). Since the decoupling transformation is eigenvalue-preserving, this implies that the original system (1) must also have an infinite eigenvalue, but we know that is not the case because the mass matrix $\mathbf{M}$ is positive definite. Thus, it is not surprising that the transformation matrix $\mathbf{S}_p$ is singular in this situation because of an apparent disconnect between the spectrum of the original system (1) and that of the decoupled form (72).

3.5.3. Odd algebraic multiplicity

Suppose system (1) possesses an even number $n_r < 2n$ of defective real eigenvalues λ_k ($k = 1, 2, \dots, n_r$), each of odd algebraic multiplicity m_k and unit geometric multiplicity, such that $m_1 + \dots + m_{n_r} = 2n$. We wish to show that a system of the type described here may be decoupled in a manner similar to that in Section 3.5.2, namely by pairing particular eigenvalues in a somewhat peculiar way. To streamline our discussion, we shall consider the case when system (1) has $n_r = 2$ defective eigenvalues λ_1 and λ_2 with algebraic multiplicities m_1 and m_2 , respectively, where $m_1 \neq m_2$ in general. The response $\mathbf{x}(t)$ may be cast in the form of Eq. (53) with $n_r = 2$ and where $\mathbf{J}_k$ ($k = 1, 2$), $\mathbf{V}_k$, and $\mathbf{c}_k$ have the structure (35) and (36), respectively. Since the defective real eigenvalues λ_1 and λ_2 have odd algebraic multiplicity, decoupling begins in essentially the same way as in the procedure outlined in Section 3.5.2. That is to say, associated with each defective eigenvalue λ_k are $\bar{m}_k = \hat{m}_k/2 = (m_k-1)/2$ critically damped, decoupled coordinates $p_j^k(t)$ ($j = 1, 2, \dots, \bar{m}_k$) governed by the equation of motion (56). The independent critically damped systems (56) corresponding to λ_1 and λ_2 account for $\bar{m}_1 + \bar{m}_2 = n-1$ degrees of freedom of the decoupled system (2). What about the remaining degree of freedom? Because the defective eigenvalues λ_1 and λ_2 are the only eigenvalues, we have no choice but to pair them if complete decoupling is to be achieved. As $\lambda_1 \neq \lambda_2$, treat the eigenvalues as real quadratic conjugates so that their pairing generates an effectively overdamped degree of freedom $p^*(t)$ governed by the equation of motion

$$\ddot{p}^*(t) - (\lambda_1 + \lambda_2)\dot{p}^*(t) + \lambda_1\lambda_2 p^*(t) = 0. \quad (74)$$

It follows from Eqs. (56) and (74) that we may form Λ and its conjugate $\hat{\Lambda}$ as

$$\Lambda = \lambda_1 \mathbf{I}_{\bar{m}_1} \oplus \lambda_1 \oplus \lambda_2 \mathbf{I}_{\bar{m}_2}, \quad \hat{\Lambda} = \lambda_1 \mathbf{I}_{\bar{m}_1} \oplus \lambda_2 \oplus \lambda_2 \mathbf{I}_{\bar{m}_2}, \quad (75)$$

which then allows us to construct the coefficient matrices $\mathbf{D}$ and $\mathbf{\Omega}$ that define the decoupled system (2) using Eq. (12). The response $\mathbf{x}(t)$ is governed by Eq. (53) with $n_r = 2$, and thus the matrices $\mathbf{J}_x$ and $\mathbf{V}_x$ are formed according to Eq. (58) with $n_r = 2$. Analogous to Eqs. (59) and (60), the $\bar{m}_k$ critically damped responses $p_j^k(t)$ corresponding to each defective eigenvalue λ_k may be cast in the compact matrix-vector form

$$\mathbf{p}_k(t) = \begin{bmatrix} p_1^k(t) \\ p_2^k(t) \\ \vdots \\ p_{\bar{m}_k}^k(t) \end{bmatrix} = \begin{bmatrix} (c_1^k + c_2^k t) e^{\lambda_k t} \\ (c_3^k + c_4^k t) e^{\lambda_k t} \\ \vdots \\ (c_{\bar{m}_k-1}^k + c_{\bar{m}_k}^k t) e^{\lambda_k t} \end{bmatrix} = \mathbf{E}_k e^{\Gamma_k t} \mathbf{c}_k, \quad (76)$$

where the matrices Γ_k and $\mathbf{E}_k$ are as given in Eq. (61). The solution $p^*(t)$ of the overdamped system governed by the equation of motion (74) is of the form

$$p^*(t) = c_{m_1}^1 e^{\lambda_1 t} + c_{m_2}^2 e^{\lambda_2 t}. \quad (77)$$

The eigensolution coefficients $c_{m_1}^1$ and $c_{m_2}^2$ associated with the defective eigenvalues λ_1 and λ_2 , respectively, are those coefficients that do not correspond to the $\bar{m}_1 + \bar{m}_2$ critically damped systems, and thus the initial conditions for $p^*(t)$ are independent of those for the critically damped solutions $p_j^k(t)$. Based on the structure of Λ , take the modal response vector $\mathbf{p}(t)$ as

$$\mathbf{p}(t) = [\mathbf{p}_1^T(t) \ p^*(t) \ \mathbf{p}_2^T(t)]^T, \quad (78)$$

and hence Eqs. (76) and (78) imply that

$$\mathbf{J}_p = \Gamma_1 \oplus \lambda_1 \oplus \Gamma_2 \oplus \lambda_2, \quad \mathbf{V}_p = [\mathbf{E}_1 \oplus \mathbf{1} \oplus \mathbf{E}_2 \mid \mathbf{e}_{\bar{m}_1+1}^{\bar{m}}], \quad (79)$$

where $\mathbf{e}_l^i$ denotes a unit vector of length l that has 1 as its i th element and 0 for the rest, and $\bar{m} = (m_1 + m_2)/2$. The decoupled system can now be readily solved, and the response $\mathbf{x}(t)$ of system (1) may be recovered from transformation (28). To summarize, if the coupled system (1) has two defective real eigenvalues λ_1 and λ_2 with odd algebraic multiplicities m_1 and m_2 , respectively, it may be decoupled into $(m_1 - 1)/2$ and $(m_2 - 1)/2$ identical, critically damped degrees of freedom with different initial conditions and one degree of freedom that is overdamped, formed by pairing the defective eigenvalues λ_1 and λ_2 . In the general case of even $n_r < 2n$, the matrices Λ , $\hat{\Lambda}$, $\mathbf{J}_p$, and $\mathbf{V}_p$ are constructed by a direct sum of $n_r/2$ sub-matrices of the form (75) and (79), respectively. Likewise, the modal response vector $\mathbf{p}(t)$ is partitioned conformably with Λ into $n_r/2$ sub-vectors with the structure (78). The matrices $\mathbf{J}_x$ and $\mathbf{V}_x$ are still given by Eq. (58). It should be noted that the decoupled system is not unique in this case since the pairing of the defective eigenvalues as real quadratic conjugates is not unique.

3.6. Relaxation of the unit geometric multiplicity constraint

How are the decoupling procedures outlined in the previous sections affected when the unit geometric multiplicity restriction on the defective eigenvalues is lifted? In the case of complex eigenvalues, easing of this constraint is a non-issue since the complex eigenvalues and their associated eigenvectors must always occur in complex conjugate pairs for real system matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$. Indeed, transformations (43) and (45) remain valid when the defective complex eigenvalues λ_k have $\rho_k > 1$ eigenvectors so long as the order m_k Jordan block $\mathbf{J}_k$ given in Eq. (35) is replaced with

$$\mathbf{J}_k = \begin{bmatrix} \lambda_k \mathbf{I}_{\rho_k-1} & 0 & 0 & \cdots & 0 \\ 0 & \lambda_k & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_k & 1 \\ 0 & \cdots & 0 & 0 & \lambda_k \end{bmatrix}. \quad (80)$$

Note that the nilpotent matrix $\mathbf{N}_k$ is redefined by Eq. (80).

When there are defective real eigenvalues that have more than a single eigenvector associated with them, decoupling becomes an even more delicate task than described earlier, and this statement is best demonstrated through a simple example. Suppose system (1) has a lone distinct real eigenvalue, necessarily making it a free eigenvalue λ_f , and a single defective real eigenvalue λ_1 of odd algebraic multiplicity m_1 and geometric multiplicity $\rho_1 = 2$ (i.e., the defective eigenvalue has two associated eigenvectors). In this case, the methodology presented in Section 3.5.2 fails to decouple system (1) since the first eigenvector $\mathbf{v}_1^1$ for λ_1 does not initiate a Jordan chain, and hence its corresponding eigensolution should be treated as if it has been generated from a distinct eigenvalue. Consequently, the defective eigenvalue λ_1 , with associated eigenvector $\mathbf{v}_1^1$, and the free eigenvalue λ_f are paired as real quadratic conjugates to yield an overdamped degree of freedom upon decoupling. Contrast this with the case of unit geometric multiplicity: an overdamped degree of freedom is formed from pairing λ_1 and λ_f , but associated with λ_1 is the generalized eigenvector $\mathbf{v}_{m_1}^1$. Lastly, since the Jordan chain for λ_1 is of even length $m_1 - 1$, the decoupling procedure in Section 3.5.1 may be utilized to generate $(m_1 - 1)/2$ identical, critically damped degrees of freedom. Of course, there are many other combinations of eigenvalues and their multiplicities to consider, but the aforementioned example sufficiently illustrates the cautious manner in which decoupling should be approached for systems with defective real eigenvalues of arbitrary (yet permissible) geometric multiplicity. Moreover, the point to take from this example is that, with careful attention to the pairing of eigenvalues, decoupling is still possible using the basic procedures outlined in Section 3.5.

4. Illustrative examples

Here we provide several examples illustrating the decoupling procedures developed in this paper for defective systems in free vibration. Without loss of generality, we take the mass matrix $\mathbf{M} = \mathbf{I}$ in all examples for convenience since we may

always convert a system $\tilde{\mathbf{M}}\ddot{\mathbf{y}}(t) + \tilde{\mathbf{C}}\dot{\mathbf{y}}(t) + \tilde{\mathbf{K}}\mathbf{y}(t) = \mathbf{0}$ (with $\tilde{\mathbf{M}}$, $\tilde{\mathbf{C}}$, and $\tilde{\mathbf{K}}$ positive definite) into system (1) with $\mathbf{M} = \mathbf{I}$ through a coordinate transformation $\mathbf{y}(t) = \tilde{\mathbf{M}}^{-1/2}\mathbf{x}(t)$, followed by multiplication on the left by $\tilde{\mathbf{M}}^{-1/2}$ (i.e., $\mathbf{C} = \tilde{\mathbf{M}}^{-1/2}\tilde{\mathbf{C}}\tilde{\mathbf{M}}^{-1/2}$, etc.).

4.1. Example 1

A non-classically damped, 2-degree-of-freedom system has damping matrix $\mathbf{C}$ and stiffness matrix $\mathbf{K}$ given by

$$\mathbf{C} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix}. \quad (81)$$

The initial conditions are prescribed as $\mathbf{x}(0) = [-1, 1]^T$ and $\dot{\mathbf{x}}(0) = [-2, 2]^T$. Solving the associated quadratic eigenvalue problem, we find that there is $n_c = 1$ pair of defective complex conjugate eigenvalues of algebraic multiplicity $m_1 = 2$ and unit geometric multiplicity:

$$\lambda_1 = -1 + i\sqrt{2}, \quad \mathbf{v}_1^1 = \begin{bmatrix} -i\sqrt{2} \\ 1 \end{bmatrix}, \quad \mathbf{v}_2^1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}. \quad (82)$$

The sole eigenvector $\mathbf{v}_1^1$ has not been subjected to any normalization scheme. Based on the methodology presented in Section 3.4, pairing of the complex conjugate eigenvalues implies that, from Eq. (40),

$$\mathbf{\Lambda} = (-1 + i\sqrt{2})\mathbf{I}_2. \quad (83)$$

It follows from Eqs. (35), (36), and (41) that the matrices $\mathbf{J}$, $\mathbf{N}$, and $\mathbf{V}$ are of the form

$$\mathbf{J} = \begin{bmatrix} -1 + i\sqrt{2} & 1 \\ 0 & -1 + i\sqrt{2} \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} -i\sqrt{2} & 3 \\ 1 & 0 \end{bmatrix}. \quad (84)$$

By Eq. (20) with conjugation strictly complex, the transformation matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ are

$$\mathbf{T}_1 = \begin{bmatrix} -1 & 3 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{T}_2 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}. \quad (85)$$

Using Eqs. (12) and (83) to construct the decoupled system's coefficient matrices $\mathbf{D}$ and $\mathbf{\Omega}$,

$$\mathbf{D} = 2\mathbf{I}_2, \quad \mathbf{\Omega} = 3\mathbf{I}_2. \quad (86)$$

Eq. (86) implies that the decoupled system consists of $m_1 = 2$ identical, underdamped single-degree-of-freedom oscillators. However, the initial conditions for each decoupled coordinate $p_j^1(t) = p_j(t)$ ($j = 1, 2$) are different: $\mathbf{p}(0) = [1, 0.5]^T$ and $\dot{\mathbf{p}}(0) = [1.5, -3]^T$ from Eq. (44). The decoupled solution $\mathbf{p}(t)$ is illustrated in Fig. 1(a), and the system response $\mathbf{x}(t)$ obtained via transformation (45) is shown in Fig. 1(b). It can be verified that the solution by direct numerical integration of the original system and that obtained by Eq. (45) are indeed the same.

4.2. Example 2

Consider a non-classically damped, 2-degree-of-freedom system for which the damping matrix $\mathbf{C}$ and stiffness matrix $\mathbf{K}$ are given by

$$\mathbf{C} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}. \quad (87)$$

Let the initial conditions be $\mathbf{x}(0) = [1, -1]^T$ and $\dot{\mathbf{x}}(0) = [-2, -1]^T$. Solution of the associated quadratic eigenvalue problem reveals that the system possesses a single eigenvalue, a defective real eigenvalue $\lambda_1 = -1$ with algebraic multiplicity $m_1 = 4$

Fig. 1. Free response of Example 1. (a) Decoupled solutions $p_j(t)$ ($j = 1, 2$). (b) System responses $x_j(t)$.

Fig. 2. Free response of Example 2. (a) Decoupled solutions $p_j(t)$ ($j=1,2$). (b) System responses $x_j(t)$.

and unit geometric multiplicity. The corresponding eigenvector $\mathbf{v}_1^1$ and generalized eigenvectors $\mathbf{v}_i^1$ ($i=2, 3, 4$) are

$$\mathbf{v}_1^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4^1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \quad (88)$$

The sole eigenvector $\mathbf{v}_1^1$ has not been subjected to any normalization scheme. Since the algebraic multiplicity of the single eigenvalue λ_1 is even, we follow the decoupling procedure presented in Section 3.5.1 with $n_r = 1$. As $\bar{m}_1 = 2$, Eq. (57) yields

$$\mathbf{\Lambda} = \hat{\mathbf{\Lambda}} = -\mathbf{I}_2. \quad (89)$$

By Eqs. (35), (36), and (58),

$$\mathbf{J}_x = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{V}_x = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}, \quad (90)$$

and it follows from Eqs. (61) and (62) that

$$\mathbf{J}_p = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{V}_p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \quad (91)$$

From Eqs. (12) and (89), the coefficient matrices $\mathbf{D}$ and $\mathbf{\Omega}$ of the decoupled system are given by

$$\mathbf{D} = 2\mathbf{I}_2, \quad \mathbf{\Omega} = \mathbf{I}_2. \quad (92)$$

Thus, the decoupled system is composed of $\bar{m}_1 = 2$ identical, critically damped single-degree-of-freedom oscillators. The initial conditions for the decoupled coordinates $p_j^1(t) = p_j(t)$ ($j=1, 2$) are $\mathbf{p}(0) = [0, -2]^T$ and $\dot{\mathbf{p}}(0) = [1, 2]^T$ from Eq. (29). The response $\mathbf{p}(t)$ of the decoupled system is illustrated in Fig. 2(a), and the system response $\mathbf{x}(t)$ recovered from transformation (28) is shown in Fig. 2(b). It can be verified that the solution by direct numerical integration of the original system and that obtained by Eq. (28) are indeed the same.

4.3. Example 3

Suppose a non-classically damped, 2-degree-of-freedom system with damping matrix $\mathbf{C}$ and stiffness matrix $\mathbf{K}$ given by

$$\mathbf{C} = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}. \quad (93)$$

The initial conditions are prescribed as $\mathbf{x}(0) = [-1, 1]^T$ and $\dot{\mathbf{x}}(0) = [2, 1]^T$. Solving the associated quadratic eigenvalue problem, we find that the system has a defective real eigenvalue $\lambda_1 = -1$ with algebraic multiplicity $m_1 = 3$ and unit geometric multiplicity whose non-normalized eigenvector $\mathbf{v}_1^1$ and generalized eigenvectors $\mathbf{v}_i^1$ ($i=2, 3$) are

$$\mathbf{v}_1^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3^1 = \begin{bmatrix} 0.5 \\ -1 \end{bmatrix}. \quad (94)$$

Fig. 3. Free response of Example 3. (a) Decoupled solutions $p_j(t)$ ($j=1,2$). (b) System responses $x_j(t)$.

The remaining eigenvalue is necessarily real and unpaired, and hence it is a free eigenvalue: $\lambda_f = -2$, $\mathbf{v}_f = [1, -1]^T$. This system may be decoupled using the procedure outlined in Section 3.5.2. Since $\hat{m}_1 = 2$ and $\bar{m}_1 = 1$, Eq. (65) implies that

$$\mathbf{\Lambda} = -\mathbf{I}_2, \quad \hat{\mathbf{\Lambda}} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}. \quad (95)$$

From Eqs. (35), (36), and (66), we have

$$\mathbf{J}_x = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}, \quad \mathbf{V}_x = \begin{bmatrix} 1 & 0 & 0.5 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}, \quad (96)$$

while Eqs. (68) and (71) yield

$$\mathbf{J}_p = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}, \quad \mathbf{V}_p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \quad (97)$$

By Eqs. (12) and (95), the decoupled system's coefficient matrices $\mathbf{D}$ and $\mathbf{\Omega}$ are

$$\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad \mathbf{\Omega} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}. \quad (98)$$

The first row of the decoupled system defined by Eq. (98) represents the $\bar{m}_1 = 1$ critically damped degree of freedom $p_1^*(t) = p_1(t)$ associated with the defective eigenvalue λ_1 , while the second row corresponds to the overdamped degree of freedom $p^*(t) = p_2(t)$ generated by pairing λ_1 with the free eigenvalue λ_f . Using Eq. (29), the modal initial conditions are $\mathbf{p}(0) = [-3, 2]^T$ and $\dot{\mathbf{p}}(0) = [6, -4]^T$. The decoupled solution $\mathbf{p}(t)$ is illustrated in Fig. 3(a), and the system response $\mathbf{x}(t)$ obtained via transformation (28) is shown in Fig. 3(b). It can be verified that the solution by direct numerical integration of the original system and that obtained by Eq. (28) are indeed the same.

4.4. Example 4

A non-classically damped, 3-degree-of-freedom system has damping matrix $\mathbf{C}$ and stiffness matrix $\mathbf{K}$ given by

$$\mathbf{C} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}. \quad (99)$$

Take as initial conditions $\mathbf{x}(0) = [1, 0, -1]^T$ and $\dot{\mathbf{x}}(0) = [2, 1, 0]^T$. Solution of the associated quadratic eigenvalue problem reveals that the system possesses a defective real eigenvalue $\lambda_1 = -1$ with algebraic multiplicity $m_1 = 4$ and geometric multiplicity $\rho_1 = 2$. The corresponding eigenvectors $\mathbf{v}_i^1$ ($i=1, 2$) and generalized eigenvectors $\mathbf{v}_j^1$ ($j=3, 4$) are

$$\mathbf{v}_1^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2^1 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3^1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \quad \mathbf{v}_4^1 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}. \quad (100)$$

The remaining eigenvalues are real and distinct:

$$\lambda_2 = -3.41, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -0.41 \\ 0.17 \end{bmatrix}, \quad \lambda_3 = -0.59, \quad \mathbf{v}_3 = \begin{bmatrix} 0.17 \\ 0.41 \\ 1 \end{bmatrix}. \quad (101)$$

The eigenvectors $\mathbf{v}_1^1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ have not been subjected to any normalization scheme. To decouple this system with mixed defective and non-defective real eigenvalues, the matrices $\mathbf{J}_x$, $\mathbf{V}_x$, etc., will need to be constructed blockwise via direct sum using the matrix structures discussed at the end of Section 3.1 for non-defective eigenvalues and presented in Section 3.5 for defective real eigenvalues. However, while the algebraic multiplicity of the defective real eigenvalue λ_1 is even, we may not employ the decoupling methodology of Section 3.5.1 because λ_1 has more than one associated eigenvector. Instead, proceed with decoupling as follows. Pair the defective eigenvalue λ_1 with corresponding eigenvector $\mathbf{v}_1^1$ and, say, the distinct eigenvalue λ_3 as real quadratic conjugates so as to yield an overdamped degree of freedom. Of course, one may pair λ_1 with λ_2 instead, if preferred. The remaining distinct eigenvalue λ_2 necessarily becomes a free eigenvalue (i.e., $\lambda_f = \lambda_2$). Moreover, the Jordan chain for λ_1 is of odd length $m_1 - 1 = 3$, and thus we may use the procedure outlined in Section 3.5.2 to generate $(m_1 - 2)/2 = 1$ critically damped degree of freedom associated with λ_1 and an overdamped degree of freedom by pairing λ_1 with λ_2 . To implement this, use Eqs. (9) and (65) to construct

$$\mathbf{A} = -\mathbf{I}_3, \quad \hat{\mathbf{A}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3.41 & 0 \\ 0 & 0 & -0.59 \end{bmatrix}. \quad (102)$$

With the assistance of Eq. (66),

$$\mathbf{J}_x = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3.41 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.59 \end{bmatrix}, \quad \mathbf{V}_x = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0.17 \\ 0 & 2 & -1 & -0.41 & 0 & 0.41 \\ 2 & -2 & 0 & 0.17 & 0 & 1 \end{bmatrix}. \quad (103)$$

In addition, Eqs. (68) and (71) imply that

$$\mathbf{J}_p = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3.41 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.59 \end{bmatrix}, \quad \mathbf{V}_p = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}. \quad (104)$$

From Eq. (102), the decoupled system's coefficient matrices $\mathbf{D}$ and $\mathbf{\Omega}$ are given by

$$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4.41 & 0 \\ 0 & 0 & 1.59 \end{bmatrix}, \quad \mathbf{\Omega} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3.41 & 0 \\ 0 & 0 & 0.59 \end{bmatrix}. \quad (105)$$

To summarize, the decoupled system is composed of (i) a single critically damped degree of freedom $p_1^1(t) = p_1(t)$ (corresponding to the first row of Eq. (105)) associated with the defective eigenvalue λ_1 ; (ii) an overdamped degree of freedom $p^*(t) = p_2(t)$ (second row of Eq. (105)) generated by pairing λ_1 with the non-defective eigenvalue λ_2 ; and (iii) an

Fig. 4. Free response of Example 4. (a) Decoupled solutions $p_j(t)$ ($j = 1, 2, 3$). (b) System responses $x_j(t)$.

overdamped degree of freedom $p_3(t)$ (third row of Eq. (105)) obtained by pairing λ_1 , with associated eigenvector $\mathbf{v}_1^1$, and the non-defective eigenvalue λ_3 . From Eq. (29), the modal initial conditions are $\mathbf{p}(0) = [0.5, 0.15, 0.65]^T$ and $\dot{\mathbf{p}}(0) = [0, 1.91, -1]^T$. The response $\mathbf{p}(t)$ of the decoupled system is illustrated in Fig. 4(a), and the system response $\mathbf{x}(t)$ recovered from transformation (28) is shown in Fig. 4(b). It can be verified that the solution by direct numerical integration of the original system and that obtained by Eq. (28) are indeed the same.

5. Conclusions

We have shown how classical modal analysis may be extended to decouple any linear dynamical system of the form (1). By formulating a decoupling transformation in a general framework that incorporates previous work [39,40], a complete solution to the problem of decoupling linear dynamical systems, defective or not, in free motion has been provided. In the course of this development, the notion of critical damping in multi-degree-of-freedom systems has also been clarified and expanded. Major results presented in this paper are summarized in the following statements:

1. Any system with defective eigenvalues can be decoupled if one does not insist on preserving geometric multiplicities. In contrast with non-defective systems, the decoupling transformation for defective systems in free vibration is time-dependent in both the configuration and state spaces.
2. All parameters required for generating the decoupled system (2) and evaluating the general decoupling transformation (28) are obtained by solving the quadratic eigenvalue problem (7), determining generalized eigenvectors from the recursive scheme (21) when eigenvalues are defective, and applying an admissible eigenvalue pairing scheme as described herein.
3. Associated with defective complex conjugate eigenvalues are underdamped degrees of freedom.
4. Defective real eigenvalues correspond to critical damping. Depending on the multiplicities of these eigenvalues, decoupling yields some combination of critically damped degrees of freedom and overdamped degrees of freedom, the latter resulting from a necessary pairing of either defective and non-defective eigenvalues or two distinct defective eigenvalues.
5. Decoupling transformation (28) is a direct generalization of classical modal analysis.

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