Exact Spin Map in a Uniform Magnetic Field with Application to Sector Bend Dipoles

C. Mitchell

June 9, 2026

Abstract

In this note, we develop in closed form the time-evolution $SO(3)$ spin map determined by the Thomas-BMT equation for a charged particle with general gyromagnetic ratio in a uniform magnetic field. This result is then applied to determine the corresponding s-evolution spin map in the curvilinear Frenet-Serret coordinate frame associated with the transport of a charged-particle beam through an ideal sector dipole (bending magnet). The map is expressed in a factorized Lie-algebraic (axis-angle) form, making it straightforward to determine the corresponding $SU(2)$ or quaternion representation. To facilitate numerical implementation, the Lie generators are expressed as functions of the six phase space variables commonly used for charged particle transport in s-based tracking codes. This capability allows exact transport of both phase space and spin variables over an entire ideal sector dipole, without the need for numerical integration. The results can also help to validate alternative spin tracking approaches.

1 Time Evolution of Spin in a Uniform Magnetic Field

The Thomas-BMT equation for the rest-frame spin 3-vector $\vec{\mathcal{S}}$ of a particle of mass m and charge q moving in a magnetic field $\vec{B}$, where $\vec{B}$ is expressed in a fixed laboratory frame, takes the following form [1, 2, 3, 4]:

$$\frac{d\vec{\mathcal{S}}}{dt} = \vec{\Omega}(t) \times \vec{\mathcal{S}}, \qquad \vec{\Omega} = -\frac{q}{m}\left[\left(G + \frac{1}{\gamma}\right)\vec{B} - G\frac{\gamma}{\gamma+1}(\vec{\beta}\cdot\vec{B})\vec{\beta}\right]. \tag{1}$$

Here G denotes the gyromagnetic anomaly of the particle, given in terms of the gyromagnetic ratio g by $G = (g-2)/2$, and t denotes the time in the laboratory frame. Consider the case of a uniform, static magnetic field $\vec{B}$. In this case, the time-dependence of the precession vector $\vec{\Omega}$ on the right-hand side of (1) is determined by the time-dependence of the relativistic factors $\vec{\beta}$ and γ along the particle orbit. Since γ is invariant in a static magnetic field, the time-dependence of $\vec{\beta}$ is determined by the relativistic Lorentz force equation as follows:

$$\frac{d\vec{p}}{dt} = q\vec{v} \times \vec{B} \Rightarrow mc\gamma\frac{d\vec{\beta}}{dt} = qc\vec{\beta} \times \vec{B} \Rightarrow \frac{d\vec{\beta}}{dt} = -\vec{\omega} \times \vec{\beta}, \quad \text{where} \quad \vec{\omega} = \frac{q\vec{B}}{m\gamma}. \tag{2}$$

Let L_j $(j = 1, 2, 3)$ denote 3×3 matrices that form a basis for the Lie algebra of $SO(3)$. We assume the specific representation [5]:

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3}$$

These matrices have the property that for any two vectors $\vec{a}$, $\vec{b} \in \mathbb{R}^3$:

$$(\vec{a} \cdot \vec{L})\vec{b} \equiv \sum_{j=1}^{3} (a_j L_j)\vec{b} = \vec{a} \times \vec{b}. \tag{4}$$

Then (5) can be rewritten in terms of the L_j $(j = 1, 2, 3)$ to give the equation:

$$\frac{d\vec{\beta}}{dt} = -(\vec{\omega} \cdot \vec{L})\vec{\beta}. \tag{5}$$

Since the matrix appearing on the right-hand side of (5) is independent of t, the solution is clearly:

$$\vec{\beta}(t) = e^{-t\vec{\omega}\cdot\vec{L}}\vec{\beta}_0, \tag{6}$$

which represents rotation about the direction of the magnetic field with angular frequency $||\vec{\omega}||$. This shows that $||\vec{\beta}||$ is an invariant of motion, as expected. Likewise, we see that the projection of $\vec{\beta}$ along the magnetic field is invariant:

$$\vec{\beta}(t) \cdot \vec{B} = \vec{\beta}(t)^T \vec{B} = \left[e^{-t\vec{\omega}\cdot\vec{L}}\vec{\beta}_0\right]^T \vec{B} = \vec{\beta}_0^T \left[e^{t\vec{\omega}\cdot\vec{L}}\right] \vec{B} = \vec{\beta}_0 \cdot \vec{B}, \tag{7}$$

where we have used the fact that the rotation $e^{t\vec{\omega}\cdot\vec{L}}$ leaves $\vec{B}$ invariant, since $\vec{\omega} \times \vec{B} = 0$. As a consequence of (6-7), the time-evolution of the precession vector $\vec{\Omega}(t)$ in (1) can be expressed in the form:

$$\begin{aligned} \vec{\Omega}(t) &= -\frac{q}{m}\left[\left(G + \frac{1}{\gamma}\right)\vec{B} - G\frac{\gamma}{\gamma+1}\left(\vec{\beta}(t)\cdot\vec{B}\right)\vec{\beta}(t)\right] && \text{(8a)} \\ &= -\frac{q}{m}\left[\left(G + \frac{1}{\gamma}\right)\vec{B} - G\frac{\gamma}{\gamma+1}\left(\vec{\beta}_0\cdot\vec{B}\right)e^{-t\vec{\omega}\cdot\vec{L}}\vec{\beta}_0\right] && \text{(8b)} \\ &= e^{-t\vec{\omega}\cdot\vec{L}}\left\{-\frac{q}{m}\left[\left(G + \frac{1}{\gamma}\right)\vec{B} - G\frac{\gamma}{\gamma+1}\left(\vec{\beta}_0\cdot\vec{B}\right)\vec{\beta}_0\right]\right\} = e^{-t\vec{\omega}\cdot\vec{L}}\vec{\Omega}(0). && \text{(8c)} \end{aligned}$$

In the third line, we have again used the fact that $\vec{B}$ is invariant under rotation by the matrix $e^{-t\vec{\omega}\cdot\vec{L}}$.

2 Exact Solution for the Spin Map

We will represent the spin evolution determined by (1) in the form:

$$\vec{\mathcal{S}}(t) = M(t; \vec{\beta}_0)\vec{\mathcal{S}}_0, \tag{9}$$

where $M(t;\vec{\beta}_0)$ is a 3×3 orthogonal matrix, whose elements depend on the evolution time t and on the initial value of $\vec{\beta}_0$ along the particle orbit. The choice of particle species sets implicitly the values of the charge q, the mass m, and the gyromagnetic anomaly G.

It follows from (1) that the spin map M satisfies the following time evolution:

$$\dot{M}(t) = (\vec{\Omega}(t)\cdot\vec{L})M(t), \quad M(0) = I, \tag{10}$$

where I is the 3×3 identity matrix. We have suppressed the dependence of M on the constant value $\vec{\beta}_0$ on both sides of (10), which we will continue to do in the following discussion.

We claim that the solution of (10) is given by:

$$M(t) = \exp(-t\vec{\omega}\cdot\vec{L})\exp\left(t\vec{\Lambda}\cdot\vec{L}\right), \qquad \vec{\Lambda} = \vec{\Omega}(0) + \vec{\omega}, \tag{11}$$

where $\vec{\Omega}(0)$ denotes the precession vector appearing in (1), evaluated at the initial time $t=0$. It is clear from (11) that $M(0) = I$. The time derivative of $M(t)$ satisfies:

$$\begin{aligned}
\dot{M}(t) &= (-\vec{\omega}\cdot\vec{L})\exp(-t\vec{\omega}\cdot\vec{L})\exp\left(t\vec{\Lambda}\cdot\vec{L}\right) + \exp(-t\vec{\omega}\cdot\vec{L})(\vec{\Lambda}\cdot\vec{L})\exp\left(t\vec{\Lambda}\cdot\vec{L}\right) && \text{(12a)}\\
&= (-\vec{\omega}\cdot\vec{L})M(t) + e^{-t\vec{\omega}\cdot\vec{L}}(\vec{\Lambda}\cdot\vec{L})e^{t\vec{\omega}\cdot\vec{L}}e^{-t\vec{\omega}\cdot\vec{L}}\exp\left(t\vec{\Lambda}\cdot\vec{L}\right) && \text{(12b)}\\
&= (-\vec{\omega}\cdot\vec{L})M(t) + \left(\left[e^{-t\vec{\omega}\cdot\vec{L}}\vec{\Lambda}\right]\cdot\vec{L}\right)e^{-t\vec{\omega}\cdot\vec{L}}\exp\left(t\vec{\Lambda}\cdot\vec{L}\right) && \text{(12c)}\\
&= \left(-\vec{\omega}\cdot\vec{L} + \left[e^{-t\vec{\omega}\cdot\vec{L}}\vec{\Omega}(0) + \vec{\omega}\right]\cdot\vec{L}\right)M(t) = \left(\vec{\Omega}(t)\cdot\vec{L}\right)M(t). && \text{(12d)}
\end{aligned}$$

In the third line, we have used the following basic identity, which describes the adjoint action of $SO(3)$ on its Lie algebra:

$$e^{\vec{a}\cdot\vec{L}}(\vec{b}\cdot\vec{L})e^{-\vec{a}\cdot\vec{L}} = (e^{\vec{a}\cdot\vec{L}}\vec{b})\cdot\vec{L}, \qquad \vec{a},\vec{b}\in\mathbb{R}^3. \tag{13}$$

In the fourth line, we used the fact that the rotation matrix $e^{-t\vec{\omega}\cdot\vec{L}}$ leaves the vector $\vec{\omega}$ invariant, together with (8). This shows that (11) is the correct solution.

3 Exact Spin Map for a Sector Bend

We now wish to determine the spin map describing transport through an ideal sector bend, whose magnetic field is given by $\vec{B} = B\hat{y}$. Figure 1 illustrates the bending magnet, showing the region of nonzero magnetic field. In this case, the spin and orbital variables are described in a Frenet-Serret frame that is attached to the design orbit, and the reference path length s is used as the independent variable. The design orbit describes a circular arc in the x-z plane that undergoes bending by an angle:

$$\phi = h\Delta s = \frac{\Delta s}{\rho_d} = \frac{\Delta s B_y}{(B\rho)_d} = \frac{\Delta s\cdot qB_y}{mc\beta_d\gamma_d} = \frac{\Delta s}{c\beta_d}\left(\frac{qB_y}{m\gamma_d}\right) = \Delta t_d\omega_d, \tag{14}$$

In the chain of equalities above, h denotes the curvature of the design orbit, ρ_d denotes the radius of curvature, $(B\rho)_d$ denotes the design value of the magnetic rigidity, and Δt_d denotes the transit

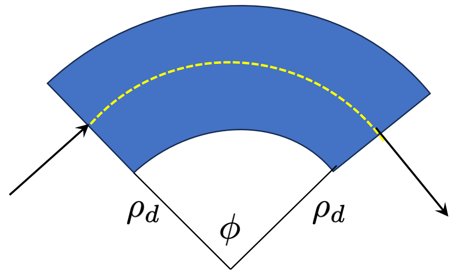

Figure 1: A sector dipole. The region of nonzero magnetic field is shown in blue. The design trajectory is shown in yellow. The arrows indicate the direction of the design momentum at the entrance and exit faces of the magnet.

time of the reference particle through the magnet. To use the result (11) to obtain the spin map for such a sector bend, we must address the following two issues:

- the rotation of the Frenet-Serret reference frame (from entrance to exit)
- the change of the independent variable (from t to s)

3.1 Rotation of the reference frame

The map in (11) assumes that all quantities are evaluated with respect to a global Cartesian coordinate system (x, y, z). It is natural to choose the global Cartesian coordinate system so that it coincides with the local Frenet-Serret coordinate system at the location at which the reference particle enters the magnet, as shown in Fig. 2. During transit through the bend, the vertical

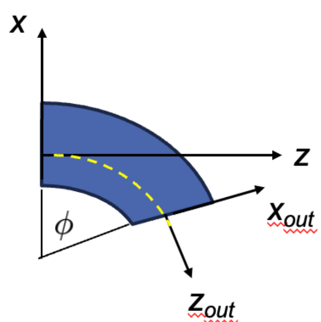

Figure 2: Schematic of a sector bend of angle ϕ, illustrating the rotation of the local coordinate system. The vertical y-axis extends out of the page. The design trajectory is shown in yellow. The reference particle moves clockwise (in the local $+z$ direction).

axis y is left fixed, but the coordinate axes x and z are rotated by the bend angle ϕ, taking $(x, z) \mapsto (x_{\text{out}}, z_{\text{out}})$.

Consider a particle with spin $\vec{\mathcal{S}}_i$ and velocity $\vec{\beta}_i$ as it enters the magnet through the plane $z = 0$ at some initial time t_i. At later time t_f it exits the magnet through the plane $z_{\mathrm{out}} = 0$, and the spin $\vec{\mathcal{S}}_f^G$ in the global (x, y, z) coordinate system at time t_f is given by (9), so:

$$\vec{\mathcal{S}}_f^G = M(t_f - t_i; \vec{\beta}_i)\vec{\mathcal{S}}_i. \tag{15}$$

Evaluating the components of the spin in the local coordinate system $(x_{\mathrm{out}}, y_{\mathrm{out}}, z_{\mathrm{out}})$ requires rotation by the angle ϕ about the y-axis, so that:

$$\vec{\mathcal{S}}_f = e^{\phi\hat{y}\cdot\vec{L}}\vec{\mathcal{S}}_f^G = e^{\phi\hat{y}\cdot\vec{L}}M(t_f - t_i; \vec{\beta}_i)\vec{\mathcal{S}}_i. \tag{16}$$

It follows that the spin evolution of the particle in the local coordinate system, from entry to exit of the bend, is given by the matrix M_B, where:

$$M_B = e^{\phi\hat{y}\cdot\vec{L}} \exp\left(-(t_f - t_i)\vec{\omega}\cdot\vec{L}\right) \exp\left((t_f - t_i)\vec{\Lambda}\cdot\vec{L}\right), \qquad \vec{\Lambda} = \vec{\Omega}(t_i) + \vec{\omega}. \tag{17}$$

Since $\vec{\omega}$ is parallel to $\hat{y}$, the two leftmost factors commute, and their exponents can be combined to give:

$$M_B = \exp\left(\{\phi\hat{y} - (t_f - t_i)\vec{\omega}\}\cdot\vec{L}\right) \exp\left((t_f - t_i)\vec{\Lambda}\cdot\vec{L}\right). \tag{18}$$

Simplifying expression (1) for the precession vector somewhat, we find that:

$$\vec{\Omega}(t_i) = -(G\gamma + 1)\vec{\omega} + G(\gamma - 1)(\hat{u}\cdot\vec{\omega})\hat{u}, \qquad \hat{u} = \frac{\vec{\beta}_i}{||\vec{\beta}_i||}, \tag{19}$$

where $\hat{u}$ is a unit vector describing the direction of the initial particle velocity at the entrance to the magnet. (The velocity is evaluated in the global Cartesian coordinate system, which coincides with the local Frenet-Serret coordinate system at the magnet entrance.) Therefore we see that:

$$\vec{\Lambda} = -G\gamma\vec{\omega} + G(\gamma - 1)(\hat{u}\cdot\vec{\omega})\hat{u}. \tag{20}$$

Simplifying further, we express the map (18) in the form:

$$M_B = \exp\left(\vec{F}_1\cdot\vec{L}\right) \exp\left(\vec{F}_2\cdot\vec{L}\right), \tag{21a}$$

where the two generators are:

$$\vec{F}_1 = (\phi - \theta)\hat{y}, \qquad \vec{F}_2 = G\theta\left\{-\gamma\hat{y} + (\gamma - 1)u_y\hat{u}\right\}, \tag{21b}$$

and we have introduced the symbol

$$\theta = \omega(t_f - t_i). \tag{21c}$$

Here θ represents the change in cyclotron phase during the particle's transit over the length of the magnet. Since the orbit (phase space) map through an ideal sector bend is known exactly, the transit time $t_f - t_i$ is easily evaluated, and the spin map (21) can be applied using two simple rotations. In the following subsection, we consider the particle orbits and provide an explicit expression for (21c).

3.2 Phase space and orbits

We would like to evaluate the spin map in terms of the 6D phase space variables, expressed in a local Frenet-Serret frame, that are used in symplectic tracking codes [8]. We denote these variables by (X, P_x, Y, P_y, T, P_t). The interpretation of the local X, Y coordinates is apparent from Fig. 2. The variable T denotes the deviation of a particle's time-of-flight from the time-of-flight along the design orbit, and the variable P_t denotes the negative of deviation in energy between a particle and the design energy [5]. We assume that the momenta P_x and P_y are normalized by the design momentum $p_d = mc\beta_d\gamma_d$, and the momentum P_t is normalized by $p_d c$, so all three momenta are dimensionless. Likewise, the time-of-flight deviation T is multiplied by the speed of light, so that all three coordinates have units of length (m).

We can express the quantities appearing in (21) in terms of these phase space variables as follows:

$$\gamma = \gamma_d(1 - \beta_d P_t), \quad u_{x,y} = \frac{P_{x,y}}{||\vec{P}||}, \quad u_z = \frac{P_s}{||\vec{P}||}, \tag{22}$$

where

$$||\vec{P}|| = \sqrt{1 - \frac{2P_t}{\beta_d} + P_t^2}, \quad P_s = \sqrt{||\vec{P}||^2 - P_x^2 - P_y^2}. \tag{23}$$

These relations can be inverted to recover the definitions of the momenta (P_x, P_y, P_t) in terms of the relativistic factors $(\gamma, \vec{\beta})$.

We will characterize the sector bend magnet by the two parameters h and s, denoting the curvature of the design orbit and the design path length through the magnet, respectively. We set $s = 0$ at the entrance of the magnet. (The plane $z = 0$ of Fig. 2.) The bend angle is recovered using $\phi = hs$. The map describing transport of the phase space variables over a sector of design path length s is known in closed form [6, 7], and the result is summarized as follows. The momenta evolve as:

$$P_x^f = P_x \cos(hs) + (P_s - (1 + hX)) \sin(hs), \qquad P_y^f = P_y, \qquad P_t^f = P_t, \tag{24a}$$

$$P_s^f = \sqrt{||\vec{P}||^2 - (P_x^f)^2 - P_y^2}, \tag{24b}$$

and setting:

$$\theta = hs + \sin^{-1}\left(\frac{P_x}{\sqrt{||\vec{P}||^2 - P_y^2}}\right) - \sin^{-1}\left(\frac{P_x^f}{\sqrt{||\vec{P}||^2 - P_y^2}}\right), \tag{24c}$$

the positions evolve as:

$$hX^f = P_s^f - 1 + P_x \sin(hs) - (P_s - (1 + hX)) \cos(hs), \tag{24d}$$

$$hY^f = hY + \theta P_y, \tag{24e}$$

$$hT^f = hT - \theta\left(P_t - \frac{1}{\beta_d}\right) - \frac{hs}{\beta_d}. \tag{24f}$$

Here, we have assumed that the design orbit follows the curvature of the dipole, so that $h = B_y/(B\rho)_d$. In (24) the superscript f denotes the final value at the location s, while the unadorned variables are evaluated at $s = 0$. It is straightforward to check that the map (24) is origin-preserving. That is, the map takes $(0,0,0,0,0,0) \mapsto (0,0,0,0,0,0)$. It is also symplectic, and the corresponding flow is generated by the Hamiltonian:

$$H(X, P_x, Y, P_y, T, P_t) = -(1+hX)\sqrt{1+P_t^2 - \frac{2P_t}{\beta_d} - P_x^2 - P_y^2} - \frac{P_t}{\beta_d} + \frac{1}{2}(1+hX)^2, \tag{25}$$

with s as the independent variable. The reader may verify that (24) is a solution of the canonical equations of motion.

The transit time of a reference particle on the design orbit is clearly given by:

$$\Delta t_d = \frac{s}{c\beta_d}. \tag{26}$$

The transit time of a general particle is then given by:

$$t_f - t_i = \Delta t_d + (T^f - T)/c = -\frac{\theta}{hc}\left(P_t - \frac{1}{\beta_d}\right). \tag{27}$$

We now note the relationship between the cyclotron frequency ω and the curvature parameter h:

$$\frac{\omega}{c} = \frac{qB_y}{mc\gamma} = \frac{qB_y}{mc\gamma_d(1-\beta_d P_t)} = \frac{B_y}{(B\rho)_d}\frac{\beta_d}{(1-\beta_d P_t)} = \frac{h\beta_d}{(1-\beta_d P_t)}. \tag{28}$$

Using (28) in (27) gives:

$$\omega(t^f - t^i) = \frac{hc\beta_d}{(1-\beta_d P_t)} \cdot \frac{\theta}{hc}\left(\frac{1}{\beta_d} - P_t\right) = \theta. \tag{29}$$

This justifies the use of the symbol θ in (21c).

3.3 Evolution equation for the spin map

We would like to show directly that the spin map (21) follows from the Thomas-BMT equation, expressed using the local Frenet-Serret frame, with s as the independent variable [4]:

$$\frac{d\vec{\mathcal{S}}}{ds} = \vec{\Omega}_s \times \vec{\mathcal{S}}, \tag{30}$$

where

$$\vec{\Omega}_s = -\frac{1+hX}{(B\rho)_d P_s}\left[(1+G\gamma)\vec{B} - G(\gamma-1)(\hat{u}\cdot\vec{B})\hat{u}\right] + h\hat{y}. \tag{31}$$

In terms of the quantities that we just defined, this expression takes the simpler form:

$$\vec{\Omega}_s = -\frac{1+hX}{P_s}\left[(1+G\gamma)h\hat{y} - G(\gamma-1)hu_y\hat{u}\right] + h\hat{y}. \tag{32}$$

Note that (32) depends on s through the evolution of the phase space variables as described in the previous section, and the vector $\hat{u}$ is evaluated in the local Frenet-Serret frame at s.

As a result, we require that the map $M_B(s)$ describing spin transport through a sector bend of design path length s satisfies the following evolution equation:

$$\dot{M}_B(s) = \left(\vec{\Omega}_s(s) \cdot \vec{L}\right) M_B(s), \qquad M_B(0) = I. \tag{33}$$

In order to check (33), we need to know the explicit evolution of the precession vector $\vec{\Omega}_s(s)$ along the particle orbit, which follows from the phase space map described in (24). That (33) is satisfied by (21) as a 3×3 matrix equation has been checked in detail using an explicit, tedious calculation in *Mathematica*. We can also show this using the following shorter, more indirect argument.

Let us return to the defining expression (17), which we write in the form:

$$M_B(s) = e^{hs\hat{y}\cdot\vec{L}} M(t_f(s) - t_i). \tag{34}$$

The initial condition in (33) is straightforward to check. Taking the derivative of (34) with respect to s gives:

$$\begin{aligned}
\frac{d}{ds} M_B(s) &= (h\hat{y} \cdot \vec{L}) e^{hs\hat{y}\cdot\vec{L}} M(t_f(s) - t_i) + e^{hs\hat{y}\cdot\vec{L}} \, \frac{dt_f(s)}{ds} \frac{d}{dt} M(t)\Big|_{t_f(s)-t_i} \\
&= (h\hat{y} \cdot \vec{L}) M_B(s) + \frac{dt_f(s)}{ds} e^{hs\hat{y}\cdot\vec{L}} (\vec{\Omega}(t) \cdot \vec{L}) M(t)\Big|_{t_f(s)-t_i} \\
&= (h\hat{y} \cdot \vec{L}) M_B(s) + \frac{dt_f(s)}{ds} e^{hs\hat{y}\cdot\vec{L}} (\vec{\Omega}(t) \cdot \vec{L}) e^{-hs\hat{y}\cdot\vec{L}} M_B(s) \\
&= (h\hat{y} \cdot \vec{L}) M_B(s) + \frac{dt_f(s)}{ds} \left([e^{hs\hat{y}\cdot\vec{L}} \vec{\Omega}(t)] \cdot \vec{L} \right) M_B(s) \\
&= \left\{ \left(h\hat{y} + \frac{dt_f(s)}{ds} [e^{hs\hat{y}\cdot\vec{L}} \vec{\Omega}(t)] \right) \cdot \vec{L} \right\} M_B(s),
\end{aligned} \tag{35}$$

where we just put $t = t_f(s) - t_i$, the particle's transit time at the location s. If we can show that:

$$\vec{\Omega}_s(s) = h\hat{y} + \frac{dt_f(s)}{ds} [e^{hs\hat{y}\cdot\vec{L}} \vec{\Omega}(t)], \tag{36}$$

then (33) immediately follows. However, using the Hamiltonian (25) we see the relation:

$$\frac{dt_f(s)}{ds} = \frac{d}{ds}(T^f/c + \Delta t_d) = \frac{1}{c}\left(\frac{\partial H}{\partial P_t} + \frac{1}{\beta_d}\right) = \left(1 + hX^f\right) \frac{(1 - \beta_d P_t^f)}{c\beta_d P_s^f}. \tag{37}$$

Then the right-hand side of (36) becomes:

$$h\hat{y} + \left(1 + hX^f\right) \frac{(1 - \beta_d P_t^f)}{c\beta_d P_s^f} \left\{ -(G\gamma + 1)\vec{\omega} + G(\gamma - 1)(\hat{u}(t) \cdot \vec{\omega}) e^{hs\hat{y}\cdot\vec{L}} \hat{u}(t) \right\} \tag{38}$$

$$= h\hat{y} - \frac{(1 + hX^f)}{P_s^f} \left\{ (1 + G\gamma) h\hat{y} - G(\gamma - 1) h u_y(t) e^{hs\hat{y}\cdot\vec{L}} \hat{u}(t) \right\} = \vec{\Omega}_s(s), \tag{39}$$

where we note that the vector $\hat{u}$ appearing in $\vec{\Omega}_s(s)$ is evaluated in the Frenet-Serret frame. This shows that (33) is satisfied.

3.4 Spin map to first order in the phase space variables

In this section, we will study the first-order dependence of the spin map (21) on the initial phase space variables describing a particle at the magnet entrance [9].

Let's first evaluate (21) for a particle on the design trajectory. Note that on the design trajectory we have $X = P_x = Y = P_y = T = P_t = 0$, so clearly:

$$\theta_d = \phi, \qquad u_{y,d} = 0. \tag{40}$$

Then the two generators in (21) become:

$$\vec{F}_{1,d} = 0, \qquad \vec{F}_{2,d} = -G\gamma_d\phi\hat{y}. \tag{41}$$

Therefore, we have for a particle on the design trajectory that the spin map M_d is simply:

$$M_d = \exp(\vec{F}_{2,d} \cdot \vec{L}). \tag{42}$$

For a *general* particle we can then write:

$$M_B = M_d M_r, \tag{43}$$

where the "remainder" factor M_r is:

$$M_r = \exp(-\vec{F}_{2,d} \cdot \vec{L}) \exp\left(\vec{F}_1 \cdot \vec{L}\right) \exp\left(\vec{F}_2 \cdot \vec{L}\right). \tag{44}$$

The generators in the two leftmost factors both point in the direction $\hat{y}$, so these factors commute. As a result, we can write this as:

$$M_r = \exp\left(\left\{\vec{F}_1 - \vec{F}_{2,d}\right\} \cdot \vec{L}\right) \exp\left(\vec{F}_2 \cdot \vec{L}\right). \tag{45}$$

Every element of $SO(3)$ can be written in terms of a single Lie generator (axis-angle form). We wish to evaluate the single generator of M_r to first order in the phase space variables (X, P_x, Y, P_y, T, P_t). Leting ϵ denote an order parameter so that $(X, P_x, Y, P_y, T, P_t) \sim O(\epsilon)$, we see that $\vec{F}_1 \sim O(\epsilon)$, and we can write:

$$\vec{F}_2 = \vec{F}_{2,d} + \delta\vec{F}_2, \qquad \delta\vec{F}_2 \sim O(\epsilon). \tag{46}$$

In fact, *to first order* in the phase space variables we have, by expanding the generators in (21):

$$(F_1)_y = -P_x(1 - \cos\phi) - (P_t/\beta_d + hX)\sin\phi, \tag{47a}$$

$$(\delta F_2)_x = 0, \tag{47b}$$

$$(\delta F_2)_y = -G\gamma_d\left\{P_x(1 - \cos\phi) + (P_t/\beta_d + hX)\sin\phi - P_t\beta_d\phi\right\}, \tag{47c}$$

$$(\delta F_2)_z = G\phi(\gamma_d - 1)P_y. \tag{47d}$$

Then we have:

$$M_r = \exp\left(\left\{\vec{F}_1 - \vec{F}_{2,d}\right\} \cdot \vec{L}\right) \exp\left(\left\{\vec{F}_{2,d} + \delta\vec{F}_2\right\} \cdot \vec{L}\right). \tag{48}$$

Since $\vec{F}_1$ and $F_{2,d}$ are both parallel to $\hat{y}$, we see that this is a 3×3 matrix equation of the following form:

$$e^{(-a\hat{y}+\epsilon b\hat{y})\cdot\vec{L}} e^{(a\hat{y}+\epsilon\vec{c})\cdot\vec{L}} = e^{\epsilon\vec{\lambda}\cdot\vec{L}}, \qquad a, b \in \mathbb{R} \quad \vec{c}, \vec{\lambda} \in \mathbb{R}^3 \tag{49}$$

where we want to determine $\vec{\lambda}$ to leading order in ϵ. The leftmost and rightmost exponentials in (49) can be trivially expanded to first order in ϵ. The nontrivial factor is the second one. This factor can be treated using Rodrigues' formula, but we note instead the following trick, which exploits the properties of the Lie algebra. We claim the following factorization holds:

$$e^{(a\hat{y}+\epsilon\vec{c})\cdot\vec{L}} = e^{\epsilon\vec{d}\cdot\vec{L}}e^{(a\hat{y}+\epsilon c_y\hat{y})\cdot\vec{L}}e^{-\epsilon\vec{d}\cdot\vec{L}} + O(\epsilon^2), \qquad \vec{d} = (\hat{y}\times\vec{c})/a. \tag{50}$$

To check this, note that:

$$\begin{aligned} e^{\epsilon\vec{d}\cdot\vec{L}}e^{(a\hat{y}+\epsilon c_y\hat{y})\cdot\vec{L}}e^{-\epsilon\vec{d}\cdot\vec{L}} &= \exp\left(\left[e^{\epsilon\vec{d}\cdot\vec{L}}(a\hat{y}+\epsilon c_y\hat{y})\right]\cdot\vec{L}\right) \\ &= \exp\left(\left[(I+\epsilon\vec{d}\cdot\vec{L})(a\hat{y}+\epsilon c_y\hat{y})\right]\cdot\vec{L}+O(\epsilon^2)\right) \\ &= \exp\left(\left[a\hat{y}+\epsilon c_y\hat{y}+\epsilon a(\vec{d}\times\hat{y})\right]\cdot\vec{L}+O(\epsilon^2)\right) \\ &= \exp\left([a\hat{y}+\epsilon c_y\hat{y}+\epsilon(\vec{c}-c_y\hat{y})]\cdot\vec{L}+O(\epsilon^2)\right) = e^{(a\hat{y}+\epsilon\vec{c})\cdot\vec{L}}+O(\epsilon^2). \end{aligned} \tag{51}$$

Using this factorization, all the factors in (49) can be trivially expanded for first-order in ϵ, and corresponding terms collected. The result is:

$$\vec{\lambda} = (b+c_y)\hat{y} + (e^{-a\hat{y}\cdot\vec{L}} - I)\vec{d}. \tag{52}$$

Written explicitly, in terms of components:

$$\lambda_x = c_x\frac{\sin a}{a} - c_z\frac{(1-\cos a)}{a}, \tag{53a}$$

$$\lambda_y = b + c_y, \tag{53b}$$

$$\lambda_z = c_x\frac{(1-\cos a)}{a} + c_z\frac{\sin a}{a}. \tag{53c}$$

This result has also been checked by evaluating the explicit 3×3 matrix representation of (49) in *Mathematica*, followed by a series expansion in ϵ. Applying this result to (48) with

$$a = \vec{F}_{2,d}\cdot\hat{y}, \qquad b = \vec{F}_1\cdot\hat{y}, \qquad \vec{c} = \delta\vec{F}_2, \tag{54}$$

and using the explicit expressions in (47), we find the result:

$$\lambda_x = P_y\left(\frac{\gamma_d-1}{\gamma_d}\right)(1-\cos(G\gamma_d\phi)), \tag{55a}$$

$$\lambda_y = -(1+G\gamma_d)\{P_x(1-\cos\phi) + (P_t/\beta_d + hX)\sin\phi\} + P_tG\beta_d\gamma_d\phi, \tag{55b}$$

$$\lambda_z = P_y\left(\frac{\gamma_d-1}{\gamma_d}\right)\sin(G\gamma_d\phi). \tag{55c}$$

Therefore, we have expressed the spin map in the approximate form:

$$M_B = \exp(-G\gamma_d\phi\hat{y}\cdot\vec{L})\exp(\vec{\lambda}\cdot\vec{L}) + O(\epsilon^2), \tag{56}$$

with $\vec{\lambda}$ given by (55).

The result (56) may also be obtained by linearizing the Thomas-BMT equation (30) (with respect to the phase space variables) around the design trajectory and applying a Magnus expansion. Following this procedure yields a spin map of the form [9]:

$$M_B^{\text{lin}} = M_d e^{A\Delta\zeta\cdot\vec{L}} + O(||\Delta\zeta||^2), \tag{57a}$$

where $\Delta\zeta = (X, P_x, Y, P_y, T, P_t)$, interpreted as a column vector, $M_d \in SO(3)$ and A is a real 3×6 matrix defined by the equations:

$$\frac{dM_d}{ds} = \left(\vec{\Omega}(s)\cdot\vec{L}\right)M_d, \qquad \frac{dA}{ds} = M_d^T D\Omega(s)R(s). \tag{57b}$$

In (57), both $\vec{\Omega}(s)$ and its Jacobian $D\Omega(s)$ are evaluated along the design orbit, and $R(s)$ denotes the 6×6 symplectic matrix that describes transport of the initial orbit vector $\Delta\zeta$ over a sector bend of design arc length s. A careful investigation confirms that the solution of (57) coincides with (56).

References

[1] L. H. Thomas, "The motion of the spinning electron," Nature **117**, 514 (1926).

[2] V. Bargmann, L. Michel, and V. L. Telegdi, "Precession of the polarization of particles moving in a homogeneous electromagnetic field," Phys. Rev. Lett. **2**, 435-436 (1959).

[3] G. H. Hoffstaetter, *High-Energy Polarized Proton Beams: A Modern View*, Springer, NY (2006).

[4] D. Abell *et al*, "Accurate and efficient spin integration for particle accelerators", Phys. Rev. ST Accel. Beams **18**, 024001 (2015).

[5] A. J. Dragt, *Lie Methods for Nonlinear Dynamics with Applications to Accelerator Physics*, University of Maryland, College Park, Nov. 2020 version, http://www.physics.umd.edu/dsat/

[6] D. L. Bruhwiler, J. R. Cary and S. G. Shasharina, "Symplectic propagation of the map, tangent map and tangent map derivative through quadrupole and combined-function dipole magnets without truncation," THP41C, Proceedings of the 6th European Particle Accelerator Conference (EPAC 98), 1171-1173 (1998).

[7] L. Michelotti, "Exact map through ideal bends (again?)," Proceedings of the IEEE Particle Accelerator Conference (1995), FERMILAB-Conf-95/171.

[8] ImpactX, https://github.com/BLAST-ImpactX/impactx

[9] C. Mitchell, "Lie algebraic spin maps and the dynamics of beam polarization," Lawrence Berkeley National Laboratory, Report: LBNL-2001756 (2026).

[10] A. E. Lobanov and O. S. Pavlova, "Solutions of the classical equation of motion for a spin in electromagnetic fields," Theor. Math. Phys. **121**, 1691-1699 (1999), https://doi.org/10.1007/BF02557213