Title

Understanding Symbols: A Situativity-Theory Analysis of Constructinf Mathematical Meaning

Journal

Proceedings of the Annual Meeting of the Cognitive Science Society, 15(0)

Authors

Greeno, James G.
Engle, Randi A.
Kerr, Laura K.
et al.

Publication Date

1993

Peer reviewed

Understanding Symbols: A Situativity-Theory Analysis of Constructing Mathematical Meaning

James G. Greeno, Randi A. Engle, Laura K. Kerr, and Joyce L. Moore
Stanford University School of Education
Stanford, CA 94305
greeno@csli.stanford.edu

Abstract

We report analyses of the construction and interpretation of mathematical symbols that refer to quantitative properties and relations of a physical system. Middle-school students solved problems that involved contructing tables, equations and graphs to represent linear functions of a device where blocks are moved varying distances by turning a handle that winds string around spools of different sizes. Previous research analyzed activities of reasoning about quantities of this system as attunement to constraints and affordances, a characterization of students' implicit understanding of concepts of variable and linear functions. This report concerns activities of representing quantitative properties and relations using mathematical notations. We are developing analyses of constructing and interpreting tables, equations, and graphs in terms of attunement to constraints and affordances of the represented system, the system of notations, and relations between the constraints of the notations and the represented domain. We present examples that illustrate concepts of semantic clumps, groups, and morphisms; descriptive and demonstrative representations; multiple referent domains; and constructions of meaning in contributions to conversational discourse.

Introduction

We are analyzing how humans understand concepts and the meanings of symbolic representations. Our approach, which we call situativity theory, focuses theoretical attention on interactions between agents, other people, and material systems in their environment (Greeno, 1992; Greeno & Moore, 1993). We draw on concepts and methods of ethnographic studies of everyday activity (Lave, 1988; Suchman, 1987), ecological psychology (Gibson, 1979/1986; Turvey, 1990; 1992), and philosophical situation theory (Barwise & Perry, 1983; Devlin, 1991) along with many precursors (Dewey, 1929/1958; Mead, 1934).

Previous studies (Greeno, Moore, & Mather, 1992; Moore & Greeno, 1991) have analyzed implicit understanding of mathematical concepts by middle- and high-school students reasoning about quantitative properties of an apparatus that we call the winch, adapted from an apparatus used by Piaget, Grize, Szeminska, & Bang (1968/1977), shown in Figure 1. Students answered questions such as where a block would be after some number of turns, or how many turns it would take for one of the blocks to catch up with the other. Their success indicated that they implicitly understood variables and linear functions, although there was no evidence that these concepts were known explicitly. In these studies, students' understanding could be characterized in terms of *attunement to constraints and affordances*, such as their attunement to the constraint that there is a constant distance the block moves each time the handle is turned and the affordance that a ruler provides for reasoning about numerical values.

Our current work focuses on how people understand symbolic representations of mathematical variables and functions. We gave students problems involving tables and equations in one experiment and graphs in another experiment. They were asked to construct and use these symbolic representations to make inferences about the winch system. Our theoretical task was to characterize how students understood the symbolic representations they constructed and to describe the processes through which those understandings were achieved.

Theory

Characterizing Meanings

Our analysis describes how symbols are constructed and interpreted and how conceptual understanding results from this activity. We use Clark and Wilkes-Gibbs's (1986) idea that reference is a collaborative achievement in conversation, and an extension of Clark and Schaefer's (1989) method for analyzing conversational turns to identify episodes in which a meaning for a symbolic expression is achieved in the conversational common ground. We employ a modification of Barwise and Perry's (1983) relational

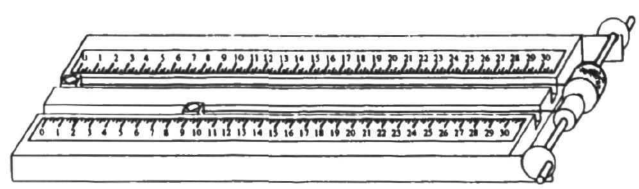

Figure 1

theory of meaning to characterize events in which a symbolic expression is given meaning. In this theory, meaning is a relation between a situation where a symbolic expression is spoken, written, or drawn, and the situation that the expression refers to.

Constraints of Representing

Understanding symbolic representations is an activity that involves attunement to constraints and affordances about (a) the domain that is referred to, (b) the domain of notational expressions, and (c) the meaning relations between notations and their referents.

Co-constrained properties. In the referent domain, attunement to constraints makes it possible to refer to co-constrained properties of a situation. For example, in the winch system the labeled size of each spool is equal to its circumference in inches, which equals the distance that a block moves each time the handle is turned with that spool. This means that saying or writing a numeral can be interpreted as referring to the size of a spool, a distance per turn, or both. Further, if a phrase is used that specifies one of these (e.g., "the 'four' spool") its referent can be extended to include the distance per turn of a block with that spool, if the speaker and listener(s) are attuned to the constraint that these quantities are equal. Similarly, symbols have co-constrained properties that provide alternative ways of referring. For example, in the graph of a linear function through the origin, the steepness of the line and its y-coordinate at x=1, are distinguishable visual features which both correspond to the slope of the function. When a property of the line is interpreted as referring to the slope of the function, either its steepness, or a y-coordinate, or both, may be the focus of attention.

Syntax, semantics, and symbolic communication. Constraints of the notations can be considered the syntax (in a broad sense) of the representational system. For example, in equations there are constraints on the sequence of numerals, letters, and operation signs that students learn, and there is a simple grammar that determines a phrase structure for any algebraic expression. In graphs, there are constraints such as the continuity of a line that is considered as an object and affordances such as considering any segment of a line as an object.

Constraints and affordances of the interpretive process involve the ways in which notations are used as symbols. These are the semantic conventions of the representational system. For example, in an equation, two adjacent symbols (e.g., "3x") represent two numbers or quantities that are combined by multiplication. In a two-dimensional graph, a line represents a function of two variables with any point's coordinates representing a pair of variable values included in the function.

Some Concepts and Distinctions

Cognitive analyses often reveal unexpected complexities. Our study has led us to realize that relations between symbols and their referents are complicated in several ways.

Clumps, groups, and morphisms. The structure of students' uses and interpretations of symbols varied considerably. They varied to the degree to which: (a) different symbols, concepts or interpretations were differentiated from each other and (b) relations between them were specified systematically. We use the term "clump" to characterize cases in which symbols or concepts were used interchangably with little specification of how they are related. At the other extreme were highly-structured "groups" in which distinct symbols or concepts were systematically related using composable and invertible operations. A further level of structuring occurred when the operations of a group in one domain (e.g. making 3 turns with a 4-spool and seeing how far the block moves) were systematically related to the operations of a group in another domain (e.g. computing 3 times 4). Such inter-domain structures are referred to as "morphisms".

Descriptive and demonstrative representations. Symbolic expressions like equations can be interpreted as descriptions; that is, as propositions about a situation that are either true or false. Interpreting an expression as a description involves identifying objects, properties, and relations that the expression designates as truth conditions and determining whether those conditions hold in a situation or situation type. In an equation, variables are generally interpreted descriptively while the entire equation asserts a relation among quantities or numbers that may be true or false.

Symbolic expressions like lines in graphs and pictures in other types of diagrams, can be interpreted as demonstrative representations, that is, as objects with properties and relations that correspond to properties and relations of objects in the situation or domain that the expression represents. Interpreting an expression demonstratively involves (a) identifying objects in the referent situation or domain that correspond to the objects in the representation, (b) identifying properties and relations of represented objects that are like properties and relations of the symbolic objects, and (c) determining whether the represented situation or class of situations has the properties and relations that it should have for the representation to be correct. In a graph, lines are generally interpreted demonstratively while numerals that mark distances along axes are generally interpreted descriptively.

Multiple referent domains. Mathematical expressions (tables, equations, and graphs) can be interpreted as referring to numbers and their properties, or to the quantitative properties and relations of material systems. Quantitative and numerical properties can be integrated more or less strongly in people's understanding. We observed differences in students' uses and interpretations of symbols as referring to either numbers and numerical operations, to quantities and their relations, or to both in an integrated fashion.

Conversational constructions of meaning. Interpretations of symbols are both social and cognitive achievements. In our studies, interpretations occurred during social interactions as pairs of students worked on problems presented in workbooks. In these settings, meanings of symbols were components of the common ground constructed during conversation. We can describe events of constructing meaning as contributions to conversation (Clark & Schaefer, 1989), involving presentations that function as proposed interpretations which are then accepted, questioned, modified, or rejected in subsequent conversational turns.

Examples

In order to illustrate these distinctions and concepts, we present some examples from two studies using the winch system (Figure 1) with two different kinds of workbooks.

Example 1

The table-and-equations workbook included problems for which students constructed and explained tables that showed positions of the block after different numbers of turns with different spool sizes, and also constructed and explained equations for calculating values of variables, such as finding the position of a block or number of turns given all other relevant values (starting position, spool size, etc.) We examine how a pair of eighth-grade students, whom we call Julie and Paula, solved problems in this workbook after they had completed all items involving tables and had constructed some equations.

Clumps, groups and morphisms. We are finding that the relations between sets containing symbols, their referents and interpretive correspondences vary over two dimensions: the degree to which items are differentiated from each other and the extent to which relations between items are specified systematically. This is illustrated by the development of Julie and Paula's use of the term "distance", symbolized in their equations by **D**, especially as they distinguish it from end position.

Before Item 8, all calculations involved situations where the block started at zero. In those cases, the distance that a block moves is equivalent to its end position, and either of those quantities is equal to the number of turns times the spool size. Paula and Julie's references to various properties of distance and end position appear to have been linked into a clump. Throughout the first 8 items, the terms "distance moved" and "end position" were used interchangeably. "Clumping" is also evidenced by Julie and Paula's use of a hybrid phrase "the distance a block moves to" which incorporates conceptual aspects of both distance moved and end position. Also there was no evidence that Julie and Paula understood the relation that made distance and end position equivalent in this case.

Paula and Julie's lack of differentiation between *distance* and *end position* became problematic in Item 8b when starting position was no longer assumed to be zero. They were asked to "write your word equation so that you taking into account where the block was before any turns were made." Julie proposed that "you subtract," then they worked on an example (4 turns of a 3-spool with a nonzero starting point) until Paula finally agreed, "So you have to—subtract...." They were able to subtract 12 from the end positions of 15 and 18 and recognize that the results would equal starting positions of 3 and 6. However, because they clumped different distance terms together, they had trouble converting their algorithm into an equation. For example, Paula originally proposed the equation "**T × S = D - D = the original distance,**" meaning to substitute different values for each distance variable. After several trials they eventually derived the equation "**D - (T × S) = the starting point,**" which we characterize as still involving some clumping as end position values rather than distance moved values were substituted for D.

In items 9 and 10, Paula and Julie were given an equation which introduced **Y** as a symbol for end position. As they followed the conventions set up by this equation, **D** and the concept of distance moved played only a minimal role in their reasoning.

In item 11, Julie and Paula began examining what they meant by different terms and symbols. This led them towards a group structure that distinguished end position (**Y**) and distance moved (**D**) while specifying the relation between them. Ironically, their progress was prompted by an oversight. When asked to write an equation "to find out how many times the handle was turned...[using] the same letters you used in your last equation" in Item 11a, Julie and Paula wrote down "**D/s = t**,", the equation they had derived in Item 7 before non-zero starting points were considered. However, item 11b asked them to apply their equation to a problem with a non-zero starting point:

> "The winch has a 6-spool on it with the block starting at 2. The block has ended up at 14. How many times was the handle turned."

To solve the problem, they would have to either modify their equation or reinterpret it.

This process began when they started listing the variables to use. After they wrote down "**D =**

distance, S = spool size, t = turns," Paula proposed that Y be included: "and Y equals ending point, even though we don't use it we should just put ...oh, we don't use it, we don't need to." Julie proposed to instead use P, which she added with Paula's implicit agreement, "oh, P is starting position." Then Julie said,

"D divided by S equals T, and we have to subtract 2"

Paula rejected this proposal by protesting against Julie's use of a specific value for P:

"Minus 2, it's not always 2 though"

Julie then changed her proposal to "minus P" (making their equation **(D / S) - P = T**), but Paula rejected this by making a proposal of her own:

"I know that Y is the *ending point* how about that? [pause] use Y [pause] I don't think we need a new variable..."

Julie expressed her disagreement by not writing Paula's suggestion down, arguing that Y was not one of the variables on their list. After nine repeated directives from Paula she eventually wrote it down. Now their equation was **(D / S) - Y = T.**

However, when they tried to substitute values into the equation, they followed their earlier clumping of **D** with end position and substituted the end position, 14, for **D**. This left them with no number to substitute for **Y**. This problem caused Paula to reinterpret distance:

"Hold on cause now the distance is 14 minus 2 which is 12 that's the distance that it traveled so we have to put 12 not 14"

Julie agreed, erasing 14 and writing in 12. The degree to which this reinterpretation had been incorporated in their scheme is evidenced by the fact that they did not consider it necessary to write out their decomposition of *distance traveled*: As Paula summed it up:

"well, the distance, I mean people can figure that out cause you have to do the ending point minus the beginning point"

This reinterpretation forms a group structure by distinguishing distance moved from end position while encapsulating the exact relationship between them.

Conversational construction of meaning. By analyzing conversational turns in terms of presentations and acceptances (Clark & Schaefer, 1989) we can identify episodes in which meanings are proposed, considered, modified, and accepted by each pair of students. In many cases, a student could simply use a symbol since its interpretation was apparently clear and acceptable to the other student. In other cases, a student would propose a symbol or interpretation to be accepted. In some cases, the other student questioned or objected to the proposal, with subsequent conversational turns needed to arrive at an agreed-to meaning to contribute to common ground.

One example was in the proposal by Julie that "you subtract," which was used during the discussion of examples and eventually accepted explicitly by Paula, "So you have to—subtract." Another example was Paula's proposal to include the variable **Y** in their variable list for Item 11b, which was rejected by Julie with a counterproposal to include **P** that Paula accepted saying "Oh, p is starting position." A more complex example was a proposal by Paula to evaluate the expression "**D/S = T**," asking "How much is distance?" The proposal to engage in this project was accepted by Julie, who said, "14." Three turns later, Paula rejected Julie's proposed interpretation of **D**, saying "Hold on cause now the distance is 14 minus 2 which is 12; that's the distance that it traveled."

Multiple referent domains. It seems to us that for these students, equations referred both to quantities in the winch system and to numbers and arithmetic operations, with their focus shifting back and forth between these comfortably. Other students interpreted referents of equations in a less integrated way, relating quantities to variables but interpreting the combinations of symbols in an equation as corresponding to numerical operations that were not integrated with combinations of quantities.

Example 2

This example comes from the triangles-and-graphs workbook where students constructed and explained graphs representing the motion of blocks with given spool sizes and starting positions. Students used a large graph board with the abscissa labeled "Turns" and the ordinate labeled "Block position". They were given plastic right triangles of three different colors and sizes: green triangles 1 inch wide and 3 inches high; blue triangles 1 inch wide and 4 inches high; and red triangles 1 inch wide and 6 inches high. When triangles were placed on the graph board as shown in Figure 2, a triangle's hypotenuse formed a line that represents a block's position over one turn using one of the spools. Placing several triangles so their hypotenuses were aligned formed a line representing how the block's position changes over several turns.

Items 1 and 2 were questions about properties of the winch, while Items 3 and 4 involved constructing representations. Item 1 said, "Connect the black block to the 3-spool and the white block to the 6-spool. With the blocks at zero, link the handles together. Without actually doing it, what do you predict would happen if you turned the handles several times?" Item 2 specified a 3-spool and a 4-spool with the 3-spool's block starting ahead, and also asked students to predict what would happen when the handle was turned. Item 3 presented the diagram shown in Figure 2 and asked, "What do you think these green triangles are a picture of? You can use the winch if you want." Item 4 presented a blank grid with the same labels as Figure 2 and asked, "Imagine [that a block] is at zero and connected to the 4-spool. Show us how you could use

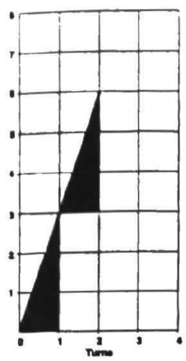

Figure 2

the blue triangles to figure out where the block would be after 2 turns. Check your answer on the winch if you want."

One pair of students, whom we call Irene and Janna, answered Item 1 by writing, "The white block would move more than the black one because the six spool is larger than the 3 spool and would take more string when turned." For Item 2, they predicted that the block connected to the 4-spool would catch up. They did not arrive at an answer to Item 3 initially. They mentioned that the triangles on the graph were "a picture of" a flag, sunglasses, and a Christmas tree before deciding to go to the next item.

In working on Item 4, Irene and Janna were attuned to the label "Turns" on the x-axis of the printed graph and interpreted it as a symbol for turns of the winch handle:

Irene: [*Restating the question*] Figure out where the block would be after 2 turns.

Janna: Oh, yeah, these are the turns [*point along the x-axis*].

Irene: Oh, yeah, see turns, 2 turns it's on the 4-spool, maybe? After 2 turns it would be there? Wherever, wherever there is [*moves finger from (0,4) to (2,0) to (2,4) to (0,0)*].

Irene and Janna then placed the blue triangles (4-spools) on the graph paper in the same way the green triangles were placed in Figure 2. They noticed that a blue triangle "goes up to the 4." They associated this information with their knowledge about turns:

Irene: Oh, yeah, because it goes up to the 4. Two turns, yeah, see that's right.

Janna: It's not.

Irene: See, cause it, that shows....

Janna: Two turns.

Irene: In two turns, it would go (*inaudible*).

At this point, Irene and Janna returned to Item 3 and wrote, "After one turn, the block is at 3 and after 2 turns it is 6." Irene said, "See different triangles are different shapes, different lengths. That's why we have to use blue for that one. You see?" Working on Item 4 again, Irene and Janna concluded:

Janna: Ok, so it'd be—

Irene: So, after 1 turn it'll be at 4 and 2 turns it would be at 8.

They then drew the lines on the graph shown in Figure 3.

The morphism they constructed between sizes of triangles and spools was apparent in the next item:

Janna: Well if this is a 4-spool [*holds up a blue triangle*], we have to find a 6-spool. [*Pick up a red triangle.*] This must be a 6-spool.

Two other seventh-grade students, whom we call Kathy and Helen, were not attuned to the co-constrained properties of the winch, spool circumference and distance per turn. To explain why a block attached to a 4-spool passed the block attached to a 3-spool in Item 2, they wrote, "Because the one at zero had a heavier block." They did not recognize that the distance a block moves and the size of its spool are co-constrained properties. Consequently, their results did not reveal an implicit understanding of linear functions. Instead, they were constrained by the total distance moved by the block and the number of turns necessary to reach the block's end position irrespective of the spool's circumference as a co-constrained property of these events.

Thus the graphical constraints they were attuned to were "Turns" (the abscissa) and the total length of the combined triangles. Co-constraints between these properties is evidenced by their demonstrative representation of the winch. As seen in Figure 4, the triangles were treated as arbitrary shapes, the proper combination of which allowed them to represent the total distance moved by the block. This interpretation of the triangle-graph representational system is also seen in their conversations about the winch:

Kathy: Ok, we go to 20 [*starting their measurements at 16 with a blue triangle on the winch grid, Kathy turns the handle 1 1/2 turns*]. Ok, two turns is half a, it's half a thing away. It's half a triangle [*referring to the position of the block pointer in relation to the top of the triangle*].

Helen: So 2 turns is like about one and a blue and a green, almost [*stack a blue and green triangle on the table*]. Cause this, I think, if I'm right, is about half. [*measuring the green triangle in relation to the blue triangle*] So it's like a blue and a green.

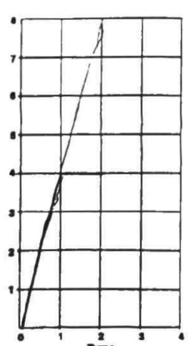

Figure 3

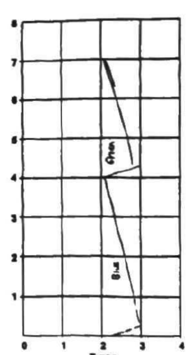

Figure 4

Clumps, groups, and morphisms with demonstrative representations. These examples differ in the students' attunements to constraints and affordances of the winch system, which probably accounts for how they differentially use the triangles as representations. Irene and Janna were attuned to the constraint that linked the circumferences of spools to the distances blocks moved on each turn. They apparently grasped the significance of the individual triangles in Figures 2 and 3—that is, that each triangle afforded representation of a single turn of the handle and that combinations of triangles afforded representations of sets of turns. In contrast, Kathy and Helen had a less differentiated attunement to constraints of the winches, recognizing only that differences in total distance resulted from differences between the spools. Their use of triangles reflected this more global attunement. They used arbitrary combinations of triangles to construct a representation with a total height that was approximately equal to the total distance that they judged a block would move in two turns.

Discussion

Our results are encouraging for analyzing the construction and interpretion of symbolic representations in terms of attunements to constraints and affordances. It is clear from these preliminary results that learning a representational system involves more than acquiring referential correspondences between symbols and their referents. A situativity-theory analysis based on attunement to constraints and affordances explains how reasoning about the represented system, constructing and using symbolic expressions, and creating correspondences between symbols and their referents all contribute interactively to conceptual understanding.

We chose the domain that we are studying partly because of its educational importance. We believed that by constructing a physical system in which quantitative relations are easily understood, we could provide a semantic interpretation of symbolic representations that could be learned more meaningfully. The results of our study support this idea. Students did learn to construct and interpret symbolic representations of quantitative properties of the winch system.

Acknowledgements

Supported by National Science Foundation Grants BNS-8718918 and MDR-9053605, a National Science Foundation Graduate Fellowship to Randi Engle, and a grant from the Stanford University Office of Technology Assessment. Richard Mander helped design the workbooks and conduct the interviews used in the analyses. Joyce Moore is now affiliated with the Cognition and Technology Group at Vanderbilt University.

References

Barwise, J., & Perry, J. (1983). *Situations and attitudes.* Cambridge, MA: MIT Press/Bradford.

Clark, H. H., & Schaefer, E. F. (1989). Contributing to discourse. *Cognitive Science, 13,* 259-294.

Clark, H. H., & Wilkes-Gibbs, D. (1986). Referring as a collaborative process. *Cognition, 22,* 1-40.

Devlin, K. (1991). *Logic and information.* Cambridge, UK: Cambridge University Press.

Dewey, J. (1958). *Experience and nature.* New York, NY: Dover (second edition, first published in 1929).

Gibson, J. J. (1986). *The ecological approach to visual perception.* Hillsdale, NJ: Lawrence Erlbaum Associates (first published in 1979).

Greeno, J. G. (June, 1992). The situation in cognitive theory: Some methodological implications of situativity. Paper presented at the Fourth Annual Convention of the American Psychological Society, San Diego, CA.

Greeno, J. G., & Moore, J. L. (1993). Situativity and symbols: Response to Vera and Simon. *Cognitive Science, 17,* 49-60.

Greeno, J. G., Moore, J. L., & Mather, R. (1992). A situativity-theory analysis of conceptual competence in quantitative reasoning. Stanford University School of Education.

Lave, J. (1988). *Cognition in practice: Mind, mathematics, and culture in everyday life.* Cambridge, UK: Cambridge University Press.

Mead, G. H. (1934). *Mind, self, and society: From the standpoint of a social behviorist.* Chicago, IL: University of Chicago Press.

Moore, J. L., & Greeno, J. G. (1991). Implicit understanding of functions in quantitative reasoning. *Proceedings of the thirteenth annual conference of the Cognitive Science Society* (pp. 221-226). Hillsdale, NJ: Lawrence Erlbaum Associates.

Piaget, J., Grize, J., Szeminska, A., & Bang, V. (1977). *The psychology and epistemology of functions.* Dordrect: D. Reidel (first published in French in 1968).

Suchman, L. (1987). *Plans and situated action.* Cambridge, UK: Cambridge University Press.

Turvey, M. T. (1990). Coordination. *American Psychologist, 45,* 938-953.

Turvey, M. T. (1992). Ecological foundations of cognition: Invariants of perception and action. In H. L. Pick, Jr., P. van den Broek, & D. C. Knill (Eds.), *Cognition: Conceptual and methodological issues* (pp. 85-117). Washington, DC: American Psychological Association.