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ALMOST-OBJECTIVE UNCERTAINTY

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ALMOST-OBJECTIVE UNCERTAINTY

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Every subjective state space with sufficient (e.g., Euclidean) structure contains *almost-objective* events that approximate, and in the limit attain, the standard properties of objectively uncertain events for all individuals with event-smooth betting preferences – whether or not they are expected utility, state-independent, or probabilistically sophisticated. These properties include well-defined and *unanimously agreed upon* revealed likelihoods, independence with respect to other subjective events, probabilistic sophistication over almost-objective acts, and linearity of state-independent and state-dependent expected utility in almost-objective likelihoods and almost-objective mixtures of subjective acts. Many real-world randomization devices are based on events of this form.

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1. INTRODUCTION

The theory of individual choice under uncertainty allows for wide variation in the mathematical specification of uncertain outcomes. Depending on the choice being studied, outcomes can take the form of univariate wealth levels, multivariate commodity bundles, consumption/leisure bundles, bundles with integer-valued (indivisible) commodities, discrete- or continuous-time consumption streams, or most generally, abstract objects $x \in \mathcal{X}$.

In addition to these different ways of representing uncertain *outcomes*, the theory also admits two different ways of representing *uncertainty itself*, namely:¹

- *Objective Uncertainty*, in which uncertainty is represented by well-defined, additive, numerical probabilities $p \in [0,1]$, and the objects of choice consist of *lotteries* $\mathbf{P} = (x_1, p_1; \dots; x_m, p_m)$ which yield outcome $x_i \in \mathcal{X}$ with probability p_i , under the standard condition $\sum_{i=1}^m p_i = 1$.
- *Subjective Uncertainty*, in which uncertainty is represented by mutually exclusive *states of nature* $s \in S$, and the objects of choice consist of *bets* or *acts* $f(\cdot) = [x_1 \text{ on } E_1; \dots; x_m \text{ on } E_m]$ which yield outcome $x_i \in \mathcal{X}$ if the realized state lies in *event* $E_i \subseteq S$, for some partition $\{E_1, \dots, E_m\}$ of S .

Each of these ways of representing uncertainty has its advantages and disadvantages. An advantage of objective uncertainty is that it allows us to draw on the tremendous body of results in probability theory, such as the Central Limit Theorem, the Law of Large Numbers, Chebyshev's Inequality, etc. Furthermore, since objective probabilities are part of the objects of choice themselves, these types of results hold *independently* of individuals' particular attitudes toward risk. Such results thus have the same character as arbitrage-based arguments in financial theory, which also hold independently of risk preferences and hence yield extremely powerful results.

But in some sense, this strength of the objective framework is also its greatest weakness: it imposes *too much* uniformity of beliefs across individuals, and in many cases, too much structure on each individual's own beliefs. Outside of the gambling hall, most real-world uncertain prospects – such as real investments, financial assets or insurance contracts – have the properties that their uncertainty does not appear in terms of pre-packaged, unanimously agreed-upon numerical probabilities, but rather, in terms of states or events. In contrast with preferences over objective lotteries, preferences over such real-world prospects are subject to the following three phenomena:

- Individuals may have different subjective likelihoods for the same event (*diverse beliefs*)
- Individuals' beliefs may not be representable by probabilities at all, with some or all events being considered “ambiguous” (*absence of probabilistic sophistication*)
- Individuals' outcome preferences may depend upon the *source of uncertainty itself* (outcome preferences may be *state-dependent*)

All this argues for modeling uncertainty via the subjective approach, which is rightly considered the more “foundational” of the two approaches.

In order to obtain as many of the benefits of probability theory as possible within the subjective framework, several researchers have proposed conditions on individual preferences over subjective acts which imply that beliefs can be represented by individual-specific *subjective probabilities* (sometimes called *personal probabilities*) over events. Examples include:

¹ Throughout this paper, we restrict attention to finite-outcome uncertain prospects.

Anscombe and Aumann (1963), who combine a subjective state space $S = \{s_1, \dots, s_m\}$ with an exogenous randomization device (a *roulette wheel*) capable of generating arbitrary objective probabilities, assume expected utility maximization, and derive von Neumann-Morgenstern outcome utilities and subjective state probabilities $\{\mu(s_1), \dots, \mu(s_m)\}$. While elegant, this two-stage approach² still ultimately relies upon exogenous objective uncertainty, via the roulette wheel.

Ramsey (1931), who considers a general family of subjective events which includes an *ethically neutral event* $\bar{E}$ such that the individual's betting preferences satisfy $[x^* \text{ on } \bar{E}; x \text{ on } \sim \bar{E}] \sim [x \text{ on } \bar{E}; x^* \text{ on } \sim \bar{E}]$ for all outcomes x^* and x , assumes expected utility maximization, and derives the utility of each outcome and the subjective probability of each event. This approach is thus fully subjective, relying just on the existence of some subjective event $\bar{E}$ whose betting properties essentially correspond to those of an exogenous objective event with probability $1/2$.

Savage (1954), who takes a general (infinite) state space S , assumes that preferences over finite-outcome acts satisfy the *Sure-Thing Principle* (separability across mutually exclusive events), consistent comparative likelihoods ($[x^* \text{ on } A; x \text{ on } \sim A] \succcurlyeq [x^* \text{ on } B; x \text{ on } \sim B]$ for $x^* \succ x$ implies $[y^* \text{ on } A; y \text{ on } \sim A] \succcurlyeq [y^* \text{ on } B; y \text{ on } \sim B]$ for all $y^* \succ y$), a version of event-continuity, and derives expected utility risk preferences and a subjective probability measure $\mu(\cdot)$ over S .

Machina and Schmeidler (1992), who adopt a Savage type setting, and derive the *probabilistically sophisticated* form $V(x_1, \mu(E_1); \dots; x_m, \mu(E_m))$ for some subjective probability measure $\mu(\cdot)$ and general (not necessarily expected utility) risk preference function $V(x_1, p_1; \dots; x_m, p_m)$. Machina and Schmeidler (1995) give a similar derivation in an Anscombe-Aumann setting.

Although such formulations succeed in bringing additive numerical probabilities into the subjective framework, the *subjective* probabilities they generate do not exhibit all of the properties of exogenous, *objective* probabilities. Since each of the above derivations requires some type of global condition on act preferences,³ they will not generate well-defined subjective probabilities for individuals whose preferences or beliefs exhibit state-dependence, ambiguity, or otherwise fail to satisfy these global conditions. Thus, subjective probabilities do not exhibit the property of *universal existence across agents* that objective probabilities exhibit. Furthermore, even when subjective probabilities do exist, they generally differ across individuals, and hence do not exhibit the property of *unanimity across agents* that objective probabilities exhibit.

But perhaps surprisingly, not all subjective events are created equal, and in particular, some subjective events turn out to be “more objective” than others. An early example of this phenomenon (described below) was offered by Poincaré (1912), who considered a Euclidean state space $S = [\underline{s}, \bar{s}] \subset R^1$ and constructed a sequence of events $\{E_n\}_{n=1}^\infty \subset S$ with the property that $\lim_{n \rightarrow \infty} \mu(E_n) = 1/2$ for every subjective probability measure $\mu(\cdot)$ over S with a bounded-derivative density function. Thus, as $n \rightarrow \infty$, the likelihoods of the events E_n exhibit a *unanimity* property not shared by general subjective events in S . But since Poincaré's results and its modern extensions assume the existence of a (suitably regular) subjective probability measure $\mu(\cdot)$, the *universal existence* property of these limiting likelihoods only extends to the class of agents who already possess such well-defined subjective probability measures over S .

² Sarin and Wakker (1997) offer a single-stage version of the Anscombe-Aumann results, which retains their assumption of both objective and subjective uncertainty, as well as expected utility risk preferences.

³ Namely, exogenously assumed expected utility risk preferences (Ramsey and Anscombe-Aumann), the Sure-Thing Principle (Savage), or a global Comparative Probability or Replacement Axiom (Machina-Schmeidler).

The purpose of this paper is to generalize the concept of such events, expand the set of “objective” properties that they arbitrarily closely approximate, and expand the universe of agents for whom these properties hold. Specifically, we show that

- Minimal (e.g., Euclidean) structural assumptions on a purely subjective state space S , and the property that act preferences are merely “smooth in the events,” imply the existence of what we term *almost-objective events* in S . Such events will approximate – and in the limit attain – all the properties of Anscombe-Aumann objective roulette wheel events for all individuals with event-smooth preferences, whether or not they possess a subjective probability measure $\mu(\cdot)$.
- These limiting properties include unanimously agreed-upon revealed likelihoods that are invariant to conditioning on any subjective event in S , state-independence and probabilistic sophistication for all bets and mixtures based on these events (whether or not the individual’s preferences are state-independent or probabilistically sophisticated in general), and linearity in these probabilities for all state-independent or state-dependent expected utility maximizers.
- Events of this form constitute the basis of most real-world “objective” randomization devices, and the bets and mixtures they generate.

Since they approximate the properties of objectively uncertain events for all event-smooth individuals, almost-objective events, and the acts and mixtures based on them, can serve as completely subjective substitutes for many of the standard uses of exogenous objective uncertainty. In addition to the applications that directly follow from their unanimous likelihood properties:

- Almost-objective events can replace an exogenously specified objective roulette wheel, to allow for a direct Anscombe-Aumann type derivation of exact subjective probability within a completely subjective framework.
- Almost-objective mixtures of subjective acts can replace objective probability mixtures, to allow for an extension of the analysis of Machina (1982) to subjective uncertainty, yielding a joint generalization of both expected utility analysis and subjective probability analysis to all event-smooth individuals, whether or not they satisfy the expected utility hypothesis or the hypothesis of probabilistic sophistication.⁴
- The fact that all event-smooth individuals have agreed-upon limiting likelihoods over almost-objective events, and probabilistically sophisticated limiting preferences over almost-objective acts, allows us to uniquely derive and compare their “objective risk preferences” from their preferences over completely subjective acts, in spite of any state-dependence, absence of probabilistic beliefs, attitudes toward ambiguity, or interpersonal differences in these attributes.

The following section provides the setting for the analysis and gives an example (“almost-ethically-neutral events”) that illustrates the general idea of the approach. Section 3 formalizes the notions of an almost-objective event, an almost-objective act, and an almost-objective mixture of subjective acts, and examines the revealed likelihood and betting properties of these constructs. Section 4 presents the above-mentioned applications, as well as a discussion of how some sources of gains from trade under uncertainty lead to bets based on almost-objective events, and others sources lead to bets on general subjective events. Section 5 offers comparisons with other results in the literature, and discusses extensions to more general types of events, more general state spaces, and more general preferences. Proofs of theorems are in an Appendix.

⁴ The basic idea of this application is sketched out in Section 4.3 and is fully developed in Machina (2002).

2. SETTING AND ILLUSTRATION

2.1 Setting

Although we shall conclude by showing how it can be developed more generally, the formal analysis of this paper is conducted within the following framework:

$\mathcal{X} = \{..., x, ...\}$	arbitrary space of <i>outcomes</i>
$S = [\underline{s}, \bar{s}] \subset R^1$	set of <i>states of nature</i> , with uniform Lebesgue measure $\lambda(\cdot)$
$\mathcal{E} = \{..., E, ...\}$	algebra of <i>events</i> (each a finite union of intervals) in S
$f(\cdot) = [x_1 \text{ on } E_1; ..., x_m \text{ on } E_m]$	finite-outcome <i>act</i> , yielding outcome x_i in event E_i , for some $\mathcal{E}$ -measurable partition $\{E_1, ..., E_m\}$ of S
$\mathcal{A} = \{..., f(\cdot), ...\}$	set of all finite-outcome, $\mathcal{E}$ -measurable acts on S
$W(\cdot)$ and $\succsim$	<i>preference function</i> and corresponding <i>preference relation</i> over $\mathcal{A}$

Following Savage (1954), an event E is said to be *null* for a preference function $W(\cdot)$ if $W(x^* \text{ on } E; f(\cdot) \text{ on } \sim E) = W(x \text{ on } E; f(\cdot) \text{ on } \sim E)$ for all x^*, x and $f(\cdot)$, and our event-continuity assumption (defined below) will imply that every event E of Lebesgue measure zero is null. Each preference function $W(\cdot)$ is also assumed to be strictly *outcome-monotonic*, in the sense that if two outcomes satisfy $x^* \succ x$ (that is, if $W(x^* \text{ on } S) > W(x \text{ on } S)$), then $W(x^* \text{ on } E; f(\cdot) \text{ on } \sim E) > W(x \text{ on } E; f(\cdot) \text{ on } \sim E)$ for all $f(\cdot)$ and all nonnull E .

It is important to note that this framework is one of completely subjective uncertainty. In particular, almost-objective events will all be subsets of S (in fact, elements of $\mathcal{E}$), almost-objective acts and almost-objective mixtures of subjective acts will all be mappings from S to $\mathcal{X}$ (elements of $\mathcal{A}$), and we shall only consider preference functions $W(\cdot)$ defined over subjective acts $f(\cdot) \in \mathcal{A}$. No “objective randomization device” will be appended to this structure.

A preference function $W(\cdot)$ over subjective acts $f(\cdot) = [x_1 \text{ on } E_1; ..., x_m \text{ on } E_m]$ in $\mathcal{A}$ may or may not be expected utility, it may or may not be state-dependent, and it may or may not be probabilistically sophisticated. Thus, it may or may not take one of the following forms:

$W_{SEU}(f(\cdot)) \equiv \int_S U(f(s)) \cdot d\mu(s)$ $\equiv \sum_{i=1}^m U(x_i) \cdot \mu(E_i)$	<i>state-independent expected utility</i> , for some utility function $U(\cdot)$ and subjective probability measure $\mu(\cdot)$
$W_{SDEU}(f(\cdot)) \equiv \int_S U(f(s) s) \cdot d\mu(s)$ $\equiv \sum_{i=1}^m \int_{E_i} U(x_i s) \cdot d\mu(s)$	<i>state-dependent expected utility</i> , for some state-dependent utility function $U(\cdot s)$ and subjective probability measure $\mu(\cdot)$
$W_{PS}(f(\cdot)) \equiv V(...; x_i, \mu(E_i); ...)$	<i>probabilistically sophisticated</i> , for some preference function $V(\cdot)$ over lotteries and subjective probability measure $\mu(\cdot)$

As mentioned, we shall also assume that each preference functions $W(\cdot)$ over $\mathcal{A}$ is “smooth in the events.” Although we postpone the formal definition of event-smoothness to Section 2.4, the above three forms will turn out to be event-smooth provided they satisfy the conditions:

$W_{SEU}(\cdot)$: the subjective probability measure $\mu(\cdot)$ has a continuous density $m(\cdot)$ over S

$W_{SDEU}(\cdot)$: for each x , its *evaluation measure* $\Phi_x(E) \equiv \int_E U(x|s) \cdot d\mu(s)$ has a continuous density⁵

$W_{PS}(\cdot)$: $\mu(\cdot)$ has a continuous density and $V(\cdot)$ is differentiable in the probabilities

⁵ Outcome-monotonicity also implies that if $x^* \succ x$ then $\Phi_{x^*}(E) > \Phi_x(E)$ for all nonnull E .

2.2 Properties of Purely Objective Events

As noted, almost-objective events, as well as the acts and mixtures based on these events, will be completely subjective constructs. However, to motivate their desired properties, we review the characteristic properties of *purely objective events*, as generated by idealized exogenous coins, dice or roulette wheels, both in isolation and in the presence of subjective uncertainty:

1. *Unanimous, Outcome-Invariant Revealed Likelihoods*: All individuals exhibit identical, outcome-invariant revealed likelihoods over purely objective events – corresponding, of course, to their objective probabilities. In contrast, betting preferences over a pair of subjective events can differ across individuals (when they have different subjective beliefs), can depend upon the prizes assigned to the two events (under state-dependence), or even depend upon the prizes assigned to mutually exclusive events, as in the Ellsberg (1961) Paradox.
2. *Independence from Subjective Realizations*: Under joint objective $\times$ subjective uncertainty, purely objective event likelihoods are independent of the realization of any given subjective event, whether or not that event happens to be assigned a subjective probability. That is, the event likelihoods for an exogenous objective coin, die or roulette wheel are invariant to whether any given subjective event E does or does not occur.
3. *Probabilistic Sophistication over Objective Lotteries*: It is almost a truism that all individuals evaluate objective lotteries $\mathbf{P} = (x_1, p_1; \dots; x_m, p_m)$ solely according to their outcomes and corresponding objective likelihoods, via a preference function of the form $V(x_1, p_1; \dots; x_m, p_m)$.
4. *Reduction of Objective $\times$ Subjective Uncertainty*: Standard *reduction of compound uncertainty* assumptions⁶ imply that all individuals – whether or not they are expected utility, state-independent or probabilistically sophisticated – evaluate an *objective mixture* $\alpha \cdot f(\cdot) + (1-\alpha) \cdot f^*(\cdot)$ of subjective acts $f(\cdot) = [x_1 \text{ on } E_1; \dots; x_m \text{ on } E_m]$ and $f^*(\cdot) = [x_1^* \text{ on } E_1^*; \dots; x_m^* \text{ on } E_m^*]$ solely according to its induced map $[\dots; (x_i, \alpha; x_j^*, 1-\alpha) \text{ on } E_i \cap E_j^*; \dots]$ from events to lotteries, so any other mixture $\hat{\alpha} \cdot \hat{f}(\cdot) + (1-\hat{\alpha}) \cdot \hat{f}^*(\cdot)$ that induces the same map will be indifferent.⁷

Besides the above properties, which apply to all individuals, exogenous objectively uncertain events also exhibit two more specialized properties, one for all probabilistically sophisticated individuals, and one for all (state-independent or state-dependent) expected utility individuals:

5. *Under Probabilistic Sophistication, Independence of Objective and Subjective Likelihoods*: Whenever subjective event likelihoods are all well-defined – that is, whenever the individual is probabilistically sophisticated with subjective probability measure $\mu(\cdot)$ – these likelihoods are invariant to the realization of exogenous objective events, and vice versa.
6. *Under Expected Utility, Linearity in Objective Likelihoods*: Under objective uncertainty, expected utility is linear in objective probabilities ($V_{EU}(x_1, p_1; \dots; x_m, p_m) \equiv \sum_{i=1}^m U(x_i) \cdot p_i$) and in objective mixtures of lotteries ($V_{EU}(\alpha \cdot \mathbf{P} + (1-\alpha) \cdot \mathbf{P}^*) \equiv \alpha \cdot V_{EU}(\mathbf{P}) + (1-\alpha) \cdot V_{EU}(\mathbf{P}^*)$). Under objective $\times$ subjective uncertainty, state-independent and state-dependent expected utility are both linear in objective mixtures of subjective acts: The state-independent expected utility of the mixture $\alpha \cdot f(\cdot) + (1-\alpha) \cdot f^*(\cdot)$ is $\alpha \cdot \int_s U(f(s)) \cdot d\mu(s) + (1-\alpha) \cdot \int_s U(f^*(s)) \cdot d\mu(s)$ and the state-dependent expected utility of this mixture is $\alpha \cdot \int_s U(f(s)|s) \cdot d\mu(s) + (1-\alpha) \cdot \int_s U(f^*(s)|s) \cdot d\mu(s)$.

As shown in Theorems 1–6 below, although almost-objective events, acts and mixtures are all completely subjective, they will approximate – and in the limit attain – each of these properties.

⁶ E.g., Anscombe and Aumann (1963), Machina and Schmeidler (1995).

⁷ For example, since the mixtures $\frac{1}{2} \cdot [x_1 \text{ on } E_1; x_2 \text{ on } E_2] + \frac{1}{2} \cdot [x_0 \text{ on } S]$ and $\frac{1}{2} \cdot [x_0 \text{ on } E_1; x_2 \text{ on } E_2] + \frac{1}{2} \cdot [x_1 \text{ on } E_1; x_0 \text{ on } E_2]$ induce the same map $[(x_1, \frac{1}{2}; x_0, \frac{1}{2}) \text{ on } E_1; (x_2, \frac{1}{2}; x_0, \frac{1}{2}) \text{ on } E_2]$ from events to lotteries, they will be indifferent.

2.3 Almost-Ethically-Neutral Events

Of course, there exists no subjective event $\bar{E}$ in the state space S that can be considered “objectively ethically neutral” with respect to *all* event-smooth preferences – that is, no $\bar{E}$ that satisfies

$$(1) \quad W(x^* \text{ on } \bar{E}; x \text{ on } \sim \bar{E}) = W(x \text{ on } \bar{E}; x^* \text{ on } \sim \bar{E}) \quad \text{all } x, x^* \in \mathcal{X}$$

for *every* event-smooth $W(\cdot)$ over $\mathcal{A}$. This is neither unexpected nor inappropriate – as noted above, one of the primary motivations for the subjective uncertainty approach is precisely to *allow* for such diversity (or even nonexistence) of beliefs.

Nevertheless, some subjective events come “closer” to being ethically neutral than others. For arbitrary positive integer n , partition the state space $S = [\underline{s}, \bar{s}]$ into the n equal-length intervals

$$(2) \quad \left[\underline{s}, \underline{s} + \frac{\lambda_s}{n}\right), \quad \dots \quad \left[\underline{s} + i \cdot \frac{\lambda_s}{n}, \underline{s} + (i+1) \cdot \frac{\lambda_s}{n}\right), \quad \dots \quad \left[\underline{s} + (n-1) \cdot \frac{\lambda_s}{n}, \bar{s}\right]$$

where $\lambda_s = \lambda(S) = \bar{s} - \underline{s}$, and define the event $\bar{E}_n \subset S$ by

$$(3) \quad \bar{E}_n = \bigcup_{i=0}^{n-1} \left[\underline{s} + i \cdot \frac{\lambda_s}{n}, \underline{s} + (i + \frac{1}{2}) \cdot \frac{\lambda_s}{n}\right]$$

that is, as the union of the *left halves* of the intervals in (2). For uniform Lebesgue measure $\lambda(\cdot)$ on S , it is clear that $\bar{E}_n$ exactly satisfies $\lambda(\bar{E}_n) \equiv \frac{1}{2} \cdot \lambda(S)$ for each n .

Consider the *subjective probabilities* of such events. Given an arbitrary continuous-density subjective probability measure $\mu(\cdot)$, say from a preference function $W_{SEU}(\cdot)$, $W_{SDEU}(\cdot)$ or $W_{PS}(\cdot)$, the events $\bar{E}_n$ generally *do not* satisfy $\mu(\bar{E}_n) = \frac{1}{2}$. However, as n grows, the events $\bar{E}_n$ and their complements $\sim \bar{E}_n = S - \bar{E}_n$ will *arbitrarily closely approximate* this property – that is, they satisfy

$$(4) \quad \lim_{n \rightarrow \infty} \mu(\bar{E}_n) = \lim_{n \rightarrow \infty} \mu(\sim \bar{E}_n) = \frac{1}{2}$$

This result can be illustrated in Figure 1, where the event $\bar{E}_n$ appears as the union of the solid intervals along the entire horizontal line (the state space S). For any subjective probability measure $\mu(\cdot)$ with a continuous density function $m(\cdot)$, the probability $\mu(\bar{E}_n)$ consists of the total shaded area lying below the density function, which for large n is seen to approach *half* of the total area below the density function – or in other words, to approach the value $\frac{1}{2}$.

Figure 1

Almost-ethically-neutral events $\bar{E}_n$ satisfy $\mu(\bar{E}_n) \approx \frac{1}{2}$ as well as $\mu(\bar{E}_n \cap E) \approx \frac{1}{2} \cdot \mu(\bar{E}_n \cap E)$ for all continuous-density $\mu(\cdot)$

As mentioned, examples like (3) date back at least to Poincaré (1912, Sects.92-93), who thought of the state space $S = [\underline{s}, \bar{s}]$ and events $\bar{E}_n, \sim\bar{E}_n$ from (3) as a spinning wheel divided into a large number of equal-sized alternating red and black sectors of angular width ε . Denoting the total number of angular rotations of the wheel by ω , Poincaré gives the following result linking ω 's subjective density $m(\cdot)$ with the probability that the spin results in red (or black):

THEOREM (POINCARÉ (1912)): For any differentiable density function $m(\cdot)$ with support lying in an interval $[0, A]$, and such that $|m'(\cdot)| < M$ over $[\underline{s}, \bar{s}]$:

$$(5) \quad |2 \cdot \text{prob}(\text{red}) - 1| < M \cdot (\bar{s} - \underline{s}) \cdot \varepsilon$$

Other authors have provided similar results, such as:

- Feller (1971, pp.62-63), who considers the case when $m(\cdot)$ is unimodal and small at its mode
- Kemperman (1975), who considers the case of k 'th order differentiable densities
- Good (1986, pp.162-164), who considers the case when $m(\cdot)$ is a mixture of distributions
- Diaconis and Engel (1986), who consider a two-dimensional state space

Similar results and/or discussions appear in Fréchet (1952, pp.3-8,16), Diaconis and Keller (1989) and the posthumously published lecture of Savage (1973), which contains a version of Figure 1.

Results such as equation (4), Poincaré's Theorem, and the above extensions show that as $n \rightarrow \infty$, the events $\bar{E}_n$ and $\sim\bar{E}_n$ will approximate the *equal subjective probability* property with respect to any suitably regular (e.g., continuous-density) subjective probability measure $\mu(\cdot)$ on S . However, the above results fall short of establishing that $\bar{E}_n$ and $\sim\bar{E}_n$ approximate the *objective ethical neutrality* property (1) for all event-smooth $W(\cdot)$, for two reasons:

- Since they *assume the existence* of a prior subjective probability measure $\mu(\cdot)$ (or density $m(\cdot)$) on S , these results can only be applied to preference functions that are *based on* such measures, such as the forms $W_{SEU}(\cdot)$, $W_{SDEU}(\cdot)$ or $W_{PS}(\cdot)$, as opposed to general event-smooth $W(\cdot)$.
- Even for a measure-based form such as $W_{SDEU}(\cdot)$, the equal-measure condition $\mu(E) = \mu(\sim E)$ for a pair of events E and $\sim E$ need not imply the ethical-neutrality property $W(x^* \text{ on } \bar{E}; x \text{ on } \sim\bar{E}) \equiv_{x, x^*} W(x \text{ on } \bar{E}; x^* \text{ on } \sim\bar{E})$, for reasons of state-dependence.

In fact, we shall be able to establish that the events $\bar{E}_n$ and $\sim\bar{E}_n$ *do* approximate the property of objective ethical neutrality, as well as other properties of exogenously objective 50:50 events, for all event-smooth preference functions $W(\cdot)$ over $\mathcal{A}$.⁸

Consider first the *revealed likelihoods properties* of the events $\bar{E}_n$ and $\sim\bar{E}_n$. Since Theorem 0 below will establish that every continuous-density evaluation measure $\Phi_x(E) \equiv \int_E U(x|s) \cdot d\mu(s)$ of a preference function $W_{SDEU}(\cdot)$ exhibits the limiting measure property $\lim_{n \rightarrow \infty} \Phi_x(\bar{E}_n) = \lim_{n \rightarrow \infty} \Phi_x(\sim\bar{E}_n) = \frac{1}{2} \cdot \Phi_x(S)$, we have that all event-smooth $W_{SEU}(\cdot)$, $W_{SDEU}(\cdot)$ or $W_{PS}(\cdot)$ will satisfy the identities⁹

$$(6) \quad \begin{aligned} \lim_{n \rightarrow \infty} W_{SEU}(x^* \text{ on } \bar{E}_n; x \text{ on } \sim\bar{E}_n) &\equiv_{x^*, x} \lim_{n \rightarrow \infty} W_{SEU}(x \text{ on } \bar{E}_n; x^* \text{ on } \sim\bar{E}_n) \equiv_{x^*, x} \frac{1}{2} \cdot U(x^*) + \frac{1}{2} \cdot U(x) \\ \lim_{n \rightarrow \infty} W_{SDEU}(x^* \text{ on } \bar{E}_n; x \text{ on } \sim\bar{E}_n) &\equiv_{x^*, x} \lim_{n \rightarrow \infty} W_{SDEU}(x \text{ on } \bar{E}_n; x^* \text{ on } \sim\bar{E}_n) \equiv_{x^*, x} \frac{1}{2} \cdot \Phi_{x^*}(S) + \frac{1}{2} \cdot \Phi_x(S) \\ \lim_{n \rightarrow \infty} W_{PS}(x^* \text{ on } \bar{E}_n; x \text{ on } \sim\bar{E}_n) &\equiv_{x^*, x} \lim_{n \rightarrow \infty} W_{PS}(x \text{ on } \bar{E}_n; x^* \text{ on } \sim\bar{E}_n) \equiv_{x^*, x} V(x^*, \frac{1}{2}; x, \frac{1}{2}) \end{aligned}$$

⁸ All the properties stated in this subsection will follow from the theorems in Section 3.

⁹ The right-hand sum in the second line of (6) can be equivalently written as $\frac{1}{2} \cdot \int_S U(x^*|s) \cdot d\mu(s) + \frac{1}{2} \cdot \int_S U(x|s) \cdot d\mu(s)$.

so that $\bar{E}_n$ and $\sim\bar{E}_n$ can be said to constitute *almost-ethically-neutral events* for all event-smooth preference functions of the form $W_{SEU}(\cdot)$, $W_{SDEU}(\cdot)$ or $W_{PS}(\cdot)$. More generally Theorem 1 will shown that $\bar{E}_n$ and $\sim\bar{E}_n$ satisfy the almost-ethically-neutrality condition

$$(7) \quad \lim_{n \rightarrow \infty} W(x^* \text{ on } \bar{E}_n; x \text{ on } \sim\bar{E}_n) = \lim_{n \rightarrow \infty} W(x \text{ on } \bar{E}_n; x^* \text{ on } \sim\bar{E}_n) \quad \text{all } x, x^* \in \mathcal{X}$$

for all event-smooth preference functions $W(\cdot)$ on $\mathcal{A}$.

In addition to unanimous, outcome-independent, and equal limiting revealed likelihoods, the almost-ethically-neutral events $\bar{E}_n$ and $\sim\bar{E}_n$ also approximate the objective property of *independence with respect to the realization of any fixed subjective event* $E \subseteq S$, in the sense that for any such E , the *joint events* $\bar{E}_n \cap E$ and $\sim\bar{E}_n \cap E$ also have unanimous, outcome-independent, equal limiting revealed likelihoods. That is, these events satisfy the conditional measure properties

$$(8) \quad \lim_{n \rightarrow \infty} \mu(\bar{E}_n \cap E) = \lim_{n \rightarrow \infty} \mu(\sim\bar{E}_n \cap E) = \frac{1}{2} \cdot \mu(E) \quad \lim_{n \rightarrow \infty} \Phi_x(\bar{E}_n \cap E) = \lim_{n \rightarrow \infty} \Phi_x(\sim\bar{E}_n \cap E) = \frac{1}{2} \cdot \Phi_x(E)$$

for all continuous-density $\mu(\cdot)$ and $\Phi_x(\cdot)$, as well as the conditional revealed likelihood identities

$$(9) \quad \lim_{n \rightarrow \infty} W \begin{pmatrix} x^* \text{ on } \bar{E}_n \cap E \\ x \text{ on } \sim\bar{E}_n \cap E \\ f(s) \text{ elsewhere} \end{pmatrix} \equiv \lim_{n \rightarrow \infty} W \begin{pmatrix} x \text{ on } \bar{E}_n \cap E \\ x^* \text{ on } \sim\bar{E}_n \cap E \\ f(s) \text{ elsewhere} \end{pmatrix} \quad \text{all } x^*, x, f(\cdot)$$

for all event-smooth $W(\cdot)$ on $\mathcal{A}$.

The limiting measure property $\lim_{n \rightarrow \infty} \mu(\bar{E}_n \cap E) = \frac{1}{2} \cdot \mu(E)$ from (8) is also illustrated in Figure 1 for an arbitrary interval event E . For any probability measure $\mu(\cdot)$ with continuous density $m(\cdot)$, the probability $\mu(\bar{E}_n \cap E)$ consists of the shaded area lying within the boundaries of the interval E and below the density function, which for large n is seen to approach half the total area within the boundaries of E and below the density function, or in other words, to approach $\frac{1}{2} \cdot \mu(E)$.

It is worth pointing out the following properties of almost-ethically-neutral events, which will also be shared by almost-objective events:¹⁰ First, since $\bar{E}_n$ and $\sim\bar{E}_n$ are subsets of S , they are completely subjective events. Second, the key assumptions needed to generate them and their properties is that the state space S have a Euclidean structure (this is somewhat generalized in Section 5.2), and that preferences are event-smooth. Third, although the *limiting event* “ $\bar{E}_\infty$ ” from (3) does not actually exist, event-smoothness will imply that the *limiting preference function values* (7) and (9) exist at $n = \infty$. Fourth, although a limit result such as (7) implies convergence of $W(\cdot)$ ’s *ordinal valuations* of the bets $[x^* \text{ on } \bar{E}_n; x \text{ on } \sim\bar{E}_n]$ and $[x \text{ on } \bar{E}_n; x^* \text{ on } \sim\bar{E}_n]$ – as measured, for example, by their certainty-equivalents – this result yields no information on $W(\cdot)$ ’s *ordinal rankings* of these bets, even for arbitrarily large n (we will, however, obtain limiting ordinal rankings once we consider events with *unequal* limiting likelihoods). Finally, as with the convergence of linear approximations or Taylor series, the *rate* of convergence in results (4)–(9) will generally depend upon the specifics of the measures $\mu(\cdot)$, $\Phi_x(\cdot)$ or the function $W(\cdot)$.

The special structure of the almost-ethically-neutral events $\bar{E}_n$ and $\sim\bar{E}_n$ suggests they might not be very relevant “in practice.” In fact, they constitute a fully operational and frequently used method of approximating objective uncertainty in a purely subjective setting. For example, if the subjective state space $S = [0^\circ, 100^\circ]$ refers to the temperature at Times Square at noon tomorrow, then the events $\bar{E}_{100}$ and $\sim\bar{E}_{100}$ correspond to the temperature being even versus odd, and $\bar{E}_{1,000}$ and $\sim\bar{E}_{1,000}$ correspond to its first decimal digit being even versus odd, etc. At this point, virtual-

¹⁰ The second, third and fourth of properties are discussed more fully below.

ly all bettors with event-smooth act preferences, regardless of differing beliefs, state-dependence or ambiguity aversion – will approach indifference between betting on $\bar{E}_{1,000}$ versus $\sim \bar{E}_{1,000}$. Similar computer-generated events based on the k 'th decimal of the second in which the return key is pressed will also generate revealed likelihoods that are unanimously viewed as equal, as well as stochastically independent of any fixed subjective event (such as tomorrow's temperature falling in the range $[40^\circ, 50^\circ]$). As seen in the Poincaré example, the events {red, black} on a roulette wheel, as well as {heads, tails} in a coin flip, are *precisely* almost-ethically-neutral events, where the subjective state s is the force with which the wheel is spun or the coin is flipped.

2.4 Event-Smooth Preferences over Subjective Acts

The previous section established almost-ethically-neutrality property (6) for all $W_{SEU}(\cdot)$, $W_{SDEU}(\cdot)$ and $W_{PS}(\cdot)$ with continuous-density subjective probability measure $\mu(\cdot)$ or continuous-density evaluation functions $\{\Phi_x(\cdot) | x \in \mathcal{X}\}$. In this section we lay the groundwork for generalizing this property, by describing the general class of “event-smooth” preference functions $W(\cdot)$ over $\mathcal{A}$.

The notion of a preference function $W(\cdot)$ being “event-differentiable” is a generalization of the standard mathematical idea of differentiability of a set function, and versions of this idea have been developed and applied by Epstein (1999) in his characterization of uncertainty aversion, and by Machina (2002) in an analysis of the robustness of the classical model of risk preferences and beliefs.¹¹ Intuitively, event-differentiability is simply the notion that $W(f(\cdot)) = W(x_1 \text{ on } E_1; \dots; x_m \text{ on } E_m)$ responds smoothly to differential changes in the events $E_1, \dots, E_m$, as opposed to changes in the outcomes $x_1, \dots, x_m$. Thus, for the two-parameter family of acts $f_{\alpha, \beta}(\cdot) = [x_1 \text{ on } [\underline{s}, \underline{s} + \alpha]; x_2 \text{ on } [\underline{s} + \alpha, \underline{s} + \beta]; x_3 \text{ on } [\underline{s} + \beta, \bar{s}]]$, $W(f_{\alpha, \beta}(\cdot))$ would be a smooth function of α and β . The following formalization event-differentiability and “event-smoothness” is based on Machina (2002). Although required for the proofs in the Appendix, these concepts will not be invoked in either the intuitive or formal exposition of the text, and a reader wishing to skip the formal details of event-smoothness can proceed to Section 3 without loss of continuity.

We define a preference function $W(\cdot)$ over $\mathcal{A}$ to be *event-continuous* if

$$(10) \quad \lim_{\delta(f(\cdot), f_0(\cdot)) \rightarrow 0} W(f(\cdot)) = W(f_0(\cdot)) \quad \text{all } f_0(\cdot) \in \mathcal{A}$$

where the distance function $\delta(\cdot, \cdot)$ between two acts is defined by

$$(11) \quad \delta(f(\cdot), f_0(\cdot)) \equiv \lambda \{s \in S | f(s) \neq f_0(s)\}$$

that is, as the Lebesgue measure of the set on which the acts differ. Just as differentiability with respect to real variables can be thought of as “local linearity” in those variables, differentiability with respect to an act $f(\cdot)$'s events $\{f^{-1}(x) | x \in \mathcal{X}\}$ can be thought of as “local additivity” in these events. Formally, $W(\cdot)$ is *event-additive* if there exists a family of (signed) *evaluation measures* $\{\Phi_x(\cdot) | x \in \mathcal{X}\}$ such that it takes the form $W(x_1 \text{ on } E_1; \dots; x_m \text{ on } E_m) \equiv \Phi_{x_1}(E_1) + \dots + \Phi_{x_m}(E_m)$, that is, where each outcome's event E_i is additively evaluated by that outcome's evaluation measure $\Phi_{x_i}(\cdot)$, and the terms are then summed. We can equivalently write this property as¹²

$$(12) \quad W(f(\cdot)) = \sum_{x \in \mathcal{X}} \Phi_x(f^{-1}(x)) \quad \text{all } f(\cdot) \in \mathcal{A}$$

By event-continuity, each evaluation measure $\Phi_x(\cdot)$ will be absolutely continuous with respect to

¹¹ See also Rosenmuller (1972), as well as Epstein and Marinacci's (2001) application to the core of large games.

¹² Our restriction to finite-outcome acts ensures that all but a finite number of terms in the following sum will be zero.

Lebesgue measure, and thus have a signed *evaluation density* $\phi_x(\cdot)$. The expected utility forms $W_{SEU}(f(\cdot)) \equiv \sum_{i=1}^m U(x_i) \cdot \mu(E_i)$ and $W_{SDEU}(f(\cdot)) \equiv \sum_{i=1}^m \int_{E_i} U(x_i|s) \cdot d\mu(s)$ are both event-additive, with

$$(13) \quad \text{for } W_{SEU}(\cdot): \begin{cases} \Phi_x(E) \equiv U(x) \cdot \mu(E) \\ \phi_x(s) \equiv U(x) \cdot m(s) \end{cases} \quad \text{for } W_{SDEU}(\cdot): \begin{cases} \Phi_x(E) \equiv \int_E U(x|s) \cdot d\mu(s) \\ \phi_x(s) \equiv U(x|s) \cdot m(s) \end{cases}$$

In fact, as shown in Machina (2002), an event-continuous $W(\cdot)$ will be event-additive *if and only if* it takes the state-independent or state-dependent expected utility form $W_{SEU}(\cdot)$ or $W_{SDEU}(\cdot)$.

Under event-additivity, the *change* in $W(\cdot)$ in going from an act $f_0(\cdot)$ to an act $f(\cdot)$ can be expressed in terms of the *changes* (expansions and contractions) $\Delta E_x^+ = f^{-1}(x) - f_0^{-1}(x)$ and $\Delta E_x^- = f_0^{-1}(x) - f^{-1}(x)$ in each outcome's event E_x , via the formula

$$(14) \quad W(f(\cdot)) - W(f_0(\cdot)) = \sum_{x \in \mathcal{X}} \Phi_x(f^{-1}(x)) - \sum_{x \in \mathcal{X}} \Phi_x(f_0^{-1}(x)) = \sum_{x \in \mathcal{X}} \Phi_x(\Delta E_x^+) - \sum_{x \in \mathcal{X}} \Phi_x(\Delta E_x^-)$$

An event-continuous $W(\cdot)$ is thus said to be *event-differentiable* at act $f_0(\cdot)$ if it is “locally” event-additive there – that is, if there exists a family of absolutely continuous *local evaluation measures* $\{\Phi_x(\cdot; f_0) | x \in \mathcal{X}\}$, with corresponding *local evaluation densities* $\{\phi_x(\cdot; f_0) | x \in \mathcal{X}\}$, such that

$$(15) \quad \begin{aligned} W(f(\cdot)) - W(f_0(\cdot)) &= \sum_{x \in \mathcal{X}} \Phi_x(f^{-1}(x); f_0) - \sum_{x \in \mathcal{X}} \Phi_x(f_0^{-1}(x); f_0) + o(\delta(f(\cdot), f_0(\cdot))) \\ &= \sum_{x \in \mathcal{X}} \Phi_x(\Delta E_x^+; f_0) - \sum_{x \in \mathcal{X}} \Phi_x(\Delta E_x^-; f_0) + o(\delta(f(\cdot), f_0(\cdot))) \end{aligned}$$

To establish the results of this paper it is necessary to impose some regularity on how much each outcome's local evaluation densities $\phi_x(\cdot; \cdot)$ can vary in their arguments s and $f(\cdot)$. Although the space of acts $\mathcal{A}$ is not compact with respect to the distance function $\delta(\cdot, \cdot)$, our regularity conditions correspond to the properties that continuous $\phi_x(\cdot; \cdot)$ functions would exhibit if their domain $S \times \mathcal{A}$ was compact, namely:

$$(16) \quad \begin{aligned} &\text{for each outcome } x: \phi_x(s; f) \text{ is jointly uniformly continuous in } s \text{ and } f(\cdot) \\ &\text{for each outcome } x: \phi_x(s; f) \text{ is bounded above and below on } S \times \mathcal{A} \\ &\text{for each pair } x^* \succ x: \Phi_{x^*}(E; f) - \Phi_x(E; f) \text{ is bounded above 0, uniformly in } f(\cdot) \\ &\text{and nonnull event } E \end{aligned}$$

We thus define an event-differentiable preference function $W(\cdot)$ on $\mathcal{A}$ to be *event-smooth* if it satisfies these properties, which can be stated more formally as¹³

$$(16)' \quad \begin{aligned} &\text{for each } x \in \mathcal{X} \text{ and } \varepsilon > 0, \text{ there exists some } \delta_{x, \varepsilon} > 0 \text{ such that } |s' - s| < \delta_{x, \varepsilon} \text{ and } \\ &\delta(f'(\cdot), f(\cdot)) < \delta_{x, \varepsilon} \text{ implies } |\phi_x(s'; f') - \phi_x(s; f)| < \varepsilon \\ &\text{for each } x \in \mathcal{X} \text{ there exist } \underline{\phi}_x \text{ and } \bar{\phi}_x \text{ such that } \underline{\phi}_x < \phi_x(s; f) < \bar{\phi}_x \text{ for all } s \in S \text{ and all } f(\cdot) \in \mathcal{A} \\ &\text{for each pair of outcomes } x^* \succ x \text{ and nonnull } E \in \mathcal{E} \text{ there exists } \underline{\Phi}_{x^*, x, E} > 0 \text{ such that} \\ &\Phi_{x^*}(E; f) - \Phi_x(E; f) > \underline{\Phi}_{x^*, x, E} \text{ for all } f(\cdot) \in \mathcal{A} \end{aligned}$$

The application of these ideas to the global analysis of event-smooth $W(\cdot)$ is given in the Appendix, prior to the proofs of the theorems.

¹³ In line with our approach of making no restrictions on the nature or extent of the outcome space, each of the values $\delta_{x, \varepsilon}$, $\underline{\phi}_x$, $\bar{\phi}_x$ and $\underline{\Phi}_{x^*, x, E}$ in (16)' is defined with respect to a *given* outcome x or outcome pair x^*, x , and we make no uniformity or boundedness assumptions on any of these values as x or x^* range over $\mathcal{X}$.

3. ALMOST-OBJECTIVE EVENTS, ACTS AND MIXTURES UNDER COMPLETELY SUBJECTIVE UNCERTAINTY

The above ideas can be extended to a general specification of “almost-objective” uncertainty, based solely on the subjective space $S = [\underline{s}, \bar{s}]$, the family of subjective acts $f(\cdot) \in \mathcal{A}$, and the family of all event-smooth preference functions $\mathcal{W}(\cdot)$ over $\mathcal{A}$.

3.1 Almost-Objective Events: Construction and Measure Properties

Although the almost-ethically-neutral events $\bar{E}_n$ from (3) were constructed from the *left halves* $[\underline{s} + i \cdot \frac{\lambda_S}{n}, \underline{s} + (i + \frac{1}{2}) \cdot \frac{\lambda_S}{n}]$ of the equal-length intervals (2), it is clear they could also have been constructed from the *central halves* $[\underline{s} + (i + \frac{1}{4}) \cdot \frac{\lambda_S}{n}, \underline{s} + (i + \frac{3}{4}) \cdot \frac{\lambda_S}{n}]$ of these intervals, or more generally, from *any* finite interval union $\wp \subset [0, 1]$ with Lebesgue measure $\frac{1}{2}$. Similarly, any events constructed from a subset $\wp \subset [0, 1]$ with $\lambda(\wp) = \frac{1}{3}$ will approximate the properties of an exogenous objective event with probability $\frac{1}{3}$, etc. Formally, for any finite interval union $\wp \subseteq [0, 1]$ and any positive integer n , we define the *almost-objective event* $\wp \times_n S \subseteq S$ by

$$(17) \quad \wp \times_n S = \bigcup_{i=0}^{n-1} \left\{ \underline{s} + (i + \omega) \cdot \frac{\lambda_S}{n} \mid \omega \in \wp \right\} \quad n = 1, 2, \dots$$

that is, as the union of the natural images of $\wp$ into each of S 's equal-length intervals from (2).

For each n , the mapping $\wp \rightarrow \wp \times_n S$ from subsets of $[0, 1]$ to almost-objective events in S is seen to satisfy the identity $\lambda(\wp \times_n S) \equiv \lambda(\wp) \cdot \lambda(S)$, as well as the following preservation properties

$$(18) \quad \begin{aligned} &\text{preservation of relative Lebesgue measure:} \quad \lambda(\wp \times_n S) / \lambda(\wp' \times_n S) \equiv \lambda(\wp) / \lambda(\wp') \\ &\text{preservation of disjointness/non-disjointness:} \quad \wp \cap \wp' = \emptyset \Leftrightarrow (\wp \times_n S) \cap (\wp' \times_n S) = \emptyset \\ &\text{preservation of finite unions and intersections:} \quad (\wp \cup \wp') \times_n S \equiv (\wp \times_n S) \cup (\wp' \times_n S) \\ &\quad \quad \quad (\wp \cap \wp') \times_n S \equiv (\wp \times_n S) \cap (\wp' \times_n S) \end{aligned}$$

Thus, each partition $\{\wp_1, \dots, \wp_m\}$ of $[0, 1]$ induces the *almost-objective partition* of S defined by

$$(19) \quad \{\wp_1, \dots, \wp_m\} \times_n S = \{\wp_1 \times_n S, \dots, \wp_m \times_n S\}$$

The above almost-objective operations on the entire subjective state space S can also be applied to any *fixed subjective event* $E \in \mathcal{E}$. Given any such event, we define its *almost-objective subevents* and *almost-objective partitions* by

$$(20) \quad \wp \times_n E = (\wp \times_n S) \cap E \quad \text{and} \quad \{\wp_1, \dots, \wp_m\} \times_n E = \{\wp_1 \times_n E, \dots, \wp_m \times_n E\}$$

Although the mapping $\wp \rightarrow \wp \times_n E$ preserves both disjointness/non-disjointness as well as finite unions and intersections in the same manner as the mapping $\wp \rightarrow \wp \times_n S$, standard “lumpiness” considerations imply that it will *not* exactly satisfy $\lambda(\wp \times_n E) = \lambda(\wp) \cdot \lambda(E)$, even when both $\wp$ and E are intervals, and hence it does not exactly preserve relative Lebesgue measure. However, almost-objective subevents will turn out to satisfy the *limiting* form of this property, namely that $\lambda(\wp \times_n E)$ converges to $\lambda(\wp) \cdot \lambda(E)$ as $n \rightarrow \infty$. More generally, the limiting measure properties (4) and (8) of *almost-ethically-neutral* events and subevents also hold for *almost-objective* events and subevents, as formalized in the following generalization of Poincaré’s Theorem:¹⁴

¹⁴ This theorem is numbered “0” so Theorems 1–6 can correspond to the six properties of purely objective events listed in Section 2.2.

THEOREM 0 (LIMITING MEASURE PROPERTIES OF ALMOST-OBJECTIVE EVENTS AND SUBEVENTS): For every finite interval union $\wp \subseteq [0,1]$ and every event $E \in \mathcal{E}$, the almost-objective events $\wp \times_n S$ and subevents $\wp \times_n E = (\wp \times_n S) \cap E$ satisfy

$$(21) \quad \lim_{n \rightarrow \infty} K(\wp \times_n S) = \lambda(\wp) \cdot K(S) \quad \text{and} \quad \lim_{n \rightarrow \infty} K(\wp \times_n E) = \lambda(\wp) \cdot K(E)$$

for every continuous-density signed measure $K(\cdot)$ over S . For any *family* of signed measures $\{K(\cdot; \tau) | \tau \in T\}$ over S with uniformly bounded and uniformly continuous densities, such convergence will be uniform.

Since almost-objective events $\wp \times_n S$ and subevents $\wp \times_n E$ are subsets of the original subjective state space S , they can be used to define “almost-objective” acts within the original subjective act space $\mathcal{A}$, as well as “almost-objective” *mixtures* of arbitrary subjective acts, which will also be elements of $\mathcal{A}$. For any partition $\{\wp_1, \dots, \wp_m\}$ of $[0,1]$ (where each $\wp_i$ is a finite interval union) and outcomes $x_1, \dots, x_m \in \mathcal{X}$, we define the *almost-objective act*

$$(22) \quad [x_1 \text{ on } \wp_1 \times_n S; \dots; x_m \text{ on } \wp_m \times_n S] \in \mathcal{A} \quad n = 1, 2, \dots$$

Similarly, for any such partition $\{\wp_1, \dots, \wp_m\}$ and subjective acts $f_1(\cdot), \dots, f_m(\cdot) \in \mathcal{A}$, we define the *almost-objective mixture*

$$(23) \quad [f_1(\cdot) \text{ on } \wp_1 \times_n S; \dots; f_m(\cdot) \text{ on } \wp_m \times_n S] \in \mathcal{A} \quad n = 1, 2, \dots$$

whose outcomewise events are accordingly given by

$$(24) \quad E_x \equiv f^{-1}(x) \equiv \wp_1 \times_n f_1^{-1}(x) \cup \dots \cup \wp_m \times_n f_m^{-1}(x) \equiv \bigcup_{j=1}^m \wp_j \times_n f_j^{-1}(x) \quad x \in \mathcal{X}$$

3.2 Revealed Likelihood Properties of Almost-Objective Events

Unanimous, Outcome-Invariant Limiting Revealed Likelihoods

In the above temperature example, it is reasonable to suppose that most individuals, regardless of their beliefs, and even if their preferences are temperature-dependent, would be indifferent between betting on the almost-objective event $[\cdot 5, \cdot 6)_{10,000} \times S$ versus $[\cdot 6, \cdot 7)_{10,000} \times S$ – that is, indifferent between betting on the second decimal of the temperature being 5 versus 6. Similarly, most individuals would be indifferent between betting on $[\cdot 5, \cdot 7)_{10,000} \times S$ versus $([0, \cdot 1) \cup [\cdot 3, \cdot 4))_{10,000} \times S$, and would prefer staking a prize on either of these events to staking it on $[\cdot 8, \cdot 95)_{10,000} \times S$. In other words, individuals’ exhibit unanimous revealed likelihoods for the almost-objective events $\wp_{10,000} \times S$, corresponding to the value of $\lambda(\wp)$. We formalize this property by

THEOREM 1 (UNANIMOUS, OUTCOME-INVARIANT LIMITING REVEALED LIKELIHOODS):

For all disjoint finite interval unions $\wp, \wp' \subseteq [0,1]$ with $\lambda(\wp) > (=) \lambda(\wp')$, the almost-objective events $\wp \times_n S$ and $\wp' \times_n S$ satisfy

$$(25) \quad \lim_{n \rightarrow \infty} W \begin{pmatrix} x^* & \text{on } \wp \times_n S \\ x & \text{on } \wp' \times_n S \\ f(s) & \text{elsewhere} \end{pmatrix} > (=) \lim_{n \rightarrow \infty} W \begin{pmatrix} x & \text{on } \wp \times_n S \\ x^* & \text{on } \wp' \times_n S \\ f(s) & \text{elsewhere} \end{pmatrix} \quad \begin{array}{l} \text{all } x^* \succ x \\ \text{all } f(\cdot) \in \mathcal{A} \end{array}$$

for every event-smooth preference function $W(\cdot)$ over $\mathcal{A}$.

Thus, whenever $\lambda(\wp) > \lambda(\wp')$, for any pair of ranked outcomes $x^* \succ x$ and any subjective act $f(\cdot) \in \mathcal{A}$, the left act in (25) will be strictly preferred to the right act for all sufficiently large n . When $\lambda(\wp) = \lambda(\wp')$, for any $x^* \succ x$ and any $f(\cdot) \in \mathcal{A}$, all event-continuous ordinal *equivalency*

measures¹⁵ of these two acts would approach equality as $n \rightarrow \infty$. Since $\lambda(\wp \times_n S) \equiv \lambda(\wp) \cdot \lambda(S)$ for all $\wp$ and n , it also follows that the unanimous limiting revealed likelihoods $\lambda(\wp)$, $\lambda(\wp')$, ... of the almost-objective events $\wp \times_n S$, $\wp' \times_n S$, ... are exactly representable by a probability measure over these events, namely the normalized uniform Lebesgue measure $\lambda(\cdot)/\lambda(S)$ over S .

Limiting One-Way Independence from Fixed Subjective Events

In addition to the property of unanimous, well-defined likelihoods, almost-objective events also approximate (and in the limit, attain) the objective property of being *independent of the realization of any fixed subjective event* E , in the same sense in which an Anscombe-Aumann objective roulette wheel is independent of their subjective horse race. Similarly, an individual's preference for betting on the second decimal of the temperature being 5 or 6 versus 7 will continue to hold even if the bet is made conditional on the temperature *also* being at least 60°. Even though individuals may differ in their subjective probabilities of the conditioning event $E = [60^\circ, 100^\circ]$, some may consider it ambiguous, and others may have temperature-dependent utility, virtually everyone with event-smooth preferences who considers E nonnull will view the pair of *joint events*

$$\{\text{second decimal of } s \text{ is 5 or 6}\} \cap \{s \geq 60^\circ\} \quad \text{versus} \quad \{\text{second decimal of } s \text{ is 7}\} \cap \{s \geq 60^\circ\}$$

as having the same likelihood ratio as the *unconditional events*

$$\{\text{second decimal of } s = 5 \text{ or } 6\} \quad \text{versus} \quad \{\text{second decimal of } s = 7\}$$

namely, a likelihood ratio of 2:1. We formalize this property by

THEOREM 2 (LIMITING ONE-WAY INDEPENDENCE OF ALMOST-OBJECTIVE EVENTS FROM FIXED SUBJECTIVE EVENTS): For all disjoint finite interval unions $\wp, \wp' \subseteq [0, 1]$ with $\lambda(\wp) > (=) \lambda(\wp')$ and nonnull events $E \in \mathcal{E}$, the events $(\wp \times_n S) \cap E$ and $(\wp' \times_n S) \cap E$ satisfy

$$(26) \quad \lim_{n \rightarrow \infty} W \begin{pmatrix} x^* & \text{on } (\wp \times_n S) \cap E \\ x & \text{on } (\wp' \times_n S) \cap E \\ f(s) & \text{elsewhere} \end{pmatrix} > (=) \lim_{n \rightarrow \infty} W \begin{pmatrix} x & \text{on } (\wp \times_n S) \cap E \\ x^* & \text{on } (\wp' \times_n S) \cap E \\ f(s) & \text{elsewhere} \end{pmatrix} \quad \begin{array}{l} \text{all } x^* \succ x \\ \text{all } f(\cdot) \in \mathcal{A} \end{array}$$

for every event-smooth preference function $W(\cdot)$ over $\mathcal{A}$.

3.3 Betting Preferences over Almost-Objective Acts and Mixtures

Limiting Probabilistic Sophistication over Almost-Objective Acts

Given that for large n all event-smooth $W(\cdot)$ approach the property of unanimous, well-defined revealed likelihoods over almost-objective *events*, it is no surprise that for large n they also approach the property of *probabilistic sophistication* over all almost-objective *acts*. That is, each event-smooth $W(\cdot)$ over $\mathcal{A}$ has an associated preference function $V_W(\cdot)$ over objective lotteries such that $W(\cdot)$'s evaluation of any almost-objective act $f(\cdot) = [x_1 \text{ on } \wp_1 \times_n S; \dots; x_m \text{ on } \wp_m \times_n S]$ is approximated – and in the limit exactly determined – solely by $V_W(\cdot)$'s evaluation of its outcomes $x_1, \dots, x_m$ and their corresponding limiting likelihoods $\lambda(\wp_1), \dots, \lambda(\wp_m)$. Formally:

¹⁵ If the outcome space $\mathcal{X}$ is a continuum, a standard equivalency measure of $f(\cdot)$ would be its *certainty equivalent*. If $\mathcal{X}$ is not necessarily a continuum but has a most and least preferred outcome $\bar{x}$ and $\underline{x}$, another equivalency measure would be any “*event-equivalent*,” such as the state $s_{f(\cdot)} \in S$ that solves $W(\underline{x} \text{ on } [\underline{s}, s_{f(\cdot)}]; \bar{x} \text{ on } (s_{f(\cdot)}, \bar{s}]) = W(f(\cdot))$.

THEOREM 3 (LIMITING PROBABILISTIC SOPHISTICATION OVER ALMOST-OBJECTIVE ACTS): For each event-smooth preference function $W(\cdot)$ over $\mathcal{A}$ there exists a preference function $V_W(\cdot)$ over objective lotteries, satisfying strict first order stochastic dominance preference, such that

$$(27) \quad \lim_{n \rightarrow \infty} W(x_1 \text{ on } \wp_1 \times_n S; \dots; x_m \text{ on } \wp_m \times_n S) \equiv V_W(x_1, \lambda(\wp_1); \dots; x_m, \lambda(\wp_m))$$

for all almost-objective acts in $\mathcal{A}$.

Thus, if two partitions $\{\wp_1, \dots, \wp_m\}$ and $\{\wp'_1, \dots, \wp'_m\}$ of $[0,1]$ satisfy $\lambda(\wp_i) = \lambda(\wp'_i)$ for each i , then for large n and n' every event-smooth individual will approach indifference between any two almost-objective acts of the form $[x_1 \text{ on } \wp_1 \times_n S; \dots; x_m \text{ on } \wp_m \times_n S]$ and $[x_1 \text{ on } \wp'_1 \times_{n'} S; \dots; x_m \text{ on } \wp'_m \times_{n'} S]$, and we refer to such acts as *almost-actuarially-equivalent*. The role of the preference function $V_W(\cdot)$ as a complete summary of $W(\cdot)$'s "objective risk preferences" is explored in Section 4.2.

Reduction of Almost-Objective $\times$ Subjective Uncertainty

Even though they are completely subjective acts, the mixtures $[f_1(\cdot) \text{ on } \wp_1 \times_n S; \dots; f_m(\cdot) \text{ on } \wp_m \times_n S]$ from (23) are seen to be the almost-objective analogues of classic Anscombe-Aumann *roulette/horse lotteries* $(f_1(\cdot), p_1; \dots; f_m(\cdot), p_m)$, which can be viewed as objective mixtures of the subjective acts $f_1(\cdot), \dots, f_m(\cdot)$. Anscombe and Aumann also considered *horse/roulette lotteries* $[\mathbf{P}_1 \text{ on } E_1; \dots; \mathbf{P}_m \text{ on } E_m]$, which can be viewed as subjective mixtures of the objective lotteries $\mathbf{P}_1, \dots, \mathbf{P}_m$, or equivalently, as mappings from subjective events to objective lotteries. As mentioned in Section 2.2, researchers have typically postulated that preferences across these distinct types of prospects are linked by the *reduction of compound uncertainty principle*, which states that all prospects that imply the same mapping from events to lotteries will be viewed as indifferent.

By definition, each horse/roulette lottery $[\mathbf{P}_1 \text{ on } E_1; \dots; \mathbf{P}_m \text{ on } E_m]$ already is a mapping from events to lotteries. For each roulette/horse lottery $(f_1(\cdot), p_1; \dots; f_m(\cdot), p_m)$, its implied mapping from events to lotteries can be derived by expressing its component acts $f_1(\cdot), \dots, f_m(\cdot)$ in terms of the *common refinement* $\{E_1^*, \dots, E_K^*\}$ of their respective underlying partitions – that is, by writing

$$(28) \quad f_1(\cdot) = [x_{1,1} \text{ on } E_1^*; \dots; x_{1,K} \text{ on } E_K^*] \quad \dots \quad f_m(\cdot) = [x_{m,1} \text{ on } E_1^*; \dots; x_{m,K} \text{ on } E_K^*]$$

so we can write the roulette/horse lottery $(f_1(\cdot), p_1; \dots; f_m(\cdot), p_m)$ as

$$(29) \quad ([x_{1,1} \text{ on } E_1^*; \dots; x_{1,K} \text{ on } E_K^*], p_1; \dots; [x_{m,1} \text{ on } E_1^*; \dots; x_{m,K} \text{ on } E_K^*], p_m)$$

and its implied mapping from events to lotteries is seen to be

$$(30) \quad [(x_{1,1}, p_1; \dots; x_{m,1}, p_m) \text{ on } E_1^*; \dots; (x_{1,K}, p_1; \dots; x_{m,K}, p_m) \text{ on } E_K^*]$$

The reduction of compound uncertainty principle thus asserts that the original roulette/horse lottery $(f_1(\cdot), p_1; \dots; f_m(\cdot), p_m)$ would be viewed as indifferent to its implied horse/roulette lottery (30), and further, that *all other* roulette/horse lotteries that yield the same (or probabilistically equivalent) implied mappings from events to lotteries also be viewed as indifferent.

In a setting of objective $\times$ subjective uncertainty, it is necessary to exogenously impose the reduction of compound uncertainty principle as an additional restriction on preferences. However in our present completely subjective framework, as $n \rightarrow \infty$ every event-smooth preference function $W(\cdot)$ will *inherently exhibit* the reduction principle for almost-objective $\times$ subjective uncertainty.

To see this, consider any almost-objective mixture $[f_1(\cdot) \text{ on } \wp_1 \times_n S; \dots; f_m(\cdot) \text{ on } \wp_m \times_n S]$. We can derive its implied mapping from events to almost-objective acts by expressing the component acts $f_1(\cdot), \dots, f_m(\cdot)$ in terms of their common refinement $\{E_1^*, \dots, E_K^*\}$ as in (28), so that we can write the mixture $[f_1(\cdot) \text{ on } \wp_1 \times_n S; \dots; f_m(\cdot) \text{ on } \wp_m \times_n S]$ as

$$(31) \quad \begin{aligned} & [x_{1,1} \text{ on } \wp_1 \times_n E_1^*; \dots; x_{1,K} \text{ on } \wp_1 \times_n E_K^*; \dots; x_{m,1} \text{ on } \wp_m \times_n E_1^*; \dots; x_{m,K} \text{ on } \wp_m \times_n E_K^*] = \\ & [x_{1,1} \text{ on } \wp_1 \times_n E_1^*; \dots; x_{m,1} \text{ on } \wp_m \times_n E_1^*; \dots; x_{1,K} \text{ on } \wp_1 \times_n E_K^*; \dots; x_{m,K} \text{ on } \wp_m \times_n E_K^*] = \\ & [[x_{1,1} \text{ on } \wp_1 \times_n S; \dots; x_{m,1} \text{ on } \wp_m \times_n S] \text{ on } E_1^*; \dots; [x_{1,K} \text{ on } \wp_1 \times_n S; \dots; x_{m,K} \text{ on } \wp_m \times_n S] \text{ on } E_K^*] \end{aligned}$$

The third line of (31) is seen to be the almost-objective analogue of the implied event-to-lottery mapping (30), with the same subjective partition $\{E_1^*, \dots, E_K^*\}$, and with (30)'s event-contingent *objective lotteries* $(x_{1,1}, p_1; \dots; x_{m,1}, p_m), \dots, (x_{1,K}, p_1; \dots; x_{m,K}, p_m)$ replaced by (31)'s event-contingent *almost-objective acts* $[x_{1,1} \text{ on } \wp_1 \times_n S; \dots; x_{m,1} \text{ on } \wp_m \times_n S], \dots, [x_{1,K} \text{ on } \wp_1 \times_n S; \dots; x_{m,K} \text{ on } \wp_m \times_n S]$. Under objective $\times$ subjective uncertainty, the original roulette/horse lottery $(f_1(\cdot), p_1; \dots; f_m(\cdot), p_m)$ and its implied mapping (30) (which is a horse/roulette lottery) are qualitatively different types of prospects, and the reduction principle ensuring their indifference – as well as the indifference of all other roulette/horse lotteries with the same implied mapping – must be exogenously imposed. But, in the present purely subjective framework, the third line of (31) is simply a *reexpression* of the original almost-objective mixture $[f_1(\cdot) \text{ on } \wp_1 \times_n S; \dots; f_m(\cdot) \text{ on } \wp_m \times_n S]$, and is thus necessarily indifferent to it. The following result further shows that as $n \rightarrow \infty$, all other almost-objective mixtures that imply a probabilistically equivalent mapping from events to almost-objective acts will also be viewed as indifferent by each event-smooth $W(\cdot)$ – again without the need to impose any exogenous reduction condition:

THEOREM 4 (REDUCTION OF ALMOST-OBJECTIVE $\times$ SUBJECTIVE UNCERTAINTY): If the almost-objective mixtures $[f_1(\cdot) \text{ on } \wp_1 \times_n S; \dots; f_m(\cdot) \text{ on } \wp_m \times_n S]$ and $[\hat{f}_1(\cdot) \text{ on } \hat{\wp}_1 \times_n S; \dots; \hat{f}_m(\cdot) \text{ on } \hat{\wp}_m \times_n S]$ imply probabilistically equivalent almost-objective acts over each event in the common refinement of $\{f_1(\cdot), \dots, f_m(\cdot), \hat{f}_1(\cdot), \dots, \hat{f}_m(\cdot)\}$, then

$$(32) \quad \lim_{n \rightarrow \infty} W(f_1(\cdot) \text{ on } \wp_1 \times_n S; \dots; f_m(\cdot) \text{ on } \wp_m \times_n S) = \lim_{n \rightarrow \infty} W(\hat{f}_1(\cdot) \text{ on } \hat{\wp}_1 \times_n S; \dots; \hat{f}_m(\cdot) \text{ on } \hat{\wp}_m \times_n S)$$

for every event-smooth preference function $W(\cdot)$ over $\mathcal{A}$.

Under Probabilistic Sophistication, Limiting Two-Way Independence of Almost-Objective and Subjective Events

Although Theorem 2 demonstrates that the limiting revealed likelihood ranking of any pair of almost-objective events $\wp \times_n S, \wp' \times_n S$ is invariant to whether or not they are conditioned on a fixed subjective event E , the opposite is generally *not* true. That is, a general event-smooth preference function $W(\cdot)$ could well prefer to bet on a subjective event E versus E' , yet reverse these preferences when they are conditioned on some almost-objective event $\wp \times_n S$, even for large n . The reason is that in the absence of any additional regularity on beliefs, subjective uncertainty admits enough phenomena, such as ambiguity, state-dependence or their conditional manifestations, to allow practically any configuration of betting preferences over arbitrary subjective events.

However, for individuals whose subjective beliefs *are* regular enough to be represented by well-defined subjective probabilities – that is, whose preference functions take the probabilistically sophisticated form $W_{PS}(f(\cdot)) \equiv V(\dots; x_i, \mu(E_i); \dots)$ – almost-objective and subjective events will exhibit the limiting property of being *mutually independent*. Thus, in addition to the first of

the following properties, which by Theorem 2 is already displayed by every event-smooth $W(\cdot)$, every event-smooth $W_{PS}(\cdot)$ will also exhibit the second property:

- for any pair of almost-objective events $\wp \times_n S$, $\wp' \times_n S$ and fixed nonnull subjective event $E \in \mathcal{E}$, the limiting revealed likelihood ranking of the joint events $(\wp \times_n S) \cap E$ versus $(\wp' \times_n S) \cap E$ is the same as for $\wp \times_n S$ versus $\wp' \times_n S$, namely, $\lambda(\wp)$ versus $\lambda(\wp')$
- for any pair of fixed subjective events $E, E' \in \mathcal{E}$ and almost-objective event $\wp \times_n S$ with $\lambda(\wp) > 0$, the limiting revealed likelihood ranking of the joint events $(\wp \times_n S) \cap E$ versus $(\wp \times_n S) \cap E'$ is the same as for E versus E' , namely, $\mu(E)$ versus $\mu(E')$

The following result subsumes these two properties into the general “limiting independence” property that a probabilistically sophisticated individual’s limiting likelihood of any joint event $(\wp \times_n S) \cap E$ is given by the unanimously-held limiting likelihood $\lambda(\wp)$ of the almost-objective event $\wp \times_n S$, multiplied by their own personal probability $\mu(E)$ of the subjective event E :

THEOREM 5 (UNDER PROBABILISTIC SOPHISTICATION, LIMITING INDEPENDENCE OF ALMOST-OBJECTIVE AND SUBJECTIVE LIKELIHOODS): For every event-smooth probabilistically sophisticated preference function $W_{PS}(\cdot)$ over $\mathcal{A}$, if the almost-objective events $\wp \times_n S$, $\wp' \times_n S$ and disjoint subjective events $E, E' \in \mathcal{E}$ satisfy $\lambda(\wp) \cdot \mu(E) > (=) \lambda(\wp') \cdot \mu(E')$, then the joint events $(\wp \times_n S) \cap E$ and $(\wp' \times_n S) \cap E'$ satisfy

$$(33) \quad \lim_{n \rightarrow \infty} W_{PS} \left(\begin{array}{l} x^* \text{ on } (\wp \times_n S) \cap E \\ x \text{ on } (\wp' \times_n S) \cap E' \\ f(s) \text{ elsewhere} \end{array} \right) > (=) \lim_{n \rightarrow \infty} W_{PS} \left(\begin{array}{l} x \text{ on } (\wp \times_n S) \cap E \\ x^* \text{ on } (\wp' \times_n S) \cap E' \\ f(s) \text{ elsewhere} \end{array} \right) \quad \begin{array}{l} \text{all } x^* \succ x \\ \text{all } f(\cdot) \in \mathcal{A} \end{array}$$

Under Expected Utility, Limiting Linearity in Almost-Objective Likelihoods

As noted in Section 2.2, under objective uncertainty expected utility preferences over lotteries $\mathbf{P} = (x_1, p_1; \dots; x_m, p_m)$ are linear in the outcomes’ *objective probabilities* $p_1, \dots, p_m$, and both state-independent and state-dependent expected utility preferences over objective probability mixtures $\alpha \cdot \mathbf{P} + (1-\alpha) \cdot \mathbf{P}^*$ of lotteries are linear in the objective mixture probabilities α and $1-\alpha$.

Under subjective uncertainty, *state-independent* expected utility $W_{SEU}(f(\cdot)) \equiv \sum_i U(x_i) \cdot \mu(E_i)$ continues to be linear in the subjective probabilities $\mu(E_1), \dots, \mu(E_m)$. However, this is *not* true of *state-dependent* expected utility $W_{SDEU}(f(\cdot)) \equiv \sum_i \int_{E_i} U(x_i | s) \cdot d\mu(s)$: even if the subjective probabilities in the acts $f(\cdot) = [x_1 \text{ on } E_1; \dots; x_m \text{ on } E_m]$, $f'(\cdot) = [x_1 \text{ on } E'_1; \dots; x_m \text{ on } E'_m]$ and $f''(\cdot) = [x_1 \text{ on } E''_1; \dots; x_m \text{ on } E''_m]$ should satisfy $\mu(E_i) = \alpha \cdot \mu(E'_i) + (1-\alpha) \cdot \mu(E''_i)$ for each i , we will typically have

$$(34) \quad W_{SDEU}(f(\cdot)) \neq \alpha \cdot W_{SDEU}(f'(\cdot)) + (1-\alpha) \cdot W_{SDEU}(f''(\cdot))$$

But even though almost-objective events are subsets of the subjective state space S , their properties for expected utility maximizers will be more similar to those of purely objective rather than regular subjective events. Thus, if the limiting likelihoods in the *almost-objective acts* $f_n(\cdot) = [x_1 \text{ on } \wp_1 \times_n S; \dots; x_m \text{ on } \wp_m \times_n S]$, $f'_n(\cdot) = [x_1 \text{ on } \wp'_1 \times_n S; \dots; x_m \text{ on } \wp'_m \times_n S]$ and $f''_n(\cdot) = [x_1 \text{ on } \wp''_1 \times_n S; \dots; x_m \text{ on } \wp''_m \times_n S]$ should satisfy $\lambda(\wp_i) = \alpha \cdot \lambda(\wp'_i) + (1-\alpha) \cdot \lambda(\wp''_i)$ for each i , then every event-smooth state-independent *and* state-dependent expected utility preference function will exhibit

$$(35) \quad \begin{aligned} \lim_{n \rightarrow \infty} W_{SEU}(f_n(\cdot)) &= \alpha \cdot \lim_{n \rightarrow \infty} W_{SEU}(f'_n(\cdot)) + (1-\alpha) \cdot \lim_{n \rightarrow \infty} W_{SEU}(f''_n(\cdot)) \\ \lim_{n \rightarrow \infty} W_{SDEU}(f_n(\cdot)) &= \alpha \cdot \lim_{n \rightarrow \infty} W_{SDEU}(f'_n(\cdot)) + (1-\alpha) \cdot \lim_{n \rightarrow \infty} W_{SDEU}(f''_n(\cdot)) \end{aligned}$$

The following result formalizes this limiting linearity property for state-independent and state-dependent expected utility preferences:

THEOREM 6 (UNDER EXPECTED UTILITY, LIMITING LINEARITY IN ALMOST-OBJECTIVE LIKELIHOODS): For every event-smooth expected utility preference function $W_{SEU}(f(\cdot)) \equiv \int_S U(f(s)) \cdot d\mu(s)$ or $W_{SDEU}(f(\cdot)) \equiv \int_S U(f(s)|s) \cdot d\mu(s)$ over $\mathcal{A}$, preferences over almost-objective acts satisfy

$$(36) \quad \begin{aligned} \lim_{n \rightarrow \infty} W_{SEU}(x_1 \text{ on } \wp_1 \times_n S; \dots; x_m \text{ on } \wp_m \times_n S) &\equiv \sum_{i=1}^m \lambda(\wp_i) \cdot U(x_i) \\ \lim_{n \rightarrow \infty} W_{SDEU}(x_1 \text{ on } \wp_1 \times_n S; \dots; x_m \text{ on } \wp_m \times_n S) &\equiv \sum_{i=1}^m \lambda(\wp_i) \cdot \int_S U(x_i|s) \cdot d\mu(s) \end{aligned}$$

and preferences over almost-objective mixtures of purely subjective acts satisfy

$$(37) \quad \begin{aligned} \lim_{n \rightarrow \infty} W_{SEU}(f_1(\cdot) \text{ on } \wp_1 \times_n S; \dots; f_m(\cdot) \text{ on } \wp_m \times_n S) &\equiv \sum_{i=1}^m \lambda(\wp_i) \cdot \int_S U(f_i(s)) \cdot d\mu(s) \\ \lim_{n \rightarrow \infty} W_{SDEU}(f_1(\cdot) \text{ on } \wp_1 \times_n S; \dots; f_m(\cdot) \text{ on } \wp_m \times_n S) &\equiv \sum_{i=1}^m \lambda(\wp_i) \cdot \int_S U(f_i(s)|s) \cdot d\mu(s) \end{aligned}$$

Ordinal Properties of Limiting Preference Function Values

Since almost-objective events $\wp_n \times S$ and therefore almost-objective acts $f_n(\cdot) = [x_1 \text{ on } \wp_1 \times_n S; \dots; x_m \text{ on } \wp_m \times_n S]$ and $f_n^*(\cdot) = [x_1^* \text{ on } \wp_1^* \times_n S; \dots; x_m^* \text{ on } \wp_m^* \times_n S]$ do not exist at $n = \infty$, we have used their *limiting preference function values* $\lim_{n \rightarrow \infty} W(f_n(\cdot))$ and $\lim_{n \rightarrow \infty} W(f_n^*(\cdot))$, which under event-smoothness *do* exist, to represent the individual's revealed beliefs and betting preferences under almost-objective (and almost objective $\times$ subjective) uncertainty. But since the limiting objects themselves – namely “ $\wp_\infty \times S$ ”, “ $f_\infty(\cdot)$ ” and “ $f_\infty^*(\cdot)$ ” – do not actually exist, it is worth pointing out what these limiting preference function values actually imply about the individual's *limiting ordinal rankings* between $f_n(\cdot)$, $f_n^*(\cdot)$ and other acts in $\mathcal{A}$, as well as what they can imply about the individual's *exact ordinal rankings* over arbitrary pairs of subjective acts in $\mathcal{A}$.

In fact, the limiting preference function values of an event-smooth $W(\cdot)$ have many of the same properties as the “limiting sensitivity rates” (i.e., derivatives) $g'(z) = \lim_{\Delta z \rightarrow 0} [g(z + \Delta z) - g(z)] / \Delta z$ of a smooth univariate function $g(\cdot)$, where the *limiting objects themselves* – namely the sensitivity rates “ $[g(z+0) - g(z)]/0$ ” – similarly do not exist, yet their limiting values $g'(z)$ yield both *limiting approximation results* for the function when taken individually, as well as *exact global results* for the function (via the Fundamental Theorem of Calculus) when taken collectively.

Taken individually, event-smoothness implies that each limiting preference function value $w = \lim_{n \rightarrow \infty} W(f_n(\cdot))$ corresponds to a unique, nonempty indifference class I_w of acts in $\mathcal{A}$. To see this, let $\underline{x}$ and $\bar{x}$ be the least and most preferred outcomes in $\{x_1, \dots, x_m\}$, so that monotonicity implies $W(\bar{x} \text{ on } S) \geq w \geq W(\underline{x} \text{ on } S)$, and event-continuity then implies some state $s_w \in S$ such that the act $f_w(\cdot) = [\underline{x} \text{ on } [\underline{s}, s_w]; \bar{x} \text{ on } (s_w, \bar{s})]$ satisfies $W(f_w(\cdot)) = w$. But since $W(f_n(\cdot))$ could converge to w from above, below, or by damped oscillation, the almost-objective acts $f_n(\cdot)$ needn't converge to any stable ranking with respect to $f_w(\cdot)$ or any other act in its indifference class I_w . However, the acts $f_n(\cdot)$ will converge to a stable strict ranking with respect to *every other act in $\mathcal{A}$* – that is, for each $f(\cdot) \in \mathcal{A} - I_w$, $f_w(\cdot) \succ (<) f(\cdot)$ will imply $f_n(\cdot) \succ (<) f(\cdot)$ for all sufficiently large n .

Taken in pairwise comparison, an *inequality* $\lim_{n \rightarrow \infty} W(f_n^*(\cdot)) > \lim_{n \rightarrow \infty} W(f_n(\cdot))$ between two limiting preference function values will imply $f_n^*(\cdot) \succ f_n(\cdot)$ for all sufficiently large n , and more generally, will be equivalent to the existence of fixed acts $\hat{f}^*(\cdot), \hat{f}(\cdot) \in \mathcal{A}$ such that $f_n^*(\cdot) \succ \hat{f}^*(\cdot) \succ \hat{f}(\cdot) \succ f_n(\cdot)$ for all sufficiently large n . Although, an *equality* $\lim_{n \rightarrow \infty} W(f_n^*(\cdot)) = w = \lim_{n \rightarrow \infty} W(f_n(\cdot))$ needn't imply a

stable limiting ranking of $f_n^*(\cdot)$ versus $f_n(\cdot)$ (since $W(f_n^*(\cdot)) - W(f_n(\cdot))$ could approach zero from above, below, or by oscillation), it does imply that $f_n^*(\cdot)$ and $f_n(\cdot)$ will converge to the *same* limiting strict ranking with respect to each act $f(\cdot) \in \mathcal{A} - I_w$.

Taken collectively, the family of limiting values $\lim_{n \rightarrow \infty} W(x_1 \text{ on } \wp_1 \times S; \dots; x_m \text{ on } \wp_m \times S)$ for all almost-objective acts, or more generally, the family of limiting preference function values for all almost-objective $\times$ subjective acts, can be used to yield exact, global results for all purely subjective acts in $\mathcal{A}$, as in Theorem 7 below, where a condition (the “Almost-Objective/Subjective Replacement Axiom”) on these limiting values is shown to imply the probabilistic sophistication property $W(f(\cdot)) \equiv V(\dots; x_i, \mu(E_i); \dots)$ over all purely subjective $f(\cdot) \in \mathcal{A}$. As outlined in Section 4.3, similar results linking limiting preference function values for almost-objective and almost-objective $\times$ subjective acts to exact global properties of preferences over purely subjective acts, including characterizations of comparative subjective likelihood, relative subjective likelihood, and comparative risk aversion, are provided in Machina (2002).

3.4 Objective vs. Subjective *Events* rather than Objective vs. Subjective *Processes*

The above results argue for a new perspective on the classic distinction between “objective” and “subjective” uncertainty. The traditional view has been to distinguish *objective processes* – such as coin flips or roulette wheels – from *subjective processes* – such as horse races or the weather. However, even in a purely subjective setting where the state space $S = [\underline{s}, \bar{s}]$ corresponds to tomorrow’s temperature, the above results show that some subjective events are “more objective” than others, and while no event $E \subseteq S$ can ever be “completely objective” in the senses of Section 2.2, as $n \rightarrow \infty$ almost-objective events come arbitrarily close to achieving each of these properties for any individual with event-smooth preferences. Similarly, in the classic case of a “subjective” horse race, regardless of individuals’ diverse or even nonexistent beliefs over how fast the horses may run, virtually everyone will exhibit 50:50 revealed likelihoods for whether a given horse will complete the race in an even versus an odd number of milliseconds.

Conversely, every so-called “*objective*” process also include completely *subjective* events. For the flip of a perfectly balanced coin, if we ask individuals their likelihoods of the event “coin lands heads,” virtually everyone would report $\frac{1}{2}$. But if we instead ask their likelihoods of the event “coin rotates more than 23 times before landing,” we would uncover a range of beliefs, as with any purely subjective event. In other words, even for a balanced coin, it is not the *process* that is “objective,” but rather the particular *event* “lands heads” defined on this process that is (almost-) objective. The alternative event “rotates more than 23 times” – even though it is defined on the same process – is clearly subjective. As noted in Section 2.3, the event “lands heads” is a periodic event defined on the underlying subjective state of nature (the force of the flip). On the other hand, “rotates more than 23 times” is an interval event, based on whether that force exceeds some specific level. This also holds for processes that depend on more than one uncertain variable: Say we determine that, besides the force of the flip, the behavior of the coin also depends on the (unknown) air pressure in the room. In this case, the event “heads” will be a periodic (non-linear checkerboard-like) event defined over a two-dimensional subjective state space, whereas “rotates more than 23 times” will be a nonlinear “half-region” of this state space. By concentrating on the distinction between almost-objective (periodic) and regular subjective (e.g., interval) *events*, and invoking theorems such as those presented above, we can drop what appears to be an artificial distinction between so-called “objective” versus “subjective” *processes*.

4. APPLICATIONS

Since almost-objective events, acts and mixtures have essentially the same properties as their objective counterparts, their advantage does not lie in generating *new* applications, but in extending the existing applications of objective uncertainty to completely subjective settings. This is similar to the analyses of Savage (1954) and Anscombe and Aumann (1963), which did not generate any new applications of probability theory, but rather, extended the settings in which it could be applied. In this section we present three extensions of “objective uncertainty” analysis to our setting of completely subjective uncertainty.¹⁶

4.1 Anscombe-Aumann without Objective Uncertainty

In their classic paper, Anscombe and Aumann (1963) provided a characterization of probabilistic sophistication that is much simpler and more direct than that of Savage (1954). Their three formal assumptions were: existence of exogenous objective uncertainty (an objective $\times$ subjective framework), “independence of order of resolution” of the objective and subjective uncertainty, and expected utility risk preferences. Machina and Schmeidler (1995) retained the objective $\times$ subjective framework and independence of order of resolution, but showed that the expected utility hypothesis could be replaced by a weaker property on preferences, as described below. By replacing objective $\times$ subjective uncertainty with almost-objective $\times$ subjective uncertainty, we can obtain a direct Anscombe-Aumann type characterization of exact probabilistic sophistication without *any* of their original three assumptions – that is, without exogenous objective uncertainty, independence of order of resolution,¹⁷ or expected utility risk preferences.

Machina and Schmeidler (1995) obtained probabilistic sophistication by replacing the expected utility hypothesis with the following property of preferences over objective $\times$ subjective prospects $[\mathbf{P}_1 \text{ on } E_1; \dots; \mathbf{P}_m \text{ on } E_m]$, where δ_M and δ_0 are degenerate lotteries yielding outcomes $M \succ 0$

HORSE/ROULETTE REPLACEMENT AXIOM: (Machina-Schmeidler (1995)): For any partition $\{E_1, \dots, E_m\}$ of S , if

$$(38) \quad \begin{bmatrix} M & \text{on } E_i \\ 0 & \text{on } E_j \\ 0 & \text{on } E_k, k \neq i, j \end{bmatrix} \sim \begin{bmatrix} \alpha \cdot \delta_M + (1-\alpha) \cdot \delta_0 & \text{on } E_i \\ \alpha \cdot \delta_M + (1-\alpha) \cdot \delta_0 & \text{on } E_j \\ \delta_0 & \text{on } E_k, k \neq i, j \end{bmatrix}$$

for some probability $\alpha \in [0,1]$ and pair of events E_i and E_j , then

$$(39) \quad \begin{bmatrix} \mathbf{P}_i & \text{on } E_i \\ \mathbf{P}_j & \text{on } E_j \\ \mathbf{P}_k & \text{on } E_k, k \neq i, j \end{bmatrix} \sim \begin{bmatrix} \alpha \cdot \mathbf{P}_i + (1-\alpha) \cdot \mathbf{P}_j & \text{on } E_i \\ \alpha \cdot \mathbf{P}_i + (1-\alpha) \cdot \mathbf{P}_j & \text{on } E_j \\ \mathbf{P}_k & \text{on } E_k, k \neq i, j \end{bmatrix}$$

for all objective lotteries $\mathbf{P}_1, \dots, \mathbf{P}_m$.

¹⁶ In addition to these applications, Joel Watson has observed that almost-objective events can be used to approximate *purified strategies* (e.g. Harsanyi (1973)) in games when players do not have common beliefs, or even knowledge of each other’s beliefs. Thus, in a game of matching pennies, if Player I announces the strategy $[\text{Heads on } [0, \frac{1}{2}] \times_n S; \text{Tails on } (\frac{1}{2}, 1] \times_n S]$ and Player II announces $[\text{Heads on } ([0, \frac{1}{4}] \cup (\frac{3}{4}, 1]) \times_n S; \text{Tails on } (\frac{1}{4}, \frac{3}{4}] \times_n S]$, then all continuous-density joint subjective probability measures on S will assign approximately $\frac{1}{4}$ probability to each combination of strategies. Eddie Dekel has observed that a similar construction can approximate *correlated equilibria*. Quiggin (2002) provides an application of almost-objective events to the value of information under ambiguity.

¹⁷ Independence of order of resolution will be unnecessary since almost-objective $\times$ subjective acts are completely subjective and hence just involve a single stage of uncertainty, namely the realization of the subjective state $s \in S$.

This axiom states that the ratio $\alpha:(1-\alpha)$ at which the individual is willing to replace a pair of distinct prizes (pure outcomes or objective lotteries) over the events E_i and E_j with an objective probability mixture of these same prizes over $E_i \cup E_j$ does not depend upon the specific prizes involved, or upon the prizes assigned to other any event E_k . Of course, the ratio $\alpha:(1-\alpha)$ will turn out to equal the individual's subjective probability ratio $\mu(E_i):\mu(E_j)$ for the two events.

To convert this axiom from objective $\times$ subjective to almost-objective $\times$ subjective uncertainty, we replace the objective lotteries $\mathbf{P}_i$ and objective probability mixtures $\alpha \cdot \mathbf{P}_i + (1-\alpha) \cdot \mathbf{P}_j$ by “unit-interval acts” $\mathcal{P}_i$ and “unit-interval mixtures” $[0, \alpha] \cdot \mathcal{P}_i + (\alpha, 1] \cdot \mathcal{P}_j$. Given a finite-interval partition $\{\wp_1, \dots, \wp_m\}$ of $[0, 1]$ and outcomes $x_1, \dots, x_m$, define the *unit-interval act* $\mathcal{P} = [x_1 \text{ on } \wp_1; \dots; x_m \text{ on } \wp_m]$. Thus $\mathcal{P} \times_n E = [x_1 \text{ on } \wp_1 \times_n E; \dots; x_m \text{ on } \wp_m \times_n E]$ is an almost-objective act on E , and each almost-objective $\times$ subjective act can be expressed as $[\mathcal{P}_1 \times_n E_1; \dots; \mathcal{P}_m \times_n E_m]$. Given two unit-interval acts $\mathcal{P} = [x_1 \text{ on } \wp_1; \dots; x_m \text{ on } \wp_m]$ and $\mathcal{P}' = [x'_1 \text{ on } \wp'_1; \dots; x'_{m'} \text{ on } \wp'_{m'}]$, define their $[0, \alpha]:(\alpha, 1]$ *unit-interval mixture* as the unit-interval act $[0, \alpha] \cdot \mathcal{P} + (\alpha, 1] \cdot \mathcal{P}' = [x_1 \text{ on } \wp_1 \times [0, \alpha]; \dots; x_m \text{ on } \wp_m \times [0, \alpha]; x'_1 \text{ on } \wp'_1 \times (\alpha, 1]; \dots; x'_{m'} \text{ on } \wp'_{m'} \times (\alpha, 1)]$, that is, as the linear horizontal contraction of $\mathcal{P}$ from its domain $[0, 1]$ to the domain $[0, \alpha]$, combined with a similar contraction of $\mathcal{P}'$ to $(\alpha, 1]$. Thus the $[0, \frac{1}{3}]:(\frac{1}{3}, 1]$ unit-interval mixture of the acts $\mathcal{P} = [x_1 \text{ on } [0, \frac{1}{2}]; x_2 \text{ on } (\frac{1}{2}, 1]]$ and $\mathcal{P}' = [x'_1 \text{ on } [0, 1]]$ is $[x_1 \text{ on } [0, \frac{1}{6}]; x_2 \text{ on } (\frac{1}{6}, \frac{1}{3}]; x'_1 \text{ on } (\frac{1}{3}, 1]]$, and the $[0, \alpha]:(\alpha, 1]$ unit-interval mixture of the constant acts $\mathcal{P}_x = [x \text{ on } [0, 1]]$ and $\mathcal{P}_{x'} = [x' \text{ on } [0, 1]]$ is simply $[0, \alpha] \cdot \mathcal{P}_x + (\alpha, 1] \cdot \mathcal{P}_{x'} = [x \text{ on } [0, \alpha]; x' \text{ on } (\alpha, 1]]$.

The resulting axiom thus states that the intervals $[0, \alpha]:(\alpha, 1]$ for which individual is willing to replace a pair of distinct prizes (pure outcomes or almost-objective acts) over the events E_i and E_j with a unit-interval mixture of these same prizes over $E_i \cup E_j$ will (in the limit) not depend upon the specific prizes involved, or upon the prizes assigned to other any event E_k :¹⁸

ALMOST-OBJECTIVE / SUBJECTIVE REPLACEMENT AXIOM: For any partition $\{E_1, \dots, E_m\}$ of S , if

$$(40) \quad W \left(\begin{array}{c} x \text{ on } E_i \\ x' \text{ on } E_j \\ x' \text{ on } E_k \quad k \neq i, j \end{array} \right) = \lim_{n \rightarrow \infty} W \left(\begin{array}{c} [[0, \alpha] \cdot \mathcal{P}_x + (\alpha, 1] \cdot \mathcal{P}_{x'}] \times_n E_i \\ [[0, \alpha] \cdot \mathcal{P}_x + (\alpha, 1] \cdot \mathcal{P}_{x'}] \times_n E_j \\ \mathcal{P}_{x'} \quad \quad \quad \text{on } E_k \quad k \neq i, j \end{array} \right)$$

for some $\alpha \in [0, 1]$, events E_i, E_j , and outcomes $x \succ x'$, then

$$(41) \quad \lim_{n \rightarrow \infty} W \left(\begin{array}{c} \mathcal{P}_i \times_n E_i \\ \mathcal{P}_j \times_n E_j \\ \mathcal{P}_k \times_n E_k \quad k \neq i, j \end{array} \right) = \lim_{n \rightarrow \infty} W \left(\begin{array}{c} [[0, \alpha] \cdot \mathcal{P}_i + (\alpha, 1] \cdot \mathcal{P}_j] \times_n E_i \\ [[0, \alpha] \cdot \mathcal{P}_i + (\alpha, 1] \cdot \mathcal{P}_j] \times_n E_j \\ \mathcal{P}_k \quad \quad \quad \times_n E_k \quad k \neq i, j \end{array} \right)$$

for all unit-interval acts $\mathcal{P}_1, \dots, \mathcal{P}_m$.

An argument parallel to that of Machina and Schmeidler (1995) then yields the following result, where $\alpha:(1-\alpha)$ will again turn out to be the individual's subjective probability ratio $\mu(E_i):\mu(E_j)$:

THEOREM 7 (ANSCOMBE-AUMANN WITHOUT OBJECTIVE UNCERTAINTY): If an outcome-monotonic, event-smooth preference function $W(\cdot)$ over $\mathcal{A}$ satisfies the Almost-Objective / Subjective Replacement Axiom, then it takes the globally probabilistically sophisticated form $W(f(\cdot)) \equiv V(\dots; x_i, \mu(E_i); \dots)$ for some finitely-additive probability measure $\mu(\cdot)$.

¹⁸ An equivalent statement of this axiom, in terms of the ordinal preference relation, would replace each preference function equality by the condition that its left and right side acts have the same limiting indifference class in $\mathcal{A}$.

4.2 Risk Preferences in the Absence of Probabilistic Sophistication

As seen in Theorem 3, each event-smooth $W(\cdot)$ over subjective acts – even if it is state-dependent or non-probabilistically sophisticated or – has an associated preference function $V_W(\cdot)$ over objective lotteries that completely represents its limiting preferences over almost-objective acts, in the sense that $\lim_{n \rightarrow \infty} W(x_1 \text{ on } \wp_1 \times_n S; \dots; x_m \text{ on } \wp_m \times_n S) = V_W(x_1, \lambda(\wp_1); \dots; x_m, \lambda(\wp_m))$. Since as $n \rightarrow \infty$ each event-smooth $W(\cdot)$ treats such events $\wp_i \times_n S$ and acts $(x_1 \text{ on } \wp_1 \times_n S; \dots; x_m \text{ on } \wp_m \times_n S)$ as if they were objective in each of the senses of Section 2.2, $V_W(\cdot)$ can be said to represent the *objective risk preferences* of the completely subjective preference function $W(\cdot)$. Among other things, this allows us to

- explicitly derive and represent an individual's attitudes toward objective risk in a completely subjective setting, even in the presence of state-dependence, nonexistence of beliefs, or ambiguity aversion or preference
- directly compare the objective risk preferences of such individuals, even if they are qualitatively different in the structure of their subjective act preferences, state-dependence/state-dependence, existence/nonexistence of beliefs, or attitudes toward ambiguity
- identify the objective risk preferences implicitly (and perhaps unintentionally) implied by different models of probabilistic or non-probabilistic beliefs under subjective uncertainty

For example, Theorem 6 implies that the objective risk preferences of state-independent expected utility $W_{SEU}(f(\cdot)) \equiv \int_S U(f(s)) \cdot d\mu(s)$ take the expected utility form $V_{W_{SEU}}(\mathbf{P}) \equiv \sum_{i=1}^m U(x_i) \cdot p_i$, and those of state-dependent expected utility $W_{SDEU}(f(\cdot)) \equiv \int_S U(f(s)|s) \cdot d\mu(s)$ also take the expected utility form $V_{W_{SDEU}}(\mathbf{P}) \equiv \sum_{i=1}^m \hat{U}(x_i) \cdot p_i$ for $\hat{U}(x) \equiv \int_S U(x|s) \cdot d\mu(s)$. Thus, if $\hat{U}(\cdot)$ should be a concave function of wealth, the subjective preference function $W_{SDEU}(\cdot)$ will have risk averse limiting preferences over all almost-objective acts, even if its underlying utility function $U(\cdot|s)$ happens to be convex in wealth over certain ranges of the state space S . For the probabilistically sophisticated form $W_{PS}(f(\cdot)) \equiv V(\dots; x_i, \mu(E_i); \dots)$, Theorem 0 implies

$$(42) \quad \begin{aligned} \lim_{n \rightarrow \infty} W_{PS}(x_1 \text{ on } \wp_1 \times_n S; \dots; x_m \text{ on } \wp_m \times_n S) &\equiv \lim_{n \rightarrow \infty} V(x_1, \mu(\wp_1 \times_n S); \dots; x_m, \mu(\wp_m \times_n S)) \\ &\equiv V(x_1, \lim_{n \rightarrow \infty} \mu(\wp_1 \times_n S); \dots; x_m, \lim_{n \rightarrow \infty} \mu(\wp_m \times_n S)) \equiv V(x_1, \lambda(\wp_1); \dots; x_m, \lambda(\wp_m)) \end{aligned}$$

so that $W_{PS}(\cdot)$'s objective risk preference function $V_{W_{PS}}(\cdot)$ is identical to underlying function $V(\cdot)$.

Two important non-probabilistically sophisticated functional forms for $W(\cdot)$, which have been used to model both the Ellsberg Paradox and general attitudes toward ambiguity, are

$$\begin{aligned} W_{maxmin}(f(\cdot)) &\equiv \min_{\tau \in T} \sum_{i=1}^m U(x_i) \cdot \mu_\tau(E_i) && \text{maxmin expected utility, for some utility function } U(\cdot) \text{ and} \\ &&& \text{family of subjective probability measures } \{\mu_\tau(\cdot) | \tau \in T\} \text{ (e.g.} \\ &&& \text{Gilboa and Schmeidler (1989))} \\ W_{Choquet}(f(\cdot)) &\equiv \int_S U(f(s)) \cdot dC(s) && \text{Choquet expected utility, for some utility function } U(\cdot) \text{ and} \\ &\equiv \sum_{i=1}^m U(x_i) \cdot [C(\cup_{j=1}^i E_j) - C(\cup_{j=1}^{i-1} E_j)] && \text{subjective capacity (nonadditive measure) } C(\cdot) \text{ (e.g. Gilboa} \\ &&& \text{(1987))} \end{aligned}$$

If $W_{maxmin}(\cdot)$ is *finitely-generated* in the sense that T is a finite collection of continuous-density probability measures over S , or the convex hull of any such finite collection, Theorem 0 implies

$$(43) \quad \begin{aligned} \lim_{n \rightarrow \infty} W_{maxmin}(x_1 \text{ on } \wp_1 \times_n S; \dots; x_m \text{ on } \wp_m \times_n S) &= \lim_{n \rightarrow \infty} [\max_{\tau \in T} \sum_{i=1}^m U(x_i) \cdot \mu_\tau(\wp_i \times_n S)] \\ &= \max_{\tau \in T} \sum_{i=1}^m U(x_i) \cdot \lim_{n \rightarrow \infty} \mu_\tau(\wp_i \times_n S) = \max_{\tau \in T} \sum_{i=1}^m U(x_i) \cdot \lambda(\wp_i) = \sum_{i=1}^m U(x_i) \cdot \lambda(\wp_i) \end{aligned}$$

so the objective risk preferences of any finitely-generated $W_{maxmin}(\cdot)$ take the expected utility form

$V_{W_{\maxmin}}(\mathbf{P}) \equiv \sum_{i=1}^m U(x_i) \cdot p_i$. For the form $W_{\text{Choquet}}(\cdot)$, define $G(p) \equiv \lim_{p \in [0,1]} C([0,p] \times_n S)$. Given outcomes $\bar{x} \succ x$, for each finite interval union $\wp \subseteq [0,1]$, event-smoothness and Theorem 3 yield

$$\begin{aligned}
(44) \quad & U(x) \cdot \lim_{n \rightarrow \infty} C(\wp \times_n S) + U(\bar{x}) \cdot [1 - \lim_{n \rightarrow \infty} C(\wp \times_n S)] = \lim_{n \rightarrow \infty} W_{\text{Choquet}}(x \text{ on } \wp \times_n S; \bar{x} \text{ on } S - \wp \times_n S) \\
& = \lim_{n \rightarrow \infty} W_{\text{Choquet}}(x \text{ on } [0, \lambda(\wp)] \times_n S; \bar{x} \text{ on } S - [0, \lambda(\wp)] \times_n S) \\
& = U(x) \cdot \lim_{n \rightarrow \infty} C([0, \lambda(\wp)] \times_n S) + U(\bar{x}) \cdot [1 - \lim_{n \rightarrow \infty} C([0, \lambda(\wp)] \times_n S)]
\end{aligned}$$

so that $\lim_{n \rightarrow \infty} C(\wp \times_n S) = \lim_{n \rightarrow \infty} C([0, \lambda(\wp)] \times_n S) = G(\lambda(\wp))$. Thus for each almost-objective act, we have

$$\begin{aligned}
(45) \quad & \lim_{n \rightarrow \infty} W_{\text{Choquet}}(x_1 \text{ on } \wp_1 \times_n S; \dots; x_m \text{ on } \wp_m \times_n S) = \lim_{n \rightarrow \infty} \sum_{i=1}^m U(x_i) \cdot [C(\bigcup_{j=1}^i \wp_j \times_n S) - C(\bigcup_{j=1}^{i-1} \wp_j \times_n S)] \\
& = \sum_{i=1}^m U(x_i) \cdot [G(\sum_{j=1}^i \lambda(\wp_j)) - G(\sum_{j=1}^{i-1} \lambda(\wp_j))]
\end{aligned}$$

so $W_{\text{Choquet}}(\cdot)$'s objective risk preferences $V_{W_{\text{Choquet}}}(\mathbf{P}) \equiv \sum_{i=1}^m U(x_i) \cdot [G(\bigcup_{j=1}^i p_j) - G(\bigcup_{j=1}^{i-1} p_j)]$ take the *rank-dependent expected utility* form of Quiggin (1982).

Formulas (43) and (45) imply that, while the finitely-generated maxmin expected utility form and the Choquet expected utility form both serve to represent *Ellsberg Paradox* type preferences, only the latter can also represent *Allais Paradox* (or Common Ratio Effect, or Certainty Effect) type preferences over almost-objective prospects. For example, consider the following betting preferences over the temperature s , where s_{tr} denotes s truncated to the nearest integer:

$$\begin{aligned}
(46) \quad & \begin{bmatrix} \$100 \text{ if } s_{tr} \text{ is divisible by 3} \\ \$0 \text{ if } s_{tr} \text{ is not, and } s \geq 65^\circ \\ \$0 \text{ if } s_{tr} \text{ is not, and } s < 65^\circ \end{bmatrix} \succ \begin{bmatrix} \$0 \text{ if } s_{tr} \text{ is divisible by 3} \\ \$100 \text{ if } s_{tr} \text{ is not, and } s \geq 65^\circ \\ \$0 \text{ if } s_{tr} \text{ is not, and } s < 65^\circ \end{bmatrix} \\
& \begin{bmatrix} \$100 \text{ if } s_{tr} \text{ is divisible by 3} \\ \$0 \text{ if } s_{tr} \text{ is not, and } s \geq 65^\circ \\ \$100 \text{ if } s_{tr} \text{ is not, and } s < 65^\circ \end{bmatrix} \prec \begin{bmatrix} \$0 \text{ if } s_{tr} \text{ is divisible by 3} \\ \$100 \text{ if } s_{tr} \text{ is not, and } s \geq 65^\circ \\ \$100 \text{ if } s_{tr} \text{ is not, and } s < 65^\circ \end{bmatrix}
\end{aligned}$$

Both the finite-family $W_{\maxmin}(\cdot)$ and $W_{\text{Choquet}}(\cdot)$ forms can represent the above Ellsberg-type preferences by suitable choice of family of probability measures $\{\mu_t(\cdot) | t \in T\}$ or capacity $C(\cdot)$ over S . However, consider the following almost-objective bets on the same uncertain variable s , where $s_{(k,k+1)}$ denotes its k 'th and $k+1$ st decimal digits, treated as a 2-digit number from 0 to 99:

$$\begin{aligned}
(47) \quad & \begin{bmatrix} \$1\text{M if } s_{(k,k+1)} \in \{0\} \\ \$1\text{M if } s_{(k,k+1)} \in \{1, \dots, 10\} \\ \$1\text{M if } s_{(k,k+1)} \in \{11, \dots, 99\} \end{bmatrix} \succ \begin{bmatrix} \$0\text{M if } s_{(k,k+1)} \in \{0\} \\ \$5\text{M if } s_{(k,k+1)} \in \{1, \dots, 10\} \\ \$1\text{M if } s_{(k,k+1)} \in \{11, \dots, 99\} \end{bmatrix} \\
& \begin{bmatrix} \$1\text{M if } s_{(k,k+1)} \in \{0\} \\ \$1\text{M if } s_{(k,k+1)} \in \{1, \dots, 10\} \\ \$0 \text{ if } s_{(k,k+1)} \in \{11, \dots, 99\} \end{bmatrix} \prec \begin{bmatrix} \$0\text{M if } s_{(k,k+1)} \in \{0\} \\ \$5\text{M if } s_{(k,k+1)} \in \{1, \dots, 10\} \\ \$0 \text{ if } s_{(k,k+1)} \in \{11, \dots, 99\} \end{bmatrix}
\end{aligned}$$

For large k , event-smooth individuals would come to view the events in these subjective acts as probabilistic, and the acts themselves as equivalent to the standard Allais Paradox lotteries, and hence most would exhibit the displayed preferences. While (45) implies that such preferences *can* be represented by the form $W_{\text{Choquet}}(\cdot)$, (43) implies they *cannot* be represented by the finitely-generated form $W_{\maxmin}(\cdot)$, which suggests that the former model is able to capture a broader range of violations of the classical expected utility/subjective probability model than the latter.

4.3 Generalized Expected Utility Analysis under Subjective Uncertainty

The approach of “generalized expected utility analysis” developed in Machina (1982)¹⁹ provides a means by which the basic concepts, tools and result of expected utility analysis under objective uncertainty can be extended to general probability-smooth preference functions $V(\cdot)$ over objective lotteries. The key ideas in this extension are that the observations that

- the classical expected utility form $V_{EU}(\mathbf{P}) \equiv \sum_{i=1}^m U(x_i) \cdot p_i$ is *linear in the probabilities*
- standard expected utility concepts and results can thus be phrased in terms of $V_{EU}(\cdot)$ ’s *probability coefficients* $\{U(\cdot)|x \in \mathcal{X}\}$, where $U(x) = \text{coefficient of prob}(x)$
- standard calculus arguments allow us to extend most of these probability coefficient results to the *probability derivatives* $U(x; \mathbf{P}) \equiv \partial V(\mathbf{P}) / \partial \text{prob}(x)$ of a general probability-smooth $V(\cdot)$

The key step in extending coefficient-based results to derivative-based results lies in integrating along the straight line path – that is, the probability mixture path $\mathbf{P}_\alpha \equiv \alpha \cdot \mathbf{P}^* + (1-\alpha) \cdot \mathbf{P}$ – between two objective lotteries $\mathbf{P}$ and $\mathbf{P}^*$. For an expected utility $V_{EU}(\cdot)$ this line integral takes the form

$$(48) \quad V_{EU}(\mathbf{P}^*) - V_{EU}(\mathbf{P}) = \int_0^1 \frac{dV_{EU}(\alpha \cdot \mathbf{P}^* + (1-\alpha) \cdot \mathbf{P})}{d\alpha} d\alpha = \int_0^1 \sum_{x \in \mathcal{X}} U(x) \cdot [\text{prob}^*(x) - \text{prob}(x)] \cdot d\alpha$$

where the integrand denotes how the outcomewise probability changes $\{\text{prob}^*(x) - \text{prob}(x) | x \in \mathcal{X}\}$ between $\mathbf{P}$ and $\mathbf{P}^*$ are evaluated by $V_{EU}(\cdot)$ ’s probability coefficients $\{U(x) | x \in \mathcal{X}\}$. For a general probability-smooth $V(\cdot)$, this line integral takes the corresponding form

$$(49) \quad V(\mathbf{P}^*) - V(\mathbf{P}) = \int_0^1 \frac{dV(\alpha \cdot \mathbf{P}^* + (1-\alpha) \cdot \mathbf{P})}{d\alpha} d\alpha = \int_0^1 \sum_{x \in \mathcal{X}} U(x; \mathbf{P}_\alpha) \cdot [\text{prob}^*(x) - \text{prob}(x)] \cdot d\alpha$$

where the integrand now denotes how these probability changes are evaluated by $V(\cdot)$ ’s probability derivatives $\{U(x; \mathbf{P}_\alpha) | x \in \mathcal{X}\}$ at each distribution along the path $\{\mathbf{P}_\alpha | \alpha \in [0, 1]\}$. Thus the condition that $U(x; \mathbf{P})$ be concave in x at each $\mathbf{P}$ is both necessary and sufficient for the integrand in (49), as well its entire integrated value $V(\mathbf{P}^*) - V(\mathbf{P})$, to be nonpositive whenever $\mathbf{P}^*$ differs from $\mathbf{P}$ by a mean-preserving increase in risk, so that the expected utility characterization of risk aversion by the concavity of a cardinal function extends to general probability-smooth $V(\cdot)$. Similar extensions hold for a large body of expected utility results under objective uncertainty.

Machina (2002) extends this approach to subjective uncertainty, via the parallel arguments:

- the classical expected utility/subjective probability forms $W_{SEU}(f(\cdot)) \equiv \sum_{i=1}^m U(x_i) \cdot \mu(E_i)$ and $W_{SDEU}(f(\cdot)) \equiv \sum_{i=1}^m \int_{E_i} U(x|s) \cdot d\mu(s)$ are both *additive in the events*
- standard expected utility/subjective probability results can be phrased in terms of these functions’ *evaluation measures* $\Phi_x(E) \equiv U(x) \cdot \mu(E)$ for $W_{SEU}(\cdot)$ and $\Phi_x(E) \equiv \int_E U(x|s) \cdot d\mu(s)$ for $W_{SDEU}(\cdot)$
- standard calculus arguments allow us to extend most of these evaluation measure results to the *event-derivatives* (local evaluation measures) $\{\Phi_x(\cdot; f) | x \in \mathcal{X}\}$ of a general event-smooth $W(\cdot)$

The analogue of (48)/(49) would be a line integral along a path $\{f_\alpha(\cdot) | \alpha \in [0, 1]\}$ from $f(\cdot)$ to $f^*(\cdot)$ whose integrand denotes how the *event changes* $\{f^{*-1}(x) - f^{-1}(x) | x \in \mathcal{X}\}$ are evaluated by $W(\cdot)$ ’s local evaluation measures $\{\Phi_x(\cdot; f_\alpha) | x \in \mathcal{X}\}$ at each act along the path. But as shown in Machina (2002), paths with this exact property cannot exist in the space $\mathcal{A}$, essentially because one cannot take exact “convex combinations” of subjective events. However, *almost-objective mixture paths* $f_\alpha(\cdot) \equiv [f^*(\cdot) \text{ on } [0, \alpha] \times_n S; f(\cdot) \text{ on } (\alpha, 1] \times_n S]$ can sufficiently closely approximate this property to yield exact, global extensions of the most basic classical results, including characterizations of comparative subjective likelihood, relative subjective likelihood, and comparative risk aversion.

¹⁹ See also the extensions and applications of Allen (1987), Bardsley (1993), Chew, Epstein and Zilcha (1988), Chew, Karni and Safra (1987), Chew and Nishimura (1992), Karni (1987, 1989) and Wang (1993).

4.4 Markets for Almost-Objective versus Purely Subjective Bets

Individuals facing subjective uncertainty have at least four sources of gains from trade: attitudes toward risk (either risk preference or differences in risk aversion), differences in beliefs, hedging against uncertain endowments, and diversification of multiple risks. The first of these sources most naturally leads to trade in almost-objective bets, and the latter three most naturally lead to trade in purely subjective bets.

For example, if individuals $W(\cdot)$ and $W^*(\cdot)$ have nonrandom endowments x_0 and x_0^* and associated risk preference functions $V_W(\cdot)$ and $V_{W^*}(\cdot)$, at least one of which is risk loving, then there exists an objective lottery $\hat{\mathbf{P}} = (\dots; \hat{x}_i, \hat{p}_i; \dots)$ with $V_W(\dots; x_0 + \hat{x}_i, \hat{p}_i; \dots) > V_W(x_0, 1)$ and $V_{W^*}(\dots; x_0^* - \hat{x}_i, \hat{p}_i; \dots) > V_{W^*}(x_0^*, 1)$, and hence an almost-objective bet $\hat{f}_n(\cdot) = [\dots; \hat{x}_i \text{ on } \hat{\phi}_i \times_n S; \dots]$ such that $W(\dots; x_0 + \hat{x}_i \text{ on } \hat{\phi}_i \times_n S; \dots) > W(x_0 \text{ on } S)$ and $W^*(\dots; x_0^* - \hat{x}_i \text{ on } \hat{\phi}_i \times_n S; \dots) > W^*(x_0^* \text{ on } S)$. This phenomenon underlies the extensive amount of real-world betting on coins, balanced roulette wheels and bingo cages, which as noted above, are examples of almost-objective bets generated from subjectively uncertain driving variables. Since the event likelihoods in such bets are unaffected by differences in subjective beliefs, trade in them is not subject to the usual difficulties that arise from differential information or moral hazard. If one individual has an almost-objective endowment and both are risk averse but not equally so, a similar opportunity for trade in almost-objective bets will arise.

Given the prevalence of betting on such almost-objective events, one might ask why *financial markets* don't offer what might be termed *almost-objective securities or options*. Such instruments would not be difficult to create – they could be defined to pay off based on the k 'th decimal of any security price, market index or publicly observed variable. But it is precisely *because* of their objective-like properties that such instruments could not serve any of the standard purposes of financial instruments, which are second, third and fourth of the above reasons for trade:

- Since they would approximate the objective property of *unanimous likelihoods*, such almost-objective instruments *would not permit gains from trade based on differences in beliefs*, which (as the saying goes) are a primary reason for horse races, as well as for trade in financial instruments such as individual securities or standard (i.e., purely subjective) options.
- Since they would approximate the objective property of *independence of the realization of subjective events*, almost-objective instruments based on a particular index or security price *could not serve to hedge* against the downside (or upside) risk in that variable, which is another primary purpose of options and certain other instruments.
- Since they would approximate *objective probability mixtures* rather than *payoff mixtures*, almost-objective mixtures of securities *could not serve to diversify* independent risks. Thus, if a probabilistically sophisticated investor believed two securities to be independent and identically distributed, he or she would view any almost-objective mixture of them as having *approximately the same distribution* as each individual security, rather than one with a lower variance.

In other words, financial markets do not trade in almost-objective securities for the same reasons that they do not trade in bets based on physical randomizing devices. Although we do not do so here, a formalization of the above arguments would involve showing that for individuals with degenerate or almost-objectively uncertain endowments, different risk preferences and identical beliefs, trade in almost-objective bets would (in the limit) weakly Pareto dominate trade in purely subjective bets, and for individuals with degenerate or purely subjectively uncertain endowments, identical risk preferences and different beliefs, trade in purely subjective bets would weakly Pareto dominate trade in almost-objective bets.

5. COMPARISON AND EXTENSIONS

5.1 Comparison with Related Results under Subjective Uncertainty

Comparison with Ghirardato-Maccheroni-Marinacci-Siniscalchi Mixtures

Ghirardato, Maccheroni, Marinacci and Siniscalchi (2001) present an alternative sense in which a purely subjective act can be thought of as a cardinal mixture of two other subjective acts. In their notation (where $x E y$ denotes the act $[x \text{ on } E; y \text{ on } \sim E]$), if the outcomes $x, y, z, c_{xEz}, c_{zEy}$ satisfy

$$(50) \quad c_{xEz} \sim x E z \quad c_{zEy} \sim z E y \quad x E y \sim c_{xEz} E c_{zEy}$$

then the outcome z can be described as a 50:50 “preference average” of x and y .²⁰ A 75:25 preference average of x and y can then be obtained by finding a 50:50 preference average of x and z , and so on. These authors then define a 50:50 mixture of two subjective acts $f(\cdot)$ and $g(\cdot)$ as any subjective act whose outcome in each state s is a 50:50 preference average of the outcomes $f(s)$ and $g(s)$ in that state, and 75:25 (and other) mixtures of $f(\cdot)$ and $g(\cdot)$ are defined similarly.

Unlike objective mixtures $\alpha f(\cdot) + (1-\alpha)g(\cdot)$ or almost-objective mixtures $[f(\cdot) \text{ on } \wp_n^\times S; g(\cdot) \text{ on } \sim \wp_n^\times S]$ – which mix objective probabilities or subjective events and are thus defined independently of act preferences – GMMS mixtures require a preference function $W(\cdot)$ with a cardinal utility function $U(\cdot)$, in order to generate individual-specific *preference mixtures* of each outcome pair $f(s), g(s)$. Thus, GMMS mixtures can be described as preference-based “subjective cardinal mixtures,” and these authors have shown how they can provide an alternative basis for axiomatizing the *SEU*, Choquet and maxmin models of preferences over subjective acts.

Comparison with Liapunov’s Convexity Theorem

Liapunov (1940) provides an important result which, though not explicitly choice-theoretic, can still be interpreted as implying the existence of a family of events with unanimously agreed-upon subjective probabilities for a given population of individuals. Most generally, it states that the range of every finite, vector-valued, non-atomic, countably additive measure $\Psi(\cdot) = (\psi_1(\cdot), \dots, \psi_K(\cdot))$ is closed and convex.²¹ In our present setting, this implies the result:

“Given a family of nonatomic probability measures $\{\mu_1(\cdot), \dots, \mu_K(\cdot)\}$ over S , for each probability $p \in [0, 1]$ there exists an event $\bar{E}_p \subseteq S$ such that $\mu_k(\bar{E}_p) = p$ for $k = 1, \dots, K$.”

This result is analogous to Theorem 1 in that it identifies events that are unanimously assigned a common likelihood by every member of some prespecified family of individuals. One aspect of this result that is *stronger* than in Theorem 1 is that the values $\mu_k(\bar{E}_p) = p$ are *exact* for each individual k and event $\bar{E}_p$, whereas the revealed likelihoods in Theorem 1 are approximate and only hold exactly in the limit. On the other hand, an aspect of this result that is *weaker* than in Theorem 1 is its restriction to a *finite* family of individual measures $\{\mu_1(\cdot), \dots, \mu_K(\cdot)\}$.²²

Another aspect of this result that is weaker than in Theorem 1 is its restriction to *measures* $\{\mu_1(\cdot), \dots, \mu_K(\cdot)\}$ as the “inputs.” Thus, for it to apply to a family $\{W_1(\cdot), \dots, W_K(\cdot)\}$ of preference functions over subjective acts, it would have to be a family of *probabilistically sophisticated* preference functions $W_k(f(\cdot)) \equiv V_k(\dots, x_i, \mu_k(E_i); \dots)$ or some other measure-based form. This contrasts with Theorem 1, which applies to all event-smooth preference functions $W(\cdot)$.

²⁰ E.g., under *SEU* or rank-dependent expected utility risk preferences, (50) would imply $U(z) = \frac{1}{2}U(x) + \frac{1}{2}U(y)$.

²¹ See also the simplified proofs of Halmos (1948) and Lindenstrauss (1966).

²² Halmos (1948, Note 1) credits Liapunov (1946) with showing that neither convexity nor closure necessarily follows in the case of an *infinite* number of measures.

5.2 Analytical Extensions

Extension to More General Events

Although the almost-ethically-neutral and almost-objective events considered above had an exactly periodic structure over the state space S , the analysis of paper will extend to more general types of events. For example, instead of events $\wp_n \times S$ based on a fixed subset $\wp \subseteq [0,1]$ and n equal-length intervals, we could define events E_n based on n unequal-length intervals, such as

$$(51) \quad [\underline{s}, \underline{s} + \psi(\frac{1}{n})], [\underline{s} + \psi(\frac{1}{n}), \underline{s} + \psi(\frac{2}{n})], \dots [\underline{s} + \psi(\frac{i}{n}), \underline{s} + \psi(\frac{i+1}{n})], \dots [\underline{s} + \psi(\frac{n-1}{n}), \bar{s}]$$

for some increasing and suitably regular function²³ $\psi(\cdot): [0,1] \rightarrow [0, \lambda_s]$, where the union $\wp_n \times S = \bigcup_{i=0}^{n-1} \{\underline{s} + (i+\omega) \cdot (\lambda_s/n) \mid \omega \in \wp\}$ is replaced by

$$(52) \quad E_n = \bigcup_{i=0}^{n-1} \{\underline{s} + \psi(\frac{i}{n}) + \omega \cdot (\psi(\frac{i+1}{n}) - \psi(\frac{i}{n})) \mid \omega \in \wp\}$$

For each n , the Lebesgue measure of E_n will continue to be exactly $\lambda(\wp) \cdot \lambda(S)$. As $n \rightarrow \infty$, the proofs of the above theorems can be appropriately adapted, and the events (52) can be thus also be described as almost-objective, with a unanimous limiting revealed likelihood of $\lambda(\wp)$.

Conversely, one could also define a sequence of events E'_n based on the n equal-length intervals but a (regularly) *variable* collection of subsets $\{\wp_{1,1}, \dots, \wp_{1,n}\}$, with the requirement that $\lambda(\wp_{n,i}) = \lambda_\wp$ for each $i = 1, \dots, n$ and each $n = 1, 2, \dots$. Replacing the events $\wp_n \times S$ by

$$(53) \quad E'_n = \bigcup_{i=0}^{n-1} \{\underline{s} + (i+\omega) \cdot \frac{\lambda_s}{n} \mid \omega \in \wp_{n,i}\} \quad n = 1, 2, \dots$$

again yields a unanimous limiting revealed likelihood, of $\lambda_\wp$. Of course, any combination of the generalizations (52) and (53) will yield similar results. With proper respect for issues of uniform convergence, the most general definition of “almost-objective” events E_n with a limiting likelihood of $\rho \in [0,1]$ would presumably be one satisfying a condition such as

$$(54) \quad \lim_{n \rightarrow \infty} \lambda(E_n \cap [a,b]) = \rho \cdot (b-a) \quad \text{all } [a,b] \subseteq [\underline{s}, \bar{s}]$$

Extension to More General State Spaces

The approach of this paper can also be generalized to more general state spaces. Given any multivariate box $S = [\underline{s}_1, \bar{s}_1] \times \dots \times [\underline{s}_K, \bar{s}_K] \subset R^K$ and any n , we could replace the n equal-length intervals in (2) by the n^K equal-volume boxes obtained by partitioning S in each dimension. Provided proper regularity is maintained, our approach would also extend to univariate or multivariate manifolds (such as the surface of a sphere), or any space that can be “tiled” by arbitrarily small but suitably comparable measure spaces.

Extension to More General Preferences

Although we have assumed event-smoothness, it is clear that both Poincaré’s Theorem and the approach of this paper will also hold for more general preferences, such as $W_{SEU}(f(\cdot)) \equiv \int_S U(f(s)) \cdot m(s) \cdot ds$ for piecewise-continuous $m(\cdot)$, or piecewise event-smooth $W(\cdot)$. However, to see that event-continuity alone is not sufficient, and that some type of absolute continuity condition is needed for both Poincaré Theorem and the analysis of this paper, observe that the standard Cantor measure²⁴ $C(\cdot)$ over $S = [\underline{s}, \bar{s}]$ will *not* satisfy the property $\lim_{n \rightarrow \infty} C(E_n) = \frac{1}{3}$ for the events $E_n = (\frac{1}{3}, \frac{2}{3})_n \times S$, since $C(E_n) = 0$ whenever n is a power of 3.

²³ For example, $\psi(\cdot)$ would be suitably regular provided $\psi'(\cdot)$ was bounded away from both 0 and $+\infty$ over $[0,1]$.

²⁴ E.g., Billingsley (1986, pp.427-429), Feller (1971, pp.35-36), Romano and Siegel (1986, pp.27-28).

APPENDIX – PROOFS OF THEOREMS

To see how event-differentiability can be applied to analyze $W(\cdot)$'s ranking of acts, define the *uniform single-sweep path* $\{f_\omega(\cdot) | \omega \in [\underline{s}, \bar{s}]\}$ from the constant act $[x_0 \text{ on } S]$ to $[x^* \text{ on } S]$ by

$$(A.1) \quad f_\omega(\cdot) \equiv [x^* \text{ on } [\underline{s}, \omega], x_0 \text{ on } (\omega, \bar{s}]] \quad \omega \in [\underline{s}, \bar{s}]$$

As ω runs from $\underline{s}$ to $\bar{s}$, the outcome x_0 is seen to be replaced by x^* over an expanding event $[\underline{s}, \omega]$ that uniformly “sweeps” across the state space $S = [\underline{s}, \bar{s}]$. At each $\hat{\omega}$, (15) and (11) imply

$$(A.2) \quad \begin{aligned} W(f_{\hat{\omega}+\Delta\omega}(\cdot)) - W(f_{\hat{\omega}}(\cdot)) &= \Phi_{x^*}([\underline{s}, \hat{\omega}+\Delta\omega]; f_{\hat{\omega}}) + \Phi_{x_0}((\hat{\omega}+\Delta\omega, \bar{s}]; f_{\hat{\omega}}) \\ &\quad - \Phi_{x^*}([\underline{s}, \hat{\omega}]; f_{\hat{\omega}}) - \Phi_{x_0}((\hat{\omega}, \bar{s}]; f_{\hat{\omega}}) + o(|\Delta\omega|) \end{aligned}$$

Dividing both sides by $\Delta\omega$ and letting $\Delta\omega \rightarrow 0$ yields

$$(A.3) \quad \left. \frac{dW(f_\omega(\cdot))}{d\omega} \right|_{\omega=\hat{\omega}} = \phi_{x^*}(\hat{\omega}; f_{\hat{\omega}}) - \phi_{x_0}(\hat{\omega}; f_{\hat{\omega}})$$

In other words, starting at the act $f_{\hat{\omega}}(\cdot)$, the differential effect of replacing x_0 by x^* at state $s = \hat{\omega}$ is given by the difference $\phi_{x^*}(\hat{\omega}; f_{\hat{\omega}}) - \phi_{x_0}(\hat{\omega}; f_{\hat{\omega}})$. From (13) it follows that for the form $W_{SEU}(\cdot)$ this difference reduces to $[U(x^*) - U(x_0)] \cdot m(\hat{\omega})$, and for $W_{SDEU}(\cdot)$ it reduces to $[U(x^*|\hat{\omega}) - U(x_0|\hat{\omega})] \cdot m(\hat{\omega})$. Applying the Fundamental Theorem of Calculus to (A.3) then yields the path integral formula

$$(A.4) \quad W(x^* \text{ on } S) - W(x_0 \text{ on } S) = \int_{\underline{s}}^{\bar{s}} \frac{dW(f_\omega(\cdot))}{d\omega} \cdot d\omega = \int_{\underline{s}}^{\bar{s}} [\phi_{x^*}(\omega; f_\omega) - \phi_{x_0}(\omega; f_\omega)] \cdot d\omega$$

which exactly characterizes $W(\cdot)$'s comparison of the acts $[x_0 \text{ on } S]$ versus $[x^* \text{ on } S]$ in terms of its local evaluation densities at the acts $f_\omega(\cdot)$ along the path $\{f_\omega(\cdot) | \omega \in [\underline{s}, \bar{s}]\}$ between them.

Define the uniform single-sweep path between a pair of *nonconstant* acts $f(\cdot)$ to $f^*(\cdot)$ in $\mathcal{A}$ by

$$(A.5) \quad f_\omega(\cdot) \equiv [f^*(\cdot) \text{ on } [\underline{s}, \omega], f(\cdot) \text{ on } (\omega, \bar{s}]]$$

in which case $W(\cdot)$'s path derivative and path integral formulas are similarly given by

$$(A.6) \quad \left. \frac{dW(f_\omega(\cdot))}{d\omega} \right|_{\omega=\hat{\omega}} = \phi_{f^*(\hat{\omega})}(\hat{\omega}; f_{\hat{\omega}}) - \phi_{f(\hat{\omega})}(\hat{\omega}; f_{\hat{\omega}})$$

$$(A.7) \quad W(f^*(\cdot)) - W(f(\cdot)) = \int_{\underline{s}}^{\bar{s}} \frac{dW(f_\omega(\cdot))}{d\omega} \cdot d\omega = \int_{\underline{s}}^{\bar{s}} [\phi_{f^*(\omega)}(\omega; f_\omega) - \phi_{f(\omega)}(\omega; f_\omega)] \cdot d\omega$$

which again characterize $W(\cdot)$'s evaluation of $f(\cdot)$ versus $f^*(\cdot)$ in terms of its local evaluation densities at the acts $f_\omega(\cdot)$ along the path $\{f_\omega(\cdot) | \omega \in [\underline{s}, \bar{s}]\}$ between them. $W(\cdot)$'s path derivative along an *almost-objective mixture path* $f_\alpha(\cdot) \equiv [f^*(\cdot) \text{ on } [0, \alpha] \times_n S; f(\cdot) \text{ on } (\alpha, 1] \times_n S]$ from $f(\cdot)$ to $f^*(\cdot)$ will be given by the “ n -sweep” version of (A.6), namely (A.21) below

Lemma: The following result gives conditions under which the integrals of a family of functions on $[\underline{s}, \bar{s}]$ can be uniformly approximated by their Riemann sums:

LEMMA: Let the family of functions $\{g(\cdot; \tau) | \tau \in T\}$ over $[\underline{s}, \bar{s}]$ be both uniformly bounded and uniformly continuous over each interval E_j of some finite interval partition $\{E_1, \dots, E_J\}$ of $[\underline{s}, \bar{s}]$. Then for each $\varepsilon > 0$ there exists some n_ε such that

$$(A.8) \quad \left| \int_{\underline{s}}^{\bar{s}} g(s; \tau) \cdot ds - \frac{\lambda_s}{n} \cdot \sum_{i=0}^{n-1} g\left(\underline{s} + \frac{(i+\alpha) \cdot \lambda_s}{n}; \tau\right) \right| < \varepsilon \quad \begin{array}{l} \text{all } n \geq n_\varepsilon \\ \text{all } \alpha \in [0, 1] \\ \text{all } \tau \in T \end{array}$$

PROOF OF LEMMA: Since the functions $\{g(\cdot; \tau) | \tau \in T\}$ are uniformly bounded over each interval $E_1, \dots, E_J$, they are uniformly bounded over $[\underline{s}, \bar{s}]$ by some values $\underline{g} < \bar{g}$. Given $\varepsilon > 0$, uniform continuity on each interval $E_1, \dots, E_J$ implies some $\delta_\varepsilon > 0$ such that if $s, s' \in E_j$ and $|s' - s| < \delta_\varepsilon$ then $|g(s'; \tau) - g(s; \tau)| < \varepsilon/(2 \cdot \lambda_s)$ for all $\tau \in T$. Select n_ε such that $\lambda_s/n_\varepsilon < \min\{\delta_\varepsilon, \varepsilon/(2 \cdot J \cdot (\bar{g} - \underline{g}))\}$.

Given arbitrary $n \geq n_\varepsilon$ and arbitrary $\alpha \in [0, 1]$ (we treat the case of $\alpha = 1$ below), partition $[\underline{s}, \bar{s}]$ into the following intervals of length λ_s/n :

$$(A.9) \quad I_0 = [\underline{s}, \underline{s} + \frac{\lambda_s}{n}), \quad I_1 = [\underline{s} + \frac{\lambda_s}{n}, \underline{s} + \frac{2\lambda_s}{n}) \quad \dots \quad I_i = [\underline{s} + \frac{i\lambda_s}{n}, \underline{s} + \frac{(i+1)\lambda_s}{n}) \quad \dots \quad I_{n-1} = [\underline{s} + \frac{(n-1)\lambda_s}{n}, \bar{s}]$$

(For $n = 1$, define $I_0 = [\underline{s}, \bar{s}]$.) Observe that, for all $i = 0, \dots, n-1$, the value $\underline{s} + \frac{(i+\alpha)\lambda_s}{n}$ lies in I_i . For each interval I_i that lies wholly within some interval E_j (and there can be up to n such intervals), the above uniform continuity property implies

$$(A.10) \quad \left| \int_{I_i} [g(s; \tau) - g(\underline{s} + \frac{(i+\alpha)\lambda_s}{n}; \tau)] \cdot ds \right| < \frac{\varepsilon}{2 \cdot \lambda_s} \cdot \frac{\lambda_s}{n} = \frac{\varepsilon}{2 \cdot n} \quad \text{all } \tau \in T$$

For each interval I_i that *does not* lie wholly within one of the intervals $E_1, \dots, E_J$ (and there can be up to $J-1$ such intervals), uniform boundedness implies

$$(A.11) \quad \left| \int_{I_i} [g(s; \tau) - g(\underline{s} + \frac{(i+\alpha)\lambda_s}{n}; \tau)] \cdot ds \right| \leq (\bar{g} - \underline{g}) \cdot \frac{\lambda_s}{n} < \frac{\varepsilon}{2 \cdot J} \quad \text{all } \tau \in T$$

We thus have

$$(A.12) \quad \left| \int_{\underline{s}}^{\bar{s}} g(s; \tau) \cdot ds - \frac{\lambda_s}{n} \cdot \sum_{i=0}^{n-1} g(\underline{s} + \frac{(i+\alpha)\lambda_s}{n}; \tau) \right| = \left| \sum_{i=0}^{n-1} \int_{I_i} [g(s; \tau) - g(\underline{s} + \frac{(i+\alpha)\lambda_s}{n}; \tau)] \cdot ds \right| \quad \text{all } \tau \in T$$

$$\leq \sum_{i=0}^{n-1} \left| \int_{I_i} [g(s; \tau) - g(\underline{s} + \frac{(i+\alpha)\lambda_s}{n}; \tau)] \cdot ds \right| < n \cdot \frac{\varepsilon}{2 \cdot n} + (J-1) \cdot \frac{\varepsilon}{2 \cdot J} < \varepsilon$$

For the case $\alpha = 1$, repeat the above argument with (A.9) replaced by the equal-length intervals

$$(A.9)' \quad I_0 = [\underline{s}, \underline{s} + \frac{\lambda_s}{n}], \quad I_1 = (\underline{s} + \frac{\lambda_s}{n}, \underline{s} + \frac{2\lambda_s}{n}] \quad \dots \quad I_i = (\underline{s} + \frac{i\lambda_s}{n}, \underline{s} + \frac{(i+1)\lambda_s}{n}] \quad \dots \quad I_{n-1} = (\underline{s} + \frac{(n-1)\lambda_s}{n}, \bar{s}]$$

Line Integral Approximation Theorem: The following result for characterizing $W(\cdot)$'s evaluation of acts $f(\cdot)$ versus $f^*(\cdot)$ differs from the characterization formula (A.7) in two respects: (i) It consists of the *limit* of a sequence of path integrals, and (ii) for all values of n and α , the sum $\sum_{x \in \mathcal{X}} [\Phi_x(f^{*-1}(x); f_\alpha^n) - \Phi_x(f^{-1}(x); f_\alpha^n)]$ in (A.13), which is also the integrand in (A.14), evaluates the *same* families of events, namely the two acts' events $\{f^{-1}(x) | x \in \mathcal{X}\}$ and $\{f^{*-1}(x) | x \in \mathcal{X}\}$.

LINE INTEGRAL APPROXIMATION THEOREM: If $W(\cdot)$ is event-smooth, then for any acts $f(\cdot), f^*(\cdot)$ and any $\varepsilon > 0$, there exists n_ε such that for each $n \geq n_\varepsilon$, $W(\cdot)$'s path derivative along the almost-objective path mixture path $\{f_\alpha^n(\cdot) \equiv [f^*(\cdot) \text{ on } [0, \alpha] \times_n S; f(\cdot) \text{ on } (\alpha, 1] \times_n S] \mid \alpha \in [0, 1]\}$ from $f(\cdot)$ to $f^*(\cdot)$ exists and satisfies

$$(A.13) \quad \left| \frac{dW(f_\alpha^n(\cdot))}{d\alpha} - \sum_{x \in \mathcal{X}} [\Phi_x(f^{*-1}(x); f_\alpha^n) - \Phi_x(f^{-1}(x); f_\alpha^n)] \right| < \varepsilon$$

at all but a finite number of values of $\alpha \in [0, 1]$. This in turn implies the line integral approximation formula

$$(A.14) \quad W(f^*(\cdot)) - W(f(\cdot)) = \lim_{n \rightarrow \infty} \int_0^1 \sum_{x \in \mathcal{X}} [\Phi_x(f^{*-1}(x); f_\alpha^n) - \Phi_x(f^{-1}(x); f_\alpha^n)] \cdot d\alpha$$

PROOF: Given arbitrary $\varepsilon > 0$, represent the acts $f_0(\cdot)$ and $f^*(\cdot)$ as

$$(A.15) \quad f(\cdot) = [x_1 \text{ on } E_1; \dots; x_J \text{ on } E_J] \quad f^*(\cdot) = [x_1^* \text{ on } E_1; \dots; x_J^* \text{ on } E_J]$$

for some interval partition $\{E_1, \dots, E_J\}$ of $[\underline{s}, \bar{s}]$, and define the two families of functions $\{g(\cdot; \hat{f}) | \hat{f}(\cdot) \in \mathcal{A}\}$ and $\{g^*(\cdot; \hat{f}) | \hat{f}(\cdot) \in \mathcal{A}\}$ over $[\underline{s}, \bar{s}]$ by

$$(A.16) \quad g(s; \hat{f}) \equiv_{s, \hat{f}(\cdot)} \phi_{f(s)}(s; \hat{f}) \quad g^*(s; \hat{f}) \equiv_{s, \hat{f}(\cdot)} \phi_{f^*(s)}(s; \hat{f})$$

Since $g(\cdot; \hat{f}) \equiv \phi_{x_j}(\cdot; \hat{f})$ and $g^*(\cdot; \hat{f}) \equiv \phi_{x_j^*}(\cdot; \hat{f})$ over each interval E_j , we have that for all $\hat{f}(\cdot) \in \mathcal{A}$:

$$(A.17) \quad \begin{aligned} \int_{\underline{s}}^{\bar{s}} g(s; \hat{f}) \cdot ds &= \sum_{j=1}^J \int_{E_j} \phi_{x_j}(s; \hat{f}) \cdot ds = \sum_{j=1}^J \Phi_{x_j}(E_j; \hat{f}) = \sum_{x \in \mathcal{X}} \Phi_x(f^{-1}(x); \hat{f}) \\ \int_{\underline{s}}^{\bar{s}} g^*(s; \hat{f}) \cdot ds &= \sum_{j=1}^J \int_{E_j} \phi_{x_j^*}(s; \hat{f}) \cdot ds = \sum_{j=1}^J \Phi_{x_j^*}(E_j; \hat{f}) = \sum_{x \in \mathcal{X}} \Phi_x(f^{*-1}(x); \hat{f}) \end{aligned}$$

By event-smoothness, the families of functions $\{g(\cdot; \hat{f}) | \hat{f}(\cdot) \in \mathcal{A}\}$ and $\{g^*(\cdot; \hat{f}) | \hat{f}(\cdot) \in \mathcal{A}\}$ are uniformly bounded over each E_j . Similarly, $\{g(\cdot; \hat{f}) | \hat{f}(\cdot) \in \mathcal{A}\}$ and $\{g^*(\cdot; \hat{f}) | \hat{f}(\cdot) \in \mathcal{A}\}$ are uniformly continuous over each E_j . A double application of the Lemma thus yields n_ε such that

$$(A.18) \quad \begin{aligned} \left| \int_{\underline{s}}^{\bar{s}} g(s; \hat{f}) \cdot ds - \frac{\lambda_s}{n} \cdot \sum_{i=0}^{n-1} g\left(\underline{s} + \frac{(i+\alpha) \cdot \lambda_s}{n}; \hat{f}\right) \right| &< \frac{\varepsilon}{2} \\ \left| \int_{\underline{s}}^{\bar{s}} g^*(s; \hat{f}) \cdot ds - \frac{\lambda_s}{n} \cdot \sum_{i=0}^{n-1} g^*\left(\underline{s} + \frac{(i+\alpha) \cdot \lambda_s}{n}; \hat{f}\right) \right| &< \frac{\varepsilon}{2} \end{aligned}$$

and hence

$$(A.19) \quad \left| \int_{\underline{s}}^{\bar{s}} [g^*(s; \hat{f}) - g(s; \hat{f})] \cdot ds - \sum_{i=0}^{n-1} \frac{\lambda_s}{n} \cdot [g^*\left(\underline{s} + \frac{(i+\alpha) \cdot \lambda_s}{n}; \hat{f}\right) - g\left(\underline{s} + \frac{(i+\alpha) \cdot \lambda_s}{n}; \hat{f}\right)] \right| < \varepsilon$$

for all $n \geq n_\varepsilon$, all $\alpha \in [0, 1]$ and all $\hat{f}(\cdot) \in \mathcal{A}$.

Given arbitrary $n \geq n_\varepsilon$, let $\{f_\alpha^n(\cdot) \equiv [f^*(\cdot) \text{ on } [0, \alpha] \times_n S; f(\cdot) \text{ on } (\alpha, 1] \times_n S] | \alpha \in [0, 1]\}$ be the almost-objective mixture path from $f(\cdot)$ to $f^*(\cdot)$. Although $W(f_\alpha^n(\cdot))$ is *continuous* in α at all $\alpha \in [0, 1]$, in general it will only be *differentiable* in α when all the “front points” of its multiple sweep, that is, when the n points

$$(A.20) \quad \underline{s} + \frac{\alpha \cdot \lambda_s}{n}, \quad \underline{s} + \frac{(1+\alpha) \cdot \lambda_s}{n}, \quad \dots \quad \underline{s} + \frac{(i+\alpha) \cdot \lambda_s}{n}, \quad \dots \quad \underline{s} + \frac{(n-1+\alpha) \cdot \lambda_s}{n}$$

are continuity points of both $f(\cdot)$ and $f^*(\cdot)$. But this will hold for all but a finite number of α in $[0, 1]$, so that except at that this finite set of values, $dW(f_\alpha^n(\cdot))/d\alpha$ exists and is given by

$$(A.21) \quad \begin{aligned} \frac{dW(f_\alpha^n(\cdot))}{d\alpha} &= \sum_{i=0}^{n-1} \frac{\lambda_s}{n} \cdot \left[\phi_{f^*\left(\underline{s} + \frac{(i+\alpha) \cdot \lambda_s}{n}\right)}\left(\underline{s} + \frac{(i+\alpha) \cdot \lambda_s}{n}; f_\alpha^n\right) - \phi_{f\left(\underline{s} + \frac{(i+\alpha) \cdot \lambda_s}{n}\right)}\left(\underline{s} + \frac{(i+\alpha) \cdot \lambda_s}{n}; f_\alpha^n\right) \right] \\ &= \sum_{i=0}^{n-1} \frac{\lambda_s}{n} \cdot \left[g^*\left(\underline{s} + \frac{(i+\alpha) \cdot \lambda_s}{n}; f_\alpha^n\right) - g\left(\underline{s} + \frac{(i+\alpha) \cdot \lambda_s}{n}; f_\alpha^n\right) \right] \end{aligned}$$

Thus at all $\alpha \in [0, 1]$ except this finite set of values, we can evaluate (A.19) at $f(\cdot) = f_\alpha^n(\cdot)$ and substitute in (A.17) and (A.21) to obtain

$$(A.22) \quad \left| \frac{dW(f_\alpha^n(\cdot))}{d\alpha} - \sum_{x \in \mathcal{X}} [\Phi_x(f^{*-1}(x); f_\alpha^n) - \Phi_x(f^{-1}(x); f_\alpha^n)] \right| < \varepsilon$$

To establish the line integral approximation formula (A.14), consider arbitrary $\varepsilon > 0$ and observe that for each $n \geq n_\varepsilon$, the fact that (A.22) holds at all but a finite number of α in $[0, 1]$ implies

$$\begin{aligned}
& \left| W(f^*(\cdot)) - W(f(\cdot)) - \int_0^1 \sum_{x \in \mathcal{X}} [\Phi_x(f^{*-1}(x); f_\alpha^n) - \Phi_x(f^{-1}(x); f_\alpha^n)] \cdot d\alpha \right| \\
(A.23) \quad &= \left| \int_0^1 \frac{dW(f_\alpha^n(\cdot))}{d\alpha} \cdot d\alpha - \int_0^1 \sum_{x \in \mathcal{X}} [\Phi_x(f^{*-1}(x); f_\alpha^n) - \Phi_x(f^{-1}(x); f_\alpha^n)] \cdot d\alpha \right| \\
&\leq \int_0^1 \left| \frac{dW(f_\alpha^n(\cdot))}{d\alpha} - \sum_{x \in \mathcal{X}} [\Phi_x(f^{*-1}(x); f_\alpha^n) - \Phi_x(f^{-1}(x); f_\alpha^n)] \right| \cdot d\alpha < \varepsilon \quad \blacksquare
\end{aligned}$$

PROOF OF THM 0: Given arbitrary family of signed measures $\{K(\cdot; \tau) | \tau \in T\}$ on S with uniformly bounded and uniformly continuous densities $\{k(\cdot; \tau) | \tau \in T\}$, finite interval union $\wp \subseteq [0, 1]$, interval $E = [s_1, s_2] \subseteq S$ and $\varepsilon > 0$, we will obtain n_ε such that $|K(\wp \times_n E; \tau) - \lambda(\wp) \cdot K(E; \tau)| < \varepsilon$ for all $n > n_\varepsilon$ and all $\tau \in T$. All assertions of the theorem will follow from this, by additivity or as special cases.

Uniform boundedness implies some $k_{\max} \in (0, \infty)$ such that $|k(s; \tau)| < k_{\max}$ for all $s \in S$ and $\tau \in T$, and uniform continuity implies some $\gamma > 0$ such that if $|s - s'| < \gamma$ then $|k(s; \tau) - k(s'; \tau)| < \varepsilon / (2 \cdot \lambda_s)$ for all $\tau \in T$. Select n_ε so that $\lambda_s / n_\varepsilon < \min\{\gamma, \varepsilon / (8 \cdot k_{\max})\}$, and consider arbitrary $n > n_\varepsilon$ and $\tau \in T$.

Of the n successive equal-length intervals

$$(A.24) \quad I_0 = [\underline{s}, \underline{s} + \lambda_s / n) \quad \dots \quad I_i = [\underline{s} + i \cdot \lambda_s / n, \underline{s} + (i+1) \cdot \lambda_s / n) \quad \dots \quad I_{n-1} = [\underline{s} + (n-1) \cdot \lambda_s / n, \bar{s}]$$

exactly one (call it I_i) will contain s_1 , exactly one (call it I_i) will contain s_2 , and each of the remaining intervals I_i will either be fully contained in E or disjoint from E , which implies both

$$(A.25) \quad K(\wp \times_n E; \tau) = \int_{I_i \cap (\wp \times_n E)} k(s; \tau) \cdot ds + \sum_{I_i \subseteq E} \int_{\{\underline{s} + \frac{(i+\omega) \cdot \lambda_s}{n} | \omega \in \wp\}} k(s; \tau) \cdot ds + \int_{I_i \cap (\wp \times_n E)} k(s; \tau) \cdot ds$$

$$(A.26) \quad \lambda(\wp) \cdot K(E; \tau) = \int_{I_i \cap E} \lambda(\wp) \cdot k(s; \tau) \cdot ds + \sum_{I_i \subseteq E} \int_{I_i} \lambda(\wp) \cdot k(s; \tau) \cdot ds + \int_{I_i \cap E} \lambda(\wp) \cdot k(s; \tau) \cdot ds$$

The first and third integrals in (A.25) and (A.26) will each be less than $k_{\max} \cdot \lambda_s / n$ in absolute value. For each interval $I_i \subseteq E$, all states in I_i are within distance γ of each other, so $|k(s; \tau) - k(s'; \tau)| < \varepsilon / (2 \cdot \lambda_s)$ for all $s, s' \in I_i$, which in turn implies

$$(A.27) \quad \left| k(s; \tau) - \frac{n}{\lambda_s} \cdot \int_{I_i} k(s'; \tau) \cdot ds' \right| < \frac{\varepsilon}{2 \cdot \lambda_s}$$

for all $s \in I_i$. Since $\lambda\{\underline{s} + (i+\omega) \cdot \lambda_s / n | \omega \in \wp\} = \lambda(\wp) \cdot \lambda_s / n$, for each $I_i \subseteq E$ we have

$$\begin{aligned}
& \left| \int_{\{\underline{s} + \frac{(i+\omega) \cdot \lambda_s}{n} | \omega \in \wp\}} k(s; \tau) \cdot ds - \int_{I_i} \lambda(\wp) \cdot k(s'; \tau) \cdot ds' \right| = \\
& \left| \int_{\{\underline{s} + \frac{(i+\omega) \cdot \lambda_s}{n} | \omega \in \wp\}} k(s; \tau) \cdot ds - \int_{\{\underline{s} + \frac{(i+\omega) \cdot \lambda_s}{n} | \omega \in \wp\}} \frac{n}{\lambda_s} \cdot ds \cdot \int_{I_i} k(s'; \tau) \cdot ds' \right| = \\
(A.28) \quad & \left| \int_{\{\underline{s} + \frac{(i+\omega) \cdot \lambda_s}{n} | \omega \in \wp\}} \left[k(s; \tau) - \frac{n}{\lambda_s} \cdot \int_{I_i} k(s'; \tau) \cdot ds' \right] \cdot ds \right| < \\
& \int_{\{\underline{s} + \frac{(i+\omega) \cdot \lambda_s}{n} | \omega \in \wp\}} \frac{\varepsilon}{2 \cdot \lambda_s} \cdot ds = \frac{\varepsilon}{2 \cdot \lambda_s} \cdot \lambda(\wp) \cdot \frac{\lambda_s}{n} \leq \frac{\varepsilon}{2 \cdot n}
\end{aligned}$$

Subtracting (A.26) from (A.25) and invoking the above inequalities on their integrals yields

$$(A.29) \quad |K(\wp \times_n E; \tau) - \lambda(\wp) \cdot K(E; \tau)| < 4 \cdot \frac{k_{\max} \cdot \lambda_s}{n} + \sum_{I_i \subseteq E} \frac{\varepsilon}{2 \cdot n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \blacksquare$$

PROOF OF THM 1: This result follows from Theorem 2, by setting E in that theorem equal to S . $\blacksquare$

PROOF OF THM 2: Given disjoint $\wp, \wp'$, nonnull event $E \in \mathcal{E}$, outcomes $x^* \succ x$ and act $f(\cdot) = [x_1 \text{ on } E_1; \dots; x_m \text{ on } E_m]$, define $\wp_0 = [0,1] - (\wp \cup \wp')$. Since we can express (26)'s left-hand acts $f_n(\cdot) = [x^* \text{ on } \wp \times_n E; x \text{ on } \wp' \times_n E; f(s) \text{ elsewhere}]$ in the form

$$(A.30) \quad \begin{aligned} & [x^* \text{ on } \wp \times_n (E \cap E_1); x \text{ on } \wp' \times_n (E \cap E_1); x_1 \text{ on } \wp_0 \times_n (E \cap E_1); x_1 \text{ on } E_1 - E; \dots \\ & \dots; x^* \text{ on } \wp \times_n (E \cap E_m); x \text{ on } \wp' \times_n (E \cap E_m); x_m \text{ on } \wp_0 \times_n (E \cap E_m); x_m \text{ on } E_m - E] \end{aligned}$$

an argument identical to Step 1 of the Proof of Theorem 4 establishes that $\lim_{n \rightarrow \infty} W(f_n(\cdot))$ exists. A similar argument establishes the existence of $\lim_{n \rightarrow \infty} W(f'_n(\cdot))$ for (26)'s right-hand acts $f'_n(\cdot) = [x \text{ on } \wp \times_n E; x^* \text{ on } \wp' \times_n E; f(s) \text{ elsewhere}]$.

For $\lambda(\wp) = \lambda(\wp')$, we establish $\lim_{n \rightarrow \infty} W(f_n(\cdot)) = \lim_{n \rightarrow \infty} W(f'_n(\cdot))$ by taking arbitrary $\varepsilon > 0$ and obtaining n_ε such that $|W(f'_n(\cdot)) - W(f_n(\cdot))| < \varepsilon$ for all $n > n_\varepsilon$. Applying event-smoothness and Theorem 0 to the families of signed measures $\{\Phi_{x^*}(\cdot; f) | f(\cdot) \in \mathcal{A}\}$ and $\{\Phi_x(\cdot; f) | f(\cdot) \in \mathcal{A}\}$ yields an n_ε such that

$$(A.31) \quad \begin{aligned} & |\Phi_{x^*}(\wp \times_n E; f) - \lambda(\wp) \cdot \Phi_{x^*}(E; f)|, \quad |\Phi_x(\wp \times_n E; f) - \lambda(\wp) \cdot \Phi_x(E; f)| \\ & |\Phi_{x^*}(\wp' \times_n E; f) - \lambda(\wp') \cdot \Phi_{x^*}(E; f)|, \quad |\Phi_x(\wp' \times_n E; f) - \lambda(\wp') \cdot \Phi_x(E; f)| \end{aligned}$$

are all less than $\varepsilon/8$ for all $n > n_\varepsilon$ and all $f(\cdot) \in \mathcal{A}$. Select arbitrary $n > n_\varepsilon$ and consider the acts $f_n(\cdot)$ and $f'_n(\cdot)$. By (A.31) and $\lambda(\wp') = \lambda(\wp)$, we have

$$(A.32) \quad |\Phi_{x^*}(\wp \times_n E; f) - \Phi_{x^*}(\wp' \times_n E; f) + \Phi_x(\wp' \times_n E; f) - \Phi_x(\wp \times_n E; f)| < \varepsilon/2 \quad \text{all } f(\cdot) \in \mathcal{A}$$

Defining $\{f_\alpha^k(\cdot) \equiv [f'_n(\cdot) \text{ on } [0, \alpha] \times_k S; f_n(\cdot) \text{ on } (\alpha, 1] \times_k S] | \alpha \in [0, 1]\}$ as the almost-objective mixture paths from $f'_n(\cdot)$ to $f_n(\cdot)$, (A.14) implies

$$(A.33) \quad \begin{aligned} W(f_n(\cdot)) - W(f'_n(\cdot)) &= \lim_{k \rightarrow \infty} \int_0^1 \sum_{x \in X} [\Phi_x(f_n^{-1}(x); f_\alpha^k) - \Phi_x(f'_n^{-1}(x); f_\alpha^k)] \cdot d\alpha = \\ & \lim_{k \rightarrow \infty} \int_0^1 [\Phi_{x^*}(\wp \times_n E; f_\alpha^k) - \Phi_{x^*}(\wp' \times_n E; f_\alpha^k) + \Phi_x(\wp' \times_n E; f_\alpha^k) - \Phi_x(\wp \times_n E; f_\alpha^k)] \cdot d\alpha \end{aligned}$$

By (A.32), the integrand in (A.33) less than $\varepsilon/2$ in absolute value for all $\alpha \in [0, 1]$ and all k , which implies $|W(f'_n(\cdot)) - W(f_n(\cdot))| < \varepsilon$.

For $\lambda(\wp) > \lambda(\wp')$, we establish $\lim_{n \rightarrow \infty} W(f_n(\cdot)) > \lim_{n \rightarrow \infty} W(f'_n(\cdot))$ by obtaining obtain $\hat{n}$ and $\delta > 0$ such that $W(f_n(\cdot)) - W(f'_n(\cdot)) \geq \delta/2$ for all $n > \hat{n}$. Recall that $\Phi_{x^*}(E; f) - \Phi_x(E; f)$ is bounded above 0 by some positive $\Phi_{x^*, x, E}$ for all $f(\cdot) \in \mathcal{A}$, and define $\delta = [\lambda(\wp) - \lambda(\wp')] \cdot \Phi_{x^*, x, E}$. Applying event-smoothness and Theorem 0 to $\{\Phi_{x^*}(\cdot; f) | f(\cdot) \in \mathcal{A}\}$ and $\{\Phi_x(\cdot; f) | f(\cdot) \in \mathcal{A}\}$ yields an $\hat{n}$ such that

$$(A.34) \quad \begin{aligned} & |\Phi_{x^*}(\wp \times_n E; f) - \lambda(\wp) \cdot \Phi_{x^*}(E; f)|, \quad |\Phi_x(\wp \times_n E; f) - \lambda(\wp) \cdot \Phi_x(E; f)| \\ & |\Phi_{x^*}(\wp' \times_n E; f) - \lambda(\wp') \cdot \Phi_{x^*}(E; f)|, \quad |\Phi_x(\wp' \times_n E; f) - \lambda(\wp') \cdot \Phi_x(E; f)| \end{aligned}$$

are all less than $\delta/8$ for all $n > \hat{n}$ and all $f(\cdot) \in \mathcal{A}$, which in turn implies

$$(A.35) \quad \begin{aligned} & \Phi_{x^*}(\wp \times_n E; f_\alpha^k) - \Phi_{x^*}(\wp' \times_n E; f_\alpha^k) + \Phi_x(\wp' \times_n E; f_\alpha^k) - \Phi_x(\wp \times_n E; f_\alpha^k) \quad \text{all } n > \hat{n}, \\ & > [\lambda(\wp) - \lambda(\wp')] \cdot (\Phi_{x^*}(E; f) - \Phi_x(E; f)) - \delta/2 \geq \delta/2 \quad \text{all } f(\cdot) \in \mathcal{A} \end{aligned}$$

Select arbitrary $n > \hat{n}$ and consider the acts $f_n(\cdot)$ and $f'_n(\cdot)$. Defining $\{f_\alpha^k(\cdot) | \alpha \in [0, 1]\}$ as the almost-objective mixture paths from $f'_n(\cdot)$ to $f_n(\cdot)$ as before, (A.14) again implies

$$(A.36) \quad \begin{aligned} W(f_n(\cdot)) - W(f'_n(\cdot)) &= \lim_{k \rightarrow \infty} \int_0^1 \sum_{x \in X} [\Phi_x(f_n^{-1}(x); f_\alpha^k) - \Phi_x(f'_n^{-1}(x); f_\alpha^k)] \cdot d\alpha = \\ & \lim_{k \rightarrow \infty} \int_0^1 [\Phi_{x^*}(\wp \times_n E; f_\alpha^k) - \Phi_{x^*}(\wp' \times_n E; f_\alpha^k) + \Phi_x(\wp' \times_n E; f_\alpha^k) - \Phi_x(\wp \times_n E; f_\alpha^k)] \cdot d\alpha \end{aligned}$$

By (A.35), the integrand in (A.36) is greater than or equal to $\delta/2$ in absolute value for all $\alpha \in [0,1]$ and all k , which implies $W(f_n(\cdot)) - W(f'_n(\cdot)) \geq \delta/2 > 0$. ■

PROOF OF THM 3: For given outcomes $x_1, \dots, x_m$ and partition $\{\wp_1, \dots, \wp_m\}$ define the acts $f_n(\cdot) = [x_1 \text{ on } \wp_1 \times_n S; \dots; x_m \text{ on } \wp_m \times_n S]$. Step 1 shows that $\{W(f_n(\cdot))\}_{n=1}^\infty$ is a Cauchy sequence in R^1 and hence converges to some $\bar{W}$. Step 2 shows that for any other partition $\{\hat{\wp}_1, \dots, \hat{\wp}_m\}$ with $\lambda(\hat{\wp}_i) = \lambda(\wp_i)$ for $i = 1, \dots, m$, the acts $\hat{f}_n(\cdot) = [x_1 \text{ on } \hat{\wp}_1 \times_n S; \dots; x_m \text{ on } \hat{\wp}_m \times_n S]$ satisfy $W(\hat{f}_n(\cdot)) - W(f_n(\cdot)) \rightarrow 0$, so $W(\hat{f}_n(\cdot))$ also converges to $\bar{W}$, which we can accordingly express as $V_W(x_1, \lambda(\wp_1); \dots; x_m, \lambda(\wp_m))$. (That $V_W(\cdot)$ is a preference function over objective lotteries follows from the preservation properties (18).) Step 3 shows that $V_W(\cdot)$ exhibits strict first order stochastic dominance preference.

Step 1: Given $\varepsilon > 0$, we need n_ε such that $|W(f_{n'}(\cdot)) - W(f_{n''}(\cdot))| < \varepsilon$ for all $n', n'' > n_\varepsilon$. Applying event-smoothness and Theorem 0 to each of the m families of signed measures $\{\Phi_{x_i}(\cdot; f) | f(\cdot) \in \mathcal{A}\}$ yields an n_ε such that, for $i = 1, \dots, m$

$$(A.37) \quad |\Phi_{x_i}(\wp_i \times_n S; f) - \lambda(\wp_i) \cdot \Phi_{x_i}(S; f)| < \varepsilon/(4m) \quad \text{all } n > n_\varepsilon, \text{ all } f(\cdot) \in \mathcal{A}$$

Select arbitrary $n', n'' > n_\varepsilon$ and consider the pair of acts $f_{n'}(\cdot)$ and $f_{n''}(\cdot)$. By (A.37) we have

$$(A.38) \quad |\Phi_{x_i}(\wp_i \times_{n'} S; f) - \Phi_{x_i}(\wp_i \times_{n''} S; f)| < \varepsilon/(2m) \quad i = 1, \dots, m, \text{ all } f(\cdot) \in \mathcal{A}$$

Defining $\{f_\alpha^k(\cdot) | \alpha \in [0,1]\}$ as the almost-objective mixture paths from $f_{n'}(\cdot)$ to $f_{n''}(\cdot)$, (A.14) yields

$$(A.39) \quad \begin{aligned} W(f_{n''}(\cdot)) - W(f_{n'}(\cdot)) &= \lim_{k \rightarrow \infty} \int_0^1 \sum_{x \in \mathcal{X}} [\Phi_x(f_{n''}^{-1}(x); f_\alpha^k) - \Phi_x(f_{n'}^{-1}(x); f_\alpha^k)] \cdot d\alpha \\ &= \lim_{k \rightarrow \infty} \int_0^1 \sum_{i=1}^m [\Phi_{x_i}(\wp_i \times_{n'} S; f_\alpha^k) - \Phi_{x_i}(\wp_i \times_{n''} S; f_\alpha^k)] \cdot d\alpha \end{aligned}$$

By (A.38), the integrand in (A.39) is less than $\varepsilon/2$ in absolute value for all $\alpha \in [0,1]$ and all k , which implies $|W(f_{n''}(\cdot)) - W(f_{n'}(\cdot))| < \varepsilon$. But since n' and n'' were arbitrary integers greater than n_ε , this implies $\{W(f_n(\cdot))\}_{n=1}^\infty$ is a Cauchy sequence, and hence converges to some $\bar{W}$.

Step 2: Take any other sequence of acts $\hat{f}_n(\cdot) = [x_1 \text{ on } \hat{\wp}_1 \times_n S; \dots; x_m \text{ on } \hat{\wp}_m \times_n S]$ where $\lambda(\hat{\wp}_i) = \lambda(\wp_i)$ for $i = 1, \dots, m$. Given arbitrary $\varepsilon > 0$, we need $\hat{n}_\varepsilon$ such that $|W(\hat{f}_n(\cdot)) - W(f_n(\cdot))| < \varepsilon$ for all $n > \hat{n}_\varepsilon$. Applying Theorem 0 again yields an $\hat{n}_\varepsilon$ such that, for $i = 1, \dots, m$, both

$$(A.40) \quad \begin{aligned} |\Phi_{x_i}(\wp_i \times_n S; f) - \lambda(\wp_i) \cdot \Phi_{x_i}(S; f)| &< \varepsilon/(4m) \\ |\Phi_{x_i}(\hat{\wp}_i \times_n S; f) - \lambda(\hat{\wp}_i) \cdot \Phi_{x_i}(S; f)| &< \varepsilon/(4m) \end{aligned} \quad \text{all } n > \hat{n}_\varepsilon, \text{ all } f(\cdot) \in \mathcal{A}$$

Select arbitrary $n > \hat{n}_\varepsilon$ and consider the acts $f_n(\cdot)$ and $\hat{f}_n(\cdot)$. By (A.40) and $\lambda(\hat{\wp}_i) = \lambda(\wp_i)$, we have

$$(A.41) \quad |\Phi_{x_i}(\hat{\wp}_i \times_n S; f) - \Phi_{x_i}(\wp_i \times_n S; f)| < \varepsilon/(2m) \quad i = 1, \dots, m, \text{ all } f(\cdot) \in \mathcal{A}$$

Defining $\{\hat{f}_\alpha^k(\cdot) | \alpha \in [0,1]\}$ as the almost-objective mixture paths from $f_n(\cdot)$ to $\hat{f}_n(\cdot)$, (A.14) implies

$$(A.42) \quad \begin{aligned} W(\hat{f}_n(\cdot)) - W(f_n(\cdot)) &= \lim_{k \rightarrow \infty} \int_0^1 \sum_{x \in \mathcal{X}} [\Phi_x(\hat{f}_n^{-1}(x); \hat{f}_\alpha^k) - \Phi_x(f_n^{-1}(x); f_\alpha^k)] \cdot d\alpha \\ &= \lim_{k \rightarrow \infty} \int_0^1 \sum_{i=1}^m [\Phi_{x_i}(\hat{\wp}_i \times_n S; \hat{f}_\alpha^k) - \Phi_{x_i}(\wp_i \times_n S; f_\alpha^k)] \cdot d\alpha \end{aligned}$$

By (A.41), the integrand in (A.42) is less than $\varepsilon/2$ in absolute value for all $\alpha \in [0,1]$ and all k , which implies $|W(\hat{f}_n(\cdot)) - W(f_n(\cdot))| < \varepsilon$. But since n was an arbitrary integer greater than $\hat{n}_\varepsilon$, this implies $W(\hat{f}_n(\cdot)) - W(f_n(\cdot))$ converges to 0, and hence that $W(\hat{f}_n(\cdot))$ converges to $\bar{W}$.

Step 3: To establish strict first order stochastic dominance preference, it suffices to show $V_W(\hat{\mathbf{P}}) > V_W(\mathbf{P})$ for any $\hat{\mathbf{P}} = (\hat{x}_1, p_1; x_2, p_2; \dots; x_m, p_m)$ and $\mathbf{P} = (x_1, p_1; x_2, p_2; \dots; x_m, p_m)$ with $\hat{x}_1 \succ x_1$ and $p_1 > 0$. Select interval partition $\{\wp_1, \dots, \wp_m\}$ of $[0,1]$ with $\lambda(\wp_i) = p_i$ for $i = 1, \dots, m$, and define $\hat{f}_n(\cdot) = [\hat{x}_1 \text{ on } \wp_1 \times_n S; x_2 \text{ on } \wp_2 \times_n S; \dots; x_m \text{ on } \wp_m \times_n S]$ and $f_n(\cdot) = [x_1 \text{ on } \wp_1 \times_n S; x_2 \text{ on } \wp_2 \times_n S; \dots; x_m \text{ on } \wp_m \times_n S]$. Event-smoothness yields $\Phi_{\hat{x}_1}(S; f) - \Phi_{x_1}(S; f) \geq \Phi_{\hat{x}_1, x_1, S} > 0$ for all $f(\cdot) \in \mathcal{A}$, and event-smoothness and Theorem 0 yield some $\hat{n}$ such that

$$(A.43) \quad |\Phi_{\hat{x}_1}(\wp_1 \times_n S; f) - \lambda(\wp_1) \cdot \Phi_{\hat{x}_1}(S; f)| \quad \text{and} \quad |\Phi_{x_1}(\wp_1 \times_n S; f) - \lambda(\wp_1) \cdot \Phi_{x_1}(S; f)|$$

are both less than $\delta = \lambda(\wp_1) \cdot \Phi_{\hat{x}_1, x_1, S} / 4$ for all $n > \hat{n}$ and all $f(\cdot) \in \mathcal{A}$. Select arbitrary $n > \hat{n}$ and consider the acts $\hat{f}_n(\cdot)$ and $f_n(\cdot)$. By (A.43) we have

$$(A.44) \quad \Phi_{\hat{x}_1}(\wp_1 \times_n S; f) - \Phi_{x_1}(\wp_1 \times_n S; f) > \lambda(\wp_1) \cdot [\Phi_{\hat{x}_1}(S; f) - \Phi_{x_1}(S; f)] - 2\delta \geq \lambda(\wp_1) \cdot \Phi_{\hat{x}_1, x_1, S} - 2\delta = 2\delta > 0$$

for all $f(\cdot) \in \mathcal{A}$. Defining $\{f_\alpha^k(\cdot) | \alpha \in [0,1]\}$ as the almost-objective mixture paths from $f_n(\cdot)$ to $\hat{f}_n(\cdot)$, (A.14) yields

$$(A.45) \quad W(\hat{f}_n(\cdot)) - W(f_n(\cdot)) = \lim_{k \rightarrow \infty} \int_0^1 [\Phi_{\hat{x}_1}(\wp_1 \times_n S; f_\alpha^k) - \Phi_{x_1}(\wp_1 \times_n S; f_\alpha^k)] \cdot d\alpha \geq 2\delta > 0$$

Since $n > \hat{n}$ was arbitrary, we thus have $V_W(\hat{\mathbf{P}}) = \lim_{n \rightarrow \infty} W(\hat{f}_n(\cdot)) > \lim_{n \rightarrow \infty} W(f_n(\cdot)) = V_W(\mathbf{P})$. ■

PROOF OF THM 4: Since $f_n^*(\cdot) = [f_1(\cdot) \text{ on } \wp_1 \times_n S; \dots; f_m(\cdot) \text{ on } \wp_m \times_n S]$ and $\hat{f}_n^*(\cdot) = [\hat{f}_1(\cdot) \text{ on } \hat{\wp}_1 \times_n S; \dots; \hat{f}_m(\cdot) \text{ on } \hat{\wp}_m \times_n S]$ imply probabilistically equivalent almost-objective acts over each event in their common refinement $\{E_1^*, \dots, E_K^*\}$, we can define $\{x_1^*, \dots, x_J^*\}$ as the union of their payoffs, and write

$$(A.46) \quad \begin{aligned} f_n^*(\cdot) &= [x_1^* \text{ on } \wp_{1,1} \times_n E_1^*; \dots; x_J^* \text{ on } \wp_{1,J} \times_n E_1^*; \dots; x_1^* \text{ on } \wp_{K,1} \times_n E_K^*; \dots; x_J^* \text{ on } \wp_{K,J} \times_n E_K^*] \\ \hat{f}_n^*(\cdot) &= [x_1^* \text{ on } \hat{\wp}_{1,1} \times_n E_1^*; \dots; x_J^* \text{ on } \hat{\wp}_{1,J} \times_n E_1^*; \dots; x_1^* \text{ on } \hat{\wp}_{K,1} \times_n E_K^*; \dots; x_J^* \text{ on } \hat{\wp}_{K,J} \times_n E_K^*] \end{aligned}$$

where for each $k = 1, \dots, K$, the partitions $\{\wp_{k,1}, \dots, \wp_{k,J}\}$ and $\{\hat{\wp}_{k,1}, \dots, \hat{\wp}_{k,J}\}$ of $[0,1]$ satisfy $\lambda(\wp_{k,j}) = \lambda(\hat{\wp}_{k,j})$ for $j = 1, \dots, J$. As in the proof of Theorem 3, we show $\lim_{n \rightarrow \infty} W(f_n^*(\cdot)) = \lim_{n \rightarrow \infty} W(\hat{f}_n^*(\cdot))$ by showing that $\{W(f_n^*(\cdot))\}_{n=1}^\infty$ is a Cauchy sequence in R^1 , and then that $W(f_n^*(\cdot)) - W(\hat{f}_n^*(\cdot)) \rightarrow 0$.

Step 1: Given $\varepsilon > 0$, we need n_ε such that $|W(f_n^*(\cdot)) - W(f_{n'}^*(\cdot))| < \varepsilon$ for all $n', n'' > n_\varepsilon$. Applying Theorem 0 to each of the $K \cdot J$ combinations of $\wp_{k,j}$, E_k^* , $\{\Phi_{x_j^*}(\cdot; f) | f(\cdot) \in \mathcal{A}\}$ yields an n_ε such that

$$(A.47) \quad |\Phi_{x_j^*}(\wp_{k,j} \times_n E_k^*; f) - \lambda(\wp_{k,j}) \cdot \Phi_{x_j^*}(E_k^*; f)| < \varepsilon / (4 \cdot K \cdot J)$$

for all $n > n_\varepsilon$, all $k = 1, \dots, K$, all $j = 1, \dots, J$, and all $f(\cdot) \in \mathcal{A}$.

Select arbitrary $n', n'' > n_\varepsilon$ and consider the acts $f_{n'}^*(\cdot)$ and $f_{n''}^*(\cdot)$. By (A.47) we have

$$(A.48) \quad |\Phi_{x_j^*}(\wp_{k,j} \times_{n'} E_k^*; f) - \Phi_{x_j^*}(\wp_{k,j} \times_{n''} E_k^*; f)| < \varepsilon / (2 \cdot K \cdot J)$$

for all $j = 1, \dots, J$, all $k = 1, \dots, K$, and all $f(\cdot) \in \mathcal{A}$. Defining $\{f_\alpha^{*\ell}(\cdot) \equiv [f_{n'}^*(\cdot) \text{ on } [0, \alpha] \times_\ell S; f_{n''}^*(\cdot) \text{ on } (\alpha, 1] \times_\ell S] | \alpha \in [0,1]\}$ as the almost-objective mixture paths from $f_{n'}^*(\cdot)$ to $f_{n''}^*(\cdot)$, (A.14) implies

$$(A.49) \quad W(f_{n''}^*(\cdot)) - W(f_{n'}^*(\cdot)) = \lim_{\ell \rightarrow \infty} \int_0^1 \sum_{k=1}^K \sum_{j=1}^J [\Phi_{x_j^*}(\wp_{k,j} \times_{n'} E_k^*; f_\alpha^{*\ell}) - \Phi_{x_j^*}(\wp_{k,j} \times_{n''} E_k^*; f_\alpha^{*\ell})] \cdot d\alpha$$

By (A.48), the integrand in (A.49) is less than $\varepsilon/2$ in absolute value for all $\alpha \in [0,1]$ and all ℓ , which implies $|W(f_n^*(\cdot)) - W(f_n^*(\cdot))| < \varepsilon$. But since n' and n'' were arbitrary integers greater than n_ε , this implies that $\{W(f_n^*(\cdot))\}_{n=1}^\infty$ is a Cauchy sequence, and hence converges to some $\bar{W}^*$.

Step 2: Given arbitrary $\varepsilon > 0$, we need $\hat{n}_\varepsilon$ such that $|W(\hat{f}_n^*(\cdot)) - W(f_n^*(\cdot))| < \varepsilon$ for all $n > \hat{n}_\varepsilon$. Applying Theorem 0 again yields an $\hat{n}_\varepsilon$ such that both

$$(A.50) \quad \begin{aligned} |\Phi_{x_j^*}(\wp_{k,j} \times E_k^*, f) - \lambda(\wp_{k,j}) \cdot \Phi_{x_j^*}(E_k^*, f)| &< \varepsilon/(4 \cdot K \cdot J) \\ |\Phi_{x_j^*}(\hat{\wp}_{k,j} \times E_k^*, f) - \lambda(\hat{\wp}_{k,j}) \cdot \Phi_{x_j^*}(E_k^*, f)| &< \varepsilon/(4 \cdot K \cdot J) \end{aligned}$$

for all $n > \hat{n}_\varepsilon$, all $k=1, \dots, K$, all $j=1, \dots, J$, and all $f(\cdot) \in \mathcal{A}$.

Select arbitrary $n > \hat{n}_\varepsilon$ and consider the acts $f_n^*(\cdot)$ and $\hat{f}_n^*(\cdot)$. By (A.50) and $\lambda(\wp_{k,j}) = \lambda(\hat{\wp}_{k,j})$ we have

$$(A.51) \quad |\Phi_{x_j^*}(\hat{\wp}_{k,j} \times E_k^*, f) - \Phi_{x_j^*}(\wp_{k,j} \times E_k^*, f)| < \varepsilon/(2 \cdot K \cdot J)$$

for all $k=1, \dots, K$, all $j=1, \dots, J$, and all $f(\cdot) \in \mathcal{A}$. Defining $\{\hat{f}_\alpha^*(\cdot) | \alpha \in [0,1]\}$ as the almost-objective mixture paths from $f_n^*(\cdot)$ to $\hat{f}_n^*(\cdot)$, (A.14) again implies

$$(A.52) \quad W(\hat{f}_n^*(\cdot)) - W(f_n^*(\cdot)) = \lim_{\ell \rightarrow \infty} \int_0^1 \sum_{k=1}^K \sum_{j=1}^J [\Phi_{x_j^*}(\hat{\wp}_{k,j} \times E_k^*; \hat{f}_\alpha^{*\ell}) - \Phi_{x_j^*}(\wp_{k,j} \times E_k^*; \hat{f}_\alpha^{*\ell})] \cdot d\alpha$$

By (A.51), the integrand in (A.52) is less than $\varepsilon/2$ in absolute value for all $\alpha \in [0,1]$ and all ℓ , which implies $|W(\hat{f}_n^*(\cdot)) - W(f_n^*(\cdot))| < \varepsilon$. But since n was an arbitrary integer greater than $\hat{n}_\varepsilon$, this implies that $W(\hat{f}_n^*(\cdot)) - W(f_n^*(\cdot))$ converges to 0. ■

PROOF OF THM 5: Given $\wp, \wp'$, disjoint E, E' , $x^* \succ x$ and $f(\cdot) = [x_1 \text{ on } E_1; \dots; x_m \text{ on } E_m]$, probabilistic sophistication of $W_{PS}(\cdot)$ implies

$$(A.53) \quad \begin{aligned} &W_{PS}(x^* \text{ on } (\wp \times S) \cap E; x \text{ on } (\wp' \times S) \cap E'; f(s) \text{ elsewhere}) \\ &= V(x^*, \mu(\wp \times E); x, \mu(\wp' \times E'); x_1, \mu(E_1 - \wp \times E - \wp' \times E'); \dots; x_m, \mu(E_m - \wp \times E - \wp' \times E')) \end{aligned}$$

Theorem 0 implies $\lim_{n \rightarrow \infty} \mu(\wp \times E) = \lambda(\wp) \cdot \mu(E)$ and $\lim_{n \rightarrow \infty} \mu(\wp' \times E) = \lambda(\wp') \cdot \mu(E)$. Disjointness of E and E' implies

$$(A.54) \quad \begin{aligned} \mu(E_i - \wp \times E - \wp' \times E') &= \mu(E_i) - \mu(E_i \cap (\wp \times E)) - \mu(E_i \cap (\wp' \times E')) \\ &= \mu(E_i) - \mu(\wp \times (E_i \cap E)) - \mu(\wp' \times (E_i \cap E')) \end{aligned} \quad i = 1, \dots, m$$

so that Theorem 0 also implies

$$(A.55) \quad \lim_{n \rightarrow \infty} \mu(E_i - \wp \times E - \wp' \times E') = \mu(E_i) - \lambda(\wp) \cdot \mu(E_i \cap E) - \lambda(\wp') \cdot \mu(E_i \cap E') \quad i = 1, \dots, m$$

Defining $\rho_i = \mu(E_i) - \lambda(\wp) \cdot \mu(E_i \cap E) - \lambda(\wp') \cdot \mu(E_i \cap E')$, continuity of $V(\cdot)$ in each probability yields

$$(A.56) \quad \begin{aligned} &\lim_{n \rightarrow \infty} W_{PS}(x^* \text{ on } (\wp \times S) \cap E; x \text{ on } (\wp' \times S) \cap E'; f(s) \text{ elsewhere}) \\ &= V(x^*, \lambda(\wp) \cdot \mu(E); x, \lambda(\wp') \cdot \mu(E'); x_1, \rho_1; \dots; x_m, \rho_m) \end{aligned}$$

Since we similarly have

$$(A.57) \quad \begin{aligned} &\lim_{n \rightarrow \infty} W_{PS}(x \text{ on } (\wp \times S) \cap E; x^* \text{ on } (\wp' \times S) \cap E'; f(s) \text{ elsewhere}) \\ &= V(x, \lambda(\wp) \cdot \mu(E); x^*, \lambda(\wp') \cdot \mu(E'); x_1, \rho_1; \dots; x_m, \rho_m) \end{aligned}$$

inequality (equality) (33) follows from first order stochastic dominance preference of $V(\cdot)$. ■

PROOF OF THM 6: We establish the bottom equation in (37), which implies each of the other assertions of the theorem. Given the subjective acts $f_1(\cdot), \dots, f_m(\cdot) \in \mathcal{A}$ and partition $\{\wp_1, \dots, \wp_m\}$ of $[0,1]$, we can express $f_1(\cdot), \dots, f_m(\cdot)$ in the form

$$(A.58) \quad f_1(\cdot) = [x_{1,1} \text{ on } \hat{E}_1; \dots; x_{1,J} \text{ on } \hat{E}_J], \dots, f_m(\cdot) = [x_{m,1} \text{ on } \hat{E}_1; \dots; x_{m,J} \text{ on } \hat{E}_J]$$

for some partition $\{\hat{E}_1, \dots, \hat{E}_J\}$ of S , so that we can express $(f_1(\cdot) \text{ on } \wp_1 \times S; \dots; f_m(\cdot) \text{ on } \wp_m \times S)$ as

$$(A.59) \quad (x_{1,1} \text{ on } \wp_1 \times \hat{E}_1; \dots; x_{1,J} \text{ on } \wp_1 \times \hat{E}_J; \dots; x_{m,1} \text{ on } \wp_m \times \hat{E}_1; \dots; x_{m,J} \text{ on } \wp_m \times \hat{E}_J)$$

Recall that every event-smooth state-dependent expected utility preference function can be expressed in form $W_{SDEU}(f(\cdot)) \equiv \int_S U(f(s)|s) \cdot d\mu(s) \equiv \sum_{x \in \mathcal{X}} \int_{f^{-1}(x)} U(x|s) \cdot m(s) \cdot ds \equiv \sum_{x \in \mathcal{X}} \Phi_x(f^{-1}(x))$, where each $\Phi_x(\cdot)$ has a bounded and continuous density $\phi_x(s) \equiv U(x|s) \cdot m(s)$ on S . Thus we can write

$$(A.60) \quad W_{SDEU}(f_1(\cdot) \text{ on } \wp_1 \times S; \dots; f_m(\cdot) \text{ on } \wp_m \times S) \equiv \sum_{i=1}^m \sum_{j=1}^J \Phi_{x_{i,j}}(\wp_i \times \hat{E}_j)$$

and Theorem 0 yields

$$(A.61) \quad \begin{aligned} \lim_{n \rightarrow \infty} W_{SDEU}(f_1(\cdot) \text{ on } \wp_1 \times S; \dots; f_m(\cdot) \text{ on } \wp_m \times S) &= \lim_{n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^J \Phi_{x_{i,j}}(\wp_i \times \hat{E}_j) = \\ \sum_{i=1}^m \lambda(\wp_i) \cdot \sum_{j=1}^J \Phi_{x_{i,j}}(\hat{E}_j) &= \sum_{i=1}^m \lambda(\wp_i) \cdot \sum_{j=1}^J \int_{\hat{E}_j} U(x_{i,j}|s) \cdot d\mu(s) = \\ \sum_{i=1}^m \lambda(\wp_i) \cdot \sum_{j=1}^J \int_{\hat{E}_j} U(f_i(s)|s) \cdot d\mu(s) &= \sum_{i=1}^m \lambda(\wp_i) \cdot \int_S U(f_i(s)|s) \cdot d\mu(s) \end{aligned}$$

PROOF OF THM 7: Select and fix outcomes $x \succ x'$. Define $V_W(\cdot)$ as from Theorem 3, and for each $E \in \mathcal{E}$, define $\mu(E) \in [0,1]$ as the unique solution to $W(x \text{ on } E; x' \text{ on } \sim E) = \lim_{n \rightarrow \infty} W(x \text{ on } [0, \mu(E)] \times_n S; x' \text{ on } (\mu(E), 1] \times_n S)$, or equivalently, to $W(x \text{ on } E; x' \text{ on } \sim E) = V_W(x, \mu(E); x', 1 - \mu(E))$, so that $\mu(\emptyset) = 0$ and $\mu(S) = 1$.

Picking an arbitrary act $f(\cdot) \in \mathcal{A}$ is equivalent to picking an arbitrary partition $\{E_1, \dots, E_m\}$ of S and then assigning arbitrary outcomes to these events. Consider an arbitrary partition $\{E_1, \dots, E_m\}$, labeled so that E_1 is nonnull. For $i = 1, \dots, m-1$, let $\alpha_i \in [0,1]$ solve

$$(A.62) \quad W \left(\begin{array}{l} x \text{ on } E_1 \cup \dots \cup E_i \\ x' \text{ on } E_{i+1} \\ x' \text{ on } E_k \text{ } k > i+1 \end{array} \right) = \lim_{n \rightarrow \infty} W \left(\begin{array}{l} [[0, \alpha_i] \cdot \mathcal{P}_x + (\alpha_i, 1] \cdot \mathcal{P}_{x'}] \times_n (E_1 \cup \dots \cup E_i) \\ [[0, \alpha_i] \cdot \mathcal{P}_x + (\alpha_i, 1] \cdot \mathcal{P}_{x'}] \times_n E_{i+1} \\ \mathcal{P}_{x'} \text{ on } E_k \text{ } k \neq i, j \end{array} \right)$$

and define

$$(A.63) \quad \begin{aligned} \tau_1 &= \alpha_{m-1} \cdot \alpha_{m-2} \cdot \dots \cdot \alpha_3 \cdot \alpha_2 \cdot \alpha_1 \\ \tau_2 &= \alpha_{m-1} \cdot \alpha_{m-2} \cdot \dots \cdot \alpha_3 \cdot \alpha_2 \cdot (1 - \alpha_1) \\ \tau_3 &= \alpha_{m-1} \cdot \alpha_{m-2} \cdot \dots \cdot \alpha_3 \cdot (1 - \alpha_2) \\ \tau_4 &= \alpha_{m-1} \cdot \alpha_{m-2} \cdot \dots \cdot (1 - \alpha_3) \\ &\vdots \\ \tau_{m-1} &= \alpha_{m-1} \cdot (1 - \alpha_{m-2}) \\ \tau_m &= (1 - \alpha_{m-1}) \end{aligned}$$

which satisfy $\tau_1 + \dots + \tau_m = 1$. For arbitrary outcomes $x_1, \dots, x_m$, repeated application of the Almost-Objective/Subjective Replacement Axiom and the definition of convex-interval mixtures yields:

$$\begin{aligned}
& W \begin{pmatrix} x_1 \text{ on } E_1 \\ x_2 \text{ on } E_2 \\ x_3 \text{ on } E_3 \\ \vdots \\ x_m \text{ on } E_m \end{pmatrix} = \lim_{n \rightarrow \infty} W \begin{pmatrix} [x_1 \text{ on } [0, \alpha_1]; x_2 \text{ on } (\alpha_1, 1]] \times_n E_1 \\ [x_1 \text{ on } [0, \alpha_1]; x_2 \text{ on } (\alpha_1, 1]] \times_n E_2 \\ x_3 \text{ on } E_3 \\ \vdots \\ x_m \text{ on } E_m \end{pmatrix} = \\
& = \lim_{n \rightarrow \infty} W \begin{pmatrix} [x_1 \text{ on } [0, \alpha_1 \cdot \alpha_2]; x_2 \text{ on } (\alpha_1 \cdot \alpha_2, \alpha_2]; x_3 \text{ on } (\alpha_2, 1]] \times_n E_1 \\ [x_1 \text{ on } [0, \alpha_1 \cdot \alpha_2]; x_2 \text{ on } (\alpha_1 \cdot \alpha_2, \alpha_2]; x_3 \text{ on } (\alpha_2, 1]] \times_n E_2 \\ [x_1 \text{ on } [0, \alpha_1 \cdot \alpha_2]; x_2 \text{ on } (\alpha_1 \cdot \alpha_2, \alpha_2]; x_3 \text{ on } (\alpha_2, 1]] \times_n E_3 \\ x_4 \text{ on } E_4 \\ \vdots \\ x_m \text{ on } E_m \end{pmatrix} = \dots \\
& \text{(A.64)} \quad = \lim_{n \rightarrow \infty} W \begin{pmatrix} [x_1 \text{ on } [0, \alpha_1 \cdot \alpha_2 \cdot \alpha_3]; x_2 \text{ on } (\alpha_1 \cdot \alpha_2 \cdot \alpha_3, \alpha_2 \cdot \alpha_3]; x_3 \text{ on } (\alpha_2 \cdot \alpha_3, \alpha_3]; x_4 \text{ on } (\alpha_3, 1]] \times_n E_1 \\ [x_1 \text{ on } [0, \alpha_1 \cdot \alpha_2 \cdot \alpha_3]; x_2 \text{ on } (\alpha_1 \cdot \alpha_2 \cdot \alpha_3, \alpha_2 \cdot \alpha_3]; x_3 \text{ on } (\alpha_2 \cdot \alpha_3, \alpha_3]; x_4 \text{ on } (\alpha_3, 1]] \times_n E_2 \\ [x_1 \text{ on } [0, \alpha_1 \cdot \alpha_2 \cdot \alpha_3]; x_2 \text{ on } (\alpha_1 \cdot \alpha_2 \cdot \alpha_3, \alpha_2 \cdot \alpha_3]; x_3 \text{ on } (\alpha_2 \cdot \alpha_3, \alpha_3]; x_4 \text{ on } (\alpha_3, 1]] \times_n E_3 \\ [x_1 \text{ on } [0, \alpha_1 \cdot \alpha_2 \cdot \alpha_3]; x_2 \text{ on } (\alpha_1 \cdot \alpha_2 \cdot \alpha_3, \alpha_2 \cdot \alpha_3]; x_3 \text{ on } (\alpha_2 \cdot \alpha_3, \alpha_3]; x_4 \text{ on } (\alpha_3, 1]] \times_n E_4 \\ \vdots \\ x_m \text{ on } E_m \end{pmatrix} \\
& = \lim_{n \rightarrow \infty} W \begin{pmatrix} [x_1 \text{ on } [0, \tau_1]; \dots; x_i \text{ on } (\sum_{j=1}^{i-1} \tau_j, \sum_{j=1}^i \tau_j]; \dots; x_m \text{ on } (\sum_{j=1}^{m-1} \tau_j, 1]] \times_n E_1 \\ \vdots \\ [x_1 \text{ on } [0, \tau_1]; \dots; x_i \text{ on } (\sum_{j=1}^{i-1} \tau_j, \sum_{j=1}^i \tau_j]; \dots; x_m \text{ on } (\sum_{j=1}^{m-1} \tau_j, 1]] \times_n E_m \end{pmatrix} \\
& = \lim_{n \rightarrow \infty} W(x_1 \text{ on } [0, \tau_1] \times_n S; \dots; x_i \text{ on } (\sum_{j=1}^{i-1} \tau_j, \sum_{j=1}^i \tau_j] \times_n S; \dots; x_m \text{ on } (\sum_{j=1}^{m-1} \tau_j, 1] \times_n S)
\end{aligned}$$

Since $\lambda((\sum_{j=1}^{i-1} \tau_j, \sum_{j=1}^i \tau_j]) = \tau_i$, Theorem 3 thus implies $W(x_1 \text{ on } E_1; \dots; x_m \text{ on } E_m) = V_W(x_1, \tau_1; \dots; x_m, \tau_m)$. Since the values $\tau_1, \dots, \tau_m$ were obtained from $E_1, \dots, E_m$ independently of the outcomes $x_1, \dots, x_m$, for each $i = 1, \dots, m$, set $x_i = x$ and $x_j = x'$ for $j \neq i$ to obtain $W(x \text{ on } E_i; x' \text{ on } \sim E_i) = V_W(x, \tau_i; x', 1 - \tau_i)$. This implies $\tau_i = \mu(E_i)$ for each i , so that $W(x_1 \text{ on } E_1; \dots; x_m \text{ on } E_m) = V_W(x_1, \mu(E_1); \dots; x_m, \mu(E_m))$.

To see that $\mu(\cdot)$ is finitely additive, observe that for arbitrary disjoint E and E' we have $V_W(x, \mu(E \cup E'); x', \mu(\sim(E \cup E'))) = W(x \text{ on } E \cup E'; x' \text{ on } \sim(E \cup E')) = W(x \text{ on } E; x \text{ on } E'; x' \text{ on } \sim(E \cup E')) = V_W(x, \mu(E); x, \mu(E'); x', \mu(\sim(E \cup E'))) = V_W(x, \mu(E) + \mu(E'); x', \mu(\sim(E \cup E')))$. ■

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