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Author

Littlejohn, R.G.

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R.G. Littlejohn

August 1984

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Differential Forms and Canonical Variables for
Drift Motion in Toroidal Geometry*

Robert G. Littlejohn
Lawrence Berkeley Laboratory and Physics Department
University of California
Berkeley, CA 94720

August 1984

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Differential Forms and Canonical Variables for Drift Motion in Toroidal Geometry

Robert G. Littlejohn
Department of Physics
University of California
Berkeley, California 94720

Received:

Abstract

Differential forms provide a simple and powerful computational tool for finding canonical variables for drift motion in toroidal geometry. The method is easily applied to relativistic drift motion.

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Recently several authors¹⁻² have made extensive use of a certain set of canonical variables to describe drift motion in toroidal geometries. However, the proof that these variables are indeed canonical is said to involve significant algebra. In this letter I will show that one can quite easily be led to these variables and simultaneously prove that they are canonical, with the use of some

elementary manipulations of differential forms. To further show the power of the method, I will generalize the results to the case of relativistic motion.

It has been shown previously³ that drift motion is completely described by a variational principle of the general form

$$\delta \int L dt = 0. \quad (1)$$

The phase space Lagrangian L is given by

$$L = \frac{e}{c} \mathbf{A}^* \cdot \dot{\mathbf{x}} + \frac{mc}{e} \mu \dot{\zeta} - H, \quad (2)$$

where H is the Hamiltonian,

$$H = e\phi + \mu B + \frac{m}{2} v_{\parallel}^2, \quad (3)$$

and $\mathbf{A}^*$ is the modified vector potential of Morozov and Solovév,⁴

$$\mathbf{A}^* = \mathbf{A} + \frac{mc}{e} v_{\parallel} \mathbf{b}. \quad (4)$$

In these formulas, $\mathbf{x}$ is the guiding center position, ζ is the gyrophase, and $\mathbf{b} = \mathbf{B}/B$. The drift equations result by carrying out the variation of Eq. (1) with respect to any six independent variables in phase space. A convenient choice is often $(\mathbf{x}, v_{\parallel}, \mu, \zeta)$, but the general covariance of Lagrangian variational principles guarantees that any coordinates can be used, such as the flux coordinates to be introduced presently.

Canonical variables are special in that the Lagrangian, when expressed in terms of these variables, has the form

$$L dt = \sum_i p_i dq_i - H dt. \quad (5)$$

Equation (2) shows that $(mc/e)\mu$ and ζ are already canonically conjugate, as one would expect, but that $v_{\parallel}$ and the components of $\mathbf{x}$ are not.

Following the notation and assumptions of Ref. 1, we take $\mathbf{B}$ to be some field with good flux surfaces, so that $\mathbf{B}$ has the form,

$$\mathbf{B} = \nabla\psi \times \nabla\theta_0, \quad (6)$$

where ψ is a single valued function of spatial position with toroidal contour surfaces (θ_0 is less well behaved). The exact magnetic field $\mathbf{B}_e$ differs from $\mathbf{B}$ by a small amount, assumed to be given by

$$\mathbf{B}_e = \mathbf{B} + \nabla \times \left(\frac{\alpha}{\lambda} \mathbf{B} \right). \quad (7)$$

We now introduce the vector potentials $\mathbf{A} = \psi \nabla\theta_0$ and $\mathbf{A}_e = \mathbf{A} + (\alpha/\lambda)\mathbf{B}$, corresponding respectively to $\mathbf{B}$ and $\mathbf{B}_e$, so that the total modified vector potential becomes

$$\mathbf{A}_e^* = \psi \nabla\theta_0 + \left(\frac{\alpha}{\lambda} + \frac{v_{||}}{\Omega} \right) \mathbf{B}. \quad (8)$$

Using now the alternative (covariant) form for $\mathbf{B}$,

$$\mathbf{B} = \lambda(\nabla\chi + \beta\nabla\psi), \quad (9)$$

and writing $\rho_{||} = v_{||}/\Omega$, we have

$$\mathbf{A}_e^* \cdot d\mathbf{x} = \psi d\theta_0 + (\alpha + \lambda\rho_{||})(d\chi + \beta d\psi). \quad (10)$$

This is the differential form which is to be transformed into $p_1 dq_1 + p_2 dq_2$. The rules of transformation are, first, that any coordinate transformation is allowed, and second, that any exact differential can be added to the given differential form with no effect on the physics. The operator d is expanded out by the chain rule, as in elementary calculus. We see that $\psi d\theta_0$ already looks like $p dq$, but we note that we also have a term in $d\psi$. This interferes with the interpretation of ψ as a generalized momentum p , because we want only dq 's,

and not dp 's, in the final result. To get rid of this term, we subtract the exact differential $d(\rho_c \beta \psi) = (\rho_c \beta) d\psi + \psi d(\rho_c \beta)$, where $\rho_c = \alpha + \lambda \rho_{\parallel}$, and we find

$$\mathbf{A}_e^* \cdot d\mathbf{x} = \psi d(\theta_0 - \rho_c \beta) + \rho_c d\chi. \quad (11)$$

This suggests that we set $\theta_c = \theta_0 - \rho_c \beta$, and thus we find

$$L dt = \frac{e}{c} (\psi d\theta_c + \rho_c d\chi) + \frac{mc}{e} \mu d\zeta - H dt, \quad (12)$$

which, apart from factors of e , m , c , shows clearly the canonical structure.

The physical factors are easily dealt with; if the action is stationary for the physical motion, any constant multiple of the action is stationary also. Therefore we can multiply L by c/e , and we find

$$\delta \int \left(\psi \dot{\theta}_c + \rho_c \dot{\chi} + \frac{mc^2}{e^2} \mu \dot{\zeta} - \frac{c}{e} H \right) dt = 0. \quad (13)$$

This calculation is easily generalized to the case of relativistic motion. In Ref. 3 it was pointed out that $\mathbf{A}^*$ is essentially the gyroaverage of the canonical momentum $\mathbf{p} = (e/c)\mathbf{A} + m\mathbf{v}$ of the particle motion. The averaging causes $\mathbf{A}$ to be evaluated at the guiding center position, and the perpendicular component of $\mathbf{v}$ to be dropped. Thus, we obtain $\mathbf{A}^*$ of Eq. (4). It is plausible, therefore, that relativistic results will follow if we use the relativistic canonical momentum, $\mathbf{p} = (e/c)\mathbf{A} + m\mathbf{u}$, where $\mathbf{u} = \gamma\mathbf{v}$ is the world velocity. Then we obtain the relativistic version of $\mathbf{A}^*$,

$$\mathbf{A}^* = \mathbf{A} + \frac{mc}{e} \mathbf{u}_{\parallel}. \quad (14)$$

Similarly, for the Hamiltonian, we must use the relativistic version of the energy,

$$H = e\phi + \sqrt{m^2 c^4 + m^2 c^2 u_{\parallel}^2 + 2mc^2 \mu B}. \quad (15)$$

It is easy to show that these versions of A^* and H , when used in Eqs. (1) and (2), give the correct relativistic drifts. These results can also be justified with full rigor.

We can now transform to flux coordinates in exactly the same way as shown previously, the only caution being that $\rho_{||} = mcu_{||}/eB$. Otherwise the canonical variables are $(\psi, \theta_c, \rho_c, \chi)$, as before.

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References.

1. R. B. White, A. H. Boozer, and Ralph Hay, *Phys. Fluids* **25**, 575(1982).
2. R. B. White and M. S. Chance, Princeton Plasma Physics Laboratory Report No. PPPL-2089, submitted to *Phys. Fluids*.
3. Robert G. Littlejohn, *J. Plasma Phys.* **29**, 111(1983).
4. A. I. Morozov and L. S. Solovév, in *Reviews of Plasma Physics*, edited by M. A. Leontovich (Consultants Bureau, New York, 1965), Vol. 2, p. 228.

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