Title
Efficiency and Fairness in the Allocation of Indivisible Goods

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Publication Date
2021

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA

Los Angeles

Efficiency and Fairness in the Allocation of Indivisible Goods

A dissertation submitted in partial satisfaction

of the requirements for the degree

Doctor of Philosophy in Economics

by

Akina Ikudo

2021

ABSTRACT OF THE DISSERTATION

Efficiency and Fairness in the Allocation of Indivisible Goods

by

Akina Ikudo
Doctor of Philosophy in Economics
University of California, Los Angeles, 2021
Professor Ichiro Obara, Chair

This dissertation studies the efficient and fair allocation of indivisible goods without monetary transfer. It is a collection of three papers and uses school-choice programs as a motivating example. I provide theoretical results that can guide the design of new allocation systems as well as tools that can be used to enhance existing systems.

In Chapter 1, I analyze how information disclosure affects social welfare using a stylized model. In my model, the utility of agents consists of a vertical "quality" component and a horizontal "idiosyncratic taste" component. The exact qualities of the objects are unknown to the agents, and the social planner seeks an information-disclosure policy that will maximize the total utility. The results show that (1) the optimal disclosure policy hides small differences in quality and reveals large differences in quality, (2) more information is disclosed when the valuations of the quality are heterogeneous, and (3) the Immediate Acceptance mechanism is more conducive for information disclosure than the Deferred Acceptance mechanism.

In Chapter 2, I study the collocation of groups of students in school-choice programs. In particular, I examine when and how stochastic assignment matrices can be decomposed into lotteries over deterministic assignments subject to collocation constraints. I first show that—regardless of the number of pairs of twins in the student body—twin collocation can be maintained in a decomposition if one extra seat can be added to each school. I then propose a decomposition algorithm based on Column Generation that can incorporate a wide variety of constraints including collocation constraints.

In Chapter 3, I propose a new notion of fairness that combines the concept of rank values and the maximin principle. An assignment is rank-egalitarian undominated (REU) if there is no other assignment that is equally or more egalitarian for any set of rank values. I show that each REU assignment can be generated as a solution to a linear programming problem that maximizes the weighted sum of expected rank values of the worst-off agents. I also provide an algorithm that generates special subsets of REU assignments that are practically important.

The dissertation of Akina Ikudo is approved.

Auyon Adnan Siddiq

Moritz Meyer-ter-Vehn

Simon Adrian Board

Ichiro Obara, Committee Chair

University of California, Los Angeles

2021

To my parents.

"Girls, we won't be able to buy you a car or pay for your weddings,

but we will have enough savings for your college education."

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ACKNOWLEDGMENTS

I am deeply grateful to my advisor, Ichiro Obara. Without his guidance and encouragement, this dissertation would not have been possible. Because of his kindness and insights, I was able to find my place in the field of Economics.

I thank Marek Pycia for introducing me to matching problems; Simon Board for helping me to learn to think like an economist; Moritz Meyer-ter-Vehn for his critical feedback that improved my work; Jay Lu for his life and career advice; Tomasz Sadzik for his helpful suggestions; Bruce Weinberg and Julia Lane for the research opportunities; Auyon Siddiq for his shared passion in Operations Research; and the members of the UCLA Economics Theory group for their feedback and support.

During my graduate studies, I had a great time with my office mates, Jesper Sørensen, Ksenia Shakhgildyan, Byeonghyeon Jeong, and Alex Graupner. I always looked forward to spending time with Moon Evans, Jin Peng, Nick Maskey, and Chris Buirley during holidays. I will miss eating at Tsujita with Luke and Akiko.

Special thanks go to Chip, Marge, Kevin, and Debra Gerber for feeding me all these years; to Matt Gerber for intriguing dinner conversations and hundreds—if not thousands—of articles he added and deleted in my writing; to my sisters for their company and rational minds which helped me grasp the essence of game theory very early in my life; and to my parents for having raised me to be a self-reliant individual and for being neutral observers as I continue to take an adventurous path.

Finally, I would like to thank Jawaad Noor for his invaluable advice: "Don't be a different person as a human and as an economist. Work on questions that are interesting to you." I followed his advice and I am glad I did.

VITA

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Ikudo, Akina, Julia Lane, Joseph Staudt, and Bruce Weinberg. "Occupational classifications: A machine learning approach." *Journal of Economic and Social Measurement* 44.2–3 (2019): 57–87.

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CHAPTER 1

Social Value of Information

1.1 Introduction

When is more information better? In individual decision-making problems, more information weakly improves the welfare of the decision maker because extraneous information can always be ignored. However, in social-choice problems, more information can decrease social welfare because the conflict of interests among individuals can be exacerbated with a greater availability of information.

In this paper, we analyze the optimal information-provision policy in indivisible-goods allocation problems. Intuitively, the allocative efficiency improves when agents have more accurate information about their own preferences. However, providing information can decrease utilitarian welfare when it undermines the diversity in preferences. We analyze this trade-off using a stylized model.

In our model, the utility of agents consists of a vertical quality component and a horizontal idiosyncratic-taste component. The quality of the objects to be allocated is a random vector and the agents know only its distribution. The social planner, who privately observes the realization of the quality, aims to maximize the total welfare by strategically disclosing the information to the agents. The key feature of the model is that all agents appreciate high-quality objects, but to various degrees.

Our model is suitable for a wide range of applications including school-choice programs. Students and parents generally prefer high-quality schools (e.g., high average test scores and high graduation rate) that also satisfy their idiosyncratic tastes (e.g., geographical proximity to their residence and the availability of particular extra-curricular activities). The social planner (usually the education board) aims to maximize the total benefit to the society by strategically providing information about the quality of participating schools. We use the school-choice terminologies for the rest of the paper.[1]

Disclosure of quality information has two opposing effects. On one hand, disclosure of quality provides the students with more accurate information about their own preferences, which facilitates assortative matching; the students with high valuations for quality are matched to high-quality schools. This increases the allocative efficiency in the vertical dimension. On the other hand, disclosure of quality causes crowding at high-quality schools, obscuring the heterogeneity in idiosyncratic tastes. As a result, it decreases the allocative efficiency in the horizontal dimension.

The trade-off between the vertical and the horizontal sorting is summarized in three main findings. First, hiding small differences in quality improves the welfare over the full disclosure. This is because the gain from improved vertical sorting is second order in quality difference (because a small fraction of students marginally improves their utility by pursuing higher-quality schools) whereas the loss from disturbed horizontal sorting is first order in quality difference (because many students

[1]Although the model is presented using the terminology in school-choice program, it is broadly applicable to any problems where the utility consists of a vertical quality component and a horizontal idiosyncratic-taste component. One example is the course-allocation problem in which all students prefer courses taught by highly rated instructors but there is some heterogeneity in the preferences for class time. Another example is matching between mentors and mentees, where all mentees prefer mentors with more experience but also care about sex, race, age, etc. of the mentor.

experience a small decrease in their utility due to externality). The concealment of information is consistent with the observation that education boards generally do not provide rankings of schools by academic performance.[2]

Second, we show that the social planner tends to disclose more information when the valuations of the quality are more heterogeneous. Intuitively, there is more to gain from the vertical sorting when students differ in their valuations of the quality. To see this, consider a society where all agents equally value the quality, i.e., they are identical in the vertical dimension. In such a society, no vertical sorting is possible, and therefore, disclosure only obfuscates the heterogeneity in idiosyncratic preferences, which leads to a loss to the society.

Third, we show that the Immediate Acceptance (IA) mechanism is more conducive for information disclosure than the Deferred Acceptance (DA) mechanism. The IA effectively punishes the students who top-rank very popular schools by forcing them to forgo their fair chance of being admitted to schools that are less popular but still highly desirable. Because of this penalty for crowding, the students pursue the highest-quality school only if they have strong enough preference given the level of congestion. In other words, the students reflect on their own preference relative to the preferences of the others, which mitigates externalities. Because the decisions made by the students are more socially conscious, the social planner is willing to share more information with the students.

[2]On its online platform, New York City Department of Education allows users to filter schools by subway lines, sizes of the school, sports teams, etc., but requires users to click on individual schools for the academic-performance information (`https://www.myschools.nyc/en/schools/high-school/`). Similarly, the school directory provided by Boston Public Schools requires users to click on individual schools, then click on School Report Card to see the information on academic performance (`https://www.bostonpublicschools.org/Page/628`).

This paper complements the growing literature on information acquisition in indivisible-goods allocation problems. Bade (2015) shows that Serial Dictatorship is the unique strategy-proof, non-bossy mechanism that provides the efficient level of incentive to acquire information. The incentive to acquire information under various allocation algorithms is further studied in Harless and Manjunath (2018) and Chen and He (2020). Using a three-school example, Artemov (2020) shows that strategy-proof mechanisms provide agents with too little incentive to acquire information about their own idiosyncratic preferences. These papers study acquisition of information on the private-value component of the utility, while this paper examines public provision of information on the common-value component of the utility. In particular, we demonstrate that full disclosure of information is not optimal even when information is free.

The general observation that more information is not necessarily good for society has been made in Hirshleifer (1978). The author demonstrates that uncertainty in the future state of the world is beneficial for society because it provides opportunity for mutual insurance. In the context of majority voting, Gersbach (1991) demonstrates that disclosing the state of the world, which affects the valuations of the project, may not be socially efficient. More recently, the role of public and private signals in coordination games is analyzed in Morris and Shin (2002). The authors show that the increased accuracy of public signals can decrease social welfare because agents may end up coordinating on noise. In this paper, we also observe the fundamental incompatibility between agents' self-interests and total utilitarianism.

Finally, this paper expands the literature on strategic information disclosure. One example with an application in education is Ostrovsky and Schwarz (2010), in which the authors show that universities can improve the average job placement of their students by providing less information on their transcripts. In this paper, we cast our model into the framework of Bayesian Persuasion (Kamenica and Gentzkow, 2011), where the sender's utility depends only on the mean of the posterior belief of the receiver. The optimality of the proposed disclosure policy is confirmed using the price-theoretic solution method developed in Dworczak and Martini (2019). As far as the author knows, this is the first paper to analyze the optimal information provision in the context of indivisible-goods allocation problems.

The paper is organized as follows. A formal model is introduced in Section 1.2, and non-optimality of full disclosure is proved in Section 1.3. For tractability, a simple two-school model is introduced in Section 1.4, and the optimal disclosure policies are analyzed in Section 1.5. The comparison of the optimal disclosure policies under the Deferred Acceptance and Immediate Acceptance mechanisms is provided in Section 1.6, followed by a conclusion in Section 1.7.

1.2 Model

We present our model using the terminologies in school-choice programs. However, our model is applicable to any allocation problem in which the utility of agents consists of a common quality component and an idiosyncratic taste component.

1.2.1 Fundamentals

There is a unit continuum of students, each demanding one seat, and a finite set S of schools, where $|S| \geq 2$. The capacity vector $q \equiv (q_s)_{s\in S}$ satisfies $\sum_{s\in S} q_s = 1$ so that there are just enough seats for all students. The utility of student i for attending school s is $u_{is} = \theta_i z_s + \varepsilon_{is}$, where $\theta_i \geq 0$ is the valuation of quality, z_s is the realized quality, and ε_{is} is the idiosyncratic taste. Let $\varepsilon_i \equiv (\varepsilon_{is})_{s \in S}$. The idiosyncratic tastes may reflect students' geographical proximity to the school, the variety of after-school programs offered by the school, etc.

The complementarity between θ and z reflects the observation that some students are more quality-sensitive than others. For example, students who have strong desire to attend college (large θ_i) are more likely to have preference for, and benefit from, attending schools that offer a greater number of AP (Advanced Placement) classes and have a higher percentage of students graduating with college scholarships.

The valuation for quality θ_i also measures how much weight the student puts on the school quality versus non-quality characteristics (idiosyncratic tastes). For example, if a student has strong preference for schools that are close to their home, their θ_i would be small in magnitude relative to ε_i, meaning that revealed quality is unlikely to change their preference order.

The student's type, $\omega_i = (\theta_i, \varepsilon_i)$, is a private value with a common prior $F(\omega)$ with support Ω. We sometimes refer to a specific student as "student i" and other times as "a type ω_i student". We denote the marginal distribution of θ_i by F_θ and the marginal distribution of ε_i by F_ε. We assume F_ε is atomless (i.e., F_ε is continuous) so that the set of students with indifference has measure zero.

The school quality, $z \equiv (z_s)_{s \in S}$, is a random vector with prior distribution $G(\tilde{z})$. We write $\tilde{z}$ to refer to the random variable, z for a realization, and $\hat{z}$ for the mean of a posterior belief about $\tilde{z}$. The school quality is independently distributed from student types, and therefore, every student has the identical prior for $\tilde{z}$. All of the above is common knowledge.

In our model, we assume the capacity and the quality of schools are fixed. In other words, schools are passive players. The fixed-capacity assumption is reasonable in the short term: there are legal limits on the number of students per classroom and the expansion of school buildings requires some time. The fixed-quality assumption is mainly imposed to keep the model tractable, but it is a reasonable assumption if the school quality depends on factors that cannot be changed quickly, such as the culture conducive for learning and the managerial ability of the school principal.

1.2.2 Allocation Mechanism

A probabilistic allocation is a matrix $(x_{is})_{\omega_i \in \Omega, s \in S}$, where x_{is} is the probability that a type ω_i student are assigned to school s. An allocation is feasible if $\sum_{s \in S} x_{is} = 1$ for each type $\omega_i \in \Omega$ and $\int_{\omega_i \in \Omega} x_{is}\, dF(\omega_i) = q_s$ for each school $s \in S$. A probabilistic allocation mechanism is a mapping from the set of preference-order profiles to the set of feasible allocations.

For simplicity, we assume all students have an equal priority and ties are broken by a single lottery. We also limit our attention to the mechanisms that always map to an allocation that is ordinally efficient (i.e., it is impossible to weakly first-order stochastically improve $x_i \equiv (x_{is})_{s \in S}$ for all $\omega_i \in \Omega$ with at least one strict improvement) and satisfy equal treatment of equals (i.e., if two students submit the same preference order, they receive the same probabilistic allocation).

We primarily focus on the Deferred Acceptance (DA) mechanism. For all possible tie-breaking outcomes, we run the following DA mechanism. In the first round, each student applies to the top school in their preference order. Each school temporarily keeps as many applicants with the highest priorities as its capacity allows and rejects the rest. In the second round, the rejected students apply to the next school in their preference order. Each school considers new applicants together with the old applicants they kept from the previous round. They temporarily keep as many applicants with the highest priorities as its capacity allows and rejects the rest. The process continues until no student is rejected or there is no more school to apply in their preference order.

In our setting, the expected DA allocation over all possible tie-breaking outcomes coincides with the Probabilistic Serial (PS) allocation described in Bogomolnaia and Moulin (2002). This is because the DA is equivalent to the Random Serial Dictatorship (RSD) when there is a single priority class, and the RSD is asymptotically equivalent to PS (Che and Kojima, 2010).

The PS works as follows. At the beginning, q_s probability share is available for each $s \in S$. Each student *eats* probability share of their most-preferred school at the same speed until one of the schools becomes exhausted. The exhausted school becomes unavailable. The process continues with the remaining schools: Each student eats probability shares of their most-preferred school from the set of remaining schools until one of the schools becomes exhausted. The algorithm terminates when all schools are exhausted.

This mechanism is strategy-proof, and therefore, it is students' best interest to truthfully report their preference orders. That is, student i ranks schools in the descending order of $\mathbb{E}[u_{is}] = \theta_i \hat{z} + \varepsilon_{is}$, where $\hat{z}$ is the mean of the posterior belief. Note that the expected utility depends only on the posterior mean rather than the entire posterior distribution. This is because, fixing the strategies of the other students, the utility is a linear function of z.

1.2.3 The Social Planner's Problem

The social planner's problem is to design a disclosure policy on $\tilde{z}$ that maximizes the sum of the expected utilities. We consider public disclosures, in which all students receive the same signal from the social planner. That the social planner can observe the realized value of $\tilde{z}$ while the students know only the prior distribution $G(\tilde{z})$ represents the the reality that the education board usually has more precise information about the school quality than the students.[3]

The timing of the game is as follows. The social planner chooses a disclosure policy on $\tilde{z}$ and announces it to the students. The school quality is drawn from the prior $G(\tilde{z})$. The social planner privately observes the realization z and maps it to a signal according to the announced disclosure policy. Upon receiving the signal, the students Bayesian-update their posterior belief $\hat{z}$. The students (strategically) report their preference orders based on $\hat{z}$. The students are probabilistically assigned to schools and the expected utilities realize.

Let $x(\hat{z})$ denote the equilibrium allocation when the posterior mean is $\hat{z}$. The allocation depends only on the belief of the students and does not depend on the actual realization z. Let $x_i(\hat{z})$ denote the allocation for a type ω_i student. The social welfare when the realization of the quality vector is z and the posterior mean is $\hat{z}$ is $w(z, \hat{z}) = \int_{\omega_i \in \Omega} x_i(\hat{z}) \cdot u_i(z)\, dF(\omega_i)$, where $u_i(z) = \theta_i z + \varepsilon_i$ is the utility vector for a type ω_i student.

[3]For example, the education board has insider knowledge about the ability of the management team and upcoming reforms that affect school qualities. Also, the education board has access to a proxy for school quality that is not easily accessible by the students such as the number of computers in classrooms and the number of books in the library. In addition, the education board can collect new data on quality at its discretion, for example, by conducting standardized tests.

Bayes consistency requires that $\mathbb{E}[\tilde{z} \mid \hat{z}] = \hat{z}$, i.e., when the students believe $\hat{z}$, the realization of $\tilde{z}$ is $\hat{z}$ on average. Because $w(z, \hat{z})$ is linear in z, we have $\mathbb{E}[w(\tilde{z}, \hat{z}) \mid \hat{z}] = w(\mathbb{E}[\tilde{z} \mid \hat{z}], \hat{z}) = w(\hat{z}, \hat{z})$. That is, the expected welfare is solely determined by the distribution of the posterior means. We define welfare function

$$W(z) \equiv w(z, z) = \int_{\omega_i \in \Omega} x_i(z) \cdot (\theta_i z + \varepsilon_i)\, dF(\omega_i).$$

This can also be interpreted as the expected welfare under the full disclosure.

Given the welfare function $W(z)$ and the prior distribution $G(\tilde{z})$, the social planner's problem is to find a garbling scheme that maximizes the expected social welfare:

$$\begin{aligned} \max_{H} \quad & \int W(z)\, dH(z) \\ \text{s.t.} \quad & \int v(z)\, dH(z) \leq \int v(z)\, dG(z) \quad \text{for all convex } v(z) \end{aligned}$$

This can be viewed as an instance of Bayesian persuasion where the payoff of the sender (social planner) depends only on the posterior mean of the receiver (the continuum of students). In solving the social planner's problem, we apply the techniques developed in Dworczak and Martini (2019).

1.2.4 Illustrative Example

Suppose there are two schools, existing (E) and new (N), with capacities $q_E = q_N = \frac{1}{2}$. We normalize the utility of attending the existing school to 0 for each student. In other words, $\varepsilon_{iE} = 0$ for all i and $z_E = 0$ with certainty. The utility of attending the new school is $u_{iN} = \theta_i z_N + \varepsilon_{iN}$ for student i. With slight abuse of notation, we drop the school subscript and simply write $u_i = \theta_i z + \varepsilon_i$ to denote the utility for the new school. Suppose one half of students have $\theta_i = 0$ and the other half have $\theta_i = 1$. Also suppose $\varepsilon_i \sim N(0, 1)$, independent across students, and independent of θ_i. The quality $\tilde{z}$ of the new school follows the prior $\tilde{z} \sim U[-6, 6]$.

Social welfare is plotted in Figure 1.1. The solid line, $W^F(z)$, represents the welfare when the value of z is fully disclosed. It has a slanted W-shape centered at $z = 0$. The dashed line, $W^0(z)$, represents the welfare when the mean of the posterior belief $\hat{z}$ is degenerate at 0 (i.e., no disclosure).

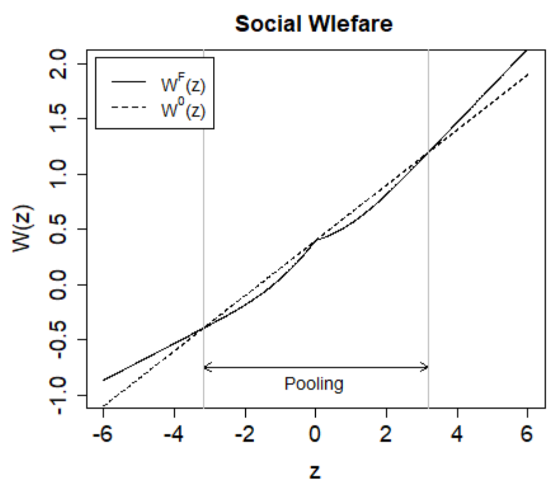


Figure 1.1: Social welfare function for $\theta \in \{0, 1\}$ and $\varepsilon \sim N(0, 1)$.

If the social planner garbles in a way such that students believe $\hat{z} = 0$, then $W^F(z)$ can be effectively replaced by $W^0(z)$. This improves the expected social welfare if $W^0(z) > W^F(z)$, which is the case for $z \in [-3.18, 3.18]$. Therefore, the social planner maps all the values of $z \in [-3.18, 3.18]$ to the same signal.[4]

[4]Optimal mapping is not unique. For example, mapping a realization z to a signal $|z|$ for each $z \in [-3.18, 3.18]$ produces the same distribution of posterior means. For simplicity, we focus on the mapping that uses the smallest number of signals.

In general, the social planner can improve the expected social welfare over full disclosure by pooling the values of z around an ideal point z^0, where no school is over-demanded and each student is assigned to their most-preferred school with certainty ($z^0 = 0$ in the example above). The intuition is as follows. Consider $z^1 > z^0$ such that $z^1 \approx z^0$. If the posterior belief is held at $\hat{z} = z^0$, no school is over-demanded, and each student is assigned to their most-preferred school according to the reported preference, which is close to the true preference because $z^1 \approx z^0$.

When the value z^1 is disclosed, some students switch from reporting $E \succ N$ to reporting $N \succ E$. We call them switchers. Because of the switchers, the new school becomes over-demanded. The welfare consequence is as follows. The total gain for the switchers is second order in $z^1 - z^0$ because both the mass of switchers and the gain per switcher are proportional to the quality difference. However, the total loss for non-switchers is first order in $z^1 - z^0$ because the mass of non-switchers is approximately constant and the loss per non-switcher increases proportionately with the mass of switchers. Therefore, when $z^1 - z^0$ is small, the loss exceeds the gain. The formal statement of this observation and its proof is provided in Section 1.3.

1.3 Non-Optimality of Full Disclosure

In this section, we show that the full disclosure of $\tilde{z}$ is not generally optimal. This is because the welfare function has a dip around the point $z^0 \in \mathbb{R}^{|S|}$ where the demand equals the supply at each school. The expected welfare can be improved by pooling the realizations of z over the dip.

To formally state the result, we first need to introduce a few definitions. We say student i demands school s if school s yields the highest utility for student i (i.e., student i top-ranks school s under strategy-proof allocation mechanisms). Formally, given the posterior mean $\hat{z}$, the demand of student i for school s is

$$d_{is}(\hat{z}) = \mathbb{1}\{\theta_i \hat{z}_s + \varepsilon_{is} > \theta_i \hat{z}_{s'} + \varepsilon_{is'} \quad \forall s' \in S\}.$$

We maintain the assumption that F_ε is atomless so that the set of students who are indifferent between schools has measure zero. That is, we have $\sum_{s\in S} d_{is}(\hat{z}) = 1$ for almost all students.

The aggregate demand for school s is

$$D_s(\hat{z}) = \int_{\omega_i \in \Omega} d_{is}(\hat{z})\, dF(\omega_i).$$

We define the demand vector $D(\hat{z}) \equiv (D_s(\hat{z}))_{s\in S}$. Because there is a unit continuum of students, we have $\sum_{s\in S} D_s(\hat{z}) = 1$ almost surely.

Recall that $q \equiv (q_s)_{s\in S}$ is the capacity vector. Suppose there is a posterior mean z^0 such that $q = D(z^0)$. That is, the capacity (supply) equals the demand at each school. Such z^0 exists if there are not too many quality-insensitive students whose preference orders cannot be altered through $\hat{z}$. In particular, we require

$$\int_{\omega_i \in \Omega} \mathbb{1}\{\theta_i = 0 \text{ and } \varepsilon_{is} > \varepsilon_{is'} \quad \forall s' \in S\}\, dF(\omega_i) \leq q_s \quad \text{for each } s \in S.$$

When $\hat{z} = z^0$, each student is assigned to their most-preferred school according to the expected utility $u_i = \theta_i \hat{z} + \varepsilon_i$. If the realization of $\tilde{z}$ is also z^0, they are assigned to their most-preferred school according to the *actual* utility $u_i = \theta_i z + \varepsilon_i$. In this sense, the sorting is perfect when $\hat{z} = z = z^0$.

We wish to show that there is a dip in the welfare function around z^0 and that pooling the values of z around z^0 improves the expected welfare over the full disclosure. In order to do so, we need to ensure that (i) there is indeed a dip in the welfare function around z^0 and (ii) pooling around z^0 is possible.

First, for the welfare function to have a dip around z^0, some students must change their reported preference orders in response to a change in belief. To see this, consider a realization $z \neq z^0$. Suppose disclosing the realized value z does not affect the preference order of any student. Then $d_i(z) = d_i(z^0)$ for all i, where $d_i(z) \equiv (d_{is}(z))_{s \in S}$. Consequently, we have $D(z) = D(z^0) = q$.

Then each student is assigned to their most-preferred school, i.e., the allocation for student i is $x_i(z) = x_i(z^0) = d_i(z^0)$ except for a measure-zero set of students. With the allocation fixed at $d_i(z^0)$, the welfare function, $W(z) = \int_{\omega_i \in \Omega} d_i(z^0) \cdot u_i(z) \, dF(\omega_i)$, changes linearly in z without a dip. Therefore, for the welfare function to be nonlinear, the demand must change when the posterior mean changes.

Definition 1. We say **the demand is responsive to a change $\vec{z}$ at z^0** if

$$\lim_{t \to 0} \frac{D(z^0 + t\vec{z}) - D(z^0)}{t} \neq \mathbf{0}.$$

We can ensure that the demand is responsive to almost any change $\vec{z}$ at z^0 by imposing appropriate restrictions on the type distribution F_ω. However, there is one exception. When $\vec{z} = \mathbf{1}\alpha$ for some $\alpha \in \mathbb{R}$, it is inevitable that the demand is not responsive. This is because the expected utilities, $u_{is} = \theta_i \hat{z}_s + \varepsilon_{is}$, increase or decrease uniformly across all schools, and therefore, student i demands the same school between $z^0 + \vec{z}$ and z^0. Consequently, $D(z^0 + \vec{z}) = D(z^0)$. The following definition takes this into account.

Definition 2. We say **the demand is responsive at z^0** if the demand is responsive to a change $\vec{z}$ at z^0 for all $\vec{z} \in \{z \in \mathbb{R}^{|S|} \mid z \neq \mathbf{1}\alpha$ for any $\alpha \in \mathbb{R}\}$.

For the demand to be responsive at z^0, there need to be quality-sensitive students who are almost indifferent between their most-preferred school and their second-most-preferred school. In other words, the conditional distribution $F_{\varepsilon|\theta>0}$ must have a strictly positive density around the set of students who are exactly indifferent. This is ensured if the set of students $\{\varepsilon_i \in supp(f_{\varepsilon|\theta>0}) \mid \theta_i > 0$ and $d_i(z^0) = \mathbf{0}\}$ is in the interior of $supp(f_{\varepsilon|\theta>0})$. It follows that the responsiveness of the demand is ensured at all $z \in \mathbb{R}^{|S|}$ if $supp(f_{\varepsilon|\theta>0})$ is $\mathbb{R}^{|S|}$.

Second, provided that the welfare function has a dip around z^0, we want to be able to pool the values of z around z^0. Intuitively, the support of $\tilde{z}$ cannot have a hole around z^0 because otherwise there is no density to be consolidated at z^0. For example, if $g(z) = 1$ for $z \in [-1, -0.5] \cup [0.5, 1]$, pooling immediately around $z = 0$ is impossible. Also, z^0 cannot be on a vertex of $supp(\tilde{z})$ because such a point cannot be a convex combination of other z's in the support. For example, if $\tilde{z} \sim U[0, 1]$, pooling around $z = 0$ is impossible.

We also want to preempt superficial pooling that has no effect on the expected welfare. Specifically, we require that the pooling region to contain some points outside the set $\{z \in supp(\tilde{z}) \mid z = z^0 + \mathbf{1}\alpha$ for some $\alpha \in \mathbb{R}\}$. Because the welfare function is linear over this set, pooling cannot be effectual if the pooling region entirely consists of points in this set.

Definition 3. We say **effectual pooling is possible around z^0** if, for any $\epsilon > 0$, there is a set $\mathcal{Z} \subseteq \{z \in supp(\tilde{z}) \mid 0 < \|z - z^0\| < \epsilon$ and $z \neq z^0 + \mathbf{1}\alpha$ for any $\alpha \in \mathbb{R}\}$ and non-negative weights $\lambda(z) \leq g(z)$, $z \in \mathcal{Z}$, such that $\frac{\int_{z \in \mathcal{Z}} z\lambda(z)\,dz}{\int_{z \in \mathcal{Z}} \lambda(z)\,dz} = z^0$.

Effectual pooling is possible around z^0, for example, if z^0 is in the interior of $supp(\tilde{z})$. Note that for a pooling to be actually effectual, there must be some z with a strictly positive weight such that $D(z) \neq D(z^0)$. We intentionally leave this out of the definition to keep the assumptions about the type distribution F_ε separate from the assumptions about $G(\tilde{z})$. Now, we are ready to present our first result.

Theorem 1. Suppose there is z^0 such that $q = D(z^0)$. If the demand is responsive at z^0 and effectual pooling is possible around z^0, then the full disclosure of $\tilde{z}$ is not optimal under the Deferred Acceptance mechanism. In particular, there is a disclosure policy that pools the values of z around z^0 that yields a larger expected welfare than the full disclosure.

Proof. Pick z^0 such that $q = D(z^0)$. Suppose the demand is responsive at z^0. Recall that $w(z, \hat{z}) \equiv \int_{\omega_i \in \Omega} x_i(\hat{z}) \cdot u_i(z)\, dF(\omega_i)$ is the welfare when the realization of $\tilde{z}$ is z and the posterior mean is $\hat{z}$. We claim that there is $\epsilon > 0$ such that

$$0 < \|z - z^0\| < \epsilon \text{ and } z \neq z^0 + \mathbf{1}\alpha \text{ for any } \alpha \in \mathbb{R} \quad \Longrightarrow \quad w(z, z) < w(z, z^0).$$

That is, the welfare can be improved by placing the posterior mean at z^0 instead of disclosing the realized value z.

If the above claim is true, then there is a pooling around z^0 that improves the expected welfare. Because effectual pooling is possible around z^0 by assumption, there is a set $\mathcal{Z} \subseteq \{z \in supp(\tilde{z}) \mid 0 < \|z - z^0\| < \epsilon \text{ and } z \neq z^0 + \mathbf{1}\alpha \text{ for any } \alpha \in \mathbb{R}\}$ and non-negative weights $\lambda(z) \leq g(z)$, $z \in \mathcal{Z}$, such that $\frac{\int_{z \in \mathcal{Z}} z\lambda(z)\, dz}{\int_{z \in \mathcal{Z}} \lambda(z)\, dz} = z^0$. If the above claim is true, we have $w(z, z) < w(z, z^0)$ for all z with $\lambda(z) > 0$. It follows that $\frac{\int_{z \in \mathcal{Z}} w(z,z)\lambda(z)\, dz}{\int_{z \in \mathcal{Z}} \lambda(z)\, dz} < \frac{\int_{z \in \mathcal{Z}} w(z,z^0)\lambda(z)\, dz}{\int_{z \in \mathcal{Z}} \lambda(z)\, dz}$.

Therefore, it remains to show that the claim is indeed true. To this end, we define the value function $V(z) \equiv w(z,z) - w(z,z^0)$. It suffices to show that

$$\lim_{t \to 0^+} \frac{V(z^0 + t\vec{z}) - V(z^0)}{t} < 0 \text{ for all } \vec{z} \in \{z \in \mathbb{R}^{|S|} \mid z \neq \mathbf{1}\alpha \text{ for any } \alpha \in \mathbb{R}\}.$$

Consider an arbitrary change $\vec{z}$ such that $\vec{z} \neq \mathbf{1}\alpha$ for any $\alpha \in \mathbb{R}$. Let $z(t) = z^0 + t\vec{z}$. With slight abuse of notation, we write $V(t)$, $x(t)$, $u(t)$, etc. to mean $V(z(t))$, $x(z(t))$, $u(z(t))$, etc. Then the value function is

$$\begin{aligned} V(t) &= \int_{\omega_i \in \Omega} x_i(t) \cdot u_i(t)\, dF(\omega_i) - \int_{\omega_i \in \Omega} x_i(0) \cdot u_i(t)\, dF(\omega_i) \\ &= \int_{\omega_i \in \Omega} \{x_i(t) - x_i(0)\} \cdot u_i(t)\, dF(\omega_i). \end{aligned}$$

For readability, we use curly and square brackets to group terms and reserve parentheses for arguments of functions.

With further abuse of notation, we write $V'(t)$, $x_i'(t)$, $u_i'(t)$, etc. to mean the right derivatives. For example, $V'(t) \equiv \lim_{t \to 0^+} \frac{V(t) - V(0)}{t}$. Using the measure theory version of the differentiation under the integral sign, we obtain

$$V'(t) = \int_{\omega_i \in \Omega} \left[x_i'(t) \cdot u_i(t) + \{x_i(t) - x_i(0)\} \cdot u_i'(t)\right] dF(\omega_i).$$

By the Dominated Convergence Theorem and the boundedness of $u_i'(t)$, we obtain

$$V'(0) = \int_{\omega_i \in \Omega} \left[x_i'(0) \cdot u_i(0)\right] dF(\omega_i).$$

We want to show that this is negative.

Consider any student i. Because $x_i(0)$ assigns student i to their most-preferred school according to $u_i(0)$, any other allocation decreases the utility of the student. In other words, for any $t > 0$, we have $x_i(t) \cdot u_i(0) \leq x_i(0) \cdot u_i(0)$. It follows that $x_i'(0) \cdot u_i(0) \leq 0$. This is true for any student, and therefore, $V'(0) \leq 0$.

We claim that there is a positive measure of students with $x_i'(0) \cdot u_i(0) < 0$. To see this, pick s^* such that $D_{s^*}'(0) > 0$. Such s^* exists because the demand is responsive at $t = 0$ and $\sum_{s \in S} D_s(0) = 1$. At $t = 0$, fraction q_{s^*} of students demand school s^*. For $t > 0$, the fraction of students who demand school s^* is $D_{s^*}(t) > q_{s^*}$, and their share of school s^* is $x_{is^*}(t) = \frac{q_{s^*}}{D_{s^*}(t)}$ per student. This has a strictly negative slope at $t = 0$ because $D_{s^*}'(0) > 0$. Furthermore, because F_ε is atomless, almost all of q_{s^*} students strictly prefer s^* to all other schools. In other words, $\int_{\omega_i \in \Omega} \mathbb{1}\{x_i'(0) \cdot u_i(0) < 0\} \, dF(\omega_i) \geq q_{s^*}$. It follows that $V'(0) < 0$. □

Note that the proof does not rely on the specific allocation under the Deferred Acceptance mechanism. Roughly speaking, the proof goes through as long as $x_i(z^0) = d_i(z^0)$ almost surely and the allocation changes when the demand changes. In particular, the Immediate Acceptance mechanism we analyze in Section 1.6 has these properties. A formal extension of Theorem 1 requires an equilibrium selection rule. We leave it for future research.

The dip in the welfare function can be explained by the trade-off between two opposing forces. On one hand, disclosing the realized value z allows quality-sensitive students to pursue high-quality schools, which improves their individual utility. On the other hand, disclosure makes high-quality schools over-demanded and causes disturbance to the sorting, which imposes negative externality to the society. When $z \approx z^0$, the latter effect dominates the former.

To see this, consider $z \approx z^0$ such that $z \neq z^0 + \mathbf{1}\alpha$ for any $\alpha \in \mathbb{R}$. Noting that $u_i(z) - u_i(z^0) = \theta_i\{z - z^0\}$, the value function can be written

$$V(z) = \int_{\omega_i \in \Omega} \left\{x_i(z) - x_i(z^0)\right\} \cdot \theta_i\{z - z^0\}\, dF(\omega_i) + \int_{\omega_i \in \Omega} \left\{x_i(z) - x_i(z^0)\right\} \cdot u_i(z^0)\, dF(\omega_i).$$

The first term captures the benefit to the switchers, who change their reports between z and z^0 in pursuit of higher-quality schools. It can also be interpreted as the benefit to the society due to assortative matching in the vertical dimension, i.e., quality-sensitive students are assigned to high-quality schools. The second term captures the negative externality due to the disturbance to the sorting. As $z \to z^0$, the first term diminishes more rapidly than the second term.

The intuition is as follows. Consider $z \approx z^0$. The switchers are almost indifferent between their most-preferred school and the second-most preferred school. Also, the mass of switchers is small because F_ε is atomless. Because individual benefit is proportional to quality difference and the mass of switchers is also proportional to quality difference, the total gain for the switchers is second order in quality difference.

For non-switchers, if student i demands school s^* regardless of whether they believe z^0 or z, then the student has a distinct preference for s^* over other schools, and therefore, a reduction in x_{is^*} decreases their utility. Although individual loss is small (because the change in allocation is small because the mass of the switchers is small), there is a strictly positive measure of non-switchers who experience negative externality. Therefore, the total loss to non-switchers is first order in quality difference. As a result, the loss exceeds gain when $z \approx z^0$.

1.4 Two-School Model

So far, we have established that the full disclosure is not optimal in general. Now, we analyze optimal disclosure policies. This section is dedicated to the development of a framework for the analysis and the actual analysis is provided in Section 1.5.

We introduce a simple two-school model in Section 1.4.1, and transform the social planner's problem using a value function. In Section 1.4.2, we discuss the sufficient condition for optimality presented in Dworczak and Martini (2019). We then derive a formula for the value function in Section 1.4.3, which plays a key role in the analysis of optimal disclosure policies we discuss in Section 1.5.

1.4.1 The Social Planner's Problem

There are two schools, existing (E) and new (N), with capacities $q_E = q_N = \frac{1}{2}$. There is a unit continuum of students, each demanding one seat. We normalize the utility from attending the existing school to 0 for each student. That is, $\varepsilon_{iE} = 0$ for all i and $z_E = 0$ with certainty. The utility from attending the new school is $u_{iN} = \theta_i z_N + \varepsilon_{iN}$ for student i. With slight abuse of notation, we drop the school subscript and simply write $u_i = \theta_i z + \varepsilon_i$ to denote the utility from attending the new school.

The type of student i is a tuple $\omega_i = (\theta_i, \varepsilon_i)$, and is drawn from a common distribution $f(\theta, \varepsilon)$, independently from other students. We assume ε is symmetric around 0 for each θ, i.e., $f(\theta, \varepsilon) = f(\theta, -\varepsilon)$. The quality of the new school (relative to that of the existing school) is a random variable with prior $G(\tilde{z})$.[5]

[5] The new school does not have to be literally new. As long as there is some uncertainty about the relative quality between the schools, our model is applicable. For example, school quality is subject to uncertainty when a school hires a new principal or launches a new program.

Given a reported preference profile, the seats are allocated through Deferred Acceptance, which returns the unique allocation $x(\hat{z})$ that is efficient and treats equals equally. Recall that the welfare function

$$W(z) = \int_{\omega_i \in \Omega} x_i(z) \cdot \{\theta_i z + \varepsilon_i\} \, dF(\omega_i)$$

can be interpreted as the welfare when realizations of $\tilde{z}$ are fully disclosed.

Suppose $\tilde{z}$ has realizations in some non-degenerate bounded interval $[\underline{z}, \overline{z}]$, and its support includes the endpoints $\underline{z}$ and $\overline{z}$. Given the welfare function $W(x)$ and the prior $G(\tilde{z})$, the social planner aims to maximize social welfare by garbling:

$$\begin{aligned} \max_{H} \quad & \int_{\underline{z}}^{\overline{z}} W(z) \, dH(z) \\ \text{s.t.} \quad & H \text{ is a mean-preserving contraction of } G \end{aligned}$$

We convert this problem to a form that is more convenient for analysis. For any mean-preserving contraction H of G, we have $\mathbb{E}_H[\tilde{z}] = \mathbb{E}_G[\tilde{z}]$. Therefore, we can subtract any linear function of z from the objective function without altering the problem. More precisely, for any constants θ_0 and ε_0, the transformed problem

$$\begin{aligned} \max_{H} \quad & \int_{\underline{z}}^{\overline{z}} [W(z) - \{\theta_0 z + \varepsilon_0\}] \, dH(z) \\ \text{s.t.} \quad & H \text{ is a mean-preserving contraction of } G \end{aligned}$$

is equivalent to the original problem. In particular, we can set $\theta_0 = q_N \bar{\theta}$ and $\varepsilon_0 = q_N \bar{\varepsilon}$, where $\bar{\theta} = \mathbb{E}[\theta]$ and $\bar{\varepsilon} = \mathbb{E}[\varepsilon \mid \varepsilon > 0]$. Essentially, we are subtracting $q_N\{\bar{\theta} z + \bar{\varepsilon}\}$, the welfare that would arise when the posterior belief is fixed at $\hat{z} = 0$.

This transformation is depicted in Figure 1.2 for the Illustrative Example from Section 1.2.4. The welfare function (solid black line on the left graph) is a slanted W-shape centered at $z = 0$. By subtracting the welfare that would arise when the belief is fixed at $\hat{z} = 0$ (dotted green line on the left graph), we can transform it to a symmetric, W-shaped function (solid black line on the right graph).

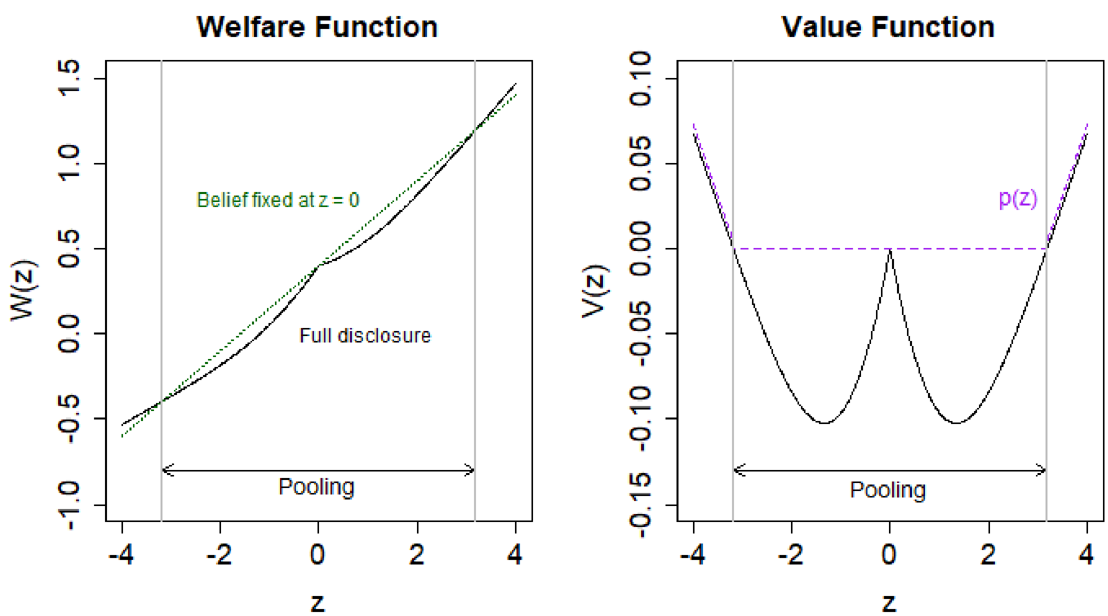


Figure 1.2: Transformation of the social planner's problem.

Formally, we define a value function to be the difference between the welfare under the full disclosure and the welfare under the no disclosure:

$$V(z) \equiv W(z) - q_N \left\{ \bar{\theta} z + \bar{\varepsilon} \right\}$$

The symmetry of value function (proved in Section 1.4.3) simplifies expositions.

1.4.2 Sufficient Condition for The Optimality

Given the value function $V(z)$ and the prior $G(\tilde{z})$, the social planner's problem is

$$\begin{aligned} \max_{H} \quad & \int_{\underline{z}}^{\overline{z}} V(z)\, dH(z) \\ \text{s.t.} \quad & H \text{ is a mean-preserving contraction of } G. \end{aligned}$$

Theorem 1 of Dworczak and Martini (2019) states the following: If there exists a distribution function H and a convex price function $p : [\underline{z}, \overline{z}] \to \mathbb{R}$ with $p(z) \geq V(z)$ for all $z \in [\underline{z}, \overline{z}]$ such that

$$\begin{aligned} \text{(i)} \quad & supp(H) \subseteq \{z \in [\underline{z}, \overline{z}] : p(z) = V(z)\}, \\ \text{(ii)} \quad & \int_{\underline{z}}^{\overline{z}} p(z)\, dH(z) = \int_{\underline{z}}^{\overline{z}} p(z)\, dG(z), \text{ and} \\ \text{(iii)} \quad & H \text{ is a mean-preserving contraction of } G, \end{aligned}$$

then H is a solution to the social planner's problem.

Condition (i) states that H can have a positive density only where the price function coincides with the value function. Condition (ii) is satisfied if the price function is linear over each pooling region. As depicted in Figure 1.2, the effect of replacing G by H is as if the value function $V(z)$ is replaced by the price function $p(z)$. When $p(z)$ is convex, further garbling cannot improve the expected value.

Note that the prior, $g(\tilde{z})$, does not have to be symmetric around 0. Continuing with the Illustrative Example, suppose $\tilde{z} \sim U[-2, 4]$. The optimal disclosure policy and the price function are depicted on the left panel in Figure 1.3. The values of $z \in [-2, 2]$ are pooled, and the values of $z \in [2, 4]$ are fully disclosed.

Similarly, $\tilde{z}$ does not have to be uniformly distributed. For example, consider $g(z) = \frac{1}{12} \cdot \mathbb{1}\{-4 < z \leq 0\} + \frac{1}{6} \cdot \mathbb{1}\{0 < z < 4\}$. When $\tilde{z}$ is not uniformly distributed, we can stretch the horizontal axis proportionate to the probability density and proceed as before. The optimal disclosure policy and the price function are depicted on the right panel in Figure 1.3. The values of $z \in [-4, 2]$ are pooled, and the values of $z \in [2, 4]$ are fully disclosed.

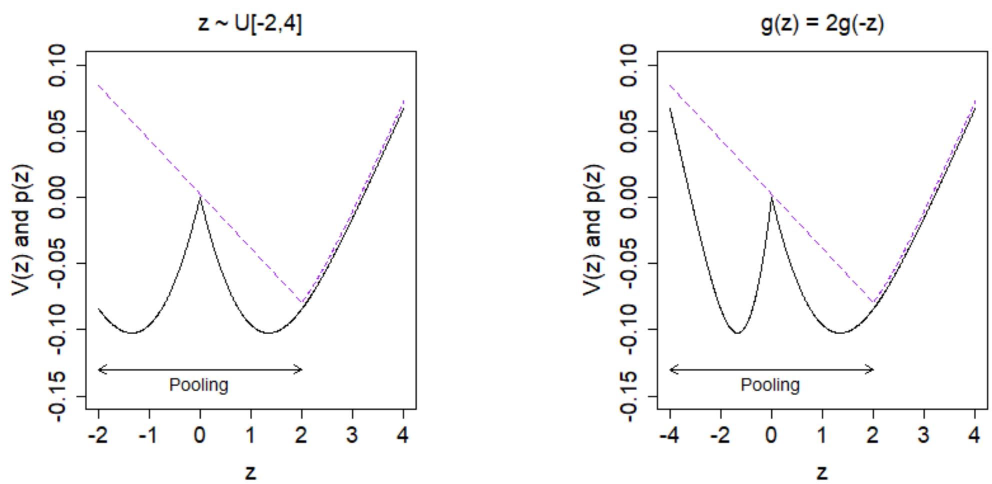


Figure 1.3: Optimal disclosure policy for $\theta \in \{0, 1\}$ for non-symmetric $g(z)$.

1.4.3 The Value Function

Now, we derive a formula for the value function. For a given value of $z > 0$, consider the set of students, $\{(\theta_i, \varepsilon_i) \mid -\theta_i z < \varepsilon_i < 0\}$. We call them *switchers* because they switch from reporting $E \succ N$ at $\hat{z} = 0$ to reporting $N \succ E$ at $\hat{z} = z$. Changes in welfare are caused by the switchers replacing non-switchers at the new school. Due to the assumption $f(\theta, \varepsilon) = f(\theta, -\varepsilon)$, the switchers are identical to those with $0 < \varepsilon_i < \theta_i z$, except that their ε_i have different signs.

Lemma 1. The value function for the two-school model is symmetric around $z = 0$ and

$$V(z) = \frac{\frac{1}{2}M(z)}{\frac{1}{2} + M(z)} \left[\left\{ \tilde{\theta}(z)z + \tilde{\varepsilon}(z) \right\} - \left\{ \bar{\theta}z + \bar{\varepsilon} \right\} \right] \qquad \text{for} \quad z \geq 0,$$

where

$$\begin{aligned} M(z) &= \mathbb{E}[\mathbb{1}\{0 < \varepsilon < \theta z\}] & & \\ \tilde{\theta}(z) &= \mathbb{E}[\theta \mid 0 < \varepsilon < \theta z] & \bar{\theta} &= \mathbb{E}[\theta] \\ \tilde{\varepsilon}(z) &= -\mathbb{E}[\varepsilon \mid 0 < \varepsilon < \theta z] & \bar{\varepsilon} &= \mathbb{E}[\varepsilon \mid \varepsilon > 0] \end{aligned}$$

The derivation is provided in Appendix 1.8.1. The value function reflects the difference between the switchers and those who are replaced by the switchers at the new school. In the formula, $M(z)$ is the mass of the switchers, $\tilde{\theta}(z)$ is the average valuation of the quality among the switchers, and $\tilde{\varepsilon}(z)$ is the average idiosyncratic taste among the switchers. The switchers partially replace non-switchers at the new school. Among those who are replaced, the average valuation of the quality is $\bar{\theta}$ and the average idiosyncratic taste is $\bar{\varepsilon}$.

It is evident in the formula that disclosure has two opposing effects on social welfare. On one hand, disclosing z allows quality-sensitive students to pursue the higher-quality school, making the overall allocation more assortative in the vertical dimension. The quantity $\left\{ \tilde{\theta}(z) - \bar{\theta} \right\} z$ represents the benefit from improved vertical sorting. On the other hand, disclosing z encourages students to demand the higher-quality school even if they do not like the higher-quality school according to their idiosyncratic taste (i.e., $\varepsilon_i < 0$). The quantity $\tilde{\varepsilon}(z) - \bar{\varepsilon}$ represents the loss from disturbed horizontal sorting. The overall effect on social welfare is determined by the trade-off between these two effects.

1.5 Optimal Disclosure Policy

In this section, we analyze the optimal disclosure policies using the two-school model. We provide a sufficient condition for the optimality of the no-disclosure policy in Section 1.5.1, and a sufficient condition for the optimality of some disclosure in Section 1.5.2. Then, in Section 1.5.3, using families of distributions, we demonstrate that heterogeneous valuations of the quality lead to more information disclosure.

1.5.1 Optimality of No Disclosure

Intuitively, the value function tends to be small when the valuations of quality θ_i are small because there is not much gain from assortative matching. Indeed, when large values of θ_i are rare, the gain from improved vertical sorting is always smaller than the loss from disturbed horizontal sorting, making it optimal to conceal information.

Proposition 1. Suppose $\mathbb{E}_G[\tilde{z}] = 0$ and θ and ε are independently distributed. If

$$\text{Thin-Tail Condition:} \qquad \mathbb{E}[\theta \mid \theta \geq \theta_1] \leq \theta_1 + \mathbb{E}[\theta] \qquad \forall \theta_1 \in supp(f_\theta)$$

is satisfied, then the no-disclosure policy is optimal.

The Thin-Tail condition is satisfied by, for example, degenerate distributions, uniform distributions, and exponential distributions. It is violated by heavy-tail distributions such as Pareto distributions and distributions with mass at $\theta = 0$, i.e., $F_\theta(0) > 0$ (their tail can be described heavy if we consider any $\theta > 0$ is a part of the tail). The Thin-Tail condition can also be violated by continuous distributions with bounded support, e.g., a beta distribution with $\alpha < 1$, for which $f_\theta(0)$ is unbounded.

The proof makes it obvious that the Thin-Tail condition is sufficient but not necessary for the optimality of the no-disclosure policy.

It may seem surprising that there is no restriction on f_ε other than the independence from θ and the symmetry around 0, i.e., $f_\varepsilon(\varepsilon) = f_\varepsilon(-\varepsilon)$. Although f_ε does not appear in the sufficient condition, it is embedded in the decision rule: Student i reports $N \succ_i E$ if and only if $\theta_i z + \varepsilon_i > 0$, or equivalently, $\theta_i > \frac{-\varepsilon_i}{z}$. If the distribution of ε_i is such that ε_i is either ε^* or $-\varepsilon^*$ for some constant $\varepsilon^* > 0$, then the absolute value of ε^* does not matter because it is canceled out by z. In fact, because θ_i and ε_i are independent, any distribution f_ε can be decomposed into a collection of distributions with support $\{\varepsilon, -\varepsilon\}$. Therefore, the shape of f_ε does not matter.

Proof. Suppose $\mathbb{E}_G[\tilde{z}] = 0$. We claim that the no-disclosure policy is optimal if $V(z) \leq 0$ for all $z \in [\underline{z}, \overline{z}]$. This follows from Theorem 1 of Dworczak and Martini (2019). Let H be degenerate at $z = 0$ and set $p(z) = V(0) = 0$ for all $z \in [\underline{z}, \overline{z}]$.

It suffices to show that $V(z) \leq 0$ for all $z > 0$ because $V(z) = V(-z)$ and $V(z) = 0$. For $z > 0$, the components of $V(z)$ can be computed as follows:

$$
\begin{aligned}
M(z) &= \mathbb{E}[\mathbb{1}\{0 < \varepsilon < \theta z\}] = \int_0^\infty f_\varepsilon(\varepsilon)\left\{1 - F_{\theta|\varepsilon}\left(\frac{\varepsilon}{z}\right)\right\} d\varepsilon \\
\tilde{\theta}(z) &= \mathbb{E}[\theta \mid 0 < \varepsilon < \theta z] = \frac{1}{M(z)}\int_0^\infty \mathbb{E}_{\theta|\varepsilon}\left[\theta \,\middle|\, \theta \geq \frac{\varepsilon}{z}\right] f_\varepsilon(\varepsilon)\left\{1 - F_{\theta|\varepsilon}\left(\frac{\varepsilon}{z}\right)\right\} d\varepsilon \\
\tilde{\varepsilon}(z) &= -\mathbb{E}[\varepsilon \mid 0 < \varepsilon < \theta z] = -\frac{1}{M(z)}\int_0^\infty \varepsilon f_\varepsilon(\varepsilon)\left\{1 - F_{\theta|\varepsilon}\left(\frac{\varepsilon}{z}\right)\right\} d\varepsilon
\end{aligned}
$$

We substitute these into the value-function formula in Lemma 1 to obtain

$$
V(z) = \frac{\frac{1}{2}}{\frac{1}{2} + M(z)}\int_0^\infty \left\{\mathbb{E}_{\theta|\varepsilon}\left[\theta \,\middle|\, \theta \geq \frac{\varepsilon}{z}\right] z - \varepsilon - \bar{\theta} z - \bar{\varepsilon}\right\} f_\varepsilon(\varepsilon)\left\{1 - F_{\theta|\varepsilon}\left(\frac{\varepsilon}{z}\right)\right\} d\varepsilon.
$$

For a given value of $\varepsilon \geq 0$, the integrand is non-positive if

$$\mathbb{E}_{\theta|\varepsilon}\left[\theta \,\middle|\, \theta \geq \frac{\varepsilon}{z}\right] \leq \frac{\varepsilon}{z} + \bar{\theta} + \frac{\bar{\varepsilon}}{z}.$$

When ε_i and θ_i are independently distributed, the inequality holds for all $\varepsilon \geq 0$ if the Thin-Tail condition is satisfied. It follows that $V(z) \leq 0$ for all $z > 0$. □

The distribution of idiosyncratic taste is irrelevant for the sufficiency for the following reason. For a given value of $\varepsilon < 0$, consider a set of students with $\varepsilon_i = \varepsilon$. They report $N \succ_i E$ if and only if $\theta_i > \frac{-\varepsilon}{z}$. According to the formula in the proof, the contribution of these students to $V(z)$ is non-positive if

$$\left\{\mathbb{E}\left[\theta \,\middle|\, \theta \geq \frac{-\varepsilon}{z}\right] - \bar{\theta}\right\} z \leq -\varepsilon + \bar{\varepsilon}.$$

The left-hand side is the gain from improved vertical sorting. It is proportional to the difference in the valuations of the quality between the switchers and those who are replaced by the switchers. It is also increasing in the quality difference between the two schools. The right-hand side is the loss from disturbed horizontal sorting. The switchers suffer from their own negative idiosyncratic taste for the new school, $-\varepsilon$. They also exert negative externalities, $\bar{\varepsilon}$, to those who are replaced by the switchers.

When $z \approx 0$, the gain from improved vertical sorting is small because the quality difference is small. As z increases, students with smaller θ_i start pursuing the new school, which decreases the average θ_i among the switchers. The Thin-Tail condition ensures that it decreases fast enough so that the gain never exceeds $-\varepsilon$, the self-inflicted loss to the switchers. Thus, the negative externality $\bar{\varepsilon}$, which depends on f_ε, is irrelevant. Of course, we can obtain a tighter condition by incorporating f_ε.

The assumption that θ and ε are independently distributed is crucial. To demonstrate this point, suppose $\theta \sim Exp(1)$ and $\varepsilon \mid \theta \sim N(0, \sigma^2(\theta))$. Figure 1.4 shows the value functions for different choices of $\sigma(\theta)$. The functions $\sigma(\theta)$ are selected so that each of them has $\bar{\varepsilon} = \sqrt{2/\pi}$. This ensures that the difference in the value functions is due to the difference in the correlation between θ and $|\varepsilon|$.

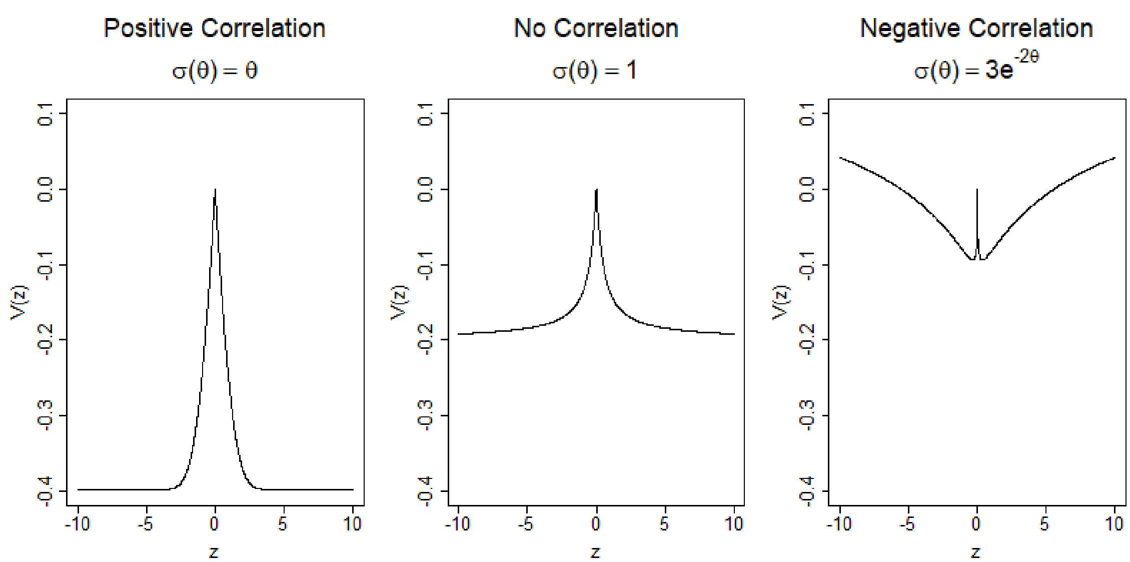


Figure 1.4: Value functions for $\theta \sim Exp(1)$ and $\varepsilon \mid \theta \sim N(0, \sigma^2(\theta))$.

The middle plot is the benchmark, where θ and $|\varepsilon|$ are uncorrelated. When θ and $|\varepsilon|$ are positively correlated (left plot), the loss from disturbed horizontal sorting is severer because the switchers tend to have a large $|\varepsilon|$. In contrast, when θ and $|\varepsilon|$ are negatively correlated (right plot), the loss from disturbed horizontal sorting is mitigated because the switchers tend to have a small $|\varepsilon|$. When $|z|$ is large enough, the gain from improved vertical sorting exceeds the loss from disturbed horizontal sorting. Clearly, the no-disclosure policy is not optimal in this case.

1.5.2 Optimality of Some Disclosure

Now, we provide a sufficient condition for the optimality of some disclosure, i.e., the mean of the posterior belief is not degenerate at the prior mean.

Proposition 2. Suppose $\mathbb{E}_G[\tilde{z}] = 0$. If each of the sets $\{z \in [\underline{z}, 0) \mid V(z) > 0\}$ and $\{z \in (0, \overline{z}] \mid V(z) > 0\}$ has strictly positive measure, then some information is disclosed under any optimal disclosure policy.

Proof. Let $\mathcal{Z}^- = \{z \in [\underline{z}, 0) \mid V(z) > 0\}$ and $\mathcal{Z}^+ = \{z \in (0, \overline{z}] \mid V(z) > 0\}$. Define $z^- = \mathbb{E}[z \mid z \in \mathcal{Z}^-]$ and $z^+ = \mathbb{E}[z \mid z \in \mathcal{Z}^+]$. Let z denote the realization of $\tilde{z}$. Consider the following disclosure policy. If $z \in \mathcal{Z}^-$, disclose it with probability p^-. If $z \in \mathcal{Z}^+$, disclose it with probability p^+. If $z \notin \mathcal{Z}^- \cup \mathcal{Z}^+$, do not disclose it. Choose $p^- > 0$ and $p^+ > 0$ so that $p^- z^- + p^+ z^+ = 0$. This ensures that the students believe $\hat{z} = 0$ when the social planner is silent. Clearly, this disclosure policy improves upon the no-disclosure policy. □

Note that having $V(z) > 0$ for some z on one side of the domain is not sufficient. To see this, consider the value function for the Illustrative Example from Section 1.2.4. Suppose $\tilde{z} = -2$ with probability $\frac{2}{3}$ and $\tilde{z} = 4$ with probability $\frac{1}{3}$. Note that $\mathbb{E}_G[\tilde{z}] = 0$. As depicted on the left panel in Figure 1.5, the expected value is negative under the full disclosure (or any partial disclosure). Therefore, it is optimal to disclose no information and achieve the expected value of zero.

Also note that having $V(z) > 0$ for some z on both sides of the domain is not necessary. Suppose $\tilde{z} = -3$ with probability $\frac{4}{7}$ and $\tilde{z} = 4$ with probability $\frac{3}{7}$. Again, we have $\mathbb{E}_G[\tilde{z}] = 0$. As depicted on the right panel in Figure 1.5, the expected value improves with disclosure.

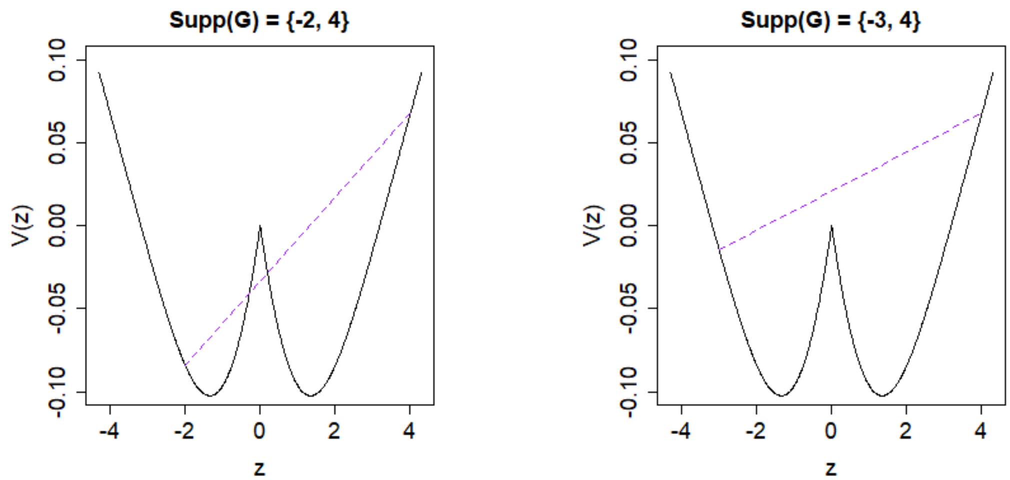


Figure 1.5: The effect of disclosure on the expected value.

In general, we cannot verify the premise of Proposition 2 without computing the value function. Therefore, the practical value of the proposition is limited. However, we can gain insights into optimal disclosure policies by analysing the limit behavior of the value function. In particular, if there is $z^* > 0$ such that $V(z) > 0$ for all $|z| \geq z^*$, then some information is disclosed, provided that the support of $G(\tilde{z})$ is wide enough. To this end, we provide two results regarding the limit values.

Lemma 2. Suppose $\mathbb{E}[\varepsilon \mid \varepsilon > 0]$ is bounded. If $0 < F_\theta(0) < 1$ then $V(z) \to \infty$ as $z \to \pm\infty$.

The lemma states that if there is a strictly positive measure of students who are completely quality-insensitive, then the benefit of disclosing z exceeds the loss for extreme values of z. The intuition is as follows. Provided that $\mathbb{E}[\varepsilon \mid \varepsilon > 0]$ is bounded, the disturbance to horizontal sorting can do only so much damage, while the gain from improved vertical sorting keeps growing as z increases. This is because

$F_\theta(0) > 0$ ensures that, on average, the switchers value the quality more than those who are replaced by the switchers. A formal proof is provided in Appendix 1.8.2.

The next lemma describes the limit behavior of the value function when there is no mass at $\theta = 0$. Although $F_\theta(0) = 0$, the density of students at $\theta = 0$ plays a crucial role in determining the limit value.

Lemma 3. Suppose $\mathbb{E}[\varepsilon \mid \varepsilon > 0]$ is bounded. If $f_{\theta|\varepsilon}(\cdot)$ is bounded for all $\varepsilon \in supp(f_\varepsilon)$ then

$$\lim_{z \to \pm\infty} V(z) = \tfrac{1}{4} \left\{ \bar{\theta}\, \mathbb{E}[\varepsilon f_{\theta|\varepsilon}(0) \mid \varepsilon > 0] - 2\bar{\varepsilon} \right\}.$$

In particular, if θ and ε are independent and $f_\theta(\cdot)$ is bounded, then

$$\lim_{z \to \pm\infty} V(z) = \tfrac{1}{4}\bar{\varepsilon} \left\{ \bar{\theta} f_\theta(0) - 2 \right\}.$$

Suppose θ and ε are dependent. Fixing $\bar{\theta}$ and $\bar{\varepsilon}$, the limit value increases as $\mathbb{E}[\varepsilon f_{\theta|\varepsilon}(0) \mid \varepsilon > 0]$ increases. Roughly speaking, this value is larger when θ and $|\varepsilon|$ are negatively correlated (refer to Figure 1.4 on page 29).

When θ and ε are independently distributed, the value function converges to a positive value if and only if $\bar{\theta} f_\theta(0) > 2$. As in Lemma 2, the gain from assortative matching keeps increasing in z as long as there are some students who do not pursue the higher-quality school. The mass of students who do not pursue the higher-quality school approaches $f_\theta(0)$ in the limit. Fixing $f_\theta(0) > 0$, the benefit from assortative matching is larger when the switchers have higher valuations for the quality, which is reflected in a larger value of $\bar{\theta}$. Therefore, a larger value of $\bar{\theta} f_\theta(0)$ is associated with a larger limit value of the value function. A formal proof of the proposition is provided in Appendix 1.8.3.

1.5.3 Heterogeneity in Valuations of Quality

Although Lemma 2 and 3 only describe the limit value of the value function, they hint that whether the society benefits from information disclosure depends on the distribution of θ. In this section, using families of distributions, we show that more information is disclosed when the valuations of the quality are heterogeneous. Throughout this section, we assume θ and ε are independently distributed and $\varepsilon \sim N(0,1)$.

1.5.3.1 $\theta \in \{0, \theta_H\}$

Suppose θ is either 0 or θ_H. We set $f_\theta(\theta_H) = \frac{0.5}{\theta_H}$ so that $\bar{\theta} = 0.5$. The variance of θ is $0.5(\theta_H - 0.5)$, and therefore, the valuations of the quality become more heterogeneous as θ_H increases. The value functions are plotted in Figure 1.6 for $\theta_H = 0.8$, 1.0, and 1.2. They are all W-shaped, and the value function increases as θ_H increases.

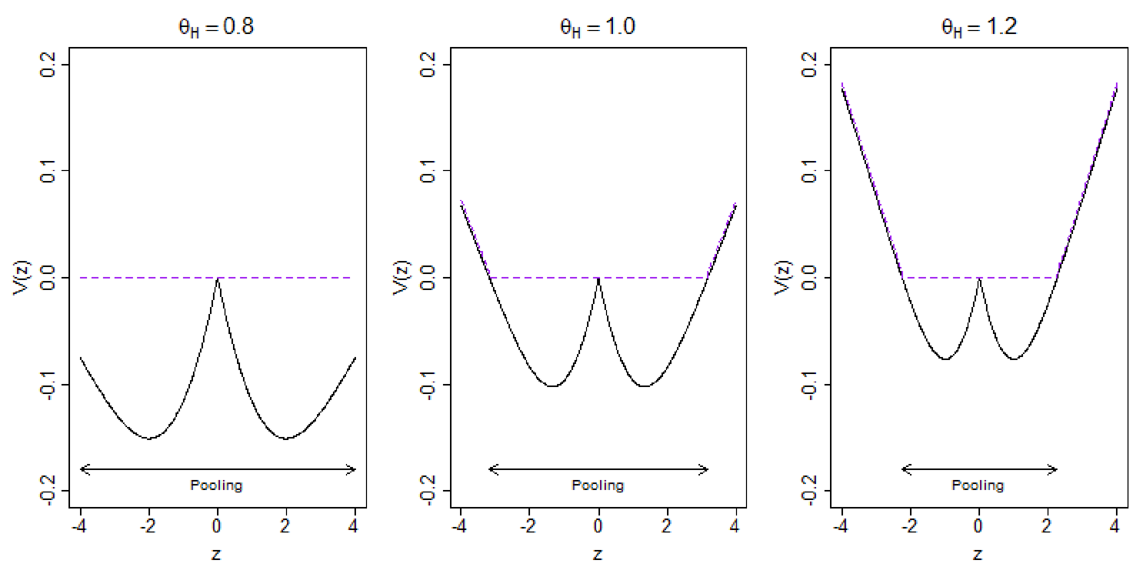


Figure 1.6: Value functions for $\theta \in \{0, \theta_H\}$.

Suppose $\tilde{z} \sim U[-4,4]$. Let z^* be the solution to $V(z^*) = 0$. An optimal disclosure policy pools the values of z in the interval $[-z^*, z^*]$ and fully discloses the values of z such that $|z| > z^*$. We can show that $V(z)$ is convex where $V(z) \geq 0$ if f_ε satisfies $-\frac{f'_\varepsilon(z)}{f_\varepsilon(z)}\bar{\varepsilon} \geq \frac{1}{2}$ for all z such that $V(z) \geq 0$, which is true for $\varepsilon \sim N(0,1)$. The pooling region shrinks as θ_H increases. In other words, more information is disclosed when students have more heterogeneous valuations for the quality.

1.5.3.2 $\theta \in \{\theta_L, \theta_H\}$

Suppose θ is either θ_L or θ_H with equal probabilities. We set $(\theta_L, \theta_H) = (1-\kappa, 1+\kappa)$ for some constant $\kappa \in [0,1)$ to maintain $\bar{\theta} = 1$,. The variance of θ increases as κ increases. The value functions are plotted in Figure 1.7 for $\kappa = 0.88$, 0.90, and 0.92. They are W-shaped around $z = 0$ and overall M-shaped. The value function increases as κ increases because a greater heterogeneity in θ increases the gain from improved vertical sorting.

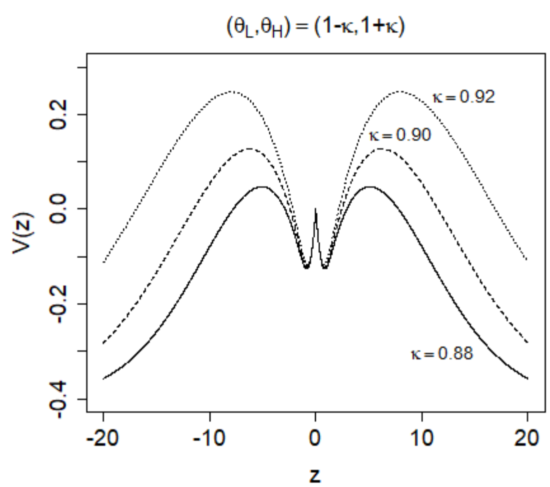


Figure 1.7: Value functions for $\theta \in \{1-\kappa, 1+\kappa\}$.

For small values of $z > 0$, we have $V(z) < 0$ as predicted by Theorem 1.

For medium values of $z > 0$, the switchers (who report $E \succ N$ when $\hat{z} = 0$ and report $N \succ E$ when $\hat{z} > 0$), consist mostly of students with $\theta_i = \theta_H$, providing a boost to the value function through assortative matching. Also, the switchers tend to have only mildly negative ε_i on average.

For extremely large values of $z > 0$, almost everyone reports $N \succ E$, reducing the gain from assortative matching. Also, the society suffers from severely disturbed horizontal sorting. $V(z)$ converges to $-\frac{\bar{\varepsilon}}{2} = -\sqrt{\frac{1}{2\pi}} \approx -0.399$ as predicted by Lemma 3.

Suppose $\tilde{z} \sim U[-20, 20]$. Consider $\kappa = 0.9$. Figure 1.8 depicts optimal pooling schemes[6] that place posterior means at $\hat{z} = \pm 6.24$, the maxima of the value function.

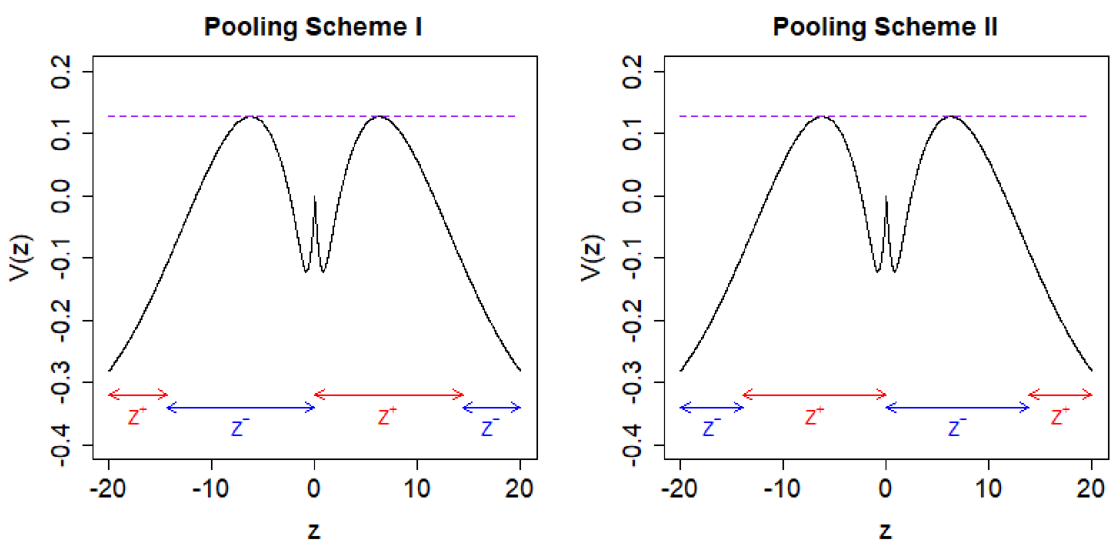


Figure 1.8: Optimal disclosure policies for $\theta \in \{0.1, 1.9\}$.

[6]Although there are infinitely many pooling schemes that maximize the expected value, none of them has convex pooling regions. Dworczak and Martini (2019) provides necessary and sufficient conditions for the existence of an optimal policy that partitions $supp(\tilde{z})$ into convex sets.

Intuitively, the social planner wants the students to pursue the higher-quality school only if their θ_i is large enough and ε_i is only moderately negative; otherwise, switchers cause too much of a disturbance to the horizontal sorting while creating only a small gain through assortative matching. By making the students believe that the quality difference is moderate, the social planner can encourage the right type of students to pursue the higher-quality school.

Going back to Figure 1.7, observe that the maxima of the value function move away from $z = 0$ as κ increases. It follows that if $\kappa' > \kappa$ then the signal associated with the pooling scheme for κ' is more informative than that for κ in the Blackwell sense: The optimized distribution of the posterior means for κ' is a mean-preserving spread of that for κ. In other words, more information is disclosed when the variance of θ is larger.

1.5.3.3 $\theta \sim$ *Lomax*

Suppose θ follows a Lomax (Pareto Type II) distribution. The Lomax distribution is a shifted Pareto distribution so that the support is $\mathbb{R}_{\geq 0}$. The Lomax distribution with the scale parameter $\lambda > 0$ and the shape parameter $\alpha > 1$ has the PDF and the mean

$$f_\theta(\theta) = \frac{\alpha}{\lambda}\left[1 + \frac{\theta}{\lambda}\right]^{-(\alpha+1)} \quad \text{and} \quad \mathbb{E}[\theta] = \frac{\lambda}{\alpha - 1}.$$

The tail probability decays polynomially, and it decays slower for a smaller α.

The value functions are plotted in Figure 1.9 for $\alpha = 1.9$, 1.8, and 1.7, with $\lambda = \alpha - 1$ so that $\bar{\theta} = 1$. They are all W-shaped, and the value function increases as α decreases. Intuitively, the benefit from assortative matching is greater when there are more students with extreme values of θ_i, as measured by $1 - F_\theta(\theta)$.

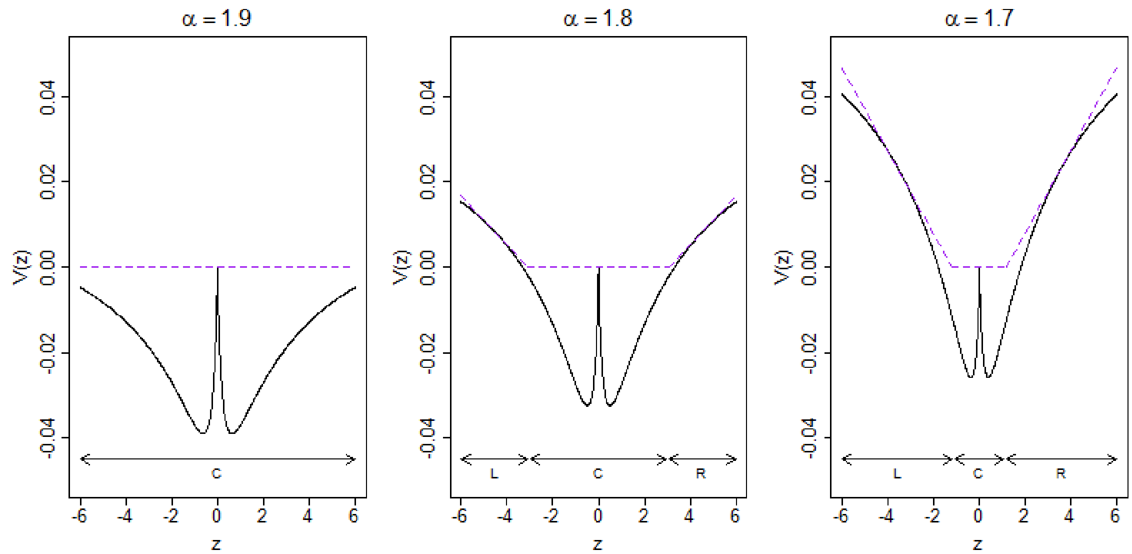


Figure 1.9: Value functions for $\theta \sim Lomax$.

Suppose $\tilde{z} \sim U[-6,6]$. An optimal disclosure policy is depicted for each value of α. As α decreases (i.e., the tail becomes heavier), the center pooling region shrinks while the left and the right pooling regions expand. This means that the social planner reveals which school has a higher quality to students more often when there are more students with extreme values of θ_i,

Comparing the disclosure policies for $\alpha = 1.7$ and $\alpha = 1.8$, neither provides more information than the other in the Blackwell sense: While information is disclosed more often for $\alpha = 1.7$, disclosed information is more extreme for $\alpha = 1.8$. More precisely, while the posterior mean differs from the prior mean for a larger subset of $supp(\tilde{z})$ for $\alpha = 1.7$, the difference between the posterior means and the prior mean is larger for $\alpha = 1.8$. This is not surprising because the variance of θ is infinity for all $\alpha \in (1,2]$, and we cannot say whether a smaller or a larger value of α corresponds to a greater heterogeneity in θ.

1.6 Comparison of Mechanisms

In this section, we analyze how the specifics of the allocation rules affect the social planner's willingness to share information with the students. In particular, we compare the optimal disclosure policies under the Deferred Acceptance (DA) mechanism and the Immediate Acceptance (IA) mechanism, the two most-widely used allocation mechanisms in school-choice programs. We introduce the continuous version of the IA in Section 1.6.1. We then provide two examples in Section 1.6.2 to demonstrate that the IA is generally more conducive for information disclosure than the DA. This is because students internalize the negative externalities under the IA.

1.6.1 Immediate Acceptance Mechanism

There is a continuum of students with equal priorities, and ties are broken through a single lottery. For each realization of the tie-breaking lottery, the (discrete version of) IA works as follows. In the first round, the students apply to the top school in their preference orders. Each school keeps the students with the highest priorities up to its capacity and rejects the rest. The assignments of the accepted students are final. In the second round, the rejected students apply to the next school in their preference orders (even if there is no seat left at that school). If there is remaining capacity, a school admits additional students with the highest priorities up to its capacity and rejects the rest. The process continues until all schools become full or students have applied to all schools in their preference orders.

Because there are infinitely many ways to break the ties for a continuum of students, this algorithm is not implementable. Therefore, we develop a simultaneous-eating version of the IA that integrates the randomization into the preference-based assignment. We say a student is *hungry* if they have not eaten a total of 1 probability share. Their *hungriness* is 1 minus the total probability share they have eaten so far. A student becomes *full* when they have eaten a total of 1 probability share.

At the beginning of the eating algorithm, q_s probability share is available for each school $s \in S$. In the first round, each student eats the probability share of their most-preferred school at a unit speed. The first round ends when all students finish eating their most-preferred school either because they become full or because their most-preferred school becomes exhausted. Some schools may exhaust earlier than other schools, and therefore, some students may have eaten less than other students.

In the second round, each student who is still hungry eats the probability share of their second-most-preferred school at a speed equal to their hungriness at the beginning of the round. There may not be any probability shares left to be eaten at some schools, but students must spend time there. The second round ends when all students finish eating their second-most-preferred school.

In general, in round k, each student who is still hungry eats the probability share of the k^{th} school in their preference order at a speed equal to their hungriness at the beginning of the round. Round k ends when all students finish eating their k^{th}-most-preferred school. The algorithm terminates after $|S|$ rounds.

One important difference between the DA and the IA is that the DA is strategy-proof while the IA is not. Indeed, in some sense, the IA is meant to be manipulated. If a student only slightly prefers a very popular school with a low admission probability to a moderately popular school with a higher admission probability, the student may strategically list the latter as his top choice. This is because the reported preference order affects the effective priority.

1.6.2 Comparison of Optimal Disclosure Policies

We compare the optimal disclosure policies between the DA and the IA under the assumption that the students play the equilibrium strategies given their posterior belief about the school quality. When the preferences of students are highly correlated, the IA tends to achieve a greater social welfare in the equilibrium than the DA because the students can communicate the intensity of their preferences through manipulation (Example 1). However, there are situations in which the equilibrium social welfare under the IA is smaller than that under the DA (Example 2).

In both examples, more information is disclosed under the IA than under the DA. This is because the equilibrium play of the IA requires the students to consider congestion at popular schools (which reflects the preference intensity of other students), leading to reduction in negative externalities.

Example 1 (Aligned Incentives)**.** There are three schools, A, B, and C, with capacities $q_A = q_B = q_C = \frac{1}{3}$, to be allocated to a unit continuum of students. The utility of attending school C is 0 for each student: $\varepsilon_{iC} = 0$ for all i and $z_C = 0$ with probability 1. The valuation of quality, θ_i, is 1 for one half of the students (quality-sensitive students) and 0 for the other half (quality-insensitive students). The idiosyncratic tastes, ε_{iA} and ε_{iB}, are distributed independently across individuals and across schools. Specifically, $(\varepsilon_{iA}, \varepsilon_{iB}) \sim U[0,1]^2$. The school qualities, z_A and z_B, are independently distributed from each other and $(z_A, z_B) \sim U[0,2]^2$.

School C is always under-demanded, and therefore, the only sensible reports are $A \succ B \succ C$ and $B \succ A \succ C$. There are other reports that produce the same outcome, but we ignore these to keep the analysis simple. Under the DA, student i reports $A \succ B \succ C$ if and only if $u_{iA} > u_{iB}$ based on the posterior mean.

Under the IA, the students consider the degree to which each school is over-demanded. Let D_A^* be the equilibrium demand for school A. The equilibrium demand for school B is $1 - D_A^*$. If a student reports $A \succ B \succ C$, they receive $\frac{q_A}{D_A^*}$ probability share of school A. If a student reports $B \succ A \succ C$, they receive $\frac{q_B}{1-D_A^*}$ probability share of school B. Thus, student i reports $A \succ B \succ C$ if and only if

$$\frac{q_A}{D_A^*} u_{iA} > \frac{q_B}{1 - D_A^*} u_{iB}, \quad \text{where} \quad D_A^* = \int \mathbb{1}\left\{\frac{q_A}{D_A^*} u_{iA} > \frac{q_B}{1 - D_A^*} u_{iB}\right\} dF(\omega_i).$$

This defines the unique equilibrium. See Appendix 1.9.1 for detailed analysis.

The welfare functions are plotted in Figure 1.10. We hold $z_A + z_B$ constant to eliminate artificial inflation and deflation of social welfare. The welfare function of the DA parallels that for the two-school model: The optimal disclosure policy has a pooling region around $z_A = z_B$ because the loss from disturbed horizontal sorting exceeds the gain from improved vertical sorting when the quality difference is small.

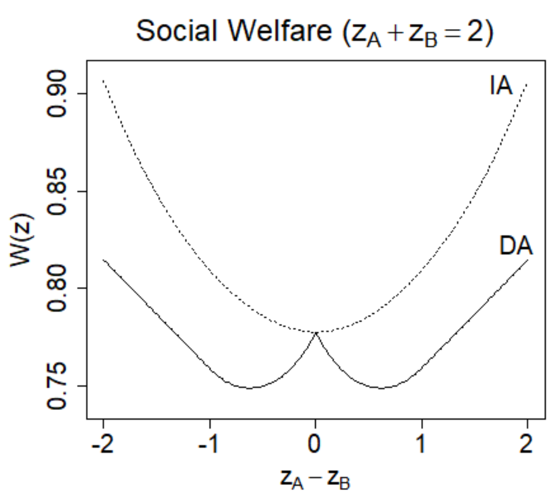


Figure 1.10: Welfare comparison when incentives are aligned.

In contrast, the welfare function of the IA is convex, and therefore, the full disclosure is optimal.[7] The social planner is more comfortable with telling the students which school is of a higher quality because the students would not pursue the most-crowded, highest-quality school unless it yields a substantially larger utility than the less-crowded, second-highest-quality school. In other words, the disturbance to horizontal sorting is mitigated under the IA.

Two comments are in order. First, the strategy-proofness is not necessarily incompatible with the full disclosure. For this example, there is a strategy-proof cardinal mechanism that is outcome-equivalent to the IA, namely, the pseudo-market for probability shares (Hylland and Zeckhauser, 1979). For an equal budget of 1, the set of prices $(p_A, p_B, p_C) = (D^*_A, 1 - D^*_A, 0)$ clears the market.

Second, the larger welfare under the IA can be explained by the alignment of the incentives between the social planner and the students. The social planner does not want the students to crowd the highest-quality school, and the students are effectively discouraged from crowding under the IA: In order to pursue a high-quality, over-demanded school, students must forgo the probability share of their second-most-preferred school.

Note that the difference in the *level* of welfare is explained by the alignment of incentives between the social planner and the students, while the difference in the *shape* of welfare functions—which determines the optimal disclosure policy—is due to the difference in how socially conscious the students are when they are making their decisions. The next example makes this point clear.

[7]This does not contradict Theorem 1 because school C is under-demanded in this example. The premise of the theorem requires that the capacity equals the demand at *all* schools.

Example 2 (Misaligned Incentives)**.** There are three schools, A, B, and C, with capacities $q_A = q_B = q_C = \frac{1}{3}$, to be allocated to a unit continuum of students. The utility of attending school A is 1 for all students: $\varepsilon_{iA} = 1$ for all i and $z_A = 0$ with probability 1. Suppose the valuation of quality, θ_i, is 1 for one half of the students (quality-sensitive students) and 0 for the other half (quality-insensitive students). The quality-sensitive students prefer school B to C. Specifically, $\varepsilon_{iB} \sim U[0,1]$ and $\varepsilon_{iC} = -1$. The quality-insensitive students prefer school C to B. Specifically, $\varepsilon_{iC} \sim U[0,1]$ and $\varepsilon_{iB} = -1$. It is known that $z_C = 0$ with certainty and $z_B \sim U[0,1]$.[8]

Under both the DA and the IA, the quality-insensitive students always report $A \succ C \succ B$ (regardless of the value of $\hat{z}_B$) because school C is under-demanded and there is no need to promote it in their preference order. The quality-sensitive students strategically choose between reporting $A \succ B \succ C$ and reporting $B \succ A \succ C$. As $\hat{z}_B$ increases, more students report $B \succ A \succ C$.

Under the DA, a quality-sensitive student i reports $A \succ B \succ C$ if and only if $u_{iA} > u_{iB}$, or equivalently, $1 > \hat{z}_B + \varepsilon_{iB}$. Therefore, the fraction of quality-sensitive students reporting $A \succ B \succ C$ in the equilibrium is $1 - \hat{z}_B$.

Under the IA, let D_A^* and D_B^* denote the equilibrium demand for school A and school B, respectively. Suppose $D_A^* \geq q_A$ and $D_B^* \geq q_B$. That is, both school A and school B are over-demanded, and therefore, each student can obtain a probability

[8]Here is one possible story that goes with this setting. School A is located in the middle of the city and it is popular among both the quality-sensitive and quality-insensitive students. The quality-sensitive students live on the East side of the city, which hosts school B. The quality-insensitive students live on the West side of the city, which hosts school C. The students can walk to the school that is located on the same side, and the utility is inversely proportional to the distance between the school and their residence. To attend the school on the other side of the city, the students have to take a bus and they don't like it. School B recently hired a new principal, and its quality is subject to uncertainty.

share of either school A or school B, not both. Under this assumption, a quality-sensitive student i reports $A \succ B \succ C$ if and only if

$$\frac{q_A}{D_A^*}u_{iA} + \left[1 - \frac{q_A}{D_A^*}\right]u_{iC} > \frac{q_B}{D_B^*}u_{iB} + \left[1 - \frac{q_B}{D_B^*}\right]u_{iC},$$

where D_A^* and D_B^*, are consistent with this decision rule. Indeed, for each $\hat{z}_B \in (0, 1]$, this is the unique equilibrium, and it satisfies $D_A^* \geq q_A$ and $D_B^* \geq q_B$.[9] Details are provided in Appendix 1.9.2. For any $\hat{z}_B \in (0, 1)$, a smaller fraction of quality-sensitive students report $A \succ B \succ C$ under the IA than under the DA.

The welfare functions are plotted in Figure 1.11. Although not apparent, the IA welfare function is convex, and therefore, the full disclosure is optimal. In contrast, the DA welfare function is concave, and therefore, the no disclosure is optimal.

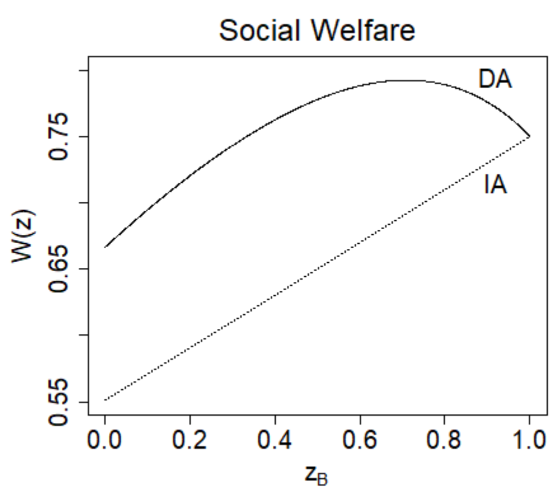


Figure 1.11: Welfare comparison when incentives are misaligned.

[9]For $\hat{z}_B = 0$, there is another equilibrium, in which all quality-sensitive students report $A \succ B \succ C$. This equilibrium coincides with that under the DA.

The IA yields a lower social welfare than the DA because the incentives of the students and the social planner are more misaligned under the IA than under the DA. When $\hat{z}_B$ is sufficiently large, quality-sensitive students want to un-crowd from school A and pursue school B. However, the social planner wants all quality-sensitive students to keep pursuing school A for the following reason. As more quality-sensitive students un-crowd from school A, the total share of school A collectively allocated to the quality-insensitive students increases, which decreases the total share of school C collectively allocated to the quality-insensitive students. Then some share of school C must be allocated to the quality-sensitive students, who experience negative utility from attending school C. Therefore, un-crowding from school A is individually optimal, but it hurts the quality-sensitive students as a whole.

Although the incentives are less aligned between the social planner and the students, more information is disclosed under the IA than under the DA. Therefore, the conventional wisdom that the alignment of the incentives between the sender (the social planner) and the receiver (the students) facilitates information disclosure seems to be valid only for comparisons within a mechanism, not across mechanisms.

As before, the IA is outcome-equivalent to the pseudo-market mechanism. For an equal budget of 1, the set of prices $(p_A, p_B, p_C) = (D_A^*, D_B^*, 0)$ clears the market.

In both examples, the existence of market-clearing prices suggests that the students internalize the externalities under the IA. Indeed, when solving for the equilibrium strategy, the students consider the demand at each school, which reflects the preferences of the others. In other words, the students are more socially conscious under the IA, and therefore, they require less intervention from the social planner in the form of information obfuscation.

1.7 Conclusion

In this paper, we analyze the role of information in indivisible-good allocation problems using a stylized model. In our model, the utilities of the agents consist of a vertical quality component and a horizontal taste component. Although the results are presented using school-choice terminology, the model is more broadly applicable, for example, to course allocation and mentor-mentee matching.

We provide three main results. First, we show that fully disclosing all information about quality is not optimal in general, and social welfare can be improved by hiding small differences in quality. This is because the efficiency gain from improved vertical sorting (assortative matching) is second order in quality difference while the efficiency loss from disturbed horizontal sorting is first order in quality difference.

Second, we show that more information is disclosed when the valuation of the quality exhibits a greater degree of heterogeneity. This is because the heterogeneity in the valuation of the quality increases the gain from assortative matching. In the extreme case where all agents have the same valuation for the quality, there is no assortative matching, and therefore, disclosure simply disturbs the horizontal sorting, leading to a loss in social welfare.

Third, we show that the Immediate Acceptance mechanism is more conducive for information disclosure than the Deferred Acceptance mechanism. Because crowding is implicitly penalized under the Immediate Acceptance mechanism, the students must consider whether their preference for high-quality schools is strong enough relative to the preference intensity of the others. This makes the decisions of the students more socially conscious, and therefore, the social planner is willing to share more information with the students.

1.8 Appendix: Proofs

1.8.1 Proof of Lemma 1

Lemma 1. The value function for the two-school model is symmetric around $z = 0$ and

$$V(z) = \frac{\frac{1}{2}M(z)}{\frac{1}{2} + M(z)} \left[\left\{ \tilde{\theta}(z)z + \tilde{\varepsilon}(z) \right\} - \left\{ \bar{\theta}z + \bar{\varepsilon} \right\} \right] \qquad \text{for} \quad z \geq 0,$$

where

$$M(z) = \mathbb{E}[\mathbb{1}\{0 < \varepsilon < \theta z\}]$$

$$\tilde{\theta}(z) = \mathbb{E}[\theta \mid 0 < \varepsilon < \theta z] \qquad \bar{\theta} = \mathbb{E}[\theta]$$

$$\tilde{\varepsilon}(z) = -\mathbb{E}[\varepsilon \mid 0 < \varepsilon < \theta z] \qquad \bar{\varepsilon} = \mathbb{E}[\varepsilon \mid \varepsilon > 0]$$

Proof. First, suppose $z > 0$. We call student i a switcher if they report $E \succ_i N$ at $\hat{z} = 0$ and report $N \succ_i E$ at $\hat{z} = z$. Because student i reports $N \succ_i E$ if and only if $\theta_i z + \varepsilon_i > 0$, student i is a switcher if and only if $-\theta_i z < \varepsilon_i < 0$. Due to the symmetry assumption $f(\theta, \varepsilon) = f(\theta, -\varepsilon)$, the set of switchers $\{(\theta_i, \varepsilon_i) \mid -\theta_i z < \varepsilon_i < 0\}$ is identical to the set of students $\{(\theta_i, \varepsilon_i) \mid 0 < \varepsilon_i < \theta_i z\}$, except that their ε_i have different signs. It follows that the mass of the switchers $M(z)$, the average valuation of the quality among the switchers $\tilde{\theta}(z)$, and the average taste for the new school among the switchers $\tilde{\varepsilon}(z)$ can be written as follows:

$$M(z) = \mathbb{E}[\mathbb{1}\{-\theta z < \varepsilon < 0\}] = \mathbb{E}[\mathbb{1}\{0 < \varepsilon < \theta z\}]$$

$$\tilde{\theta}(z) = \mathbb{E}[\theta \mid -\theta z < \varepsilon < 0] = \mathbb{E}[\theta \mid 0 < \varepsilon < \theta z]$$

$$\tilde{\varepsilon}(z) = \mathbb{E}[\varepsilon \mid -\theta z < \varepsilon < 0] = -\mathbb{E}[\varepsilon \mid 0 < \varepsilon < \theta z]$$

Because exactly half of students have $\varepsilon_i > 0$, the total mass of students reporting $N \succ E$ is $\frac{1}{2} + M(z)$. The probability share of school N is equally divided among these students, who have the average valuation of the quality and the average taste

$$\mathbb{E}[\theta \mid \varepsilon > -\theta z] = \frac{\frac{1}{2}\bar{\theta} + M(z)\,\tilde{\theta}(z)}{\frac{1}{2} + M(z)} \quad \text{and} \quad \mathbb{E}[\varepsilon \mid \varepsilon > -\theta z] = \frac{\frac{1}{2}\bar{\varepsilon} + M(z)\,\tilde{\varepsilon}(z)}{\frac{1}{2} + M(z)}.$$

Therefore, the average value of school N is

$$\mathbb{E}[\theta z + \varepsilon \mid \varepsilon > -\theta z] = \frac{\frac{1}{2}\bar{\theta} + M(z)\,\tilde{\theta}(z)}{\frac{1}{2} + M(z)} z + \frac{\frac{1}{2}\bar{\varepsilon} + M(z)\,\tilde{\varepsilon}(z)}{\frac{1}{2} + M(z)}$$

and the social welfare is $W(z) = q_E\,0 + q_N \mathbb{E}[\theta z + \varepsilon \mid \varepsilon > -\theta z]$. Given this, the value function, $V(z) = W(z) - q_N(\bar{\theta} z + \bar{\varepsilon})$, simplifies to

$$V(z) = \frac{\frac{1}{2}M(z)}{\frac{1}{2} + M(z)} \left[\left\{ \tilde{\theta}(z) z + \tilde{\varepsilon}(z) \right\} - \left\{ \bar{\theta} z + \bar{\varepsilon} \right\} \right].$$

Next, suppose $z < 0$. We call student i a switcher if they report $N \succ_i E$ at $\hat{z} = 0$ and report $E \succ_i N$ at $\hat{z} = z$. Because student i reports $E \succ_i N$ if and only if $\theta_i z + \varepsilon_i < 0$, student i is a switcher if and only if $0 < \varepsilon_i < -\theta_i z$. The mass of switchers is $\mathbb{E}[\mathbb{1}\{0 < \varepsilon < -\theta z\}]$, which equals $M(-z)$. Among the switchers, the average valuation of the quality is $\mathbb{E}[\theta \mid 0 < \varepsilon < -\theta z]$, which equals $\tilde{\theta}(-z)$, and the average idiosyncratic taste is $\mathbb{E}[\varepsilon \mid 0 < \varepsilon < -\theta z]$, which equals $-\tilde{\varepsilon}(-z)$.

Because exactly half of students have $\varepsilon_i < 0$, the total mass of students reporting $E \succ N$ is $\frac{1}{2} + M(-z)$. Among the students reporting $E \succ N$, the average valuation of the quality is

$$\mathbb{E}[\theta \mid \varepsilon < -\theta z] = \frac{\frac{1}{2}\bar{\theta} + M(-z)\,\tilde{\theta}(-z)}{\frac{1}{2} + M(-z)}.$$

The average idiosyncratic taste is

$$\mathbb{E}[\varepsilon \mid \varepsilon < -\theta z] = \frac{\frac{1}{2}(-\bar{\varepsilon}) + M(-z)\{-\tilde{\varepsilon}(-z)\}}{\frac{1}{2} + M(-z)}.$$

The mass of students reporting $N \succ E$ is $\frac{1}{2} - M(-z)$. Among these students, the average valuation of the quality is

$$\mathbb{E}[\theta \mid \varepsilon > -\theta z] = \frac{\frac{1}{2}\bar{\theta} - M(-z)\,\tilde{\theta}(-z)}{\frac{1}{2} - M(-z)}.$$

The average idiosyncratic taste is

$$\mathbb{E}[\varepsilon \mid \varepsilon > -\theta z] = \frac{\frac{1}{2}\bar{\varepsilon} - M(-z)\{-\tilde{\varepsilon}(-z)\}}{\frac{1}{2} - M(-z)}.$$

School N is under-demanded, and therefore, each of $\frac{1}{2} - M(-z)$ students reporting $N \succ E$ receives 1 probability share of school N. The remaining probability share, $M(-z)$, is equally allocated to the students reporting $E \succ N$. The average value of school N is the weighted sum between the two groups, and the social welfare is

$$W(z) = q_N \left\{ \frac{\frac{1}{2} - M(-z)}{\frac{1}{2}}\, \mathbb{E}[\theta z + \varepsilon \mid \varepsilon > -\theta z] + \frac{M(-z)}{\frac{1}{2}}\, \mathbb{E}[\theta z + \varepsilon \mid \varepsilon < -\theta z] \right\}.$$

Given this, the value function simplifies to

$$V(z) = \frac{\frac{1}{2}M(-z)}{\frac{1}{2} + M(-z)} \left[\left\{ \{\tilde{\theta}(-z)\}(-z) + \tilde{\varepsilon}(-z) \right\} - \left\{ \bar{\theta}(-z) + \bar{\varepsilon} \right\} \right].$$

This completes the proof. □

1.8.2 Proof of Lemma 2

Lemma 2. Suppose $\mathbb{E}[\varepsilon \mid \varepsilon > 0]$ is bounded. If $0 < F_\theta(0) < 1$ then $V(z) \to \infty$ as $z \to \pm\infty$.

Proof. Because the value function is symmetric around $z = 0$, we only need to prove the claim for $z > 0$. As stated in Lemma 1, the value function is

$$V(z) = \frac{\frac{1}{2}M(z)}{\frac{1}{2} + M(z)} \left[\left\{ \tilde{\theta}(z) - \bar{\theta} \right\} z + \tilde{\varepsilon}(z) - \bar{\varepsilon} \right].$$

We prove $V(z) \to \infty$ as $z \to \infty$ by showing (1) The mass of switchers, $M(z)$, converges to a strictly positive value as $z \to \infty$; (2) The difference in the average valuations of the quality between the switchers and those who are replaced by the switchers, $\tilde{\theta}(z) - \bar{\theta}$, converges to a strictly positive value as $z \to \infty$; and (3) The average taste for the new school among the switchers, $\tilde{\varepsilon}(z)$, is bounded below;.

First, we show that $M(z)$ converges to a positive value as $z \to \infty$. Recall that

$$M(z) = \mathbb{E}[\mathbb{1}\{0 < \varepsilon < \theta z\}] = \int_0^\infty f_\varepsilon(\varepsilon) \left\{ 1 - F_{\theta|\varepsilon}\left(\frac{\varepsilon}{z}\right) \right\} d\varepsilon.$$

The integrand approaches $f_\varepsilon(\varepsilon)\left\{1 - F_{\theta|\varepsilon}(0)\right\}$ as $z \to \infty$ and is dominated by $f_\varepsilon(\varepsilon)$. Thus, by the Dominated Convergence Theorem, we have

$$\lim_{z\to\infty} M(z) = \int_0^\infty f_\varepsilon(\varepsilon) \left\{ 1 - F_{\theta|\varepsilon}(0) \right\} d\varepsilon = \tfrac{1}{2} \left\{ 1 - F_\theta(0) \right\} > 0,$$

where the last inequality follows from $f(\theta, \varepsilon) = f(\theta, -\varepsilon)$.

Second, we show that $\tilde{\theta}(z) - \bar{\theta}$ converges to a positive value as $z \to \infty$. We have

$$\tilde{\theta}(z) = \mathbb{E}[\theta \mid 0 < \varepsilon < \theta z] = \frac{1}{M(z)} \int_0^\infty \theta \left\{ F_{\varepsilon|\theta}(\theta z) - \tfrac{1}{2} \right\} f_\theta(\theta)\, d\theta.$$

The integrand approaches $\frac{1}{2}\theta f_\theta(\theta)$ as $z \to \infty$ and is dominated by $\frac{1}{2}\theta f_\theta(\theta)$. Thus, by the Dominated Convergence Theorem, we have

$$\lim_{z\to\infty} \tilde{\theta}(z) = \frac{\frac{1}{2}\bar{\theta}}{\frac{1}{2}\left\{1 - F_\theta(0)\right\}} = \frac{\bar{\theta}}{1 - F_\theta(0)}.$$

Consequently,

$$\lim_{z\to\infty} \left\{ \tilde{\theta}(z) - \bar{\theta} \right\} = \frac{F_\theta(0)}{1 - F_\theta(0)}\, \bar{\theta} > 0.$$

Third, we show that $\tilde{\varepsilon}(z)$ is bounded below. Recall that

$$\tilde{\varepsilon}(z) = -\mathbb{E}[\varepsilon \mid 0 < \varepsilon < \theta z] = -\frac{1}{M(z)} \int_0^\infty \varepsilon f_\varepsilon(\varepsilon) \left\{ 1 - F_{\theta|\varepsilon}\left(\frac{\varepsilon}{z} \right) \right\} d\varepsilon.$$

The integrand approaches $\varepsilon f_\varepsilon(\varepsilon)\left\{1 - F_{\theta|\varepsilon}(0)\right\}$ as $z \to \infty$ and is dominated by $\varepsilon f_\varepsilon(\varepsilon)$. Thus, by the Dominated Convergence theorem, we have

$$\begin{aligned}
\lim_{z\to\infty} \tilde{\varepsilon}(z) &= -\frac{1}{\frac{1}{2}\left\{1 - F_\theta(0)\right\}} \int_0^\infty \varepsilon f_\varepsilon(\varepsilon) \left\{1 - F_{\theta|\varepsilon}(0)\right\} d\varepsilon \\
&> -\frac{1}{\frac{1}{2}\left\{1 - F_\theta(0)\right\}} \int_0^\infty \varepsilon f_\varepsilon(\varepsilon) d\varepsilon \\
&= -\frac{\bar{\varepsilon}}{1 - F_\theta(0)}.
\end{aligned}$$

Given these, $\left\{ \tilde{\theta}(z) - \bar{\theta} \right\} z$ is unbounded, while all other terms are bounded. It follows that $V(z) \to \infty$ as $z \to \infty$. □

1.8.3 Proof of Lemma 3

Lemma 3. Suppose $\mathbb{E}[\varepsilon \mid \varepsilon > 0]$ is bounded. If $f_{\theta|\varepsilon}(\cdot)$ is bounded for all $\varepsilon \in supp(f_\varepsilon)$ then

$$\lim_{z \to \pm\infty} V(z) = \tfrac{1}{4}\left\{\bar{\theta}\,\mathbb{E}[\varepsilon f_{\theta|\varepsilon}(0) \mid \varepsilon > 0] - 2\bar{\varepsilon}\right\}.$$

In particular, if θ and ε are independent and $f_\theta(\cdot)$ is bounded, then

$$\lim_{z \to \pm\infty} V(z) = \tfrac{1}{4}\bar{\varepsilon}\left\{\bar{\theta} f_\theta(0) - 2\right\}.$$

Proof. Because the value function is symmetric around $z = 0$, we only need to prove the claim for $z > 0$. From Lemma 1, the value function is

$$V(z) = \frac{\frac{1}{2}M(z)}{\frac{1}{2} + M(z)}\left[\left\{\tilde{\theta}(z) - \bar{\theta}\right\} z + \tilde{\varepsilon}(z) - \bar{\varepsilon}\right].$$

We separately compute the limit values of $M(z)$, $\tilde{\varepsilon}(z)$, and $\left\{\tilde{\theta}(z) - \bar{\theta}\right\} z$.

First, we show that $M(z) \to \frac{1}{2}$ as $z \to \infty$. Recall that

$$M(z) = \mathbb{E}[\mathbb{1}\{0 < \varepsilon < \theta z\}] = \int_0^\infty f_\varepsilon(\varepsilon)\left\{1 - F_{\theta|\varepsilon}\left(\frac{\varepsilon}{z}\right)\right\} d\varepsilon.$$

The integrand approaches $f_\varepsilon(\varepsilon)\left\{1 - F_{\theta|\varepsilon}(0)\right\} = f_\varepsilon(\varepsilon)$ as $z \to \infty$ and is dominated by $f_\varepsilon(\varepsilon)$. Thus, by the Dominated Convergence Theorem, we have

$$\lim_{z \to \infty} M(z) = \int_0^\infty f_\varepsilon(\varepsilon)\, d\varepsilon = \tfrac{1}{2}.$$

Second, we show that $\tilde{\varepsilon}(z) \to -\bar{\varepsilon}$ as $z \to \infty$. Recall that

$$\tilde{\varepsilon}(z) = -\mathbb{E}[\varepsilon \mid 0 < \varepsilon < \theta z] = -\frac{1}{M(z)} \int_0^\infty \varepsilon f_\varepsilon(\varepsilon) \left\{1 - F_{\theta|\varepsilon}\left(\frac{\varepsilon}{z}\right)\right\} d\varepsilon.$$

The integrand approaches $\varepsilon f_\varepsilon(\varepsilon)\left\{1 - F_{\theta|\varepsilon}(0)\right\} = \varepsilon f_\varepsilon(\varepsilon)$ as $z \to \infty$ and is dominated by $\varepsilon f_\varepsilon(\varepsilon)$. Thus, by the Dominated Convergence Theorem, we have

$$\lim_{z\to\infty} \tilde{\varepsilon}(z) = -\frac{1}{1/2} \int_0^\infty \varepsilon f_\varepsilon(\varepsilon) d\varepsilon = -\bar{\varepsilon}.$$

Third, we show that $\left\{\tilde{\theta}(z) - \bar{\theta}\right\} z$ converges to $\bar{\theta}\,\mathbb{E}\left[\varepsilon f_{\theta|\varepsilon}(0) \mid \varepsilon > 0\right]$ as $z \to \infty$. We can write

$$M(z) = \mathbb{E}[\mathbb{1}\{0 < \varepsilon < \theta z\}] = \int_0^\infty \int_{\frac{\varepsilon}{z}}^\infty f_{\theta|\varepsilon}(\theta) f_\varepsilon(\varepsilon) d\theta \, d\varepsilon$$

$$\tilde{\theta}(z) = \mathbb{E}[\theta \mid 0 < \varepsilon < \theta z] = \frac{1}{M(z)} \int_0^\infty \int_{\frac{\varepsilon}{z}}^\infty \theta f_{\theta|\varepsilon}(\theta) f_\varepsilon(\varepsilon) d\theta \, d\varepsilon$$

and therefore,

$$\left\{\tilde{\theta}(z) - \bar{\theta}\right\} z = \frac{z}{M(z)} \int_0^\infty \int_{\frac{\varepsilon}{z}}^\infty \left\{\theta - \bar{\theta}\right\} f_{\theta|\varepsilon}(\theta) f_\varepsilon(\varepsilon) d\theta \, d\varepsilon.$$

As z goes to infinity, $\frac{z}{M(z)}$ approaches ∞, while the integral approaches 0. Thus, the limit value is indeterminate. In preparation for applying L'Hôpital's rule, we let $y = \frac{1}{z}$ and write

$$\lim_{z\to\infty} \left\{\tilde{\theta}(z) - \bar{\theta}\right\} z = \lim_{y\to 0} \frac{1}{yM(1/y)} \int_0^\infty \int_{\varepsilon y}^\infty \left\{\theta - \bar{\theta}\right\} f_{\theta|\varepsilon}(\theta) f_\varepsilon(\varepsilon) d\theta \, d\varepsilon.$$

For the denominator, we have

$$\frac{d}{dy}\left[yM(1/y)\right] = M(1/y) + y\,\frac{d}{dy}\left[\int_0^\infty \int_{\varepsilon y}^\infty f_{\theta|\varepsilon}(\theta) f_\varepsilon(\varepsilon) d\theta\, d\varepsilon\right]$$
$$= M(1/y) + y \int_0^\infty (-\varepsilon) f_{\theta|\varepsilon}(\varepsilon y) f_\varepsilon(\varepsilon)\, d\varepsilon,$$

which approaches $\frac{1}{2}$ as $y \to 0$.

For the numerator, we have

$$\frac{d}{dy}\left[\int_0^\infty \int_{\varepsilon y}^\infty \left\{\theta - \bar{\theta}\right\} f_{\theta|\varepsilon}(\theta) f_\varepsilon(\varepsilon) d\theta\, d\varepsilon\right] = \int_0^\infty -\varepsilon \left\{\varepsilon y - \bar{\theta}\right\} f_{\theta|\varepsilon}(\varepsilon y) f_\varepsilon(\varepsilon)\, d\varepsilon.$$

The integrand converges to $\varepsilon\bar{\theta} f_{\theta|\varepsilon}(0) f_\varepsilon(\varepsilon)$ as $y \to 0$. Furthermore, because $f_{\theta|\varepsilon}(\cdot)$ is bounded for each ε, there is $\bar{f}$ such that $f_{\theta|\varepsilon}(\theta) \leq \bar{f}$ for all θ and ε. Then the integrand is bounded by $\varepsilon\bar{\theta}\bar{f} f_\varepsilon(\varepsilon)$, and this function is integrable. Thus, by the Dominated Convergence Theorem, the derivative of the numerator approaches

$$\bar{\theta}\int_0^\infty \varepsilon f_{\theta|\varepsilon}(0) f_\varepsilon(\varepsilon) = \tfrac{1}{2}\bar{\theta}\,\mathbb{E}\left[\varepsilon f_{\theta|\varepsilon}(0) \mid \varepsilon > 0\right]$$

as $y \to 0$.

By applying L'Hôpital's rule, we obtain

$$\lim_{z\to\infty}\left\{\tilde{\theta}(z) - \bar{\theta}\right\} z = \bar{\theta}\,\mathbb{E}\left[\varepsilon f_{\theta|\varepsilon}(0) \mid \varepsilon > 0\right],$$

and the stated limit value follows. □

1.9 Appendix: Equilibrium Analysis

1.9.1 Details of Example 1

There are three schools, A, B, and C, with capacities $q_A = q_B = q_C = \frac{1}{3}$, to be allocated to a unit continuum of students. The utility of attending school C is 0 for each student: $\varepsilon_{iC} = 0$ for all i and $z_C = 0$ with probability 1. The valuation of quality, θ_i, is 1 for one half of the students (quality-sensitive students) and 0 for the other half (quality-insensitive students). The idiosyncratic tastes, ε_{iA} and ε_{iB}, are distributed independently across individuals and across schools. Specifically, $(\varepsilon_{iA}, \varepsilon_{iB}) \sim U[0,1]^2$. The school qualities, z_A and z_B, are independently distributed from each other and $(z_A, z_B) \sim U[0,2]^2$.

For any realizations of $(\varepsilon_{iA}, \varepsilon_{iB})$ and (z_A, z_B), school C is unanimously the worst school, and therefore, it is always under-demanded. Thus, the only sensible reports are $A \succ B \succ C$ and $B \succ A \succ C$. We ignore other outcome-equivalent preference orders to keep the analysis simple.

Under the DA, student i reports $A \succ B \succ C$ if and only if $u_{iA} > u_{iB}$, where $u_{iA} = \theta_i \hat{z}_A + \varepsilon_{iA}$ and $u_{iB} = \theta_i \hat{z}_B + \varepsilon_{iB}$. When $\hat{z}_A = \hat{z}_B$, exactly one half of students report $A \succ B \succ C$ because ε_{Ai} and ε_{Bi} are symmetric. As $|\hat{z}_A - \hat{z}_B|$ increases, the fraction of the quality-sensitive students pursuing the highest-quality school increases, and it reaches 1 when $|\hat{z}_A - \hat{z}_B| = 1$.

Without loss of generality, suppose $\hat{z}_A > \hat{z}_B$. In the equilibrium, the allocation for students who report $A \succ B \succ C$ is $x_{iA} = \frac{q_A}{D_A^*}$, $x_{iB} = q_A + q_B - \frac{q_A}{D_A^*}$, and $x_{iC} = q_C$, where $D_A^* = \int \mathbb{1}\{u_{iA} > u_{iB}\} \, dF(\omega_i)$. The allocation for students who report $B \succ A \succ C$ is $x_{iA} = 0$, $x_{iB} = q_A + q_B$, and $x_{iC} = q_C$.

The resulting welfare is plotted in Figure 1.10 in the main text. The welfare function has a local peak at $z_A = z_B$, where school A and B are equally over-demanded. Indeed, quality vectors (z_A, z_B) that satisfy $z_A = z_B$ are similar to the ideal point z^0 we discuss in Section 1.3. The optimal disclosure policy pools the values of $z_A - z_B$ in the interval $[-1.34, 1.34]$.

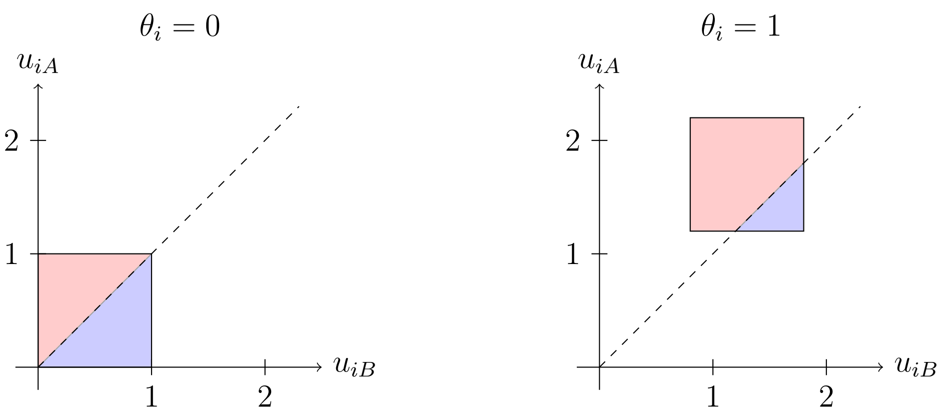
Under the IA, the students report strategically. Suppose $\hat{z}_A > \hat{z}_B$. We assume both school A and B are over-demanded in the equilibrium, and later confirm this assumption. When both school A and B are over-demanded, each student receives a probability share of either school A or B, not both (i.e., at the beginning of the second round of the simultaneous-eating algorithm, their second-most-preferred school is already exhausted). The allocation for students who report $A \succ B \succ C$ is $x_{iA} = \frac{q_A}{D_A^*}$, $x_{iB} = 0$, and $z_{iC} = 1 - \frac{q_A}{D_A^*}$. The allocation for students who report $B \succ A \succ C$ is $x_{iA} = 0$, $x_{iB} = \frac{q_B}{D_B^*}$, and $z_{iC} = 1 - \frac{q_B}{D_B^*}$. Given this, student i reports $A \succ B \succ C$ if and only if

$$\frac{q_A}{D_A^*} u_{iA} > \frac{q_B}{D_B^*} u_{iB} \quad \Leftrightarrow \quad \frac{u_{iA}}{u_{iB}} > \frac{D_A^*}{D_B^*},$$

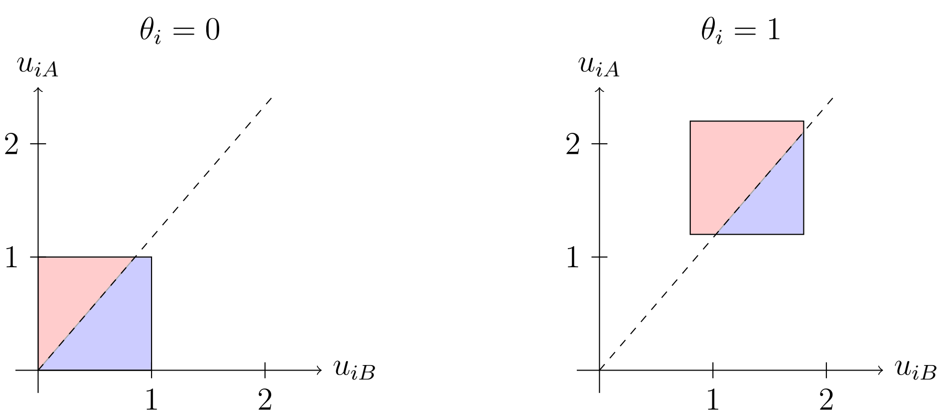
where D_A^* and D_B^* are consistent with this decision rule. It can be confirmed that, there is a unique equilibrium for each value of $(\hat{z}_A, \hat{z}_B)$, and school A and B are indeed over-demanded in the equilibrium, i.e., $D_A^* \geq q_A$ and $D_B^* \geq q_B$.

The resulting welfare is plotted in Figure 1.10 in the main text. The welfare function is convex under the IA. The disturbance to the horizontal sorting is mitigated under the IA because the IA discourages the students who are relatively indifferent between school A and B from pursuing the highest-quality school. As a result, the gain from improved vertical sorting always exceeds the loss from disturbed horizontal sorting. It follows that the full-disclosure policy is optimal.

The equilibrium strategies under the DA and the IA are contrasted in Figure 1.12 for $(\hat{z}_A, \hat{z}_B) = (1.2, 0.8)$. The utility vector for the quality-insensitive students is $(u_{iA}, u_{iB}) = (\varepsilon_{iA}, \varepsilon_{iB}) \sim U[0,1]^2$, a unit square. For the quality-sensitive students, the square is shifted up by $z_{iA} = 1.2$ and to the right by $z_{iB} = 0.8$. The students shaded in red report $A \succ B \succ C$ while the students shaded in blue report $B \succ A \succ C$ in the equilibrium.



(a) Deferred Acceptance



(b) Immediate Acceptance

Figure 1.12: The equilibrium strategy when $(\hat{z}_A, \hat{z}_B) = (1.2, 0.8)$.

Under the DA, the slope of the dividing line between the students reporting $A \succ B \succ C$ and the students reporting $B \succ A \succ C$ is 1, and the fraction of students reporting $A \succ B \succ C$ is 0.660. Under the IA, the fraction of students reporting $A \succ B \succ C$ is 0.539, and the slope of the dividing line is $0.539/0.451 = 1.17$. In other words, the students pursue school A if and only if $u_{iA}/u_{iB} > 1.17$. Notice that the quality-insensitive students actively avoid reporting $A \succ B \succ C$. This is because they can secure a larger probability share by reporting $B \succ A \succ C$.

Under both the DA and the IA, as $|\hat{z}_A - \hat{z}_B|$ increases, more students pursue the highest-quality school, but at different rates. Under the DA, all quality-sensitive students pursue the highest-quality school when $|\hat{z}_A - \hat{z}_B| \geq 1$. Under the IA, students are discouraged from crowding, and it is not until $|\hat{z}_A - \hat{z}_B| \geq 1.78$ that all quality-sensitive students pursue the highest-quality school. Also, under the IA, the fraction of quality-insensitive students who actively avoid the highest-quality school increases as $|\hat{z}_A - \hat{z}_B|$ increases, keeping the middle-quality school over-demanded.

1.9.2 Details of Example 2

There are three schools, A, B, and C, with capacities $q_A = q_B = q_C = \frac{1}{3}$, to be allocated to a unit continuum of students. The utility of attending school A is 1 for all students: $\varepsilon_{iA} = 1$ for all i and $z_A = 0$ with probability 1. Suppose the valuation of quality, θ_i, is 1 for one half of the students (quality-sensitive students) and 0 for the other half (quality-insensitive students). The quality-sensitive students prefer school B to C. Specifically, $\varepsilon_{iB} \sim U[0, 1]$ and $\varepsilon_{iC} = -1$. The quality-insensitive students prefer school C to B. Specifically, $\varepsilon_{iC} \sim U[0, 1]$ and $\varepsilon_{iB} = -1$. It is known that $z_C = 0$ with certainty and $z_B \sim U[0, 1]$.

Under the DA, the quality-insensitive students truthfully report $A \succ C \succ B$. A quality-sensitive student i reports $B \succ A \succ C$ if and only if $u_{iB} > u_{iA}$, equivalently, $\hat{z}_B + \varepsilon_{iB} \geq 1$. The cutoff type, $\varepsilon^{DA} = 1 - \hat{z}_B$, is indifferent between reporting $B \succ A \succ C$ and reporting $A \succ B \succ C$. The fraction of students who un-crowd from A is 0 when $\hat{z}_B = 0$, it increases linearly in $\hat{z}_B$, and it reaches 1 when $\hat{z}_B = 1$.

In the DA equilibrium (in which all students report truthfully), the demand for school A is $D_A^* = \frac{1}{2} + \frac{1}{2}\varepsilon^{DA}$ and the demand for school B is $D_B^* = \frac{1}{2}(1 - \varepsilon^{DA})$. The equilibrium allocation under the DA is shown in the table below, where $\alpha = \frac{q_A}{D_A^*}$, the per-student share of school A, and $\beta = \frac{q_B - \alpha D_B^*}{1/2}$, the per-student share of school B in the second round of simultaneous eating.

Report	x_{iA}	x_{iB}	x_{iC}
$A \succ B \succ C$	α	β	$1 - \alpha - \beta$
$B \succ A \succ C$	0	$\alpha + \beta$	$1 - \alpha - \beta$
$A \succ C \succ B$	α	0	$1 - \alpha$

Under the IA, for the quality-insensitive students, truthfully reporting $A \succ C \succ B$ is the optimal strategy because, for any belief $\hat{z}_B \in [0, 1]$, school C is least demanded, and therefore, it is effectively free. For the quality-sensitive students, let ε^{IA} be the cutoff type: the students with $\varepsilon_{iB} \leq \varepsilon^{IA}$ report $A \succ B \succ C$ and the students with $\varepsilon_{iB} > \varepsilon^{IA}$ report $B \succ A \succ C$.

Suppose both school A and B are over-demanded in the equilibrium. Then the expected utility from reporting $A \succ B \succ C$ for student i is

$$\frac{q_A}{D_A} 1 + \left(1 - \frac{q_A}{D_A}\right)(-1), \quad \text{where } D_A = \frac{1}{2} + \frac{1}{2}\varepsilon^{IA}.$$

The expected utility from reporting $B \succ A \succ C$ is

$$\frac{q_B}{D_B}(\varepsilon_{iB} + \hat{z}_B) + \left(1 - \frac{q_B}{D_B}\right)(-1), \quad \text{where } D_B = \frac{1}{2}(1 - \varepsilon^{IA}).$$

The cutoff type ε^{IA} is indifferent between the two reports, and it is identified by

$$\varepsilon^{IA} = \frac{1}{2}\left[-4 - \hat{z}_B + \sqrt{20 + 4\hat{z}_B + \hat{z}_B^2}\right].$$

At $\hat{z}_B = 0$, we have $\varepsilon^{IA} = \sqrt{5} - 2 \approx 0.236$. In other words, fraction 0.764 of quality-sensitive students report $B \succ A \succ C$ when $\hat{z}_B = 0$ (recall that this fraction is 0 under the DA). The fraction of quality-sensitive students who report $B \succ A \succ C$ increases as $\hat{z}_B$ increases, and it reaches 1 when $\hat{z}_B = 1$. This is consistent with the assumption that $D_B \geq q_B = \frac{1}{3}$ for all $\hat{z}_B \in [0, 1]$.

For $\hat{z}_B \in (0, 1]$, this is the unique equilibrium. A quality-sensitive student i reports $A \succ_i B \succ_i C$ if $\varepsilon_{iB} \leq \varepsilon^{IA}$ and reports $B \succ_i A \succ_i C$ if $\varepsilon_{iB} > \varepsilon^{IA}$. The equilibrium allocation under the IA is shown in the table below, where D_A^* and D_B^* are equilibrium demands.

Report	x_{iA}	x_{iB}	x_{iC}
$A \succ B \succ C$	$\frac{q_A}{D_A^*}$	0	$1 - \frac{q_A}{D_A^*}$
$B \succ A \succ C$	0	$\frac{q_B}{D_B^*}$	$1 - \frac{q_B}{D_B^*}$
$A \succ C \succ B$	$\frac{q_A}{D_A^*}$	0	$1 - \frac{q_A}{D_A^*}$

When $\hat{z}_B = 0$, there is another equilibrium, in which all quality-sensitive students report $A \succ B \succ C$ and all quality-insensitive students report $A \succ C \succ B$.

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CHAPTER 2

Decomposition with Collocation Constraints

2.1 Introduction

Randomization is an essential part of fair resource allocation when the objects to be allocated are indivisible. When there is only one type of object to be allocated, a simple lottery restores fairness. For example, the Diversity Immigrant Visa program allocates U.S. permanent residency through what is commonly known as the green card lottery. When there are multiple types of objects to be allocated, the randomization needs to be used in conjunction with an arrangement algorithm that considers the preferences of the applicants. For example, school-choice programs aim to achieve fair allocation of seats at public schools through matching algorithms.[1]

There are two ways to integrate a randomization and a preference-based arrangement. In most allocation algorithms used in real life, the randomization precedes the preference-based arrangement: A lottery is first used to randomly assign an artificial priority order over the applicants, then a deterministic assignment algorithm is applied respecting the priority order and the preferences of the applicants. One prominent algorithm is the Random Serial Dictatorship (RSD), in which applicants

[1]The seminal paper Abdulkadiroğlu and Sönmez (2003) introduced school-choice programs to the mechanism design literature, and it has significantly expanded since then.

are randomly ordered first, then they sequentially pick their favorite object. In contrast, stochastic assignment mechanisms, such as Probabilistic Serial (PS) rule, integrate the randomization with the preference-based arrangement, and directly specify which applicant receives which object with what probability.

Although the PS has superior efficiency over the RSD,[2] it is rarely used in reality. There are two likely reasons. First, an average person can understand the working of the RSD very easily, but they may have difficulty understanding the working of the PS because it involves probability, a concept unfamiliar to many people. Therefore, the PS is less likely to be adopted if decision makers are concerned about participant alienation. Second, the RSD can be more easily implemented than the PS because the RSD returns a deterministic assignment, which is ready for implementation, whereas the PS returns a stochastic assignment, which requires some post-processing before implementation. Kesten et al. (2017) offers a remedy to these shortcomings of the PS by reformulating the PS as a variant of the RSD.

We argue that stochastic assignments are actually preferred to deterministic assignments as an output from allocation algorithms because they offer a greater flexibility in implementation. We explain this perspective using an example in school-choice programs. Suppose there are four students, 1 through 4, and two schools, A and B, each with 2 seats. Every student demands one seat and prefers school A to B. Also suppose students 1 and 2 are twins and their parents wish them to be assigned to the same school.

[2]This observation is first formalized in Bogomolnaia and Moulin (2001), in which the authors introduce the notion of stochastic-dominance efficiency (in their original term, ordinal efficiency) and develop the Probabilistic Serial rule. Erdil and Ergin (2008) makes a similar observation in the context of two-sided matching: An arbitrary tie-breaking in Deferred Acceptance algorithm—which coincides with the RSD when there is only one priority class—introduces artificial stability constraint, which in turn compromises the efficiency.

Suppose we employ the RSD disregarding the collocation of the twins. There are $4! = 24$ possible orderings of the students, of which the twins are assigned to the same school in only 8 orderings. In other words, the twins are separated into different schools with $\frac{2}{3}$ probability. To ensure twin collocation, the RSD algorithm needs to be substantially modified, which may render the modified RSD more enigmatic to the participants of the program.[3]

In contrast, the PS returns a stochastic assignment matrix which simply specifies that each student should be assigned to school A with $\frac{1}{2}$ probability. To implement this, we flip a coin to decide which two students are to be assigned to school A, the twins or students 3 and 4. Notice that twin collocation is achieved without any cost: We run the PS as it is, take the resulting stochastic assignment matrix, and decompose it into a lottery over deterministic assignments—which needs to be done anyway—keeping in mind the twin collocation. Therefore, the implementation is much simpler for the PS.

In this paper, we take the stochastic assignment matrices as given and aim to accommodate social desiderata such as collocation of twins during the decomposition step. In the first half of the paper, we examine the feasibility of twin collocation in school-choice programs. We show that any stochastic assignment matrix can be decomposed with guaranteed twin collocation if one extra seat can be added to each school. This is true for any number of pairs of twins in the student body. We then examine the related problem of collocating students from the same community. We

[3]Alternatively, we can keep the RSD algorithm as it is and compute the stochastic assignment matrix by enumerating all possible orderings of the students and corresponding deterministic assignments, then decompose the stochastic assignment matrix into a new lottery over deterministic assignments. However, this approach is computationally prohibitive even for a moderate number of students. For example, with 20 students, there are $20! \approx 10^{18}$ possible orderings.

show that two extra seats at each school is sufficient to guarantee each student a company of at least one peer from their community, provided that students from the same community have the same stochastic allocation.

These results are qualitatively similar to Nguyen and Vohra (2018), in which the authors show that stability can be restored in the National Resident Matching problem with couples (who have joint preferences over pairs of hospitals) if the capacity of each hospital can be increased by two, not exceeding four in total. However, twin collocation in one-sided matching we study here (where twins must be assigned to the same school) is fundamentally different from couple collocation in two-sided matching (where couples prefer to be assigned to different hospitals in the same city). Our result shows that twin collocation may be impossible even in large markets, while Kojima et al. (2013) shows that there always exists a stable matching as long as the number of couples is small relative to the market size.

In the second half of the paper, we ask the question "When twin collocation can be guaranteed, how do we find the lottery decomposition that implements the given stochastic assignment?" We first show that straightforward modifications of existing algorithms fail. We then propose a new algorithm based on Column Generation, a technique commonly used in the field of operations research.

A similar effort is made in Ashlagi and Shi (2014), in which the authors provide a decomposition heuristic that maximizes the average number of peers from the same community assigned to the same school. Their solution technique is a clever generalization of the existing decomposition algorithm that is tailored specifically toward community cohesion, whereas our algorithm based on Column Generation is more general and can be used to accommodate any quantifiable assignment objectives.

Here are some examples of social desiderata that can be integrated into the decomposition step. Given a stochastic assignment matrix, an education board can explicitly balance the racial composition at each school instead of leaving it to chance.[4] The total cost of providing bus transportation can be minimized by integrating the bus-route planning into the decomposition step. Sibling priorities can be exercised more effectively if, in any given year, students who have younger siblings are deliberately assigned to many different schools so that their younger siblings have a guaranteed seat at the same school in subsequent years.

This paper is similar to Balbuzanov (2019) in that both aim to accommodate all types of constraints. The author converts an arbitrary set of constraints to an equivalent set of additive constraints with upper bounds. Then the simultaneous-eating algorithm (Bogomolnaia and Moulin, 2001) is applied subject to these constraints. Essentially, all permissible deterministic assignments are implicitly listed first, then the weights on these assignments are adjusted by varying the eating speeds. In contrast, our decomposition algorithm can be added to existing stochastic assignment mechanisms without any alterations to the assignment mechanisms.

This paper is organized as follows. We formally introduce allocation problems in Section 2.2 and define decomposability of stochastic assignment matrices. We introduce collocation constraints in Section 2.3 and examine the feasibility of decomposition subject to these constraints. We demonstrate the limitations of existing algorithms in Section 2.4 and propose a new decomposition algorithm in Section 2.5. We conclude in Section 2.6.

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[4] In this paper, we take the stochastic assignment as given, while in the controlled-choice literature, hard or soft constraints are imposed on the assignments to be implemented. Also, in the controlled-choice literature, the primary focus is on the preservation of stability, which does not exist in one-sided matching we study here.

2.2 Preliminary

We write our model using the terminology in school-choice programs, but the model can be applied to any indivisible-goods allocation problems. Let $I = \{1, ..., n\}$ denote the set of students and $S = \{s_1, ..., s_m\}$ the set of schools. Let q_s denote the capacity (i.e., the number of seats) at school $s \in S$. We assume $\sum_{s \in S} q_s \geq n$. Each student has strict preference over schools, and demands exactly one seat. Let $\mathcal{P}$ denote the set of strict preferences and $\mathcal{P}^I$ the set of preference profiles.

A stochastic assignment matrix $X \equiv (x_{is})_{i \in I, s \in S}$ specifies the probability x_{is} with which student i is assigned to school s for each $i \in I$ and $s \in S$. A stochastic assignment matrix $X \equiv (x_{is})_{i \in I, s \in S}$ is feasible if it satisfies the following constraints:

$$x_{is} \geq 0 \qquad \forall i \in I,\ s \in S \qquad \text{(C1)}$$

$$\sum_{s \in S} x_{is} = 1 \qquad \forall i \in I \qquad \text{(C2)}$$

$$\sum_{i \in I} x_{is} \leq q_s \qquad \forall s \in S \qquad \text{(C3)}$$

Let $\mathcal{X} \equiv \{X \in \mathbb{R}^{nm} \mid (C1), (C2), (C3)\}$ denote the set of all feasible stochastic assignments. An ordinal assignment mechanism is a mapping from $\mathcal{P}^I$ to $\mathcal{X}$. When a mechanism returns a matrix that contains non-integral values, the matrix needs to be decomposed into a lottery over deterministic assignment matrices, which satisfy

$$x_{is} \in \{0, 1\} \qquad \forall i \in I,\ s \in S \qquad \text{(C1}'\text{)}$$

Let $\mathcal{Z} \equiv \{X \in \mathbb{R}^{nm} \mid (C1'), (C2), (C3)\}$ denote the set of all feasible deterministic assignments.

By an extension of the Birkhoff-von Neumann theorem (Birkhoff, 1946; Von Neumann, 1953), it can be proved that each $X \in \mathcal{X}$ has a lottery decomposition over $\mathcal{Z}$. That is, for each $X \in \mathcal{X}$, there is a set of non-negative weights $(w^Z)_{Z \in \mathcal{Z}}$ such that $X = \sum_{Z \in \mathcal{Z}} w^Z Z$. We say **the decomposition includes Z** or **Z is in the decomposition** if $w^Z > 0$.

We are interested in the decomposability of stochastic assignment matrices when additional constraints are imposed. Given a set $\mathcal{C}$ of constraints, let $\mathcal{X}^{\mathcal{C}}$ denote the subset of $\mathcal{X}$ that satisfy $\mathcal{C}$, and let $\mathcal{Z}^{\mathcal{C}}$ denote the subset of $\mathcal{Z}$ that satisfy $\mathcal{C}$.

Definition 1. For each $X \in \mathcal{X}^{\mathcal{C}}$, we say **$X$ is decomposable respecting $\mathcal{C}$** if there is a set of non-negative weights $(w^Z)_{Z \in \mathcal{Z}^{\mathcal{C}}}$ such that $X = \sum_{Z \in \mathcal{Z}^{\mathcal{C}}} w^Z Z$.

Budish et al. (2013) identifies the sets of constraints that can be imposed on assignment matrices without compromising decomposability. The authors consider *additive constraints* of the form $\underline{q}_H \leq \sum_{(i,s) \in H} x_{is} \leq \overline{q}_H$, where H is a subset of student-school pairs $\{(i,s) \in I \times S\}$ and is called a constraint set. A collection $\mathcal{H}$ of constraint sets form a hierarchy (Laminar set family) if each pair of constraint sets $H_1, H_2 \in \mathcal{H}$ are either disjoint ($H_1 \cap H_2 = \emptyset$) or related by inclusion ($H_1 \subseteq H_2$ or $H_2 \subseteq H_1$). The authors show that assignment matrices are decomposable respecting a set of additive constraints on a collection $\mathcal{H}$ of constraint sets if and only if $\mathcal{H}$ forms a bi-hierarchy, i.e., $\mathcal{H}$ can be partitioned into two hierarchies.[5]

Although this result is useful, there are many practically important constraints that take non-additive forms. In this paper, we address one of such constraints, namely, collocation of groups of students.

[5]Building on this result, Akbarpour and Nikzad (2020) shows that one may add soft constraints that do not conform to the bi-hierarchy, and the soft constraints can be approximately satisfied.

2.3 Decomposition with Collocation Constraints

In this section, we show that assignment matrices are in general not decomposable respecting collocation constraints, and non-decomposability can persist in large markets. However, stochastic assignment matrices can be decomposed into a lottery over deterministic assignments that violate the capacity constraints by a small amount. In Section 2.3.1, we analyze the twin constraints, which require twins to be assigned to the same school. In Section 2.3.2, we analyze the company constraints, which require that each student has a company of at least one peer from their community at their assigned school.

2.3.1 Twin Constraints

There are many situations in which collocation of groups of students is desirable. For example, parents of twins and triplets often want their children to attend the same school. Two families living next to each other may want their children to attend the same school so that they can have joint tutoring sessions. If there are a couple of students in the city who are especially talented in chess, collocating them to the same school can facilitate their interaction with each other.

For exposition purposes, we call the groups of students to be collocated twins, triplets, etc. For a pair of twins i and i', the twin constraint requires the following:

$$x_{is} = x_{i's} \qquad\qquad \forall s \in S.$$

We first show that assignment matrices are in general not decomposable respecting the twin constraint.

Example 1 (Twin constraint)**.** Suppose there are four students, 1 through 4, and two schools, A and B, with capacities $q_A = q_B = 2$. Students 1 and 2 are twins and they must be assigned to the same school. The table below shows the preference profile and the stochastic assignment matrix we wish to implement.

	Preference	x_{iA}	x_{iB}
Student 1	$A \succ B$	$\frac{2}{3}$	$\frac{1}{3}$
Student 2	$A \succ B$	$\frac{2}{3}$	$\frac{1}{3}$
Student 3	$A \succ B$	$\frac{2}{3}$	$\frac{1}{3}$
Student 4	$B \succ A$	0	1

In any decomposition, student 4 must be assigned to school B in all deterministic assignments because $x_{4B} = 1$. Also, a decomposition must include some deterministic assignment in which students 1 and 2 are assigned to school B because $x_{1B} > 0$ and $x_{2B} > 0$. However, students 1 and 2 cannot be assigned to school B together because there are only two seats at school B and one of them is always occupied by student 4. Therefore, this assignment matrix is not decomposable respecting the twin constraint.

This example can be scaled up while maintaining the same structure in the preference profile. That is, neither increasing the total number of students nor increasing the number of seats at each school resolves the non-decomposability.

Proposition 1. Suppose there are $m \geq 2$ schools, each with capacity $\underline{q} \geq 1$ or larger. Also suppose there are $\sum_{s \in S} q_s$ students, two of which form a pair of twins. For any finite $\underline{q}$, there is a preference profile for which the PS assignment matrix is not decomposable respecting the twin constraint.

Proof. Arbitrarily label the schools $s_1, ..., s_m$. Consider the following preference profile. The twins and additional $q_{s_1} - 1$ students prefer school s_1 to s_2 to all other schools. There are $q_{s_2} - 1$ students who prefer s_2 to all other schools. For $j \in \{3, ..., m\}$, there are exactly q_{s_j} students who prefer s_j to all other schools.

Under the PS, each student who prefers school s_1 to s_2 to all other schools is assigned to school s_1 with probability $\frac{q_{s_1}}{q_{s_1}+1}$ and assigned to school s_2 with probability $1-\frac{q_{s_1}}{q_{s_1}+1}$. All other students are assigned to their most-preferred school with certainty. In particular, $q_{s_2} - 1$ students must be assigned to school s_2 in all deterministic assignments in a decomposition. This is incompatible with the requirement that the twins must be assigned to school s_2 together in some deterministic assignments because there is only one unoccupied seat. □

It takes only one pair of twins to cause non-decomposability, and the non-decomposability persists in large markets. Thus, if we insist on collocation of twins, the capacity constraints must be relaxed. It turns out that—no matter how many pairs of twins there are—we only need one extra seat at each school.

Proposition 2. Regardless of the number of pairs of twins, any stochastic assignment matrix in which each pair of twins has the identical assignments can be decomposed into a lottery over deterministic assignments that collocate the pairs of twins and violate the capacity constraint by at most one at each school.

Proof. Suppose we want to implement $X = (x_{is})_{i \in I, s \in S}$. Let each pair of twins select a representative for the pair. We partition the set of students I into three subsets: the set of non-twin students $\tilde{I}$, the set of representative twins T, and the set of non-representative twins T'. We temporarily remove T', find a decomposition of X restricted to $\tilde{I} \cup T$, then add back T'.

We formulate the problem of decomposing the restricted matrix $(x_{is})_{i\in\tilde{I}\cup T, s\in S}$ as follows. Let $r_s = \sum_{i\in T'} x_{is}$ be the total share of school s assigned to non-representative twins. Because r_s may not be integral, the maximum number of seats we can reserve for T' without causing infeasibility is $\lfloor r_s \rfloor$, the largest integer not exceeding r_s. We modify the capacity constraint as follows:

$$\sum_{i\in\tilde{I}\cup T} x_{is} \leq q_s - \lfloor r_s \rfloor \qquad \forall s \in S.$$

We also impose an upper limit on the number of representative twins that can be assigned to each school:

$$\sum_{i\in T} x_{is} \leq \lceil r_s \rceil \qquad \forall s \in S,$$

where $\lceil r_s \rceil$ is the smallest integer that is at least as large as r_s.

These two new constraints together with (C1) and (C2) form a bi-hierarchy of additive constraints, and therefore, by Theorem 1 of Budish et al. (2013), there is a decomposition. Pick any decomposition. Now, we add back the non-representative twins. At school $s \in S$, they require at most $\lceil r_s \rceil$ seats, and we have reserved $\lfloor r_s \rfloor$ seats. Therefore, we only need one extra seat. □

More generally, given a set of students $\{i_1, ..., i_k\} \subseteq I$ to be collocated, the collocation constraints for the group can be written

$$x_{i_1 s} = x_{i_j s} \qquad \forall s \in S, \qquad j = 2, ..., k.$$

Proposition 2 extends to collocation of triplets, quadruplets, and so forth.

Proposition 3. For arbitrary $k \in \mathbb{N}$, any stochastic assignment matrix in which each set of k-multiples has the identical assignments can be decomposed into a lottery over deterministic assignments that collocate the sets of multiples and violate the capacity constraint by at most $k-1$ seats at each school.

Proof. Suppose we want to implement $X = (x_{is})_{i \in I, s \in S}$. We call each set of multiples a family. Let each family of multiples select one representative sibling. Let $\tilde{I}$ denote the set of single students and let F denote the set of representative siblings. We temporarily remove the remaining $k-1$ non-representative siblings from each family.

For each $s \in S$, let $r_s = \sum_{i \in F} x_{is}$. We reserve seats for the removed siblings by modifying the capacity constraint to $\sum_{i \in \tilde{I} \cup F} x_{is} \leq q_s - \lfloor (k-1) r_s \rfloor$. We also impose $\sum_{i \in F} x_{is} \leq \lceil r_s \rceil$ to ensure that no particular school has a high concentration of representative siblings. These two constraints together with (C1) and (C2) form a bi-hierarchy of additive constraints, and therefore, $(x_{is})_{i \in \tilde{I} \cup F, s \in S}$ is decomposable. Pick any decomposition. To add back the non-representative twins, we need at most $(k-1) \left\lceil \sum_{i \in F} x_{is} \right\rceil$ seats at school s, and we have reserved $\left\lfloor (k-1) \sum_{i \in F} x_{is} \right\rfloor$ seats. Therefore, the excess demand for seats is at most $k-1$ at each school. □

Indeed, there can be sets of multiples of different sizes. For example, collocation of twins and triplets can be guaranteed if deterministic assignments are allowed to violate the capacity constraint by up to two seats at each school.

Proposition 4. Suppose each set of multiples consists of at most k students. Then, any stochastic assignment matrix in which each set of multiples has the identical assignments can be decomposed into a lottery over deterministic assignments that collocate the sets of multiples and violate the capacity constraint by at most $k-1$ seats at each school.

The basic idea for the proof remains the same: We remove all but one sibling from each set of multiples; find a decomposition of the assignment matrix restricted to the single students and the representative siblings with appropriate constraints imposed; and add back the removed siblings. The proof is provided in Appendix 2.7.1.

2.3.2 Company Constraints

Related to twin constraints are company constraints, which require that each student has a familiar face at their assigned school. Suppose the set I of students can be partitioned into communities. Let $c(i)$ denote the community that student i belongs to. The company constraint requires that

$$x_{is} \leq \sum_{\substack{j \neq i \\ c(j)=c(i)}} x_{js} \qquad \forall i \in I,\ s \in S.$$

In other words, each student is guaranteed a company of at least one peer from their community at their assigned school. Equivalently, each school has either zero or multiple students from each community.

Rationale for company constraints can be psychological, logistical, or financial. When racial or religious minorities are assigned to the same school or classroom, they may be able to provide emotional support for each other. To reduce the cost of bus transportation, a dozen students living in a remote area can be deliberately assigned to a few schools in small groups rather than to twelve different schools. Similarly, immigrant students have a better chance of receiving necessary language support if they are not scattered over many different schools because there may be a limited number of language specialists in the city.

Unfortunately, assignment matrices are not always decomposable respecting the company constraint. Furthermore, non-decomposability persists even when there are plenty of seats at each school.

Example 2 (Company constraint)**.** Suppose there are three students, 1, 2, and 3, and three schools, A, B, and C, each with three seats. Suppose all students are from the same community. We wish to implement the following assignment matrix[6]:

	Preference	A	B	C
Student 1	$A \succ B \succ C$	$\frac{1}{2}$	$\frac{1}{2}$	0
Student 2	$B \succ C \succ A$	0	$\frac{1}{2}$	$\frac{1}{2}$
Student 3	$A \succ C \succ B$	$\frac{1}{2}$	0	$\frac{1}{2}$

This assignment matrix satisfies the company constraint: $x_{is} \leq \sum_{j \neq i} x_{js}$ for all $i \in \{1, 2, 3\}$ for all $s \in \{A, B, C\}$. However, decomposition is impossible because all three students must be placed at the same school in any deterministic assignment to respect the company constraint.

Adding extra seats does not resolve the non-decomposability because the non-decomposability is caused by the cyclic structure of the assignment matrix. By a cycle, we mean a sequence of non-zero probabilities linked either through a student or a school. In this example, $x_{1A} \overset{1}{\longleftrightarrow} x_{1B} \overset{B}{\longleftrightarrow} x_{2B} \overset{2}{\longleftrightarrow} x_{2C} \overset{C}{\longleftrightarrow} x_{3C} \overset{3}{\longleftrightarrow} x_{3A} \overset{A}{\longleftrightarrow} x_{1A}$ is a cycle. We suspect that cycles that involve an odd number of students play some role in non-decomposability.

[6]This assignment is neither the expected RSD assignment nor the PS assignment. However, it can be generated by an unfair Serial Dictatorship that randomly picks one of the two orderings of the students, $(1, 2, 3)$ or $(3, 1, 2)$, with equal probabilities. It can also be generated by an extension of the PS if each student is indifferent between their two most-preferred schools.

Example 3 (Cycles)**.** Suppose there are four schools, A, B, C, and D, each with three seats. Suppose all students belong to the same community, and we wish to guarantee each student at least one peer at their assigned school. Consider the following assignment matrix:

	A	B	C	D
Student 1	$\frac{1}{2}$	$\frac{1}{2}$	0	0
Student 2	0	$\frac{1}{2}$	$\frac{1}{2}$	0
Student 3	0	0	$\frac{1}{2}$	$\frac{1}{2}$
Student 4	$\frac{1}{2}$	0	0	$\frac{1}{2}$
Student 5	$\frac{1}{2}$	0	$\frac{1}{2}$	0

First, suppose the community consists of students 1 through 4. There is a unique decomposition that respects the community constraint: Randomize over the assignments "students 1 and 4 to school A and students 2 and 3 to school C" and "students 1 and 2 to school B and students 3 and 4 to school D". Notice that there is only one cycle involving all four students.

Next, suppose the community consists of students 1 through 5. With the addition of student 5, the assignment matrix becomes non-decomposable respecting the company constraint. A decomposition must include a deterministic assignment in which student 1 is assigned to school B, which requires student 2 to be also assigned to school B. Because none of students 3, 4, and 5 can be assigned to school B, these three students must be assigned to some other school together. However, there is no school to which all three are assigned with a positive probability. Notice that students 1, 2, and 5 form a cycle. Students 3, 4, and 5 also form a cycle.

This example suggests that there is something special about cycles involving an odd number of students. In fact, for additive constraints, the equivalence between absence of odd cycles, bi-hierarchy structure of the constraint sets, and decomposability of assignment matrices is proved in Budish et al. (2013).[7] Therefore, it would not be surprising if cycles involving an odd number of students were responsible for non-decomposability of assignment matrices respecting collocation constraints. However, a further investigation of this topic may be of little practical value because commonly used assignment mechanisms such as the RSD and the PS do not generate cycles in their assignment matrices when the preferences of the students are strict. We leave this topic for future research.

We end this section with one positive result. As demonstrated in Example 2 and 3, the company constraint can cause non-decomposability that persists even when there are infinitely many seats at each school; however, if students from the same community have the same stochastic assignment, the problem reduces to that of collocating twins and triplets. Specifically, we can partition each community of students into arbitrary groups of two and three, then apply Proposition 4 to obtain the following result.

Corollary 1. Suppose each community has at least two students. Also suppose all students within each community have identical stochastic assignments. Then stochastic assignment matrices are decomposable respecting the company constraint without violating the capacity constraint by more than two seats at each school.

[7] They define odd cycles as follows. Recall that a constraint set is a subset of student-school pairs $\{(i,s) \in I \times S\}$. A sequence of ℓ constraint sets $(H_1, ..., H_\ell)$ in $\mathcal{H}$ is an odd cycle if ℓ is odd and there exists a sequence of linking student-school pairs $h_1, ..., h_\ell$ such that $h_j \in H_j \cap H_{j+1}$ (where $\ell + 1 \equiv 1$) and $h_j \notin S_k$ for any $k \neq j, j+1$. According to this definition, the cycles that involve an odd number of students in our examples can be termed pseudo-odd cycles because the number of constraint sets that constitute the cycles is not divisible by 4.

2.4 Limitations of Existing Algorithms

So far, we have established that assignment matrices are not generally decomposable respecting collocation constraints. We now ask the question, "When assignment matrices are decomposable respecting collocation constraints, how do we actually find a lottery over deterministic assignments?" In this section, we attempt to accommodate collocation constraints through modifications of existing algorithms, which turn out to be futile. An alternative approach is proposed in Section 2.5.

2.4.1 Modification of Existing Decomposition Algorithm

We begin with a proof sketch for a generalized version of the Birkhoff-von Neumann theorem, which provides the basis for the existing decomposition algorithm. Consider any feasible stochastic assignment matrix $X^0 \in \mathcal{X}$. If any column sum of X^0 is non-integral, add dummy students to I so that $\sum_{i \in I} x^0_{is}$ is integral for each $s \in S$.

Pick any $Z^0 \in \mathcal{Z}$ such that $z^0_{is} = 1 \implies x^0_{is} > 0$ for all $i \in I$ and $s \in S$. The existence of such Z^0 can be proved by formulating the problem as a network-flow problem and invoking the Integrality Theorem. Let $w = \min\{x^0_{is} \mid z^0_{is} = 1\}$. Then X^0 can be partially decomposed as

$$X^0 = wZ^0 + (1-w)X^1, \quad \text{where} \quad X^1 = \frac{X^0 - wZ^0}{1-w}.$$

Because X^0 and Z^0 are in $\mathcal{X}$ and all their column sums are integral, X^1 is also in $\mathcal{X}$ and all its column sums are integral. Furthermore, X^1 has one less non-zero elements than X^0. Therefore, by repeating this process, we can obtain a decomposition in a finite number of iterations.

We show that simply imposing the collocation constraint when finding a new component $Z \in \mathcal{Z}$ of a decomposition—and keeping the rest of the algorithm unchanged—is inadequate. With such a crude modification, the algorithm can fail to find a decomposition when there exists one.

Example 4 (Greedy algorithm)**.** Suppose there are five students, 1 through 5, and three schools, A, B and C, with capacities $q_A = q_B = 2$ and $q_C = 1$. Students 1 and 2 prefer school A to B to C, while students 3, 4, and 5 prefer school A to C to B. The RSD and the PS both return assignment $\bar{X}$ shown below. We wish to implement $\bar{X}$ with the collocation constraint on students 1 and 2. This is possible because $\bar{X}$ can be decomposed as follows:

$$\bar{X} = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} & 0 \\ \frac{2}{5} & \frac{3}{5} & 0 \\ \frac{2}{5} & \frac{4}{15} & \frac{1}{3} \\ \frac{2}{5} & \frac{4}{15} & \frac{1}{3} \\ \frac{2}{5} & \frac{4}{15} & \frac{1}{3} \end{bmatrix} = \frac{2}{15}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{2}{15}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \frac{2}{15}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$+ \frac{1}{5}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

We refer to the assignments on the top row as Z^{tl}, Z^{tc}, and Z^{tr}, and the assignments on the bottom row as Z^{bl}, Z^{bc}, and Z^{br}, from left to right. These six assignments are the only feasible deterministic assignments that satisfy the twin constraint.

We show that the greedy algorithm that puts the maximum possible weight on a newly found deterministic assignment in each iteration fails to find a decomposition. Suppose the greedy algorithm finds Z^{tl} first. The maximum weight the algorithm can put on Z^{tl} is $\min\{\frac{2}{5}, \frac{4}{15}, \frac{1}{3}\} = \frac{4}{15}$, and $\bar{X}$ can be partially decomposed as

$$\bar{X} = \tfrac{4}{15} Z^{tl} + \tfrac{11}{15} X' \quad \text{where} \quad X' = \left(\bar{X} - \tfrac{4}{15} Z^{tr}\right)/(\tfrac{11}{15}).$$

Because $x'_{3B} = x'_{4B} = 0$, in any decomposition of X', the weights on Z^{tc} and Z^{tr} must be zero. This is incompatible with satisfying $x'_{5B} = \frac{4}{11}$. Therefore, for the greedy algorithm to work, it cannot start with Z^{tl}. By similar arguments, we can show that neither Z^{tc} nor Z^{tr} can be the first assignment to be found by the algorithm.

Suppose, instead, the greedy algorithm finds Z^{bl} first. The maximum weight the algorithm can put on Z^{bl} is $\min\{\frac{3}{5}, \frac{2}{5}, \frac{1}{3}\} = \frac{1}{3}$, and $\bar{X}$ can be partially decomposed as

$$\bar{X} = \tfrac{1}{3} Z^{bl} + \tfrac{2}{3} X'' \quad \text{where} \quad X'' = \left(\bar{X} - \tfrac{1}{3} Z^{bl}\right)/(\tfrac{2}{3}).$$

Because $x''_{5C} = 0$, in any decomposition of X'', the weight on Z^{tl} must be zero. Then, to satisfy $x''_{1A} = x''_{2A} = \frac{3}{5}$, the total weights on Z^{tc} and Z^{tr} must be $\frac{3}{5}$. This is incompatible with satisfying $x''_{5B} = \frac{2}{5}$. Therefore, for the greedy algorithm to work, it cannot start with Z^{bl}. By similar arguments, we can show that neither Z^{bc} nor Z^{br} can be the first assignment to be found by the algorithm.

Therefore, regardless of which $Z \in \mathcal{Z}$ enters the decomposition first, the greedy algorithm fails to find a decomposition.

This example suggests that a modification to the existing decomposition algorithm required to accommodate collocation constraints is not as simple as it seems.

2.4.2 Modification of Existing Lottery Mechanism

Alternatively, we may try to accommodate collocation constraints by modifying existing assignment mechanisms that return deterministic assignments as an output. This eliminates the need for decomposition altogether.

Example 5 (Successful modification of RSD)**.** Suppose there are five students, 1 through 5, and two schools, A and B, with capacities $q_A = 3$ and $q_B = 2$. Students 1 and 2 are twins and they must be collocated. Every student prefers school A to B. We wish to implement the expected RSD assignment, $(x_{iA}, x_{iB}) = (\frac{3}{5}, \frac{2}{5}) \quad \forall i \in I$.

One possible modification of the RSD is to treat the twins as one student. The twins receive one lottery number, and when it is their turn to claim seats, they choose their most-preferred school that has two or more remaining seats. For the twins to be able to claim two seats at school A, their lottery number has to be the largest or the second largest, which happens with probability $\frac{1}{2}$. Therefore, the expected assignment for the twins is $(x_{1A}, x_{1B}) = (\frac{1}{2}, \frac{1}{2})$. The twins are disadvantaged.

We complement this with another variant of the RSD that gives the twins an advantage. We give each twin their own lottery number. When it is the turn of the twin with the larger lottery number, let the twin with the smaller lottery number skip the line. For the twins to be able to claim two seats at school A, one of the two lottery numbers needs to be the largest or the second largest, which happens with probability $\frac{7}{10}$. Therefore, the expected assignment for the twins is $(x_{1A}, x_{1B}) = (\frac{7}{10}, \frac{3}{10})$.

The desired expected assignment $(x_{1A}, x_{1B}) = (\frac{3}{5}, \frac{2}{5})$ can be obtained by implementing the one-lottery-number scheme with probability $\frac{1}{2}$ and the two-lottery-number scheme with probability $\frac{1}{2}$. Therefore, for this specific example, it is possible to modify the RSD in simple ways to accommodate the twin collocation.

Unfortunately, randomizing over the one-lottery-number and the two-lottery-numbers schemes does not always work as demonstrated in the following example.

Example 6 (Unsuccessful modification of RSD)**.** Suppose there are six students, 1 through 6, and three schools, A, B, and C, each with two seats. Students 1 and 2 are twins and they must be assigned to the same school. Students 1, 2, 3, and 4 prefer school A to B to C, while students 5 and 6 prefer school B to A to C. Suppose we wish to implement the expected RSD assignment:

$$(x_{iA}, x_{iB}, x_{iC}) = \begin{cases} (\frac{1}{2}, \frac{1}{6}, \frac{1}{3}) & \text{for } i = 1, 2, 3, 4 \\ (0, \frac{2}{3}, \frac{1}{3}) & \text{for } i = 5, 6 \end{cases}$$

We show that the one-lottery-number scheme which treats the twins as one student generates an assignment matrix that cannot be a part of any decomposition. Consider the following priority order of the students: (3, 1/2, 5, 6, 4). First, student 3 claims a seat at school A. Then, the twins pass up school A because it has only one seat left, and claim two seats at school B. Next, student 5 claims a seat at school A. Finally, students 6 and 4 take the remaining seats at school C. This assignment cannot be a part of any decomposition because student 5 is not supposed to be assigned to school A with a positive probability.

Of course, this example does not prove that the RSD assignment cannot be implemented respecting twin constraints through some randomization over variants of the RSD. However, even if there are variants of the RSD that can collectively reproduce the RSD assignment, it is likely that required modifications to the RSD are convoluted. For this reason, we seek an alternative approach in the next section.

2.5 Column Generation

Given a collection $\mathcal{C}$ of constraints, suppose we wish to decompose a stochastic assignment $\bar{X} \in \mathcal{X}^{\mathcal{C}}$ into a lottery over deterministic assignments in $\mathcal{Z}^{\mathcal{C}}$. One possible approach is to list all assignments in $\mathcal{Z}^{\mathcal{C}}$, then find a set of weights $(w^Z)_{Z \in \mathcal{Z}^{\mathcal{C}}}$ that constitutes a decomposition. This can be achieved by solving the following problem:

$$\min_{w} \quad \left| \sum_{Z \in \mathcal{Z}^{\mathcal{C}}} w^Z Z - \bar{X} \right|,$$

where $|\cdot|$ denotes the absolute value function. Any solution to this problem with the minimized objective value of 0 is a decomposition. Although simple, this approach is impractical because the number of assignments in $\mathcal{Z}^{\mathcal{C}}$ increases rapidly as the number of students increases.

Instead of listing all assignments in $\mathcal{Z}^{\mathcal{C}}$ at once, we search for $Z \in \mathcal{Z}^{\mathcal{C}}$ one at a time, using an algorithm based on Column Generation.[8] The algorithm has three components: a master problem, a subproblem, and a pool of feasible deterministic assignments. We solve the master problem and the subproblem alternately: We attempt to find a decomposition using the current pool of assignments by solving the master problem, then we update the pool by solving the subproblem.

Given $\bar{X} \in \mathcal{X}^{\mathcal{C}}$ to be decomposed into a lottery over assignments in $\mathcal{Z}^{\mathcal{C}}$, the algorithm works as follows. We first identify one or more assignments in $\mathcal{Z}^{\mathcal{C}}$ to form the initial pool $\mathcal{Z}^1 \subset \mathcal{Z}^{\mathcal{C}}$ of deterministic assignments.

[8]Column Generation is a technique commonly used in the field of operations research. The idea is initially proposed in Ford Jr and Fulkerson (1958) and is related to Dantzig and Wolfe (1960). Interested readers can refer to, for example, Desrosiers and Lübbecke (2005), which provides a good introduction to the topic.

In the first iteration, the master problem searches for a convex combination of the assignments in $\mathcal{Z}^1$ that makes the resulting stochastic assignment X^1 as close to $\bar{X}$ as possible. The shadow prices (dual variables) from the master problem indicate which elements of X^1 differ from $\bar{X}$. Using this information, the subproblem searches for a new deterministic assignment $Z^* \in \mathcal{Z}^{\mathcal{C}}$ that helps close the gap between X^1 and $\bar{X}$. We add the newly found assignment to the pool: $\mathcal{Z}^2 = \mathcal{Z}^1 \cup \{Z^*\}$.

In the second iteration, the master problem searches for a convex combination of the assignments in $\mathcal{Z}^2$ that makes the resulting stochastic assignment X^2 as close to $\bar{X}$ as possible. Using the information contained in the shadow prices from the master problem, we identify a new deterministic assignment to be added to the pool.

The process continues similarly. As the pool expands, the discrepancy between the best convex combination of the assignments in the current pool and $\bar{X}$ diminishes. The algorithm terminates when there is no discrepancy or when a further expansion of the pool does not reduce the discrepancy.

We walk through the algorithm using the assignment matrix in Example 4, which is reproduced in the table below. Recall that $I = \{1, 2, 3, 4, 5\}$ and students 1 and 2 are twins. Also recall that $S = \{A, B, C\}$ with capacities $q_A = q_B = 2$ and $q_C = 1$. The table also lists all feasible deterministic assignments that collocate the twins.

i	Preference	$\bar{x}_{iA}$	$\bar{x}_{iB}$	$\bar{x}_{iC}$	Z^{tl}	Z^{tc}	Z^{tr}	Z^{bl}	Z^{bc}	Z^{br}
1	$A \succ B \succ C$	$\frac{2}{5}$	$\frac{3}{5}$	0	A	A	A	B	B	B
2	$A \succ B \succ C$	$\frac{2}{5}$	$\frac{3}{5}$	0	A	A	A	B	B	B
3	$A \succ C \succ B$	$\frac{2}{5}$	$\frac{4}{15}$	$\frac{1}{3}$	B	B	C	A	A	C
4	$A \succ C \succ B$	$\frac{2}{5}$	$\frac{4}{15}$	$\frac{1}{3}$	B	C	B	A	C	A
5	$A \succ C \succ B$	$\frac{2}{5}$	$\frac{4}{15}$	$\frac{1}{3}$	C	B	B	C	A	A

Suppose our initial pool is $\mathcal{Z}^1 = \{Z^{tl}, Z^{bl}\}$.

In the first iteration, the master problem (MP^1) searches for optimal weights (w^{tl}, w^{bl}) that minimize the discrepancy between $X^1 = w^{tl}Z^{tl} + w^{bl}Z^{bl}$ and $\bar{X}$. We use slack variables, $(\delta^+_{is}, \delta^-_{is})_{i\in I, s\in S}$, to record the element-wise upward and downward deviations of X^1 from $\bar{X}$. Because $\sum_{s\in S}\bar{x}_{is} = 1$ for all $i \in I$, it is sufficient to record the discrepancy for only two schools out of three.

$$
\begin{aligned}
(\mathrm{MP}^1) \qquad & \min_{w,\delta^+,\delta^-} && \sum_{i\in I}\sum_{s\in\{A,B\}}(\delta^+_{is} + \delta^-_{is}) && \\
& \text{s.t.} && w^{tl}z^{tl}_{is} + w^{bl}z^{bl}_{is} + \delta^+_{is} \geq \bar{x}_{is} && \forall i \in I,\ s \in \{A, B\} \\
& && w^{tl}z^{tl}_{is} + w^{bl}z^{bl}_{is} - \delta^-_{is} \leq \bar{x}_{is} && \forall i \in I,\ s \in \{A, B\} \\
& && w^{tl} + w^{bl} = 1 && \\
& && w^{tl}, w^{bl} \geq 0 && \\
& && \delta^+_{is}, \delta^-_{is} \geq 0 && \forall i \in I,\ s \in \{A, B\}
\end{aligned}
$$

The solution to (MP^1) is $(w^{tl}, w^{bl}) = \left(\frac{2}{5}, \frac{3}{5}\right)$.

Let $(\lambda^+_{is}, \lambda^-_{is})_{i\in I, s\in\{A,B\}}$ denote the shadow prices associated with the first two sets of constraints. Of the ten constraints that keep track of upward deviations, two of them have non-zero shadow prices: $\lambda^+_{5A} = \lambda^+_{5B} = 1$, which implies that student 5 should be assigned to school A and B more often. Of the ten constraints that keep track of downward deviations, four of them have non-zero shadow prices: $\lambda^-_{3A} = \lambda^-_{4A} = \lambda^-_{3B} = \lambda^-_{4B} = -1$, which implies that students 3 and 4 should be assigned to school A and B less often. In other words, if there were assignments in $\mathcal{Z}^1$ that assign student 5 to school A or B and student 3 or 4 to school C, the discrepancy between X^1 and $\bar{X}$ would have been smaller.

Given this information, we search for an assignment that complements the current pool $\mathcal{Z}^1$. Specifically, we solve the following subproblem:

$$
\begin{aligned}
\text{(SP)} \qquad & \max_{z} && \sum_{i \in I} \sum_{s \in \{A,B\}} (\lambda_{is}^{+} + \lambda_{is}^{-}) z_{is} && \\
& \text{s.t.} && \sum_{i \in I} z_{is} \leq q_s && \forall s \in S \\
& && \sum_{s \in S} z_{is} = 1 && \forall i \in I \\
& && z_{1s} - z_{2s} = 0 && \forall s \in S \\
& && z_{is} \in \{0, 1\} && \forall i \in I,\, s \in S
\end{aligned}
$$

There are multiple optimal solutions to (SP). Suppose we find Z^{tc}. We update the pool accordingly: $\mathcal{Z}^2 = \mathcal{Z}^1 \cup \{Z^{tc}\} = \{Z^{tl}, Z^{bl}, Z^{tc}\}$.

In the second iteration, the master problem (MP^2) searches for the optimal weights on these three deterministic assignments:

$$
\begin{aligned}
(\text{MP}^2) \qquad & \min_{w, \delta^{+}, \delta^{-}} && \sum_{i \in I} \sum_{s \in \{A,B\}} (\delta_{is}^{+} + \delta_{is}^{-}) && \\
& \text{s.t.} && w^{tl} z_{is}^{tl} + w^{bl} z_{is}^{bl} + w^{tc} z_{is}^{tc} + \delta_{is}^{+} \geq \bar{x}_{is} && \forall i \in I,\, s \in \{A, B\} \\
& && w^{tl} z_{is}^{tl} + w^{bl} z_{is}^{bl} + w^{tc} z_{is}^{tc} - \delta_{is}^{-} \leq \bar{x}_{is} && \forall i \in I,\, s \in \{A, B\} \\
& && w^{tl} + w^{bl} + w^{tc} = 1 && \\
& && w^{tl}, w^{bl}, w^{tc} \geq 0 && \\
& && \delta_{is}^{+}, \delta_{is}^{-} \geq 0 && \forall i \in I,\, s \in \{A, B\}
\end{aligned}
$$

The solution to (MP^2) is $(w^{tl}, w^{bl}, w^{tc}) = \left(\frac{2}{15}, \frac{3}{5}, \frac{4}{15}\right)$.

Guided by the shadow prices, the subproblem finds Z^{tr} and it is added to the pool. Continuing similarly, Z^{bc} and Z^{br} are subsequently found and added to the pool. At the end of the fourth iteration, the pool $\mathcal{Z}^5$ contains all six deterministic assignments, and in the fifth iteration, the master problem finds the decomposition

$$X^5 = \tfrac{2}{15}Z^{tl} + \tfrac{2}{15}Z^{tc} + \tfrac{2}{15}Z^{tr} + \tfrac{1}{5}Z^{bl} + \tfrac{1}{5}Z^{bc} + \tfrac{1}{5}Z^{br}.$$

The corresponding shadow prices are all zero, indicating that $X^5 = \bar{X}$. Although $\mathcal{Z}^5 = \mathcal{Z}^{\mathcal{C}}$ in this example, the final pool is usually much smaller than the set of all permissible deterministic assignments.

In general, given an arbitrary set $\mathcal{C}$ of constraints, stochastic assignment $\bar{X} \in \mathcal{X}^{\mathcal{C}}$ can be decomposed into a lottery over assignments in $\mathcal{Z}^{\mathcal{C}}$ as follows. We start with an initial pool $\mathcal{Z}^1 \subset \mathcal{Z}^{\mathcal{C}}$ of deterministic assignments. In iteration $k \geq 1$, the master problem searches for a convex combination of the assignments in $\mathcal{Z}^k$ that minimizes the discrepancy between the resulting stochastic assignment and $\bar{X}$:

$$\begin{aligned}
\min_{w,\delta^+,\delta^-} \quad & \sum_{i\in I}\sum_{s\in S}(\delta^+_{is} + \delta^-_{is}) & \\
\text{s.t.} \quad & \sum_{Z\in\mathcal{Z}^k} w^Z z_{is} + \delta^+_{is} \geq \bar{x}_{is} & \forall i \in I,\, s \in S \\
& \sum_{Z\in\mathcal{Z}^k} w^Z z_{is} - \delta^-_{is} \leq \bar{x}_{is} & \forall i \in I,\, s \in S \\
& \sum_{Z\in\mathcal{Z}^k} w^Z = 1 & \\
& w^Z \geq 0 & \forall Z \in \mathcal{Z}^k \\
& \delta^+_{is}, \delta^-_{is} \geq 0 & \forall i \in I,\, s \in S
\end{aligned}$$

Using the information contained in the shadow prices $(\lambda^+_{is}, \lambda^-_{is})_{i\in I, s\in S}$, the subproblem identifies a deterministic assignment that helps reduce the discrepancy:

$$Z^* \in \arg\max_{Z\in\mathcal{Z}^{\mathcal{C}}} \sum_{i\in I}\sum_{s\in S}(\lambda^+_{is} + \lambda^-_{is})z_{is}$$

We update the pool $\mathcal{Z}^{k+1} = \mathcal{Z}^k \cup Z^*$ and proceeds to the next iteration. The algorithm terminates when all shadow prices from the master problem are zero (i.e., a decomposition is found) or when the subproblem returns one of the deterministic assignments that are already in the pool (i.e., decomposition is infeasible).

Although we have not tested this algorithm using real data, algorithms based on Column Generation generally run efficiently. We reiterate that our algorithm can be used in conjunction with any stochastic assignment mechanism without altering the assignment mechanism. This provides opportunities to improve welfare without any cost. Thus, stochastic assignment mechanisms such as the PS should be given serious consideration as an alternative to lottery mechanisms such as the RSD.

2.6 Conclusion

In school-choice programs, it is often desirable to assign groups of students, such as twins, to the same school. In this paper, we take a stochastic assignment as given, and attempt to achieve as much collocation as possible when decomposing the stochastic assignment into a lottery over deterministic assignments.

With hard capacity constraints, stochastic assignments cannot always be decomposed respecting twin collocation. However, twin collocation can be guaranteed in a decomposition if the capacity constraint can be relaxed by one seat at each school.

This is true for any number of pairs of twins in the student body. Similarly, if we are allowed to violate the capacity constraint by up to two seats at each school, we can guarantee each student that they have a company of at least one peer from their community at their assigned school, provided that the students in the same community have the same stochastic assignment.

When a stochastic assignment can be decomposed respecting collocation of groups of students, straightforward modifications of existing algorithms may fail to find a decomposition. Therefore, we propose a new decomposition algorithm based on Column Generation. The algorithm is versatile and can be used to accommodate a wide range of constraints. A run-time comparison with Balbuzanov (2019) would be an interesting topic for future research.

2.7 Appendix: Proofs

2.7.1 Proof of Proposition 4

Proposition 4. Suppose each set of multiples consists of at most k students. Then, any stochastic assignment matrix in which each set of multiples has the identical assignments can be decomposed into a lottery over deterministic assignments that collocate the sets of multiples and violate the capacity constraint by at most $k-1$ seats at each school.

Proof. Suppose we want to implement $X = (x_{is})_{i \in I, s \in S}$. First, we partition students into families of multiples and let each family select one representative. Let F denote the set of representative students. Let F^j be the set of representatives from the families of j-multiples for $j = 1, ..., k$ so that $F = \bigcup_{j=1}^{k} F^j$.

For $j = 2, ..., k$, from each family of j-multiples, we temporarily remove the $j-1$ non-representative siblings. We then find a lottery decomposition of $(x_{is})_{i\in F, s\in S}$, the assignment matrix restricted to the representative students. Finally, we add back the removed siblings to the school to which their representative siblings are assigned.

For each $j \in \{2, ..., k\}$, let $r_s^j = \sum_{i\in F^j} x_{is}$. We impose the following constraints:

$$x_{is} \geq 0 \qquad \forall i \in F, \forall s \in S \tag{C1}$$

$$\sum_{s\in S} x_{is} = 1 \qquad \forall i \in F \tag{C2}$$

$$\sum_{i\in F} x_{is} \leq q_s - \left\lfloor \sum_{j=2}^{k} (j-1) r_s^j \right\rfloor \qquad \forall s \in S \tag{C3$'$}$$

$$\sum_{\ell=j}^{k} \sum_{i\in F^\ell} x_{is} \leq \left\lceil \sum_{\ell=j}^{k} r_s^\ell \right\rceil \qquad \forall s \in S,\ j = 2, ..., k \tag{C4}$$

Constraint (C3$'$) reserves seats for the removed siblings. Constraint (C4) ensures that no school has a high concentration of representative students who have many siblings. These constraints form a bi-hierarchy, and therefore, there is a decomposition.

It remains to show that we can add back the removed siblings without violating the capacity constraint by more than $k-1$ seats. At school $s \in S$, the greatest number of extra seats are required when constraints (C4) are all binding. That is, the multiples are maximally concentrated at school s when a deterministic assignment Z is such that

$$\sum_{i\in F^j} z_{is} = \left\lceil \sum_{\ell=j}^{k} r_s^\ell \right\rceil - \left\lceil \sum_{\ell=j+1}^{k} r_s^\ell \right\rceil \qquad j = 2, ..., k-1$$

$$\sum_{i\in F^k} z_{is} = \lceil r_s^k \rceil$$

In this case, the total number of seats demanded by the removed siblings is

$$Q^D = \sum_{j=2}^{k-1}(j-1)\left(\left\lceil \sum_{\ell=j}^{k} r_s^\ell \right\rceil - \left\lceil \sum_{\ell=j+1}^{k} r_s^\ell \right\rceil\right) + (k-1)\left\lceil r_s^k \right\rceil .$$

Letting $R_s^j = \sum_{\ell=j}^{k} r_s^\ell$ for $j \in \{2, ..., k\}$ and $R_s^{k+1} = 0$, we can write the demand as

$$Q^D = \sum_{j=2}^{k}(j-1)\left(\left\lceil R_s^j \right\rceil - \left\lceil R_s^{j+1} \right\rceil\right).$$

The total supply of seats is the sum of the reserved seats and the extra $k-1$ seats granted by the premise of the proposition. We want to show that the total supply is at least as large as the maximum possible demand. To this end, first re-write the number of reserved seats as

$$\left\lfloor \sum_{j=2}^{k}(j-1)r_s^j \right\rfloor = \left\lfloor \sum_{j=2}^{k}\sum_{\ell=j}^{k} r_s^\ell \right\rfloor \geq \sum_{j=2}^{k} \left\lfloor R_s^j \right\rfloor = \sum_{j=2}^{k-1} \left\lfloor R_s^j \right\rfloor + \left\lfloor r_s^k \right\rfloor .$$

Letting $\sigma_s^j = \lfloor R_s^j \rfloor - \lfloor r_s^j \rfloor - \lfloor R_s^{j+1} \rfloor$ for $j \in \{2, ..., k-1\}$, we can write

$$\left\lfloor R_s^j \right\rfloor = \left\lfloor r_s^j \right\rfloor + \sigma_s^j + \left\lfloor R_s^{j+1} \right\rfloor = \sum_{\ell=j}^{k-1}\left(\left\lfloor r_s^\ell \right\rfloor + \sigma_s^\ell\right) + \left\lfloor r_s^k \right\rfloor .$$

Thus, the total number of reserved seats can be written

$$\sum_{j=2}^{k-1}\left\{\sum_{\ell=j}^{k-1}\left(\left\lfloor r_s^\ell \right\rfloor + \sigma_s^\ell\right) + \left\lfloor r_s^k \right\rfloor\right\} + \left\lfloor r_s^k \right\rfloor = \sum_{j=2}^{k-1}(j-1)\left(\left\lfloor r_s^j \right\rfloor + \sigma_s^j\right) + (k-1)\left\lfloor r_s^k \right\rfloor .$$

Combining this with the extra $k-1$ seats granted by the premise of the proposition, we obtain the total supply

$$Q^S = \sum_{j=2}^{k-1} (j-1) \left(\lfloor r_s^\ell \rfloor + \sigma_s^\ell \right) + (k-1) \left(\lfloor r_s^k \rfloor + 1 \right) = \sum_{j=2}^{k} (j-1) \left(\lfloor r_s^j \rfloor + \sigma_s^j \right),$$

where $\sigma_s^j = \lfloor R_s^j \rfloor - \lfloor r_s^j \rfloor - \lfloor R_s^{j+1} \rfloor$ for $j \in \{2, ..., k-1\}$ and $\sigma_s^k = 1$.

We want to show that $Q^D \leq Q^S$, or equivalently,

$$\sum_{j=2}^{k} (j-1) \left(\lceil R_s^j \rceil - \lceil R_s^{j+1} \rceil \right) \leq \sum_{j=2}^{k} (j-1) \left(\lfloor r_s^j \rfloor + \sigma_s^j \right).$$

We keep track of excess demand by defining

$$\gamma_s^j = \left(\lceil R_s^j \rceil - \lceil R_s^{j+1} \rceil \right) - \left(\lfloor r_s^j \rfloor + \sigma_s^j \right), \qquad j \in \{2, ..., k\}.$$

Intuitively, the inequality $Q^D \leq Q^S$ holds if each excess demand $\gamma_s^{j'} > 0$ for some $j' \in \{2, ..., k-1\}$ is canceled by excess supply $\gamma_s^{j^*} < 0$ for some $j^* > j'$. To this end, we first show that $\gamma_s^j = 0$, 1, or -1 for each $j \in \{2, ..., k-1\}$.

For any $j \in \{2, ..., k-1\}$, noting that $R_s^j = r_s^j + R_s^{j+1}$, we have

$$\text{either} \quad \lceil R_s^j \rceil = \lceil r_s^j \rceil + \lceil R_s^{j+1} \rceil - 1 \quad \text{or} \quad \lceil R_s^j \rceil = \lceil r_s^j \rceil + \lceil R_s^{j+1} \rceil.$$

Thus, $\lceil R_s^j \rceil - \lceil R_s^{j+1} \rceil$ is either $\lceil r_s^j \rceil - 1$ or $\lceil r_s^j \rceil$. We also have

$$\text{either} \quad \lfloor R_s^j \rfloor = \lfloor r_s^j \rfloor + \lfloor R_s^{j+1} \rfloor \quad \text{or} \quad \lfloor R_s^j \rfloor = \lfloor r_s^j \rfloor + \lfloor R_s^{j+1} \rfloor + 1.$$

Recalling that $\sigma_s^j = \lfloor R_s^j \rfloor - \lfloor r_s^j \rfloor - \lfloor R_s^{j+1} \rfloor$, it follows that σ^j is either 0 or 1.

Suppose $\lceil R_s^j \rceil = \lceil r_s^j \rceil + \lceil R_s^{j+1} \rceil - 1$. Then r_s^j cannot be integral, so $\lceil r_s^j \rceil = \lfloor r_s^j \rfloor + 1$. Hence, $\gamma_s^j = (\lceil r_s^j \rceil - 1) - \left(\lfloor r_s^\ell \rfloor + \sigma_s^j\right) = -\sigma_s^j$, and therefore, γ_s^j is either 0 or -1.

Suppose $\lceil R_s^j \rceil = \lceil r_s^j \rceil + \lceil R_s^{j+1} \rceil$. There are three possibilities: (i) r_s^j is integral; (ii) r_s^j is non-integral and R_s^{j+1} is non-integral; and (iii) r_s^j is non-integral and R_s^{j+1} is integral. In case (i), we have $\lfloor R_s^j \rfloor = \lfloor r_s^j \rfloor + \lfloor R_s^{j+1} \rfloor$, and therefore, $\gamma_s^j = \lceil r_s^j \rceil - \lfloor r_s^j \rfloor = 0$. In case (ii), R_s^j must be non-integral as well because otherwise $\lceil R_s^j \rceil = \lceil r_s^j \rceil + \lceil R_s^{j+1} \rceil$ does not hold. Then we have $\lceil R_s^j \rceil = \lfloor R_s^j \rfloor + 1$, $\lceil r_s^j \rceil = \lfloor r_s^j \rfloor + 1$, and $\lceil R_s^{j+1} \rceil = \lfloor R_s^{j+1} \rfloor + 1$, and therefore, $\lfloor R_s^j \rfloor = \lfloor r_s^j \rfloor + \lfloor R_s^{j+1} \rfloor + 1$. Thus, $\gamma_s^j = \lceil r_s^j \rceil - (\lfloor r_s^j \rfloor + 1) = 0$. In case (iii), we have $\lfloor R_s^j \rfloor = \lfloor r_s^j \rfloor + \lfloor R_s^{j+1} \rfloor$, and therefore, $\gamma_s^j = \lceil r_s^j \rceil - \lfloor r_s^j \rfloor = 1$.

We have shown that $\gamma_s^j = 0$, 1, or -1 for each $j \in \{2, ..., k-1\}$. As to γ_s^k, using the definitions $R_s^{k+1} = 0$ and $\sigma_s^k = 1$, we obtain $\gamma_s^k = \lceil r_s^j \rceil - (\lfloor r_s^j \rfloor + 1)$. Therefore, $\gamma_s^k = 0$ if r_s^k is non-integral and $\gamma_s^k = -1$ if r_s^k is integral.

We claim that each excess demand is promptly canceled by excess supply, i.e., for each $j' \in \{2, ..., k-1\}$ with $\gamma_s^{j'} = 1$, there is $j^* \in \{j'+1, ..., k\}$ such that $\gamma_s^j = 0$ for all $j \in \{j'+1, ..., j^*-1\}$ and $\gamma_s^{j^*} = -1$. If this claim is true, excess demand is more than canceled and we have $Q^D \leq Q^S$.

It remains to show that the claim is true. Consider any $j' \in \{2, ..., k-1\}$ such that $\gamma_s^{j'} = 1$. This is possible only if $R_s^{j'+1}$ is integral. If r_s^j is integral for all subsequent $j \in \{j'+1, ..., k\}$, then $\gamma_s^j = 0$ for all $j \in \{j'+1, k-1\}$ and $\gamma_s^k = -1$, so we are done. Otherwise, let $j^* \in \{j'+1, ..., k\}$ be the smallest index for which $r_s^{j^*}$ is non-integral. Then, we have $\gamma_s^j = 0$ for all $j \in \{j'+1, ..., j^*-1\}$. Furthermore, $R_s^{j^*} = R_s^{j'+1} - \sum_{j=j'+1}^{j^*-1} r_s^j$, so $R_s^{j^*}$ is integral. It follows that $R_s^{j^*+1} = R_s^{j^*} - r_s^{j^*}$ is non-integral (and therefore, j^* cannot be k). Thus, we have $\left\lceil R_s^{j^*} \right\rceil = \left\lceil r_s^{j^*} \right\rceil + \left\lceil R_s^{j^*+1} \right\rceil - 1$ and $\left\lfloor R_s^{j^*} \right\rfloor = \left\lfloor r_s^{j^*} \right\rfloor + \left\lfloor R_s^{j^*+1} \right\rfloor + 1$, which implies $\gamma_s^{j^*} = -1$. □

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CHAPTER 3

Rank-Egalitarian Assignments

3.1 Introduction

Resource allocation programs such as school-choice programs often use rank distribution as a performance measure. For example, Newark Enrolls, a public school-choice program in New Jersey, reports "70.2% of rising 9^{th} grade students gained access to one of their top 3 choices" during the 2018–2019 matching cycle.[1] Teach for America, a nonprofit organization that sends college graduates to at-risk schools as teachers, summarizes their placement as follows: 85% of applicants in 2018 were placed in one of their top three regional choices.[2] National Resident Matching Program, which assigns medical school students to hospitals for postgraduate training, notes "The percentage of all U.S. seniors who matched to their first-choice programs was 47.1 %, the lowest on record; however, almost three-quarters (72.5%) of U.S. seniors matched to one of their top three choices."[3] Despite its wide use as a performance measure, rank distribution is rarely considered in the design of resource-allocation algorithms.

[1] `https://www.nps.k12.nj.us/mdocs-posts/ues_match-pdf/`

[2] `https://www.teachforamerica.org/how-to-join/placement`

[3] `https://mk0nrmpcikgb8jxyd19h.kinstacdn.com/wp-content/uploads/2019/04/NRMP-Results-and-Data-2019_04112019_final.pdf`

In this paper, we propose a resource-allocation algorithm that optimizes the rank distribution while maintaining fairness. There are two main elements. The first is rank values, which are similar to points in Borda counts in elections. For example, the value of matching an agent to their first, second, and third choice may be 10, 9, and 8, respectively. Unlike Borda counts, the rank values need not decrease linearly as one goes down a preference list. For example, if there is only a small difference in values of receiving one's second and third choice, the rank values can be set to 10, 7, and 6 for the first, second, and third choice, respectively. Rank-value based allocation algorithm is first proposed in Featherstone (2020). One novel feature of our algorithm is that it can accommodate multiple sets of rank values should there be disagreement among social planners regarding *the* right rank values.[4]

The second element is the maximin principle. Rawl's Difference Principle (Rawls, 1971) states that "Social and economic inequalities are to be arranged so that they are to be of the greatest benefit to the least-advantaged members of society."[5] In applications such as school-choice programs and kidney-exchange programs, the matching outcome has significant and prolonged effects on the lives of individuals, and we believe the maximin principle is the most suitable notion of fairness.[6]

[4]The idea of examining multiple instances of rank values has some similarities to Doğan et al. (2018), in which the authors define social-welfare efficiency based on the ex-ante efficiency dominance relation over the set of all cardinal-utility profiles consistent with the ordinal preferences. Our framework allows social planners to specify the set of relevant or important rank values instead of defaulting to the set of all permissible rank values.

[5]To implement the maximin principle, we must assume inter-agent comparability of welfare. Whether such comparison is possible or should be made is discussed in detail in, for example, Elster and Roemer (1993).

[6]In other applications, envy-freeness may be more relevant. An assignment is envy-free if every agent feels their allocation is at least as good as the allocation of any other agent. As noted in Brams and King (2005), maximin and envy-freeness are incompatible.

There are a few algorithms based on the maximin principle in the matching literature. Bogomolnaia and Moulin (2004) analyzes the marriage problem where each potential partner is either acceptable or unacceptable. They propose an egalitarian algorithm that maximizes, in the leximin order, the probability of being matched to an acceptable partner. In the context of pairwise kidney exchange, Roth et al. (2005) proposes an algorithm that generates a Lorenz-dominant assignment, where the utility of an agent is 1 if he receives a compatible kidney and 0 if he does not. These algorithms are special cases of rank-value based algorithms tailored for binary preferences. We extend the allocation algorithm to a general preference domain.

To the best of our knowledge, the current paper is the first to combine general rank-value based algorithms with the maximin principle in the context of probabilistic assignment.[7] We make three contributions to the existing literature. First, we propose a new concept of rank-fairness. An assignment is said to be rank-egalitarian undominated (REU) over a set of instances of rank values if there is no other assignment that guarantees weakly larger expected rank values to the worst-off agents at each instance of rank values. Second, we show that each REU assignment is an optimal solution to a linear programming problem in which a weighted sum of the minimum expected rank values is maximized. Conversely, optimal solutions to such linear programming problems are REU. Third, we propose algorithms to generate subsets of REU assignments that are practically important.

This paper is organized as follows. In Section 3.2, we define the allocation problem and introduce notation. We provide an illustrative example in Section 3.3 to highlight

[7]In the fields of computer science and operations research—where the objects of interests are deterministic rules—the complexity of the rank-value based maximin problems have been studied. See, for example, Baumeister et al. (2017).

the main idea, and formally define the REU assignments in Section 3.4. We then show the connection between REU assignments and linear programming problems in Section 3.5 and propose an algorithm to generate REU assignments in Section 3.6, followed by a conclusion in Section 3.7.

3.2 Preliminary

Let $I = \{1, ..., n\}$ denote the set of agents and $O = \{o_1, ..., o_m\}$ the set of object types. A generic agent is denoted by i and a generic object by o. Let q_o denote the number of copies of object type $o \in O$. We assume $\sum_{o \in O} q_o \geq n$. This is without loss of generality because we can always expand the set O to include a null object with n copies. Agents have unit demands and strict preferences, $\succ_i$, over object types. When it is clear from the context, we drop the agent subscript and simply write $\succ$. Let $\mathcal{P}$ denote the set of preference rankings and $\mathcal{P}^I$ the set of preference profiles.

Let x_{io} denote the probability that agent i receives object o. A probabilistic assignment $x = [x_{io}]_{i \in I, o \in O}$ is feasible if the following three conditions are satisfied:

$$x_{io} \geq 0 \qquad \forall i \in I, o \in O \qquad \text{(C1)}$$

$$\sum_{o \in O} x_{io} = 1 \qquad \forall i \in I \qquad \text{(C2)}$$

$$\sum_{i \in I} x_{io} \leq q_o \qquad \forall o \in O \qquad \text{(C3)}$$

We denote the set of feasible assignments by $X = \{x \in \mathbb{R}^{nm} \mid (C1), (C2), (C3)\}$. An ordinal assignment algorithm is a mapping from the set of preference profiles to the set of feasible assignments.

The algorithms we introduce shortly select assignments based on rank values. An instance[8] of rank values $v = (v_1, ..., v_m)$ is a vector of length m such that $v_1 \geq v_2 \geq \cdots \geq v_m$, where r^{th} element, v_r, indicates the value of assigning an object ranked r^{th} by an agent. Rank values can be interpreted as utility of the social planner or proxy for utilities of agents. Let $V = \{v \in \mathbb{R}^m \mid 1 = v_1 \geq \cdots \geq v_m = 0\}$ denote the set of all permissible instances of rank values. The normalization, $v_1 = 1$ and $v_m = 0$, is without loss of generality, and the results in this paper do not depend on them.

Given a preference profile $\succ = (\succ_i)_{i \in I} \in \mathcal{P}^I$, let $r_i(o; \succ)$ denote the rank of object o according to the preference ranking of agent i. For readability, we suppress the dependence on the preference profile and simply write $r_i(o)$. For example, if $A \succ_1 B \succ_1 C$, we have $r_1(A) = 1$, $r_1(B) = 2$, and $r_1(C) = 3$, and the corresponding rank values for A, B, and C can be written $v_{r_1(A)}$, $v_{r_1(B)}$, and $v_{r_1(C)}$.

Given a preference profile $\succ \in \mathcal{P}^I$, an assignment $x \in X$, and an instance of rank values $v \in V$, the **Expected Rank Value** for agent $i \in I$ is defined as

$$\Gamma_i(x, v; \succ) = \sum_{o \in O} v_{r_i(o)} x_{io}$$

and the **Minimum Expected Rank Value** over the set of agents is defined as

$$\underline{\Gamma}(x, v; \succ) = \min_{i \in I} \Gamma_i(x, v; \succ).$$

For readability, we suppress the dependence on the preference profile and simply write $\Gamma_i(x, v)$ and $\underline{\Gamma}(x, v)$.

[8]By an instance, we mean a tuple. It should not be understood as an occurrence. As we discuss later, social planners select rank values. To avoid wordiness, we may refer to an instance of rank values as simply rank values when the distinction is unimportant.

3.3 Motivating Example

In this section, we compare two rank-value based allocations: Utilitarian Rank-Value Allocation and Egalitarian Rank-Value Allocation. We show that the Egalitarian allocation is fairer than the Utilitarian allocation, where the notion of fairness is determined by the selection of rank values. We then generalize the Egalitarian Rank-Value Allocation to accommodate various beliefs held by a group of decision makers. The purpose of this section is to illustrate the main idea of the paper using an example. Formal definitions are deferred to Section 3.4.

3.3.1 Comparison of Rank-Value Allocations

The first allocation, the Utilitarian Rank-Value Allocation, maximizes the sum of the expected rank values.[9] Given an instance of rank values $v \in V$, the algorithm solves the Utilitarian Rank-Value Maximization Problem:

$$\text{(URVMP)} \qquad \max_{x \in X} \sum_{i \in I} \Gamma_i(x, v)$$

The second allocation, the Egalitarian Rank-Value Allocation, aims to help the worst-off agent as much as possible by raising the lower bound on the expected rank value. Specifically, it solves the Egalitarian Rank-Value Maximization Problem:

$$\text{(ERVMP)} \qquad \max_{x \in X} \min_{i \in I} \Gamma_i(x, v)$$

We illustrate the difference between the two allocations using an example.

[9] Featherstone (2020) introduces the algorithm as the Rank-Value Mechanism.

Example 1. Consider a problem of assigning four objects (A, B, C, and D) to four agents (1, 2, 3, and 4) with the following preference profile.

Profile I

Agent 1: $A \succ B \succ C \succ D$
Agent 2: $A \succ C \succ B \succ D$
Agent 3: $C \succ B \succ A \succ D$
Agent 4: $B \succ A \succ D \succ C$

The central question is what object should be assigned to agent 4. On one hand, agent 4 should receive object B because he is the only agent who top-ranks object B. On the other hand, agent 4 should receive object D because he is the only agent who does not bottom-rank object D. The assignments for two different choices of rank values are shown in the table below.

Rank Value	Agent	Utilitarian				Egalitarian			
		A	B	C	D	A	B	C	D
$v^{convex} = \left(1, \frac{1}{2}, \frac{1}{6}, 0\right)$	Agent 1	$1-\alpha$	0	0	α	$\frac{15}{31}$	$\frac{12}{31}$	0	$\frac{4}{31}$
	Agent 2	α	0	0	$1-\alpha$	$\frac{16}{31}$	0	$\frac{10}{31}$	$\frac{5}{31}$
	Agent 3	0	0	1	0	0	0	$\frac{21}{31}$	$\frac{10}{31}$
	Agent 4	0	1	0	0	0	$\frac{19}{31}$	0	$\frac{12}{31}$
$v^{concave} = \left(1, \frac{5}{6}, \frac{1}{2}, 0\right)$	Agent 1	0	1	0	0	$\frac{11}{27}$	$\frac{12}{27}$	0	$\frac{4}{27}$
	Agent 2	1	0	0	0	$\frac{16}{27}$	0	$\frac{6}{27}$	$\frac{5}{27}$
	Agent 3	0	0	1	0	0	0	$\frac{21}{27}$	$\frac{6}{27}$
	Agent 4	0	0	0	1	0	$\frac{15}{27}$	0	$\frac{12}{27}$

α is an arbitrary constant such that $0 \leq \alpha \leq 1$.

For the convex rank values, $v^{convex} = \left(1, \frac{1}{2}, \frac{1}{6}, 0\right)$, the Utilitarian allocation prioritizes giving the top-ranked objects to as many agents as possible, which may involve unfair treatment of agents 1 and 2. In contrast, the Egalitarian allocation ensures that each agent has a fair chance of receiving their more-preferred objects.

For the concave rank values, $v^{concave} = \left(1, \frac{5}{6}, \frac{1}{2}, 0\right)$, the Utilitarian allocation prioritizes avoiding assigning the bottom-ranked objects to agents, and therefore, agent 4 is forced to take object D. In contrast, the Egalitarian allocation ensures no particular agent receives a large share of their less-preferred objects. In this sense, the Egalitarian allocation treats agents more fairly than the Utilitarian allocation.

3.3.2 Selection of Rank Values

As demonstrated in Example 1, different choices of rank values lead to assignments that emphasize different aspects of rank fairness. The question is then what is the right instance of rank values to be used in the algorithm? One possibility is to elicit cardinal utilities from agents and compute a "representative" utility.[10] This may be reasonable if the agents have similar cardinal utilities, but accurately eliciting cardinal utilities is notoriously difficult, if at all possible. Alternatively, we may let the social planner choose the rank values based on his personal belief about what is best for the society. However, it seems unreasonable to ask a single individual to bear the burden of identifying *the* right rank values, which dictate who lift themselves out of poverty through education or who extend their lives through organ transplantation.

[10]This is the approach used in Bronfman et al. (2015). They conducted a survey to measure the cardinal utilities and estimated the rank values $v_r = (m + 1 - r)^2$, $r = 1, ..., m$, where r is the rank and m is the number of object types. Using the Random Serial Dictatorship assignment as a starting point, they implement trading cycles that improve the expected rank values.

Instead, we generalize the rank-value algorithms to allow for simultaneous consideration of multiple instances of rank values. Consider a group of three social planners faced with the assignment problem in Example 1. Suppose the first social planner insists that each agent be given the maximum chance of receiving their top-ranked object, and suggests using the rank values, $v^1 = (1, 0, 0, 0)$. If we follow his recommendation, we can guarantee each agent $\frac{1}{2}$ probability of receiving their top-ranked object. The second social planner suggests $v^2 = (1, 1, 0, 0)$, which guarantees each agent $\frac{3}{4}$ probability of receiving one of their two top-ranked objects. The third social planner suggests $v^3 = (1, 1, 1, 0)$ so that no one receives their bottom-ranked object.

To enable a simultaneous consideration of multiple instances of rank values, we modify (ERVMP) as follows:

$$\max_{x \in X} \quad \lambda^1 \underline{\Gamma}(x, v^1) + \lambda^2 \underline{\Gamma}(x, v^2) + \lambda^3 \underline{\Gamma}(x, v^3),$$

where λ^1, λ^2, and λ^3 are non-negative weights that reflect the importance of each instance of rank values (they could be also influenced by the strength of belief or power dynamics of the social planners). To illustrate how the weights affect the solution, we fix $\lambda^1 = \lambda^2 = 1$ and vary λ^3. The resulting assignments are shown in the table below.

Agent	Preference	$\lambda^3 \in [1,3)$				$\lambda^3 \in (3,6)$				$\lambda^3 \in (6,\infty)$			
		A	B	C	D	A	B	C	D	A	B	C	D
Agent 1	$A \succ B \succ C \succ D$	$\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{3}$	0	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0
Agent 2	$A \succ C \succ B \succ D$	$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	0	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0
Agent 3	$C \succ B \succ A \succ D$	0	0	$\frac{3}{4}$	$\frac{1}{4}$	0	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0
Agent 4	$B \succ A \succ D \succ C$	0	$\frac{3}{4}$	0	$\frac{1}{4}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	1

When $\lambda^3 = 3$ or 6, any convex combination of the adjacent assignments also solves the maximization problem. The three assignments in the table and their convex combinations define the Pareto-frontier of $(\underline{\Gamma}(x, v^1), \underline{\Gamma}(x, v^2), \underline{\Gamma}(x, v^3))$. They are maximally egalitarian in the sense that no assignment can unambiguously improve the expected rank values of the worst-off agents.

Note that maximizing $\frac{1}{3}\underline{\Gamma}(x, v^1) + \frac{1}{3}\underline{\Gamma}(x, v^2) + \frac{1}{3}\underline{\Gamma}(x, v^3)$ is different from maximizing $\underline{\Gamma}(x, \overline{v})$, where $\overline{v} = \frac{1}{3}(v^1 + v^2 + v^3) = \left(1, \frac{2}{3}, \frac{1}{3}, 0\right)$. Indeed, the allocation associated with the latter is dominated by the allocation associated with the former in regard to $(\underline{\Gamma}(x, v^1), \underline{\Gamma}(x, v^2), \underline{\Gamma}(x, v^3))$ as shown in the table below.

Agent	Preference	$\max_x \frac{1}{3}\sum_{j=1}^{3} \underline{\Gamma}(x, v^j)$				$\max_x \underline{\Gamma}(x, \frac{1}{3}\sum_{j=1}^{3} v^j)$			
		A	B	C	D	A	B	C	D
Agent 1	$A \succ B \succ C \succ D$	$\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{10}{22}$	$\frac{9}{22}$	0	$\frac{3}{22}$
Agent 2	$A \succ C \succ B \succ D$	$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{12}{22}$	0	$\frac{6}{22}$	$\frac{4}{22}$
Agent 3	$C \succ B \succ A \succ D$	0	0	$\frac{3}{4}$	$\frac{1}{4}$	0	0	$\frac{16}{22}$	$\frac{6}{22}$
Agent 4	$B \succ A \succ D \succ C$	0	$\frac{3}{4}$	0	$\frac{1}{4}$	0	$\frac{13}{22}$	0	$\frac{9}{22}$

3.4 Rank-Egalitarian Dominance

In this section, we formally define the dominance relationship between assignments based on egalitarian rank-fairness. Let $\widetilde{V} \subseteq V$ be the set of relevant or important rank values. The set $\widetilde{V}$ can be a singleton or the set of all permissible rank values, i.e., $\widetilde{V} = V = \{v \in \mathbb{R}^m \mid 1 = v_1 \geq \cdots \geq v_m = 0\}$. In the example above with three social planners, $\widetilde{V} = \{v^1, v^2, v^3\}$. For an arbitrary choice of $\widetilde{V}$, we have the following definitions.

Definition 1. Fix a preference profile. Given a set $\widetilde{V} \subseteq V$ of rank values and assignments $x, x' \in X$, we say $\boldsymbol{x'}$ **rank-egalitarian (RE) dominates** $\boldsymbol{x}$ **over** $\widetilde{\boldsymbol{V}}$ if $\underline{\Gamma}(x', v) \geq \underline{\Gamma}(x, v)$ for all $v \in \widetilde{V}$ and $\underline{\Gamma}(x', v') > \underline{\Gamma}(x, v')$ for at least one $v' \in \widetilde{V}$.

Definition 2. Fix a preference profile. Given a set $\widetilde{V} \subseteq V$ of rank values, we say an assignment $x \in X$ is **rank-egalitarian undominated (REU) over** $\widetilde{\boldsymbol{V}}$ if there does not exist another assignment $x' \in X$ that RE-dominates x over $\widetilde{V}$.

We denote the set of all REU assignments over $\widetilde{V}$ by $\mathcal{X}^{REU}(\widetilde{V})$. Because $\underline{\Gamma}(x, v)$ depends on the preference profile, so does the definition of REU over $\widetilde{V}$. We write $\mathcal{X}^{REU}(\widetilde{V})$ instead of $\mathcal{X}^{REU}(\widetilde{V}; \succ)$ for readability, but readers should keep in mind that $\mathcal{X}^{REU}(\widetilde{V})$ is defined for each preference profile.

The concept of REU is similar to that of Pareto dominance. Pareto dominance concerns the trade-off among agents, while the REU deals with the trade-off across possible instances of rank values. If an assignment is REU over $\widetilde{V}$, it is impossible to make the worst-off agents better off at some $v \in \widetilde{V}$ without making the worst-off agents worse off at another $v' \in \widetilde{V}$. Note that the set of worst-off agents according to v is generally different from the set of worst-off agents according to v'. The REU assignments are maximally fair in the sense that there is no other assignment that unambiguously makes worst-off agents better off.

3.4.1 Efficiency

We first show that the REU assignments are not necessarily efficient. Consider Profile I in Example 1, reproduced in the table below. Let $\widetilde{V} = \{v^3\} = \{(1, 1, 1, 0)\}$. Then, any allocation in which agent 4 receives object D with certainty is REU over $\widetilde{V}$. In particular, the Inefficient assignment in the table is REU over $\widetilde{V}$.

Agent	Profile I	Efficient				Inefficient			
		A	B	C	D	A	B	C	D
Agent 1	$A \succ B \succ C \succ D$	0	1	0	0	0	0	1	0
Agent 2	$A \succ C \succ B \succ D$	1	0	0	0	0	1	0	0
Agent 3	$C \succ B \succ A \succ D$	0	0	1	0	1	0	0	0
Agent 4	$B \succ A \succ D \succ C$	0	0	0	1	0	0	0	1

Although the Inefficient assignment is Pareto efficient in terms of $\Gamma_i(\cdot, v^3)$, it is inefficient in terms of $\Gamma_i(\cdot, v^1)$ and $\Gamma_i(\cdot, v^2)$. Because (ERVMP) concerns only the expected rank values of the worst-off agents, it allows an inefficient allocation of objects among better-off agents. In other words, it is oblivious to the allocation of A, B, and C among agents 1, 2, and 3.

To avoid this kind of inefficiency, we impose the stochastic-dominance efficiency[11], and express it in terms of rank values. To this end, we first list all the extreme points of $V = \{v \in \mathbb{R}^m \mid 1 = v_1 \geq \cdots \geq v_m = 0\}$, the set of all permissible rank values. Let $v^j \in V$ denote the vector of j ones followed by $m - j$ zeros. Formally, $v^j = \left(\mathbb{1}_{\{r \leq j\}}\right)_{r=1,...,m}$ for $j = 1, ..., m-1$. We call these rank values dichotomous because one possible interpretation is that each object is either acceptable or unacceptable. Let $V^D = \{v^j \mid j = 1, ..., m-1\}$ denote the set of all **dichotomous rank values**.

Definition 3. We say an assignment $x \in X$ is **stochastic-dominance (sd) Pareto efficient** if there does not exist another assignment $x' \in X$ such that (i) for each agent $i \in I$, we have $\Gamma_i(x', v) \geq \Gamma_i(x, v)$ for all $v \in V^D$; and (ii) there is at least one agent $i' \in I$ such that $\Gamma_{i'}(x', v') > \Gamma_{i'}(x, v')$ for some $v' \in V^D$.

[11] Bogomolnaia and Moulin (2001) introduced the concept and termed it ordinal efficiency.

Note that we have an equivalent definition when V^D is replaced by V because $\Gamma_i(x, v)$ is linear in v and V^D is the set of extreme points of V.

Definition 4. Fix a preference profile. Given a set $\widetilde{V} \subseteq V$ of rank values, we say an assignment $x \in X$ is **rank-egalitarian undominated* (REU*) over $\widetilde{\boldsymbol{V}}$** if x is REU over $\widetilde{V}$ and sd-Pareto efficient.

We denote the set of all REU* assignments over $\widetilde{V}$ by $\mathcal{X}^{REU^*}(\widetilde{V})$. Obviously, we have $\mathcal{X}^{REU^*}(\widetilde{V}) \subseteq \mathcal{X}^{REU}(\widetilde{V})$ for any $\widetilde{V} \subseteq V$. We briefly discuss (non-)relation between $\mathcal{X}^{REU^*}(\widetilde{V})$ for various choices of $\widetilde{V}$ in Section 3.5.2.

3.4.2 Incentive

The following example demonstrates that a mechanism that always maps to a REU assignment cannot be strategy-proof.

Example 2. Consider Profile II and Profile II′ in the table below. Given any $v_2 \in (0, 1)$, the assignment in the table, where $\alpha = \frac{1}{2-v_2^2}$ and $\beta = \frac{1-v_2}{2-v_2^2}$, is uniquely REU over $\widetilde{V} = \{(1, v_2, 0)\}$ for each profile.

Agent	Profile II	A	B	C	Profile II′	A	B	C
Agent 1	$A \succ B \succ C$	$\frac{1}{2}$	$\frac{1}{2}$	0	$A \succ B \succ C$	α	β	$1 - \alpha - \beta$
Agent 2	$A \succ C \succ B$	$\frac{1}{2}$	0	$\frac{1}{2}$	$A \succ C \succ B$	$1 - \alpha$	0	α
Agent 3	$B \succ C \succ A$	0	$\frac{1}{2}$	$\frac{1}{2}$	$B \succ A \succ C$	0	$1 - \beta$	β

Under Profile II, agent 3 can first-order stochastically improve his allocation by misreporting $B \succ A \succ C$ to induce Profile II′. Therefore, ensuring REU assignments over $\widetilde{V}$ necessarily leads to a violation of strategy-proofness.

3.5 Necessary and Sufficient Conditions for REU

We now show a connection between the set of REU assignments and the set of optimal solutions to a linear programming problem which maximizes the weighted sum of the minimum expected rank values. We first focus our attention on finite sets, $\ddot{V} \subseteq V$, of rank values. We use the symbol $\ddot{V}$ instead of $\widetilde{V}$ to emphasize that the set is a collection of finitely many instances of rank values.

Theorem 1. Fix a preference profile. Let a finite set $\ddot{V} \subseteq V$ of rank values be given. If $x \in X$ is an optimal solution to the Weighted Egalitarian Rank-Value Maximization Problem

$$\text{(WERVMP)} \qquad \max_{x \in X} \sum_{v \in \ddot{V}} \lambda^v \underline{\Gamma}(x, v)$$

for some set of strictly positive weights $(\lambda^v)_{v \in \ddot{V}}$, then $x \in \mathcal{X}^{REU}(\ddot{V})$. Furthermore, if x is a unique optimal solution to (WERVMP), then $x \in \mathcal{X}^{REU^*}(\ddot{V})$. Conversely, if $x \in \mathcal{X}^{REU}(\ddot{V})$, then x is an optimal solution to (WERVMP) for some set of non-negative weights $(\lambda^v)_{v \in \ddot{V}}$, not all zero.

The relation between $\mathcal{X}^{REU}(\widetilde{V})$ and (WERVMP) is not one-to-one: One REU assignment can be an optimal solution to multiple (WERVMP), and one (WERVMP) can have multiple optimal solutions that are REU. For example, in Section 3.3.2, we show that the sets of weights, $\lambda^1 = \lambda^2 = 1$ and $\lambda^3 \in [1, 3)$, produce the same solution, and when $\lambda^3 = 3$, there are multiple optimal solutions to (WERVMP).

We prove Theorem 1 using two lemmas. Lemma 1 establishes the convexity of the set of attainable $(\underline{\Gamma}(x, v))_{v \in \ddot{V}}$ and Lemma 2 connects the Separating Hyperplane Theorem to the optimality of a solution to a linear programming problem.

Lemma 1. Let a finite set $\ddot{V} \subseteq V$ of rank values be given. The set of attainable vectors of minimum expected rank values is convex. That is,

$$U = \left\{ u \in \mathbb{R}^{|\ddot{V}|} \;\middle|\; u \leq \left(\underline{\Gamma}(x,v)\right)_{v \in \ddot{V}} \text{ for some } x \in X \right\} \text{ is convex.}$$

Proof. Consider any $u, u' \in U$. By definition, there is $x, x' \in X$ such that $u \leq (\underline{\Gamma}(x,v))_{v \in \ddot{V}}$ and $u' \leq (\underline{\Gamma}(x',v))_{v \in \ddot{V}}$. Then, for any $\alpha \in (0,1)$, we have

$$\begin{aligned}
\alpha u + (1-\alpha)u' &\leq \alpha(\underline{\Gamma}(x,v))_{v \in \ddot{V}} + (1-\alpha)(\underline{\Gamma}(x',v))_{v \in \ddot{V}} \\
&= \alpha \left(\min_{i \in I} \Gamma_i(x,v) \right)_{v \in \ddot{V}} + (1-\alpha) \left(\min_{i \in I} \Gamma_i(x',v) \right)_{v \in \ddot{V}} \\
&\leq \left(\min_{i \in I} \{ \alpha \Gamma_i(x,v) + (1-\alpha)\Gamma_i(x',v) \} \right)_{v \in \ddot{V}} \\
&= \left(\min_{i \in I} \Gamma_i(\alpha x + (1-\alpha)x', v) \right)_{v \in \ddot{V}} \\
&= (\underline{\Gamma}(\alpha x + (1-\alpha)x', v))_{v \in \ddot{V}}
\end{aligned}$$

The second to the last equality holds because the expected rank value is linear in x. Because X is convex, $\alpha x + (1-\alpha)x' \in X$, and therefore, $\alpha u + (1-\alpha)u' \in U$. □

Lemma 2. Let a finite set $\ddot{V} \subseteq V$ of rank values be given. Given an assignment $\overline{x} \in X$, if the sets,

$$\begin{aligned}
U &= \left\{ u \in \mathbb{R}^{|\ddot{V}|} \;\middle|\; u \leq (\underline{\Gamma}(x,v))_{v \in \ddot{V}} \text{ for some } x \in X \right\} \text{ and} \\
U^* &= \left\{ u \in \mathbb{R}^{|\ddot{V}|} \;\middle|\; u > (\underline{\Gamma}(\overline{x},v))_{v \in \ddot{V}} \right\},
\end{aligned}$$

are disjoint, then $\overline{x}$ is an optimal solution to (WERVMP) for some set of non-negative weights $(\lambda^v)_{v \in \ddot{V}}$, not all zero.

Proof. Suppose U and U^* are disjoint for some assignment $\overline{x} \in X$. We construct the set of non-negative weights $(\lambda^v)_{v \in \ddot{V}}$ for which $\overline{x}$ solves (WERVMP).

By Lemma 1, U is convex. Because U^* is an open hyper-rectangle, it is convex. Then, by the Separating Hyperplane Theorem, there exists a non-zero vector $a \in \mathbb{R}^{|\ddot{V}|}$ and a constant $b \in \mathbb{R}$ such that $a \cdot u \leq b$ for all $u \in U$ and $a \cdot u > b$ for all $u \in U^*$.

We claim that each element of a is non-negative. To see this, suppose $a_j < 0$ for some $j \in \{1, ..., |\ddot{V}|\}$. Take an arbitrary $u \in U^*$, and replace the j^{th} element of u by $u_j + (2a \cdot u - b)/(-a_j)$ and call this new vector u'. Clearly, $u' \in U^*$. Furthermore, $a \cdot u' = a \cdot u + a_j(2a \cdot u - b)/(-a_j) = -(a \cdot u - b) < 0$, violating the inequality implied by the Separating Hyperplane Theorem. Therefore, $a \geq 0$.

Now, let $\overline{u} = (\underline{\Gamma}(\overline{x}, v))_{v \in \ddot{V}}$. We claim that $a \cdot \overline{u} = b$. Consider a sequence of vectors $u^k = \overline{u} + (\frac{1}{k}, ..., \frac{1}{k})$ for $k \in \mathbb{N}$. Because $u^k \in U^*$, we have $a \cdot u^k > b$. Take k to infinity and we obtain $a \cdot \overline{u} \geq b$. But $\overline{u} \in U$, so $a \cdot \overline{u} \leq b$. Thus, $a \cdot \overline{u} = b$. Therefore, $\overline{x}$ solves (WERVMP) for $(\lambda^v)_{v \in \ddot{V}} = a$. □

If the set U of attainable vectors of minimum rank values is strictly convex, then (WERVMP) has a unique solution, and the solution lies on the Pareto-frontier of U. However, if the boundary of U is linear, there are multiple solutions to (WERVMP), and some of the solutions may not lie on the Pareto-frontier of U. Therefore, the efficiency cannot be guaranteed without the uniqueness. We now prove Theorem 1.

Proof of Theorem 1. Given a finite set $\ddot{V} \subseteq V$, suppose $\overline{x} \in X$ solves (WERVMP) for some set of strictly positive weights $(\lambda^v)_{v \in \ddot{V}}$. Suppose, toward a contradiction, that $\overline{x} \notin \mathcal{X}^{REU}(\ddot{V})$. Then there is $x' \in X$ such that $\underline{\Gamma}(x', v) \geq \underline{\Gamma}(\overline{x}, v)$ for all $v \in \ddot{V}$ and $\underline{\Gamma}(x', v') > \underline{\Gamma}(\overline{x}, v')$ for some $v' \in \ddot{V}$. Then, because the weights, $(\lambda^v)_{v \in \ddot{V}}$, are strictly positive, we have $\sum_{v\in\ddot{V}} \lambda^v \underline{\Gamma}^v(x',v) > \sum_{v\in\ddot{V}} \lambda^v \underline{\Gamma}^v(\overline{x},v)$, contradicting the assumption that $\overline{x}$ is an optimal solution to (WERVMP). Therefore, $\overline{x} \in \mathcal{X}^{REU}(\ddot{V})$.

Now, suppose $\overline{x} \in X$ is a unique optimal solution to (WERVMP). If $\overline{x} \notin \mathcal{X}^{REU^*}(\ddot{V})$, then $\overline{x}$ is sd-Pareto inefficient and there is another assignment $x' \in X$ such that $\Gamma_i(x',v) \geq \Gamma_i(\overline{x},v)$ for all $i \in I$ for all $v \in V$. It follows that $\underline{\Gamma}(x',v) \geq \underline{\Gamma}(\overline{x},v)$ for all $v \in \ddot{V}$ because $\ddot{V} \subseteq V$. But then $\sum_{v\in\ddot{V}} \lambda^v \underline{\Gamma}^v(x',v) \geq \sum_{v\in\ddot{V}} \lambda^v \underline{\Gamma}^v(\overline{x},v)$, contradicting the assumption that $\overline{x}$ is the unique optimal solution to (WERVMP). Therefore, the unique maximizer $\overline{x}$ must be REU* over $\widetilde{V}$.

Conversely, suppose $\overline{x} \in \mathcal{X}^{REU}(\ddot{V})$. Let

$$\begin{aligned} U &= \left\{ u \in \mathbb{R}^{|\ddot{V}|} \;\middle|\; u \leq \left(\underline{\Gamma}(x,v)\right)_{v\in\ddot{V}} \text{ for some } x \in X \right\} \text{ and} \\ U^* &= \left\{ u \in \mathbb{R}^{|\ddot{V}|} \;\middle|\; u > \left(\underline{\Gamma}(\overline{x},v)\right)_{v\in\ddot{V}} \right\}. \end{aligned}$$

The sets U and U^* are disjoint because if there is $u' \in U \cap U^*$ then there must be $x' \in X$ such that $\left(\underline{\Gamma}(x',v)\right)_{v\in\ddot{V}} \geq u' > \left(\underline{\Gamma}(\overline{x},v)\right)_{v\in\ddot{V}}$, contradicting the assumption that $\overline{x} \in \mathcal{X}^{REU}(\ddot{V})$. The rest follows from Lemma 2. □

Given Theorem 1, for a finite set $\ddot{V} \subseteq V$ of rank values, it is possible to generate all assignments in $\mathcal{X}^{REU}(\ddot{V})$ by systematically varying the weights in (WERVMP). However, the analysis of $\mathcal{X}^{REU}(\widetilde{V})$ for an arbitrary $\widetilde{V} \subseteq V$, which may be an infinite set, requires a generalization of the theorem. This is because (WERVMP) has hidden constraints of the form $\underline{\Gamma}(x,v) \geq \Gamma_i(x,v)$ for each $i \in I$ for each $v \in \widetilde{V}$. Obviously, (WERVMP) cannot be properly formulated if $\widetilde{V}$ is an infinite set. It turns out that we need only finitely many instances of rank values to construct (WERVMP) even when $\widetilde{V}$ is an infinite set.

Theorem 2. Fix a preference profile. Let a set $\widetilde{V} \subseteq V$ of rank values be given. If there is a finite subset $\ddot{V} \subseteq \widetilde{V}$ of rank values and a set of non-negative weights $(\lambda^v)_{v \in \ddot{V}}$, not all zero, such that an assignment $x \in X$ is a unique optimal solution to the Weighted Egalitarian Rank-Value Maximization Problem

$$\text{(WERVMP)} \qquad \max_{x \in X} \sum_{v \in \ddot{V}} \lambda^v \underline{\Gamma}(x, v)$$

then $x \in \mathcal{X}^{REU^*}(\widetilde{V})$. Conversely, if $x \in \mathcal{X}^{REU^*}(\widetilde{V})$, then, for any finite union $\breve{V} \subseteq V$ of finite polytopes that encloses $\widetilde{V}$, there is a finite set $\ddot{V} \subseteq \breve{V}$ and a set of strictly positive weights $(\lambda^v)_{v \in \ddot{V}}$ such that x is an optimal solution to (WERVMP).

Theorem 2 is similar to Theorem 1, but there are some important differences. For $x \in X$ to be REU over $\widetilde{V}$, in Theorem 1 where $\widetilde{V}$ is a finite set, it is sufficient for x to be a solution to (WERVMP), whereas in Theorem 2 where $\widetilde{V}$ can be an infinite set, x must be a unique solution. This is because x being REU over $\ddot{V} \subset \widetilde{V}$ does not necessarily imply x being REU over $\widetilde{V}$ as we see in Section 3.5.2.

For necessity, if $\widetilde{V}$ is a finite union of finite polytopes, we let $\breve{V} = \widetilde{V}$ and the set $\ddot{V}$ of instances of rank values that constitute (WERVMP) is a subset of $\widetilde{V}$. However, if the boundary of $\widetilde{V}$ has a curved surface, no matter what enclosure $\breve{V}$ we select, there is always a point that is inside $\breve{V}$ and outside $\widetilde{V}$. Therefore, we can only guarantee that instances of rank values appearing in (WERVMP) reside inside $\breve{V}$ and not necessarily inside $\widetilde{V}$. We do not anticipate this to be an issue because practically meaningful $\widetilde{V}$ is unlikely to have a curved surface.

Although Theorem 2 guarantees the existence of (WERVMP) that generates REU* assignment, it stays agnostic about the identity of $\ddot{V}$. Therefore, Theorem 2 does not provide a machinery to produce REU* assignments. In contrast, if $\widetilde{V}$ is a finite set as in Theorem 1, the REU* assignments can be systematically generated by varying the weights in (WERVMP). We defer the discussion on how to generate REU* assignment over an infinite set $\widetilde{V}$ to Section 3.6.

The proof of Theorem 2 relies on the division of $\widetilde{V}$ into subsets of instances of rank values such that a particular agent is worst off in each subset. Let $\widehat{V} \subseteq V$ be a finite convex polytope. Given a preference profile $\succ \in \mathcal{P}^I$ and an assignment $\overline{x} \in X$, let $W_i(\overline{x}, \widehat{V}; \succ)$ denote the set of $v \in \widehat{V}$ for which agent i is worst off under assignment $\overline{x}$. We suppress the dependence on the preference profile and simply write $W_i(\overline{x}, \widehat{V})$. Formally,

$$W_i(\overline{x}, \widehat{V}) = \left\{ v \in \widehat{V} \;\middle|\; \Gamma_i(\overline{x}, v) \leq \Gamma_{i'}(\overline{x}, v) \quad \forall i' \in I \right\}.$$

Because $W_i(\overline{x}, \widehat{V})$ is an intersection of $\widehat{V}$ and $n-1$ half-space, each $W_i(\overline{x}, \widehat{V})$ is convex and has finitely many extreme points. For each $i \in I$, let $\ddot{W}_i(\overline{x}, \widehat{V})$ denote the set of extreme points of $W_i(\overline{x}, \widehat{V})$. With these notations, we are now ready to prove Theorem 2.

Proof of Theorem 2. Suppose $\overline{x} \in X$ is a unique solution to (WERVMP) for a finite set $\ddot{V} \subseteq \widetilde{V}$ of rank values and non-negative weights, $(\lambda^v)_{v \in \ddot{V}}$, not all zero. Suppose, toward a contradiction, that $\overline{x} \notin \mathcal{X}^{REU^*}(\widetilde{V})$. Then, either $\overline{x} \notin \mathcal{X}^{REU}(\widetilde{V})$ or $\overline{x}$ is sd-Pareto inefficient. Either way, there is $x' \in X$ such that $\underline{\Gamma}(x', v) \geq \underline{\Gamma}(\overline{x}, v)$ for all $v \in \widetilde{V}$. Then, in particular, $\underline{\Gamma}(x', v) \geq \underline{\Gamma}(\overline{x}, v)$ for all $v \in \ddot{V}$ because $\ddot{V} \subseteq \widetilde{V}$. It follows that $\sum_{v \in \ddot{V}} \lambda^v \underline{\Gamma}(x', v) \geq \sum_{v \in \ddot{V}} \lambda^v \underline{\Gamma}(\overline{x}, v)$, contradicting the assumption that $\overline{x}$ is the unique optimal solution to (WERVMP). Therefore, $\overline{x} \in \mathcal{X}^{REU^*}(\widetilde{V})$.

Conversely, suppose $\overline{x} \in \mathcal{X}^{REU^*}(\widetilde{V})$. Consider any finite union $\breve{V} \subseteq V$ of finite polytopes that encloses $\widetilde{V}$. That is, $\widetilde{V} \subseteq \breve{V}$. Let $\mathcal{V}$ be a decomposition of $\breve{V}$ into finite convex polytopes. That is, $\bigcup_{\widehat{V} \in \mathcal{V}} \widehat{V} = \breve{V}$, where each $\widehat{V}$ is a finite convex polytope. We further decompose each $\widehat{V}$ into $W_i(\overline{x}, \widehat{V})$, $i \in I$. Let $\ddot{V} = \bigcup_{\widehat{V} \in \mathcal{V}} \bigcup_{i \in I} \ddot{W}_i(\overline{x}, \widehat{V})$ be the set of all extreme points of this decomposition, and define

$$U = \left\{ u \in \mathbb{R}^{|\ddot{V}|} \;\middle|\; u \leq (\underline{\Gamma}(x, v))_{v \in \ddot{V}} \text{ for some } x \in X \right\} \text{ and}$$
$$U^* = \left\{ u \in \mathbb{R}^{|\ddot{V}|} \;\middle|\; u > (\underline{\Gamma}(\overline{x}, v))_{v \in \ddot{V}} \right\}.$$

If U and U^* are disjoint, the existence of (WERVMP) follows from Lemma 2, and the weights can be made strictly positive by dropping the instances of rank values with zero weights. Therefore, it remains to show that U and U^* are disjoint.

Suppose, toward a contradiction, that there is $u' \in U \cap U^*$. Then there is $x' \in X$ such that $(\underline{\Gamma}(x', v))_{v \in \ddot{V}} \geq u' > (\underline{\Gamma}(\overline{x}, v))_{v \in \ddot{V}}$. Pick any $\widehat{V} \in \mathcal{V}$. Pick any $i' \in I$ with non-empty $W_{i'}(\overline{x}, \widehat{V})$. Pick any $\overline{v} \in W_{i'}(\overline{x}, \widehat{V})$. There is a set of non-negative weights, $(\alpha^w)_{w \in \ddot{W}_{i'}(\overline{x}, \widehat{V})}$, such that $\overline{v} = \sum_{w \in \ddot{W}_{i'}(\overline{x}, \widehat{V})} \alpha^w w$. Using the linearity of $\Gamma_i(x, v)$ in v and the concavity of the min function, we obtain $\underline{\Gamma}(x', \overline{v}) \geq \sum_{w \in \ddot{W}_{i'}(\overline{x}, \widehat{V})} \alpha^w \underline{\Gamma}(x', w)$. We also have $\sum_{w \in \ddot{W}_{i'}(\overline{x}, \widehat{V})} \alpha^w \underline{\Gamma}(\overline{x}, w) = \underline{\Gamma}(\overline{x}, \overline{v})$ because agent i' is worst off for all $w \in \ddot{W}_{i'}(\overline{x}, \widehat{V})$ and $\Gamma_i(x, v)$ is linear in v. Because $\underline{\Gamma}(x', w) > \underline{\Gamma}(\overline{x}, w)$ for all $w \in \ddot{W}_{i'}(\overline{x}, \widehat{V})$, it follows that $\underline{\Gamma}(x', \overline{v}) > \underline{\Gamma}(\overline{x}, \overline{v})$.

Because this is true for arbitrary $\overline{v} \in W_{i'}(\overline{x}, \widehat{V})$, we have $\underline{\Gamma}(x', v) > \underline{\Gamma}(\overline{x}, v)$ for all $v \in W_{i'}(\overline{x}, \widehat{V})$. Because this is true for arbitrary i' and $\widehat{V} = \bigcup_{i' \in I} W_{i'}(\overline{x}, \widehat{V})$, we have $\underline{\Gamma}(x', v) > \underline{\Gamma}(\overline{x}, v)$ for all $v \in \widehat{V}$. Because this is true for arbitrary $\widehat{V} \in \mathcal{V}$ and $\widetilde{V} \subseteq \bigcup_{\widehat{V} \in \mathcal{V}} \widehat{V}$, we have $\underline{\Gamma}(x', v) > \underline{\Gamma}(\overline{x}, v)$ for all $v \in \widetilde{V}$. This contradicts the assumption that $\overline{x} \in \mathcal{X}^{REU^*}(\widetilde{V})$. Therefore, U and U^* are disjoint. □

3.5.1 Uniqueness of Solutions

Because the uniqueness of solutions to (WERVMP) plays a key role in Theorem 2, we wish to understand it better. However, whether there is a unique solution to (WERVMP) depends on the interaction of preference profile and the instance of rank values, making it difficult to predict the uniqueness of solutions before actually solving (WERVMP).[12] The following example demonstrates this point.

Example 3. Consider the following preference profile and assignments. Both assignments solve (ERVMP) $\max_{x \in X} \underline{\Gamma}(x, v)$ for $v = (1, \frac{2}{3}, \frac{1}{3}, 0)$.

Agent	Preference	A	B	C	D	A	B	C	D
Agent 1	$A \succ B \succ C \succ D$	$\frac{4}{11}$	$\frac{5}{11}$	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{4}{11}$	$\frac{1}{2}$	0	$\frac{3}{22}$
Agent 2	$A \succ C \succ D \succ B$	$\frac{1}{11}$	0	$\frac{10}{11}$	0	$\frac{1}{11}$	0	$\frac{10}{11}$	0
Agent 3	$A \succ B \succ D \succ C$	$\frac{6}{11}$	0	0	$\frac{5}{11}$	$\frac{6}{11}$	0	0	$\frac{5}{11}$
Agent 4	$B \succ C \succ D \succ A$	0	$\frac{6}{11}$	0	$\frac{5}{11}$	0	$\frac{1}{2}$	$\frac{1}{11}$	$\frac{9}{22}$

In fact, for any rank values $v = (1, v_2, v_3, 0)$ such that

$$\frac{v_2 - v_3}{1 - v_2} = \frac{v_3}{v_2 - v_3},$$

agents 1 and 4 can trade objects B, C, and D without affecting their expected rank values. Indeed, there are multiple optimal solutions to (ERVMP) for these rank values except when v_2 solves $v_2^3 + v_2^2 = 1$. At this v_2, the equation $v_2^2 + (1 - v_3)v_2 = 1$ is also satisfied, and it is optimal to allocate the entire C to agent 2.

[12] After solving (WERVMP), we can determine whether the obtained solution is unique by solving one more linear programming problem as described in Appa (2002).

In this example, a small perturbation to the rank value restores the uniqueness of the solution to (ERVMP). However, for some preference profiles, there are multiple solutions to (ERVMP) for any choice of rank value. This prompts us to develop an algorithm that does not rely on the uniqueness of solutions to (ERVMP), which we explore in Section 3.6. Our current understanding of the uniqueness of solutions to (ERVMP) is limited to the following.

Proposition 1. (i) For $n = m = 3$, if no pair of agents have the same preferences, there is a unique solution to (ERVMP) for any rank values $v = (1, v_2, 0)$ such that $0 < v_2 < 1$. (ii) For $n \geq 4$, each agent having a distinct preference is not sufficient for the uniqueness of solution to (ERVMP).

The proof of Proposition 1 (i) is provided in Appendix 3.8.1. Example 3 serves as a proof of Proposition 1 (ii). Whether the multiplicity of optimal solutions is ubiquitous in real-world problems is an interesting topic for future research.

3.5.2 (Non-)Relation Among REU Assignments

The following is an immediate consequence of Theorem 2.

Corollary 1. If $\widetilde{V} \subseteq V$ is a finite union of finite polytopes, we have

$$\mathcal{X}^{REU^*}(\widetilde{V}) \subseteq \bigcup_{\ddot{V} \subseteq \widetilde{V} \,|\, |\ddot{V}| < \infty} \mathcal{X}^{REU^*}(\ddot{V}).$$

Obviously, it is practically impossible to enumerate all finite subsets of $\widetilde{V}$ even when we have a good bound for $|\ddot{V}|$. Unfortunately, we cannot say much beyond this about the relation between $\mathcal{X}^{REU^*}(\widetilde{V})$ for various choices of $\widetilde{V} \subseteq V$.

Proposition 2. There are preference profiles for which the following relations do not hold:

(i) For $V', V'' \subseteq V$, if $V' \subseteq V''$ then $\mathcal{X}^{REU^*}(V') \subseteq \mathcal{X}^{REU^*}(V'')$.

(ii) $\mathcal{X}^{REU^*}(V^D) \subseteq \mathcal{X}^{REU^*}(V)$.

(iii) $\mathcal{X}^{REU^*}(V) \subseteq \mathcal{X}^{REU^*}(V^D) \cup \left(\bigcup_{v \in V} \mathcal{X}^{REU^*}(\{v\})\right)$.

Recall that $V^D = \left\{ \left(\mathbb{1}_{\{r \leq j\}}\right)_{r=1,\ldots,m} \middle| j = 1, ..., m-1 \right\}$ is the set of all dichotomous rank values, which are the extreme points of the set V of all permissible rank values. Even though V^D is representative of V, there is no clear relation between the sets of REU* assignments between the two. Given the non-relation between REU assignments, we focus on generating some REU assignments instead of all of them.

Example 3 provides a proof of (i). The assignment on the right is REU* over $\{(1, \frac{2}{3}, \frac{1}{3}, 0)\}$, but it is not REU* over $\{(1, \frac{2}{3}, \frac{1}{3}, 0), (1, 1, 1, 0)\}$ because it is RE-dominated by the assignment on the left. The proofs of (ii) and (iii) are provided in Appendix 3.8.2 and 3.8.3, respectively.

3.6 Generating REU Assignments

Now, we address the question of how to generate REU assignments over an arbitrary $\widetilde{V} \subseteq V$. One possible approach is to pick an arbitrary $\ddot{V} \subseteq \widetilde{V}$, construct (WERVMP) with some weights, and hope that the optimal solution is unique. This approach is simple, but inefficient. Also, when there is a small group of agents who rank unpopular objects highly, we may never obtain a unique optimal solution for any choice of $\ddot{V}$ because (WERVMP) is oblivious to better-off agents.

Here, we propose an alternative way of checking whether an assignment is REU. It directly examines whether there is another assignment that weakly improves the minimum expected rank value for every $v \in \widetilde{V}$, with a strict improvement for at least one $v \in \widetilde{V}$. The following proposition is stated for the case where $\widetilde{V}$ is a finite convex polytope, but it can be easily modified for $\widetilde{V}$ that is a finite union of finite polytopes. The proof is similar to that for Theorem 2, and is deferred to Appendix 3.8.4.

Proposition 3. (Test of RE dominance) Let a finite convex polytope $\widehat{V} \subseteq V$ and an assignment $\overline{x} \in X$ be given. Let $\ddot{V} = \bigcup_{i \in I} \ddot{W}_i(\overline{x}, \widehat{V})$. Let $I' \subseteq I$ be the set of agents for which the interior of $W_i(\overline{x}, \widehat{V})$ is non-empty. For each $i \in I'$, pick an arbitrary point $\overline{w}_i$ in the interior of $W_i(\overline{x}, \widehat{V})$. Solve the following problem:

$$\begin{aligned} \max_{x \in X} \quad & \sum_{i \in I'} s_i \\ \text{s.t.} \quad & \underline{\Gamma}(x, v) \geq \underline{\Gamma}(\overline{x}, v) && \forall v \in \ddot{V} \\ & \underline{\Gamma}(x, \overline{w}_i) \geq \underline{\Gamma}(\overline{x}, \overline{w}_i) + s_i && \forall i \in I' \\ & s_i \geq 0 && \forall i \in I' \end{aligned}$$

The maximized objective value is zero if and only if $\overline{x} \in \mathcal{X}^{REU}(\widehat{V})$.

When $\overline{x}$ is not REU over $\widehat{V}$, the test provides $v \in \widehat{V}$ at which the minimum expected rank value can be improved, namely, $\overline{w}_{i'}$. By adding $\overline{w}_{i'}$ to $\ddot{V}$, we can contract the set of optimal solutions to (WERVMP). The following corollary states that we can indeed construct $\ddot{V}$ iteratively. Instead of maximizing the weighted sum of the minimum expected rank values, we can sequentially set the lower bound on the minimum expected rank value evaluated at each $v \in \ddot{V}$.

Corollary 2. Theorem 2 holds when (WERVMP) for $\ddot{V}$ is replaced by Constrained Egalitarian Rank-Value Maximization Problem with some rank value $v' \in \ddot{V}$ and constants $(\underline{\Gamma}^*(v))_{v \in \ddot{V} \setminus \{v'\}}$:

$$\text{(CERVMP)} \qquad \max_{x \in X} \quad \underline{\Gamma}(x, v')$$
$$\text{s.t.} \quad \underline{\Gamma}(x, v) \geq \underline{\Gamma}^*(v) \qquad \forall v \in \ddot{V} \setminus \{v'\}$$

The equivalence is established as follows. For each (CERVMP), an equivalent (WERVMP) can be obtained by adding the constraints to the objective function weighted by the shadow prices. For each (WERVMP), to construct an equivalent (CERVMP), select any $v' \in \ddot{V}$ to be used in the objective function, and set $\underline{\Gamma}^*(v) = \underline{\Gamma}(x^*, v)$ for the remaining $v \in \ddot{V}$, where x^* is an optimal solution to (WERVMP).

Given Corollary 2, we can search for REU assignments in a sequential manner. Specifically, the following algorithm finds a REU assignment over arbitrary $\widetilde{V} \subseteq V$.

Algorithm to generate a REU assignment over $\widetilde{V}$.

1. Set $X^0 = X$ and select an arbitrary assignment $\overline{x} \in X^0$.
2. If $\overline{x} \in \mathcal{X}^{REU}(\widetilde{V})$, stop.
   Otherwise, find $v \in \widetilde{V}$ such that $\underline{\Gamma}(x', v) > \underline{\Gamma}(\overline{x}, v)$ for some $x' \in X^0$.
3. Set $X^1 = \arg\max_{x \in X^0} \underline{\Gamma}(x, v)$.
4. Set $X^0 = X^1$. Select some $\overline{x} \in X^0$. Go to Step 2.

The resulting $\overline{x}$ is REU over $\widetilde{V}$.

In Step 2, if $\overline{x}$ is the unique element in X^1, we have $\overline{x} \in \mathcal{X}^{REU^*}(\widetilde{V})$ by Theorem 2. If there are multiple elements in X^1, we may use Proposition 3, which either confirms that $\overline{x} \in \mathcal{X}^{REU}(\widetilde{V})$ or provides v to be added to $\ddot{V}$. Restricting the domain of the search to X^0 in Step 3 is equivalent to solving (CERVMP) with added constraints.

We show how the algorithm works using the preference profile in Example 3, which is reproduced here. Suppose we want to find a REU assignment over $\widetilde{V} = V$.

Agent	Preference	x^*				x'			
		A	B	C	D	A	B	C	D
Agent 1	$A \succ B \succ C \succ D$	$\frac{4}{11}$	$\frac{5}{11}$	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{4}{11}$	$\frac{1}{2}$	0	$\frac{3}{22}$
Agent 2	$A \succ C \succ D \succ B$	$\frac{1}{11}$	0	$\frac{10}{11}$	0	$\frac{1}{11}$	0	$\frac{10}{11}$	0
Agent 3	$A \succ B \succ D \succ C$	$\frac{6}{11}$	0	0	$\frac{5}{11}$	$\frac{6}{11}$	0	0	$\frac{5}{11}$
Agent 4	$B \succ C \succ D \succ A$	0	$\frac{6}{11}$	0	$\frac{5}{11}$	0	$\frac{1}{2}$	$\frac{1}{11}$	$\frac{9}{22}$

In the first iteration, in Step 1, let's set the initial $\overline{x}$ to be the equal division, $x_{io} = \frac{1}{4}$ for all $i \in I$ and $o \in O$. In Step 2, clearly, $\overline{x} \notin \mathcal{X}^{REU}(\widetilde{V})$, so let's pick $v^l = \left(1, \frac{2}{3}, \frac{1}{3}, 0\right)$. In Step 3, we solve (ERVMP) $\max_{x \in X} \underline{\Gamma}(x, v)$, which has infinitely many solutions, including x^* and x' in the table. The maximized objective value is $\frac{23}{33}$. In Step 4, let the new $\overline{x}$ be x'.

In the second iteration, in Step 2, because $\overline{x}$ is not REU over $\widetilde{V}$, we pick another $v \in \widetilde{V}$, say $v^3 = (1, 1, 1, 0)$. In Step 3, we solve the following (CERVMP):

$$\max_{x \in X} \quad \underline{\Gamma}(x, v^3) \qquad \text{s.t.} \quad \underline{\Gamma}(x, v^l) \geq \tfrac{23}{33}$$

Assignment x^* in the table is the unique solution to this problem. Therefore, we have found a REU assignment over $\widetilde{V}$.

Proposition 4. For any set $\widetilde{V} \subseteq V$ of instances of rank values, the algorithm to generate a REU assignment over $\widetilde{V}$ terminates in at most $n(m-1)+1$ iterations.

Proof. Let $\widetilde{V} \subseteq V$ be given. Because we choose one instance of rank values in each iteration, we will have chosen $n(m-1)$ instances of rank values at the end of iteration $n(m-1)$. No instance of rank values is chosen twice because an improvement in $\underline{\Gamma}(x, v)$ is impossible for any v that has previously been chosen. Let $\ddot{V}$ be the set of instances of rank values that have been chosen by the end of iteration $n(m-1)$.

Given $\overline{x} \in X^0$ at the end of iteration $n(m-1)$, divide $\widetilde{V}$ into $W_i(\overline{x}, \widetilde{V})$, $i \in I$. Because $\underline{\Gamma}(\overline{x}, v) = \Gamma_i(\overline{x}, v)$ for all $v \in W_i(\overline{x}, \widetilde{V})$ and $\Gamma_i(x, v)$ is linear in v for each $i \in I$, the surface $\underline{\Gamma}(\overline{x}, v)$ plotted over the space $\widetilde{V}$ is piecewise linear; i.e., $\underline{\Gamma}(\overline{x}, v)$ is a hyperplane over each $W_i(\overline{x}, \widetilde{V})$. Moreover, the hyperplane over $W_i(\overline{x}, \widetilde{V})$ is pinned at each $v' \in W_i(\overline{x}, \widetilde{V}) \cap \ddot{V}$ to the constrained-maximized value of $\underline{\Gamma}(\cdot, v')$. Because the space $\widetilde{V}$ is $m-2$ dimension (due to the normalization $v_1 = 1$ and $v_m = 0$), the graph of $\underline{\Gamma}(\overline{x}, v)$ over $\widetilde{V}$ is $m-1$ dimension, and it takes $m-1$ instances of rank values to completely fix the hyperplane.

If we are at the end of iteration $n(m-1)$, it must be that each $W_i(\overline{x}, \widetilde{V})$ contains exactly $m-1$ instances of rank values in $\ddot{V}$, and each hyperplane is completely fixed. Therefore, it is impossible to improve $\underline{\Gamma}(\overline{x}, v)$ at any $v \in \widetilde{V}$. □

Note that the proposed algorithm is just one possible way of producing REU assignments, and there are REU assignments that cannot be produced by the algorithm. One such example is assignment x^* in Example 5 in Appendix 3.8.3. It requires simultaneous consideration of multiple instances of rank values rather than sequential one.

Once we find REU assignments, we can obtain REU* assignments by imposing sd-efficiency as follows. Suppose $x \in \mathcal{X}^{REU}(\ddot{V})$ is an optimal solution to (WERVMP) with $\ddot{V}$ for some strictly positive weights, $(\lambda^v)_{v\in\ddot{V}}$. Let $\underline{\Gamma}^*$ be the maximized objective value. Pick an instance $v' \in V$ of rank values that is strictly decreasing, and solve the following problem:

$$\max_{x\in X} \quad \sum_{i\in I} \Gamma_i(x, v') \qquad \text{s.t.} \quad \sum_{v\in\ddot{V}} \lambda^v \underline{\Gamma}(x, v) \geq \underline{\Gamma}^*$$

Any optimal solution to this problem is REU* over $\ddot{V}$. This is merely one of many possible ways to impose sd-Pareto efficiency.

3.6.1 Special REU Assignments

Although social planners are free to specify any set $\widetilde{V} \subseteq V$ of instances of rank values, some are practically more meaningful than others. In particular, the set of all dichotomous rank values, $V^D = \left\{ \left(\mathbb{1}_{\{r\leq j\}}\right)_{r=1,...,m} \,\middle|\, j = 1, ..., m-1 \right\}$, permits simple interpretation: An assignment in $\mathcal{X}^{REU^*}(V^D)$ maximizes a weighted sum of the guaranteed minimum probability each agent receives an object ranked r^{th} or higher for $r = 1, ..., m-1$.

If it is important to impartially allocate high-ranked objects, in Step 2 of the algorithm, we can select $v^1 = (1, 0, 0, ..., 0)$ in the first iteration, $v^2 = (1, 1, 0, ..., 0)$ in the second iteration, and so forth. The resulting assignment maximizes the probability that each agent receives their top-ranked object, then that for the two top-ranked objects, and so forth. This assignment is similar to the Probabilistic Serial Rule (Bogomolnaia and Moulin, 2001), but they are not identical.

If it is important to impartially allocate low-ranked objects, we may first select $v^{m-1} = (1, ...1, 1, 0)$, then $v^{m-2} = (1, ...1, 0, 0)$, and work our way up to v^1. The resulting assignment minimizes the probability that each agent receives their bottom-ranked object, then that for the two bottom-ranked objects, and so forth. This assignment is ex-post egalitarian: the maximum rank of the objects that are assigned with a positive probability is minimized.[13] Unfortunately, it creates an incentive to misreport. If an agent moderately prefers an unpopular object, he would hide that information to avoid ending up with the object. This issue can be mitigated by, for example, imposing equal-division lower bound.[14]

More generally, the order in which elements of V^D are added to (CERVMP) can be arbitrarily chosen. For example, if the two most important criteria are maximizing the guaranteed probability that each agent receives their top-ranked object and that for one of three top-ranked objects, then v^1 and v^3 should enter (CERVMP) first. By changing the order in which $v \in V^D$ enters (CERVMP), we can trace the Pareto-frontier of $(\underline{\Gamma}(x, v))_{v \in V^D}$.

3.7 Conclusion

In this paper, we introduce the rank-value based fairness notion, rank-egalitarian dominance, and the set of rank-fair assignments. An assignment is rank-egalitarian undominated (REU) over a set of instances of rank values if there is no other assign-

[13]This idea is presented in Proll (1972), and is termed the bottleneck assignment.

[14]Equal division $\overline{\overline{x}}$ assigns each agent an equal probability share of each object; i.e., $\overline{\overline{x}}_{io} = q_o/n$. When $\sum_{o \in O} q_o > n$, let each agent discard the probability share of their lowest-ranked objects until the total probability share reduces to 1. An assignment $x \in X$ satisfies equal-division lower bound if $\underline{\Gamma}(x, v) \geq \underline{\Gamma}(\overline{\overline{x}}, v)$ for all $v \in V^D$.

ment with the minimum expected rank values weakly larger at each instance of rank values and strictly larger at some instance of rank values. Defining rank-fairness over a set of instances of rank values relieves the social planner from the burden of having to identify *the* right instance of rank values.

We then provide the necessary and sufficient conditions for an assignment to be REU. Each REU assignment is a solution to a linear programming problem that maximizes the weighted sum of the minimum expected rank values evaluated at each instance of rank values in the set. Conversely, if an assignment solves such a problem, then it is REU. We also show that REU assignments can be sought sequentially, and propose an algorithm that generates subsets of REU assignments that may be of interest to market designers.

We hope the foundation we laid here facilitates future discussions of rank-value based algorithms, which have potential to improve rank-fairness and rank-efficiency. In particular, much work is needed to understand the incentive properties of rank-value based algorithms, both theoretically and empirically.

3.8 Appendix: Proofs

3.8.1 Proof of Proposition 1 (i)

Proposition 1.(i) For $n = m = 3$, if no pair of agents have the same preferences, there is a unique solution to (ERVMP) for any rank values $v = (1, v_2, 0)$ such that $0 < v_2 < 1$.

Proof. If each agent top-ranks different objects, the optimal assignment is trivial and unique. Thus, assume this is not the case. For $n = m = 3$, there are only two non-trivial preference profiles (modulo relabeling of objects and agents) for which no two agents have the identical preference:

Profile II	Profile II′
Agent 1: $A \succ B \succ C$	Agent 1: $A \succ B \succ C$
Agent 2: $A \succ C \succ B$	Agent 2: $A \succ C \succ B$
Agent 3: $B \succ A \succ C$	Agent 3: $B \succ C \succ A$

Consider Profile II. Given any rank values $v = (1, v_2, 0)$, where $0 < v_2 < 1$, we claim that any solution to (ERVMP) satisfies either $x_{3A} = 0$ or $x_{1B} = x_{2B} = 0$. To see this, suppose $x_{3A} > 0$ and $x_{1B} > 0$. Then agent 3 and 1 can trade $\eta = \min\{x_{3A}, x_{1B}\}$ units of object A and B, and they will be strictly better off. Subsequently, agent 1 can give $\frac{1}{2}\eta(1-v_2)$ units of object A to agent 2 and receive the same amount of B or C from agent 2, which makes agent 2 better off. Agent 1 is also better off because the expected rank value has increased by at least $\frac{1}{2}\eta(1-v_2)$. Because the trade makes every agent better off, the initial assignment cannot be a solution to (ERVMP). A similar argument holds for the case where $x_{3A} > 0$ and $x_{2B} > 0$. Therefore, it must be that if x solves (ERVMP), then either $x_{3A} = 0$ or $x_{1B} = x_{2B} = 0$.

Suppose $x_{1B} = x_{2B} = 0$. Then $x_{3B} = 1$. Agents 1 and 2 are equally well off when $x_{1A} = x_{2C} = \frac{1}{2-v_2}$ and $x_{1C} = x_{2A} = \frac{1-v_2}{2-v_2}$, which yields the expected rank value of $\frac{1}{2-v_2}$. Because the expected rank value is 1 for agent 3, agents 1 and 2 are worse off than agent 3. But then agent 3 can help agent 1 by taking less share of object B and more share of object C. Subsequently, agent 1 can help agent 2 by taking less share of object A and more share of object C. Thus, $x_{1B} = x_{2B} = 0$ cannot be a part of an optimal solution. It follows that $x_{3A} = 0$.

Similarly, if x solves (ERVMP), the allocation of object B and C must satisfy either $x_{1C} = x_{3C} = 0$ or $x_{2B} = 0$. If $x_{1C} > 0$ and $x_{2B} > 0$, then agent 1 can trade C for B with agent 2 and subsequently give some fraction of B to agent 3. If $x_{3C} > 0$ and $x_{2B} > 0$, then agent 3 can trade C for B with agent 2, and subsequently, agent 2 can give some fraction of A to agent 1 (if agent 2 does not own any fraction of A, it means $x_{1A} = 1$, and agent 1 is best off, so there is no need to improve the expected rank value of agent 1). In any case, there is a sequence of trades that improves the minimum expected rank value. Therefore, it must be that any solution x to (ERVMP) satisfies either $x_{1C} = x_{3C} = 0$ or $x_{2B} = 0$.

Suppose $x_{1C} = x_{3C} = 0$. Then $x_{2C} = 1$. It follows that $x_{1A} = x_{3B} = 1$, and agent 2 is strictly worse off than agents 1 and 3. But agent 1 can help agent 2 by taking less share of object A and more share of object C. Therefore, $x_{1C} = x_{3C} = 0$ cannot be a part of an optimal solution. Hence, we have $x_{2B} = 0$.

Therefore, an optimal solution must be in the following form:

$$\begin{array}{lll} x_{1A} = \alpha & x_{1B} = \beta & x_{1C} = 1 - \alpha - \beta \\ x_{2A} = 1 - \alpha & x_{2B} = 0 & x_{2C} = \alpha \\ x_{3A} = 0 & x_{3B} = 1 - \beta & x_{3C} = \beta \end{array}$$

If one of agent 1 or agent 2 is worst off, then both must be worst off because otherwise the better-off agent can help the worst-off agent by trading A for C. If one of agent 1 or agent 3 is worst off, then both must be worst off because otherwise the better-off agent can help the worst-off agent by trading B for C. Thus, all of them must be equally worse off, which uniquely pins down $\alpha = \frac{1}{2-v_2^2}$ and $\beta = \frac{1-v_2}{2-v_2^2}$. In other words, the solution to (ERVMP) is unique. The proof for Profile II′ is similar. □

3.8.2 Proof of Proposition 2 (ii)

Proposition 2. (ii) $\mathcal{X}^{REU^*}(V^D) \not\subseteq \mathcal{X}^{REU^*}(V)$ for some preference profile.

Example 4. Consider the following preference profile and assignments. We claim that $x^0 \in \mathcal{X}^{REU^*}(V^D) \setminus \mathcal{X}^{REU^*}(V)$.

Agent	Preference	x^0				x^ε			
		A	B	C	D	A	B	C	D
Agent 1	$A \succ B \succ C \succ D$	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{3}{14}$	$\frac{1}{14}$	$\frac{3}{7}+\varepsilon$	$\frac{2}{7}-\frac{\varepsilon}{2}$	$\frac{3}{14}-\frac{\varepsilon}{2}$	$\frac{1}{14}$
Agent 2	$B \succ A \succ C \succ D$	0	$\frac{5}{7}$	$\frac{3}{14}$	$\frac{1}{14}$	0	$\frac{5}{7}+\frac{\varepsilon}{2}$	$\frac{3}{14}-\frac{\varepsilon}{2}$	$\frac{1}{14}$
Agent 3	$A \succ C \succ D \succ B$	$\frac{4}{7}$	0	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{4}{7}-\varepsilon$	0	$\frac{1}{7}+\varepsilon$	$\frac{2}{7}$
Agent 4	$C \succ D \succ B \succ A$	0	0	$\frac{3}{7}$	$\frac{4}{7}$	0	0	$\frac{3}{7}$	$\frac{4}{7}$

First, $x^0 \in \mathcal{X}^{REU}(V^D)$ because x^0 solves the following (WERVMP):

$$\max_{x \in X} \quad \underline{\Gamma}(x, v^1) + \underline{\Gamma}(x, v^2) + 2\underline{\Gamma}(x, v^3)$$

It can be easily confirmed that x^0 is sd-Pareto efficient. Thus, $x^0 \in \mathcal{X}^{REU^*}(V^D)$.

Next, we show $x^0 \notin \mathcal{X}^{REU^*}(V)$. Let $d_{ir}(x) = \sum_{o \in O} x_{io} \cdot \mathbb{1}_{\{r_i(o) \leq r\}}$ denote the probability that agent i receives an object ranked r^{th} or higher under assignment x. Let $d_i(x) = (d_{i1}(x), ..., d_{im}(x))$. Under x^0, agent 1 is worst off at every $v \in V$ because $d_1(x^0) = (\frac{3}{7}, \frac{5}{7}, \frac{13}{14}, 1)$ is first-order stochastically dominated by $d_2(x^0)$, $d_3(x^0)$, and $d_4(x^0)$. Now, consider x^ε with $0 < \varepsilon \leq \frac{1}{14}$. Observe that $d_i(x^\varepsilon)$ first-order stochastically dominates $d_1(x^0)$ for each $i = 1, ..., 4$. Therefore, $\underline{\Gamma}(x^\varepsilon, v) \geq \underline{\Gamma}(x^0, v)$ for all $v \in V$. Furthermore, $\underline{\Gamma}(x^\varepsilon, v) > \underline{\Gamma}(x^0, v)$ at $v = (1, \frac{1}{2}, 0, 0)$. Thus, x^0 is RE-dominated by x^ε over V, i.e., $x^0 \notin \mathcal{X}^{REU}(V)$. This means $x^0 \notin \mathcal{X}^{REU^*}(V)$.

3.8.3 Proof of Proposition 2 (iii)

Proposition 2. (iii) $\mathcal{X}^{REU^*}(V) \not\subseteq \mathcal{X}^{REU^*}(V^D) \cup \left(\bigcup_{v \in V} \mathcal{X}^{REU^*}(\{v\})\right)$ for some preference profile.

Example 5. Consider the following preference profile and assignments. We claim that $x^* \in \mathcal{X}^{REU^*}(V)$, but it is not REU over V^D or over any singleton set.

Agent	Preference	x^*				x'			
		A	B	C	D	A	B	C	D
Agent 1	$A \succ B \succ C \succ D$	$\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{4}{9}$	0
Agent 2	$B \succ A \succ C \succ D$	0	$\frac{3}{4}$	0	$\frac{1}{4}$	0	$\frac{5}{9}$	$\frac{1}{3}$	$\frac{1}{9}$
Agent 3	$A \succ C \succ D \succ B$	0	0	1	0	$\frac{1}{3}$	0	$\frac{2}{9}$	$\frac{4}{9}$
Agent 4	$A \succ B \succ D \succ C$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{9}$	0	$\frac{4}{9}$

First, $x^* \in \mathcal{X}^{REU^*}(V)$ because it uniquely solves the following (WERVMP):

$$\max_{x \in X} \quad \underline{\Gamma}\left(x, \left(1, 1, \tfrac{1}{2}, 0\right)\right) + \underline{\Gamma}\left(x, \left(1, \tfrac{2}{3}, \tfrac{1}{3}, 0\right)\right)$$

Next, $x^* \notin \mathcal{X}^{REU^*}(V^D)$ because x^* is RE-dominated by x' over V^D.

Lastly, we show that x^* is not REU over $\{v\}$ for any $v \in V$. Suppose, toward a contradiction, that $x^* \in \mathcal{X}^{REU^*}(\{v^*\})$ for some $v^* = (1, v_2, v_3, 0) \in V$.

We claim that $v_2 \neq 1$. To see this, suppose $v_2 = 1$. Then $v_3 \neq 0$ because otherwise agent 4 would have received more of object A and B. Also, $v_3 \neq 1$ because otherwise agents 1 and 2 would not have received any fraction of object D. Thus, $0 < v_3 < 1$. Then the minimum expected rank value under x^* is $\min\{\frac{3}{4}, \frac{1}{2} + \frac{1}{2}v_3\}$. It can be improved by the assignment below:

i	Preference	A	B	C	D	Γ_i
1	$A \succ B \succ C \succ D$	$\frac{1}{2} - \frac{v_3}{16}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8} + \frac{v_3}{16}$	$\frac{3}{4} + \frac{1}{16}v_3$
2	$B \succ A \succ C \succ D$	0	$\frac{3}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{4} + \frac{1}{8}v_3$
3	$A \succ C \succ D \succ B$	0	0	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{3}{4} + \frac{1}{4}v_3$
4	$A \succ B \succ D \succ C$	$\frac{1}{2} + \frac{v_3}{16}$	0	0	$\frac{1}{2} - \frac{v_3}{16}$	$\frac{1}{2}(1 + v_3) + \frac{v_3}{16}(1 - v_3)$

Therefore, $x^* \notin \mathcal{X}^{REU}(\{v^*\})$. Thus, our assumption that $v_2 = 1$ must be false.

Also, it must be that $v_2 \neq 0$ because otherwise agent 3 would have received some fraction of object A. Thus, $0 < v_2 < 1$. In this case, the minimum expected rank value under x^* is $\min\{\frac{1}{2} + \frac{1}{4}v_2, v_2, \frac{1}{2} + \frac{1}{2}v_3\}$. It can be improved by the following assignment with $\alpha = \frac{v_2}{2}$ and $\beta = \frac{1-v_2}{8}$:

i	A	B	C	D	Γ_i
1	$\frac{1}{2} - \alpha\beta$	$\frac{1}{4} + \beta$	0	$\frac{1}{4} - (1 - \alpha)\beta$	$\frac{1}{2} + \frac{1}{4}v_2 + \frac{v_2(1-v_2)}{16}$
2	0	$\frac{3}{4} - \beta$	$\frac{\alpha\beta}{2}$	$\frac{1}{4} + \beta - \frac{\alpha\beta}{2}$	$\frac{1}{2} + \frac{1}{4}\left(\frac{1+v_2}{2}\right) + \frac{\alpha\beta}{2}v_3$
3	$\frac{\alpha\beta}{2}$	0	$1 - \frac{\alpha\beta}{2}$	0	$v_2 + \frac{\alpha\beta}{2}(1 - v_2)$
4	$\frac{1}{2} + \frac{\alpha\beta}{2}$	0	0	$\frac{1}{2} - \frac{\alpha\beta}{2}$	$\frac{1}{2}(1 + v_3) + \frac{\alpha\beta}{2}(1 - v_3)$

Therefore, $x^* \notin \mathcal{X}^{REU}(\{v^*\})$. Hence, there is no $v^* \in V$ for which $x^* \in \mathcal{X}^{REU}(\{v^*\})$.

3.8.4 Proof of Proposition 3

Proposition 3. (Test of RE dominance) Let a finite convex polytope $\widehat{V} \subseteq V$ and an assignment $\overline{x} \in X$ be given. Let $\ddot{V} = \bigcup_{i \in I} \ddot{W}_i(\overline{x}, \widehat{V})$. Let $I' \subseteq I$ be the set of agents for which the interior of $W_i(\overline{x}, \widehat{V})$ is non-empty. For each $i \in I'$, pick an arbitrary point $\overline{w}_i$ in the interior of $W_i(\overline{x}, \widehat{V})$. Solve the following problem:

$$\begin{aligned}
\max_{x \in X} \quad & \sum_{i \in I'} s_i \\
\text{s.t.} \quad & \underline{\Gamma}(x, v) \geq \underline{\Gamma}(\overline{x}, v) && \forall v \in \ddot{V} \\
& \underline{\Gamma}(x, \overline{w}_i) \geq \underline{\Gamma}(\overline{x}, \overline{w}_i) + s_i && \forall i \in I' \\
& s_i \geq 0 && \forall i \in I'
\end{aligned}$$

The maximized objective value is zero if and only if $\overline{x} \in \mathcal{X}^{REU}(\widehat{V})$.

Proof. Let $\widehat{V} \subseteq V$ and $\overline{x} \in X$ be given. To simplify the notation, instead of writing $W_i(\overline{x}, \widehat{V})$ and $\ddot{W}_i(\overline{x}, \widehat{V})$, we write W_i and $\ddot{W}_i$. We show that the maximized objective value is strictly positive if and only if $\overline{x} \notin \mathcal{X}^{REU}(\widehat{V})$.

Suppose $\overline{x} \notin \mathcal{X}^{REU}(\widehat{V})$. Then there is $x^* \in X$ and $v' \in V$ such that $\underline{\Gamma}(x^*, v) \geq \underline{\Gamma}(\overline{x}, v)$ for all $v \in \widehat{V}$ and $\underline{\Gamma}(x^*, v') > \underline{\Gamma}(\overline{x}, v')$. We claim that $v' \in W_{i'}$ for some $i' \in I'$. Recall that $\widehat{V} = \bigcup_{i \in I} W_i$. Because each W_i is a closed convex set, W_i with an empty interior can be dropped, and we obtain $\widehat{V} = \bigcup_{i \in I'} W_i$. Pick $i' \in I'$ such that $v' \in W_{i'}$.

Because x^* RE-dominates $\overline{x}$ over $\widehat{V}$, for each $w \in \ddot{W}_{i'}$, we have $\underline{\Gamma}(x^*, w) \geq \underline{\Gamma}(\overline{x}, w)$, which is equivalent to $\Gamma_i(x^*, w) \geq \Gamma_{i'}(\overline{x}, w)$ for all $i \in I$. Then, for each $i \in I$, there must be $\dot{w}_i \in \ddot{W}_{i'}$ such that $\Gamma_i(x^*, \dot{w}_i) > \Gamma_{i'}(\overline{x}, \dot{w}_i)$ because otherwise—due to the linearity of $\Gamma_i(x, v)$ in v—we have $\Gamma_i(x^*, v') = \Gamma_{i'}(\overline{x}, v')$, which implies $\underline{\Gamma}(x^*, v') = \underline{\Gamma}(\overline{x}, v')$, a contradiction. Now, because $\overline{w}_{i'}$ is an interior point of $W_{i'}$, there is a set of strictly positive weights $(\alpha^w)_{w \in \ddot{W}_{i'}}$ such that $\overline{w}_{i'} = \sum_{w \in \ddot{W}_{i'}} \alpha^w w$. It follows that $\Gamma_i(x^*, \overline{w}_{i'}) > \Gamma_{i'}(\overline{x}, \overline{w}_{i'})$ for each $i \in I$, and therefore, $\underline{\Gamma}(x^*, \overline{w}_{i'}) > \underline{\Gamma}(\overline{x}, \overline{w}_{i'})$. Thus, $s_{i'} > 0$, and the objective value is strictly positive.

Conversely, suppose the objective value is strictly positive for some feasible solution x^*. Then, there is $i^* \in I$ such that $s_{i^*} > 0$. Equivalently, $\underline{\Gamma}(x^*, \overline{w}_{i^*}) > \underline{\Gamma}(\overline{x}, \overline{w}_{i^*})$. Therefore, if $\underline{\Gamma}(x^*, v) \geq \underline{\Gamma}(\overline{x}, v)$ for all $v \in \widehat{V}$ then x^* RE-dominates $\overline{x}$ over $\widehat{V}$, and we are done.

Indeed, for any feasible solution x, we have $\underline{\Gamma}(x, v) \geq \underline{\Gamma}(\overline{x}, v)$ for all $v \in \widehat{V}$. Consider any $i' \in I$. For any $\overline{v} \in W_{i'}$, there is a set of non-negative weights $(\alpha^w)_{w \in \ddot{W}_{i'}}$ such that $\overline{v} = \sum_{w \in \ddot{W}_{i'}} \alpha^w w$. First, because $\Gamma_i(x, v)$ is linear in v and the min function is concave, $\underline{\Gamma}(x, \overline{v}) \geq \sum_{w \in \ddot{W}_{i'}} \alpha^w \underline{\Gamma}(x, w)$. Second, because $\underline{\Gamma}(x, v) = \Gamma_{i'}(x, v)$ for all $v \in W_{i'}$, we have $\sum_{w \in \ddot{W}_{i'}} \alpha^w \underline{\Gamma}(\overline{x}, w) = \underline{\Gamma}(\overline{x}, \overline{v})$. Third, any feasible solution x satisfies the constraint $\underline{\Gamma}(x, v) \geq \underline{\Gamma}(\overline{x}, v)$ for all $v \in \widehat{V}$, and therefore, $\underline{\Gamma}(x, w) \geq \underline{\Gamma}(\overline{x}, w)$ for all $w \in \ddot{W}_{i'}$ in particular. These three relations imply that $\underline{\Gamma}(x, \overline{v}) \geq \underline{\Gamma}(\overline{x}, \overline{v})$ for any $\overline{v} \in W_{i'}$. Because i' is arbitrary and $\widehat{V} = \bigcup_{i' \in I} W_{i'}$, we have $\underline{\Gamma}(x, v) \geq \underline{\Gamma}(\overline{x}, v)$ for all $v \in \widehat{V}$. □

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