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A NOTE ON UNITARY OPERATORS IN C^*-ALGEBRAS

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1. Introduction. We show that, in any C^*-algebra $\mathfrak{A}$, convex linear combinations of unitary operators are uniformly dense in the unit sphere of $\mathfrak{A}$. In other terms, the unit sphere in $\mathfrak{A}$ is the closed convex hull of its normal extreme points, even though non-normal extreme points will in general be present. This fact has several useful technical implications. For example, it follows that the norm of a linear mapping ϕ between C^*-algebras can be computed using only normal operators, that is, from the effect of ϕ on abelian *-subalgebras. In addition, we show that a linear mapping between C^*-algebras which conserves the identity and sends unitary operators into unitary operators is a C^*-homomorphism.

2. The main result. Let $\mathfrak{A}$ be a C^*-algebra, that is, a uniformly closed self-adjoint algebra of operators on some complex Hilbert space H. Throughout, we assume that $\mathfrak{A}$ contains the identity operator I. $U(\mathfrak{A})$ will denote the set of unitary operators in $\mathfrak{A}$, and $co(U(\mathfrak{A}))$ the convex hull of $U(\mathfrak{A})$.

LEMMA 1. *In any von Neumann algebra M, co $(U(M))$ is weakly dense in the unit sphere of M.*

Proof. This follows readily from the known fact that, in a von Neumann algebra M with no finite summands, the weak closure of $U(M)$ is the unit sphere ([3, Theorem 1 et seq.]). For completeness, however, we include a proof of the lemma.

Let C denote the weak closure of $co(U(M))$. To show that C is the unit sphere, by Krein–Mil'man, it suffices to show that C contains all extreme points of the unit sphere. Using [5, Theorem 1], it follows readily that these are the partial isometries V in M such that, for some central projection D, $V^*V \geq D$ and $VV^* \geq I - D$. Therefore, replacing M by appropriate direct summands and noting that $C^* = C$, it suffices to consider the case $V^*V = I$. In addition, we can assume that $VV^* = P \neq I$. Given vectors x_i, y_i $(i = 1, \cdots, n)$ and $\epsilon > 0$, we will exhibit a unitary U in M such that $|((U - V)x_i, y_i)| < \epsilon$, for all i.

Let $\mathfrak{M}$ be the range of $I - P$. Then the $V^n\mathfrak{M}$ are mutually orthogonal $(n \geq 0)$ and the restriction of V to the orthogonal complement $\mathfrak{N}$ of $\bigoplus_{n=0}^{\infty} V^n\mathfrak{M}$ is unitary. Let Q_n be the projection on $V^n\mathfrak{M}$, and choose n such that $\|\sum_{k>n} Q_k x_i\| < \epsilon/2(1 + \max \|y_i\|)$, for all i. Let $U = V$ on the subspace $\mathfrak{N} \oplus \mathfrak{M} \oplus \cdots \oplus V^n\mathfrak{M}$, $= V^{*(n+1)}$ on $V^{n+1}\mathfrak{M}$, and $= I$ on $\bigoplus_{k>n+1} V^k\mathfrak{M}$. Then

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U is unitary, $U \varepsilon M$, and $|((U - V)x_i, y_i)| = |(\sum_{k>n} (U - V)Q_k x_i, y_i)| \leq 2 \, ||\sum_{k>n} Q_k x_i|| \, ||y_i|| < \epsilon$. The lemma follows.

THEOREM 1. *In any C^*-algebra $\mathfrak{A}$, $co(U(\mathfrak{A}))$ is uniformly dense in the unit sphere of $\mathfrak{A}$.*

Proof. Any C^*-algebra is $*$-isomorphic to a C^*-algebra all of whose states are weakly continuous. In fact, let ϕ be the universal representation of $\mathfrak{A}$. By definition, $\phi = \bigoplus_\rho \phi_\rho$, where ρ ranges over the entire state space of $\mathfrak{A}$ and ϕ_ρ is the $*$-representation of $\mathfrak{A}$ determined by ρ. Then each state of $\phi(\mathfrak{A})$ is canonical (see, for example, [11]). So we may assume in proof that all states of $\mathfrak{A}$ are weakly continuous.

Suppose $T \varepsilon \mathfrak{A}$, $||T|| \leq 1$, and that T does not lie in the uniform closure of $co(U(\mathfrak{A}))$. By a standard separation theorem, there will exist a continuous linear functional σ on $\mathfrak{A}$, a real c, and an $\epsilon > 0$ such that

$$\text{Re}\, \sigma(A) \leq c < c + \epsilon \leq \text{Re}\, \sigma(T), \tag{2.1}$$

for all A in $co(U(\mathfrak{A}))$. The functional σ will be a finite linear combination of states of $\mathfrak{A}$, each assumed weakly continuous. Let M be the strong closure of $\mathfrak{A}$. By the Glimm–Kadison variant of the Kaplansky density theorem [4, Theorem 2]. $U(\mathfrak{A})$ is strongly dense in $U(M)$. Therefore, (2.1) holds for all A in $co(U(M))$. Again by weak continuity, (2.1) holds for all A in the weak closure of $co(U(M))$. In view of Lemma 1, this is a contradiction. The theorem is proved.

For abelian C^*-algebras (namely, for $C(X)$, X compact Hausdorff), Theorem 1 has been proved by Phelps [8]. Little is known about the pre-closed convex hull $co(U(\mathfrak{A}))$. For a von Neumann algebra M, $co(U(M))$ coincides with the unit sphere if and only if M is finite; this follows readily from results in [5]. For a general C^*-algebra, $co(U(\mathfrak{A}))$ contains the open sphere about 0 of radius $\frac{1}{2}$. [To see this, let the C^*-algebra $\mathfrak{A}$ act on H, and let A be a regular operator in $\mathfrak{A}$ of norm 1. If $A = U|A|$ is the polar decomposition of A in $L(H)$, then U is unitary, $|A| \varepsilon \mathfrak{A}$, and so $U = A(|A|)^{-1} \varepsilon \mathfrak{A}$. It is standard that $|A|$ is a convex linear combination of two unitary operators in $\mathfrak{A}$, and therefore the same applies to A. Now, if T is any operator in $\mathfrak{A}$ of norm $< \frac{1}{2}$, then $\pm T + \frac{1}{2}I$ is regular, since $||I - (\pm T + \frac{1}{2}I)|| < 1$, and one has

$$T = \tfrac{1}{2}[(T + \tfrac{1}{2}I) + (T - \tfrac{1}{2}I)].$$

It follows that T is a convex linear combination of four unitaries.]

3. Applications. Each C^*-algebra $\mathfrak{A}$ is the linear span of its unitary group $U(\mathfrak{A})$. For each $A \varepsilon \mathfrak{A}$, we define

$$||A||_U = \inf \textstyle\sum_i |\lambda_i|, \tag{3.1}$$

taken over all representations $A = \sum_{i=1}^n \lambda_i U_i$ of A as a finite linear combination of unitaries. A simple calculation shows that $||A||_U$ is a normed algebra norm on $\mathfrak{A}$ such that $||A|| \leq ||A||_U$.

LEMMA 2. *For all* A, $||A|| = ||A||_U$.

Proof. Availing ourselves of the comments following Theorem 1, we see that if $||A|| = 1$, then for each $\epsilon > 0$, $||(1/(2 + \epsilon))A||_U \leq 1$. It follows that $\frac{1}{2} ||A||_U \leq ||A|| \leq ||A||_U$, for all A, so that the two norms are equivalent. By the theorem, each A with $||A|| = 1$ is the $||$-limit of a sequence with $||A_n||_U = 1$. Since the A_n must also converge to A in $||_U$-norm, it follows that $||A||_U = 1$. In general, therefore, $||A|| = ||A||_U$.

Recall that a mapping ϕ between C^*-algebras is termed positive if $\phi(A) \geq 0$ whenever $A \geq 0$.

COROLLARY 1. *The norm of a linear mapping ϕ of a C^*-algebra $\mathfrak{A}$ in a normed linear space $\mathfrak{B}$ is $\sup_{U \varepsilon U(\mathfrak{A})} ||\phi(U)||$. Moreover, if $\mathfrak{B}$ is a C^*-algebra and $\phi(I) = I$, then ϕ is positive if and only if $||\phi|| = 1$.*

Proof. Let $K = \sup_{U \varepsilon U(\mathfrak{A})} ||\phi(U)||$. If $A \, \varepsilon \, \mathfrak{A}$, $A = \sum_i \lambda_i U_i$, then $||\phi(A)|| \leq (\sum |\lambda_i|)K$, so $||\phi(A)|| \leq K \, ||A||_U = K \, ||A||$, by Lemma 2.

We turn to the second statement. That $||\phi|| = 1$ entails the positivity of ϕ when $\phi(I) = I$ is well known: if x is a unit vector in the representation space of $\mathfrak{B}$ and if $\sigma(A) = (\phi(A)x, x)$, then σ is a linear functional on $\mathfrak{A}$ of norm 1 with value 1 at I, and any such functional is a state [2; 25]; therefore, $A \geq 0$ forces $\sigma(A) \geq 0$, so that $\phi(A) \geq 0$. Furthermore, the converse is known when the algebra $\mathfrak{A}$ is abelian. In this case, by results of Stinespring [9], ϕ is completely positive, and therefore has the form $\phi(A) = V^*\rho(A)V$, where V is an isometry and ρ is a $*$-representation of $\mathfrak{A}$. Therefore, $||\phi(A)|| \leq ||\rho(A)|| \leq ||A||$. This fact and the first paragraph show that, for any C^*-algebra $\mathfrak{A}$, the positivity of ϕ entails $||\phi(A)|| \leq ||A||$, for all A. Since $\phi(I) = I$, one therefore has $||\phi|| = 1$.

Using this, one can put Bonsall's minimal norm theorem [1] in the following sharper form: in order that a normed algebra norm $||_0$ on a C^*-algebra coincide with the C^*-norm, it is necessary and sufficient that $||A||_0 \leq ||A||$ on all abelian $*$-subalgebras of $\mathfrak{A}$. Similarly, it follows that an identity-conserving linear order isomorphism between C^*-algebras is an isometry. This reduces [6, Corollary 5] to [5, Theorem 7], which asserts that such a mapping is a C^*-isomorphism.

By definition, a C^*-homomorphism of a C^*-algebra $\mathfrak{A}$ in a C^*-algebra $\mathfrak{B}$ is a $*$-linear mapping ϕ of $\mathfrak{A}$ into $\mathfrak{B}$ such that $\phi(A^2) = \phi(A)^2$, for all self-adjoint A in $\mathfrak{A}$. As is known (see Størmer [10]), such a mapping is the sum of a $*$-homomorphism and a $*$-anti-homomorphism; more precisely, there is a central projection E in the W^*-envelope of $\phi(\mathfrak{B})$ such that $A \to \phi(A)E$ is a $*$-homomorphism and $A \to \phi(A)(I - E)$ is a $*$-anti-homomorphism.

COROLLARY 2. *Let ψ be a linear mapping of the C^*-algebra $\mathfrak{A}$ in the C^*-algebra $\mathfrak{B}$ such that $\psi(U(\mathfrak{A})) \subseteq U(\mathfrak{B})$. Then, ψ has a factorization $\psi(A) = U\phi(A)$, where $U \, \varepsilon \, U(\mathfrak{B})$ and ϕ is a C^*-homomorphism.*

Proof. We set $U = \psi(I)$, $\phi(A) = U^{-1}\psi(A)$. Then ϕ conserves unitaries and $\phi(I) = I$. Application of Corollary 1 shows that $||\phi|| = 1$ and, in turn, that ϕ is positive (and hence, a *-mapping).

It remains to show that $\phi(A^2) = \phi(A)^2$, for all self-adjoint A. In any case, by Kadison's generalized Schwarz inequality, [6], one has $\phi(A^2) \geq \phi(A)^2$. Applying this to a self-adjoint operator of the form $U + U^*(U \in U(\mathfrak{A}))$, one obtains after expansion $\phi(U^2) + \phi(U^2)^* \geq \phi(U)^2 + \phi(U^*)^2$. The same inequality holds with iU replacing U. But this reverses the preceding inequality in U. Since, up to a scale factor, each self-adjoint A has the form $U + U^*$, we have proved that $\phi(A^2) = \phi(A)^2$, for all self-adjoint A, and the corollary is proved.

In particular, if M is a finite factor and ψ is a unitary-conserving mapping of M on itself, then the ϕ of the corollary is either a *-isomorphism or a *-anti-isomorphism. For factors of type I_n, this has been proved by Marcus [7].

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