Title

Quark-lepton mass relations from modular flavor symmetry

Journal

Journal of High Energy Physics, 2024(2)

ISSN

1126-6708

Authors

Chen, Mu-Chun
King, Stephen F
Medina, Omar
et al.

Publication Date

2024

DOI

10.1007/jhep02(2024)160

Quark-lepton mass relations from modular flavor symmetry

Mu-Chun Chen,[1, *] Stephen F. King,[2, †] Omar Medina,[3, ‡] and José W. F. Valle[3, §]

[1] *Department of Physics and Astronomy, University of California, Irvine, CA 92697-4575 USA*

[2] *School of Physics and Astronomy, University of Southampton, Southampton SO17 1BJ, United Kingdom*

[3] *Instituto de Física Corpuscular (IFIC), Universidad de Valencia-CSIC, Paterna (Valencia) E-46980, Spain*

The so-called Golden Mass Relation provides a testable correlation between charged-lepton and down-type quark masses, that arises in certain flavor models that do not rely on Grand Unification. Such models typically involve broken family symmetries. In this work, we demonstrate that realistic fermion mass relations can emerge naturally in modular invariant models, without relying on *ad hoc* flavon alignments. We provide a model-independent derivation of a class of mass relations that are experimentally testable. These relations are determined by both the Clebsch-Gordan coefficients of the specific finite modular group and the expansion coefficients of its modular forms, thus offering potential probes of modular invariant models. As a detailed example, we present a set of viable mass relations based on the $\Gamma_4 \cong S_4$ symmetry, which have calculable deviations from the usual Golden Mass Relation.

I. INTRODUCTION

In the Standard Model (SM), fermion masses and mixings arise from the Yukawa interaction of quarks and leptons with the Higgs field. Although the fields of the three families have identical SM gauge group quantum numbers, they exhibit largely distinct masses. Such hierarchical structure of masses across the three families appears rather enigmatic [1–3]. Moreover, the mixing pattern of quarks and leptons encoded in the CKM and lepton mixing matrices is quite different, and unexplained from first principles in the SM.

Understanding the pattern of fermion masses and mixings presents a two-fold puzzle for particle physics. While some success has been achieved towards predicting fermion mixings through the imposition of family symmetries [4–8], less progress has been made concerning the formulation of a fully convincing theory of fermion mass hierarchies, though there have been many proposals in this direction [9–18].

The idea of relating quark and lepton masses has a long history. Since the $SU(5)$ model proposed by Georgi and Glashow [19], that places quarks and leptons within a common representation, it has become usual to expect quark and lepton mass relations to emerge from gauge unification [20–27].

However, despite many efforts that started rather early on [28], no truly definitive theory relating quarks and leptons has ever been devised. The so-called flavor puzzle became more acute after the discovery of neutrino oscillations [29, 30] and the need to account for neutrino masses and mixings as well.

Interestingly, viable relations between quark and lepton masses can also emerge in flavor symmetry models, even in the absence of genuine gauge unification. This is the case for the so-called approximate *golden* quark-lepton mass

* muchunc@uci.edu
† s.f.king@soton.ac.uk
‡ Omar.Medina@ific.uv.es
§ valle@ific.uv.es

relation [31–35]

$$\frac{m_b}{\sqrt{m_s m_d}} \approx \frac{m_\tau}{\sqrt{m_\mu m_e}}\,, \tag{1}$$

that has been obtained both with discrete as well as continuous family symmetry groups. This relation has also been obtained in the context of orbifold extensions of the SM [36–40]. Given the experimental uncertainty of down-quark mass measurements,

$$\frac{m_d}{m_s} \approx \frac{1}{20}\,, \qquad \frac{m_s}{m_b} \approx \frac{1}{50}\,, \qquad \frac{m_e}{m_\mu} \approx \frac{1}{200}\,, \qquad \text{and} \qquad \frac{m_\mu}{m_\tau} \approx \frac{1}{17}\,, \tag{2}$$

one can readily verify that this mass relation is consistent with experimental data. Cleary this is just one relation and, by itself, does not exhaust the complexity of the flavor problem. However, given its success and simplicity, one may argue that it could constitute part of the ultimate theory of flavor. Note also that it is consistent with the Georgi-Jarlskog mass relations [28],

$$\frac{m_e}{m_d} \approx \frac{1}{3}\,, \qquad \frac{m_\mu}{m_s} \approx 3\,, \qquad \frac{m_\tau}{m_b} \approx 1\,, \tag{3}$$

which was predicted to hold at the GUT scale, with quark masses increased by a factor of about 3 at low energies, due to renormalization group (RG) running (with the largest contribution coming from QCD). We emphasize that the combination in Eq. (1) (satisfied by the Georgi-Jarlskog relations) is rather stable under renormalization group evolution [41, 42]. As a consequence the Golden mass relation, which holds at the electroweak scale could potentially hold also at high energy scales, even all the way up to the gauge unification scale $M_{\mathrm{GUT}} \sim 10^{16}$ GeV (See discussion in Subsection III B).

A common drawback of flavor symmetry model predictions, such as Eq. (1), is that they usually rely on *ad hoc* flavor symmetry breaking and vacuum alignment assumptions, for example through *flavons* in the scalar sector [4, 43–54]. By contrast, in modular invariant models where the flavor symmetry is nonlinearly realized [55] (for a recent review see [56]), minimal, realistic, and uniquely defined symmetry breaking patterns can be obtained without the need for flavons. Moreover, these symmetries could unveil a possible connection between the SM and strings or extra dimensional field theories [57–68].

In this work we point out that modular flavor symmetries can naturally yield viable correlations between the SM fermion masses without invoking flavons nor Grand Unification. This illustrates the potential of these symmetries towards the formulation of a successful and experimentally testable flavor theory of fermion masses. Mass relations can emerge from the symmetry structure of the vector-valued modular forms that uniquely parametrize the breaking of modular invariance [55, 56, 69, 70]. In particular, using modular invariance we obtain analytically a generalization of the *golden* quark-lepton mass relation in Eq. (1).

This work is structured as follows: In Section II we present the general method, i.e. a model-independent derivation of fermion mass relations in modular invariant models that contain few parameters. In Section III we present an explicit example for the $\Gamma \cong S_4$ modular group, in which we derive viable mass relations of down-quarks and charged leptons. We also contrast these to experimental data. In Section IV we argue that, though the chosen example is not intended to be a complete flavor model, the general method may be useful to build more comprehensive modular symmetry models of flavor, with significantly fewer free parameters than the SM.

II. MASS RELATIONS FROM MODULAR SYMMETRY

To begin with, let us assume an $\mathcal{N}=1$ supersymmetric theory, whose action is given as

$$\mathcal{S} = \int d^4x d^2\theta d^2\bar{\theta}\, \mathcal{K}(\bar{\tau}, \bar{\psi}, \tau, \psi) + \int d^4x d^2\theta \mathcal{W}(\tau, \psi) + h.c\,, \tag{4}$$

where $\mathcal{K}$ is the Kähler potential and $\mathcal{W}$ is the superpotential. These are functions of the chiral superfields τ and ψ.

A. Modular invariance

Besides the SM gauge symmetry, we require the action $\mathcal{S}$ to be invariant under the modular group $SL(2,\mathbb{Z}) \equiv \Gamma$, so it remains unchanged under the modular transformation [50, 55, 57, 58]

$$\tau \xrightarrow{\gamma} \frac{a\tau + b}{c\tau + d} \qquad \text{where} \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{with} \quad ad - cb = 1 \tag{5}$$

where the matrix γ has integer entries and belongs to the modular group Γ, generated by the elements

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6}$$

that obey the relations $S^2 = R$, $(ST)^3 = R^2 = \mathbf{1}$, and $RT = TR$.

The set of matter chiral superfields ψ of the Minimal Supersymmetric Standard Model (MSSM) present in the action in Eq. (4) transform as weighted representations $\psi \sim \bigoplus_\alpha (\mathbf{r}_\alpha, -k_\alpha)$ under the action of $\gamma \in \Gamma$.

$$\psi_\alpha \xrightarrow{\gamma} (c\tau + d)^{-k_\alpha} \rho_{\mathbf{r}_\alpha}(\gamma)\, \psi_\alpha\,. \tag{7}$$

Here α labels different irreducible representations $\rho_{\mathbf{r}}(\gamma)$ of $\Gamma_{\rm F}$, a finite subgroup of Γ that plays the role of the flavor symmetry. The corresponding automorphic factor $(c\tau + d)^{-k_\alpha}$ depends on the weight k_α.

The superpotential $\mathcal{W}$, which is assumed to be invariant under a modular transformation, can be written as a power series in the superfields ψ_α

$$\mathcal{W} = \sum_n \left(Y_{\alpha_1 \ldots \alpha_n}(\tau)\psi_{\alpha_1} \ldots \psi_{\alpha_n}\right)_{\mathbf{1}}\,. \tag{8}$$

Note that given the transformation properties of τ and ψ_α under the modular group in Eqs. (5) and (7) the Yukawa couplings $Y_{\alpha_1 \ldots \alpha_n}(\tau)$ are requited to be the vector-valued modular forms [69–71] of the finite modular group $\Gamma_{\rm F}$

$$Y_{\alpha_1 \ldots \alpha_n}(\gamma\tau) = (c\tau + d)^{k_{Y_n}} \rho_Y(\gamma) Y_{\alpha_1 \ldots \alpha_n}(\tau), \tag{9}$$

with weight k_{Y_n} and representation $\rho_Y(\gamma)$ of $\Gamma_{\rm F}$, such that each term of the superpotential in Eq. (8) is invariant under a modular transformation.

The structure of the Kähler potential $\mathcal{K}$, and its covariance under a modular transformation is of great importance once a model is specified [72, 73]. Nonetheless, since our aim in this paper is not to build a particular modular invariant model but rather to demonstrate a proof of principle that the mass relations can be a result of modular invariance, we will not include further discussion around it.

The superpotential terms in $\mathcal{W}$ involving the SM fermions can be written compactly as modular invariant fermion bilinears,

$$\mathcal{W} \supset \psi_i\, M(\tau)_{ij}\, \psi^c_j\,, \tag{10}$$

with $i, j = 1, \ldots, 3$ denoting the three families of the SM. Here it is understood that $M(\tau)$ includes the Higgs doublet chiral superfields $\Phi_{u,d}$ of the MSSM. Their vacuum expectation values (VEVs) induce the spontaneous breaking of the SM electroweak symmetry, while the VEV of the scalar component of the field τ, the modulus, characterizes the breaking of modular invariance. Altogether, the $\Phi_{u,d}$ and τ VEVs give rise to the mass matrices of the SM fermions.

We now turn to the issue of modular symmetry breaking and residual symmetries. Note that, as seen from Eqs. (5) and (9), modular invariance is nonlinearly realized, and any value of τ will break this symmetry. In Fig. 1 we display the fundamental domain of the modular group action on τ. Furthermore, there are three special values of the modulus τ at which the modular group Γ breaks down into different preserved residual discrete symmetries [55, 74–77]. We generically denote these values as $\tau_{\rm sym}$ and refer to them as symmetry points. These are associated to some unbroken combination of the generators of the modular group in Eq. (6). As seen from Eqs. (5) and (6) the R generator is unbroken for any value of τ, so that a $\mathbb{Z}_2^R$ symmetry is always preserved. Therefore the three symmetry points are

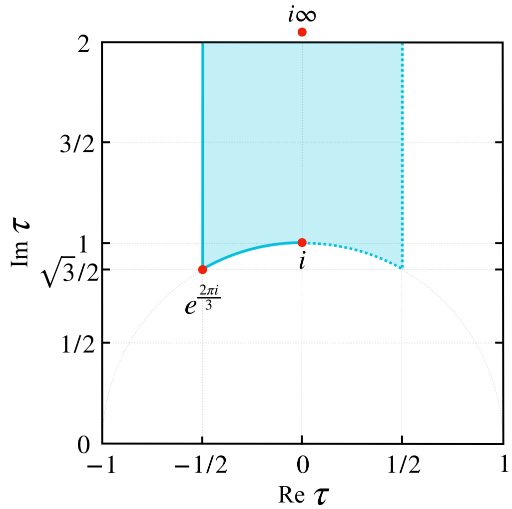


FIG. 1: This figure shows the fundamental domain for the action of the modular group in cyan. We chose to include the solid line border. The three symmetry points, $\tau_{\rm sym} = i\infty,\ i,\ \omega$, are marked in red. An arbitrary value of τ in the complex plane can be mapped to a point within this domain via a modular transformation in Eq. (5).

- $\tau_{\rm sym} = i\infty$, invariant under T, preserving $\mathbb{Z}_N^T \otimes \mathbb{Z}_2^R$.
- $\tau_{\rm sym} = i$, invariant under S, preserving $\mathbb{Z}_4^S$, with $\mathbb{Z}_2^R$ as a subgroup.
- $\tau_{\rm sym} = \omega \equiv \exp(2\pi i/3)$, invariant under ST, preserving $\mathbb{Z}_3^{ST} \otimes \mathbb{Z}_2^R$.

Notice that for $\tau_{\rm sym} = i\infty$ the preserved symmetry $\mathbb{Z}_N^T \otimes \mathbb{Z}_2^R$ is determined by the order N of the T generator ($T^N = \mathbb{I}$) of the corresponding finite modular group.

When the modulus τ is near any of the symmetry points $\tau_{\rm sym}$, certain appealing properties of the modular forms in Eq. (9) become manifest. For instance, they can give rise to hierarchical patterns of fermion masses [77–80], a property that is fundamental to this work. Our main result, outlined below in this section, emerges from these residual symmetries in a model-independent manner. Our derivation holds as long as we are close to any of the three symmetry points $\tau_{\rm sym} = i\infty,\ i,\ \omega$. It is convenient to define deviation parameters from each of the three symmetry points

$$\epsilon_{i\infty}(\tau) = e^{i\frac{2\pi\tau}{N}}, \qquad \epsilon_i(\tau) = \frac{\tau - i}{\tau + i}, \qquad \text{and} \qquad \epsilon_\omega(\tau) = \frac{\tau - \omega}{\tau - \omega^2}. \tag{11}$$

Note that one can write the modular form multiplets in Eq. (9) in terms of these variables. This is useful since the expansion the vector-valued modular forms in terms of $\epsilon_{i\infty}$, ϵ_i or ϵ_ω is related to their "charge" under the respective residual symmetry group, as stressed in [77].

We now turn to the main result of our proposal, which is a model-independent derivation of how correlations between fermion masses can emerge naturally from modular flavor symmetries. We denote a generic pair of matter superfield multiplets ψ and ψ^c resembling the three SM fermion families as:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad \text{and} \quad \psi^c = \begin{pmatrix} \psi_1^c \\ \psi_2^c \\ \psi_3^c \end{pmatrix}. \tag{12}$$

In general, one can write a general fermion mass matrix M_ψ as a function of n parameters $a_1, \ldots, a_n$ present in the superpotential

$$\mathcal{W} \supset \psi M_\psi(a_1, \ldots, a_n) \psi^c\ , \tag{13}$$

so that the three fermion masses are functions of these parameters:

$$m_1(a_1,\dots,a_n),\quad m_2(a_1,\dots,a_n),\quad \text{and}\quad m_3(a_1,\dots,a_n). \tag{14}$$

Given the chiral structure of the MSSM gauge interactions, it is convenient to use the Hermitian and positive semi-definite matrix H_ψ instead of the original mass matrix M_ψ in Eq. (13),

$$H_\psi \equiv M_\psi M_\psi^\dagger, \tag{15}$$

this eliminates unphysical mixing parameters associated with the $\mathrm{SU(2)_L}$ singlet superfields ψ^c. The squared masses of the three fermions are obtained by performing a unitary basis tranformation U_ψ over the ψ superfields:

$$D_\psi^2 = U_\psi^\dagger H_\psi U_\psi,\qquad \text{where}\qquad D_\psi^2 \equiv \mathrm{Diag}(m_1^2, m_2^2, m_3^2). \tag{16}$$

Without loss of generality we assume the following ordering of the masses,

$$m_1 \le m_2 \le m_3. \tag{17}$$

Solving for the individual masses in Eq. (14) is, in general, a highly non-trivial model-dependent task. In contrast, we can use a set of basis invariants of $H_\psi(a_1,\dots,a_n)$ to extract useful information concerning the SM fermion masses in terms of the $a_1,\dots,a_n$ parameters (see for example [81]), i.e.

$$\mathrm{Tr}[H_\psi] = m_1^2 + m_2^2 + m_3^2, \tag{18}$$

$$\frac{1}{2}\left\{(\mathrm{Tr}[H_\psi])^2 - \mathrm{Tr}[H_\psi H_\psi]\right\} = m_1^2\,m_3^2 + m_2^2\,m_3^2 + m_1^2\,m_2^2, \tag{19}$$

$$\mathrm{Det}[H_\psi] = m_1^2\,m_2^2\,m_3^2. \tag{20}$$

In these equations, the left-hand side are polynomials in the underlying parameters $a_1,\dots,a_n$, with mass-dimension of 2, 4, and 6, respectively.

B. Mass matrices at the symmetry points

Close to the three symmetry points $\tau_{\mathrm{sym}} = i\infty,\ i,\ \omega$ one can conveniently write $H_\psi(a_1,\dots,\epsilon(\tau))$ as a function of the dimensionless parameter $\epsilon(\tau)$ [77–80]. In a given model the definition of ϵ will depend on the corresponding symmetry point τ_{sym} according to the definitions in Eq. (11). We will write the expressions in terms of $\epsilon(\tau)$ instead of τ.

In a modular invariant model, the number of independent parameters of the superpotential is reduced, since each term must be invariant under a modular transformation, i.e.

$$\mathcal{W} \supset \sum_i \alpha_i \left(Y_{\mathbf{r}}^{(k)}(\epsilon)\psi^\dagger \Phi_{u,d}\psi^c\right)_{\mathbf{1}}, \tag{21}$$

where $\Phi_{u,d}$ denote the two MSSM Higgs doublets, while $Y_{\mathbf{r}}^{(k)}(\epsilon)$ represents the vector-valued modular forms of weight k. The index i labels all the modular invariant contractions of $Y_{\mathbf{r}}^{(k)}(\epsilon)$ with the MSSM superfields and α_i are independent numerical coefficients. To match with our notation in Eq. (13), we will consider the dimensionful quantities $a_i \equiv \alpha_i v_{u,d}/\sqrt{2}$ as the mass matrix parameters, along with ϵ describing the deviation from one of the three residual symmetry points. Note that in top-down constructions the superpotential coefficients α_i are expected to be correlated quantities [60, 65, 66, 82–89], nonetheless in bottom-up constructions they are taken to be independent.

At the symmetry point (in the $\epsilon \to 0$ limit) a discrete residual symmetry is preserved. For a given finite modular flavor group, and for certain weighted representation assignments for ψ and ψ^c, the rank of H_ψ will be reduced in the symmetric limit, due to the preservation of the corresponding residual symmetry [77–80],

$$\lim_{\epsilon\to 0} \mathrm{Det}[H_\psi(a_1,\dots,\epsilon)] = m_1^2\,m_2^2\,m_3^2 = 0. \tag{22}$$

This condition implies that at least the first family is massless in the symmetric limit. Thus, the small mass m_1 would result from the deviation of the modulus τ from the residual symmetry point $\tau_{\rm sym}$, providing a symmetry-based explanation for the lightness of the first-family fermion. As detailed in [77], certain weighted representation assignments can render m_1, m_2, and even m_3 massless at the symmetry point. In our derivation, we focus on the minimal scenario where only m_1 is massless in this limit, though the approach is directly applicable to the other cases.

C. Conditions for mass relations

If the number of coefficients α_i is either one (α_1) or two (α_1, α_2) in Eq. (21), the mass matrix resulting from the superpotential will depend on at most three independent parameters, one of which is dimensionless (ϵ). In this case the system in Eqs. (18)-(20) is a solvable polynomial system of three variables.

This implies that in a given model, the H_ψ matrix will lead to a correlation among the three fermion masses, provided it satisfies the following two conditions:

1. The superpotential (21) must contain (at most) two independent coefficients, α_1 and α_2, so that H_ψ depends only on two dimensionful parameters: $a_1 = (\alpha_1 v_{u,d}/\sqrt{2})$ and $a_2 = (\alpha_2 v_{u,d}/\sqrt{2})$, in addition to ϵ. This condition is necessary, otherwise, Eqs. (18)-(20) form an unconstrained polynomial system.

2. The three-dimensional matrix H_ψ reduces its rank at the symmetry point, as indicated by Eq. (22), i.e.

$$\text{rank}\left[\lim_{\epsilon\to 0} H_\psi(a_1, a_2, \epsilon)\right] < \text{rank}[H_\psi(a_1, a_2, \epsilon)]. \tag{23}$$

In general, since H_ψ is positive semi-definite, one can write Eq. (20) as a power expansion in the dimensionless parameter $|\epsilon|$ around the symmetry point $\tau_{\rm sym}$, leading to

$$\text{Det}[H_\psi(a_1, \ldots, \epsilon)] = \sum_{m=0}^{\infty} f_m(a_1, \ldots, a_n)|\epsilon|^{2m}. \tag{24}$$

If the second condition above is satisfied by H_ψ then $f_0(a_1, \ldots, a_n) = 0$ in the expasion. Therefore, when both conditions hold we can write the last equation as

$$\text{Det}[H_\psi(a_1, a_2, \epsilon)] = m_1^2\, m_2^2\, m_3^2 \equiv f_\psi(a_1, a_2)\,|\epsilon|^\eta + \mathcal{O}\left(|\epsilon|^{2\eta}\right) + \ldots\,, \qquad \text{where} \quad \eta > 0. \tag{25}$$

where we defined $f_\psi(a_1, a_2) \equiv f_\eta(a_1, a_2)$, which is the coefficient of the leading term in the expansion in Eq. (24). From dimensional analysis, we know that f_ψ is a homogeneous order-six polynomial in the parameters a_1 and a_2. However, since ϵ is dimensionless, the value of η can not be inferred solely from dimensional analysis.

Using Eqs. (17) and (22), we can write (18) and (19) in the symmetric limit

$$\lim_{\epsilon\to 0} \text{Tr}[H_\psi] \equiv h_\psi(a_1, a_2) = m_2^2 + m_3^2, \tag{26}$$

$$\lim_{\epsilon\to 0} \frac{1}{2}\left\{(\text{Tr}[H_\psi])^2 - \text{Tr}[H_\psi H_\psi]\right\} \equiv g_\psi(a_1, a_2) = m_2^2\, m_3^2. \tag{27}$$

The two homogeneous polynomials $h_\psi(a_1, a_2)$ and $g_\psi(a_1, a_2)$ are of order 2 and 4 respectively. This is a two-equation system of two variables a_1 and a_2. At the exact symmetry point these polynomials give solutions of the form

$$\tilde{a}_1(m_2, m_3)\,, \quad \text{and} \quad \tilde{a}_2(m_2, m_3)\,, \tag{28}$$

which relate the model parameters a_1, a_2 with the fermion masses m_2 and m_3. See Appendix B for further discussion about the polynomial system solutions.

The solutions in the last equation, that hold at the symmetry point ($\epsilon \to 0$), together with Eq. (25) lead us to define the polynomial

$$f(m_2, m_3) \equiv f_\psi(\tilde{a}_1(m_2, m_3), \tilde{a}_2(m_2, m_3)), \tag{29}$$

Close to the symmetry point we can approximate Eq. (25) to its leading term in $|\epsilon| \ll 1$. This yields an expression relating the three fermion masses to $|\epsilon|$

$$\frac{m_1^2\, m_2^2\, m_3^2}{f(m_2, m_3)} \approx |\epsilon|^\eta \,. \tag{30}$$

This is our central result, it allows us to identify potentially viable correlations between the masses of the SM fermions in modular invariant models. In a specific model, the polynomial $f(m_2, m_3)$ is determined by both the Clebsch-Gordan coefficients of the finite modular group $\Gamma_{\rm F}$ and the vector-valued modular forms in the superpotential terms of Eq. (21). It is important to note that Eq. (30) serves as a valid approximation near the symmetry points $\tau_{\rm sym} = i\infty$, i, ω; otherwise, additional terms of the expansion in Eq. (25) must be considered. Notice that Eq. (30) is fully independent of the phase of $\epsilon(\tau)$.

III. EXAMPLE OF VIABLE MASS RELATIONS

To illustrate the general derivation in the previous section and emphasize key points, we now turn to a specific example. Let's consider $\Gamma_4 \cong S_4$ as the finite modular group. There are several flavor models based on this group or its double cover (see e.g. [75, 90–97]). Detailed information about the group properties, the chosen basis for the representations, and the expansion of the relevant modular forms are given in Appendix A.

The Γ_4 weighted-representation assignments of the down-sector superfields that result in a quark-lepton mass relation are given in Table I. For simplicity we omit transformation properties under the $\mathrm{SU(3)_c \otimes SU(2)_L \otimes U(1)_Y}$ gauge symmetry, as they are the standard ones of the MSSM [98]. As mentioned in the previous section, the first necessary condition for predicting a fermion mass relation is related to the number of invariant contractions in the superpotential (see Eq. (21)). In the case of Γ_4 at weight 2, there are five modular forms arranged in two multiplets:

$$Y^{(2)}_{\mathbf{2}}(\tau) = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix}, \quad Y^{(2)}_{\mathbf{3}}(\tau) = \begin{pmatrix} Y_3(\tau) \\ Y_4(\tau) \\ Y_5(\tau) \end{pmatrix}. \tag{31}$$

Hence there are only two independent contractions of the superfields with the modular forms,

	Q	D^c	L	E^c	Φ_d	$Y^{(2)}_{\mathbf{2}}$	$Y^{(2)}_{\mathbf{3}}$
Γ_4	$\mathbf{3}$	$\mathbf{3'}$	$\mathbf{3}$	$\mathbf{3'}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$
k	-1	-1	-2	0	0	2	2

TABLE I: This table contains the Γ_4 weighted-representation assignments that render a quark-lepton mass relation. MSSM gauge symmetry transformations are omitted in this table.

$$\mathbf{3} \otimes \mathbf{3'} \otimes Y^{(2)}_{\mathbf{2}} \Longrightarrow \mathbf{1} \qquad \text{and} \qquad \mathbf{3} \otimes \mathbf{3'} \otimes Y^{(2)}_{\mathbf{3}} \Longrightarrow \mathbf{1}\,. \tag{32}$$

Hence, the relevant superpotential terms in Eq. (21) for down-quarks and charged-leptons have the following structure:

$$\begin{aligned} \mathcal{W}^{\Gamma_4}_{\Phi_d} \supset\; & \alpha^d_1 \left(Q\Phi_d D^c Y^{(2)}_{\mathbf{2}}(\tau)\right)_{\mathbf{1}} + \alpha^d_2 \left(Q\Phi_d D^c Y^{(2)}_{\mathbf{3}}(\tau)\right)_{\mathbf{1}} \\ & + \alpha^e_1 \left(L\Phi_d E^c Y^{(2)}_{\mathbf{2}}(\tau)\right)_{\mathbf{1}} + \alpha^e_2 \left(L\Phi_d E^c Y^{(2)}_{\mathbf{3}}(\tau)\right)_{\mathbf{1}} . \end{aligned} \tag{33}$$

There are four coefficients in Eq. (33): α^d_1, α^d_2, α^e_1, and α^e_2. In our bottom-up setup, these are independent parameters. Although they are potentially complex, we assume the preservation of CP symmetry so these coefficients are real [99]. We define the following set of dimensionful parameters

$$a^d_1 \equiv \frac{\alpha^d_1 v_d}{\sqrt{2}}, \quad a^d_2 \equiv \frac{\alpha^d_2 v_d}{\sqrt{2}}, \quad a^e_1 \equiv \frac{\alpha^e_1 v_d}{\sqrt{2}}, \quad \text{and} \quad a^e_2 \equiv \frac{\alpha^e_2 v_d}{\sqrt{2}}. \tag{34}$$

in terms of the standard MSSM VEVs $\langle\Phi_{u,d}\rangle \equiv v_{u,d}/\sqrt{2}$.

The mass matrices for down-type quarks and charged-leptons, denoted as M_d and M_e respectively, are given as:

$$M_f = \begin{pmatrix} -a_1^f Y_2 & -a_2^f Y_4 & a_2^f Y_5 \\ -a_2^f Y_4 & \frac{1}{\sqrt{2}}\left(\sqrt{3}a_1^f Y_1 - 2a_2^f Y_3\right) & \frac{1}{2}a_1^f Y_2 \\ a_2^f Y_5 & \frac{1}{2}a_1^f Y_2 & \frac{1}{\sqrt{2}}\left(\sqrt{3}a_1^f Y_1 + 2a_2^f Y_3\right) \end{pmatrix}, \qquad \text{for} \quad f = d, e. \tag{35}$$

For the sake of compactness, we have omitted the explicit τ dependence of the modular forms $Y_i(\tau)$, with $i = 1, .., 5$. Notice that M_d and M_e, as described in Eq. (35), have a common structure arising from their shared covariance properties under the gauge as well as the Γ_4 symmetries. Both mass matrices satisfy our first condition in Section II C.

As prescribed in the previous section we define the Hermitian matrices

$$H_d \equiv M_d M_d^\dagger, \qquad \text{and} \qquad H_e \equiv M_e M_e^\dagger. \tag{36}$$

For τ close to the T-symmetric point $\tau_T \equiv i\infty$ these matrices fulfill the second condition outlined in Eq. (23). Consequently, they will lead to mass correlations that we will now discuss. Following Eq. (11), throughout this section we define the $\epsilon(\tau)$ parameter:

$$\epsilon(\tau) \equiv q^{\frac{1}{4}} = e^{\frac{2\pi i\tau}{4}}, \qquad \text{thus} \qquad \lim_{\tau\to i\infty} \epsilon(\tau) = 0. \tag{37}$$

In this limit the superpotential preserves a residual $\mathbb{Z}_4^T$ symmetry, generated by T in Table II of Appendix A.

Given that our general derivation applies near the residual symmetry points, we can truncate the expansion of the modular forms given in Appendix A to their leading order in ϵ:

$$Y_{\mathbf{2}}^{(2)}(\epsilon) \approx \begin{pmatrix} 1 + 24\epsilon^4 \\ -8\sqrt{3}\epsilon^2 \end{pmatrix}, \quad Y_{\mathbf{3}}^{(2)}(\epsilon) \approx \begin{pmatrix} 1 - 8\epsilon^4 \\ -4\sqrt{2}\epsilon \\ -16\sqrt{2}\epsilon^3 \end{pmatrix}. \tag{38}$$

It is manifest that the T-symmetric limit $\epsilon \to 0$ corresponds to the following alignment of the vector-valued modular forms

$$\lim_{\epsilon\to 0} Y_{\mathbf{2}}^{(2)}(\epsilon) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \lim_{\epsilon\to 0} Y_{\mathbf{3}}^{(2)}(\epsilon) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{39}$$

At this symmetry point we obtain the following mass spectrum for charged-leptons and down-quarks

$$\lim_{\epsilon\to 0} \begin{pmatrix} m_\tau \\ m_\mu \\ m_e \end{pmatrix} = \begin{pmatrix} m_\tau\left(a_1^e, a_2^e\right) \\ m_\mu\left(a_1^e, a_2^e\right) \\ 0 \end{pmatrix}, \qquad \lim_{\epsilon\to 0} \begin{pmatrix} m_b \\ m_s \\ m_d \end{pmatrix} = \begin{pmatrix} m_b\left(a_1^d, a_2^d\right) \\ m_s\left(a_1^d, a_2^d\right) \\ 0 \end{pmatrix}, \tag{40}$$

meaning that in the limit $\tau \to i\infty$ the down-quark and the electron become massless, hence their mass is only generated through a deviation from this symmetry point. In this limit, the masses of the the second and third family fermions are functions of the parameters defined in Eq. (34).

Following the general derivation in Section II we determine the polynomials defined in Eqs. (26), and (27) for this example:

$$h_d(a_1^d, a_2^d) = \frac{3}{2}a_1^{d^2} + 2a_2^{d^2} = m_s^2 + m_b^2, \qquad g_d(a_1^d, a_2^d) = \frac{1}{16}\left(3a_1^{d^2} - 4a_2^{d^2}\right)^2 = m_s^2 m_b^2, \tag{41}$$

and analogously for charged-leptons $h_e(a_1^e, a_2^e)$ and $g_e(a_1^e, a_2^e)$ which in this case have the equivalent structure.

We can obtain solutions for these two polynomial systems as described in Eq. (28). For a detailed list of all the different solutions to this polynomial system please refer to Appendix B. Notably, the following set of solutions (obtained for $\epsilon \to 0$) produces viable mass relations:

$$\tilde{a}^d_{1,\pm}(m_s, m_b) = \frac{m_b \pm m_s}{\sqrt{3}}, \quad \tilde{a}^d_{2,\pm}(m_s, m_b) = \frac{m_b \mp m_s}{2}, \tag{42}$$

$$\tilde{a}^e_{1,\pm}(m_\mu, m_\tau) = \frac{m_\tau \pm m_\mu}{\sqrt{3}}, \quad \tilde{a}^e_{2,\pm}(m_\mu, m_\tau) = \frac{m_\tau \mp m_\mu}{2}. \tag{43}$$

When the modulus value τ departs from a given symmetry point, in this case $\tau_T = i\infty$, there is a common scaling dependence on the parameter $\epsilon(\tau)$ that generates both m_d and m_e. Referring to Eq. (25) we can express the equations for the determinants of H_e and H_d as an expansion in the ϵ parameter:

$$\mathrm{Det}\,[H_d] = m_b^2 m_s^2 m_d^2 \approx \frac{4}{27}\left(3a_1^d + 2\sqrt{3}a_2^d\right)^4 \left(3a_1^d - 4\sqrt{3}a_2^d\right)^2 |\epsilon|^4 + \mathcal{O}\left(|\epsilon|^8\right). \tag{44}$$

$$\mathrm{Det}\,[H_e] = m_\tau^2 m_\mu^2 m_e^2 \approx \frac{4}{27}\left(3a_1^e + 2\sqrt{3}a_2^e\right)^4 \left(3a_1^e - 4\sqrt{3}a_2^e\right)^2 |\epsilon|^4 + \mathcal{O}\left(|\epsilon|^8\right). \tag{45}$$

Therefore close to the symmetry point ($|\epsilon| \ll 1$) we can plug in Eqs. (42) and (43) into Eqs. (44), and (45) to obtain

$$m_b^2 m_s^2 m_d^2 \approx f_\pm(m_b, m_s)\,|\epsilon|^4, \qquad m_\tau^2 m_\mu^2 m_e^2 \approx f_\pm(m_\tau, m_\mu)\,|\epsilon|^4, \tag{46}$$

where $f_\pm(m_3, m_2)$ are two polynomials of order 6 that depend on masses of the third and second generation fermions. The viable polynomial obtained $f_\pm(m_3, m_2)$ is given by

$$f_\pm(m_3, m_2) = 64 m_3^4 \left(m_3 \mp 3 m_2\right)^2. \tag{47}$$

We stress that the coefficients (marked in red in the last equation) in the polynomials are determined not only by the scaling properties of the modular forms in Eq. (38) with respect to $|\epsilon|$ (away from a symmetry point), but they also involve Clebsch-Gordan coefficients of the S_4 group and the coefficients of the Fourier expansion of the modular forms (also marked in red in Eq. (38)). The expansion coefficients have been proven to contain mathematical information (see e.g. [100]) whose significance in models with modular flavor symmetry remains somewhat underexplored, as the focus so far has primarily centered on the ϵ scaling properties [77–80, 101].

Looking at Eq. (46) it is straightforward to obtain the predicted mass relations for this example

$$\frac{m_b^2 m_s^2 m_d^2}{f_\pm(m_b, m_s)} \approx \frac{m_\tau^2 m_\mu^2 m_e^2}{f_\pm(m_\tau, m_\mu)} \tag{48}$$

which can be simplified to obtain

$$\frac{m_s m_d}{m_b\,(m_b \pm 3m_s)} \approx \frac{m_\mu m_e}{m_\tau\,(m_\tau \pm 3m_\mu)}. \tag{49}$$

Different possible signs in the denominators correspond to different solutions in Eqs. (42) and (43). There are four distinct mass relations emerging from Eq. (49), which we label as follows:

$$\text{R1:} \quad \frac{m_s m_d}{m_b\,(m_b - 3m_s)} \approx \frac{m_\mu m_e}{m_\tau\,(m_\tau - 3m_\mu)}. \tag{50}$$

$$\text{R2:} \quad \frac{m_s m_d}{m_b\,(m_b - 3m_s)} \approx \frac{m_\mu m_e}{m_\tau\,(m_\tau + 3m_\mu)}. \tag{51}$$

$$\text{R3:} \quad \frac{m_s m_d}{m_b\,(m_b + 3m_s)} \approx \frac{m_\mu m_e}{m_\tau\,(m_\tau - 3m_\mu)}. \tag{52}$$

$$\text{R4:} \quad \frac{m_s m_d}{m_b\,(m_b + 3m_s)} \approx \frac{m_\mu m_e}{m_\tau\,(m_\tau + 3m_\mu)}. \tag{53}$$

To scrutinize the viability of these mass relations we performed a scan over the following set of parameters determining the masses of down quarks and charged leptons

$$\{a^{d,e}_{1,2}, |\epsilon|\} \Longrightarrow \{m_{d,s,b,e,\mu,\tau}\}. \tag{54}$$

Notice that the light down-quark masses m_d and m_s are determined through lattice QCD computations, and exhibit the largest uncertainties amongst all fermion masses in Eqs. (50)-(53). In Figs. 2 and 3 we display the results of our parameter scan, each point plotted falls within the experimental 3-σ region for m_b, m_e, m_μ, and m_τ. In Fig. 2 we display the curves corresponding to all four mass relations R1 to R4 derived above with solid lines of different colors, illustrating how the mass relations differing only on signs in the denominators can be substantially different. In Fig. 3 we show a close-up plot of R1 which is the most favored experimentally. These curves use the experimental central values for m_b, m_e, m_μ, and m_τ, allowing us to compare the analytical mass relations (curves) with the numerical results from the parameter scan (points). The scatter plot and the plotted curves show excellent agreement, indicating that the approximations we used to derive the mass relations are accurate enough.

A. Zooming in the golden quark-lepton mass relation

We will now take a closer look at the derived mass relations in Eqs. (50)-(53) in comparison to the golden quark-lepton mass relation obtained from traditional flavor symmetries [31–35]. In the usual derivation of the latter, using

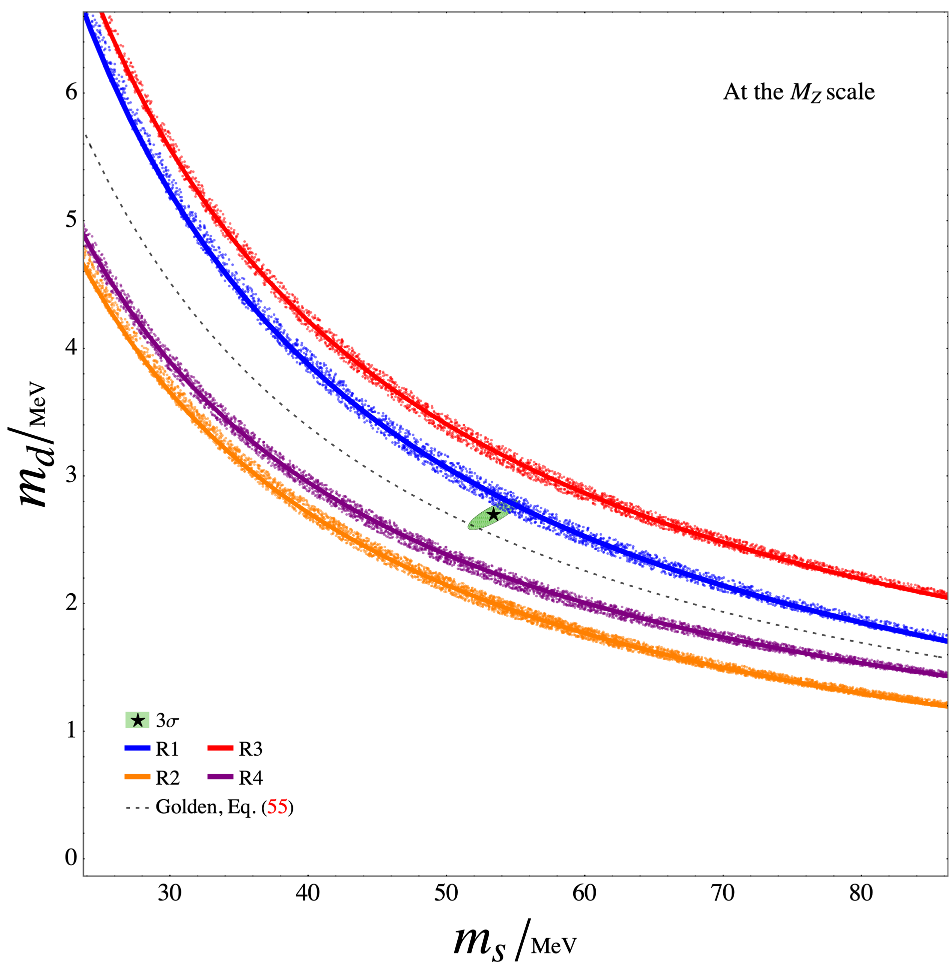


FIG. 2: This plot displays the results of our parameter scan $\{a^{d,e}_{1,2}, |\epsilon|\} \Longrightarrow \{m_{d,s,b,e,\mu,\tau}\}$ in the $m_d - m_s$ plane. Each point lies inside the experimental 3-σ range for m_b, m_e, m_μ, and m_τ. The allowed 3-σ region for m_d and m_s is shown in light-green [42]. We include the curves of the mass relations R1 to R4 from Eqs. (50) - (53) as solid lines. There can be sizable differences between these mass relations. For comparison we also plot (dashed) the *golden* quark-lepton mass relation in Eq. (55) obtained in "non-modular" flavor models [31–35].

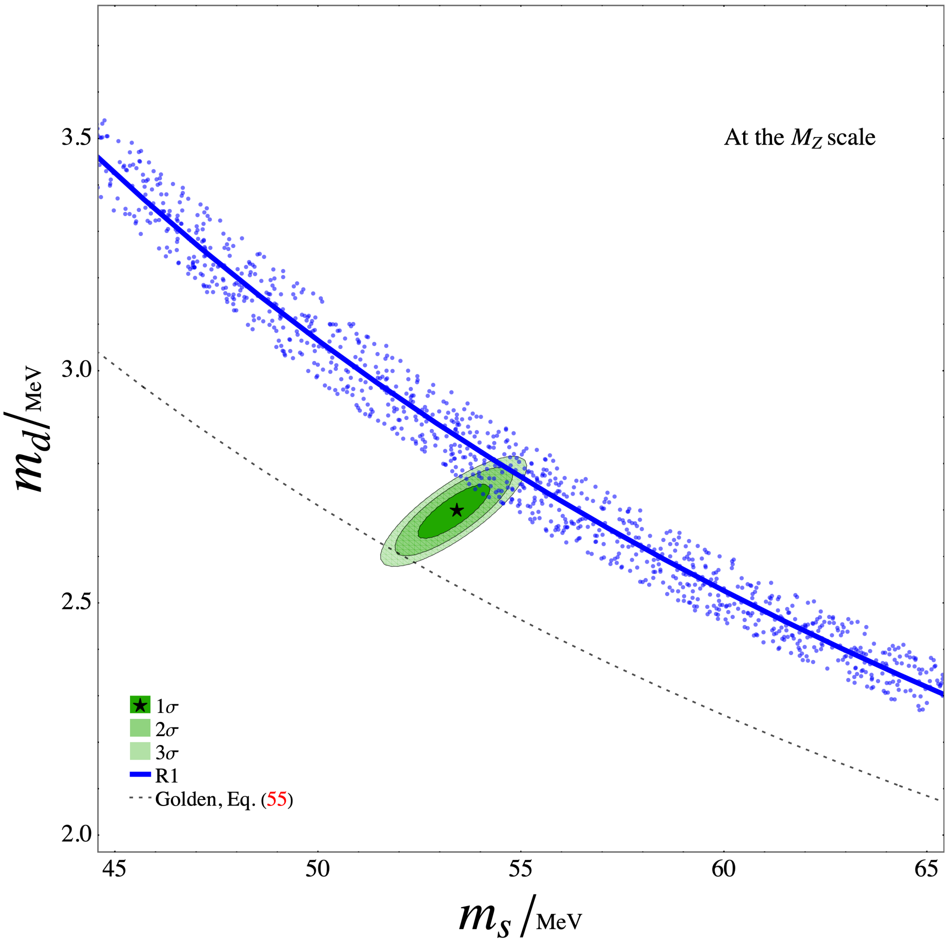


FIG. 3: This is a close-up of Fig. 2 where we only display the results for the R1 mass relation in Eq. (50) which is the experimentally most favored one. The allowed 1- 2- and 3-σ regions for m_d and m_s are shown in different shades of green [42]. The dashed curve in the plot shows the *golden* quark-lepton mass prediction in Eq. (55) obtained in "non-modular" flavor models [31–35]. An improved determination of m_s and m_d could probe our mass relations.

flavons for example, there are small corrections which are usually neglected, i.e.

$$\frac{m_b\, m_s\, m_d}{m_b^3 + \mathcal{O}(m_s^3)} \approx \frac{m_\tau\, m_\mu\, m_e}{m_\tau^3 + \mathcal{O}(m_\mu^3)}. \tag{55}$$

These corrections in the denominators of Eq. (55) are very small. In contrast, the corrections in Eq. (49) arising from Clebsch-Gordan coefficients of the Γ_4 symmetry and those of the modular forms expansion around the symmetry point, are typically much larger. They can yield substantial differences, as shown in Figs. 2 and 3, where we also display the prediction from Eq. (55).

Indeed, the mass relations in Eqs. (50)-(53) have an interesting property. They involve polynomials $f_\pm(m_3, m_2)$, Eq. (47), which include a term proportional to m_3^6. Given the strong hierarchy between the masses of fermions in the second and third families ($m_3 \gg m_2$), these polynomials $f_\pm(m_3, m_2)$ can be approximated using their leading term:

$$\frac{m_2}{m_3} \ll 1 \quad \Longrightarrow \quad f_\pm(m_3, m_2) \approx 64\, m_3^6. \tag{56}$$

This illustrates that there is a class of polynomials $f(m_3, m_2)$, defined in Eq. (29), which have a leading term proportional to m_3^6. And when used to relate charged-leptons and down-type quarks, this class of polynomials will approximate the *golden* quark-lepton mass relation in Eq. (55) for largely hierarchical fermion masses $m_1 \ll m_2 \ll m_3$.

In fact, by using this insight, one can infer that the polynomials in Eq. (47) will be roughly consistent with experimental data even without performing a quantitative parameter scan.

What is important to note is that the mass relations in Eqs. (50)-(53) are indeed valid for the values of quark and lepton masses (up to theoretical or experimental uncertainty). Our general derivation in Section II and the above example demonstrate that a class of viable mass predictions for down-quarks and charged-leptons can arise from modular invariant models near the symmetry points. Although the obtained mass relation can be similar to the non-modular *golden* mass relation, these predictions can yield testable deviations from it. As a result, improving determinations of light-quark masses could help constrain modular invariant flavor models, where the mass hierarchies of quarks and leptons emerge from deviations from residual symmetry points.

B. Stability under renormalization group evolution

Mass relations are generally very sensitive to renormalization group (RG) running. For instance, the Georgi-Jarlskog mass relations in Eq. (3), derived at the GUT scale [28], undergo significant evolution to fit experimental quark and lepton masses at lower scales, such as M_Z. In contrast, the mass relations in Eqs. (50)-(53), derived from Eq. (30), involve ratios of down-quark to charged-lepton masses. Thus, they are expected to be fairly stable under RG running. This follows from the fact that the dominant contribution to the Yukawa couplings' β-functions comes from gauge couplings, which effectively cancel in the mass ratios. However, for the bottom quark mass m_b there is a contribution involving the top quark Yukawa coupling y_t, making it necessary to analyze more closely the stability of our fermion mass relations.

To illustrate this point with an example we define the quantity $F_{\rm R1}(\mu_E)$ and rewrite the mass relation R1 in Eq. (50) with the down-quark and lepton masses given as functions of the scale $\mu_{\rm E}$,

$$F_{\rm R1}(\mu_E) \equiv \frac{m_s m_d m_\tau (m_\tau - 3m_\mu)}{m_\mu m_e m_b (m_b - 3m_s)} \approx 1. \tag{57}$$

For definiteness we perform the RG running of the mass relation R1 in the constrained MSSM (CMSSM) [102–104], using the Mathematica Package REAP [105] with the extension SusyTC [24] so as to account for SUSY threshold corrections [41, 106, 107]. To quantify the RG running of the mass relation R1 we choose the mass of the Z-boson (M_Z) as the initial scale [41, 42], where it holds quite well, as seen from Figs. 2 and 3. For definiteness we choose as benchmark the following values for the SUSY breaking scale $M_{\rm SUSY} = 10$ TeV and $\tan\beta = {}^{v_u}/{}_{v_d} = 50$. Concerning the four CMSSM soft SUSY-breaking parameters, we take the common scalar mass $m_0 = 12$ TeV, fermion mass $m_{1/2} = 10$ TeV, and the trilinear coupling $A_0 = -15$ TeV.

With these assumptions, we plot in Fig. 4 the evolution of $F_{\rm R1}(\mu_E)$ in Eq. (57) from the electroweak scale M_Z to the GUT scale $M_{\rm GUT} = 2 \times 10^{16}$ GeV, using the central values of the experimental quark and lepton masses as input. We also display as a reference the case $F_{\rm R1}(\mu_E) = 1$, and the *10%* deviation band. It is clear from this figure that the mass relation is quite stable under RG running, and holds all the way from the electroweak scale to the GUT scale, given the large uncertainty of the light quark masses m_d and m_s.

IV. DISCUSSION AND OUTLOOK

We have given a model-independent method to derive mass relations for the SM fermions in modular flavor symmetry models containing few parameters. They are determined by the Clebsch-Gordan coefficients of the specific modular flavor group as well as the expansion coefficients of the corresponding vector-valued modular forms around the symmetry points.

We illustrated our results with a detailed example in Section III, where we obtained viable mass relations for charged leptons and down-type quarks from the $\Gamma_4 \cong S_4$ flavor symmetry. We showed that these mass relations are

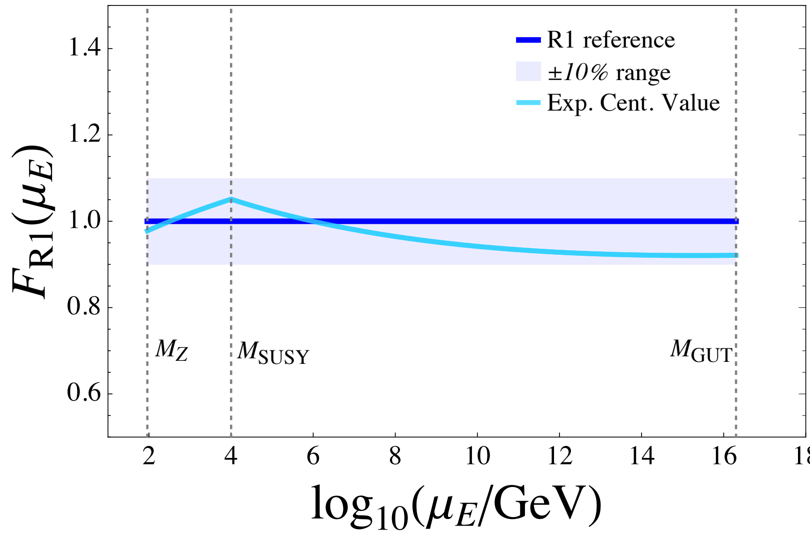


FIG. 4: $F_{\rm R1}(\mu_E)$ from Eq. (57) is plotted in cyan, with the values of the masses running from the electroweak scale M_Z to the GUT scale $M_{\rm GUT} = 2 \times 10^{16}$ GeV, using the REAP and SusyTC packages [24, 105]. We take central values of quark and lepton masses [41, 42]. The dark blue solid line $F_{\rm R1}(\mu_E) = 1$ is taken as reference and corresponds to the R1 mass relation. The plot shows how its changes remain within the light-blue band corresponding to 10% deviation, showing the potential validity of R1 from the electroweak to the GUT scale, given the uncertainty of the light quark masses m_d and m_s.

experimentally testable, and distinguishable from their non-modular counterpart the so-called golden mass relation given by Eq. (55) [31–35]. Since the mass relations can have sizable differences from one modular flavor group to another, they could be used to experimentally probe different models.

In Appendix C we point out the existence of a second class of mass relations that could yield viable predictions regarding neutrino masses and the up-quark sector. Nonetheless, these are model-dependent. We presented a particular example to highlight the differences with respect to our general derivation in Section III. This example is very suggestive, as it relates the smallness of the solar squared mass splitting to the lightness of the up quark.

We want to stress that the examples presented in Section III and Appendix C are not meant to be understood as complete flavor models, rather as illustrations of the use of our general method for deriving mass relations. Note that, as discussed in the Introduction (Section I), the main significance of the mass relations in Eqs. (50)-(53) and Eq. (55) is that they are compatible with experimental data of the SM fermion masses, which is a non-trivial fact. Our two examples appear to be complementary, since in Section III we only deal with the down sector of the MSSM, while in Appendix C we discuss the up sector and neutrinos. Since we present them independently, we can not make any meaningful statement concerning CKM mixing, which by definition involves both up and down quark sectors simultaneously. Although one can complete each of the models trivially, we can not preserve both predictions simultaneously in a simple manner. In other words, it is straightforward to extend the model of the down sector in Section III to include also the up-type quarks and neutrinos, while preserving the mass relations in Eqs. (50)-(53), at the expense of losing the prediction for the up sector in Appendix C.

It has been proven challenging to build experimentally viable quark and lepton modular flavor models while keeping the number of input parameters to a minimum. We expect that the results obtained in this work, largely model-independent, may prove useful also for more comprehensive modular invariant models and for top-down constructions. We demonstrated that viable mass relations for the SM fermions can emerge in modular flavor symmetry in a general manner relying only on the modular flavor group and its vector-valued modular forms, rather than *ad hoc* flavon alignments. The mass relations can differentiate amongst models. This was illustrated with one explicit example where the mass relations predicted are distinguishable from the one obtained in a traditional flavor symmetry model with flavons. The differences may be resolved experimentally and this may, perhaps, shed light on the symmetry underlying the flavor problem.

ACKNOWLEDGMENTS

We would like to thank Stefan Antusch, Salvador Centelles Chuliá, Xueqi Li, Xiang-Gan Liu, and Michael Ratz for insightful discussions. SFK also thanks CERN for hospitality and Peter Stangl for very helpful discussions. We also thank Peter Stangl for providing us with the tools to significantly improve our plots using more updated values of light-quark masses. OM thanks the Department of Physics and Astronomy at UC, Irvine, for their hospitality during his visit. SFK thanks IFIC, Valencia for their hospitality during his visit where this work began. Work supported by the Spanish grants PID2020-113775GB-I00 (AEI/10.13039/501100011033) and Prometeo CIPROM/2021/054 (Generalitat Valenciana). OM acknowledges financial support from the Generalitat Valenciana through Programa Santiago Grisolía (No. GRISOLIA/2020/025) and the research visit grant CIBEFP/2022/63. The research of MCC was supported, in part, by the U.S. National Science Foundation under Grant No. PHY-1915005. SFK acknowledges the STFC Consolidated Grant ST/L000296/1 and the European Union's Horizon 2020 Research and Innovation programme under Marie Sklodowska-Curie grant agreement HIDDeN European ITN project (H2020-MSCA-ITN-2019//860881-HIDDeN).

Appendix A: The Γ_4 Group - Basis and Modular Forms

The $\Gamma_4 \simeq S_4$ group is of order 24, all its elements can be written in terms of two generators S and T following its presentation equation:

$$S_4 \simeq \{S, T | S^2 = T^4 = (ST)^3 = e\}. \tag{A1}$$

The S_4 group has five irreducible representations: two singlets $\mathbf{1}$, $\mathbf{1'}$, one doublet $\mathbf{2}$, and two triplets $\mathbf{3}$, $\mathbf{3'}$. In this work we use the basis displayed in Table II. In this basis the T generator is diagonal for all irreducible representations, and both generators are symmetric matrices.

	S	T
$\mathbf{1}, \mathbf{1'}$	± 1	± 1
$\mathbf{2}$	$\frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$\mathbf{3}, \mathbf{3'}$	$\pm\frac{1}{2}\begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -1 & 1 \\ \sqrt{2} & 1 & -1 \end{pmatrix}$	$\pm\begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}$

TABLE II: Generators of S_4 in the symmetric T-diagonal basis.

The S_4 representation products relevant for the examples outlined in Section III and Appendix C are given by

$$\mathbf{3}\otimes\mathbf{3}'\Longrightarrow\begin{cases}\mathbf{1}'\sim\alpha_1\beta_1+\alpha_2\beta_3+\alpha_3\beta_2\\ \mathbf{2}\ \sim\begin{pmatrix}\frac{\sqrt{3}}{2}\left(\alpha_2\beta_2+\alpha_3\beta_3\right)\\ -\alpha_1\beta_1+\frac{1}{2}\left(\alpha_2\beta_3+\alpha_3\beta_2\right)\end{pmatrix}\\ \mathbf{3}\ \sim\begin{pmatrix}\alpha_3\beta_3-\alpha_2\beta_2\\ \alpha_1\beta_3+\alpha_3\beta_1\\ -\alpha_1\beta_2-\alpha_2\beta_1\end{pmatrix}\\ \mathbf{3}'\sim\begin{pmatrix}\alpha_3\beta_2-\alpha_2\beta_3\\ \alpha_2\beta_1-\alpha_1\beta_2\\ \alpha_1\beta_3-\alpha_3\beta_1\end{pmatrix}\end{cases}\qquad \mathbf{3}\otimes\mathbf{3}\Longrightarrow\begin{cases}\mathbf{1}\ \sim\alpha_1\beta_1+\alpha_2\beta_3+\alpha_3\beta_2\\ \mathbf{2}\ \sim\begin{pmatrix}\alpha_1\beta_1-\frac{1}{2}\left(\alpha_2\beta_3+\alpha_3\beta_2\right)\\ \frac{\sqrt{3}}{2}\left(\alpha_2\beta_2+\alpha_3\beta_3\right)\end{pmatrix}\\ \mathbf{3}\ \sim\begin{pmatrix}\alpha_3\beta_2-\alpha_2\beta_3\\ \alpha_2\beta_1-\alpha_1\beta_2\\ \alpha_1\beta_3-\alpha_3\beta_1\end{pmatrix}\\ \mathbf{3}'\sim\begin{pmatrix}\alpha_3\beta_3-\alpha_2\beta_2\\ \alpha_1\beta_3+\alpha_3\beta_1\\ -\alpha_1\beta_2-\alpha_2\beta_1\end{pmatrix}\end{cases}\tag{A2}$$

$$\mathbf{2}\otimes\mathbf{2}\Longrightarrow\begin{cases}\mathbf{1}\ \sim\alpha_1\beta_1+\alpha_2\beta_2\\ \mathbf{1}'\sim\alpha_1\beta_2-\alpha_2\beta_1\\ \mathbf{2}\ \sim\begin{pmatrix}\alpha_2\,\beta_2-\alpha_1\,\beta_1\\ \alpha_1\,\beta_2+\alpha_2\,\beta_1\end{pmatrix}\end{cases}\tag{A3}$$

The vector-valued modular forms of the Γ_4 group at weight 2 are

$$Y^{(2)}_{\mathbf{2}}(\tau)=\begin{pmatrix}Y_1(\tau)\\ Y_2(\tau)\end{pmatrix},\quad Y^{(2)}_{\mathbf{3}}(\tau)=\begin{pmatrix}Y_3(\tau)\\ Y_4(\tau)\\ Y_5(\tau)\end{pmatrix}.\tag{A4}$$

these can be written as power expansions

$$Y_i(\tau)=\sum_{n=0}^{\infty}c_{i,n}q^{\frac{n}{4}}\,,\qquad \text{with}\qquad q\equiv e^{2\pi i\tau},\tag{A5}$$

where $c_{i,n}$ are constant coefficients. We obtained the explicit expansion following the derivation in [108].

$$\begin{aligned}Y_1(\tau)&=1+24q+24q^2+96q^3+24q^4+144q^5+96q^6+192q^7+\dots,\\ Y_2(\tau)&=q^{\frac{1}{2}}(-8\sqrt{3}-32\sqrt{3}q-48\sqrt{3}q^2-64\sqrt{3}q^3-104\sqrt{3}q^4-96\sqrt{3}q^5+\dots),\\ Y_3(\tau)&=1-8q+24q^2-32q^3+24q^4-48q^5+96q^6-64q^7+\dots,\\ Y_4(\tau)&=q^{\frac{1}{4}}(-4\sqrt{2}-24\sqrt{2}q-52\sqrt{2}q^2-56\sqrt{2}q^3-72\sqrt{2}q^4-128\sqrt{2}q^5)\dots,\\ Y_5(\tau)&=q^{\frac{3}{4}}(-16\sqrt{2}-32\sqrt{2}q^2-48\sqrt{2}q^3-96\sqrt{2}q^4-80\sqrt{2}q^5+\dots)\,.\end{aligned}\tag{A6}$$

Appendix B: Polynomial system solutions

In Eqs. (26) and (27) of Section II we defined the polynomials $h_\psi(a_1,a_2)$ and $g_\psi(a_1,a_2)$, these are homogeneous polynomials of order 2 and 4 respectively. Ultimately the solutions of the polynomial system

$$\begin{aligned}h_\psi(a_1,a_2)&=m_2^2+m_3^2,\\ g_\psi(a_1,a_2)&=m_2^2\,m_3^2,\end{aligned}\tag{B1}$$

determine $f(m_3,m_2)$ in Eq. (30), which is the main result of this work.

By naive counting one could infer that this polynomial system has eight solutions, the product of the orders of the two polynomials. Nonetheless the actual number of independent and physically inequivalent solutions of the polynomial system in (B1) cannot be determined generically without the explicit form of $h_\psi(a_1,a_2)$ and $g_\psi(a_1,a_2)$.

To illustrate this, consider the mass matrix M_f in Eq. (35), for this particular case the polynomials take the form

$$h_\psi(a_1,a_2) = \frac{3}{2}{a_1}^2 + 2{a_2}^2 = m_2^2 + m_3^2, \qquad g_\psi(a_1,a_2) = \frac{1}{16}\left(3{a_1}^2 - 4{a_2}^2\right)^2 = m_2^2\, m_3^2. \tag{B2}$$

This polynomial system has in fact only three independent and physically inequivalent solutions

$$\textbf{a)} \qquad \tilde{a}_{1,+}(m_2,m_3) = \frac{m_3+m_2}{\sqrt{3}}, \qquad \tilde{a}_{2,-}(m_2,m_3) = \frac{m_3-m_2}{2}, \tag{B3}$$

$$\textbf{b)} \qquad \tilde{a}_{1,-}(m_2,m_3) = \frac{m_3-m_2}{\sqrt{3}}, \qquad \tilde{a}_{2,+}(m_2,m_3) = \frac{m_3+m_2}{2}, \tag{B4}$$

$$\textbf{c)} \qquad \hat{a}_{1,+}(m_2,m_3) = \frac{m_3+m_2}{\sqrt{3}}, \qquad \hat{a}_{2,-}(m_2,m_3) = -\frac{m_3-m_2}{2}, \tag{B5}$$

These solutions, following Eqs. (29) and (30) lead to the following three mass relations around the $\tau_{\text{sym}} = i\infty$, in terms of ϵ defined in Eq. (37)

$$\textbf{a)} \quad \frac{m_2 m_1}{8m_3(m_3-3m_2)} \approx |\epsilon|^2\,, \qquad \textbf{b)} \quad \frac{m_2 m_1}{8m_3(m_3+3m_2)} \approx |\epsilon|^2\,, \qquad \textbf{c)} \quad \frac{m_3 m_1}{8m_2(m_2+3m_3)} \approx |\epsilon|^2\,. \tag{B6}$$

Solutions **a)** and **b)** yield viable predictions when relating down-type quark and charged-lepton masses, as they feature a leading power of m_3^2 in the denominator (refer to Subsection III A for more details). In a complementary fashion, solution **c)** provides novel viable predictions for the up-sector masses, as demonstrated by Eq. (C10) in Appendix C.

Appendix C: Another class of mass relations

Our derivation in Section II is general, and model-independent. We now consider a second class of mass relations that can emerge in modular invariant models. This happens when an H_ψ matrix, defined in Eq. (15), satisfies the first condition listed in Subsection II C, while it does not satisfy the second. Meaning that at the symmetric point $\epsilon \to 0$ the rank of the matrix is not reduced.

$$\text{rank}\left[\lim_{\epsilon\to 0} H_\psi(a_1,a_2,\epsilon)\right] = \text{rank}[H_\psi(a_1,a_2,\epsilon)]. \tag{C1}$$

This implies that, in the limit, the three non-vanishing masses m_1, m_2 and m_3 can be written as functions of only two parameters a_1, and a_2. This yields algebraic relations amongst the three masses at the symmetry point. This relation can be obtained by solving the polynomial system in Eqs. (26), and (27). However, it cannot be derived in a model independent way, as it does not only depend on the representation and weight of the fields, but also on the specific symmetry point.

1. Example of a mass relation for the up-type quarks and neutrinos

We regard it illustrative to give an explicit example of how to obtain the second kind of mass relations which could also potentially yield interesting and viable predictions.

	Q	U^c	L	N^c	Φ_u	$Y_{\mathbf{2}}^{(2)}$	$Y_{\mathbf{3}}^{(2)}$
S_4	$\mathbf{3}$	$\mathbf{3'}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$
k	-1	-1	-2	0	0	2	2

TABLE III: Γ_4 weight and representation assignments that lead to a correlation between neutrino and up-quark masses. Gauge MSSM transformation properties are omitted.

For definiteness, we assume that neutrinos acquire a mass through a Type-I seesaw mechanism, including thus the N^c superfields for the three heavy right-handed (RH) neutrinos. The representation and weight assignments are given in Table III, leading to the following up-sector MSSM superpotential terms

$$\begin{aligned}\mathcal{W}_{\Phi_u}^{\Gamma_4} \supset\ & \alpha_1^u \left(Q\Phi_u U^c Y_{\mathbf{2}}^{(2)}(\tau)\right)_{\mathbf{1}} + \alpha_2^u \left(Q\Phi_u U^c Y_{\mathbf{3}}^{(2)}(\tau)\right)_{\mathbf{1}} \\ & + \alpha_1^\nu \left(L\Phi_u N^c Y_{\mathbf{2}}^{(2)}(\tau)\right)_{\mathbf{1}} + \alpha_2^\nu \left(L\Phi_u N^c Y_{\mathbf{3}}^{(2)}(\tau)\right)_{\mathbf{1}} + \frac{m_N}{2} N^c N^c.\end{aligned} \tag{C2}$$

Here the modular forms are the ones given in Eq. (31). The Dirac and RH neutrino mass blocks are given by

$$M_\nu^D = \begin{pmatrix} a_1^\nu Y_1 & -a_2^\nu Y_5 & a_2^\nu Y_4 \\ a_2^\nu Y_5 & \frac{\sqrt{3}}{2} a_1^f Y_2 & -\frac{1}{2}\left(a_1^\nu Y_1 + 2a_2^\nu Y_3\right) \\ -a_2^f Y_4 & -\frac{1}{2}\left(a_1^\nu Y_1 - 2a_2^\nu Y_3\right) & \frac{\sqrt{3}}{2} a_1^\nu Y_2 \end{pmatrix}, \qquad \text{and,} \qquad M_N = m_N \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{C3}$$

The effective mass of the light neutrinos is given by the standard seesaw formula

$$M_\nu \approx -M_\nu^D M_N^{-1} (M_\nu^D)^\top. \tag{C4}$$

We choose these particular representations to illustrate a relevant point. The Hermitian matrix $H_\nu \equiv M_\nu M_\nu^\dagger$ fulfills the first property in subsection II C, with only two dimensionful parameters; a_1^ν, and a_2^ν. However, at the symmetry point $\tau_{\text{sym}} = i\infty$ its rank is not reduced, Eq. (C1).

In this example, for simplicity we choose the up-quark mass matrix M_u to have the same structure that M_f in Eq. (35) thus around $\tau_{\text{sym}} = i\infty$ we have the following expression relating up-quark masses

$$\frac{m_t^2 m_c^2 m_u^2}{f(m_t, m_c)} \approx |\epsilon|^4 \tag{C5}$$

where the polynomial can be either one of the three solutions listed in Eq. (B6) in Appendix B. The last equation indicates that at this symmetric point the masses of the three families satisfy

$$\lim_{\epsilon\to 0} \begin{pmatrix} m_t \\ m_c \\ m_u \end{pmatrix} = \begin{pmatrix} m_t\left(a_1^u, a_2^u\right) \\ m_c\left(a_1^u, a_2^u\right) \\ 0 \end{pmatrix}, \qquad \lim_{\epsilon\to 0} \begin{pmatrix} m_3^\nu \\ m_2^\nu \\ m_1^\nu \end{pmatrix} = \begin{pmatrix} m_3^\nu\left(a_1^\nu, a_2^\nu\right) \\ m_2^\nu\left(a_1^\nu, a_2^\nu\right) \\ m_1^\nu\left(a_1^\nu, a_2^\nu\right) \end{pmatrix}, \tag{C6}$$

Notice that m_u vanishes at the symmetric limit as expected from the relation in Eq. (C5). However, since neither of the neutrino masses vanish, the three neutrino masses are not independent. In fact at the symmetry point two neutrino masses are degenerate:

$$\lim_{\tau\to i\infty} m_2^\nu = m_1^\nu \qquad \Longrightarrow \qquad \lim_{\tau\to i\infty} \Delta m_{21}^2 = 0. \tag{C7}$$

Therefore in this example the smallest of the required neutrino mass squared splitting, the solar Δm_{21}^2, must result from a departure from the symmetry point $\tau_T = i\infty$ yielding $\Delta m_{21}^2 \propto |\epsilon|^2$. [1] For this illustrative example we restrict ourselves to the region Im $\epsilon = 0$, hence for values $|\epsilon| \ll 1$ we obtain the simplified expression

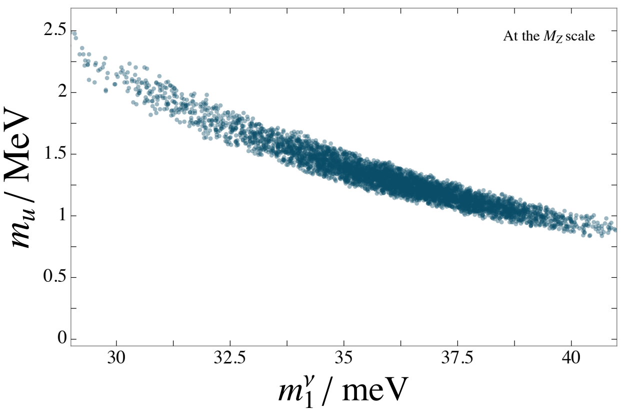
$$\Delta m_{21}^2 \approx f_\pm^\nu(m_3^\nu, m_1^\nu)|\epsilon|^2, \tag{C8}$$

This expression is analogous to Eq. (30) but manifestly different. The function $f_\pm^\nu(m_3^\nu, m_1^\nu)$ has mass-dimension 2 and is given by

$$f_\pm^\nu(m_3^\nu, m_1^\nu) = \frac{8m_1^\nu}{\Delta m_{31}^2}\left[8{m_1^\nu}^3 + 16{m_1^\nu}^2 m_3^\nu + 14 m_1^\nu {m_3^\nu}^2 - 3{m_3^\nu}^3 \pm 3\sqrt{4m_1^\nu + m_3^\nu}\left(4m_1^\nu {m_3^\nu}^{\frac{3}{2}} + {m_3^\nu}^{\frac{5}{2}}\right)\right], \tag{C9}$$

---

[1] Note that in this example the relation between the three neutrino masses depends on the value of $\epsilon(\tau)$, in contrast to the one obtained in reference [109].



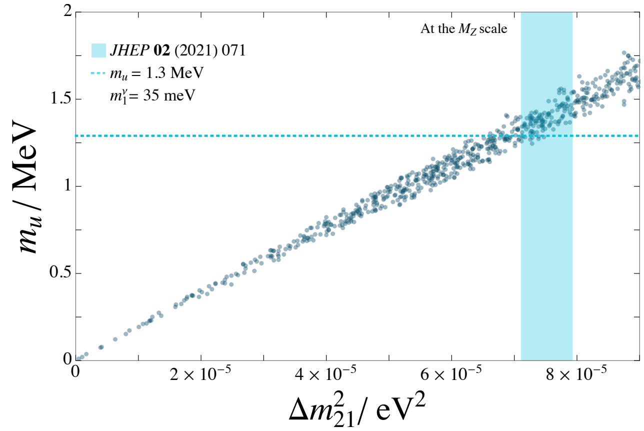


FIG. 5: Predicted quark-lepton mass correlations. In the left panel we show m_u versus the lightest neutrino mass m_1^ν, each point lies inside the experimental 3-σ range for m_t, m_c, Δm_{21}^2, and Δm_{31}^2 from PDG [110] and [111] respectively. This correlation is the prediction in Eq. (C10). The right panel assumes normal ordering, fixing $m_1^\nu = 35$ meV. It illustrates how both m_u and Δm_{21}^2 are generated simultaneously, when departing from the symmetry point as predicted by Eqs. (C5) and (C8).

where Δm_{31}^2 is the atmospheric neutrino squared mass difference. Notice that last equation is valid for both normal (NO) and inverted ordering (IO) of neutrino masses since the polynomial form is common.

A viable correlation between the up-quark and neutrino masses can be obtained by using Eqs. (C5), (C9), the plus sign function f_+^ν and the third solution in Eq. (B6), i.e.

$$\frac{\Delta m_{21}^2}{f_+^\nu(m_3^\nu, m_1^\nu)} \approx \frac{m_t m_c m_u}{8m_c^2(m_c + 3m_t)}. \tag{C10}$$

This correlation becomes manifest when we scan our parameter space $\{a_{1,2}^{u,\nu}, |\epsilon|\} \Longrightarrow \{m_{u,c,t}, m_{1,2,3}^\nu\}$ as shown in Fig. 5. Indeed, this figure shows a correlation in the m_u vs. squared solar mass splitting plane that results by taking points $\{m_t, m_c, \Delta m_{21}^2, \Delta m_{31}^2 \text{ (NO)}\}$ within their 3σ ranges given the PDG [110] and Ref. [111], respectively. This result is very suggestive, as it correlates the smallness of the solar squared mass splitting to the lightness of m_u, which is the quark mass with the largest uncertainty [41].

This example is relevant as it shows that there is a (second) kind of mass relations that can emerge in modular symmetric models that is not included in our general derivation in Section II. We want to highlight that this example, and in particular Eqs. (C7) and (C8), illustrates how mass matrices satisfying Eq. (C1) could lead to viable predictions for the neutrino sector.

[1] Z.-z. Xing, "Quark Mass Hierarchy and Flavor Mixing Puzzles," *Int. J. Mod. Phys. A* **29** (2014) 1430067, arXiv:1411.2713 [hep-ph].

[2] F. Feruglio, "Pieces of the Flavour Puzzle," *Eur. Phys. J. C* **75** no. 8, (2015) 373, arXiv:1503.04071 [hep-ph].

[3] Z.-z. Xing, "Flavor structures of charged fermions and massive neutrinos," *Phys. Rept.* **854** (2020) 1–147, arXiv:1909.09610 [hep-ph].

[4] S. F. King and C. Luhn, "Neutrino Mass and Mixing with Discrete Symmetry," *Rept. Prog. Phys.* **76** (2013) 056201, arXiv:1301.1340 [hep-ph].

[5] S. F. King, A. Merle, S. Morisi, Y. Shimizu, and M. Tanimoto, "Neutrino Mass and Mixing: from Theory to Experiment," *New J. Phys.* **16** (2014) 045018, arXiv:1402.4271 [hep-ph].

[6] S. F. King, "Models of Neutrino Mass, Mixing and CP Violation," *J. Phys. G* **42** (2015) 123001, arXiv:1510.02091 [hep-ph].

[7] S. F. King, "Unified Models of Neutrinos, Flavour and CP Violation," *Prog. Part. Nucl. Phys.* **94** (2017) 217–256, arXiv:1701.04413 [hep-ph].

[8] F. Feruglio and A. Romanino, "Lepton flavor symmetries," *Rev. Mod. Phys.* **93** no. 1, (2021) 015007, arXiv:1912.06028 [hep-ph].

[9] C. D. Froggatt and H. B. Nielsen, "Hierarchy of Quark Masses, Cabibbo Angles and CP Violation," *Nucl. Phys. B* **147** (1979) 277–298.

[10] Y. Koide, "A Fermion - Boson Composite Model of Quarks and Leptons," *Phys. Lett. B* **120** (1983) 161–165.

[11] M. Leurer, Y. Nir, and N. Seiberg, "Mass matrix models," *Nucl. Phys. B* **398** (1993) 319–342, arXiv:hep-ph/9212278.

[12] L. E. Ibanez and G. G. Ross, "Fermion masses and mixing angles from gauge symmetries," *Phys. Lett. B* **332** (1994) 100–110, arXiv:hep-ph/9403338.

[13] K. S. Babu and S. M. Barr, "Large neutrino mixing angles in unified theories," *Phys. Lett. B* **381** (1996) 202–208, arXiv:hep-ph/9511446.

[14] L. Randall and R. Sundrum, "A Large mass hierarchy from a small extra dimension," *Phys. Rev. Lett.* **83** (1999) 3370–3373, arXiv:hep-ph/9905221.

[15] D. E. Kaplan and T. M. P. Tait, "New tools for fermion masses from extra dimensions," *JHEP* **11** (2001) 051, arXiv:hep-ph/0110126.

[16] M.-C. Chen, D. R. T. Jones, A. Rajaraman, and H.-B. Yu, "Fermion Mass Hierarchy and Proton Stability from Non-anomalous U(1)(F) in SUSY SU(5)," *Phys. Rev. D* **78** (2008) 015019, arXiv:0801.0248 [hep-ph].

[17] A. J. Buras, C. Grojean, S. Pokorski, and R. Ziegler, "FCNC Effects in a Minimal Theory of Fermion Masses," *JHEP* **08** (2011) 028, arXiv:1105.3725 [hep-ph].

[18] S. Weinberg, "Models of Lepton and Quark Masses," *Phys. Rev. D* **101** no. 3, (2020) 035020, arXiv:2001.06582 [hep-th].

[19] H. Georgi and S. L. Glashow, "Unity of All Elementary Particle Forces," *Phys. Rev. Lett.* **32** (1974) 438–441.

[20] G. Lazarides, Q. Shafi, and C. Wetterich, "Proton Lifetime and Fermion Masses in an SO(10) Model," *Nucl. Phys. B* **181** (1981) 287–300.

[21] G. Altarelli and F. Feruglio, "A Simple grand unification view of neutrino mixing and fermion mass matrices," *Phys. Lett. B* **451** (1999) 388–396, arXiv:hep-ph/9812475.

[22] G. Ross and M. Serna, "Unification and fermion mass structure," *Phys. Lett. B* **664** (2008) 97–102, arXiv:0704.1248 [hep-ph].

[23] S. Antusch, S. F. King, and M. Spinrath, "GUT predictions for quark-lepton Yukawa coupling ratios with messenger masses from non-singlets," *Phys. Rev. D* **89** no. 5, (2014) 055027, arXiv:1311.0877 [hep-ph].

[24] S. Antusch and C. Sluka, "Predicting the Sparticle Spectrum from GUTs via SUSY Threshold Corrections with SusyTC," *JHEP* **07** (2016) 108, arXiv:1512.06727 [hep-ph].

[25] S. Antusch, C. Hohl, and V. Susič, "Yukawa ratio predictions in non-renormalizable SO(10) GUT models," *JHEP* **02** (2020) 086, arXiv:1911.12807 [hep-ph].

[26] S. Antusch, K. Hinze, and S. Saad, "Viable quark-lepton Yukawa ratios and nucleon decay predictions in SU(5) GUTs with type-II seesaw," *Nucl. Phys. B* **986** (2023) 116049, arXiv:2205.01120 [hep-ph].

[27] S. Antusch, K. Hinze, and S. Saad, "Quark-lepton Yukawa ratios and nucleon decay in SU(5) GUTs with type-III seesaw," *Nucl. Phys. B* **991** (2023) 116195, arXiv:2301.03601 [hep-ph].

[28] H. Georgi and C. Jarlskog, "A New Lepton - Quark Mass Relation in a Unified Theory," *Phys. Lett. B* **86** (1979) 297–300.

[29] T. Kajita, "Nobel Lecture: Discovery of atmospheric neutrino oscillations," *Rev. Mod. Phys.* **88** no. 3, (2016) 030501.

[30] A. B. McDonald, "Nobel Lecture: The Sudbury Neutrino Observatory: Observation of flavor change for solar neutrinos," *Rev. Mod. Phys.* **88** no. 3, (2016) 030502.

[31] S. Morisi, E. Peinado, Y. Shimizu, and J. W. F. Valle, "Relating quarks and leptons without grand-unification," *Phys. Rev. D* **84** (2011) 036003, arXiv:1104.1633 [hep-ph].

[32] F. Bazzocchi, S. Morisi, E. Peinado, J. W. F. Valle, and A. Vicente, "Bilinear R-parity violation with flavor symmetry," *JHEP* **01** (2013) 033, arXiv:1202.1529 [hep-ph].

[33] S. F. King, S. Morisi, E. Peinado, and J. W. F. Valle, "Quark-Lepton Mass Relation in a Realistic A_4 Extension of the Standard Model," *Phys. Lett. B* **724** (2013) 68–72, arXiv:1301.7065 [hep-ph].

[34] C. Bonilla, S. Morisi, E. Peinado, and J. W. F. Valle, "Relating quarks and leptons with the T_7 flavour group," *Phys. Lett. B* **742** (2015) 99–106, arXiv:1411.4883 [hep-ph].

[35] M. Reig, J. W. F. Valle, and F. Wilczek, "SO(3) family symmetry and axions," *Phys. Rev. D* **98** no. 9, (2018) 095008, arXiv:1805.08048 [hep-ph].

[36] F. J. de Anda, J. W. F. Valle, and C. A. Vaquera-Araujo, "Flavour and CP predictions from orbifold compactification," *Phys. Lett. B* **801** (2020) 135195, arXiv:1910.05605 [hep-ph].

[37] F. J. de Anda, I. Antoniadis, J. W. F. Valle, and C. A. Vaquera-Araujo, "Scotogenic dark matter in an orbifold theory of flavor," *JHEP* **10** (2020) 190, arXiv:2007.10402 [hep-ph].

[38] F. J. de Anda, N. Nath, J. W. F. Valle, and C. A. Vaquera-Araujo, "Probing the predictions of an orbifold theory of flavor," *Phys. Rev. D* **101** no. 11, (2020) 116012, arXiv:2004.06735 [hep-ph].

[39] F. J. de Anda, O. Medina, J. W. F. Valle, and C. A. Vaquera-Araujo, "Scotogenic Majorana neutrino masses in a predictive orbifold theory of flavor," *Phys. Rev. D* **105** no. 5, (2022) 055030, arXiv:2110.06810 [hep-ph].

[40] F. J. de Anda, O. Medina, J. W. F. Valle, and C. A. Vaquera-Araujo, "Revamping Kaluza-Klein dark matter in an orbifold theory of flavor," *Phys. Rev. D* **108** no. 3, (2023) 035046, arXiv:2212.09174 [hep-ph].

[41] S. Antusch and V. Maurer, "Running quark and lepton parameters at various scales," *JHEP* **11** (2013) 115, arXiv:1306.6879 [hep-ph].

[42] D. M. Straub, "flavio: a Python package for flavour and precision phenomenology in the Standard Model and beyond," arXiv:1810.08132 [hep-ph].

[43] K. S. Babu, E. Ma, and J. W. F. Valle, "Underlying A(4) symmetry for the neutrino mass matrix and the quark mixing matrix," *Phys. Lett. B* **552** (2003) 207–213, arXiv:hep-ph/0206292.

[44] C. I. Low and R. R. Volkas, "Tri-bimaximal mixing, discrete family symmetries, and a conjecture connecting the quark and lepton mixing matrices," *Phys. Rev. D* **68** (2003) 033007, arXiv:hep-ph/0305243.

[45] M.-C. Chen and S. F. King, "A4 See-Saw Models and Form Dominance," *JHEP* **06** (2009) 072, arXiv:0903.0125 [hep-ph].

[46] J. Barry and W. Rodejohann, "Deviations from tribimaximal mixing due to the vacuum expectation value misalignment in A_4 models," *Phys. Rev. D* **81** (2010) 093002, arXiv:1003.2385 [hep-ph]. [Erratum: Phys.Rev.D 81, 119901 (2010)].

[47] G. Altarelli and F. Feruglio, "Discrete Flavor Symmetries and Models of Neutrino Mixing," *Rev. Mod. Phys.* **82** (2010) 2701–2729, arXiv:1002.0211 [hep-ph].

[48] G. Altarelli and F. Feruglio, "Tri-bimaximal neutrino mixing, A(4) and the modular symmetry," *Nucl. Phys. B* **741** (2006) 215–235, arXiv:hep-ph/0512103.

[49] G. Altarelli and F. Feruglio, "Tri-bimaximal neutrino mixing from discrete symmetry in extra dimensions," *Nucl. Phys. B* **720** (2005) 64–88, arXiv:hep-ph/0504165.

[50] R. de Adelhart Toorop, F. Feruglio, and C. Hagedorn, "Finite Modular Groups and Lepton Mixing," *Nucl. Phys. B* **858** (2012) 437–467, arXiv:1112.1340 [hep-ph].

[51] F. Feruglio, C. Hagedorn, Y. Lin, and L. Merlo, "Tri-bimaximal Neutrino Mixing and Quark Masses from a Discrete Flavour Symmetry," *Nucl. Phys. B* **775** (2007) 120–142, arXiv:hep-ph/0702194. [Erratum: Nucl.Phys.B 836, 127–128 (2010)].

[52] G. Chauhan, P. S. B. Dev, I. Dubovyk, B. Dziewit, W. Flieger, K. Grzanka, J. Gluza, B. Karmakar, and S. Zięba,

"Phenomenology of Lepton Masses and Mixing with Discrete Flavor Symmetries," arXiv:2310.20681 [hep-ph].

[53] M.-C. Chen, M. Fallbacher, Y. Omura, M. Ratz, and C. Staudt, "Predictivity of models with spontaneously broken non-Abelian discrete flavor symmetries," *Nucl. Phys. B* **873** (2013) 343–371, arXiv:1302.5576 [hep-ph].

[54] F. Feruglio, C. Hagedorn, and L. Merlo, "Vacuum Alignment in SUSY A4 Models," *JHEP* **03** (2010) 084, arXiv:0910.4058 [hep-ph].

[55] F. Feruglio, *Are neutrino masses modular forms?*, pp. 227–266. 2019. arXiv:1706.08749 [hep-ph].

[56] G.-J. Ding and S. F. King, "Neutrino Mass and Mixing with Modular Symmetry," arXiv:2311.09282 [hep-ph].

[57] S. Ferrara, . D. Lust, and S. Theisen, "Target Space Modular Invariance and Low-Energy Couplings in Orbifold Compactifications," *Phys. Lett. B* **233** (1989) 147–152.

[58] S. Ferrara, D. Lust, A. D. Shapere, and S. Theisen, "Modular Invariance in Supersymmetric Field Theories," *Phys. Lett. B* **225** (1989) 363.

[59] J. Lauer, J. Mas, and H. P. Nilles, "Twisted sector representations of discrete background symmetries for two-dimensional orbifolds," *Nucl. Phys. B* **351** (1991) 353–424.

[60] D. Cremades, L. E. Ibanez, and F. Marchesano, "Computing Yukawa couplings from magnetized extra dimensions," *JHEP* **05** (2004) 079, arXiv:hep-th/0404229.

[61] T. Kobayashi, S. Nagamoto, and S. Uemura, "Modular symmetry in magnetized/intersecting D-brane models," *PTEP* **2017** no. 2, (2017) 023B02, arXiv:1608.06129 [hep-th].

[62] T. Kobayashi, S. Nagamoto, S. Takada, S. Tamba, and T. H. Tatsuishi, "Modular symmetry and non-Abelian discrete flavor symmetries in string compactification," *Phys. Rev. D* **97** no. 11, (2018) 116002, arXiv:1804.06644 [hep-th].

[63] A. Baur, H. P. Nilles, A. Trautner, and P. K. S. Vaudrevange, "Unification of Flavor, CP, and Modular Symmetries," *Phys. Lett. B* **795** (2019) 7–14, arXiv:1901.03251 [hep-th].

[64] A. Baur, M. Kade, H. P. Nilles, S. Ramos-Sanchez, and P. K. S. Vaudrevange, "Siegel modular flavor group and CP from string theory," *Phys. Lett. B* **816** (2021) 136176, arXiv:2012.09586 [hep-th].

[65] A. Baur, H. P. Nilles, S. Ramos-Sanchez, A. Trautner, and P. K. S. Vaudrevange, "Top-down anatomy of flavor symmetry breakdown," *Phys. Rev. D* **105** no. 5, (2022) 055018, arXiv:2112.06940 [hep-th].

[66] H. P. Nilles and S. Ramos-Sanchez, "The Flavor Puzzle: Textures and Symmetries," 8, 2023. arXiv:2308.14810 [hep-ph].

[67] Y. Olguin-Trejo, R. Pérez-Martínez, and S. Ramos-Sánchez, "Charting the flavor landscape of MSSM-like Abelian heterotic orbifolds," *Phys. Rev. D* **98** no. 10, (2018) 106020, arXiv:1808.06622 [hep-th].

[68] K. Ishiguro, T. Kobayashi, and H. Otsuka, "Landscape of Modular Symmetric Flavor Models," *JHEP* **03** (2021) 161, arXiv:2011.09154 [hep-ph].

[69] G.-J. Ding, F. Feruglio, and X.-G. Liu, "Automorphic Forms and Fermion Masses," *JHEP* **01** (2021) 037, arXiv:2010.07952 [hep-th].

[70] X.-G. Liu and G.-J. Ding, "Modular flavor symmetry and vector-valued modular forms," *JHEP* **03** (2022) 123, arXiv:2112.14761 [hep-ph].

[71] X.-G. Liu and G.-J. Ding, "Neutrino Masses and Mixing from Double Covering of Finite Modular Groups," *JHEP* **08** (2019) 134, arXiv:1907.01488 [hep-ph].

[72] M.-C. Chen, S. Ramos-Sánchez, and M. Ratz, "A note on the predictions of models with modular flavor symmetries," *Phys. Lett. B* **801** (2020) 135153, arXiv:1909.06910 [hep-ph].

[73] M.-C. Chen, V. Knapp-Perez, M. Ramos-Hamud, S. Ramos-Sanchez, M. Ratz, and S. Shukla, "Quasi–eclectic modular flavor symmetries," *Phys. Lett. B* **824** (2022) 136843, arXiv:2108.02240 [hep-ph].

[74] Y. Reyimuaji and A. Romanino, "Can an unbroken flavour symmetry provide an approximate description of lepton masses and mixing?," *JHEP* **03** (2018) 067, arXiv:1801.10530 [hep-ph].

[75] P. P. Novichkov, J. T. Penedo, S. T. Petcov, and A. V. Titov, "Modular S_4 models of lepton masses and mixing," *JHEP* **04** (2019) 005, arXiv:1811.04933 [hep-ph].

[76] G.-J. Ding, S. F. King, X.-G. Liu, and J.-N. Lu, "Modular S_4 and A_4 symmetries and their fixed points: new predictive examples of lepton mixing," *JHEP* **12** (2019) 030, arXiv:1910.03460 [hep-ph].

[77] P. P. Novichkov, J. T. Penedo, and S. T. Petcov, "Fermion mass hierarchies, large lepton mixing and residual modular symmetries," *JHEP* **04** (2021) 206, arXiv:2102.07488 [hep-ph].

[78] F. Feruglio, V. Gherardi, A. Romanino, and A. Titov, "Modular invariant dynamics and fermion mass hierarchies around $\tau = i$," *JHEP* **05** (2021) 242, arXiv:2101.08718 [hep-ph].

[79] F. Feruglio, "Universal Predictions of Modular Invariant Flavor Models near the Self-Dual Point," *Phys. Rev. Lett.* **130** no. 10, (2023) 101801, arXiv:2211.00659 [hep-ph].

[80] F. Feruglio, "Fermion masses, critical behavior and universality," *JHEP* **03** (2023) 236, arXiv:2302.11580 [hep-ph].

[81] M. P. Bento, J. P. Silva, and A. Trautner, "The Basis Invariant Flavor Puzzle," arXiv:2308.00019 [hep-ph].

[82] A. Baur, H. P. Nilles, A. Trautner, and P. K. S. Vaudrevange, "A String Theory of Flavor and $\mathscr{CP}$," *Nucl. Phys. B* **947** (2019) 114737, arXiv:1908.00805 [hep-th].

[83] H. P. Nilles, S. Ramos–Sánchez, and P. K. S. Vaudrevange, "Eclectic flavor scheme from ten-dimensional string theory – I. Basic results," *Phys. Lett. B* **808** (2020) 135615, arXiv:2006.03059 [hep-th].

[84] A. Baur, M. Kade, H. P. Nilles, S. Ramos-Sanchez, and P. K. S. Vaudrevange, "The eclectic flavor symmetry of the $\mathbb{Z}_2$ orbifold," *JHEP* **02** (2021) 018, arXiv:2008.07534 [hep-th].

[85] Y. Almumin, M.-C. Chen, V. Knapp-Pérez, S. Ramos-Sánchez, M. Ratz, and S. Shukla, "Metaplectic Flavor Symmetries from Magnetized Tori," *JHEP* **05** (2021) 078, arXiv:2102.11286 [hep-th].

[86] A. Baur, H. P. Nilles, S. Ramos-Sanchez, A. Trautner, and P. K. S. Vaudrevange, "The first string-derived eclectic flavor model with realistic phenomenology," *JHEP* **09** (2022) 224, arXiv:2207.10677 [hep-ph].

[87] T. Kobayashi, H. P. Nilles, F. Ploger, S. Raby, and M. Ratz, "Stringy origin of non-Abelian discrete flavor symmetries," *Nucl. Phys. B* **768** (2007) 135–156, arXiv:hep-ph/0611020.

[88] H. Abe, K.-S. Choi, T. Kobayashi, and H. Ohki, "Non-Abelian Discrete Flavor Symmetries from Magnetized/Intersecting Brane Models," *Nucl. Phys. B* **820** (2009) 317–333, arXiv:0904.2631 [hep-ph].

[89] T. Kai, K. Ishiguro, H. Okada, and H. Otsuka, "Flavor, CP and Metaplectic Modular Symmetries in Type IIB Chiral Flux Vacua," arXiv:2305.19155 [hep-th].

[90] J. T. Penedo and S. T. Petcov, "Lepton Masses and Mixing from Modular S_4 Symmetry," *Nucl. Phys. B* **939** (2019) 292–307, arXiv:1806.11040 [hep-ph].

[91] X.-G. Liu, C.-Y. Yao, and G.-J. Ding, "Modular invariant quark and lepton models in double covering of S_4 modular group," *Phys. Rev. D* **103** no. 5, (2021) 056013, arXiv:2006.10722 [hep-ph].

[92] B.-Y. Qu, X.-G. Liu, P.-T. Chen, and G.-J. Ding, "Flavor mixing and CP violation from the interplay of an S4 modular group and a generalized CP symmetry," *Phys. Rev. D* **104** no. 7, (2021) 076001, arXiv:2106.11659 [hep-ph].

[93] Y. Abe, T. Higaki, J. Kawamura, and T. Kobayashi, "Quark and lepton hierarchies from S4' modular flavor symmetry," *Phys. Lett. B* **842** (2023) 137977, arXiv:2302.11183 [hep-ph].

[94] I. de Medeiros Varzielas, M. Levy, J. T. Penedo, and S. T. Petcov, "Quarks at the modular S_4 cusp," *JHEP* **09** (2023) 196, arXiv:2307.14410 [hep-ph].

[95] Y. Abe, T. Higaki, J. Kawamura, and T. Kobayashi, "Quark masses and CKM hierarchies from S_4' modular flavor symmetry," arXiv:2301.07439 [hep-ph].

[96] P. P. Novichkov, J. T. Penedo, and S. T. Petcov, "Double cover of modular S_4 for flavour model building," *Nucl. Phys. B* **963** (2021) 115301, arXiv:2006.03058 [hep-ph].

[97] J. C. Criado, F. Feruglio, and S. J. D. King, "Modular Invariant Models of Lepton Masses at Levels 4 and 5," *JHEP* **02** (2020) 001, arXiv:1908.11867 [hep-ph].

[98] S. P. Martin, "A Supersymmetry primer," *Adv. Ser. Direct. High Energy Phys.* **18** (1998) 1–98, arXiv:hep-ph/9709356.

[99] P. P. Novichkov, J. T. Penedo, S. T. Petcov, and A. V. Titov, "Generalised CP Symmetry in Modular-Invariant Models of Flavour," *JHEP* **07** (2019) 165, arXiv:1905.11970 [hep-ph].

[100] J. H. Bruinier, G. Van der Geer, G. Harder, and D. Zagier, *The 1-2-3 of modular forms: lectures at a summer school in Nordfjordeid, Norway*. Springer Science & Business Media, 2008.

[101] S. T. Petcov, "On the Normalisation of the Modular Forms in Modular Invariant Theories of Flavour," arXiv:2311.04185 [hep-ph].

[102] L. J. Hall, J. D. Lykken, and S. Weinberg, "Supergravity as the Messenger of Supersymmetry Breaking," *Phys. Rev. D* **27** (1983) 2359–2378.

[103] D. Ghosh, M. Guchait, S. Raychaudhuri, and D. Sengupta, "How Constrained is the cMSSM?," *Phys. Rev. D* **86** (2012) 055007, arXiv:1205.2283 [hep-ph].

[104] S. Gupta and S. K. Gupta, "An information theoretic exploration of constrained MSSM," *Nucl. Phys. B* **965** (2021) 115336, arXiv:2008.00415 [hep-ph].

[105] S. Antusch, J. Kersten, M. Lindner, M. Ratz, and M. A. Schmidt, "Running neutrino mass parameters in see-saw scenarios," *JHEP* **03** (2005) 024, arXiv:hep-ph/0501272.

[106] R. Hempfling, "Yukawa coupling unification with supersymmetric threshold corrections," *Phys. Rev. D* **49** (1994) 6168–6172.

[107] T. Blazek, S. Raby, and S. Pokorski, "Finite supersymmetric threshold corrections to CKM matrix elements in the large tan Beta regime," *Phys. Rev. D* **52** (1995) 4151–4158, arXiv:hep-ph/9504364.

[108] G.-J. Ding, X.-G. Liu, and C.-Y. Yao, "A minimal modular invariant neutrino model," *JHEP* **01** (2023) 125, arXiv:2211.04546 [hep-ph].

[109] S. Centelles Chuliá, R. Kumar, O. Popov, and R. Srivastava, "Neutrino Mass Sum Rules from Modular $\mathcal{A}_4$ Symmetry," arXiv:2308.08981 [hep-ph].

[110] **Particle Data Group** Collaboration, R. L. Workman *et al.*, "Review of Particle Physics," *PTEP* **2022** (2022) 083C01.

[111] P. F. de Salas, D. V. Forero, S. Gariazzo, P. Martínez-Miravé, O. Mena, C. A. Ternes, M. Tórtola, and J. W. F. Valle, "2020 global reassessment of the neutrino oscillation picture," *JHEP* **02** (2021) 071, arXiv:2006.11237 [hep-ph].