Two models of default from finance and a model of invasion from ecology

by

Alexandru Hening

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Committee in charge:

Professor Steven N. Evans, Chair
Professor Lawrence C. Evans
Professor Cari Kaufman

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Abstract

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My thesis consists of three different projects.

1) Investors are exposed to credit risk due to the possibility that one or more counterparties in a financial agreement will default; that is, not honor their obligations to make certain payments. It is usually not enough to consider the default of a single firm because of the effect of contagion - the default of one firm is dependent of the other firms in the economy.

   This project considers static models of default that have appeared in the mathematical finance literature. These models are constructed from an underlying graph with a set of nodes V representing firms. They give a probability distribution on $\{0,1\}^V$, where a 1 in the k^{th} coordinate indicates that the k^{th} firm has defaulted at the end of a particular time period. The drawback of these models is that they are static - they do not try to say anything about the distribution of the default times of a group of firms. It is therefore of interest to try to give these models Markovian dynamics. In Chapter 1, much of which has appeared in [EH11], we show in several natural cases that this is not possible.

2) In ecology, the extinction of a population can be described as the first passage through some threshold value for the diffusion process which represents the number of individuals. Similarly, in finance, the default time of a counterparty is sometimes modeled as the first passage time of a credit index process below a barrier. It is therefore relevant to consider the following question: If we know the distribution of the default time can we find a unique barrier which gives this distribution? This is known as the Inverse First Passage Time (IFPT) problem in the literature. We consider a more general 'smoothed' version of the inverse first passage time problem in which the first passage time is replaced by the first instant that the time spent below the barrier exceeds an independent exponential random variable. In Chapter 2, which is based on [EEH12], we

show that any smooth distribution results from some unique continuously differentiable barrier.

3) A fundamental problem in ecology is to understand when it is possible for one species to invade the range of another, established species. Mathematical models for invasibility have contributed significantly to the understanding of the epidemiology of infectious disease outbreaks ([CLSJG05]) and ecological processes ([LM96], [Cas01]).

   There is widespread empirical evidence that invasions can occur when there is significant heterogeneity in space and time in the range of the resident species. This heterogeneity can arise due to variability in abiotic factors (e.g. precipitation, temperature or sunlight) or biotic factors (presence of other competitors or predators). There have only been a a few studies that try to explain how spatio-temporal heterogeneity facilitates invasibility (see, for example, [SLS09]).

   Using ideas from [ERSS], we propose in Chapter 3, which is an expansion of [EHS], a general model of the invasion process with a view to understanding what factors make invasion possible. We consider a stochastic differential equation (SDE) model of a resident population that is living in an environment consisting of n patches and is subject to an attempted invasion by another species.

To my family

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Chapter 1

Non-existence of Markovian time dynamics for graphical models of correlated default

1.1 Introduction

Investors are exposed to *credit risk* due to the possibility that one or more counterparties in a financial agreement will *default*; that is, not honor their obligations to make certain payments. Some examples of default are a consumer or business not making a due payment on a loan, a manufacturer or retailer not paying for goods already received from a supplier, a bond issuer not making coupon or principal payments, or an insolvent financial institution not returning deposited funds to its customers upon demand.

Some credit risk is present in virtually any financial agreement, and a key ingredient in its satisfactory management is a model that produces a sufficiently accurate probability for a given default event. Consequently, there is a large theoretical and applied literature on this topic [BR02, BOW02, DS03, dSR04, Gie04b, Sch04, ZP07, Wag08]. Roughly speaking, models of default lie on a spectrum between the *structural* and *reduced form* ones. For the example of a firm defaulting on its debt obligations, a structural model might include explicit descriptions of the dynamics of the firm's assets, capital holdings and debt structure, whereas a reduced form model would not seek to incorporate the details of the actual mechanism by which the firm is led to default but rather it might typically be something of a "black box" that treats the time of default as a random time with an associated exogenous intensity process having a rather simple structure characterized by a small number of parameters which may have little direct economic interpretation. Although structural models are perhaps theoretically more satisfying because in principle they provide a means of testing how well the factors that cause default are understood, they are often perceived as being too complex and parameter-rich for them to be fitted adequately: defaults are uncommon and even firms within the same sector of the economy can be quite heterogeneous, so there can be insufficient "independent replication" upon which to base statistically sound parameter estimates.

The difficulty of modeling default probabilities is compounded for complex financial instruments such as *collateralized debt obligations* (CDOs) and other structured asset-backed securities that are constructed by, in essence, bundling together a group of borrowers. It is then no longer sufficient to determine the default probability for a single "firm" – rather, it becomes necessary to model the joint probabilities that various subsets of firms in the basket will default, and it is usually not appropriate to treat the defaults of different firms as statistically independent events. The most obvious reason for this absence of independence is that firms are subject to the same background economic environment. Moreover, situations such as the interconnectedness between manufacturers, their parts suppliers and the retailers who sell their products can cause problems for one firm in such a network to spread to others via a process that is usually described as *contagion*. A small sampling of the substantial empirical and modeling work on this phenomenon of *correlated default* is [DL01b, DL01a, HW01, SS01, Zho01, FM03, Gie03, LG03, Gie04a, GW04, GW06, DDKS07, EGG07, JZ07, Yu07, ES09, GGD09, CDGH10].

In this chapter we investigate a particularly appealing class of models for correlated default in [FGMS08] (see also [MV05, KMH06], where special cases of this model were introduced). The basic model in [FGMS08] does not attempt to describe the time course of defaults for some group of firms. Rather, it is a *one period* model that gives the probability any given subset of the firms will have defaulted at some time during a prescribed time interval.

The ingredients of the model in [FGMS08] are a finite (undirected, simple) graph G with vertex set V and edge set E and two vectors of parameters $\alpha = (\alpha_v)_{v\in V} \in \mathbb{R}^V$ and $\beta = (\beta_e)_{e\in E} \in \mathbb{R}^E$. Each vertex $v \in V$ represents a firm and the graph structure provided by the edges is intended to capture the network of interdependencies between the firms. Write I_v, $v \in V$, for the indicator random variable of the event that firm v defaults; that is, I_v takes the value 1 if firm v defaults and the value 0 otherwise. The probability of a given pattern $\varepsilon = (\varepsilon_v)_{v\in V} \in \{0,1\}^V$ of defaults is

$$\mathbb{P}\{I_v = \varepsilon_v,\ v \in V\} := \frac{1}{Z}\exp(H(\varepsilon)), \tag{1.1.1}$$

where the *Hamiltonian* H is given by

$$H(\varepsilon) := \sum_{u\in V} \alpha_u \varepsilon_u + \sum_{\{v,w\}\in E} \beta_{\{v,w\}}\varepsilon_v\varepsilon_w \tag{1.1.2}$$

and the *partition function* Z is the normalizing constant that ensures the sum over $\{0,1\}^V$ of the probabilities is one. The parameter α_u, $u \in V$, is clearly some measure of the individual propensity of firm u to default. The parameter $\beta_{\{v,w\}}$, $\{v,w\} \in E$, captures in some way the dependence between the defaults of firm v and firm w: if this parameter is positive, then the joint default of both firms is favored, whereas it is discouraged when the parameter is negative. We write $\mathcal{P}(G, \alpha, \beta)$ for the distribution of the random binary vector I.

Note that if we set $Y_v = 2I_v - 1$, $v \in V$, then $(Y_v)_{v \in V} \in \{\pm 1\}^V$ and for $(\sigma_v)_{v \in V} \in \{\pm 1\}^V$ we have

$$\begin{aligned}\mathbb{P}\{Y_v = \sigma_v,\ v \in V\} &= \mathbb{P}\{I_v = (1+\sigma_v)/2,\ v \in V\} \\ &= \frac{1}{Z} \exp\left(\frac{1}{2}\sum_{u \in V} \alpha_u (1+\sigma_u) + \frac{1}{4}\sum_{\{v,w\} \in E} \beta_{\{v,w\}}(1+\sigma_v)(1+\sigma_w)\right) \\ &= \frac{1}{\tilde{Z}} \exp\left(\sum_{u \in V} \gamma_u \sigma_u + \sum_{\{v,w\} \in E} \delta_{\{v,w\}} \sigma_v \sigma_w\right)\end{aligned}$$

for suitable parameters $(\gamma_v)_{v \in V} \in \mathbb{R}^V$ and $(\delta_e)_{e \in E} \in \mathbb{R}^E$ and a corresponding normalization constant $\tilde{Z}$. Thus, the random vector of *spins* $(Y_v)_{v \in V}$ is described by the usual *Ising model* associated with the graph $G = (V, E)$.

It is shown in [FGMS08] that this class of correlated default models is as flexible as one could possibly hope: if J is an arbitrary $\{0,1\}^V$-valued random variable, then there is a choice of the parameters $(\alpha_v)_{v \in V}$ and $(\beta_e)_{e \in E}$ such that I_u has the same distribution as J_u for all $u \in V$ and for all $\{v, w\} \in E$ the pair (I_v, I_w) has the same distribution as (J_v, J_w). Moreover, it is observed in [FGMS08] that it is possible to fit such a model to data using existing techniques such as iterative proportional fitting, various convex optimization techniques, or a number of other "off-the-shelf" numerical optimization methods suitable for large-scale computation.

A significant drawback of the class of models in [FGMS08] is that they don't provide a description of the time dynamics of default: they just give the probability that a given subset of firms have defaulted during some fixed time period without saying anything about the distribution of the times at which the defaults occurred. If we let $[0, T]$ be the time period of interest, then we would like there to be a $\{0,1\}^V$-valued stochastic process $(I(t))_{0 \le t \le T}$ such that

- $I_v(t) = 1$ if and only if firm $v \in V$ has defaulted by time t, so that $I_v(0) = 0$ and the sample paths of $(I_v(t))_{0 \le t \le T}$ are right-continuous and non-decreasing (once a firm defaults it does not "undefault"),
- $\#\{v \in V : I_v(t) \neq I_v(t-)\} \le 1$ for any $t \in [0, T]$ (two or more firms do not default simultaneously – we use the notation $\#B$ denote the cardinality of the set B),
- $I(T)$ is has distribution $\mathcal{P}(G, \alpha, \beta)$.

Furthermore, since $\mathcal{P}(G, \alpha, \beta)$ is supposed to be an appropriate description for the pattern of defaults during $[0, T]$, it is reasonable to require that

- $I(t)$ has distribution $\mathcal{P}(G, \alpha(t), \beta(t))$ for suitable parameters $\alpha(t)$ and $\beta(t)$ when $0 < t < T$.

In this chapter we investigate whether such a process exists within the simplest and perhaps most natural class of models, namely the time-homogeneous Markov chains. Recast in the language of the equivalent Ising model, we are thus asking if it is possible to begin at time 0 with a configuration in which every spin is -1 and then flip spins one at a time from -1 to $+1$ according to Markovian dynamics so that the configuration of spins at time T is distributed according to a prescribed Ising model and at all other times the configuration is described by some Ising model.

We can certainly construct such a chain if $\beta = 0$, so that $\mathcal{P}(G, \alpha, \beta) = \mathcal{P}(G, \alpha, 0)$ is the distribution of a vector $(I_v)_{v \in V}$ of independent $\{0, 1\}$-valued random variables with

$$\mathbb{P}\{I_v = 0\} = \frac{1}{1 + \exp(\alpha_v)}.$$

We simply takes the processes $(I_v(t))_{t \geq 0}$ to be independent, with

$$\mathbb{P}\{I_v(t) = 0\} = \exp(-\lambda_v t),$$

where the jump rate λ_v is chosen so that

$$\exp(-\lambda_v T) = \frac{1}{1 + \exp(\alpha_v)}.$$

Thus, $\lambda_v = \frac{1}{T} \log(1 + \exp(\alpha_v))$ and $I(t)$ has distribution $\mathcal{P}(G, \alpha(t), 0)$, where

$$\frac{1}{1 + \exp(\alpha_v(t))} = \exp(-\lambda_v t) = \exp\left(-\frac{t}{T} \log(1 + \exp(\alpha_v))\right),$$

so that

$$\alpha_v(t) = \log\left((1 + \exp(\alpha_v))^{\frac{t}{T}} - 1\right)$$

for $0 < t < T$.

After establishing some general facts in Section 1.2, we investigate in Sections 1.3, 1.4 and 1.5 whether it is possible to construct a time-homogeneous Markov chain for non-zero β in the following cases:

(I) G is the complete graph K_N in which there are N vertices with each vertex connected to every other one, $\alpha_u(t) = \alpha_v(t)$ for $u, v \in V$, $0 < t \leq T$, and $\beta_e(t) = \beta_f(t)$ for $e, f \in E$, $0 < t \leq T$;

(II) G is the complete bipartite graph $K_{M,N}$ in which V is partitioned into two disjoint subsets $\hat{V}$ and $\check{V}$ of cardinality M and N such that every vertex in $\hat{V}$ is connected to every vertex in $\check{V}$ and there are no other edges, $\alpha_u(t) = \alpha_v(t)$ for $u, v \in V$, $0 < t \leq T$, and $\beta_e(t) = \beta_f(t)$ for $e, f \in E$, $0 < t \leq T$;

(III) G is again the complete bipartite graph $K_{M,N}$, $\alpha_u(t) = \alpha_v(t)$ for $u, v \in \hat{V}$, $0 < t \leq T$, $\alpha_u(t) = \alpha_v(t)$ for $u, v \in \check{V}$, $0 < t \leq T$, and $\beta_e(t) = \beta_f(t)$ for $e, f \in E$, $0 < t \leq T$.

In Model I there is complete symmetry: each firm has the same individual propensity to default and the interdependence between any two firms is the same as that between any other two. Model II and III both describe a situation in which there are two types of firms (say, for example, car manufacturers and auto parts suppliers) and there is only interdependence between firms of different types. In Model II all firms have the same individual propensity to default, whereas in Model III this propensity can depend on the type of the firm.

We conclude in all three cases (with a minor technical restriction for Model III) that it is impossible to construct a time-homogeneous Markov chain with the desired properties unless β is zero; that is, unless the firms behave independently.

1.2 Generalities

It will be notationally more convenient to identify a vector $\varepsilon = (\varepsilon_v)_{v \in V} \in \{0,1\}^V$ with the subset $A = \{v \in V : \varepsilon_v = 1\} \subseteq V$ and regard $\mathcal{P}(G, \alpha, \beta)$ as a probability measure on subsets of V rather than $\{0,1\}^V$. If we extend the definition of $\beta_{\{u,v\}}$ by declaring that $\beta_{\{u,v\}} = 0$ when $\{u,v\} \notin E$ and write $\beta_{\{u,v\}}$ more simply as β_{uv}, then our Hamiltonian, now thought of as function defined on subsets of V, is given by

$$H(A) := \sum_{u \in A} \alpha_u + \sum_{\{u,v\} \subset A} \beta_{uv}. \tag{1.2.1}$$

If we write $\mathbb{P}^H$ for the probability measure $\mathcal{P}(G, \alpha, \beta)$, then

$$\mathbb{P}^H(\{A\}) := \frac{1}{Z} \exp(H(A)), \tag{1.2.2}$$

where

$$Z := \sum_{B \subseteq V} \exp(H(B)).$$

We are interested in the existence of a time-homogeneous Markov chain $X = (X_t)_{t \geq 0}$ that has as its state-space the collection of subsets of V and has the following properties, where we write $Q(A, B)$ for the jump rate from state A to state B:

- $Q(A, B) = 0$ unless $B = A \cup \{v\}$ for some $v \notin A$;
- when $X(0) = \emptyset$, the distribution of $X(T)$ is $\mathbb{P}^H$;
- there are parameter vectors $\alpha(t)$ and $\beta(t)$ for $0 < t \leq T$ such that if we set

  $$H_t(A) := \sum_{u \in A} \alpha_u(t) + \sum_{\{u,v\} \subset A} \beta_{uv}(t),$$

  then $X(t)$ has distribution $\mathbb{P}^{H_t}$ when $X(0) = \emptyset$.

If such a Markov chain exists, we say that the default model *admits time-homogeneous Markovian dynamics.*

Write $A \to B$ if $B = A \cup \{v\}$ for some $v \notin A$. The Kolmogorov forward equations for the chain X with initial state $\emptyset$ become

$$\frac{d}{dt}\mathbb{P}^{H_t}(B) = \sum_{A \to B} \mathbb{P}^{H_t}(A)Q(A,B) + \mathbb{P}^{H_t}(B)Q(B,B), \tag{1.2.3}$$

where, as usual, we put $Q(B,B) := -\sum_{C \neq B} Q(B,C)$.

Denoting the partition function associated with the Hamiltonian H_t by $Z_t := \sum_{C \in E} e^{H_t(C)}$, we have

$$\frac{d}{dt}\mathbb{P}^{H_t}(B) = \frac{d}{dt}\frac{e^{H_t(B)}}{Z_t}. \tag{1.2.4}$$

To further simplify notation, set $R_B = -Q(B,B)$. Because $H_t(\emptyset) = 0$, we see from (1.2.3) and (1.2.4) for $B = \emptyset$ that $t \mapsto Z_t$ is differentiable with

$$-Z_t' = -R_\emptyset Z_t.$$

We require

$$1 = \lim_{t \downarrow 0} \mathbb{P}^{H_t}(\emptyset) = \lim_{t \downarrow 0} \frac{1}{Z_t},$$

and so

$$Z_t = e^{R_\emptyset t}. \tag{1.2.5}$$

It now follows from (1.2.4) that $t \mapsto H_t(B)$ is differentiable for all $B \subseteq V$ with

$$\begin{aligned} \frac{d}{dt}\mathbb{P}^{H_t}(B) &= \frac{d}{dt}\frac{e^{H_t(B)}}{Z_t} \\ &= \frac{Z_t e^{H_t(B)} H_t'(B) - Z_t' e^{H_t(B)}}{Z_t^2}, \end{aligned}$$

and thus (1.2.3) can be re-written as

$$Z_t e^{H_t(B)} H_t'(B) - Z_t' e^{H_t(B)} = \sum_{A \to B} Q(A,B) e^{H_t(A)} Z_t - R_B e^{H_t(B)} Z_t. \tag{1.2.6}$$

Substituting (1.2.5) into (1.2.6) gives

$$H_t'(B) = \sum_{A \to B} Q(A,B) e^{H_t(A) - H_t(B)} + R_\emptyset - R_B. \tag{1.2.7}$$

Note for $u \in V$ that $H_t(\{u\}) = \alpha_u(t)$, and so $t \mapsto \alpha_u(t)$, $t > 0$, is differentiable. Similarly, note for $u, v \in V$ with $u \neq v$ that $H_t(\{u, v\}) = \alpha_u(t) + \alpha_v(t) + \beta_{uv}(t)$, and so $t \mapsto \beta_{uv}(t)$, $t > 0$, is differentiable. Hence, (1.2.7) can be re-written as

$$\begin{aligned}
&\sum_{u \in B} \alpha'_u(t) + \sum_{\{u,v\} \subseteq B} \beta'_{uv}(t) \\
&\quad = \sum_{u \in B} Q(B \setminus \{u\}, B) \exp\left(-\alpha_u(t) - \sum_{v \in B \setminus \{u\}} \beta_{uv}(t) \right) + R_\emptyset - R_B.
\end{aligned} \tag{1.2.8}$$

For $u \in V$, set $Q_u = Q(\emptyset, \{u\})$ and $R_u := R_{\{u\}} = -Q(\{u\}, \{u\})$. Equation (1.2.8) for $B = \{u\}$ is

$$\alpha'_u(t) = Q_u e^{-\alpha_u(t)} + R_\emptyset - R_u. \tag{1.2.9}$$

Hence, by the method of variation of parameters (also called variation of constants),

$$\alpha_u(t) = \log\left(\frac{Q_u}{R_\emptyset - R_u} \left(e^{(R_\emptyset - R_u)t} - 1 \right) \right) \tag{1.2.10}$$

and

$$\alpha'_u(t) = \frac{R_\emptyset - R_u}{1 - e^{-(R_\emptyset - R_u)t}} \tag{1.2.11}$$

when $R_\emptyset \neq R_u$. If $R_\emptyset = R_u$, then

$$\alpha_u(t) = \log\left(Q_u t\right) \tag{1.2.12}$$

and

$$\alpha'_u(t) = \frac{1}{t}. \tag{1.2.13}$$

Note that each function α_u, $u \in V$, is completely determined by the rates $Q_u = Q(\emptyset, \{u\})$, $R_\emptyset = \sum_{v \in V} Q(\emptyset, \{v\})$, and $R_u = \sum_{v \in V \setminus \{u\}} Q(\{u\}, \{u, v\})$, and hence the vector of functions $(\alpha_u)_{u \in V}$ is completely determined by the collection of rates $\{Q(\emptyset, \{u\}) : u \in V\} \cup \{Q(\{u\}, \{u, v\}) : u, v \in V, u \neq v\}$.

For $u, v \in V$, set $Q_{uv} := Q(\{u\}, \{u, v\})$ and $R_{uv} := R_{\{u,v\}} = -Q(\{u, v\}, \{u, v\})$. Equation (1.2.8) for $B = \{u, v\}$ is, upon substituting from (1.2.11),

$$\begin{aligned}
\beta'_{uv}(t) &= Q_{vu} e^{-\alpha_u(t) - \beta_{uv}(t)} + Q_{uv} e^{-\alpha_v(t) - \beta_{vu}(t)} \\
&\quad - \alpha'_u(t) - \alpha'_v(t) - R_{uv} + R_\emptyset \\
&= \frac{Q_{vu}}{Q_u} \frac{R_\emptyset - R_u}{1 - e^{(R_\emptyset - R_u)t}} e^{-\beta_{uv}(t)} + \frac{Q_{uv}}{Q_v} \frac{R_\emptyset - R_v}{1 - e^{(R_\emptyset - R_v)t}} e^{-\beta_{vu}(t)} \\
&\quad - \frac{R_\emptyset - R_u}{1 - e^{-(R_\emptyset - R_u)t}} - \frac{R_\emptyset - R_v}{1 - e^{-(R_\emptyset - R_v)t}} \\
&\quad + R_\emptyset - R_{uv}
\end{aligned} \tag{1.2.14}$$

when $R \neq R_u$ and $R \neq R_v$. Analogous results hold when $R = R_u$ or $R = R_v$. Recall that $\beta_{uv}(t) = \beta_{\{u,v\}}(t) = \beta_{vu}(t)$, and so (1.2.14) is an ordinary differential equation for the function β_{uv} if we treat the rates of the Markov chain as given. In particular, the two vectors of functions $(\alpha_u)_{u\in V}$ and $(\beta_{uv})_{u,v\in V, u\neq v}$ are completely determined by the collection of rates $\{Q(\emptyset, \{u\}) : u \in V\} \cup \{Q(\{u\}, \{u, v\}) : u, v \in V,\ u \neq v\} \cup \{Q(\{u, v\}, \{u, v, w\}) : u, v, w \in V,\ u \neq v \neq w \neq u\}$.

In principle, we could attempt to find values for these rates such that $(\alpha_u(T))_{u\in V}$ and $(\beta_{uv}(T))_{u,v\in V,u\neq v}$ have the required value, substitute the resulting values of $\alpha_u(t)$ and $\beta_{uv}(t)$ into (1.2.8) (using (1.2.11) or (1.2.13) for the values of $\alpha'_u(t)$ and (1.2.14) or its analogues when $R = R_u$ or $R = R_v$ for the values of $\beta'_{uv}(t)$) and hope to either find values for the remaining rates so that (1.2.8) holds for all $B \subseteq V$ or show that this is impossible no matter what our initial choice of rates was. This seems to be a rather forbidding task in general, but we are able to carry it out in the three special cases described in the Introduction.

1.3 Model I: complete symmetry

Recall Model I from the Introduction. The graph G is K_N, the complete graph on N vertices for some N, and there are functions α and β such that

$$\begin{cases} \alpha_u(t) = \alpha(t) \text{ for all } u \in V \\ \beta_{uv}(t) = \beta(t) \text{ for all } u, v \in V\ u \neq v. \end{cases} \tag{1.3.1}$$

Proposition 1.3.1. *Model I with $N \geq 4$ admits time-homogeneous Markovian dynamics if and only if the firms default independently.*

Proof. We observed in the Introduction that the general default model admits Markovian dynamics when firms default independently. So we need to establish a converse for the special case of Model I with $N \geq 4$.

Suppose that a collection of rates exists such that (1.2.8) holds for all subsets B. When $\#B \geq 1$, (1.2.8) becomes

$$\begin{aligned} \#B\alpha'(t) + \binom{\#B}{2}\beta'(t) = \left[\sum_{u\in B} Q(B \setminus \{u\}, B)\right] \exp\left(-\alpha(t) - (\#B - 1)\beta(t)\right) \\ + R_\emptyset - R_B. \end{aligned} \tag{1.3.2}$$

If we average (1.3.2) over all $\binom{N}{k}$ choices of sets B with $\#B = k$ for some $k \geq 1$ and set

$$\lambda_\ell = \binom{N}{\ell}^{-1} \sum_{A\subseteq V, \#A=\ell} R_A, \quad 0 \leq \ell \leq N,$$

we get the equations

$$k\alpha'(t) + \binom{k}{2}\beta'(t) = (\lambda_0 - \lambda_k) \\ + \lambda_{k-1}\frac{k}{N-k+1}e^{-\alpha(t)-(k-1)\beta(t)}, \quad 1 \le k \le N. \tag{1.3.3}$$

Note that $\lambda_\ell > 0$ for $0 \le \ell \le N-1$ and $\lambda_N = 0$.

Equation (1.3.3) for $k = 1$ and $k = 2$ yields

$$\begin{cases} \alpha'(t) = (\lambda_0 - \lambda_1) + \dfrac{\lambda_0}{N}e^{-\alpha(t)} \\ \beta'(t) = (\lambda_0 - \lambda_2) + \lambda_1\dfrac{2}{N-1}e^{-\alpha(t)-\beta(t)} - 2(\lambda_0 - \lambda_1) - 2\dfrac{\lambda_0}{N}e^{-\alpha(t)}. \end{cases} \tag{1.3.4}$$

Substituting the values for $\alpha'(t)$ and $\beta'(t)$ from (1.3.4) into (1.3.3) gives a system of equations of the form

$$\lambda_{k-1}\frac{k}{N-k+1}e^{-(k-1)\beta(t)} - \lambda_1\frac{2}{N-1}e^{-\beta(t)} = a_k e^{\alpha(t)} + b_k, \quad 1 \le k \le N, \tag{1.3.5}$$

for appropriate constants a_k and b_k, $1 \le k \le N$, that depend on the constants λ_ℓ, $0 \le \ell \le N$.

We claim that the continuous function β is constant. Suppose that this is not so. Note that a_k can be non-zero for at most one value of $k \in \{1, \ldots, N\}$, because if $a_{k'} \neq 0$ and $a_{k''} \neq 0$ for $1 \le k' < k'' \le N$, then

$$\frac{\lambda_{k'-1}\frac{k'}{N-k'+1}e^{-(k'-1)\beta(t)} - \lambda_1\frac{2}{N-1}e^{-\beta(t)} - b_{k'}}{a'_k} \\ = \frac{\lambda_{k''-1}\frac{k''}{N-k''+1}e^{-(k''-1)\beta(t)} - \lambda_1\frac{2}{N-1}e^{-\beta(t)} - b_{k''}}{a''_k},$$

and letting t vary over an open interval J such that the image $\{\beta(t) : t \in J\}$ contains an open interval we would conclude that two polynomials of different degrees coincided over an open interval. Because $N \ge 4$, we thus must have $a_k = 0$ for some $k \ge 3$. Observe for such a k that

$$\lambda_{k-1}\frac{k}{N-k+1}e^{-(k-1)\beta(t)} - \lambda_1\frac{2}{N-1}e^{-\beta(t)} = b_k,$$

and again we would conclude that two polynomials of different degrees coincided over an open interval. Therefore, the function β must be a constant, say β^*. Of course, β^* is the pre-specified value for $\beta(T)$.

We now show that $\beta^* = 0$. Equation (1.3.3) now becomes

$$k\alpha'(t) = (\lambda_0 - \lambda_k) + \lambda_{k-1}\frac{k}{N-k+1}e^{-\alpha(t)-(k-1)\beta^*}$$

and hence, by (1.3.4),

$$k\left[(\lambda_0-\lambda_1)+\frac{\lambda_0}{N}e^{-\alpha(t)}\right]=(\lambda_0-\lambda_k)+\lambda_{k-1}\frac{k}{N-k+1}e^{-\alpha(t)-(k-1)\beta^*}. \tag{1.3.6}$$

Each side of (1.3.6) is first degree polynomial in $e^{-\alpha(t)}$ for every $k\in\{1,\ldots,N\}$. It is apparent from the differential equation in (1.3.4) that the function α is not constant (indeed, we solved this equation explicitly in (1.2.10)). Consequently, the coefficients of these two polynomial coincide and hence

$$\begin{cases} k(\lambda_0-\lambda_1)=(\lambda_0-\lambda_k) \\ k\dfrac{\lambda_0}{N}=\lambda_{k-1}\dfrac{k}{N-k+1}e^{-(k-1)\beta^*} \end{cases} \tag{1.3.7}$$

for $1\le k\le N$. Re-arranging (1.3.7), we conclude that

$$\lambda_k=k(\lambda_1-\lambda_0)+\lambda_0$$

for $1\le k\le N$ and

$$\lambda_k=\frac{(N-k)}{N}\lambda_0e^{k\beta^*}$$

for $0\le k\le N-1$. Because $N\ge 4$, this is impossible unless $\beta^*=0$. □

1.4 Model II: Two classes with common individual propensity to default

Recall Model II from the Introduction. The graph G is $K_{M,N}$, the complete bipartite graph with vertex set the disjoint union $V=\hat{V}\sqcup\check{V}$, where $\hat{V}$ has M vertices, $\check{V}$ has N vertices, and there are functions α and β such that

$$\begin{cases} \alpha_u(t)=\alpha(t) \text{ for all } u\in V \\ \beta_{uv}(t)=\beta(t) \text{ for all } u\in\hat{V},\ v\in\check{V}. \end{cases} \tag{1.4.1}$$

Proposition 1.4.1. *Model II with $M\ge 3$ or $N\ge 3$ admits time-homogeneous Markovian dynamics if and only if the firms default independently.*

Proof. As in the proof of Proposition 1.3.1, it suffices from the remarks made in the Introduction about the general model to show that if the model admits time-homogeneous Markovian dynamics, then the firms default independently.

Symmetry considerations similar to those in the proof of Proposition 1.3.1 show that if (1.2.8) holds for some choice of jump rates, then there are constants $\lambda^{\rightarrow}_{m,n}$ and $\lambda^{\uparrow}_{m,n}$,

$0 \le m \le M$ and $0 \le n \le N$, with $\lambda^{\rightarrow}_{M,n} = 0$ for $0 \le n \le N$, $\lambda^{\uparrow}_{m,N} = 0$ for $0 \le m \le M$, and $\lambda^{\rightarrow}_{m,n}$ and $\lambda^{\uparrow}_{m,n}$ strictly positive otherwise such that

$$\begin{aligned}(m+n)\alpha'(t) + mn\beta'(t) = r - \lambda^{\rightarrow}_{m,n} - \lambda^{\uparrow}_{m,n} \\ + \frac{m}{M-m+1}\lambda^{\rightarrow}_{m-1,n}e^{-\alpha(t)-n\beta(t)} \\ + \frac{n}{N-n+1}\lambda^{\uparrow}_{m,n-1}e^{-\alpha(t)-m\beta(t)},\end{aligned} \tag{1.4.2}$$

where we set $r := \lambda^{\uparrow}_{0,0} + \lambda^{\rightarrow}_{0,0}$ and adopt the convention that $\lambda^{\rightarrow}_{-1,n} = 0$, $0 \le n \le N$, and $\lambda^{\uparrow}_{m,-1} = 0$, $0 \le m \le M$. We leave the straightforward details to the reader.

Setting $(m,n) = (1,0)$ in (1.4.2) gives

$$\alpha'(t) = r - (\lambda^{\rightarrow}_{1,0} + \lambda^{\uparrow}_{1,0}) + \frac{\lambda^{\rightarrow}_{0,0}}{M}e^{-\alpha(t)}. \tag{1.4.3}$$

Similarly, setting $(m,n) = (0,1)$ in (1.4.2) gives

$$\alpha'(t) = r - (\lambda^{\rightarrow}_{0,1} + \lambda^{\uparrow}_{0,1}) + \frac{\lambda^{\uparrow}_{0,0}}{N}e^{-\alpha(t)}. \tag{1.4.4}$$

In particular, we have the identity

$$\frac{\lambda^{\rightarrow}_{0,0}}{M} = \frac{\lambda^{\uparrow}_{0,0}}{N}.$$

Setting $(m,n) = (1,1)$ in (1.4.2) and substituting in the expression for $\alpha'(t)$ from (1.4.3) gives

$$\begin{aligned}\beta'(t) &= r - (\lambda^{\uparrow}_{1,1} + \lambda^{\rightarrow}_{1,1}) + \left(\frac{\lambda^{\rightarrow}_{0,1}}{M} + \frac{\lambda^{\uparrow}_{1,0}}{N}\right)e^{-\alpha(t)-\beta(t)} - 2\alpha'(t) \\ &= r - (\lambda^{\uparrow}_{1,1} + \lambda^{\rightarrow}_{1,1}) + \left(\frac{\lambda^{\rightarrow}_{0,1}}{M} + \frac{\lambda^{\uparrow}_{1,0}}{N}\right)e^{-\alpha(t)-\beta(t)} \\ &\quad - 2\left(r - (\lambda^{\rightarrow}_{1,0} + \lambda^{\uparrow}_{1,0}) + \frac{\lambda^{\rightarrow}_{0,0}}{M}e^{-\alpha(t)}\right).\end{aligned} \tag{1.4.5}$$

Further substituting the above expressions for $\alpha'(t)$ and $\beta'(t)$ from (1.4.3) and (1.4.5) into (1.4.2) for a general pair (m,n) leads to a system of equations of the form

$$\begin{gathered}a_{m,n}e^{\alpha(t)} + b_{m,n} = c_{m,n}e^{-n\beta(t)} + d_{m,n}e^{-m\beta(t)} - e_{m,n}e^{-\beta(t)}, \\ 0 \le m \le M,\ 0 \le n \le N,\end{gathered} \tag{1.4.6}$$

where the various coefficients are given by

$$\begin{cases} a_{m,n} = (m+n-2mn)(r - \lambda^{\rightarrow}_{1,0} - \lambda^{\uparrow}_{1,0}) + mn(r - \lambda^{\uparrow}_{1,1} - \lambda^{\rightarrow}_{1,1}) \\ \qquad\quad - r + \lambda^{\uparrow}_{m,n} + \lambda^{\rightarrow}_{m,n} \\ b_{m,n} = (m+n-2mn)\dfrac{\lambda^{\rightarrow}_{0,0}}{M} \\ c_{m,n} = \dfrac{m\lambda^{\rightarrow}_{m-1,n}}{M-m+1} \\ d_{m,n} = \dfrac{n\lambda^{\uparrow}_{m,n-1}}{N-n+1} \\ e_{m,n} = mn\left(\dfrac{\lambda^{\rightarrow}_{0,1}}{M} + \dfrac{\lambda^{\uparrow}_{1,0}}{N}\right). \end{cases}$$

Note that $c_{m,n} > 0$ whenever $m > 0$ and $d_{m,n} > 0$ whenever $n > 0$.

We claim that the continuous function β is constant. Assume without loss of generality that $M \geq 3$ and suppose that the function β is not constant.

Consider the two cases:

(i) One of $a_{2,1}$ or $a_{3,1}$ is zero.

(ii) Both $a_{2,1}$ and $a_{3,1}$ are non-zero.

Case (i) is impossible, because for either $m = 2$ or $m = 3$ we would have $0 = d_{m,1}e^{-m\beta(t)} + (c_{m,1} - e_{m,1})e^{-\beta(t)} - b_{m,1}$ and conclude that either a quadratic or cubic polynomial was identically zero on some open interval.

Turning to Case (ii), note first that if $a_{m,n} \neq 0$ we have

$$e^{\alpha(t)} = \frac{c_{m,n}e^{-n\beta(t)} + d_{m,n}e^{-m\beta(t)} - e_{m,n}e^{-\beta(t)} - b_{m,n}}{a_{m,n}} \tag{1.4.7}$$

and so we would have a quadratic and a cubic polynomial that agreed on an open interval.

Therefore, the function β must be a constant, say β^*. Of course, β^* is the pre-specified value for $\beta(T)$. We now show that $\beta^* = 0$.

Equation (1.4.2) becomes

$$\begin{aligned} &(m+n)\left(r - (\lambda^{\rightarrow}_{1,0} + \lambda^{\uparrow}_{1,0}) + \frac{\lambda^{\rightarrow}_{0,0}}{M}e^{-\alpha(t)}\right) \\ &= r - (\lambda^{\rightarrow}_{m,n} + \lambda^{\uparrow}_{m,n}) + \frac{m\lambda^{\rightarrow}_{m-1,n}}{M-m+1}e^{-\alpha(t)-n\beta^*} + \frac{n\lambda^{\uparrow}_{m,n-1}}{N-n+1}e^{-\alpha(t)-m\beta^*}. \end{aligned} \tag{1.4.8}$$

For a fixed pair (m, n), each side of (1.4.8) is a first degree polynomial in $\alpha(t)$ and, since $\alpha(t)$ is continuous and non-constant, we can equate coefficients. If we also record our boundary conditions and conventions from above we arrive at the following system of equations for $0 \le m \le M$ and $0 \le n \le N$

$$\begin{cases} \dfrac{(m+n)}{M}\lambda^{\rightarrow}_{0,0} = \dfrac{m}{M-m+1}\lambda^{\rightarrow}_{m-1,n}e^{-n\beta^*} \\ \qquad\qquad + \dfrac{n}{N-n+1}\lambda^{\uparrow}_{m,n-1}e^{-m\beta^*} \\ (m+n)(r-\lambda^{\rightarrow}_{1,0}-\lambda^{\uparrow}_{1,0}) = r-\lambda^{\rightarrow}_{m,n}-\lambda^{\uparrow}_{m,n} \\ \lambda^{\uparrow}_{m,N} = 0 \\ \lambda^{\rightarrow}_{M,n} = 0 \\ \lambda^{\uparrow}_{m,-1} = 0 \\ \lambda^{\rightarrow}_{-1,n} = 0. \end{cases} \tag{1.4.9}$$

Because $\lambda^{\rightarrow}_{M,N} = \lambda^{\uparrow}_{M,N} = 0$, we see from the second equation of (1.4.9) for $(m,n) = (M,N)$ that

$$r - \lambda^{\rightarrow}_{1,0} - \lambda^{\uparrow}_{1,0} = \frac{r}{M+N}, \tag{1.4.10}$$

and we can substitute this value into the second equation of (1.4.9) for general (m,n) to conclude that

$$\lambda^{\rightarrow}_{m,n} + \lambda^{\uparrow}_{m,n} = r\left(1 - \frac{m+n}{M+N}\right), \quad 0 \le m \le M,\ 0 \le n \le N.$$

Setting $m = 0$ in the first equation of (1.4.9) and noting the fifth equation, we get

$$\lambda^{\uparrow}_{0,n-1} = \frac{N-n+1}{M}\lambda^{\rightarrow}_{0,0}, \quad 1 \le n \le N.$$

A similar argument leads to

$$\lambda^{\rightarrow}_{m-1,0} = \frac{M-m+1}{M}\lambda^{\rightarrow}_{0,0}, \quad 1 \le m \le M.$$

Combining these observations, we arrive at the following system of equations

$$\begin{cases} \dfrac{n\lambda^{\uparrow}_{m,n-1}}{N-n+1}e^{(n-m)\beta^*} - \dfrac{m\lambda^{\uparrow}_{m-1,n}}{M-m+1} \\ \quad = \dfrac{\lambda^{\rightarrow}_{0,0}}{M}\Big[(m+n)e^{n\beta^*} - \dfrac{m(M+N-m-n+1)}{M-m+1}\Big], \\ \qquad 1 \le m \le M,\ 0 \le n \le N, \\ \lambda^{\rightarrow}_{m,n} + \lambda^{\uparrow}_{m,n} = r\left(1 - \dfrac{m+n}{M+N}\right), \quad 0 \le m \le M,\ 0 \le n \le N, \\ \lambda^{\uparrow}_{m,-1} = 0, \quad 0 \le m \le M, \\ \lambda^{\rightarrow}_{-1,n} = 0, \quad 0 \le n \le N, \\ \lambda^{\rightarrow}_{M,n} = 0, \quad 0 \le n \le N, \\ \lambda^{\uparrow}_{m,N} = 0, \quad 0 \le m \le M, \\ \lambda^{\uparrow}_{0,n} = \dfrac{N-n}{M}\lambda^{\rightarrow}_{0,0}, \quad 0 \le n \le N, \\ \lambda^{\rightarrow}_{m,0} = \dfrac{M-m}{M}\lambda^{\rightarrow}_{0,0}, \quad 0 \le m \le M. \end{cases} \tag{1.4.11}$$

Note that if $\lambda^{\rightarrow}_{m,n}$ and $\lambda^{\uparrow}_{m,n}$ satisfy (1.4.9), then they also satisfy (1.4.11). It will thus suffice to show that if $\beta^* \neq 0$, then there do not exist $\lambda^{\rightarrow}_{m,n}$ and $\lambda^{\uparrow}_{m,n}$ satisfying (1.4.11).

Setting $(m,n) = (1,1)$ in the first equation of (1.4.11) gives

$$\frac{\lambda^{\uparrow}_{1,0}}{N} - \frac{\lambda^{\uparrow}_{0,1}}{M} = \frac{\lambda^{\rightarrow}_{0,0}}{M}\left(2e^{\beta^*} - \frac{M+N-1}{M}\right),$$

while the fifth equation of (1.4.11) forces $\lambda^{\uparrow}_{0,1} = \frac{\lambda^{\rightarrow}_{0,0}}{M}(N-1)$. Thus,

$$\lambda^{\uparrow}_{1,0} = \frac{N}{M}\lambda^{\rightarrow}_{0,0}(2e^{\beta^*} - 1). \tag{1.4.12}$$

If instead we set $(m,n) = (2,0)$ in the first equation from (1.4.11), we obtain

$$\lambda^{\uparrow}_{1,0} = \frac{N}{M}\lambda^{\rightarrow}_{0,0}. \tag{1.4.13}$$

Comparing (1.4.12) and (1.4.13), we conclude that

$$2e^{\beta^*} - 1 = 1,$$

and hence $\beta^* = 0$. $\square$

1.5 Two classes with different individual propensity to default

Recall Model III from the Introduction. As with Model II, the graph G is $K_{M,N}$, the complete bipartite graph with vertex set the disjoint union $V = \hat{V} \sqcup \check{V}$, where $\hat{V}$ has M vertices and $\check{V}$ has N vertices. Now, however, there are functions $\hat{\alpha}$, $\check{\alpha}$ and β such that

$$\begin{cases} \alpha_u(t) = \hat{\alpha}(t) \text{ for all } u \in \hat{V} \\ \alpha_v(t) = \check{\alpha}(t) \text{ for all } v \in \check{V} \\ \beta_{uv}(t) = \beta(t) \text{ for all } u \in \hat{V},\ v \in \check{V}. \end{cases} \tag{1.5.1}$$

Proposition 1.5.1. *Consider Model III with $M \geq 4, N \geq 3$ or $M \geq 3, N \geq 4$. Suppose that the prescribed values of $\hat{\alpha}(T)$ and $\check{\alpha}(T)$ are distinct. If the prescribed value of $\beta(T)$ is non-zero and sufficiently small, then the model does not admit time-homogeneous Markovian dynamics.*

Proof. Another symmetry argument similar to those in the proofs of Proposition 1.3.1 and Proposition 1.4.1 shows that if (1.2.8) holds for some choice of jump rates, then there are constants $\lambda^{\rightarrow}_{m,n}$ and $\lambda^{\uparrow}_{m,n}$, $0 \leq m \leq M$ and $0 \leq n \leq N$, with $\lambda^{\rightarrow}_{M,n} = 0$ for $0 \leq n \leq N$, $\lambda^{\uparrow}_{m,N} = 0$ for $0 \leq m \leq M$, and $\lambda^{\rightarrow}_{m,n}$ and $\lambda^{\uparrow}_{m,n}$ strictly positive otherwise such that

$$\begin{aligned} m\hat{\alpha}'(t) \quad &+ \quad n\check{\alpha}'(t) + mn\beta'(t) = r - (\lambda^{\rightarrow}_{m,n} + \lambda^{\uparrow}_{m,n}) + \\ &+ \quad \frac{m\lambda^{\rightarrow}_{m-1,n}}{M-m+1} e^{-\hat{\alpha}(t)-n\beta(t)} + \frac{n\lambda^{\uparrow}_{m,n-1}}{N-n+1} e^{-\check{\alpha}(t)-m\beta(t)}, \end{aligned} \tag{1.5.2}$$

where we set $r := \lambda^{\uparrow}_{0,0} + \lambda^{\rightarrow}_{0,0}$ and adopt the convention that $\lambda^{\rightarrow}_{-1,n} = 0$, $0 \leq n \leq N$, and $\lambda^{\uparrow}_{m,-1} = 0$, $0 \leq m \leq M$.

Applying (1.5.2) with $(m,n) = (1,0)$ and $(m,n) = (0,1)$ gives

$$\begin{cases} \hat{\alpha}'(t) = r - (\lambda^{\uparrow}_{1,0} + \lambda^{\rightarrow}_{1,0}) + \dfrac{\lambda^{\rightarrow}_{0,0}}{M} e^{-\hat{\alpha}(t)} \\ \check{\alpha}'(t) = r - (\lambda^{\uparrow}_{0,1} + \lambda^{\rightarrow}_{0,1}) + \dfrac{\lambda^{\uparrow}_{0,0}}{N} e^{-\check{\alpha}(t)}. \end{cases} \tag{1.5.3}$$

Similarly, applying (1.5.2) with $(m,n) = (1,1)$ and then substituting in the expressions for $\hat{\alpha}(t)$ and $\check{\alpha}(t)$ from (1.5.3) gives

$$\begin{aligned} \beta'(t) &= r - (\lambda^{\uparrow}_{1,1} + \lambda^{\rightarrow}_{1,1}) + \frac{\lambda^{\rightarrow}_{0,1}}{M} e^{-\hat{\alpha}(t)-\beta(t)} + \frac{\lambda^{\uparrow}_{1,0}}{N} e^{-\check{\alpha}(t)-\beta(t)} - \hat{\alpha}'(t) - \check{\alpha}'(t) \\ &= -r - (\lambda^{\uparrow}_{1,1} + \lambda^{\rightarrow}_{1,1}) - (\lambda^{\uparrow}_{0,1} + \lambda^{\rightarrow}_{0,1}) - (\lambda^{\uparrow}_{1,0} + \lambda^{\rightarrow}_{1,0}) + \frac{\lambda^{\rightarrow}_{0,1}}{M} e^{-\hat{\alpha}(t)-\beta(t)} \\ &\quad + \frac{\lambda^{\uparrow}_{1,0}}{N} e^{-\check{\alpha}(t)-\beta(t)} - \frac{\lambda^{\rightarrow}_{0,0}}{M} e^{-\hat{\alpha}(t)} - \frac{\lambda^{\uparrow}_{0,0}}{N} e^{-\check{\alpha}(t)}. \end{aligned} \tag{1.5.4}$$

Substituting the expressions for the $\hat{\alpha}$, $\check{\alpha}$ and β from (1.5.3) and (1.5.4) into (1.5.2) for general (m,n) produces a system of equations of the form

$$\begin{aligned} &a_{m,n}e^{\hat{\alpha}(t)} + b_{m,n} + c_{m,n}e^{\hat{\alpha}(t)-\check{\alpha}(t)} + d_{m,n}e^{-\beta(t)} + e_{m,n}e^{\hat{\alpha}(t)-\check{\alpha}(t)-\beta(t)} \\ &\quad = f_{m,n}e^{-n\beta(t)} + g_{m,n}e^{\hat{\alpha}(t)-\check{\alpha}(t)-m\beta(t)}, \end{aligned} \tag{1.5.5}$$

where

$$\begin{cases} a_{m,n} = (m-mn)(r-\lambda^{\uparrow}_{1,0}-\lambda^{\rightarrow}_{1,0}) + (n-mn)(r-\lambda^{\uparrow}_{0,1}-\lambda^{\rightarrow}_{0,1}) \\ \qquad\quad + mn(r-\lambda^{\uparrow}_{1,1}-\lambda^{\rightarrow}_{1,1}) - r + \lambda^{\rightarrow}_{m,n} + \lambda^{\uparrow}_{m,n} \\ b_{m,n} = (m-mn)\dfrac{\lambda^{\rightarrow}_{0,0}}{M} \\ c_{m,n} = (n-mn)\dfrac{\lambda^{\uparrow}_{0,0}}{N} \\ d_{m,n} = mn\dfrac{\lambda^{\rightarrow}_{0,1}}{M} \\ e_{m,n} = mn\dfrac{\lambda^{\uparrow}_{1,0}}{N} \\ f_{m,n} = \dfrac{m\lambda^{\rightarrow}_{m-1,n}}{M-m+1} \\ g_{m,n} = \dfrac{n\lambda^{\uparrow}_{m,n-1}}{N-n+1}. \end{cases}$$

Observe that because $\lambda^{\rightarrow}_{m,n}$ is strictly positive for $1 \le m \le M-1$ and $1 \le n \le N$ and $\lambda^{\uparrow}_{m,n}$ is strictly positive for $1 \le m \le M$ and $1 \le n \le N-1$, both $f_{m,n}$ and $g_{m,n}$ are strictly positive for $1 \le m \le M$ and $1 \le n \le N$.

We claim that the continuous function β is constant. Assume without loss of generality that $M \ge 4$, $N \ge 3$ and suppose that the function β is not constant.

Re-arrange (1.5.5) to get

$$\begin{aligned} &\left[a_{m,n} + c_{m,n}e^{-\check{\alpha}(t)} + e_{m,n}e^{-\check{\alpha}(t)}e^{-\beta(t)} - g_{m,n}e^{-\check{\alpha}(t)}e^{-m\beta(t)}\right] e^{\hat{\alpha}(t)} \\ &\quad = f_{m,n}e^{-n\beta(t)} - d_{m,n}e^{-\beta(t)} - b_{m,n}. \end{aligned} \tag{1.5.6}$$

Because $f_{m,n}$ is strictly positive for $1 \le m \le M$ and $1 \le n \le N$, there is an open interval J such that the image $\{\beta(t) : t \in J\}$ contains an open interval and the right-hand side of (1.5.6) is non-zero for all $t \in J$, $1 \le m \le M$ and $2 \le n \le N$, and hence the same is true for the left-hand side.

Taking (1.5.6) with the indices (m,n) replaced by another pair (i,j), we see that if $t \in J$,

$1 \le m, i \le M$ and $2 \le n, j \le N$, then

$$\begin{aligned}&\left[f_{m,n}e^{-n\beta(t)} - d_{m,n}e^{-\beta(t)} - b_{m,n}\right]\\ &\quad\times\left[a_{i,j} + c_{i,j}e^{-\check{\alpha}(t)} + e_{i,j}e^{-\check{\alpha}(t)}e^{-\beta(t)} - g_{i,j}e^{-\check{\alpha}(t)}e^{-i\beta(t)}\right]\\ &\quad = \left[f_{i,j}e^{-j\beta(t)} - d_{i,j}e^{-\beta(t)} - b_{i,j}\right]\\ &\qquad\times\left[a_{m,n} + c_{m,n}e^{-\check{\alpha}(t)} + e_{m,n}e^{-\check{\alpha}(t)}e^{-\beta(t)} - g_{m,n}e^{-\check{\alpha}(t)}e^{-m\beta(t)}\right].\end{aligned} \tag{1.5.7}$$

Re-arranging (1.5.7) gives

$$p(e^{-\beta(t)}; m, n, i, j)e^{-\check{\alpha}(t)} = q(e^{-\beta(t)}; m, n, i, j), \tag{1.5.8}$$

where

$$\begin{aligned}p(z; m, n, i, j) := {}&(c_{i,j} + e_{i,j}z - g_{i,j}z^i)(f_{m,n}z^n - d_{m,n}z - b_{m,n})\\ &- (c_{m,n} + e_{m,n}z - g_{m,n}z^m)(f_{i,j}z^j - d_{i,j}z - b_{i,j})\end{aligned}$$

and

$$q(z; m, n, i, j) := a_{m,n}(f_{i,j}z^j - d_{i,j}z - b_{i,j}) - a_{i,j}(f_{m,n}z^n - d_{m,n}z - b_{m,n}).$$

Suppose now that $2 \le m, i \le M$ and $2 \le n, j \le N$. The leading term of the polynomial $p(z; m, n, i, j)$ is $-g_{i,j}f_{m,n}z^{i+n}$ if $i + n > m + j$ and $g_{m,n}f_{i,j}z^{m+j}$ if $i + n < m + j$ (recall that $f_{m,n}, g_{m,n}, f_{i,j}, g_{i,j}$ are all strictly positive). Therefore, by taking a subinterval of J if necessary, when $i + n \ne m + j$ we may suppose that J retains the properties required of it above and, moreover, that both sides of (1.5.8) are non-zero for all $t \in J$. In particular, either $a_{m,n} \ne 0$ or $a_{i,j} \ne 0$ and the polynomial $q(z; m, n, i, j)$ has degree either n or j when $n \ne j$.

Consider two 4-tuples (m', n', i', j') and (m'', n'', i'', j'') with

$$\begin{cases} 2 \le m', m'', i', i'' \le M \\ 2 \le n', n'', j', j'' \le N \\ i' + n' \ne m' + j' \\ i'' + n'' \ne m'' + j'' \\ n' \ne j' \\ n'' \ne j''. \end{cases} \tag{1.5.9}$$

We conclude from (1.5.8) that

$$p(z; m', n', i', j')q(z; m'', n'', i'', j'') = p(z; m'', n'', i'', j'')q(z; m', n', i', j') \tag{1.5.10}$$

for all z in an open interval. The left-hand side of (1.5.10) is a polynomial in z of degree either $((i' + n') \vee (m' + j')) + n''$ or $((i' + n') \vee (m' + j')) + j''$, whereas the right-hand size has degree either $((i''+n'')\vee(m''+j''))+n'$ or $((i''+n'')\vee(m''+j''))+j'$. For $(m',n',i',j')=(2,2,2,3)$ and $(m'',n'',i'',j'')=(4,2,4,3)$ we have

$$\begin{cases} i'+n'=4\neq 5=m'+j' \\ i''+n''=6\neq 7=m''+j'' \\ n'=2\neq 3=j' \\ n''=2\neq 3=j'' \\ ((i'+n')\vee(m'+j'))+n''=5+2=7 \\ ((i'+n')\vee(m'+j'))+j''=5+3=8 \\ ((i''+n'')\vee(m''+j''))+n'=7+2=9 \\ ((i''+n'')\vee(m''+j''))+j'=7+3=10, \end{cases}$$

and so the possible degrees of the left-hand side of (1.5.10) are 7 and 8, whereas the possible degrees of the right-hand side are 9 and 10.

Therefore, the function β must be a constant, say β^*. Note that β^* is just the pre-specified value for $\beta(T)$. We now show that $\beta^*=0$.

For the moment, consider Model III with $\beta(T)=0$, so that the function β must be identically zero and the firms evolve independently. In this special case, we know from the Introduction that

$$\exp(\hat{\alpha}(t))=(1+\exp(\hat{\alpha}(T)))^{\frac{t}{T}}-1 \tag{1.5.11}$$

and

$$\exp(\check{\alpha}(t))=(1+\exp(\check{\alpha}(T)))^{\frac{t}{T}}-1. \tag{1.5.12}$$

It follows from the linear independence of the functions $\exp(c_1\cdot),\dots,\exp(c_h\cdot)$ when $c_1,\dots,c_h$ are distinct that in this case the functions $\exp(\hat{\alpha})$, $\exp(\check{\alpha})$ and $\exp(\hat{\alpha}+\check{\alpha})$ are linearly independent when $\hat{\alpha}(T)\neq\check{\alpha}(T)$.

Now return to the case of a general value for $\beta^*=\beta(T)$. Equation (1.5.2) becomes

$$\begin{aligned} m\hat{\alpha}'(t)+n\check{\alpha}'(t)&=r-(\lambda^{\rightarrow}_{m,n}+\lambda^{\uparrow}_{m,n}) \\ &\quad+\frac{m\lambda^{\rightarrow}_{m-1,n}}{M-m+1}e^{-\hat{\alpha}(t)-n\beta^*}+\frac{n\lambda^{\uparrow}_{m,n-1}}{N-n+1}e^{-\check{\alpha}(t)-m\beta^*}. \end{aligned} \tag{1.5.13}$$

We can solve (1.5.3) for the $\hat{\alpha}$ and $\check{\alpha}$ as in Section 1.2 to get

$$\begin{cases} \hat{\alpha}(t)=\log\left(\dfrac{\lambda^{\rightarrow}_{0,0}}{M(r-\lambda^{\uparrow}_{1,0}-\lambda^{\rightarrow}_{1,0})}(e^{(r-\lambda^{\uparrow}_{1,0}-\lambda^{\rightarrow}_{1,0})t}-1)\right) \\ \check{\alpha}(t)=\log\left(\dfrac{\lambda^{\uparrow}_{0,0}}{N(r-\lambda^{\uparrow}_{0,1}-\lambda^{\rightarrow}_{0,1})}(e^{(r-\lambda^{\uparrow}_{0,1}-\lambda^{\rightarrow}_{0,1})t}-1)\right) \end{cases} \tag{1.5.14}$$

Substituting (1.5.14) into (1.5.13) gives

$$\begin{aligned} & m(r-\lambda^{\uparrow}_{1,0}-\lambda^{\rightarrow}_{1,0})+n(r-\lambda^{\uparrow}_{0,1}-\lambda^{\rightarrow}_{0,1})+(\lambda^{\rightarrow}_{m,n}+\lambda^{\uparrow}_{m,n})-r \\ & +\left(\frac{m\lambda^{\rightarrow}_{0,0}}{M}-\frac{m\lambda^{\rightarrow}_{m-1,n}e^{-n\beta^*}}{M-m+1}\right)\frac{M(r-\lambda^{\uparrow}_{1,0}-\lambda^{\rightarrow}_{1,0})}{\lambda^{\rightarrow}_{0,0}(e^{(r-\lambda^{\uparrow}_{1,0}-\lambda^{\rightarrow}_{1,0})t}-1)} \\ & +\left(\frac{n\lambda^{\uparrow}_{0,0}}{N}-\frac{n\lambda^{\uparrow}_{m,n-1}e^{-m\beta^*}}{N-n+1}\right)\frac{N(r-\lambda^{\uparrow}_{0,1}-\lambda^{\rightarrow}_{0,1})}{\lambda^{\uparrow}_{0,0}(e^{(r-\lambda^{\uparrow}_{0,1}-\lambda^{\rightarrow}_{0,1})t}-1)}=0. \end{aligned} \tag{1.5.15}$$

It follows from the observations above and a compactness argument that if $\hat{\alpha}(T)\neq\check{\alpha}(T)$ and $\beta^*=\beta(T)$ is sufficiently close to zero, then the functions $\exp(\hat{\alpha})$, $\exp(\check{\alpha})$ and $\exp(\hat{\alpha}+\check{\alpha})$ are linearly independent. Suppose that this is the case. Equation (1.5.15) is of the form

$$a+be^{-\hat{\alpha}(t)}+ce^{-\check{\alpha}(t)}=0$$

for suitable constants a,b,c, and hence $a=b=c=0$. Thus,

$$\begin{aligned} \frac{m\lambda^{\rightarrow}_{0,0}}{M}-\frac{m\lambda^{\rightarrow}_{m-1,n}e^{-n\beta^*}}{M-m+1} &= 0 \\ \frac{n\lambda^{\uparrow}_{0,0}}{N}-\frac{n\lambda^{\uparrow}_{m,n-1}e^{-m\beta^*}}{N-n+1} &= 0 \\ m(r-\lambda^{\rightarrow}_{1,0}-\lambda^{\uparrow}_{1,0})+n(r-\lambda^{\rightarrow}_{0,1}-\lambda^{\uparrow}_{0,1})+(\lambda^{\rightarrow}_{m,n}+\lambda^{\uparrow}_{m,n})-r &= 0 \end{aligned}$$

for $(m,n)\in\{0,\dots,M\}\times\{0,\dots,N\}$, and so, after some algebra,

$$\lambda^{\rightarrow}_{0,0}\frac{M-m}{M}e^{n\beta^*}+\lambda^{\uparrow}_{0,0}\frac{N-n}{N}e^{m\beta^*}=r-m(r-\lambda^{\rightarrow}_{1,0}-\lambda^{\uparrow}_{1,0})-n(r-\lambda^{\rightarrow}_{0,1}-\lambda^{\uparrow}_{0,1})$$

for $(m,n)\in\{0,\dots,M\}\times\{0,\dots,N\}$. In particular, considering $(m,n)=(k,k)$ for $0\leq k\leq M\wedge N$ leads to a system of the form

$$Ake^{k\beta^*}+Be^{k\beta^*}+Ck+D=0$$

for suitable constants A,B,C,D with $A>0$ and $B>0$. A straight line can intersect the graph of the function $t\mapsto Ate^{\beta^*t}+Be^{\beta^*t}$ at most twice if $\beta^*>0$ and at most three times if $\beta^*<0$, and since $M\wedge N\geq 3$ we must have $\beta^*=0$. □

Chapter 2

The Inverse First Passage Time Problem

2.1 Introduction

Counterparty risk has to be taken into account when pricing a transaction or portfolio, and it is necessary to model the occurrence of default jointly with the behavior of asset values.

The default time is sometimes modeled as the first passage time of a credit index process below a barrier. Black and Cox [BC76] were among the first to use this approach. They define the time of default as the first time the ratio of the value of a firm and the value of its debt falls below a constant level, and they model debt as a zero-coupon bond and the value of the firm as a geometric Brownian motion. In this case, the default time has the distribution of the first-passage time of a Brownian motion (with constant drift) below a certain barrier.

Hull and White [HW01] model the default time as the first time a Brownian motion hits a given time-dependent barrier. They show that this model gives the correct market credit default swap and bond prices if the time-dependent barrier is chosen so that the first passage time of the Brownian motion has a certain distribution derived from those prices. Given a distribution for the default time, it is usually impossible to find a closed-form expression for the corresponding time-dependent barrier, and numerical methods have to be used.

We adopt a perspective similar to that of [HW01]. Namely, we model the default time as

$$\tau := \inf\left\{ t > 0 : \lambda \int_0^t \psi(Y_s - b(s))\, ds > U \right\} \tag{2.1.1}$$

where the diffusion Y is some credit index process, U is an independent mean one exponentially distributed random variable, $0 \leq \psi \leq 1$ is a suitably smooth, non-increasing function with $\lim_{x\to-\infty}\psi(x)=1$ and $\lim_{x\to+\infty}\psi(x)=0$, and $\lambda>0$ is a rate parameter. Then,

$$\mathbb{P}\{\tau>t\}=\mathbb{E}\left[\exp\left(-\lambda\int_0^t\psi(Y_s-b(s))\,ds\right)\right]. \tag{2.1.2}$$

The random time τ is a "smoothed-out" version of the stopping time of Hull and White; instead of of killing Y as soon at it crosses some sharp, time-dependent boundary, we kill Y at rate $\lambda\psi(y-b(t))$ if it is in state $y\in\mathbb{R}$ at time $t\geq 0$. That is,

$$\lim_{\Delta t\downarrow 0}\mathbb{P}\{\tau\in(t,t+\Delta t)\,|\,(Y_s)_{0\leq s\leq t},\,\tau>t\}/\Delta t=\lambda\psi(Y_t-b(t)).$$

When the credit index value Y_t is large, corresponding to a time t when the counterparty is in sound financial health, the killing rate $\lambda\psi(Y_t-b(t))$ is close to 0 and default in an ensuing short period of time is unlikely, whereas the killing rate is close to its maximum possible value, λ, when Y_t is low and default is more probable. Note that if we consider a family of $[0,1]$-valued, non-increasing functions ψ that converges to the indicator function of the set $\{x\in\mathbb{R}:x<0\}$ and λ tends to ∞, then the corresponding stopping time τ converges to the Hull and White stopping time $\inf\{t>0:Y_t<b(t)\}$.

The hazard rate of the random time τ is

$$\begin{aligned}
\frac{\mathbb{P}\{\tau\in dt\mid\tau>t\}}{dt} &:= \lim_{\Delta t\downarrow 0}\frac{\mathbb{P}\{\tau\in(t,t+\Delta t)\}}{\Delta t\mathbb{P}\{\tau>t\}}\\
&= \lim_{\Delta t\downarrow 0}\frac{\mathbb{P}\left\{\lambda\int_0^t\psi(Y_s-b(s))ds\leq U\leq\lambda\int_0^{t+\Delta t}\psi(Y_s-b(s))\,ds\right\}}{\Delta t\mathbb{P}\left\{\lambda\int_0^t\psi(Y_s-b(s))\,ds\leq U\right\}}\\
&= \lim_{\Delta t\downarrow 0}\frac{\mathbb{E}\left[e^{-\lambda\int_0^t\psi(Y_s-b(s))\,ds}-e^{-\lambda\int_0^{t+\Delta t}\psi(Y_s-b(s))\,ds}\right]}{\Delta t\mathbb{E}\left[\exp\left(-\lambda\int_0^t\psi(Y_s-b(s))\,ds\right)\right]}\\
&= \frac{\lambda\mathbb{E}\left[\psi(Y_t-b(t))\exp\left(-\lambda\int_0^t\psi(Y_s-b(s))\,ds\right)\right]}{\mathbb{E}\left[\exp\left(-\lambda\int_0^t\psi(Y_s-b(s))\,ds\right)\right]}.
\end{aligned} \tag{2.1.3}$$

On the other hand, suppose that ζ is a non-negative random variable with survival function $t\mapsto G(t):=\mathbb{P}\{\zeta>t\}$. Writing g for the derivative of G, the corresponding hazard rate is

$$-\frac{g(t)}{G(t)}=-\frac{d}{dt}\log G(t).$$

As a result, a necessary condition for a function b to exist such that the corresponding random time τ has the same distribution as ζ is that

$$0<-g(t)<\lambda G(t),\,t\geq 0. \tag{2.1.4}$$

We show in Theorem 2.3.1 that if Y is a Brownian motion with a given suitable random initial condition, the assumption (2.1.4) holds, and the survival function G is twice continuously differentiable, then there is a unique differentiable function b such that the stopping time τ has the same distribution as ζ. In particular, we establish that the function b can be determined by solving a system consisting of a parabolic linear PDE with coefficients depending on b and a non-linear ODE for b with coefficients depending on the solution of the PDE. Note from (2.1.2) that changing the function b on a set with Lebesgue measure zero does not affect the distribution of τ, and so we have to be careful when we talk about the uniqueness of b. This minor annoyance does not appear if we restrict to continuous b.

In Theorem 2.5.1 we give an analogue of the existence part of the above result when ψ is the indicator of the set $\{x \in \mathbb{R} : x < 0\}$.

Having proven the existence and uniqueness of a barrier b, we consider the pricing of certain contingent claims in Section 2.6. For simplicity, we take the asset price $(X_t)_{t\geq 0}$ to be a geometric Brownian motion

$$\frac{dX_t}{X_t} = \mu dt + \sigma dW_t,$$

where W is a standard Brownian motion. We take the credit index $(Y_t)_{t\geq 0}$ to be given by

$$dY_t = dB_t$$

where B is another standard Brownian motion, and take the default time to be given by (2.1.1), where the exponential random variable U is independent of the asset price X and the credit index Y. We assume that the Brownian motions W and B are correlated; that is, that their covariation is $[B, W]_t = \rho t$ for some constant $\rho \in [-1, 1]$. We consider claims with a payoff of the form $F(X_T)1\{\tau > T\}$ for some fixed maturity T. We show how it is possible to compute conditional expected values such as

$$\mathbb{E}\left[F(X_T)1\{\tau > T\} \,\middle|\, (X_s)_{0\leq s\leq t},\, \tau > t\right].$$

In Section 2.7 we report the results of some experiments where we solved the PDE/ODE system for the barrier b numerically. Lastly, in Section 2.8, we follow [DP11] to demonstrate how it is possible to use market data on credit default swap prices to determine the survival function G.

2.2 The FPT and IFPT problems

We present a discussion of the literature dealing with first passage times of diffusions across time-dependent barriers.

Consider a Brownian motion $(B_t)_{t\geq 0}$ defined on a filtered probability space $(\Omega, \mathbb{P}, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ which satisfies the usual conditions. Define the diffusion $(Y_t)_{t\geq 0}$ via the SDE

$$dY_t = \mu(Y_t, t)\, dt + \sigma(Y_t, t)\, dB_t,$$

where we assume that the coefficients $\mu : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ and $\sigma : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ are such that the SDE has a unique strong solution.

For a Borel function $b : \mathbb{R}_+ \to \bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, the first passage time of the diffusion process Y below the barrier b is the stopping time

$$\tilde{\tau} = \inf\{t > 0 : Y_t < b(t)\}. \tag{2.2.1}$$

The following two problems related to this notion have been discussed in the literature.

The First Passage Time problem (FPT): For a given barrier $b : \mathbb{R}_+ \to \bar{\mathbb{R}}$, compute the survival function G of the first time that X goes below b; that is, find

$$G(t) := \mathbb{P}\{\tilde{\tau} > t\}, \quad t \geq 0. \tag{2.2.2}$$

The Inverse First Passage Time problem (IFPT): For a given survival function G, does there exist a barrier b such that $G(t) = \mathbb{P}\{\tilde{\tau} > t\}$ for all $t \geq 0$?

The First Passage Time problem started with Bachelier [Bac00] who examined the first passage of a Brownian motion to a constant boundary. Paul Lévy generalized the problem to a linear boundary. Kolmogorov clarified the connection between probability theory and analysis in [Kol31] and this started the PDE approach to first passage time problems. Examples of the PDE approach are results by Petrowsky [Pet34] and by Khinchine [Khi33]. There are not many closed form results regarding the first passage time problem and those which exist are mostly confined to Brownian motion. Therefore, people have studied other aspects of the FPT such as the asymptotic behavior of the FPT distribution (see for example [Pes02a, Nov14]).

Upper and Lower Boundaries In the case when the diffusion Y is a standard Brownian motion B it is natural to try to find the value of $\mathbb{P}\{\tilde{\tau} > 0\}$. By Blumenthal's 0-1 law, because the event $\{\tilde{\tau} > 0\}$ lies in $\cap_{t>0}\mathcal{F}_t$, we have that $\mathbb{P}\{\tilde{\tau} > 0\} \in \{0, 1\}$. A continuous function $b : \mathbb{R}_+ \to \mathbb{R}$ is a *lower boundary* function for B if $\mathbb{P}\{\tilde{\tau} > 0\} = 1$ and a *upper boundary* function for B if $\mathbb{P}\{\tilde{\tau} > 0\} = 0$. Kolmogorov's test [IM65] says that if b is continuous, decreasing and $b(s)/\sqrt{s}$ is increasing then b is a lower boundary function for B if and only if

$$-\int_0^\infty \frac{b(s)}{s^{3/2}} \phi(b(s)/\sqrt{s})\, ds < \infty$$

where ϕ is the standard normal density. One can therefore note that $-\sqrt{2t \log\log 1/t}$ is an upper function for B and $-\sqrt{(2+\epsilon)t \log\log 1/t}$ is a lower function for W for every $\epsilon > 0$.

A large class of first passage time problems may be solved within a PDE framework. Let $u(x,t) = \frac{\partial}{\partial x}\mathbb{P}\{Y_t \leq x, \tilde{\tau} > t\}$ be the sub-probability density of the diffusion Y killed at $\tilde{\tau}$. Then, by the Kolmogorov forward equation, u satisfies

$$\begin{cases} u_t(x,t) = \frac{1}{2}(\sigma^2 u)_{xx} - (\mu u)_x, \quad x > b(t),\, t > 0, \\ u(x,t) = 0, \quad x \leq b(t),\, t > 0, \\ u(x,0) = f(x), \quad x \in \mathbb{R}, \end{cases} \tag{2.2.3}$$

where f is the probability density of Y_0. For nice enough functions b this system has a unique solution and we can express the survival probability

$$G(t) = \mathbb{P}\{\tilde{\tau} > t\} = \int_{b(t)}^{\infty} u(x,t)\,dx, \quad t \geq 0.$$

This approach is used in [Ler86, Val09] to get closed form solutions for some classes of boundaries. An integral equation technique is used in [Pes02a, Pes02b, PS06, Val09] to find the derivative $g(t) = G'(t)$ in the FPT problem for a Brownian motion. Writing $\Psi(z) := \int_z^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$, the derivative g satisfies a Volterra integral equation of the first kind of the form

$$\Psi\left(\frac{b(t)}{\sqrt{t}}\right) = -\int_0^t \Psi\left(\frac{b(t)-b(s)}{\sqrt{t-s}}\right) g(s)\,ds.$$

This and other such integral equations can be used to find g numerically.

A. Shiryaev is generally credited with introducing the IFPT problem in 1976 (we have not been able to find an explicit reference). The IFPT problem is significantly more challenging than the FPT problem.

Most authors have investigated numerical methods for finding the boundary. Details can be found in [HW00, HW01, IK02, ZS09]. It is shown in [AZ01] that for sufficiently smooth boundaries the density $u(x,t)$ and the boundary $b(t)$ are a solution of the following free boundary problem

$$\begin{cases} u_t(x,t) = \frac{1}{2}(\sigma^2 u)_{xx} - (\mu u)_x, \quad x > b(t),\, t > 0, \\ u(x,t) = 0, \quad x \leq b(t),\, t > 0, \\ u(x,0) = f(x), \quad x \in \mathbb{R}, \\ G(t) = \int_{b(t)}^{\infty} u(x,t)\,dx, \quad t \geq 0. \end{cases} \tag{2.2.4}$$

where f is again the probability density of Y_0. The existence and uniqueness of a viscosity solution of (2.2.4) is established in [CCCS11] along with upper and lower bounds on the asymptotic behavior of b. This chapter also shows that this b does in fact produce a boundary that gives the survival function G. To our knowledge it has not be proven that a strong solution to the system (2.2.4) exists, nor that there is a smooth b solving the IFPT.

A variation of the IFPT is studied in [DP11]. There the barrier is fixed at zero (i.e. $b \equiv 0$) and it is the volatility parameter $\sigma(\cdot,\cdot)$ that is allowed to vary. The authors show that this problem admits an explicit solution for every differentiable survival function.

2.3 Global Existence and Uniqueness

Suppose for the remainder of this chapter that $Y_t := Y_0 + B_t$ where $(B_t)_{t\geq 0}$ is a standard Brownian motion and Y_0 is a random variable, independent of B and with density f. In this case, (2.1.2) is

$$G(t) = \int_{\mathbb{R}} \mathbb{E}\left[\exp\left(-\lambda \int_0^t \psi(x + B_z - b(z))\, dz\right)\right] f(x)\, dx$$

which, by time reversal, becomes

$$G(t) = \int_{\mathbb{R}} \mathbb{E}\left[\exp\left(-\lambda \int_0^t \psi(x + B_{t-z} - b(z))dz\right) f(x + B_t)\right] dx.$$

Set

$$u(x,t) := \mathbb{E}\left[\exp\left(-\lambda \int_0^t \psi(x + B_{t-z} - b(z))dz\right) f(x + B_t)\right]. \tag{2.3.1}$$

That is, u is the sub-probability density of Y killed at the random time τ. It is well known that if u is smooth enough, then u is the unique solution of the PDE

$$\begin{cases} u_t(x,t) = \frac{1}{2}u_{xx}(x,t) - \lambda\psi(x - b(t))u(x,t), & x \in \mathbb{R},\, t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}. \end{cases}$$

Any solution to this PDE satisfies

$$\lim_{x\to\pm\infty} u(x,t) = \lim_{x\to\pm\infty} u_x(x,t) = 0, \quad t > 0. \tag{2.3.2}$$

Our question as to whether we can find a "barrier" b giving us the survival function G is now equivalent to whether the system

$$\begin{cases} u_t(x,t) = \frac{1}{2}u_{xx}(x,t) - \lambda\psi(x - b(t))u(x,t),\ x \in \mathbb{R},\ t > 0, \\ u(x,0) = f(x),\ x \in \mathbb{R}, \\ \int_{\mathbb{R}} u(x,t)\, dx = G(t),\ t \geq 0, \end{cases} \tag{2.3.3}$$

has solutions (u, b). Differentiating the third equation from (2.3.3) with respect to t and then using the first equation together with an integration by parts, we get that

$$-\, g(t) = \lambda \int_{\mathbb{R}} \psi(x - b(t))u(x,t)\, dx, \tag{2.3.4}$$

where we recall that $g(t) = G'(t)$. A second differentiation in t followed by another integration by parts yields

$$
\begin{aligned}
g'(t) - \lambda^2 \int_{\mathbb{R}} \psi^2(x - b(t))u(x,t)dx &= \lambda \int_{\mathbb{R}} \psi_x(x - b(t))u(x,t)b'(t)\,dx \\
&\quad - \lambda/2 \int_{\mathbb{R}} \psi(x - b(t))u_{xx}(x,t)\,dx \\
&= \lambda \int_{\mathbb{R}} \psi_x(x - b(t))u(x,t)b'(t)\,dx \\
&\quad + \lambda/2 \int_{\mathbb{R}} \psi_x(x - b(t))u_x(x,t)\,dx \qquad (2.3.5) \\
&= \lambda \int_{\mathbb{R}} \psi_x(x - b(t))u(x,t)b'(t)\,dx \\
&\quad - \lambda/2 \int_{\mathbb{R}} \psi_{xx}(x - b(t))u(x,t)\,dx.
\end{aligned}
$$

Note that (2.3.5) may be rearranged to give an ODE for b of the form $b'(t) = \Theta(b(t), t)$, where the function Θ is constructed from the function u (which, of course, depends in turn on b). Re-writing this integral equation in the form $b(t) = b(0) + \int_0^t \Theta(b(s), s)\,ds$ leads to the following theorem, our main result.

Theorem 2.3.1. *Suppose the following.*

- *The survival function G is twice continuously differentiable with first and second derivatives g and g' and $0 < -g(t) < \lambda G(t)$ for all $t \geq 0$ for some constant $\lambda > 0$.*
- *The initial density f satisfies $\int_{\mathbb{R}} f(x)\,dx = 1$, $f(x) > 0$ for all $x \in \mathbb{R}$, $f \in C^2(\mathbb{R})$, and the functions f, f', f'' are bounded.*
- *The function ψ is non-increasing and belongs to $C^3(\mathbb{R})$, and for some $h > 0$, $\psi(x) = 1$ for $x \leq -h$ and $\psi(x) = 0$ for $x \geq h$.*

Then, there exists a unique continuously differentiable function $b : [0, \infty) \to \mathbb{R}$ such that the following three equations hold

$$G(t) = \int_{\mathbb{R}} \mathbb{E}\left[\exp\left(-\lambda \int_0^t \psi(x + B_u - b(u))du\right)\right] f(x)\,dx, \qquad (2.3.6)$$

$$-g(t) = \lambda \int_{\mathbb{R}} \mathbb{E}\left[\exp\left(-\lambda \int_0^t \psi(x + B_u - b(u))du\right)\psi(x + B_t - b(t))\right] f(x)\,dx, \qquad (2.3.7)$$

and

$$b(t) = b(0) + \int_0^t \left(\frac{g'(s) - \lambda^2 \int_{\mathbb{R}} \mathbb{E}\left[\psi^2(x + B_s - b(s))e^{-\lambda \int_0^s \psi(x+B_r-b(r))dr}\right] f(x)\,dx}{\lambda \int_{\mathbb{R}} \mathbb{E}\left[\psi_x(x + B_s - b(s))e^{-\lambda \int_0^s \psi(x+B_r-b(r))dr}\right] f(x)\,dx} + \frac{\lambda/2 \int_{\mathbb{R}} \mathbb{E}\left[\psi_{xx}(x + B_s - b(s))e^{-\lambda \int_0^s \psi(x+B_r-b(r))dr}\right] f(x)\,dx}{\lambda \int_{\mathbb{R}} \mathbb{E}\left[\psi_x(x + B_s - b(s))e^{-\lambda \int_0^s \psi(x+B_r-b(r))dr}\right] f(x)\,dx} \right) ds \qquad (2.3.8)$$

for all $t \geq 0$*.*

Proof. From now on we assume for ease of notation that $\lambda = 1$. The modifications necessary for general λ are straightforward. The proof will be via a sequence of lemmas, all of them assuming the hypotheses of Theorem 2.3.1 (with $\lambda = 1$). We start with the following simple observation.

Lemma 2.3.2. *Suppose that*

$$G(t) = \int_{\mathbb{R}} u(x,t)\,dx$$

for some continuous function $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ *such that* $u(x,t) > 0$ *for* $x \in \mathbb{R},\ t \geq 0$*. Then, for each* $t \geq 0$ *there exists a unique* $b(t) \in \mathbb{R}$ *such that*

$$-g(t) = \int_{\mathbb{R}} \psi(x - b(t))u(x,t)\,dx.$$

Proof. Set

$$F(t,z) = \int_{\mathbb{R}} \psi(x - z)u(x,t)\,dx.$$

Then,

$$\lim_{z\to-\infty} F(t,z) = \int_{\mathbb{R}} u(x,t)\,dx = G(t),$$

$$\lim_{z\to+\infty} F(t,z) = 0,$$

and, by assumption,

$$0 < -g(t) < G(t).$$

Furthermore, F is continuous and strictly decreasing in z. So, by the intermediate value property, we can find a unique $b(t) \in \mathbb{R}$ such that $F(t,b(t)) = -g(t)$. □

Lemma 2.3.3 (Global Uniqueness)**.** *Suppose there exist continuous functions* b_1, b_2 *such that equations* (2.3.6)*,* (2.3.7) *and* (2.3.8) *are satisfied for* $b = b_1$ *and* $b = b_2$*. Then,* $b_1(t) = b_2(t)$ *for all* $t \geq 0$*.*

Proof. Recall that we are assuming $\lambda = 1$ to simplify notation.

Suppose that b_1 and b_2 are two continuous solutions of (2.3.6), (2.3.7) and (2.3.8). It follows from Lemma 2.3.2 and (2.3.7) that $b_1(0) = b_2(0)$. Set $V := \inf\{t \geq 0 : b_1(t) \neq b_2(t)\}$ and suppose that $V < \infty$.

Define $\tilde{f} : \mathbb{R} \to \mathbb{R}$ by

$$\tilde{f}(y)\,dy := \int_{\mathbb{R}} \mathbb{E}\left[\mathbf{1}\{x + B_V \in dy\} e^{-\int_0^V \psi(x+B_r-b(r))\,dr}\right] f(x)\,dx,$$

where $b(t) = b_1(t) = b_2(t)$ for $0 \leq t \leq V$. Define functions $\tilde{b}_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1, 2$, by $\tilde{b}_i(t) = b_i(V + t)$, $t \geq 0$. Then, $\tilde{b}_1(0) = \tilde{b}_2(0) = b(V)$, and

$$\begin{aligned}\tilde{b}_i(t) &= \tilde{b}_i(0) \\ &+ \int_0^t \left(\frac{g'(s+V) - \int_{\mathbb{R}} \mathbb{E}\left[\psi^2(x + B_s - \tilde{b}_i(s)) e^{-\int_0^s \psi(x+B_r-\tilde{b}_i(r))\,dr}\right] \tilde{f}(x)\,dx}{\int_{\mathbb{R}} \mathbb{E}\left[\psi_x(x + B_s - \tilde{b}_i(s)) e^{-\int_0^s \psi(x+B_r-\tilde{b}_i(r))\,dr}\right] \tilde{f}(x)\,dx} \right. \\ &\left. + \frac{1/2 \int_{\mathbb{R}} \mathbb{E}\left[\psi_{xx}(x + B_s - \tilde{b}_i(s)) e^{-\int_0^s \psi(x+B_r-\tilde{b}_i(r))\,dr}\right] \tilde{f}(x)\,dx}{\int_{\mathbb{R}} \mathbb{E}\left[\psi_x(x + B_s - \tilde{b}_i(s)) e^{-\int_0^s \psi(x+B_r-\tilde{b}_i(r))\,dr}\right] \tilde{f}(x)\,dx} \right) ds.\end{aligned}$$

Fix $\epsilon > 0$ and set

$$K := \min_{i=1,2} \inf_{0 \leq s \leq \epsilon} \int_{\mathbb{R}} \mathbb{E}\left[\psi_x(x + B_s - \tilde{b}_i(s)) e^{-\int_0^s \psi(x+B_r-\tilde{b}_i(r))\,dr}\right] \tilde{f}(x)\,dx > 0.$$

By the triangle inequality, for $0 \leq t \leq \epsilon$,

$$|\tilde{b}_1(t) - \tilde{b}_2(t)| \leq I + II + III,$$

where

$$\begin{aligned} I := K^{-2} \int_0^t |g'(s+V)| \int_{\mathbb{R}} \mathbb{E}\Big[\Big|&\psi_x(x + B_s - \tilde{b}_2(s)) e^{-\int_0^s \psi(x+B_r-\tilde{b}_2(r))\,dr} \\ &- \psi_x(x + B_s - \tilde{b}_1(s)) e^{-\int_0^s \psi(x+B_r-\tilde{b}_1(r))\,dr}\Big|\Big] \tilde{f}(x)\,dx\,ds,\end{aligned}$$

$$
\begin{aligned}
II := K^{-2} \int_0^t \int_{\mathbb{R}} & \mathbb{E}\left[\psi^2(x+B_s-\tilde{b}_1(s))e^{-\int_0^s \psi(x+B_r-\tilde{b}_1(r))\,dr}\right]\tilde{f}(x)\,dx \\
& \times \int_{\mathbb{R}} \mathbb{E}\Big[\Big|\psi_x(x+B_s-\tilde{b}_2(s))e^{-\int_0^s \psi(x+B_r-\tilde{b}_2(r))\,dr} \\
& \quad - \psi_x(x+B_s-\tilde{b}_1(s))e^{-\int_0^s \psi(x+B_r-\tilde{b}_1(r))\,dr}\Big|\Big]\tilde{f}(x)\,dx\,ds \\
& + K^{-2} \int_0^t \int_{\mathbb{R}} \mathbb{E}\Big[\Big|\psi^2(x+B_s-\tilde{b}_1(s))e^{-\int_0^s \psi(x+B_r-\tilde{b}_1(r))\,dr} \\
& \quad - \psi^2(x+B_s-\tilde{b}_2(s))e^{-\int_0^s \psi(x+B_r-\tilde{b}_2(r))\,dr}\Big|\Big]\tilde{f}(x)\,dx \\
& \quad \times \int_{\mathbb{R}} \mathbb{E}\left[\left|\psi_x(x+B_s-\tilde{b}_1(s))e^{-\int_0^s \psi(x+B_r-\tilde{b}_1(r))\,dr}\right|\right]\tilde{f}(x)\,dx\,ds,
\end{aligned}
$$

and

$$
\begin{aligned}
III := \frac{1}{2}K^{-2} \int_0^t \int_{\mathbb{R}} & \mathbb{E}\left[\left|\psi_{xx}(x+B_s-\tilde{b}_1(s))e^{-\int_0^s \psi(x+B_r-\tilde{b}_1(r))\,dr}\right|\right]\tilde{f}(x)\,dx \\
& \times \int_{\mathbb{R}} \mathbb{E}\Big[\Big|\psi_x(x+B_s-\tilde{b}_2(s))e^{-\int_0^s \psi(x+B_r-\tilde{b}_2(r))\,dr} \\
& \quad - \psi_x(x+B_s-\tilde{b}_1(s))e^{-\int_0^s \psi(x+B_r-\tilde{b}_1(r))\,dr}\Big|\Big]\tilde{f}(x)\,dx\,ds \\
& + \frac{1}{2}K^{-2} \int_0^t \int_{\mathbb{R}} \mathbb{E}\Big[\Big|\psi_{xx}(x+B_s-\tilde{b}_1(s))e^{-\int_0^s \psi(x+B_r-\tilde{b}_1(r))\,dr} \\
& \quad - \psi_{xx}(x+B_s-\tilde{b}_2(s))e^{-\int_0^s \psi(x+B_r-\tilde{b}_2(r))\,dr}\Big|\Big]\tilde{f}(x)\,dx \\
& \quad \times \int_{\mathbb{R}} \mathbb{E}\left[\left|\psi_x(x+B_s-\tilde{b}_1(s))\right|e^{-\int_0^s \psi(x+B_r-\tilde{b}_1(r))\,dr}\right]\tilde{f}(x)\,dx\,ds.
\end{aligned}
$$

Consider the integrand in I. Note that

$$
\begin{aligned}
& \left|\psi_x(x+B_s-\tilde{b}_2(s))e^{-\int_0^s \psi(x+B_r-\tilde{b}_2(r))\,dr} - \psi_x(x+B_s-\tilde{b}_1(s))e^{-\int_0^s \psi(x+B_r-\tilde{b}_1(r))\,dr}\right| \\
& \quad \le |\psi_x(x+B_s-\tilde{b}_2(s))|\left|e^{-\int_0^s \psi(x+B_r-\tilde{b}_2(r))\,dr} - e^{-\int_0^s \psi(x+B_r-\tilde{b}_1(r))\,dr}\right| \\
& \qquad + e^{-\int_0^s \psi(x+B_r-\tilde{b}_1(r))\,dr}|\psi_x(x+B_s-\tilde{b}_2(s)) - \psi_x(x+B_s-\tilde{b}_1(s))| \\
& \quad \le \|\psi_x\|_{L^\infty} s \|\psi_x\|_{L^\infty} \sup_{0\le r\le s} |b_2(r)-b_1(r)| \\
& \qquad + \|\psi_{xx}\|_{L^\infty} \sup_{0\le r\le s} |b_2(r)-b_1(r)|.
\end{aligned}
$$

Similar arguments for the integrands in II and III using the boundedness and global

Lipschitz properties of ψ, ψ_x, and ψ_{xx} establish that, for a suitable constant C,

$$\sup_{0\le s\le t} |\tilde{b}_1(s) - \tilde{b}_2(s)| \le C \int_0^t \sup_{0\le r\le s} |\tilde{b}_1(r) - \tilde{b}_2(r)|\, ds$$

for $0 \le t \le \epsilon$. It follows from Grönwall's inequality that $\tilde{b}_1(t) = \tilde{b}_2(t)$ for $0 \le t \le \epsilon$, and so $b_1(t) = b_2(t)$ for $0 \le t \le V + \epsilon$, contrary to the definition of V and the assumption that V is finite. □

Lemma 2.3.4 (Global Existence)**.** *Define S to be the supremum of the set of T such that the equations* (2.3.6)*,* (2.3.7) *and* (2.3.8) *have a continuous solution on* $[0, T]$*. Then,* $S = +\infty$*.*

Proof. Suppose to the contrary that $S < +\infty$. From Lemma 2.3.3, the equations have a unique solution on $[0, S)$. By time-reversal, equation (2.3.6) is equivalent to

$$G(t) = \int_{\mathbb{R}} \mathbb{E}\left[\exp\left(-\int_0^t \psi(x + B_{t-u} - b(u))du\right) f(x + B_t)\right] dx. \tag{2.3.9}$$

Similarly, (2.3.7) is equivalent to

$$\begin{aligned} &- g(t) \\ &= \int_{\mathbb{R}} \mathbb{E}\left[\exp\left(-\int_0^t \psi(x + B_{t-u} - b(u))\, du\right) \psi(x - b(t)) f(x + B_t)\right] dx. \end{aligned} \tag{2.3.10}$$

For $0 \le t < S$ put

$$u(x,t) := \mathbb{E}\left[\exp\left(-\int_0^t \psi(x + B_{t-u} - b(u))du\right) f(x + B_t)\right]. \tag{2.3.11}$$

Consider $t_1 < t_2 < \ldots \uparrow S$. It follows from the continuity of the sample paths of B that as $t_n \uparrow S$

$$\begin{aligned} &\exp\left(-\int_0^{t_n} \psi(x + B_{t_n-u} - b(u))\, du\right) f(x + B_{t_n}) \\ &\quad \to \exp\left(-\int_0^S \psi(x + B_{S-u} - b(u))\, du\right) f(x + B_S) \end{aligned}$$

almost surely for each $x \in \mathbb{R}$, and so

$$u(x, t_n) \to \mathbb{E}\left[\exp\left(-\int_0^S \psi(x + B_{S-u} - b(u))\, du\right) f(x + B_S)\right] =: u(x, S).$$

Because

$$u(x,t) \le \mathbb{E}[f(x + B_t)],$$

it follows from dominated convergence that

$$\int_{\mathbb{R}} u(x,S)\,dx = \lim_n \int_{\mathbb{R}} u(x,t_n)\,dx = \lim_n G(t_n) = G(S).$$

Also,

$$\lim_n \int_{\mathbb{R}} \psi(x-b(t_n))u(x,t_n)\,dx = -\lim_n g(t_n) = -g(S).$$

Because $0 < -g(S) < G(S)$ and

$$u(x,S) \geq e^{-S}\mathbb{E}[f(x+B_S)] > 0, \quad x \in \mathbb{R},$$

there is, by Lemma 2.3.2, a unique $b^* \in \mathbb{R}$ such that

$$\int_{\mathbb{R}} \psi(x-b^*)u(x,S)\,dx = -g(t).$$

We claim that $b(t_n) \to b^*$. If this was not the case, then, by passing to a subsequence we would have $b(t_n)$ converging to some other extended real c and hence, by dominated convergence,

$$\begin{aligned} -g(t) &= -\lim_n g(t_n) \\ &= \lim_n \int_{\mathbb{R}} \psi(x-b(t_n))u(x,t_n)\,dx \\ &= \int_{\mathbb{R}} \psi(x-c)u(x,S)\,dx, \end{aligned}$$

contradicting the definition of b^* (where we used the natural definitions $\psi(-\infty) := 1, \psi(+\infty) := 0$). Using dominated convergence in (2.3.8) we get that there exists a continuous b such that all three equations hold on $[0,S]$.

All we need to do now is show that we can extend the existence from $[0,S]$ to $[0,S+\delta]$ for some $\delta > 0$. This amounts to proving existence on $[0,\delta]$ starting at a different initial condition – replacing the original probability density f by the density of the probability measure

$$\int_{\mathbb{R}} \mathbb{E}\left[\exp\left(-\int_0^S \psi(x+B_u-b(u))\,du\right), B_S \in \bullet\right] f(x)\,dx/G(S).$$

This will follow if we can establish the local existence, that is the existence for some $\delta > 0$, of a solution of the following PDE/ODE system

$$\begin{cases} \tilde{u}_t(x,t) = \frac{1}{2}\tilde{u}_{xx}(x,t) - \psi(x-\tilde{b}(t))\tilde{u}(x,t),\ x \in \mathbb{R},\ 0 < t < \delta, \\ \tilde{u}(x,0) = u(x,S)/G(S),\ x \in \mathbb{R}, \\ \tilde{b}(0) = b(S), \\ \tilde{b}'(t) = \dfrac{(g(S+t)+g'(S+t))/G(S) - \int_{\mathbb{R}}[\psi^2(x-\tilde{b}(t)) - \psi(x-\tilde{b}(t))]\tilde{u}(x,t)\,dx}{\int_{\mathbb{R}}\psi_x(x-\tilde{b}(t))\tilde{u}(x,t)\,dx} \\ \qquad - \dfrac{1/2\int_{\mathbb{R}}\psi_x(x-\tilde{b}(t))\tilde{u}_x(x,t)\,dx}{\int_{\mathbb{R}}\psi_x(x-\tilde{b}(t))\tilde{u}(x,t)\,dx},\ 0 < t < \delta. \end{cases}$$

We note that the expression for $\tilde{b}'(t)$ is not the analogue of the one for $b'(t)$ that arises immediately from differentiating (2.3.8), which in turn arose from rearranging (2.3.5) and integrating. However, adding $0 = \int_{\mathbb{R}}\psi(x-b(t))u(x,t)\,dx - g(t)$ to the right-hand side of (2.3.5) and then solving for $b'(t)$ leads to an expression of this form. Note that

$$u(x,S) = \mathbb{E}\left[\exp\left(-\int_0^S \psi(x+B_{S-u}-b(u))du\right)f(x+B_S)\right] > 0,$$

and, by dominated convergence, that $u(\cdot,S) \in C^2(\mathbb{R})$ with $\|u(\cdot,S)\|_{L^\infty(\mathbb{R})}$, $\|u_x(\cdot,S)\|_{L^\infty(\mathbb{R})}$, $\|u_{xx}(\cdot,S)\|_{L^\infty(\mathbb{R})}$ all finite. Therefore, we can apply Theorem 2.4.14 below to get that there is a time $\delta > 0$ and a unique pair $\tilde{u}, \tilde{b}$ satisfying the PDE/ODE system above with $\tilde{u}$ twice continuously differentiable in x on $\mathbb{R}$ and once continuously differentiable in t on $[0,\delta]$, i.e. $\tilde{u} \in C_x^2(\mathbb{R})C_t^1([0,\delta])$, and with $\tilde{b} \in C^1([0,\delta])$. Thus, we have proven that we have a unique continuous b satisfying equations (2.3.6), (2.3.7) and (2.3.8) on $[0, S+\delta]$. This contradicts the maximality of S. As a result, $S = \infty$ and we are done. □

This completes the proof of Theorem 2.3.1. □

Remark 2.3.5. Note that one needs the global uniqueness proof from Lemma 2.3.3 because it is not a priori clear that all the solutions to equations (2.3.6)-(2.3.8) are solutions to (2.3.3). Therefore, one cannot apply the local existence result 2.4.14 to prove the uniqueness of the solution of (2.3.6)-(2.3.8).

Remark 2.3.6. Equation (2.3.8) shows that b has a finite right derivative at 0. In the standard Inverse First Passage Problem this usually fails (for example when G is exponential).

As a corollary we get the global existence and uniqueness of the PDE/ODE system.

Corollary 2.3.7. *Suppose that the conditions of Theorem 2.3.1 hold. Then, the system*

$$\begin{cases} u_t(x,t) = \dfrac{1}{2}u_{xx}(x,t) - \psi(x-b(t))u(x,t), \\ u(x,0) = f(x), \quad x \in \mathbb{R}, \\ -g(0) = \displaystyle\int_{\mathbb{R}} \psi(x-b(0))f(x)\,dx, \\ b'(t) = \dfrac{g(t)+g'(t) - \int_{\mathbb{R}}[\psi^2(x-b(t)) - \psi(x-b(t))]u(x,t)\,dx}{\int_{\mathbb{R}} \psi_x(x-b(t))u(x,t)\,dx} \\ \qquad - \dfrac{1/2\int_{\mathbb{R}} \psi_x(x-b(t))u_x(x,t)dx}{\int_{\mathbb{R}} \psi_x(x-b(t))u(x,t)\,dx}, \quad t>0, \end{cases} \tag{2.3.12}$$

has a unique solution $(u,b) \in C_x^2(\mathbb{R})C_t^1(\mathbb{R}_+) \times C_t^1(\mathbb{R}_+)$.

2.4 Local Existence and Uniqueness

We now consider the PDE/ODE system (2.3.12). We have already used the standard notation F_x and F_{xx} to denote the first and second derivatives of a function F of one variable or the first and second partial derivatives with respect to the variable x of a function F of several variables. Because we repeatedly deal with the function $(x,t) \mapsto \psi(x-b(t))$, it will be convenient to recycle notation and define a function $\psi_b : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ by $\psi_b(x,t) = \psi(x-b(t))\}$. We will then set $\psi_{x,b} := \partial_x \psi_b$ and $\psi_{xx,b} := \partial_{xx}\psi_b$. We will continue to use the notation ψ_x and ψ_{xx} with its old meaning, but there should be no confusion between the different objects ψ_b and ψ_x. Similarly, we set $\phi := \psi^2 - \psi = -\psi(1-\psi)$ and put $\phi_b(x,t) = \phi(x-b(t))$. Lastly, for two functions f,g and fixed $t \geq 0$ define $\langle f,g\rangle = \int_{\mathbb{R}} f(x,t)g(x,t)\,dx$.

In the notation we have introduced, we wish to consider the system

$$\begin{cases} u_t(x,t) = \dfrac{1}{2}u_{xx}(x,t) - \psi(x-b(t))u(x,t), \quad x \in \mathbb{R}, t>0, \\ u(x,0) = f(x), \quad x \in \mathbb{R}, \\ b(0) = b_0, \\ b'(t) = \dfrac{g(t)+g'(t) - \langle \phi_b, u\rangle - 1/2\langle \psi_{x,b}, u_x\rangle}{\langle \psi_{x,b}, u\rangle}, \quad t>0, \end{cases} \tag{2.4.1}$$

for some $b_0 \in \mathbb{R}$. (In the proof of Theorem 2.3.1 we choose b_0 to satisfy $-g(0) = \int_{\mathbb{R}} \psi(x - b_0)f(x)\,dx$, but we may take an arbitrary value for b_0 and still obtain a local existence and uniqueness result.)

We have assumed in the statement of Theorem 2.3.1 that $f \in C^2(\mathbb{R})$ and $\psi \in C^3(\mathbb{R})$ with $\|\psi\|_{L^\infty} = 1$, $\|\psi\|_{L^\infty} =: B$, $\|\psi_{xx}\|_{L^\infty} =: C$, and $\|\psi_{xxx}\|_{L^\infty} =: F$ for finite constants B, C, F. Furthermore, we have assumed for some $h > 0$ that $\psi(x) = 1$ for $x \leq -h$, that $\psi(x) = 0$

for $x \geq h$, and that $\psi \geq 0$ and $\psi_x \leq 0$ for all $x \in \mathbb{R}$. Set $\int_{\mathbb{R}} |\psi_x(x)|\, dx =: D$ and note that $0 < D < \infty$. It is immediate that $\|\phi\|_{L^\infty} \leq 1$ and $\|\phi_x\|_{L^\infty} = \|\psi_x(1-2\psi)\|_{L^\infty} \leq \|\psi_x\|_{L^\infty} = B$. Moreover, the functions ϕ and ϕ_x are supported on $[-h, h]$ and $0 < \int_{\mathbb{R}} |\phi(x)|\, dx =: E < \infty$.

Definition 2.4.1. For $T > 0$, let $(\mathcal{L}^T, \|\cdot\|_T)$ be the Banach space consisting of pairs of functions (u, b) such that $u \in C_x^2(\mathbb{R})C_t([0,T])$, $b \in C([0,T])$ and

$$\begin{aligned}\|(u,b)\|_T &:= \|u\|_{L_x^\infty(\mathbb{R})L_t^\infty([0,T])} \\ &\quad + \|u_x\|_{L_x^\infty(\mathbb{R})L_t^\infty([0,T])} + \|u_{xx}\|_{L_x^\infty(\mathbb{R})L_t^\infty([0,T])} \\ &\quad + \|b\|_{L^\infty([0,T])} \\ &< \infty.\end{aligned} \tag{2.4.2}$$

Definition 2.4.2. Given constants $M, N, P, A, L > 0$, $b_0 \in \mathbb{R}$ and $T > 0$, define the closed subset $\Gamma^T_{MNPALb_0} \subset \mathcal{L}^T$ by

$$\begin{aligned}\Gamma^T_{MNPALb_0} := \Bigg\{ &(u,b) \in \mathcal{L}^T : \\ &\|u\|_{L_x^\infty L_t^\infty([0,T])} \leq M, \\ &\|u_x\|_{L_t^\infty([0,T])L_x^\infty} \leq N, \\ &\|u_{xx}\|_{L_t^\infty([0,T])L_x^\infty} \leq P, \\ &b(0) = b_0, \\ &\|b\|_{L^\infty([0,T])} \leq A/2, \\ &\inf_{x \in [-A,A],\, t \in [0,T]} u(x,t) \geq L \Bigg\}.\end{aligned} \tag{2.4.3}$$

The following is the main result of this section.

Theorem 2.4.3. *Suppose that the assumptions of Theorem 2.3.1 hold. Suppose also that the constants $M, N, P, A, L > 0$ and $b_0 \in \mathbb{R}$ are such that*

- $|b_0| \leq A/4$,
- $f(x) \geq 4L > 0$ *for* $x \in [-A, A]$,
- $\|f\|_{L^\infty(\mathbb{R})} \leq M/2$,
- $\|f_x\|_{L^\infty(\mathbb{R})} \leq N/2$,
- $\|f_{xx}\|_{L^\infty(\mathbb{R})} \leq P/2$.

Then, for $T > 0$ sufficiently small there is a contractive map $\Phi : \Gamma^T_{MNPALb_0} \to \Gamma^T_{MNPALb_0}$ defined by $\Phi(v, b) = (u, c)$, where

$$\begin{cases} u_t(x,t) = \frac{1}{2}u_{xx}(x,t) - \psi(x - b(t))v(x,t), \quad x \in \mathbb{R}, t > 0, \\ u(x,0) = f(x), \quad x \in \mathbb{R}, \\ c'(t) = \frac{g(t) + g'(t) - \langle \phi_b, v\rangle - 1/2\langle \psi_{x,b}, v_x\rangle}{\langle \psi_{x,b}, v\rangle}, \quad 0 < t \le T, \\ c(0) = b_0. \end{cases} \tag{2.4.4}$$

We will prove Theorem 2.4.3 in a series of lemmas. Each lemma will assume the hypotheses of Theorem 2.4.3 and the bounds established in the previous lemmas.

Remark 2.4.4. Since f is continuous and positive, for any $A > 0$ there exists $L > 0$ such that $f(x) \ge 4L$ for $x \in [-A, A]$. Therefore, we are not restricting the possible values of $b(0)$ by the above assumptions. We will also assume without loss of generality that $h \le A/4$.

Lemma 2.4.5 (Boundedness of u)**.** *Suppose that $(u, c) = \Phi((v, b))$, with $(v, b) \in \Gamma^T_{MNPALb_0}$. Then, there exists a time $T > 0$ such that*

$$\|u\|_{L^\infty_x L^\infty_t([0,T])} \le M.$$

Proof. Using Duhamel's formula (see (2.9.2)),

$$\begin{aligned} |u(x,t)| &= \left| \int_{\mathbb{R}} G(y,t) f(x-y)\, dy - \int_0^t \int_{\mathbb{R}} G(x-y, t-s)\psi_{c(s)}(y) v(y,s)\, dy\, ds \right| \\ &\le \int_{\mathbb{R}} G(y,t) f(x-y)\, dy + \int_0^t \int_{\mathbb{R}} G(x-y,t-s)|\psi_{c(s)}(y)||v(y,s)|\, dy\, ds \\ &\le M/2 \int_{\mathbb{R}} G(y,t)\, dy + M \int_0^t \int_{\mathbb{R}} G(x-y,t-s)\, dy\, ds \\ &\le M/2 + Mt \\ &\le M \end{aligned}$$

when $t \le \frac{1}{2}$, where

$$G(x,t) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}, \quad x \in \mathbb{R},\ t > 0.$$

□

Lemma 2.4.6 (Boundedness of u_x)**.** *Suppose that $(u, c) = \Phi((v, b))$ with $(v, b) \in \Gamma^T_{MNPALb_0}$. Then, there exists a time $T > 0$ such that*

$$\|u_x\|_{L^\infty_t([0,T]) L^\infty_x} \le N.$$

Proof. Since u_x solves

$$\begin{cases} \left(\partial_t - \dfrac{\partial_{xx}}{2}\right) u_x = -\psi_{x,c} v - \psi_c v_x, & x \in \mathbb{R},\ t > 0, \\ u_x(x,0) = f_x(x), \end{cases}$$

we have via Duhamel's formula that

$$\begin{aligned} |u_x(x,t)| &= \left| \int_{\mathbb{R}} G(y,t) f_x(x-y)\, dy \right. \\ &\quad \left. + \int_0^t \int_{\mathbb{R}} G(x-y,t-s)(-\psi_{x,c} v - \psi_c v_x)(y,s)\, dy\, ds \right| \\ &\le \int_{\mathbb{R}} G(y,t)|f_x(x-y)|\, dy + \int_0^t \int_{\mathbb{R}} G(x-y,t-s)|\psi_{x,c}||v(y,s)|\, dy\, ds \\ &\quad + \int_0^t \int_{\mathbb{R}} G(x-y,t-s)|\psi(y-c(s))||v_x(y,s))|\, dy\, ds \\ &\le \frac{N}{2} + MB \int_0^t \int_{\mathbb{R}} G(x-y,t-s)\, dy\, ds + N \int_0^t \int_{\mathbb{R}} G(x-y,t-s)\, dy\, ds \\ &\le \frac{N}{2} + MBt + Nt. \end{aligned}$$

Thus,

$$\|u_x\|_{L_t^\infty([0,T])L_x^\infty} \le \frac{N}{2} + (MB+N)T \le N$$

whenever $T \le T^*$, where

$$T^* = \frac{N}{2(MB+N)}.$$

□

Lemma 2.4.7 (Boundedness of u_{xx}). *Suppose that* $(u,c) = \Phi((v,b))$ *with* $(v,b) \in \Gamma^T_{MNPALb_0}$. *Then, there exists a time* $T > 0$ *such that*

$$\|u_{xx}\|_{L_t^\infty([0,T])L_x^\infty} \le P.$$

Proof. Noting that u_{xx} solves

$$\begin{cases} \left(\partial_t - \dfrac{\partial_{xx}}{2}\right) u_{xx} = -\psi_{xx,c} v - 2\psi_{x,c} v_x - \psi_c v_{xx}, & x \in \mathbb{R},\ t > 0, \\ u_{xx}(x,0) = f_{xx}(x), \end{cases}$$

analogous manipulations to those from Lemma 2.4.6 yield the result. □

Lemma 2.4.8 (Lower bound for u and boundedness of c' and c)**.** *Suppose that* $(u, c) = \Phi((v, b))$ *with* $(v, b) \in \Gamma^T_{MNPALb_0}$. *Then, there exists a time* $T > 0$ *such that*

$$u \geq L \text{ on } x \in [-A, A],\ t \in [0, T], \tag{2.4.5}$$

and $c(t) \in [-A/2, A/2]$ *for* $t \in [0, T]$.

Proof. Recall that $b(0) \in [-A/4, A/4]$. Then, it is immediate that

$$\left| \int_{\mathbb{R}} \psi_x(x - b(t)) v(x, t)\, dx \right| = \left| \int_{\mathbb{R}} \psi_x(y) v(y + b(t))\, dy \right| \geq DL, \quad t \in [0, T], \tag{2.4.6}$$

because on the support $[-h, h]$ of ψ_x we have $y \in [-h, h] \subseteq [-A/4, A/4]$ which together with the bound on $b(t)$ implies $y + b(t) \in [-A, A]$. Therefore, $v(y + b(t)) \geq L$ for $t \in [0, T]$ which, since $\psi_x \leq 0$, yields

$$\int_{\mathbb{R}} \psi_x(y) v(y + b(t))\, dy \leq L \int_{\mathbb{R}} \psi_x(y)\, dy = -LD < 0, \quad t \in [0, T].$$

We see from these bounds that

$$|c'(t)| \leq \frac{\sup_{[0,t]}(|g + g'|) + ME + ND/2}{LD}$$

and, by integrating,

$$|c(t)| \leq |c(0)| + \frac{\sup_{[0,t]}(|g + g'|) + ME + ND/2}{LD} t.$$

Thus, there is $T > 0$ such that for $t \in [0, T]$,

$$|c(t)| \in [-A/2, A/2].$$

Using the assumptions, equation (2.9.2) gives

$$\begin{aligned} u(x, t) &= \int_{\mathbb{R}} G(y, t) f(x - y)\, dy - \int_0^t \int_{\mathbb{R}} G(x - y, t - s) \psi_{c(s)}(y) v(y, s)\, dy\, ds \\ &\geq 4L \int_{x-A}^{x+A} G(y, t)\, dy - M \int_0^t \int_{\mathbb{R}} G(x - y, t - s)\, dy\, ds \\ &\geq 4L \int_{x-A}^{x+A} G(y, t)\, dy - Mt. \end{aligned}$$

If $0 \leq x \leq A$ then $x - A \leq 0$ and $x + A \geq A > 0$ so for small enough t we have

$$\int_{x-A}^{x+A} G(y, t)\, dy \geq \int_0^A G(y, t)\, dy \geq \frac{1}{3}.$$

If $-A \le x < 0$ then $x + A \ge 0$ and $x - A \le -A < 0$. So, for small enough t,

$$\int_{x-A}^{x+A} G(y,t)\,dy \ge \int_{-A}^{0} G(y,t)\,dy \ge \frac{1}{3}.$$

Therefore, there exists a time $T > 0$ such that whenever $t \in [0,T]$ and $x \in [-A, A]$,

$$\begin{aligned} u(x,t) &\ge \frac{4}{3}L - Mt \\ &\ge L. \end{aligned}$$

□

Lemma 2.4.9. *For a sufficiently small time $T > 0$, the set $\Gamma^T_{MNPALb_0}$ is mapped into itself by Φ.*

Proof. The above lemmas provided the necessary bounds. Now, note that if we start with $(v, b) \in \Gamma^T_{MNPALb_0}$, then we first get the function c from the last two equations in (2.4.4) by simply integrating. The integration is well-defined because the denominator is bounded in absolute value below by $DL > 0$ and the numerator is bounded above. Thus, $c \in C^1([0,t])$. Next, having c in hand we get the function u from the first two equations of (2.4.4). We note that, by Duhamel's formula, the function u has actually more than the desired smoothness, namely, $u \in C^2_x(\mathbb{R})C^1_t([0,T])$. □

Lemma 2.4.10. *Suppose that $(v_1, b_1), (v_2, b_2) \in \Gamma^T_{MNPALb_0}$. Set $(u_1, c_1) = \Phi((v_1, b_1))$ and $(u_2, c_2) = \Phi((v_2, b_2))$. For any $\epsilon > 0$ there exists $T > 0$ such that*

$$\|c_2 - c_1\|_{L^\infty_t([0,T])} \le \epsilon \|(v_2, b_2) - (v_1, b_1)\|_T. \tag{2.4.7}$$

Proof. Note that the functions c_1, c_2 satisfy

$$\begin{cases} c_1'(t) = \dfrac{g(t) + g'(t) - \langle \phi_{b_1}, v_1\rangle - 1/2\langle \psi_{x,b_1}, \partial_x v_1\rangle}{\int_{\mathbb{R}} \langle \psi_{x,b_1}, v_1\rangle}, & t > 0, \\ c_2'(t) = \dfrac{g(t) + g'(t) - \langle \phi_{b_2}, v_2\rangle - 1/2\langle \psi_{x,b_2}, \partial_x v_2\rangle}{\int_{\mathbb{R}} \langle \psi_{x,b_2}, v_2\rangle}, & t > 0. \end{cases} \tag{2.4.8}$$

Subtracting the two equations gives

$$
\begin{aligned}
c_2'(t) - c_1'(t) \quad = \quad & [g(t) + g'(t)] \left(\frac{\langle \psi_{x,b_1}, v_1 \rangle - \langle \psi_{x,b_1}, v_2 \rangle}{\langle \psi_{x,b_1}, v_1 \rangle \langle \psi_{x,b_2}, v_2 \rangle} + \frac{\langle \psi_{x,b_1}, v_2 \rangle - \langle \psi_{x,b_2}, v_2 \rangle}{\langle \psi_{x,b_1}, v_1 \rangle \langle \psi_{x,b_2}, v_2 \rangle} \right) \\
& + \frac{\left(\langle \phi_{b_1}, v_1 \rangle - \langle \phi_{b_2}, v_1 \rangle \right) \langle \psi_{x,b_2}, v_2 \rangle}{\langle \psi_{x,b_1}, v_1 \rangle \langle \psi_{x,b_2}, v_2 \rangle} + \frac{\left(\langle \phi_{b_2}, v_1 \rangle - \langle \phi_{b_2}, v_2 \rangle \right) \langle \psi_{x,b_2}, v_2 \rangle}{\langle \psi_{x,b_1}, v_1 \rangle \langle \psi_{x,b_2}, v_2 \rangle} \\
& + \frac{\left(\langle \phi_{b_2}, v_2 \rangle - \langle \phi_{b_1}, v_2 \rangle \right) \langle \phi_{b_2}, v_2 \rangle}{\langle \psi_{x,b_1}, v_1 \rangle \langle \psi_{x,b_2}, v_2 \rangle} + \frac{\left(\langle \phi_{b_1}, v_2 \rangle - \langle \phi_{b_1}, v_1 \rangle \right) \langle \phi_{b_2}, v_2 \rangle}{\langle \psi_{x,b_1}, v_1 \rangle \langle \psi_{x,b_2}, v_2 \rangle} \\
& + \frac{\left(\langle \psi_{x,b_1}, \partial_x v_1 \rangle - \langle \psi_{x,b_2}, \partial_x v_1 \rangle \right) \langle \psi_{x,b_2}, v_2 \rangle}{2 \langle \psi_{x,b_1}, v_1 \rangle \langle \psi_{x,b_2}, v_2 \rangle} \\
& + \frac{\left(\langle \psi_{x,b_2}, \partial_x v_1 \rangle - \langle \psi_{x,b_2}, \partial_x v_2 \rangle \right) \langle \psi_{x,b_2}, v_2 \rangle}{2 \langle \psi_{x,b_1}, v_1 \rangle \langle \psi_{x,b_2}, v_2 \rangle} \\
& + \frac{\left(\langle \psi_{x,b_2}, v_2 \rangle - \langle \psi_{x,b_1}, v_2 \rangle \right) \langle \psi_{x,b_2}, \partial_x v_2 \rangle}{2 \langle \psi_{x,b_1}, v_1 \rangle \langle \psi_{x,b_2}, v_2 \rangle} \\
& + \frac{\left(\langle \psi_{x,b_1}, v_2 \rangle - \langle \psi_{x,b_1}, v_1 \rangle \right) \langle \psi_{x,b_2}, \partial_x v_2 \rangle}{2 \langle \psi_{x,b_1}, v_1 \rangle \langle \psi_{x,b_2}, v_2 \rangle}.
\end{aligned}
$$

Using the fact that the functions ψ, ψ_x and ϕ are Lipschitz, that v_1 and v_2 are bounded, and that their first derivatives are bounded, we find that

$$
\begin{aligned}
\|c_2' - c_1'\|_{L_t^\infty([0,T])} \quad \leq \quad & \frac{\sup_{[0,T]} |g + g'| \|v_1 - v_2\|_{L_x^\infty L_t^\infty([0,T])}}{L^2 D^2} \\
& + \frac{\sup_{[0,T]} |g + g'| M C (A + 2h) \|b_2 - b_1\|_{L_t^\infty([0,T])}}{L^2 D^2} \\
& + \frac{D M^2 B (A + 2h) \|b_2 - b_1\|_{L_t^\infty([0,T])}}{L^2 D^2} \\
& + \frac{D M E \|v_2 - v_1\|_{L_x^\infty L_t^\infty([0,T])}}{L^2 D^2} \\
& + \frac{E M^2 B (A + 2h) \|b_2 - b_1\|_{L_t^\infty([0,T])}}{L^2 D^2} \\
& + \frac{M E^2 \|v_2 - v_1\|_{L_x^\infty L_t^\infty([0,T])}}{L^2 D^2} \\
& + \frac{N M D C (A + 2h) \|b_2 - b_1\|_{L_t^\infty([0,T])}}{2 L^2 D^2} \\
& + \frac{M D^2 \|\partial_x v_2 - \partial_x v_1\|_{L_x^\infty L_t^\infty([0,T])}}{2 L^2 D^2} \\
& + \frac{N M D C (A + 2h) \|b_2 - b_1\|_{L_t^\infty([0,T])}}{2 L^2 D^2} \\
& + \frac{N D^2 \|v_2 - v_1\|_{L_x^\infty L_t^\infty([0,T])}}{2 L^2 D^2}.
\end{aligned}
$$

Integrating and recalling that $c_1(0) = c_2(0) = b_0$ leads to

$$\begin{aligned} \left| \int_0^t (c_2'(s) - c_1'(s))\, ds \right| &= |c_2(t) - c_1(t) - (c_2(0) - c_1(0))| \\ &\leq \int_0^t |c_2'(s) - c_1'(s)|\, ds \\ &\leq t\|c_2' - c_1'\|_{L_t^\infty([0,t])}. \end{aligned}$$

Hence,

$$\|c_2 - c_1\|_{L_t^\infty([0,T])} \leq T\|c_2' - c_1'\|_{L_t^\infty([0,T])},$$

and by the above bound on $\|c_2' - c_1'\|_{L_t^\infty([0,T])}$ for any $\epsilon > 0$ we can choose T small enough that

$$\|c_2 - c_1\|_{L_t^\infty([0,T])} \leq \epsilon\|(v_2, b_2) - (v_1, b_1)\|_T.$$

□

Lemma 2.4.11. *Suppose that* $(v_1, b_1), (v_2, b_2) \in \Gamma^T_{MNPALb_0}$. *Set* $(u_1, c_1) = \Phi((v_1, b_1))$ *and* $(u_2, c_2) = \Phi((v_2, b_2))$. *For any* $\epsilon > 0$ *there exists* $T > 0$ *such that*

$$\|u_2 - u_1\|_{L_x^\infty L_t^\infty([0,T])} \leq \epsilon\|(v_2, b_2) - (v_1, b_1)\|_T. \tag{2.4.9}$$

Proof. The following equations hold

$$\begin{cases} \left(\partial_t - \dfrac{\partial_{xx}}{2}\right) u_1 = -\psi(x - c_1(t))v_1, & x \in \mathbb{R},\, t > 0, \\ \left(\partial_t - \dfrac{\partial_{xx}}{2}\right) u_2 = -\psi(x - c_2(t))v_2, & x \in \mathbb{R},\, t > 0, \\ u_1(x,0) = f(x), & x \in \mathbb{R}, \\ u_2(x,0) = f(x), & x \in \mathbb{R}. \end{cases} \tag{2.4.10}$$

By Duhamel's formula we have

$$u_1 = G * (f\delta_{t=0}) + G * (-\psi_{c_1} v_1) \tag{2.4.11}$$

and

$$u_2 = G * (f\delta_{t=0}) + G * (-\psi_{c_2} v_2), \tag{2.4.12}$$

where we recall that $*$ denotes convolution on $\mathbb{R}_+ \times \mathbb{R}$. Subtracting the two equations gives

$$u_1 - u_2 = G * ((\psi_{c_2} - \psi_{c_1})v_1 + \psi_{c_2}(v_2 - v_1)).$$

Bounding in terms of the sup norm and using the fact that

$$|\psi(x - c_1(t)) - \psi(x - c_2(t))| \leq \|\psi_x\|_{L_x^\infty}|c_1(t) - c_2(t)|,$$

we have

$$\begin{aligned}
|u_1(x,t)-u_2(x,t)| &\leq \int_0^t \int_{\mathbb{R}} G(x-y,t-s)|\psi_{c_1}(y,s)-\psi_{c_2}(y,s)||v_1(y,s)|\,dy\,ds \\
&\quad + \int_0^t \int_{\mathbb{R}} G(x-y,t-s)|\psi_{c_2}(y,s)||v_2(y,s)-v_1(y,s)|\,dy\,ds \\
&\leq \|\psi_x\|_{L^\infty_x}\|v_1\|_{L^\infty L^\infty_t([0,T])}\|c_1-c_2\|_{L^\infty_x} t \\
&\quad + \|\psi\|_{L^\infty_x}\|v_1-v_2\|_{L^\infty_x L^\infty_t([0,t])} t \\
&= BM\|c_1-c_2\|_{L^\infty_x} t + \|v_1-v_2\|_{L^\infty_x L^\infty_t([0,t])} t.
\end{aligned}$$

Thus,

$$\|u_1-u_2\|_{L^\infty_x L^\infty_t([0,T])} \leq B\|c_1-c_2\|_{L^\infty_x} T + \|v_1-v_2\|_{L^\infty_x L^\infty_t([0,T])} T.$$

so for small enough T we see that (2.4.9) holds. □

Lemma 2.4.12. *Suppose that* $(v_1,b_1),(v_2,b_2) \in \Gamma^T_{MNPALb_0}$. *Set* $(u_1,c_1)=\Phi((v_1,b_1))$ *and* $(u_2,c_2)=\Phi((v_2,b_2))$. *For any* $\epsilon>0$ *there exists* $T>0$ *such that*

$$\|\partial_x u_1 - \partial_x u_2\|_{L^\infty_x L^\infty_t([0,T])} \leq \epsilon\|(v_2,b_2)-(v_1,b_1)\|_T. \tag{2.4.13}$$

Proof. Differentiating (2.4.10) with respect to x

$$\begin{cases}
\left(\partial_t - \frac{\partial_{xx}}{2}\right)\partial_x u_1(x,t) = -\psi_{x,c_1}(x,t)v_1(x,t) - \psi_{c_1}(x,t)\partial_x v_1(x,t), \\
\qquad x\in\mathbb{R}, t>0, \\
\left(\partial_t - \frac{\partial_{xx}}{2}\right)\partial_x u_2(x,t) = -\psi_{x,c_2}(x,t)v_2(x,t) - \psi_{c_2}(x,t)\partial_x v_2(x,t), \\
\qquad x\in\mathbb{R}, t>0, \\
\partial_x u_1(x,0) = f_x(x), \quad x\in\mathbb{R}, \\
\partial_x u_2(x,0) = f_x(x), \quad x\in\mathbb{R},
\end{cases} \tag{2.4.14}$$

Via Duhamel's formula,

$$\partial_x u_1 = G*(f_x\delta_{t=0}) + G*(-\psi_x(\cdot - c_1(\cdot))v_1 - \psi(\cdot - c_2(\cdot))\partial_x v_1) \tag{2.4.15}$$

and

$$\partial_x u_1 = G*(f_x\delta_{t=0}) + G*(-\psi_x(\cdot - c_2(\cdot))v_2 - \psi(\cdot - c_2(\cdot))\partial_x v_2). \tag{2.4.16}$$

Subtracting and rearranging,

$$\begin{aligned}(\partial_x u_1 - \partial_x u_2)(x,t) &= \int_0^t \int_{\mathbb{R}} G(x-y,t-s)[\psi_{x,c_2} v_2(y,s) - \psi_{x,c_1} v_1(y,s)]\, dy\, ds \\ &\quad + \int_0^t \int_{\mathbb{R}} G(x-y,t-s)[\psi_{c_2} \partial_x v_2(y,s) - \psi_{c_1} \partial_x v_1(y,s)]\, dy\, ds \\ &= \int_0^t \int_{\mathbb{R}} G(x-y,t-s)[\psi_{x,c_2} v_2(y,s) - \psi_{x,c_2} v_1(y,s)]\, dy\, ds \\ &\quad + \int_0^t \int_{\mathbb{R}} G(x-y,t-s)[\psi_{x,c_2} v_1(y,s) - \psi_{x,c_1} v_1(y,s)]\, dy\, ds \\ &\quad + \int_0^t \int_{\mathbb{R}} G(x-y,t-s)[\psi_{c_2} \partial_x v_2(y,s) - \psi_{c_2} \partial_x v_1(y,s)]\, dy\, ds \\ &\quad + \int_0^t \int_{\mathbb{R}} G(x-y,t-s)[\psi_{c_2} \partial_x v_1(y,s) - \psi_{c_1} \partial_x v_1(y,s)]\, dy\, ds.\end{aligned}$$

Using estimates similar to those in the proof of Lemma 2.4.11

$$\begin{aligned}\|\partial_x u_1 - \partial_x u_2\|_{L_x^\infty L_t^\infty([0,T])} &\leq BM\|v_2 - v_1\|_{L_x^\infty L_t^\infty([0,T])} T \\ &\quad + CM\|c_2 - c_1\|_{L_t^\infty([0,T])} T \\ &\quad + \|\partial_x v_2 - \partial_x v_1\|_{L_x^\infty L_t^\infty([0,T])} T \\ &\quad + BN\|c_2 - c_1\|_{L_t^\infty([0,T])} T \\ &= BMT\|v_2 - v_1\|_{L_x^\infty L_t^\infty([0,T])} \\ &\quad + (CM + BN)T\|c_2 - c_1\|_{L_t^\infty([0,T])} \\ &\quad + T\|\partial_x v_2 - \partial_x v_1\|_{L_x^\infty L_t^\infty([0,T])}.\end{aligned}$$

so for T small we recover (2.4.13). □

Lemma 2.4.13. *Suppose that* $(v_1, b_1), (v_2, b_2) \in \Gamma^T_{MNPALb_0}$. *Set* $(u_1, c_1) = \Phi((v_1, b_1))$ *and* $(u_2, c_2) = \Phi((v_2, b_2))$. *For any* $\epsilon > 0$ *there exists* $T > 0$ *such that*

$$\|\partial_{xx} u_1 - \partial_{xx} u_2\|_{L_x^\infty L_t^\infty([0,T])} \leq \epsilon \|(v_2, b_2) - (v_1, b_1)\|_T. \tag{2.4.17}$$

Proof. Differentiating (2.4.10) twice with respect to x

$$\begin{cases} \left(\partial_t - \dfrac{\partial_{xx}}{2}\right) \partial_{xx} u_1 = -\psi_{xx,c_1} v_1 - 2\psi_{x,c_1} \partial_x v_1 - \psi_{c_1} \partial_{xx} v_1, \\ \qquad x \in \mathbb{R},\ t > 0, \\ \left(\partial_t - \dfrac{\partial_{xx}}{2}\right) \partial_{xx} u_2 = -\psi_{xx,c_2} v_2 - 2\psi_{x,c_2} \partial_x v_2 - \psi_{c_2} \partial_{xx} v_2, \\ \qquad x \in \mathbb{R},\ t > 0, \\ \partial_{xx} u_1(x,0) = f_{xx}(x), \quad x \in \mathbb{R}, \\ \partial_{xx} u_2(x,0) = f_{xx}(x), \quad x \in \mathbb{R}. \end{cases} \tag{2.4.18}$$

Duhamel's formula and similar manipulations to Lemmas 2.4.11 and 2.4.12 give

$$\begin{aligned}
\|\partial_{xx}u_1 - \partial_{xx}u_2\|_{L_x^\infty L_t^\infty([0,T])} &\leq CM\|v_2 - v_1\|_{L_t^\infty([0,T])L_x^\infty}T \\
&\quad + FM\|c_2 - c_1\|_{L_t^\infty([0,T])}T \\
&\quad + 2B\|\partial_x v_2 - \partial_x v_1\|_{L_x^\infty L_t^\infty([0,T])}T \\
&\quad + 2CN\|c_2 - c_1\|_{L_t^\infty([0,T])}T \\
&\quad + \|\partial_{xx}v_2 - \partial_{xx}v_1\|_{L_x^\infty L_t^\infty([0,T])}T \\
&\quad + BP\|c_2 - c_1\|_{L_t^\infty([0,T])}T \\
&= CMT\|v_2 - v_1\|_{L_x^\infty L_t^\infty([0,T])} \\
&\quad + 2BT\|\partial_x v_2 - \partial_x v_1\|_{L_x^\infty L_t^\infty([0,T])} \\
&\quad + T\|\partial_{xx}v_2 - \partial_{xx}v_1\|_{L_x^\infty L_t^\infty([0,T])} \\
&\quad + (FM + 2CN + BP)T\|c_2 - c_1\|_{L_t^\infty([0,T])}.
\end{aligned}$$

so when $T > 0$ is small (2.4.17) holds. □

Theorem 2.4.14. *[Local existence and uniqueness] Suppose that the conditions of Theorem 2.3.1 hold. Then, there exists a time $T > 0$ such that the system*

$$\begin{cases}
u_t(x,t) = \frac{1}{2}u_{xx}(x,t) - \psi(x - b(t))u(x,t), \quad x \in \mathbb{R},\, t > 0, \\
u(x,0) = f(x), \quad x \in \mathbb{R}, \\
b'(t) = \dfrac{g(t) + g'(t) - \langle \phi_b, u\rangle - 1/2\langle \psi_{x,b}, u_x\rangle}{\langle \psi_{x,b}, u\rangle}, \quad t > 0, \\
b(0) = b_0,
\end{cases}$$

has a unique solution $(u,b) \in C_x^2(\mathbb{R})C_t^1([0,T]) \times C^1([0,T])$.

Proof. Note there exist strictly positive constants A, M, N and P such that $b_0 \in \left[-\frac{A}{4}, \frac{A}{4}\right]$, $f(x) \geq L > 0$, when $x \in [-A, A]$, $\|f\|_{L^\infty(\mathbb{R})} \leq M$, $\|f_x\|_{L^\infty(\mathbb{R})} \leq N/2$, and $\|f_x\|_{L^\infty(\mathbb{R})} \leq P/2$. Putting all the estimates from the above lemmas together we have that, if $0 < \epsilon < 1$ is fixed, then for $T > 0$ small enough

$$\|(u_2, c_2) - (u_1, c_1)\| \leq \epsilon\|(v_2, b_2) - (v_1, b_1)\|.$$

Thus, there exists a $T > 0$ such that the map $\Phi : \Gamma^T_{MNPALb_0} \to \Gamma^T_{MNPALb_0}$ is a contraction. Since $\Gamma^T_{MNPALb_0}$ is a closed subset of the Banach space $\mathcal{L}^T$, the Contraction Mapping Theorem gives that there exists a unique fixed point, that is, a pair $(u,b) \in C_x^2(\mathbb{R})C_t([0,T]) \times C([0,T])$

with $b(0) = b_0$ such that

$$\begin{cases} u_t(x,t) = \dfrac{1}{2}u_{xx}(x,t) - \psi(x-b(t))u(x,t) \\ u(x,0) = f(x) \\ b'(t) = \dfrac{g(t) + g'(t) - \langle \phi_b, u\rangle - 1/2\langle \psi_{x,b}, u_x\rangle}{\langle \psi_{x,b}, u\rangle} \\ b(0) = b_0. \end{cases} \tag{2.4.19}$$

We can now argue that our fixed point (u, b) has more smoothness than it seems *a priori*. The third equation in (2.4.19) implies that b must be continuously differentiable with a bounded derivative. This, together with the first equation from (2.4.19) then tells us that u has a continuous derivative in time. Therefore, we must have $(u, b) \in C_x^2(\mathbb{R})C_t^1([0,T]) \times C^1([0,T])$. □

Corollary 2.4.15. *Assume the hypotheses of Theorem 2.4.14 and the extra conditions*

$$\begin{cases} G(0) = \displaystyle\int_{\mathbb{R}} f(x)\,dx, \\ -g(0) = \displaystyle\int_{\mathbb{R}} \psi(x-b(0))f(x)\,dx, \\ 0 < -g(t) < G(t), \quad t \in [0,T]. \end{cases} \tag{2.4.20}$$

Then, there exists a time $T > 0$ such that the system

$$\begin{cases} u_t(x,t) = \dfrac{1}{2}u_{xx}(x,t) - \psi(x-b(t))u(x,t), \quad x \in \mathbb{R},\, 0 < t < T, \\ u(x,0) = f(x), \quad x \in \mathbb{R}, \\ G(t) = \displaystyle\int_{\mathbb{R}} u(x,t)\,dx, \quad t \in [0,T], \end{cases}$$

has a unique solution $(u,b) : \mathbb{R} \times [0,T] \to \mathbb{R}$. Furthermore, $u \in C_x^2(\mathbb{R})C_t^1([0,T])$ and $b \in C^1([0,T])$.

Proof. First note that by Lemma 2.3.2 we have that $b(0)$ is uniquely determined. From Theorem 2.4.14 we have that there exist unique u, b with $u \in C_x^2(\mathbb{R})C_t^1([0,T])$ and $b \in C^1([0,T])$ satisfying the PDE and having everywhere in $[0,T]$

$$b'(t) = \frac{g(t) + g'(t) - \langle \phi_b, u\rangle - 1/2\langle \psi_{x,b}, u_x\rangle}{\langle \psi_{x,b}, u\rangle}.$$

Set $F(t) := G(t) - \int_{\mathbb{R}} u(x,t)\,dx$ and note that the first two conditions from (2.4.20) yield, together with the PDE, $F_t(0) = F(0) = 0$. The function F belongs to $C^1([0,T])$ and F_t belongs to $C([0,T])$. The above equation for b' is equivalent, after using the PDE, to

$$F_{tt}(t) - F_t(t) = 0, \quad t \in [0,T].$$

Integrating and using the fundamental theorem of calculus, we get

$$F_t(t) - F(t) = F_t(0) - F(0) = 0, \quad t \in [0,T].$$

The unique solution to this differential equation is $F(t) = Ce^t$ for some constant $C \in \mathbb{R}$. This together with $F(0) = 0$ yields $F(t) = 0$ for $t \in [0,T]$. Thus,

$$G(t) = \int_{\mathbb{R}} u(x,t)\,dx, \quad t \in [0,T].$$

Then, taking a derivative and using the PDE,

$$-g(t) = \int_{\mathbb{R}} \psi(x - b(t))u(x,t)\,dx, \quad t \in [0,T].$$

Because $|\psi(x)| \leq 1$ for $x \in \mathbb{R}$, $\psi = 0$ for $x \geq h$ and $u(x,t) > 0$ we see that

$$0 < \int_{\mathbb{R}} \psi(x - b(t))u(x,t)\,dx = -g(t) < \int_{\mathbb{R}} u(x,t)\,dx = G(t).$$

□

2.5 Discontinuous killing

Next, we consider the existence of a barrier when killing is done non-smoothly. That is, we ask whether there exists a function b such that, for a given G

$$G(t) = \int_{\mathbb{R}} \mathbb{E}\left[\exp\left(-\int_0^t \mathbf{1}_{(-\infty,0]}(x + B_u - b(u))\,du\right) f(x)\right] dx \tag{2.5.1}$$

Note that $\int_0^t \mathbf{1}_{(-\infty,0]}(x+B_u-b(u))\,du$ is the time during the interval $[0,t]$ spent by a Brownian motion started at x below the barrier b.

Theorem 2.5.1. *There exists a function b such that, for a given, twice continuously differentiable G satisfying $0 < -g(t)/G(t) < 1$, $t \geq 0$, equation (2.5.1) holds for all $t \geq 0$.*

Proof. Let ϕ be a smooth decreasing function supported on $[0,1]$ with $\int_{\mathbb{R}} \phi(x)\,dx = 1$. Put

$$\underline{\psi}_\epsilon(x) = \int_x^\infty \phi((y - \epsilon)/\epsilon))(1/\epsilon)\,dy$$

and

$$\overline{\psi}_\epsilon(x) = \int_x^\infty \phi(y/\epsilon))(1/\epsilon)dy,$$

so that

$$\underline{\psi}_\epsilon(x) \le 1\{x \le 0\} \le \overline{\psi}_\epsilon. \tag{2.5.2}$$

Note also that

$$\underline{\psi}_\epsilon(x) \text{ increases with } \epsilon \text{ for all } x \tag{2.5.3}$$

and

$$\overline{\psi}_\epsilon(x) \text{ decreases with } \epsilon \text{ for all } x. \tag{2.5.4}$$

Let $\underline{b}_\epsilon$ and $\overline{b}_\epsilon$ be the two barriers corresponding to $\underline{\psi}_\epsilon(x)$ and $\overline{\psi}_\epsilon$. The existence and uniqueness of these two barriers follows by Theorem 2.3.1. From (2.5.2) we have that

$$\overline{b}_\epsilon(t) \le \underline{b}_\epsilon(t)$$

for all t and from (2.5.3), (2.5.4) that

$$\overline{b}_\epsilon(t) \text{ is increasing in } \epsilon \text{ for each } t$$

and

$$\underline{b}_\epsilon(t) \text{ is decreasing in } \epsilon \text{ for each } t.$$

Put

$$\overline{b}_*(t) = \lim_{\epsilon \downarrow 0} \overline{b}_\epsilon(t)$$

and

$$\underline{b}_*(t) = \lim_{\epsilon \downarrow 0} \underline{b}_\epsilon(t).$$

Then,

$$\overline{b}_*(t) \le \underline{b}_*(t), \tag{2.5.5}$$

and both of these functions give a stopping time with the correct distribution for the case where ψ is the indicator of $(-\infty, 0]$. Because of (2.5.5), it must be the case that $\overline{b}_*(t) = \underline{b}_*(t)$ for Lebesgue almost all t. □

2.6 Pricing Claims

Suppose that the asset price $(X_t)_{t\ge0}$ is a geometric Brownian motion given by

$$\frac{dX_t}{X_t} = \mu dt + \sigma dW_t, \tag{2.6.1}$$

where $(W_t)_{t\ge0}$ is a standard Brownian motion. We model default using a diffusion $(Y_t)_{t\ge0}$ where

$$dY_t = dB_t, \tag{2.6.2}$$

with $(B_t)_{t\geq 0}$ another standard Brownian motion. We assume that the Brownian motions W and B are correlated with correlation $-1 \leq \rho \leq 1$; that is, the cross-variation of the two processes satisfies

$$[B, W]_t = \rho t, \quad t \geq 0.$$

We can assume without loss of generality that for two independent Brownian motions B', B'' we have

$$\begin{cases} W_t = B'_t, \\ B_t = \rho B'_t + \sqrt{1-\rho^2} B''_t. \end{cases}$$

In the following we will look at pricing contingent claims with a fixed maturity $T > 0$ and payoff of the form

$$F(X_T) 1\{\tau > T\}$$

for the random time

$$\tau := \inf \left\{ t > 0 : \lambda \int_0^t \psi(Y_s - b(s))\, ds > U \right\},$$

where U is an independent exponentially distributed random variable with mean one.

Note that

$$\mathbb{E}^x[F(X_T) 1\{\tau > T\}] = \mathbb{E}^x \left[F(X_T) \exp\left(-\lambda \int_0^T \psi(Y_s - b(s))\, ds \right) \right].$$

More generally, we will be interested in expressions of the form

$$\begin{aligned} &\mathbb{E}^x \left[F(X_T) 1\{\tau > T\} \;\middle|\; (X_s)_{0\leq s\leq t},\, \tau > t \right] \\ &\quad = \mathbb{E}^x \left[F(X_T) \exp\left(-\lambda \int_t^T \psi(Y_s - b(s))\, ds \right) \;\middle|\; (X_s)_{0\leq s\leq t},\, \tau > t \right], \end{aligned}$$

which we interpret as the price of the payoff at time $0 \leq t \leq T$ given that default has not yet occurred.

Consider the Markov process $Z = (X, Y, V)$ where X, Y are as above, and V is a process that, when started at v is at $v + t$ after t units of time, that is, $V_t = V_0 + t$. The generator of Z is

$$A = (1/2)\sigma^2 x^2 D_x^2 + \mu x D_x + (1/2) D_y^2 + \rho\sigma x D_x D_y + D_v.$$

We want to compute

$$\mathbb{E}^{(x,y)} \left[F(X_T) e^{-\int_0^T \lambda\psi(Y_s - b(s))\, ds} \right] = \mathbb{E}^{(x,y,0)} \left[F(X_T) e^{-\int_0^T \lambda\psi(Y_s - b(V_s))\, ds} \right].$$

The Feynman-Kac formula says that the solution to the PDE

$$\begin{cases} D_t u(x,y,v,t) = Au(x,y,v,t) - \lambda\psi(y - b(v)) u(x,y,v,t), \\ \quad u(x,y,v,0) = F(x), \end{cases} \tag{2.6.3}$$

satisfies

$$\mathbb{E}^{(x,y)}\left[F(X_T)\exp\left(-\int_0^T \lambda\psi(Y_s-b(s))ds\right)\right] = u(x,y,0,T).$$

Thus, if we assume the Brownian motion Y has an random starting point Y_0 with density f that is independent of $(Y_t - Y_0)_{t\geq 0}$, then

$$\mathbb{E}^x\left[F(X_T)\exp\left(-\int_0^T \lambda\psi(Y_s-b(s))ds\right)\right] = \int_{\mathbb{R}} u(x,y,0,T)f(y)\,dy.$$

Using this and the Markov property, one can find the function $K(x,y,t)$ satisfying

$$K(X_t,Y_t,t) = \mathbb{E}^x\left[F(X_T)\exp\left(-\lambda\int_t^T \psi(Y_s-b(s))\,ds\right)\,\middle|\,(X_s)_{0\leq s\leq t},\,(Y_s)_{0\leq s\leq t},\,\tau>t\right].$$

The price at time t, given that we know the history of the price process X_t and that default has not happened up to time t, is

$$\begin{aligned}\mathbb{E}\left[F(X_T)1\{\tau>T\}\,\middle|\,(X_s)_{0\leq s\leq t},\,\tau>t\right] &= \mathbb{E}\left[K(X_t,Y_t,t)\,\middle|\,(X_s)_{0\leq s\leq t},\,\tau>t\right]\\ &= \frac{\mathbb{E}[K(X_t,Y_t,t)1\{\tau>t\}\mid (X_s)_{0\leq s\leq t}]}{\mathbb{E}[1\{\tau>t\}\mid (X_s)_{0\leq s\leq t}]}.\end{aligned}$$

It follows from the SDE for X that

$$B'_t = W_t = \frac{1}{\sigma}\left[\log X_t - \log X_0 + \left(\frac{\sigma^2}{2}-\mu\right)t\right],$$

so if we observe the asset price X, then we can reconstruct the standard Brownian motion B'. On the other hand,

$$X_t = X_0\exp\left(\sigma B'_t - \left(\frac{\sigma^2}{2}-\mu\right)t\right).$$

Now,

$$\begin{aligned}&\mathbb{E}[K(X_t,Y_t,t)1\{\tau>t\}\mid (X_s)_{0\leq s\leq t}]\\ &\quad= \mathbb{E}\Bigg[K\left(X_0\exp\left(\sigma B'_t-\left(\frac{\sigma^2}{2}-\mu\right)t\right), Y_0+\rho B'_t+\sqrt{1-\rho^2}B''_t, t\right)\\ &\qquad\times \mathbf{1}\left\{\int_0^t \psi\left(Y_0+\rho B'_s+\sqrt{1-\rho^2}B''_s-b(s)\right)\,ds\leq U\right\}\\ &\qquad\Bigg|\,X_0,\,(B'_s)_{0\leq s\leq t}\Bigg].\end{aligned}$$

We therefore want to be able to compute for a function $c : \mathbb{R}_+ \to \mathbb{R}$ the conditional expected value

$$\begin{aligned}
&\mathbb{E}\Big[K\left(X_0 \exp\left(\sigma c(t) - \left(\frac{\sigma^2}{2} - \mu\right) t\right), Y_0 + \rho c(t) + \sqrt{1-\rho^2} B''_t, t\right) \\
&\quad \times \mathbf{1}\left\{\int_0^t \psi\left(Y_0 + \rho c(s) + \sqrt{1-\rho^2} B''_s - b(s)\right) ds \le U\right\} \Big| X_0\Big] \\
&= \mathbb{E}\Big[K\left(X_0 \exp\left(\sigma c(t) - \left(\frac{\sigma^2}{2} - \mu\right) t\right), Y_0 + \rho c(t) + \sqrt{1-\rho^2} B''_t, t\right) \\
&\quad \times \exp\left(-\int_0^t \psi\left(Y_0 + \rho c(s) + \sqrt{1-\rho^2} B''_s - b(s)\right) ds\right) \Big| X_0\Big],
\end{aligned}$$

with $(B''_t)_{t\ge 0}$ a standard Brownian motion independent of X_0. We can do this using Feynman-Kac.

The denominator in the formula for the price at time t is a special case of the numerator we have just calculated with $K \equiv 1$, and it can be dealt with in the same way.

We have thus observed that computing the price of a contingent claim reduces to solving certain PDEs with coefficients depending on the path of the asset price.

2.7 Numerical Results

In this section we present the results of some numerical experiments. We solved the PDE/ODE system (2.3.12) using the pseudo-spectral Implicit-Explicit Fourth Order Runge-Kutta scheme ARK4(3)6L[2]SA-ERK, taking 8192 nodes and period 16, developed in [KC03]. For the function ψ we used the Fejér kernel of order 512 applied to the indicator of the set $\{x \in \mathbb{R} : x < 0\}$; in other words ψ is the Cesàro sum of the truncated Fourier series of order 512 of the indicator of the set $\{x \in \mathbb{R} : x < 0\}$. The time horizon was taken to be $T = 8$, the initial distribution of the credit index process Y was taken to be normal ($Y_0 \sim N(0, \sigma^2)$ with standard deviation $\sigma = 0.25$), and the time to default was taken to have an exponential distribution ($G(t) = e^{-\nu t}$ with rates $\nu = 0.0625, 0.125, 0.25, 0.5$).

For the first experiment, we fix the killing parameter to $\lambda = 1$. We show the resulting barriers b in Figure 2.1. We also show the relative error between the survival function $G(t)$ and the numerically computed value of $\int_{\mathbb{R}} u(x,t)\,dx$ (recall (2.3.3)), and the relative error between the hazard rate $-g(t)/G(t)$ and the numerically computed value of $\int_{\mathbb{R}} \psi(x - b(t))u(x,t)\,dx / \int_{\mathbb{R}} u(x,t)\,dx$ (recall (2.3.4)).

For the second experiment, we take the exponential rate to be $\nu = 0.125$ and the standard deviation to be $\sigma = 0.25$. We look at the graphs for when the killing parameter is $\lambda = 1, 10, 50, 200$. The barriers, together with the relative errors in the survival functions and hazard rates are given in Figure 2.2.

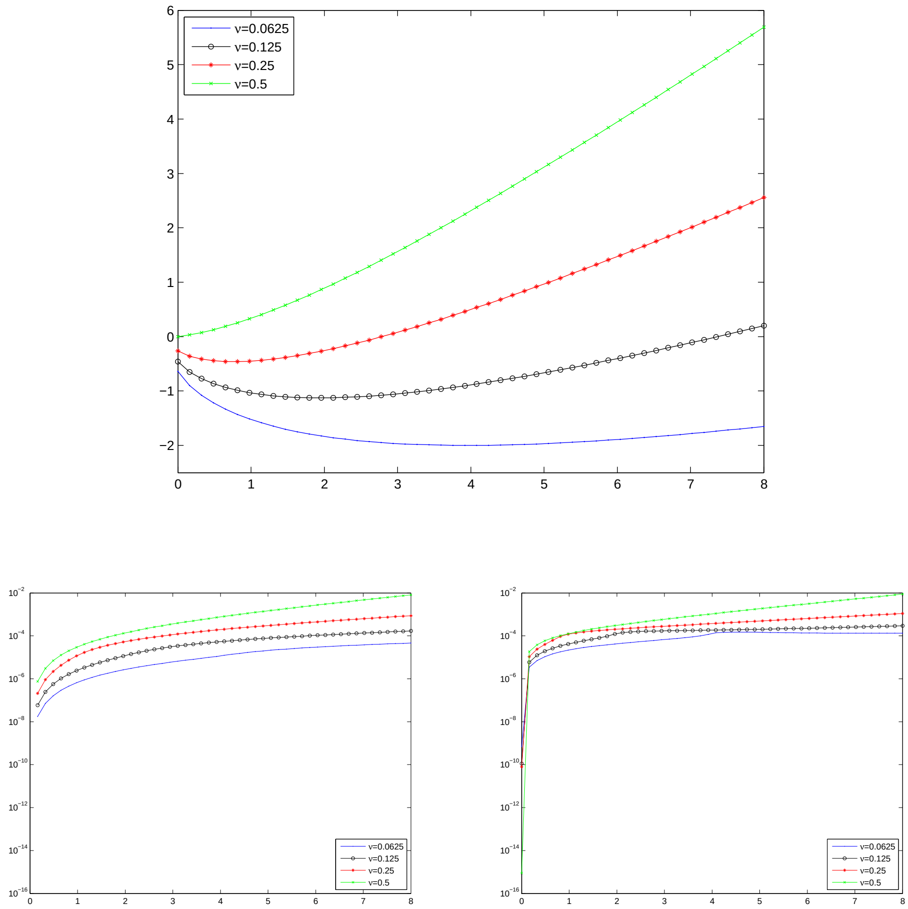

Figure 2.1: This figure displays the results of the numerical experiments described in Section 2.7. We fix the standard deviation for the initial distribution of the credit index process Y to be $\sigma = 0.25$ and the killing parameter to be $\lambda = 1$. The first row gives the barriers for the rate parameters $\nu = 0.0625, 0.125, 0.25, 0.5$ of the exponential default time distribution. The first (resp. second) panels in the second row give the relative errors between the actual survival function values $G(t)$ (resp. the actual hazard function values $-g(t)/G(t)$) and the numerically computed ones – see the text for details.

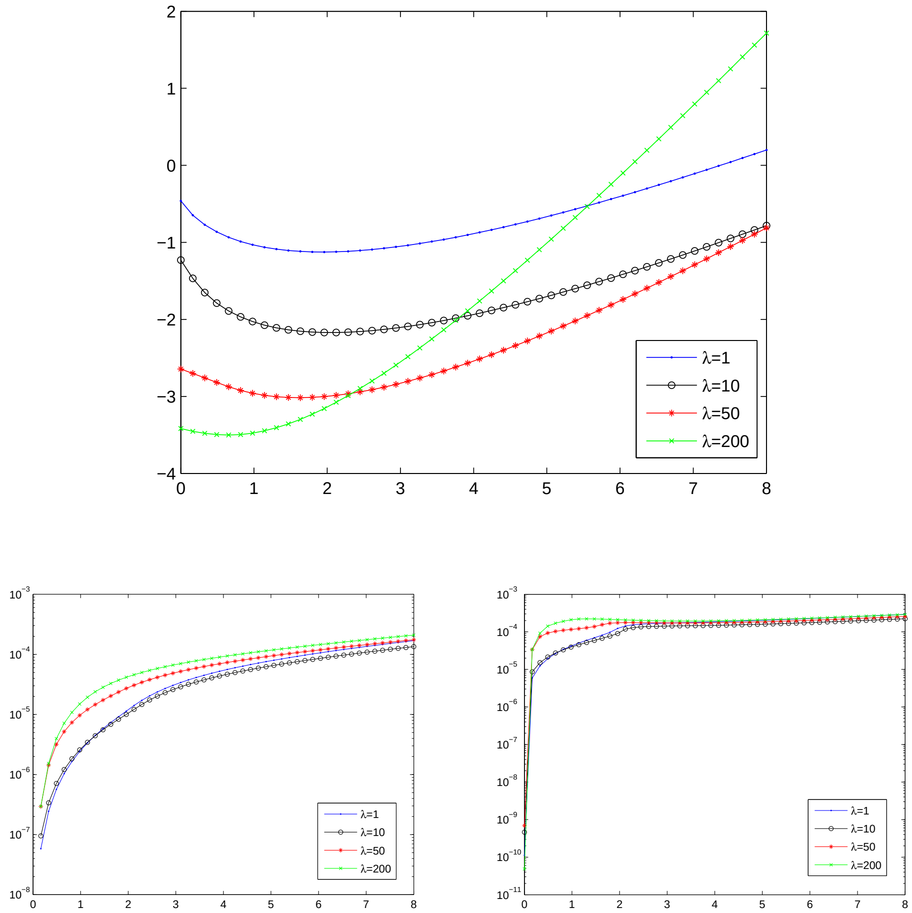

Figure 2.2: In this figure we fixed the standard deviation to $\sigma = 0.25$ and the rate parameter to $\nu = 0.125$. The first row gives the barriers for the killing parameters $\lambda = 1, 10, 50, 200$. The first and second panels in the second row give the relative errors for the survival function (resp. the hazard function).

2.8 Calibrating the default distribution using CDS rates

For the sake of completeness, we review briefly the scheme proposed in [DP11] for determining the distribution of the time to default.

A credit default swap (CDS) is a contract between two parties. The buyer of the swap makes a number of predetermined payments until the moment of default. The seller is liable to pay the unrecovered value of the underlying bond in the event of a default before maturity. Normalizing the notional value of the bond to 1, the seller's contingent payment is $1 - R$, where $R \in (0,1)$ is the recovery rate, which we take to be constant. The premium payments are made at a set of times $\{t_i\}$. The maturities are a subset of the premium payment times; that is, they are of the form $T_0 = 0$, $T_j = t_{k(j)}$, $j = 1, \dots, n$. For $j = 1, \dots, n$ there is an upfront premium π_j^0 and a running premium rate π_j^1 (having accrual factors δ_i). Denote the price at time zero of a zero coupon risk-free bond with maturity t_j by $p_0(t_j)$. It follows from standard non-arbitrage arguments that

$$\pi_j^0 + \pi_j^1 \sum_{i=k(j-1)}^{k(j)-1} \delta_i p_0(t_i) G(t_i) = (1-R) \sum_{i=k(j-1)+1}^{k(j)} p_0(t_i)(G(t_{i-1}) - G(t_i)), \tag{2.8.1}$$

where $G(t) = \mathbb{P}\{\tau > t\}$ is the tail of the distribution of the time to default.

Suppose now that the default distribution has piecewise constant hazard rate; that is, that

$$G(t) = \exp\left(-\int_0^t h(s)\,ds\right), \quad t \geq 0,$$

where $h(s) = h_i$ for $s \in [T_i, T_{i+1})$. Given the market data $(\pi_1^0, \pi_1^1), (\pi_2^0, \pi_2^1), \dots$ we can find, using equation (2.8.1), the constants $h_0, h_1, \dots$.

We use the following procedure to find the barrier b. Set $\nu = h_0$ and $G(t) = e^{-\nu t}$. Given the initial density f, which we can choose to be any strictly positive function f that is twice continuously differentiable with bounded f, f' and f'', we want to find a barrier such that for $0 \leq t \leq T = T_1$ we have

$$e^{-\nu t} = \mathbb{E}\left[\int_{\mathbb{R}} f(x) \exp\left(-\lambda \int_0^t \psi(x + B_s - b(s))\,ds\right) dx\right].$$

This can be achieved by solving the ODE/PDE system (2.3.12). Next, set $\nu_1 = h_1$, $T = T_2 - T_1$, $f_1(x) = \mathbb{E}\left[f(x)\exp\left(-\lambda \int_0^{T_1} \psi(x + B_s - b(s))\,ds\right)\right]$ and find a barrier with $b_1(0) = b(T_1)$ such that on $0 \leq t < T = T_2 - T_1$ we have

$$e^{-\nu_1 t} = \mathbb{E}\left[\int_{\mathbb{R}} f_1(x) \exp\left(-\lambda \int_0^t \psi(x + B_s - b_1(s))\,ds\right) dx\right].$$

This procedure can be repeated until we find a function b on $[0, T_n]$ that is continuously differentiable everywhere, except perhaps the finite number of points $T_1, \dots, T_n$.

2.9 Duhamel's formula

For the sake of reference, we provide a statement of Duhamel's formula. Given functions $v : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ and $b : \mathbb{R}_+ \to \mathbb{R}$, the solution of

$$\begin{cases} \left(\partial_t - \frac{\partial_{xx}}{2}\right) u = -\psi_b v, & x \in \mathbb{R}, t > 0, \\ u(x,0) = f(x) & x \in \mathbb{R}, \end{cases} \tag{2.9.1}$$

is given by

$$\begin{aligned} u(x,t) &= [G * (f\delta_{t=0})](x,t) + [G * (-\psi_b v)](x,t) \\ &= \int_{\mathbb{R}} G(x-y,t) f(y)\, dy - \int_0^t \int_{\mathbb{R}} G(x-y,t-s)\psi_{b(s)}(y) v(y,s)\, dy\, ds, \end{aligned} \tag{2.9.2}$$

where

$$G(x,t) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}, \quad x \in \mathbb{R},\ t > 0. \tag{2.9.3}$$

Chapter 3

Invasibility in spatio-temporally heterogeneous environments

3.1 Introduction

Environmental conditions such as light, precipitation and food availability are usually functions of space and time. Organisms are influenced by environmental conditions and are constantly faced with deciding whether or not they should change location. If an individual disperses, it may go to a location with poorer conditions. If, on the other hand, the individual chooses to stay in the same place then it may face worsening local environmental conditions due to temporal fluctuations. There have been extensive field and simulation studies regarding the dispersal of a population in a heterogeneous environment and how this influences the persistence of a population [JY98, RHB05, MG07]. Population growth is inherently stochastic due to numerous unpredictable causes. The simplest continuous time model for a single population would be one of the form $d\hat{X}_t = \mu\hat{X}_t dt + \sigma\hat{X}_t dB_t$ where $\hat{X}t$ denotes the population size at time t, μ is the mean growth rate, $\mathbb{E}[\hat{X}_{t+\Delta t} - \hat{X}_t \mid \hat{X}_t = z] \approx z\mu\Delta t$, σ^2 is the variance of fluctuations in the growth rate, $\mathbb{E}[(\hat{X}_{t+\Delta t} - \hat{X}_t - z\mu\Delta t)^2 \mid \hat{X}_t = z] \approx z^2\sigma^2\Delta t$, and B_t is a Brownian motion. Due to its simplicity, this model has been used in the literature for evaluating the risk of extinction [DMS91, LES03]. There has been a lot of work on understanding the joint effects of temporal and spatial heterogeneity on population persistence and the evolution of dispersal, but most of it is not mathematically rigorous [GH02, MH92, Has83, LCH84]. There also has been mathematical work for the case of spatial heterogeneity by itself [HMP01, CC91, CCL06].

A general model which addresses the spatio-temporal effects of heterogeneity is discussed in [ERSS]. The authors assume that there are n distinct patches and that the population can disperse from one patch to the other while continuously experiencing uncertainty in environmental conditions in space and time. They use a system of coupled stochastic differential equations (SDE) driven by Brownian motions which are not perfectly correlated, so that good years in a region (patch) do not necessarily correspond to good years everywhere. In [ERSS] the authors try to answer some important questions arising in population biology: For diffusively dispersing populations, when is there selection for higher versus lower dispersal rate? How do different spatial scales of environmental heterogeneity influence the persistence of a population? If there are no constraints on the dispersal strategy, then which one maximizes the population growth rate?

If $(X_t^1, \ldots, X_t^n)$ denotes the populations of the n patches at time t then adding dispersal to the regional dynamics, following [ERSS], leads to the following system of stochastic differential equations

$$dX_t^i = X_t^i(\mu_i dt + dE_t^i) + \sum_{j=1}^n D_{ji} X_t^j dt, i = 1, \ldots, n,$$

where $(E^1, \ldots, E^n)$ is a vector of correlated Brownian motions with covariance matrix $\Gamma^T\Gamma$, $D_{ij} \geq 0, i \neq j$ is the per-capita rate at which the population from patch i disperses to patch j and $-D_{ii} := \sum_{j\neq i} D_{ij}$ is the total per-capita immigration rate out of patch i. The covariance matrix $\Gamma^T\Gamma$ captures the spatial dependence between the temporal fluctuations in patch quality and the drift μ_i is the mean per-capita growth rate in patch i.

The model from [ERSS] does not account for an important biological feature: negative density-dependent feedbacks. At the within-patch scale, individual per-capita growth rates often are reduced by increasing local population density due to the effect of competition for resources.

Generalizing and extending the model from [ERSS] to include competition of individuals for resources will lead to studying stochastic differential equations of the form

$$dX_t^i = \mu_i X_t^i dt - \kappa_i (X_t^i)^2 dt + X_t^i dE_t^i + \sum_{j=1}^n D_{ji} X_t^j dt,\ i = 1, \ldots, n, \tag{3.1.1}$$

where the term $-\kappa_i(X_t^i)^2 dt$ accounts for negative density dependence which may arise due to competition for resources. We will study this system of SDE under the simplifying assumption that for all $t \geq 0$ the total population X_t is spread through the patches via

$$X_t^i = \alpha_i X_t,\ i = 1, \ldots, n,$$

where $\alpha_i \in [0, 1]$ and $\sum_{i=1}^n \alpha_i = 1$.

Assume we have two species whose total populations M_t, N_t are spread out via $M_t^i = \alpha_i M_t$ and $N_t^i = \beta_i N_t$. We show below that we can model the interaction of the two species by the coupled system of SDE

$$\begin{aligned} dM_t &= M_t \left[\mu \cdot \alpha - \langle \alpha, \beta \rangle N_t - \langle \alpha, \alpha \rangle M_t\right] dt + M_t \sqrt{\alpha^T \Gamma^T \Gamma \alpha}\, dU_t \\ dN_t &= N_t \left[\mu \cdot \beta - \langle \alpha, \beta \rangle M_t - \langle \beta, \beta \rangle N_t\right] dt + N_t \sqrt{\beta^T \Gamma^T \Gamma \beta}\, dV_t. \end{aligned}$$

As a first step, in Theorem 3.2.1 we find necessary and sufficient conditions for the existence of a stationary distribution for the one dimensional SDE

$$d\bar{M}_t = \left(\sum_i \mu_i \alpha_i \bar{M}_t - \sum_i \kappa_i (\alpha_i \bar{M}_t)^2 \right) dt + \bar{M}_t \sqrt{\alpha^T \Gamma^T \Gamma \alpha}\, dW_t.$$

A similar SDE describes the process $(\bar{N})$. Proposition 3.3.10 tells us that, in some sense, if we start the diffusion (M, N) at a point (x, y) with $x, y > 0$ then the process (M, N) converges weakly to a stationary distribution on $[0, \infty) \times [0, \infty)$. One would like to know in which cases one, both or none of the two populations go extinct. By looking at the Lyapunov exponents of the linearized SDE

$$d\hat{N}_t := \hat{N}_t \left[\mu \cdot \beta - \langle \alpha, \beta \rangle \bar{M}_t \right] dt + \hat{N}_t \sqrt{\beta^T \Gamma^T \Gamma \beta}\, dV_t,$$

we show in Proposition 3.3.11 that if the Lyapunov exponent of $\hat{M}$ is negative, $L_{\hat{M}} < 0$ then almost surely $\lim_{t \to \infty} M_t = 0$ i.e. M goes extinct. This enables us to prove in Theorem 3.3.8 that when $L_{\hat{M}} < 0$ the probability measures

$$\frac{1}{t} \int_0^t \mathbb{P}^{(x,y)} \{ (M_s, N_s) \in \cdot \}\, ds$$

converge weakly as $t \to \infty$ to $\delta_0 \otimes \rho$, where ρ is the unique stationary distribution of $\bar{N}$ concentrated on $(0, \infty)$. In Theorem 3.3.14 we say something about the case $L_{\hat{M}} > 0, L_{\hat{N}} > 0$. When both Lyapunov exponents are positive one can show that there exists $\epsilon > 0$ such that for all $s > 0$ there exists $t \geq s$ such that $M_t \geq \epsilon, N_t \geq \epsilon$. In particular with probability one M_t and N_t do not die out.

The species N is trying to invade M so it tries to maximize its Lyapunov exponent. Theorem 3.4.1 gives us the value of $\max_\beta L_{\hat{N}}$, the maximal Lyapunov exponent of $\hat{N}$ for a fixed α. Finally, in Section 3.5 we show some results in the simplified case when we have only $n = 2$ patches.

3.2 The Model

Suppose we have n patches and that the total population of the resident species at time t, $\tilde{M}_t$, is spread through these patches via

$$\tilde{M}_t^i = \alpha_i \tilde{M}_t,$$

where $\alpha_i > 0$ for $1 \leq i \leq n$ and $\sum_i \alpha_i = 1$. Assume to begin with that the population in the i^{th}patch satisfies the SDE

$$d\tilde{M}_t^i = \mu_i \tilde{M}_t^i\, dt + \tilde{M}_t^i\, dE_t^i,$$

where $E^i = \sum_j \gamma_{ji} B_t^j$ for a standard Brownian motion $(B^1, \ldots, B^n)^T$ on $\mathbb{R}^n$ and $\Gamma := (\gamma_{ij})$ is an $n \times n$ matrix.

This model may be thought as the limit as $\delta \to \infty$ of a model of the form

$$d\tilde{M}_t^i = \mu_i \tilde{M}_t^i \, dt + \delta \sum_{j=1}^n q_{ji} \tilde{M}_t^j \, dt + \tilde{M}_t^i \, dE_t^i,$$

where δq_{ji}, $j \neq i$ is the per-capita dispersal rate from patch j into patch i, $-\delta q_{ii} = \delta \sum_{k \neq i} q_{ik}$ is the total per-capita rate of dispersal out of patch i and $\alpha := (\alpha_1, \ldots, \alpha_n)$ satisfies $\sum_{j=1}^n \alpha_j q_{ji} = 0$ for $1 \leq i \leq n$; that is, the probability vector α is the stationary distribution of the continuous time Markov chain with infinitesimal generator matrix Q. The covariance matrix $\Gamma^T \Gamma$ captures the spatial dependence between the temporal fluctuations in patch quality and the drift μ_i is the mean per-capita growth rate in patch i.

Returning to our initial model, since $\tilde{M}_t = \sum_i \tilde{M}_t^i$, we have

$$\begin{aligned} d\tilde{M}_t &= \left(\sum_i \alpha_i \mu_i \tilde{M}_t \right) dt + \sum_i \alpha_i \tilde{M}_t \, dE_t^i \\ &= \left(\sum_i \alpha_i \mu_i \right) \tilde{M}_t \, dt + \tilde{M}_t \sum_{i,j} \alpha_i \gamma_{ji} \, dB_t^j \\ &= (\alpha \cdot \mu) \tilde{M}_t \, dt + \tilde{M}_t \sqrt{\alpha^T \Gamma^T \Gamma \alpha} \, dW_t, \end{aligned}$$

where $\mu := (\mu_1, \ldots, \mu_n)$ and W is a one-dimensional standard Brownian motion. The last line in the above follows because the quadratic variation of the process $\sum_{i,j} \alpha_i \gamma_{ji} B^j$ satisfies $d[\sum_{i,j} \alpha_i \gamma_{ji} B^j]_t = \alpha^T \Gamma^T \Gamma \alpha \, dt$.

Suppose now that we introduce competition between the individuals in each patch. The SDE for the total population $\bar{M}_t$ becomes

$$d\bar{M}_t = \left(\sum_i \mu_i \alpha_i \bar{M}_t - \sum_i \kappa_i (\alpha_i \bar{M}_t)^2 \right) dt + \bar{M}_t \sqrt{\alpha^T \Gamma^T \Gamma \alpha} \, dW_t. \tag{3.2.1}$$

Note that if we set $X_t := \log \bar{M}_t$, then by Itô's lemma,

$$dX_t = \left(\sum_i \mu_i \alpha_i - \sum_i \kappa_i \alpha_i^2 \exp(X_t) - \frac{1}{2} \alpha^T \Gamma^T \Gamma \alpha \right) dt + \sqrt{\alpha^T \Gamma^T \Gamma \alpha} \, dW_t.$$

It is clear that if $\bar{M}_0 \in \mathbb{R}_{++} := (0, \infty)$, then $\bar{M}_t \in \mathbb{R}_{++}$ for all $t \geq 0$ almost surely. Note that if we define the process

$$d\tilde{X}_t = \left(\sum_i \mu_i \alpha_i - \frac{1}{2} \alpha^T \Gamma^T \Gamma \alpha \right) dt + \sqrt{\alpha^T \Gamma^T \Gamma \alpha} \, dW_t.$$

then $\tilde{X}$ is just a Brownian motion with drift so it does not explode. By the comparison theorem for 1-dimensional SDE (see Theorem V.43.1 from [RW00]) if $\tilde{X}_0 = X_0$ then $X_t \leq \tilde{X}_t$ for all $t \geq 0$. Therefore, X does not explode to $+\infty$. Equivalently one can use the Feller test for explosion (see Theorem 5.5.29 in [KS91]) to show that X does not explode to $\pm\infty$.

Of course, if $\bar{M}_0 = 0$, then $\bar{M}_t = 0$ for all $t \geq 0$.

We can be explicit about conditions under which the one-dimensional diffusion process $\{\bar{M}_t\}_{t\geq 0}$ has a stationary distribution concentrated on $\mathbb{R}_{++}$. For ease of notation, we introduce the inner product $\langle\cdot,\cdot\rangle$ defined via $\langle x, y\rangle := \sum_{i=1}^n \kappa_i x_i y_i$ and adopt the notation $\mu \cdot \alpha := \sum_{i=1}^n \mu_i \alpha_i$ for the usual inner product.

Theorem 3.2.1. *The diffusion process $\{\bar{M}_t\}_{t\geq 0}$ defined by* (3.2.1) *has a stationary distribution concentrated on $\mathbb{R}_{++}$ if and only if $\mu \cdot \alpha - \frac{\alpha^T \Gamma^T \Gamma \alpha}{2} > 0$, in which case that stationary distribution is unique and has the Gamma density $x \mapsto \frac{1}{\Gamma(k)\theta^k} x^{k-1} e^{-\frac{x}{\theta}}$ with parameters*

$$\theta := \frac{\alpha^T \Gamma^T \Gamma \alpha}{2\langle \alpha, \alpha\rangle}$$

and

$$k := \frac{2\alpha \cdot \mu}{\alpha^T \Gamma^T \Gamma \alpha} - 1.$$

Proof. The diffusion process $\bar{M}$ has state space the interval $I := \mathbb{R}_{++}$ and is of the form

$$d\bar{M}_t = b(\bar{M}_t)\, dt + \sigma(\bar{M}_t)\, dW_t,$$

where $b(z) = Az - Bz^2$ and $\sigma(z) = Cz$ with $A = \alpha \cdot \mu$, $B = \langle \alpha, \alpha\rangle$ and $C = \sqrt{\alpha^T \Gamma^T \Gamma \alpha}$. General facts about one-dimensional diffusions, in particular about the scale function and speed measure, can be found in Chapter 23 of [Kal02] and Chapter V.6-7 of [RW00]. It follows that a choice for the scale function is the function

$$\begin{aligned} s(x) &= \int_c^x \exp\left(-\int_a^y \frac{2b(z)}{\sigma^2(z)}\, dz\right) dy \\ &= \int_c^x \left(\frac{y}{a}\right)^{-2A/C^2} e^{\frac{2B}{C^2}(y-a)}\, dy \end{aligned} \tag{3.2.2}$$

for arbitrary $a, c \in I$ (recall that the scale function is only defined up to affine transformations). If we set $\tilde{\sigma} = (\sigma s') \circ s^{-1}$, then

$$ds(\bar{M}_t) = \tilde{\sigma}(s(\bar{M}_t))\, d\tilde{W}_t$$

and the diffusion process $s(\bar{M})$ is in natural scale on the state space $s(I)$ with speed measure

m that has density $\frac{1}{\tilde{\sigma}^2}$. The total mass of the speed measure is

$$\begin{aligned} m(I) &= \int_{s(I)} \frac{1}{\tilde{\sigma}^2(x)}\,dx = \int_{s(I)} \frac{1}{((\sigma s')\circ s^{-1})^2(x)}\,dx = \int_0^\infty \frac{1}{\sigma^2(u)s'(u)}\,du \\ &= \int_0^\infty \frac{1}{(Cu)^2\left(\frac{u}{a}\right)^{-2A/C^2} e^{\frac{2B}{C^2}(u-a)}}\,du \\ &= \frac{1}{C^2a^{2A/C}} \int_0^\infty u^{\frac{2A}{C^2}-2} e^{-\frac{2B}{C^2}(u-a)}\,du. \end{aligned} \tag{3.2.3}$$

By Theorem 23.15 of [Kal02], the diffusion process $\bar{M}$ has a stationary distribution concentrated on $\mathbb{R}_{++}$ if and only if the process $s(\bar{M})$ has $(-\infty,+\infty)$ as its state space and the speed measure has finite total mass or $s(\bar{M})$ has a finite interval as its state space and the boundaries are reflecting. The introduction of an extra negative drift to geometric Brownian motion cannot make zero a reflecting boundary, so we are interested in conditions under which $s(I) = (-\infty,\infty)$ and the speed measure has finite total mass. We see from (3.2.2) and (3.2.3) that this happens if and only if $2A/C^2 > 1$.

The diffusion $s(\bar{M})$ has an stationary distribution with density $f := \frac{1}{m(I)\tilde{\sigma}^2}$ on $s(I) = (-\infty,+\infty)$, and so the stationary distribution of $\bar{M}$ is the distribution on I that has density

$$\begin{aligned} g(x) &= f(s(x))s'(x) \\ &= \frac{1}{m(I)\tilde{\sigma}^2(s(x))}s'(x) \\ &= \frac{1}{m(I)\sigma^2(x)s'(x)} \\ &= \frac{1}{m(I)x^2C^2\left(\frac{x}{a}\right)^{-2A/C^2} e^{\frac{2B}{C^2}(x-a)}}, \quad x \in I. \end{aligned}$$

Note that this has the form of a Gamma(k,θ) density with parameters $\theta := \frac{C^2}{2B}$ and $k = \frac{2A}{C^2}-1$. Therefore,

$$g(x) = \frac{1}{\Gamma(k)\theta^k}x^{k-1}e^{-\frac{x}{\theta}} = \frac{1}{\Gamma\left(\frac{2A}{C^2}-1\right)\left(\frac{C^2}{2B}\right)^{\frac{2A}{C^2}-1}} x^{\frac{2A}{C^2}-2} e^{\frac{-2Bx}{C^2}}$$

for $x \in I$. □

The next proposition tells us that the diffusion $(\bar{M}_t)_{t\geq 0}$ satisfies a Law of Large Numbers. The argument is standard, but we include it for completeness.

Proposition 3.2.2. *Assume $\mu\cdot\alpha - \frac{\alpha^T\Gamma^T\Gamma\alpha}{2} > 0$. The process $(\bar{M}_t)_{t\geq 0}$ given by (3.2.1) satisfies*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \bar{M}_s\,ds = \int_{\mathbb{R}_{++}} x\,d\pi(x) = \left(\frac{2\alpha\cdot\mu}{\alpha^T\Gamma^T\Gamma\alpha} - 1\right)\frac{\alpha^T\Gamma^T\Gamma\alpha}{2\langle\alpha,\alpha\rangle} \quad a.s.\ \mathbb{P}^x \quad \textit{for all } x \in \mathbb{R}_{++},$$

where π is the unique stationary distribution of $\bar{M}$ concentrated on $\mathbb{R}_{++}$. The above quantity is equal to $\mathbb{E}_\pi[\bar{M}_t]$, the expected value of $\bar{M}_t$ for any t when M_0 has the stationary distribution π.

Proof. By Theorem 3.2.1 we have a unique stationary distribution π which is concentrated on $\mathbb{R}_{++}$. Theorem 20.21 from [Kal02] implies that the shift-invariant σ-field is trivial for all starting points. The ergodic theorem for stationary stochastic processes then tells us that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \bar{M}_s \, ds = \mathbb{E}_\pi[\bar{M}_t] \text{ a.s. } \mathbb{P}^\pi.$$

Now observe by the existence of everywhere positive transition densities (see Theorem V.50.11 from [RW00]) and the Markov property that if some tail event happens almost surely for some starting point, then it happens almost surely for every starting point. As a result,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \bar{M}_s \, ds = \mathbb{E}_\pi[\bar{M}_t] \text{ a.s. } \mathbb{P}^x \text{ for all } x \in \mathbb{R}_{++}.$$

By Theorem 3.2.1 it is easily seen that

$$\int_{\mathbb{R}_{++}} x d\pi(x) = \left(\frac{2\alpha \cdot \mu}{\alpha^T \Gamma^T \Gamma \alpha} - 1 \right) \frac{\alpha^T \Gamma^T \Gamma \alpha}{2\langle \alpha, \alpha \rangle}.$$

□

3.3 Conditions for invasibility

Suppose now that a new species with total population size given by the process $\{N_t\}_{t\geq 0}$ tries to invade the habitat of the resident species. We assume that the size of the invader population in patch i at time t is $N_t^i = \beta_i N_t$ for all $t \geq 0$. We now write M_t for the total population size of the resident species at time t and let $M_t^i = \alpha_i M_t$ be the size of the resident population in patch i at time t. The appropriate coupled system of SDEs for the processes $(M^1, \ldots, M^n)$ and $(N^1, \ldots, N^n)$ is

$$\begin{aligned} M_t^i &= \mu_i M_t^i \, dt - \kappa_i M_t^i (M_t^i + N_t^i) \, dt + M_t^i \, dE_t^i \\ N_t^i &= \mu_i N_t^i \, dt - \kappa_i N_t^i (M_t^i + N_t^i) \, dt + N_t^i \, dE_t^i. \end{aligned}$$

We re-express the SDE for the two-dimensional diffusion process (M, N) of total population sizes by noting that

$$\begin{aligned} d[M, M]_t &= M_t^2 \alpha^T \Gamma^T \Gamma \alpha \, dt \\ d[N, N]_t &= N_t^2 \beta^T \Gamma^T \Gamma \beta \, dt \\ d[M, N]_t &= N_t M_t \alpha^T \Gamma^T \Gamma \beta \, dt. \end{aligned}$$

Therefore, the diffusion process (M, N) is given by

$$dM_t = M_t\left[\mu\cdot\alpha - \langle\alpha,\beta\rangle N_t - \langle\alpha,\alpha\rangle M_t\right] dt + M_t\sqrt{\alpha^T\Gamma^T\Gamma\alpha}\, dU_t \quad (3.3.1)$$

$$dN_t = N_t\left[\mu\cdot\beta - \langle\alpha,\beta\rangle M_t - \langle\beta,\beta\rangle N_t\right] dt + N_t\sqrt{\beta^T\Gamma^T\Gamma\beta}\, dV_t, \quad (3.3.2)$$

where (U, V) is a (non-standard) Brownian motion with covariance structure $d[U, U]_t = dt$, $d[V, V]_t = dt$, and $d[U, V]_t = \frac{\alpha^T\Gamma^T\Gamma\beta}{\sqrt{\alpha^T\Gamma^T\Gamma\alpha}\sqrt{\beta^T\Gamma^T\Gamma\beta}}\, dt$. Note that if $\frac{\alpha^T\Gamma^T\Gamma\beta}{\sqrt{\alpha^T\Gamma^T\Gamma\alpha}\sqrt{\beta^T\Gamma^T\Gamma\beta}} = 1$, then we are in a singular case and we have $U = V$. We do not consider this case in what follows.

Definition 3.3.1. We say that the species described by N can invade M successfully if

$$\lim_{\varepsilon\downarrow 0}\liminf_{t\to\infty}\mathbb{P}\{N_t > \delta \,|\, N_0 = \varepsilon\} > 0$$

for some $\delta > 0$.

Question 3.3.2. What are necessary and sufficient conditions for N to be able to successfully invade the habitat of M?

We partially answer this question in Theorem 3.3.8 and Theorem 3.3.14 below.

Proposition 3.3.3. *Define the process* $\hat{N}$ *via*

$$d\hat{N}_t := \hat{N}_t\left[\mu\cdot\beta - \langle\alpha,\beta\rangle\bar{M}_t\right] dt + \hat{N}_t\sqrt{\beta^T\Gamma^T\Gamma\beta}\, dV_t, \quad (3.3.3)$$

where $\bar{M}$ *is given by* (3.2.1) *with* $W = U$. *Suppose that* $\mu\cdot\alpha - \frac{\alpha^T\Gamma^T\Gamma\alpha}{2} > 0$, *so the Markov process* $\bar{M}$ *has a stationary distribution concentrated on* $\mathbb{R}_{++}$. *Then, the limit* $L_{\hat{N}}(\alpha,\beta) := \lim_{t\to\infty}\frac{\log\hat{N}_t}{t}$ *exists almost surely and is given by*

$$L_{\hat{N}}(\alpha,\beta) = \mu\cdot\beta - \left(\frac{2\alpha\cdot\mu}{\alpha^T\Gamma^T\Gamma\alpha} - 1\right)\frac{\alpha^T\Gamma^T\Gamma\alpha}{2\langle\alpha,\alpha\rangle}\langle\alpha,\beta\rangle - \frac{1}{2}\beta^T\Gamma^T\Gamma\beta. \quad (3.3.4)$$

Proof. Note from (3.3.3) that

$$d\log\hat{N}_t = \left(\sum\mu_i\beta_i - \kappa_i\alpha_i\beta_i\bar{M}_t\right) dt + \sqrt{\beta^T\Gamma^T\Gamma\beta}\, dV_t + \frac{1}{2}\left(-\frac{1}{\hat{N}_t^2}\right)\hat{N}_t^2(\beta^T\Gamma^T\Gamma\beta)\, dt.$$

By Proposition 3.2.2,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \bar{M}_s\, ds = \nu_{\bar{M}}(\alpha),$$

where

$$\nu_{\bar{M}}(\alpha) := \left(\frac{2\alpha\cdot\mu}{\alpha^T\Gamma^T\Gamma\alpha} - 1\right)\frac{\alpha^T\Gamma^T\Gamma\alpha}{2\langle\alpha,\alpha\rangle}. \quad (3.3.5)$$

Therefore,

$$\lim_{t\to\infty}\frac{\log\hat{N}_t}{t} = \mu\cdot\beta - \langle\alpha,\beta\rangle\nu_{\bar{M}}(\alpha) - \frac{1}{2}\beta^T\Gamma^T\Gamma\beta. \quad (3.3.6)$$

□

Interchanging the roles of the resident and the invader in Proposition 3.3.3, define the pair of processes $(\hat{M}, \bar{N})$ via

$$\begin{aligned} d\hat{M}_t &= \hat{M}_t \left[\mu \cdot \alpha - \langle \alpha, \beta \rangle \bar{N}_t\right] \, dt + \hat{M}_t \sqrt{\alpha^T \Gamma^T \Gamma \alpha} \, dU_t & (3.3.7) \\ d\bar{N}_t &= \bar{N}_t(\mu \cdot \beta - \langle \beta, \beta \rangle \bar{N}_t) \, dt + \bar{N}_t \sqrt{\beta^T \Gamma^T \Gamma \beta} \, dV_t. & (3.3.8) \end{aligned}$$

It follows from Proposition 3.3.3 that if the process $\bar{N}$ has a stationary distribution concentrated on $\mathbb{R}_{++}$ (which, by Theorem 3.2.1, occurs if and only if $\mu \cdot \beta - \frac{\beta^T \Gamma^T \Gamma \beta}{2} > 0$), then

$$L_{\hat{M}}(\alpha, \beta) := \lim_{t \to \infty} \frac{\log \hat{M}_t}{t} = \mu \cdot \alpha - \left(\frac{2\beta \cdot \mu}{\beta^T \Gamma^T \Gamma \beta} - 1 \right) \frac{\beta^T \Gamma^T \Gamma \beta}{2\langle \beta, \beta \rangle} \langle \alpha, \beta \rangle - \frac{1}{2} \alpha^T \Gamma^T \Gamma \alpha. \quad (3.3.9)$$

Proposition 3.3.4. *Suppose that the processes $\bar{M}$ and $\bar{N}$ both have stationary distributions concentrated on $\mathbb{R}_{++}$, that is, $\mu \cdot \alpha - \frac{\alpha^T \Gamma^T \Gamma \alpha}{2} > 0$ and $\mu \cdot \beta - \frac{\beta^T \Gamma^T \Gamma \beta}{2} > 0$. Then, $L_{\hat{N}}(\alpha, \beta) < 0$ implies that $L_{\hat{M}}(\alpha, \beta) > 0$, and $L_{\hat{M}}(\alpha, \beta) < 0$ implies that $L_{\hat{N}}(\alpha, \beta) > 0$.*

Proof. By symmetry, it suffices to prove the first claim. Set $A := \mu \cdot \alpha - \frac{\alpha^T \Gamma^T \Gamma \alpha}{2}$ and $B := \mu \cdot \beta - \frac{\beta^T \Gamma^T \Gamma \beta}{2}$. By the assumption that the processes $\bar{M}$ and $\bar{N}$ both have stationary distributions concentrated on $\mathbb{R}_{++}$, we get by Theorem 3.2.1 that $A > 0$ and $B > 0$. Note that

$$L_{\hat{N}}(\alpha, \beta) = B - A \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}$$

and

$$L_{\hat{M}}(\alpha, \beta) = A - B \frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle}.$$

Assume that $L_{\hat{N}}(\alpha, \beta) < 0$ and $L_{\hat{M}}(\alpha, \beta) \leq 0$. From the Cauchy-Schwarz inequality $\langle x, y \rangle \leq \langle x, x \rangle^{1/2} \langle y, y \rangle^{1/2}$ we get

$$\begin{aligned} B\langle \alpha, \alpha \rangle^{1/2} \langle \beta, \beta \rangle^{1/2} &\geq B\langle \alpha, \beta \rangle \geq A\langle \beta, \beta \rangle \\ A\langle \alpha, \alpha \rangle^{1/2} \langle \beta, \beta \rangle^{1/2} &\geq A\langle \alpha, \beta \rangle > B\langle \alpha, \alpha \rangle. \end{aligned}$$

The above inequalities yield the contradiction $B\langle \alpha, \alpha \rangle^{1/2} \geq A\langle \beta, \beta \rangle^{1/2}$ and $B\langle \alpha, \alpha \rangle^{1/2} < A\langle \beta, \beta \rangle^{1/2}$. □

For ease of notation, we re-write the joint dynamics of M and N as

$$\begin{aligned} dM_t &= (\mu \cdot \alpha M_t - M_t(aM_t + cN_t)) \, dt + \sigma_M M_t \, dU_t & (3.3.10) \\ dN_t &= (\mu \cdot \beta N_t - N_t(cM_t + bN_t)) \, dt + \sigma_N N_t \, dV_t, \end{aligned}$$

where $a := \langle \alpha, \alpha \rangle$, $b := \langle \beta, \beta \rangle$, $c := \langle \alpha, \beta \rangle$, $\sigma_M := \sqrt{\alpha^T \Gamma^T \Gamma \alpha}$, and $\sigma_N := \sqrt{\beta^T \Gamma^T \Gamma \beta}$.

Set $\mathbb{R}_+ := [0, \infty)$. The next theorem gives us the existence and uniqueness of solutions to the system (3.3.10) as well as some very useful comparison results.

Theorem 3.3.5. *The SDE from 3.3.10 has unique strong solutions and* $M_t, N_t \in L^p(\mathbb{P}^{(x,y)})$ *for all* $p > 0$ *for all* $(x, y) \in \mathbb{R}^2_{++}$. *Suppose the processes* $\{(M_t, N_t)\}_{t\geq 0}$ *and* $\{(\bar{M}_t, \bar{N}_t)\}_{t\geq 0}$ *are equal at* $t = 0$. *Then,*

$$M_t \leq \bar{M}_t$$

and

$$N_t \leq \bar{N}_t$$

for all $t \geq 0$.

Proof. The uniqueness and existence of strong solutions is fairly standard, see, for example, Theorem 2.1 in [LM09]. One notes that the drift coefficients are locally Lipschitz so strong solutions exist and are unique up to the explosion time. It is easy to show this explosion time is almost surely infinite (see Theorem 2.1 in [LM09]). Next, suppose that $M_0 = \bar{M}_0$. We adapt the comparison principle of Ikeda and Watanabe (Chapter VI Theorem 1.1 from [IW89]) proved by the local time techniques of Le Gall (see Theorem 1.4 from [LG83] and Theorem V.43.1 in [RW00]) to show that $\bar{M}_t - M_t \geq 0$ for all $t \geq 0$.

Define $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ by $\rho(x) = |x|^2$. Note that

$$\begin{aligned}\int_0^t \rho(|\bar{M}_s - M_s|)^{-1} \mathbb{1}\{\bar{M}_s - M_s > 0\}\, d[\bar{M} - M]_s &= \int_0^t [\rho(|\bar{M}_s - M_s|)^{-1} \\ &\quad (\sigma_M \bar{M}_s - \sigma_M M_s)^2 \mathbb{1}\{\bar{M}_s - M_s > 0\}]\, ds \\ &\leq \sigma_M^2 t.\end{aligned}$$

Since $\int_{0+} \rho(u)^{-1}\, du = \infty$, by Proposition V.39.3 from [RW00] the local time at 0 of $M - \bar{M}$ is zero for all $t \geq 0$. Put $x^+ := x \vee 0$. By Tanaka's formula (see equation IV.43.6 in [RW00]),

$$\begin{aligned}(M_t - \bar{M}_t)^+ &= \int_0^t \mathbb{1}\{M_s - \bar{M}_s > 0\}(\sigma_M M_s - \sigma_M \bar{M}_s)\, dU_t \\ &\quad + \int_0^t \mathbb{1}\{M_s - \bar{M}_s > 0\} \left[(\mu \cdot \alpha M_s - M_s(aM_s + cN_s)) - (\mu \cdot \alpha \bar{M}_s - a\bar{M}_s^2)\right] ds.\end{aligned}$$

For $K > 0$ define the stopping time

$$T_K := \inf\{t > 0 : M_t \geq K \text{ or } \bar{M}_t \geq K\ \}$$

and the stopped processes $M_t^K = M_{T_K \wedge t}, \bar{M}_t^K = \bar{M}_{T_K \wedge t}$ Then, stopping the processes at T_K

and taking expectations yields

$$\begin{aligned}
0 \leq \mathbb{E}(M_t^K - \bar{M}_t^K)^+ &= \mathbb{E}\int_0^{t\wedge T_K} \mathbb{1}\{M_s - \bar{M}_s > 0\}[(\mu\cdot\alpha M_s - M_s(aM_s + cN_s)) \\
&\quad - (\mu\cdot\alpha\bar{M}_s - a\bar{M}_s^2)\,ds \\
&= \mathbb{E}\int_0^{t\wedge T_K} \mathbb{1}\{M_s - \bar{M}_s > 0\}\left[\mu\cdot\alpha(M_s - \bar{M}_s) - a(M_s^2 - \bar{M}_s^2) - cM_sN_s\right] ds \\
&\leq \int_0^{t\wedge T_K} \mathbb{1}\{M_s - \bar{M}_s > 0\}\mu\cdot\alpha(M_s - \bar{M}_s)\,ds \\
&\leq \mu\cdot\alpha\mathbb{E}\int_0^{t\wedge T_K} (M_s - \bar{M}_s)^+\,ds \\
&\leq \mu\cdot\alpha\mathbb{E}\int_0^{t} (M_s^K - \bar{M}_s^K)^+\,ds
\end{aligned}$$

By Gronwall's Lemma, see Exercise V.11.11 in [RW00], $\mathbb{E}(M_t^K - \bar{M}_t^K)^+ = 0$ for all $t \geq 0$, so $M_t^K \leq \bar{M}_t^K$ for all $t \geq 0$. Now let $K \to \infty$ to get that, remembering that $\bar{M}$ does not explode, $M_t \leq \bar{M}_t$ for all $t \geq 0$. Since we have shown before that $\bar{M}$ is dominated by a geometric Brownian motion which has moments of all orders we get that $M_t, N_t \in L^p(\mathbb{P}^{(x,y)})$ for all $t, p > 0$ and for all $(x, y) \in \mathbb{R}^2_{++}$.

□

Remark 3.3.6. Note that the SDE for $\hat{M}$,$\hat{N}$, $\tilde{M}$ and $\tilde{N}$ have unique strong solutions and $\hat{M}_t, \hat{N}_t, \tilde{M}_t, \tilde{N}_t \in L^p$ for all $t \geq 0, p > 0$ and for all starting points $(x, y) \in \mathbb{R}_{++}$. This follows by arguments similar to those that are in Theorem 2.1 from [LM09] and in Theorem 3.3.5 by noting that our SDE always look like

$$\begin{aligned}
dX_t &= X_t\left[\lambda_1 - \lambda_2 Y_t - \lambda_3 X_t\right] dt + X_t\sigma_X\,dU_t \\
dY_t &= Y_t\left[\lambda_4 - \lambda_5 X_t - \lambda_6 Y_t\right] dt + Y_t\sigma_Y\,dV_t \\
X_0 &= x \\
Y_0 &= y
\end{aligned}$$

for $\lambda_1, \ldots, \lambda_6 \in \mathbb{R}_+$ and $x, y \in \mathbb{R}_{++}$.

The next proposition tells us that none of our processes hit zero in finite time.

Proposition 3.3.7. *If $(M_0, N_0) \in \mathbb{R}^2_{++}$, then $(M_t, N_t) \in \mathbb{R}^2_{++}$ for all $t \geq 0$ almost surely. A similar conclusion holds for the various processes with hats and bars. Similarly, all of the other processes we work with, $\hat{M}, \hat{N}, \tilde{N}, \check{N}, \ldots$ etc live in $\mathbb{R}^2_{++}$ for all $t \geq 0$.*

Proof. As an example of the method of proof, we look at the process M given by (3.3.10). Taking logs and using Itô's lemma,

$$d\log M_t = \left(\mu\cdot\alpha - (aM_t + cN_t) - \frac{1}{2}\sigma_M^2\right) dt + \sigma_M\,dU_t.$$

Therefore,

$$\log M_t = \int_0^t \left(\mu \cdot \alpha - (aM_s + cN_s) - \frac{1}{2}\sigma_M^2 \right) ds + \sigma_M U_t.$$

can't go to $-\infty$ in finite time because M_t and N_t do not blow up. □

Theorem 3.3.8. *Suppose that $\bar{M}$ and $\bar{N}$ both have stationary distributions concentrated on $\mathbb{R}_{++}$ and that $L_{\hat{M}}(\alpha, \beta) < 0$, that is $\mu \cdot \alpha - \frac{\alpha^T\Gamma^T\Gamma\alpha}{2} > 0$, $\mu \cdot \beta - \frac{\beta^T\Gamma^T\Gamma\beta}{2} > 0$ and $\mu \cdot \alpha - \left(\frac{2\beta\cdot\mu}{\beta^T\Gamma^T\Gamma\beta} - 1\right)\frac{\beta^T\Gamma^T\Gamma\beta}{2\langle\beta,\beta\rangle}\langle\alpha, \beta\rangle - \frac{1}{2}\alpha^T\Gamma^T\Gamma\alpha < 0$. Then, for $(x, y) \in \mathbb{R}^2_{++}$, the probability measures*

$$\frac{1}{t}\int_0^t \mathbb{P}^{(x,y)}\{(M_s, N_s) \in \cdot\}\, ds$$

converge weakly as $t \to \infty$ to $\delta_0 \otimes \rho$, where ρ is the unique stationary distribution of $\bar{N}$ concentrated on $\mathbb{R}_{++}$.

Remark 3.3.9. In Theorem 3.1 of [ZC13] the authors claim to show that the system of SDE describing (M, N) always has a unique stationary distribution. We note that their use of moments just checks tightness in $\mathbb{R}^2_+$ and not in $\mathbb{R}^2_{++}$. It doesn't stop mass going off to the boundary, which is exactly what can happen in our case. Their proof only shows the existence of a stationary distribution on $\mathbb{R}^2_+$ - it does not show the existence of a stationary distribution on $\mathbb{R}^2_{++}$.

Furthermore, their proof for the uniqueness of a stationary distribution on $\mathbb{R}^2_+$ breaks down because their assumption of irreducibility is false. (M, N) is irreducible on $\mathbb{R}^2_{++}$ but is not irreducible on $\mathbb{R}^2_+$ since for example $P_t((0, 0), U) := \mathbb{P}^{(0,0)}\{(M_t, N_t) \in U\} = 0$ for any open subset U which lies in the interior of $\mathbb{R}^2_+$. If we work on $\mathbb{R}^2_+$, it is not true that the diffusion (M, N) has a unique stationary distribution. We can obtain infinitely many stationary distributions on $\mathbb{R}^2_+$ of the form $(u\pi_{\bar{M}} + v\delta_0) \otimes \delta_0$ where $\pi_{\bar{M}}$ is the stationary distribution of $\bar{M}$ and $u, v \in \mathbb{R}_+$ satisy $u + v = 1$.

Proof. We prepare for the proof with some preliminary results.

Proposition 3.3.10. *Fix $(x, y) \in \mathbb{R}^2_{++}$. Any sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $t_n \to \infty$ has a subsequence $\{u_n\}_{n\in\mathbb{N}}$ such that the probability measure*

$$\frac{1}{u_n}\int_0^{u_n} \mathbb{P}^{(x,y)}\{(M_s, N_s) \in \cdot\}\, ds$$

converges in the topology of weak convergence of probability measures on $\mathbb{R}^2_+$. Any such limit is a stationary distribution for the process (M, N) thought of as a process with state space $\mathbb{R}^2_+$.

Proof. Set $\varphi(x,y) := x+y$ so that $\varphi \geq 0$ for $x,y>0$. Put $\psi(x,y) = \mu\cdot\alpha x + \mu\cdot\beta y - x(ax+cy) - y(cx+by)$. Note that ψ is bounded above on the quadrant $x,y\geq 0$ and $\lim_{\|(x,y)\|\to\infty}\psi(x,y) = -\infty$. Using Itô's lemma we get

$$\varphi(M_t,N_t) - \int_0^t \psi(M_s,N_s)\,ds \;=\; \int_0^t \sigma_N N_s\,dV_s + \int_0^t \sigma_M M_s\,dU_s$$

Therefore, $\varphi(M_t,N_t) - \int_0^t \psi(M_s,N_s)\,ds$ is a martingale. Applying Theorem 9.9 of [EK05] completes the proof. $\square$

The following result is essentially Theorem 10 in [LWW11]. We include the proof for completeness.

Proposition 3.3.11. *Suppose that $\bar{M}$ and $\bar{N}$ both have stationary distributions concentrated on $\mathbb{R}_{++}$ and that $L_{\hat{M}}(\alpha,\beta) < 0$, that is $\mu\cdot\alpha - \frac{\alpha^T\Gamma^T\Gamma\alpha}{2} > 0$, $\mu\cdot\beta - \frac{\beta^T\Gamma^T\Gamma\beta}{2} > 0$ and $\mu\cdot\alpha - \left(\frac{2\beta\cdot\mu}{\beta^T\Gamma^T\Gamma\beta} - 1\right)\frac{\beta^T\Gamma^T\Gamma\beta}{2\langle\beta,\beta\rangle}\langle\alpha,\beta\rangle - \frac{1}{2}\alpha^T\Gamma^T\Gamma\alpha < 0$. Then, $\lim_{t\to\infty} M_t = 0$ $\mathbb{P}^{(x,y)}$-a.s. for all $(x,y)\in\mathbb{R}^2_{++}$.*

Proof. Using Ito's lemma and (3.3.9),

$$\begin{aligned}
b\frac{\log\left(\frac{M_t}{M_0}\right)}{t} - c\frac{\log\left(\frac{N_t}{N_0}\right)}{t} &= b\left(\mu\cdot\alpha - \frac{\sigma_M^2}{2}\right) - c\left(\mu\cdot\beta - \frac{\sigma_N^2}{2}\right) - (ab-c^2)\frac{\int_0^t M_s\,ds}{t} \\
&\quad + b\sigma_M\frac{U_t}{t} - c\sigma_N\frac{V_t}{t} \\
&= bL_{\hat{M}} - (ab-c^2)\frac{\int_0^t M_s\,ds}{t} + b\sigma_M\frac{U_t}{t} - c\sigma_N\frac{V_t}{t}
\end{aligned}$$

By the Cauchy-Schwarz inequality, $(ab-c^2) = \langle\alpha,\alpha\rangle\langle\beta,\beta\rangle - (\langle\alpha,\beta\rangle)^2 \geq 0$, and so

$$\frac{\log\left(\frac{M_t}{M_0}\right)}{t} \;\leq\; \frac{c}{b}\frac{\log\left(\frac{N_t}{N_0}\right)}{t} + L_{\hat{M}} + \sigma_M\frac{U_t}{t} - \frac{c}{b}\frac{V_t}{t}$$

Observe that $\bar{N}$ was defined by

$$d\bar{N}_t = (\mu\cdot\beta\bar{N}_t - b\bar{N}_t^2)\,dt + \sigma_N\bar{N}_t\,dV_t.$$

Following the proof of Theorem 3.2.1, $\mathbb{E}_{\pi_N}[\bar{N}_t] = \frac{1}{b}\left(\mu\cdot\beta - \frac{\sigma_N^2}{2}\right)$ where π_N is the stationary distribution of $\bar{N}$. By Proposition 3.2.2, we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \bar{N}_s\,ds = \mathbb{E}_{\pi_N}[\bar{N}_t] = \frac{1}{b}\left(\mu\cdot\beta - \frac{\sigma_N^2}{2}\right). \tag{3.3.11}$$

It follows from Theorem 3.3.5 that $N_t \le \bar{N}_t$ for all $t \ge 0$. Thus,

$$\begin{aligned}\limsup_{t\to\infty} \frac{\log N_t}{t} &\le \limsup_{t\to\infty} \frac{\log \bar{N}_t}{t} \\ &= \left(\mu\cdot\beta - \frac{\sigma_N^2}{2}\right) - b\lim_{t\to\infty}\frac{1}{t}\int_0^t \bar{N}_s\,ds + \sigma_N \lim_{t\to\infty}\frac{V_t}{t} \\ &= \left(\mu\cdot\beta - \frac{\sigma_N^2}{2}\right) - b\mathbb{E}[\bar{N}] \\ &= 0.\end{aligned}$$

Next, since U and V are Brownian motions, $\lim_{t\to\infty}\frac{U_t}{t} = \lim_{t\to\infty}\frac{V_t}{t} = 0$, and $\limsup_{t\to\infty}\frac{\log N_t}{t} \le 0$, so

$$\limsup_{t\to\infty}\frac{\log M_t}{t} \le L_{\hat{M}} < 0.$$

In particular, $\lim_{t\to\infty} M_t = 0$. □

We can now finish the proof of Theorem 3.3.8. Fix $\epsilon > 0$ and $\eta > 0$ sufficiently small. Define the stopping time

$$T_\epsilon := \inf\{t \ge 0 : M_t \ge \epsilon\}.$$

and the stopped process $N_t^\epsilon := N_{t\wedge T_\epsilon}$. By Proposition 3.3.11, there exists $T > 0$ such that

$$\mathbb{P}^{(x,y)}\{M_t \le \epsilon \text{ for all } t \ge T\} \ge 1-\eta$$

Define the process $\check{N}$ via

$$d\check{N}_t = \check{N}_t[(\mu\cdot\beta - c\epsilon) - b\check{N}_t]\,dt + \sigma_N \check{N}_t dV_t$$

and the stopped process $\check{N}_t^\epsilon := \check{N}_{t\wedge T_\epsilon}$. Start the process $\check{N}$ at time T with the condition $\check{N}_T = N_T$. We want to show that the process $\check{N}^\epsilon$ is dominated by the process N^ϵ, that is $N_t^\epsilon \ge \check{N}_t^\epsilon$ for all $t \ge T$. By the strong Markov property, we can assume $T = 0$.

The proof is very similar to the one from Theorem 3.3.5. With the notation from the proof of Theorem 3.3.5 it is trivial to check that

$$\begin{aligned}\int_0^t \rho(|\check{N}_s^\epsilon - N_s^\epsilon|)^{-1}\mathbb{1}\{\check{N}_s^\epsilon - N_s^\epsilon > 0\}\,d[\check{N}^\epsilon - N^\epsilon]_s &= \int_0^t [\rho(|\check{N}_s^\epsilon - N_s^\epsilon|)^{-1} \\ &\quad (\sigma_N\check{N}_s^\epsilon - \sigma_N N_s^\epsilon)^2\mathbb{1}\{\check{N}_s^\epsilon - N_s^\epsilon > 0\}]\,ds \\ &\le \sigma_N^2 t,\end{aligned}$$

so the local time of the process $\check{N}^\epsilon - N^\epsilon$ at zero is identically zero. Then, using Tanaka's formula,

$$\begin{aligned}(\check{N}_t^\epsilon - N_t^\epsilon)^+ &= \int_0^{t\wedge T_\epsilon} \mathbb{1}\{\check{N}_s - N_s > 0\}(\sigma_N\check{N}_s - \sigma_N N_s)\,dV_t \\ &\quad + \int_0^{t\wedge T_\epsilon} \mathbb{1}\{\check{N}_s - N_s > 0\}\left[((\mu\cdot\beta - c\epsilon)\check{N}_s - b\check{N}_s^2) - (\mu\cdot\beta N_s - N_s(cM_s + bN_s))\right]\,ds.\end{aligned}$$

Taking expectations,

$$\begin{aligned}\mathbb{E}[(\check{N}^\epsilon_t - N^\epsilon_t)^+] &= \mathbb{E}\int_0^{t\wedge T_\epsilon} \mathbb{1}\{\check{N}_s - N_s > 0\}[(\mu\cdot\beta(\check{N}_s - N_s) - (c\epsilon\check{N}_s - cN_sM_s) \\ &\quad - b(\check{N}_s^2 - N_s^2))\,ds] \\ &\le \mu\cdot\beta\mathbb{E}\int_0^{t\wedge T_\epsilon} (\check{N}_s - N_s)^+\,ds \\ &\le \mu\cdot\beta\mathbb{E}\int_0^{t} (\check{N}^\epsilon_s - N^\epsilon_s)^+\,ds.\end{aligned}$$

By Gronwall's Lemma, $\mathbb{E}[(\check{N}^\epsilon_t - N^\epsilon_t)^+] = 0$. As a result, remembering we assumed $T = 0$, we have $\check{N}^\epsilon_t \le N^\epsilon_t$ for all $t \ge T$. For ϵ small enough we know that $\check{N}$ has a stationary distribution concentrated on $\mathbb{R}_{++}$. For any sequence $a_n \to \infty$, if the Cesaro averages $\frac{1}{a_n}\int_0^{a_n} \mathbb{P}^{(x,y)}\{(M_s, N_s) \in \cdot\}\,ds$ converge weakly, then the limit is a distribution of the form $\delta_0 \otimes \varphi$, where φ is a mixture of the unique stationary distribution ρ of $\bar{N}$ concentrated on $\mathbb{R}_{++}$ and the point mass at 0. By the above, the limit of $\frac{1}{a_n}\int_0^{a_n} \mathbb{P}^{(x,y)}\{(M_s, N_s) \in \cdot\}\,ds$ cannot have any mass at $(0,0)$ because $\check{N}_t \le N_t$ on the event $\{M_t \le \epsilon \text{ for all } t \ge T\}$ which has probability $\mathbb{P}^{(x,y)}\{M_t \le \epsilon \text{ for all } t \ge T\} \ge 1 - \eta$. Since $\eta > 0$ was arbitrary, we conclude that $\varphi = \rho$, as required. □

Remark 3.3.12. Theorem 3.3.8 partially answers Question 3.3.2. Namely, we show that invasion is possible under the assumptions of this theorem.

Proposition 3.3.13. *Suppose that the processes $\bar{M}$ and $\bar{N}$ both have stationary distributions concentrated on $\mathbb{R}_{++}$ and that $L_{\tilde{N}}(\alpha,\beta) > 0$, that is $\mu\cdot\alpha - \frac{\alpha^T\Gamma^T\Gamma\alpha}{2} > 0$, $\mu\cdot\beta - \frac{\beta^T\Gamma^T\Gamma\beta}{2} > 0$ and $\mu\cdot\beta - \frac{\beta^T\Gamma^T\Gamma\beta}{2} - \left(\mu\cdot\alpha - \frac{\alpha^T\Gamma^T\Gamma\alpha}{2}\right)\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} > 0$. Then, there is an $\epsilon > 0$ such that*

$$\mathbb{P}^{(x,y)}\{\forall s \ge 0,\ \exists t \ge s : N_t \ge \epsilon\} = 1.$$

for all $(x,y) \in \mathbb{R}_{++}$.

Proof. Recall that

$$d\bar{M}_t = (\mu\cdot\alpha\bar{M}_t - a\bar{M}_t^2)\,dt + \sigma_M\bar{M}_t\,dU_t.$$

Define the process $\tilde{N}_t$ via

$$d\tilde{N}_t = (\xi\tilde{N}_t - c\bar{M}_t\tilde{N}_t)\,dt + \sigma_N\tilde{N}_t\,dV_t,$$

where $\xi > 0$. Note that if ξ is close enough to $\mu\cdot\beta$, then, by (3.3.4), we have

$$L_{\tilde{N}}(\alpha,\beta) := \lim_{t\to\infty}\frac{\log\tilde{N}_t}{t} > 0 \tag{3.3.12}$$

so that almost surely $\tilde{N}_t \not\to 0$. Let us compare the drifts of N_t and $\tilde{N}_t$. We want, for y small, to have $\mu \cdot \beta y - y(cx + by) \geq \xi y - ycx$. This is equivalent to

$$(\mu \cdot \beta - \xi)y \geq by^2 \tag{3.3.13}$$

for small $y \geq 0$. It is possible to choose a ξ and a $\epsilon > 0$ such that ξ is close to $\mu \cdot \beta$ and (3.3.13) is satisfied for $0 \leq y \leq 2\epsilon$.

Set $S_\epsilon := \inf\{t \geq 0 : \tilde{N}_t \geq 2\epsilon\}$. If $(x, y) \in \mathbb{R}^2_{++}$ with $y \leq \epsilon$, then, by (3.3.12), $\mathbb{P}^{(x,y)}\{S_\epsilon < \infty\} = 1$.

We next show that $N_t \geq \tilde{N}_t$ for $0 \leq t \leq S_\epsilon$. This is again very similar to the proof of Theorem 3.3.5. The local time of $\tilde{N} - N$ at 0 will be zero and taking expectations in Tanaka's formula

$$\begin{aligned}\mathbb{E}[(\tilde{N}_{t\wedge S_\epsilon} - N_{t\wedge S_\epsilon})^+] &= \mathbb{E}\int_0^{t\wedge S_\epsilon} \mathbb{1}\{\tilde{N}_s - N_s > 0\}[(\xi\tilde{N}_s - c\tilde{N}_s\bar{M}_s) \\ &\quad - (\mu \cdot \beta N_s - cN_sM_s - bN_s^2)]\, ds \\ &\leq \mathbb{E}\int_0^{t\wedge S_\epsilon} \mathbb{1}\{\tilde{N}_s - N_s > 0\}[(\xi\tilde{N}_s - (\mu \cdot \beta N_s - bN_s^2))]\, ds.\end{aligned}$$

Now use the fact that

$$\mu \cdot \beta N_t - bN_t^2 \geq \xi N_t$$

when $\tilde{N}_t > N_t$ and $0 \leq t \leq S_\epsilon$ to get

$$\begin{aligned}\mathbb{E}[(\check{N}_{t\wedge S_\epsilon} - N_{t\wedge S_\epsilon})^+] &\leq \xi\mathbb{E}\int_0^{t\wedge S_\epsilon} \mathbb{1}\{\tilde{N}_s - N_s > 0\}(\tilde{N}_s - N_s)\, ds \\ &\leq \xi\mathbb{E}\int_0^t (\tilde{N}_{s\wedge S_\epsilon} - N_{s\wedge S_\epsilon})^+\, ds.\end{aligned}$$

By Gronwall's Lemma $\mathbb{E}[(\check{N}_{t\wedge S_\epsilon} - N_{t\wedge S_\epsilon})^+] = 0$, so $N_t \geq \tilde{N}_t$ for $0 \leq t \leq S_\epsilon$. Define $T^1_\epsilon := \inf\{t > 0 : N_t \leq \epsilon\}$ and $T^2_\epsilon := \inf\{t > T^1_\epsilon : N_t \geq 2\epsilon\}$. By the strong Markov property, for any $(x, y) \in \mathbb{R}^2_{++}$ we have that $T^2_\epsilon < \infty$ $\mathbb{P}^{(x,y)}$-almost surely on the event $T^1_\epsilon < \infty$. Define $T^3_\epsilon, T^4_\epsilon, \ldots$ recursively by $T^{2n+1}_\epsilon := \inf\{t > T^{2n}_\epsilon : N_t \leq \epsilon\}$, and $T^{2n+2}_\epsilon := \inf\{t > T^{2n+1}_\epsilon : N_t \geq 2\epsilon\}$ and repeat the above argument to obtain the desired result. □

Theorem 3.3.14. *Suppose that the processes $\bar{M}$ and $\bar{N}$ both have stationary distributions concentrated on $\mathbb{R}_{++}$ and that $L_{\hat{N}}(\alpha, \beta) > 0, L_{\hat{M}}(\alpha, \beta) > 0$. That is, assume that $\mu \cdot \alpha - \frac{\alpha^T\Gamma^T\Gamma\alpha}{2} > 0$, $\mu \cdot \beta - \frac{\beta^T\Gamma^T\Gamma\beta}{2} > 0$, $\mu \cdot \beta - \frac{\beta^T\Gamma^T\Gamma\beta}{2} - \left(\mu \cdot \alpha - \frac{\alpha^T\Gamma^T\Gamma\alpha}{2}\right)\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} > 0$ and $\mu \cdot \alpha - \frac{\alpha^T\Gamma^T\Gamma\alpha}{2} - \left(\mu \cdot \beta - \frac{\beta^T\Gamma^T\Gamma\beta}{2}\right)\frac{\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle} > 0$.*

The process (M, N) has smooth strictly positive transition densities and for $(x, y) \in \mathbb{R}^2_{++}$ and any sequence (t_n) such that $t_n \to \infty$ there exists a subsequence $(u_n) \subseteq (t_n)$ such that the probability measures

$$\frac{1}{u_n} \int_0^{u_n} \mathbb{P}^{(x,y)}\{(M_s, N_s) \in \cdot\}\, ds$$

converge weakly to a distribution on $\mathbb{R}_+ \times \mathbb{R}_+$. Furthermore, there exists $\epsilon > 0$ such that

$$\mathbb{P}^{(x,y)}\{\forall s \geq 0,\, \exists t \geq s : M_t \geq \epsilon\} = 1$$

and

$$\mathbb{P}^{(x,y)}\{\forall s \geq 0,\, \exists t \geq s : N_t \geq \epsilon\} = 1$$

for all $(x, y) \in \mathbb{R}_{++}$.

Proof. Note that the infinitesimal generator of $(\log M, \log N)$ thought of as a process on $\mathbb{R}^2$ is uniformly elliptic with smooth coefficients and so it has smooth transition densities (see, for example, Section 3.3.4 of [Str08]). Moreover, an application of a suitable minimum principle for the Kolmogorov forward equation (see, for example, Theorem 5 in Section 2 of Chapter 2 of [Fri64]) shows that the transition densities are everywhere strictly positive. It follows that (M, N) thought of as a process on $\mathbb{R}^2_{++}$ has smooth transition densities that are everywhere positive.

An argument analogous to the one from Proposition 3.3.10 shows that subsequences of the Cesaro averages $\frac{1}{t_n} \int_0^{t_n} \mathbb{P}^{(x,y)}\{(M_s, N_s) \in \cdot\}\, ds$ can be chosen to converge to a distribution on $\mathbb{R}_+ \times \mathbb{R}_+$. Then, the comparison argument from Proposition 3.3.13 applied to both M and N combined with the assumptions $L_{\hat{M}}(\alpha, \beta) > 0$ and $L_{\hat{N}}(\alpha, \beta) > 0$ give the last claim of the theorem. □

Remark 3.3.15. In Theorem 3.3.14 we can prove that when the two Lyapunov exponents are strictly positive, $L_{\hat{M}}(\alpha, \beta) > 0$ and $L_{\hat{N}}(\alpha, \beta) > 0$ then almost surely $M_t \not\to 0$ and $N_t \not\to 0$. We are not able to prove the stronger version of invasibility from Definition 3.3.1.

3.4 A maximization problem

Since the population described by the process N is trying to invade the habitat occupied by the population described by the process M, it can "choose" its dispersal strategy β in order to maximize the Lyapunov exponent $L_{\hat{N}}(\alpha, \beta)$.

The next result tells us how to determine this maximal Lyapunov exponent. Define the probability simplex $\Delta := \{(\beta_1, \ldots, \beta_n) \in \mathbb{R}^n_+ : \sum_{i=1}^n \beta_i = 1\}$.

Theorem 3.4.1. *Assume that the matrix $S = \Gamma^T \Gamma$ is positive definite and let $T = S^{-1}$. Fix $\alpha \in \Delta$ and assume that $L_{\hat{N},\max}(\alpha) := \max\{L_{\hat{N}}(\alpha, \beta) : \beta \in \Delta\}$ is attained at some $\beta^*(\alpha)$ in* *the interior of* Δ*. Then,* $\beta^*(\alpha)$ *is unique and*

$$\begin{aligned} L_{\hat{N},\max}(\alpha) &= \frac{1}{2}\Big(\mu^T T\mu + \nu_{\bar{M}}(\alpha)^2\kappa^T T\kappa - 2\nu_{\bar{M}}(\alpha)\mu^T T(\alpha \diamond \kappa) \\ &\quad - \frac{(\mu^T T\mathbf{1} - \nu_{\bar{M}}(\alpha)\mathbf{1}^T T(\alpha \diamond \kappa) - 1)^2}{\mathbf{1}^T T\mathbf{1}}\Big), \end{aligned} \tag{3.4.1}$$

where $\nu_{\bar{M}}(\alpha)$ *is as in* (3.3.5)*,* $\mathbf{1}$ *is a column vector with* 1 *in every entry, and* $\alpha \diamond \kappa$ *is the column vector* $(\alpha_1\kappa_1, \ldots, \alpha_n\kappa_n)^T$*.*

Proof. For simplicity, set $C_i := \mu_i - \nu_{\bar{M}}(\alpha)\alpha_i\kappa_i$ and $g(\beta) = \sum_{i=1}^n \beta_i - 1$. For a fixed value of α we use Lagrange multipliers to maximize

$$L_{\hat{N}}(\alpha, \beta) = \sum_{i=1}^n C_i\beta_i - \frac{1}{2}\sum_{i,j=1}^n S_{ij}\beta_i\beta_j$$

subject to the constraint $g(\beta) = 0$.

The relevant partial derivatives are

$$\begin{aligned} \frac{\partial L_{\hat{N}}}{\partial \beta_l} &= C_l - \sum_{j=1}^n S_{lj}B_j \\ \frac{\partial g}{\partial \beta_l} &= 1. \end{aligned}$$

for $l = 1, \ldots, n$. We need to solve the system

$$\begin{aligned} \frac{\partial L_{\hat{N}}}{\partial \beta_l} &= \lambda\frac{\partial g}{\partial \beta_l} \quad l = 1, \ldots, n \\ g(\beta) &= 0, \end{aligned}$$

where λ is the Lagrange multiplier variable. Using the expressions we found for the partial derivatives, this becomes

$$\begin{aligned} C_l - \sum_{j=1}^n S_{lj}\beta_j &= \lambda, \quad l = 1, \ldots, n \\ \sum_{i=1}^n \beta_i &= 1. \end{aligned}$$

If we take β and C to be column vectors and write $\mathbf{1}$ for the n-dimensional column vector whose entries are all equal to 1, we get the system

$$\begin{aligned} S\beta &= C - \lambda\mathbf{1} \\ \sum_{i=1}^n \beta_i &= 1. \end{aligned}$$

Because the matrix S is positive definite it has an inverse, $T := S^{-1}$, which is also positive definite. Our system then becomes

$$\begin{aligned} \beta &= TC - \lambda T\mathbf{1} \\ \sum_{i=1}^{n} \beta_i &= 1, \end{aligned}$$

so that $\beta_l = \sum_{j=1}^{n} T_{lj} C_j - \lambda \sum_{j=1}^{n} T_{lj}$. The constraint $g(\beta) = 0$ forces

$$\lambda = \frac{\sum_{i,j=1}^{n} T_{ij} C_i - 1}{\sum_{i,j=1}^{n} T_{ij}}.$$

Thus, $\beta \mapsto L_{\hat{N}}(\alpha, \beta)$ achieves its maximum at the vector $\tilde{\beta}(\alpha)$, where

$$\tilde{\beta}_l(\alpha) := \sum_{j=1}^{n} T_{lj} C_j - \sum_{j=1}^{n} T_{lj} \left(\frac{\sum_{i,j=1}^{n} T_{ij} C_i - 1}{\sum_{i,j=1}^{n} T_{ij}} \right), \; l = 1, \ldots, n. \tag{3.4.2}$$

provided this vector is in the interior of Δ.

The corresponding maximal value is

$$
\begin{aligned}
L_{\hat{N},\max}(\alpha) &= \sum_{i=1}^{n} C_l \beta_l^*(\alpha) - \frac{1}{2}\sum_{i,j=1}^{n} S_{ij}\beta_i^*(\alpha)\beta_j^*(\alpha) \\
&= \sum_{l,j=1}^{n} T_{lj}C_jC_l - \lambda \sum_{l,j=1}^{n} C_l T_{lj} - \frac{1}{2}\left[\sum_{l,m,j,i=1}^{n} S_{lm}T_{lj}T_{mi}(\lambda^2 + C_jC_i - \lambda(C_j + C_i))\right] \\
&= \sum_{l,j=1}^{n} T_{lj}C_jC_l - \lambda \sum_{l,j=1}^{n} C_l T_{lj} - \frac{1}{2}\left[\sum_{j,i=1}^{n} T_{ij}(\lambda^2 + C_jC_i - \lambda(C_j + C_i))\right] \\
&= \frac{1}{2}\sum_{l,j=1}^{n} (T_{lj}C_jC_l - \lambda^2 T_{jl}) \\
&= \frac{1}{2}\left(C^T T C - \frac{(C^T T \mathbf{1}^T - 1)^2}{\mathbf{1}^T T \mathbf{1}}\right) \\
&= \frac{1}{2}\sum_{l,j=1}^{n} (T_{lj}(\mu_j - \nu_{\bar{M}}(\alpha)\alpha_j\kappa_j)(\mu_l - \nu_{\bar{M}}(\alpha)\alpha_l\kappa_l) - \lambda^2 T_{jl}) \\
&= \frac{1}{2}\sum_{l,j=1}^{n} \left[T_{lj}(\mu_j\mu_l + \nu_{\bar{M}}(\alpha)^2\kappa_j\kappa_l) - 2\nu_{\bar{M}}(\alpha)T_{lj}\mu_j\alpha_l\kappa_l - \lambda^2 T_{jl}\right] \\
&= \frac{1}{2}\sum_{l,j=1}^{n} [T_{lj}(\mu_j\mu_l + \nu_{\bar{M}}(\alpha)^2\kappa_j\kappa_l) - 2\nu_{\bar{M}}(\alpha)T_{lj}\mu_j\alpha_l\kappa_l \\
&\quad - \left(\frac{\sum_{e,f=1}^{n} T_{ef}(\mu_e - \nu_{\bar{M}}(\alpha)\alpha_e\kappa_e) - 1}{\sum_{e,f=1}^{n} T_{ef}}\right)^2 T_{jl}] \\
&= \frac{1}{2}\sum_{l,j=1}^{n} \left[T_{lj}(\mu_j\mu_l + \nu_{\bar{M}}(\alpha)^2\kappa_j\kappa_l) - 2\nu_{\bar{M}}(\alpha)T_{lj}\mu_j\alpha_l\kappa_l\right] \\
&\quad - \frac{1}{2}\left(\frac{(\sum_{l,j=1}^{n} T_{lj}(\mu_l - \nu_{\bar{M}}(\alpha)\alpha_l\kappa_l) - 1)^2}{\sum_{l,j=1}^{n} T_{jl}}\right),
\end{aligned}
$$

which is the same as the expression in (3.4.1). □

Remark 3.4.2. If $\max_\beta L_{\hat{N}}(\alpha, \beta)$ is attained in the interior of one of the faces of the convex polytope Δ, that is, in one of the convex sets of the form $\{\beta \in \Delta : \beta_i = 0,\ i \in I,\ \beta > 0,\ i \notin I\}$, where $I \subset \{1, \ldots, n\}$, then it is necessary to perform a similar Lagrange multiplier computation on that set to determine the optimal β.

3.5 The two patch ($n = 2$) case

Since the formula for the maximal Lyapunov exponent is fairly complicated, we look at the simplest case when there are only two patches in our habitat.

First note that for $n = 2$ the matrix S is positive definite if and only if $S_{11} > 0$ and $S_{11}S_{22} - S_{12}^2 > 0$.

With a slight abuse of notation, we now write $(\alpha, 1-\alpha)$ for the vector we would have previously written as $\alpha = (\alpha_1, \alpha_2)$ and $\nu_{\bar{M}}(\alpha)$ for the quantity that we would have previously written as $\nu_{\bar{M}}(\alpha, 1-\alpha)$. We have

$$\nu_{\bar{M}}(\alpha) = \frac{\mu_1\alpha + \mu_2(1-\alpha) + \frac{1}{2}\left(-S_{11}\alpha^2 - 2(1-\alpha)\alpha S_{12} - (1-\alpha)^2 S_{22}\right)}{\kappa_1\alpha^2 + \kappa_2(1-\alpha)^2}.$$

and

$$\nu_{\bar{N}}(\beta) = \frac{\mu_1\beta + \mu_2(1-\beta) + \frac{1}{2}\left(-S_{11}\beta^2 - 2(1-\beta)\beta S_{12} - (1-\beta)^2 S_{22}\right)}{\kappa_1\beta^2 + \kappa_2(1-\beta)^2}.$$

Note that then the numerator of the two above equations is a quadratic in α (respectively β) and the coefficient of α^2 (respectively β^2) is $\left(-\frac{S_{11}}{2} + S_{12} - \frac{S_{22}}{2}\right)$ which is strictly negative because $|S_{12}| < \sqrt{S_{11}S_{22}}$ implies

$$\begin{aligned} S_{11} + S_{22} - 2S_{12} &> S_{11} + S_{22} \mp 2\sqrt{S_{11}S_{22}} \\ &= (\sqrt{S_{11}} \mp \sqrt{S_{22}})^2 \\ &\geq 0. \end{aligned}$$

where we have $-$ if $S_{12} > 0$ and $+$ if $S_{12} \leq 0$. Set $D = S_{11} - 2S_{12} + S_{22}$.

Thus, there exists an stationary distribution for $\bar{N}$ and for $\bar{M}$ for all $\alpha \in [0,1]$ (respectively $\beta \in [0,1]$) if and only if

$$\begin{aligned} \kappa_2\nu_{\bar{M}}(0) &= \mu_2 - \frac{S_{22}}{2} > 0 \\ \kappa_1\nu_{\bar{M}}(1) &= \mu_1 - \frac{S_{11}}{2} > 0. \end{aligned}$$

Assume the maximum Lyapunov exponent is attained, when α is fixed, for $\beta = \beta^*(\alpha)$. There are two separate cases

1) $\beta^*(\alpha)$ is in the interior of $[0,1]$. Then $\beta^*(\alpha) = \tilde{\beta}(\alpha)$ is given by equation (3.4.2) which in our case is

$$\tilde{\beta}(\alpha) = \frac{(-1+\alpha)\kappa_2(-2\mu_1 + 2S_{12} - S_{22} + \alpha^2 D) + \alpha\kappa_1(-2\mu_2 + S_{22} + \alpha^2 D)}{2(\kappa_2 - 2\alpha\kappa_2 + \alpha^2(\kappa_1 + \kappa_2))D}. \quad (3.5.1)$$

The maximum occurs in $(0,1)$ if and only if $\tilde{\beta}(\alpha) \in (0,1)$. Then by Theorem 3.4.1 the expression for $L_{\hat{N},\max}(\alpha)$ is

$$L_{\hat{N},\max}(\alpha) = \frac{[\alpha\kappa_1(2\mu_2 - S_{22} + \alpha^2 D) + (-1+\alpha)\kappa_2(2\mu_1 - S_{11} + D - 2\alpha D + \alpha^2 D)]^2}{8(\kappa_2 - 2\alpha\kappa_2 + \alpha^2(\kappa_1+\kappa_2))^2 D}.$$

Note that the denominator is strictly positive and the numerator is nonnegative, so $L_{\hat{N},\max}(\alpha) \geq 0$.

2) $\beta^*(\alpha)$ is on the boundary of $[0,1]$, that is $\beta^*(\alpha) \in \{0,1\}$. Note that this happens if and only if $\tilde{\beta}(\alpha) \notin (0,1)$. In this case the expression for $L_{\hat{N},\max}(\alpha)$ is

$$L_{\hat{N},\max}(\alpha) \;=\; \max\{L_{\hat{N}}(\alpha,0), L_{\hat{N}}(\alpha,1)\}$$

where

$$L_{\hat{N}}(\alpha,0) = \frac{\alpha[\alpha\kappa_1(2\mu_2 - S_{22}) + (1-\alpha)\kappa_2(-2\mu_1 + S_{11} - D + \alpha D)]}{2(\kappa_1\alpha^2 + \kappa_2(1-\alpha)^2)}$$

and

$$L_{\hat{N}}(\alpha,1) = \frac{(1-\alpha)[(1-\alpha)\kappa_2(2\mu_1 - S_{11}) + \alpha\kappa_1(-2\mu_2 + S_{22} - \alpha D)]}{2(\kappa_1\alpha^2 + \kappa_2(1-\alpha)^2)}$$

Some computations show that the following identities hold

$$\begin{aligned} L_{\hat{N}}(\alpha,1) &= \left(\tilde{\beta}(\alpha) - \frac{1+\alpha}{2}\right)(1-\alpha)D \\ L_{\hat{N}}(\alpha,0) &= \left(\frac{\alpha}{2} - \tilde{\beta}(\alpha)\right)\alpha D. \end{aligned}$$

As a result we have that if $\tilde{\beta}(\alpha) \geq 1$ then

$$\begin{aligned} L_{\hat{N}}(\alpha,1) &\geq \frac{(1-\alpha)^2 D}{2} \geq 0 \\ L_{\hat{N}}(\alpha,0) &\leq -\left(\frac{1-\alpha}{2}\right)\alpha D \leq 0 \end{aligned}$$

so we have $L_{\hat{N},\max} = L_{\hat{N}}(\alpha,1) \geq 0$. Likewise, if $\tilde{\beta}(\alpha) \leq 0$ then

$$\begin{aligned} L_{\hat{N}}(\alpha,1) &\leq -\left(\frac{1+\alpha}{2}\right)\alpha D \leq 0 \\ L_{\hat{N}}(\alpha,0) &\geq \frac{\alpha^2 D}{2} \geq 0 \end{aligned}$$

so we have $L_{\hat{N},\max} = L_{\hat{N}}(\alpha,0) \geq 0$.

The above treatment shows that no matter if the maximum is attained in the interior or on the boundary of $[0,1]$ we will always have $L_{\hat{N},\max} \geq 0$. It also shows that

$$\beta^*(\alpha) = 0 \vee (\tilde{\beta}(\alpha) \wedge 1).$$

Remark 3.5.1. There are examples where the maximum is not achieved in the interior. If we choose S to be the identity matrix, $\mu_1 = 2, \mu_2 = 1.5, \kappa_1 = 1$ and $\kappa_2 = 3$ then we get that $\bar{M}$ and $\bar{N}$ always have stationary distributions for $\alpha, \beta \in [0,1]$. However,

$$\tilde{\beta}(\alpha) = \frac{3(-1+\alpha)(-5+2\alpha^2) + \alpha(-2+2\alpha^2)}{4(3-6\alpha+4\alpha^2)}$$

will not lie in $(0,1)$ for all α. For example $\beta^*(0.5) = 1.5$.

We know from Proposition 3.3.4, Theorem 3.3.8 and Theorem 3.3.14 that if $L_{\hat{N},\max}(\alpha) > 0$, then

$$\mathbb{P}^{(x,y)}\{N_t \not\to 0 \text{ as } t \to \infty\} = 1$$

and, if in addition $L_{\hat{M}}(\alpha, \beta^*(\alpha)) < 0$, then

$$\mathbb{P}^{(x,y)}\{M_t \to 0 \text{ and } N_t \not\to 0 \text{ as } t \to \infty\} = 1.$$

Thus, the population described by the process M should, if possible, "choose" its dispersal strategy to be α_* so that

$$L_{\hat{N},\min,\max} := \min_\alpha L_{\hat{N},\max}(\alpha) = \min_\alpha \max_\beta L_{\hat{N}}(\alpha,\beta) = L_{\hat{N}}(\alpha_*, \beta^*(\alpha_*)) = 0.$$

We now show that we can always find such an α_*. Note that if there exists a solution $\bar{\alpha}$ to the equation $\alpha = \beta^*(\alpha)$ then we get $L_{\hat{N}}(\bar{\alpha}, \beta^*(\bar{\alpha})) = L_{\hat{N}}(\bar{\alpha}, \bar{\alpha}) = 0$. It thus suffices to find solutions to

$$\alpha = 0 \vee (\tilde{\beta}(\alpha) \wedge 1)$$

in $[0,1]$. It is enough to show that there exists a solution in the open interval $(0,1)$. We prove that that there is some $\alpha \in (0,1)$ which satisfies the fixed point problem

$$\tilde{\beta}(\alpha) = \alpha.$$

This is always possible because by the definition of $\tilde{\beta}(\alpha)$ there are solutions if and only if the cubic polynomial

$$p(\alpha) := \alpha\kappa_1(2\mu_2 - S_{22} + \alpha^2 D) + (-1+\alpha)\kappa_2(2\mu_1 - S_{11} + D - 2\alpha D + \alpha^2 D).$$

has a root in $(0,1)$. However,

$$p(0) = -\kappa_2(2\mu_1 - S_{11} + D) < 0$$

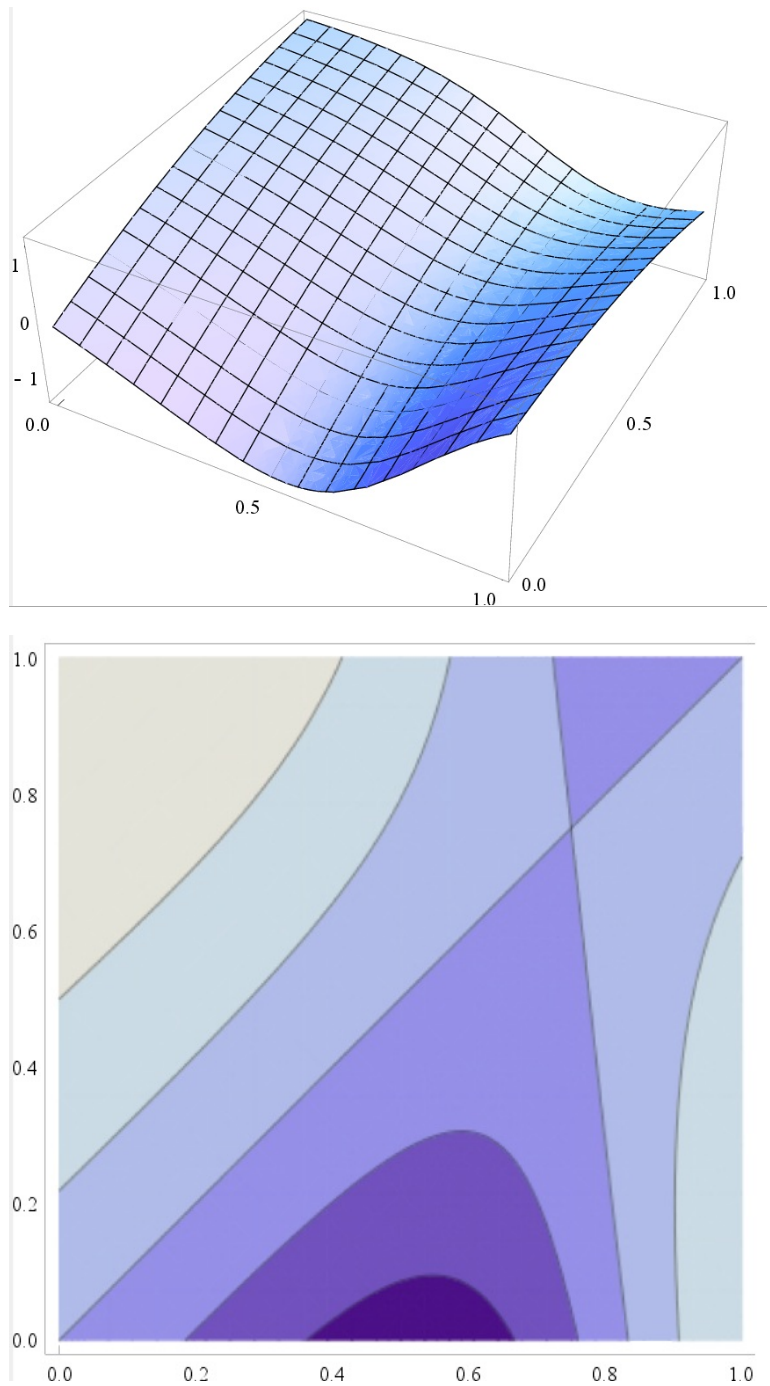

Figure 3.1: In this figure we have the 3D and contour graphs of the Lyapunov exponent $L_{\hat{N}}(\alpha, \beta)$ as a function of $(\alpha, \beta) \in [0,1] \times [0,1]$ when S is the identity matrix, $\mu_1 = 2, \mu_2 = 1.5, \kappa_1 = 1$ and $\kappa_2 = 3$. One can see that there is a saddle point on the line $\alpha = \beta$, which is expected from our discussion of α_* and β^* from the text.

and

$$p(1) = \kappa_1(2\mu_2 - S_{22} + D) > 0,$$

and so there is $\alpha \in (0, 1)$ such that $\tilde{\beta}(\alpha) = \alpha$. Therefore, we have shown that there exists $\alpha_* \in (0, 1)$ such that

$$L_{\hat{N}}(\alpha_*, \beta^*(\alpha_*)) = 0.$$

Remark 3.5.2. It is not possible to have solutions solutions to

$$\alpha = 0 \vee (\tilde{\beta}(\alpha) \wedge 1)$$

for $\alpha \in \{0, 1\}$ for the following reasons: If $\alpha = 0$ were a solution we would need $0 \vee (\tilde{\beta}(0) \wedge 1) = 0$, that is $\tilde{\beta}(0) \leq 0$. This is impossible because

$$\tilde{\beta}(0) = \frac{2\mu_1 - S_{11} + D}{2D} > 0.$$

If $\alpha = 1$ were a solution we would need $0 \vee (\tilde{\beta}(1) \wedge 1) = 1$, that is $\tilde{\beta}(1) \geq 1$. This is again impossible because

$$\tilde{\beta}(1) = 1 - \frac{2\mu_2 - S_{22}}{2D} < 1.$$

Remark 3.5.3. If both populations use the same dispersal strategy, that is if $\alpha = \beta$, then our diffusion becomes singular and one population size is always a fixed multiple of the other.

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