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A Computer Assisted Study of Go on $M \times N$ Boards†

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ABSTRACT

The game of Go invites analysis. The rules seem few and simple, suggesting that the game may have helpful theorems. Tens of millions of people play and skill has developed over centuries to extraordinary levels. Thus, computer analysis can be tested against analysis by highly skilled human players.

We study $M \times N$ boards, rather than the usual 19×19 . We begin with the computer-assisted complete tree calculation for the tiniest boards. The analysis extends to slightly larger boards with the aid of various lemmas and concepts of connectedness and symmetry. The usual rules are incomplete. Though difficulties rarely arise from this on a 19×19 board, they are frequent on small boards. We therefore extend and complete the rules in a way which, we believe, preserves their spirit.

We give bounds on the value of the game and on its combinatorial magnitude. We discuss a heuristic strategy based on potential. The flow chart for our computer program is included and may be easily modified for use in human-machine symbiosis. We offer conjectures about the optimal strategies and the value of the game, for somewhat larger boards than those solved.

We suggest small-board Go as a progressive testing ground, as M and N increase, for: (1) techniques to yield the complete solution, (2) the power of learning programs and of (positional) evaluation functions, (3) human-machine symbiosis, and (4) human-machine confrontation.

1. INTRODUCTION

The game of Go is believed to have originated in China about four thousand years ago. It has been intensely cultivated in Japan for more than a thousand years and is now played throughout the world. For the history of Go, refer

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to Falkener (1961) or Smith (1956). Playing technique is discussed in Goodell (1957), Kishikawa, Korschelt (1880), Lasker (1960), Morris and Morris (1951), Smith (1956), and Takagawa (1958). See also Carlson (1966) and Zobrist (1970).

For those who worry about whether the study of games is useful, we remark that an interest in games of chance led Cardano, Fermat, and Pascal to initiate the theory of probability. Providing a theoretical framework for poker was one objective of von Neumann's theory of games. Recently one of us was faced with the problem of determining the optimum amount to bet on positive expectation situations in casino blackjack. By using the results of Kelly (1953) and Breiman (1961), a theory of resource allocation (synonyms are "bet sizing" and "portfolio selection") was developed. This theory (Thorp 1969) supplants and refutes the theory of portfolio selection that is generally accepted by economists (Markowitz 1959).

We find Go a particularly promising game of skill to analyze. Tens of millions of people play it and it has been developed to an extremely high level of skill. This means that computer analysis can be checked or tested against analysis by highly skilled human players. Also, the rules appear few and simple, which suggests that the game may have significant theorems.

There are several ways to study Go with the aid of a computer. Remus (1962) wrote a computer program to learn "good" strategies and simulated the game on a smaller board. Another approach is to combine positional evaluation functions and tree computations, as has been done for chess.

We study the game on an $M \times N$ board rather than the usual 19×19 board. First we give the complete tree calculation for tiny boards. Then we extend the analysis to larger boards. Previous studies of bridge and poker via tiny decks are close in spirit to our approach.

The many authors suggesting the consideration of small board or of $M \times N$ Go include Olmstead and Robinson (1964, page 89), Gardner (1962) and (1969), Lasker (1960, page 78), Rosenthal (1954), and Thorp and Walden (1964). It is therefore somewhat surprising that this paper and our previous one seem to be the first attempts to analyze systematically $M \times N$ Go for small values of M and N . This paper continues our work in Thorp and Walden (1964), which may be read at this point.

All the computer-based approaches to Go can be modified by combining man and machine. For instance, our computer program can be used like a "super slide rule" to assist a human player in actual play. This might fit in with the project announced in Engelbart (1968). One wonders why this has not been done in chess (as suggested, for instance, in Birkhoff 1969).

The contention is sometimes made that two chess players who consult generally do not do as well as the stronger player would have done alone. If so, it is perhaps because, with two disparate consulting players, the contribution of the weaker player is nearly subsumed by that of the stronger

player. Thus, the weaker player is redundant and only impedes the stronger player. With two equal players we might expect their different competing strategies to interfere with each other.

But in a man-machine symbiosis, skills are complementary. Also disputes do not arise: the man uses the machine when he wishes to, and not otherwise. The difference is that between calculating with a super slide rule and calculating with a committee.

2. TOPOLOGY

The game of Go is played on a rectangular board marked with two mutually perpendicular sets of parallel lines. The standard board has 19 lines in each direction. This produces 361 points of intersection. The two players, Black and White, move alternately beginning with Black. A move consists either of placing a piece (stone) on a vacant point of intersection, or of passing. The object of the game is to capture stones and to surround territory.

Two vertices are *adjacent* if they are on the same horizontal or vertical line and there are no vertices between them. In particular a vertex is adjacent to itself. Classify vertices into three types: black, white, and vacant. Vertices v and w are *connected* if there is a chain $(v_1, \dots, v_n)$ of vertices of the same type such that $v = v_1$, v_i is adjacent to v_{i+1} , $1 \leq i \leq n-1$, and $v_n = w$. We say that v and w are *joined* by the *path* $(v_1, v_2, \dots, v_n)$. The v_i need not be distinct. In particular, we allow $v_i = v$ for $1 \leq i \leq n$ so a vertex is connected to itself. Connectedness is an equivalence relation (i.e., it is reflexive, symmetric, and transitive). Thus, it partitions the board into disjoint equivalence classes. We call such an equivalence class a *group*. A group of stones is also called an *army*.

If all the vertices adjacent to a vertex in a given group are of the same type as the given vertex, that vertex is in the interior of the group. If the adjacent vertices are not all of the same type, the vertex is a *boundary point* of the group. A vacant vertex adjacent to a black or to a white stone is a *breathing space*, *degree of freedom*, or *liberty* (Remus 1962, Good 1965) for the group to which that stone belongs. (See Olmstead 1964, page 90, for a similar topologic discussion.)

Rule 1 (Capture). If a group of stones has exactly one breathing space and the other player is permitted to move there (he may not be, because of the Ko rule or a similar rule prohibiting all cycles of even length—see below) then if he does so, he captures that group. He removes it from the board and the stones are his “captives.” At the end of the game players receive one point for each captive.

Rule 2 (Suicide Is Illegal). If a player's group of stones has exactly one breathing space, a move there by the player (which would deprive his own group of breathing space and cause it to be removed, i.e., suicide) is illegal unless it deprives an opposing group of its breathing space. In this instance, the opposing group is removed instead. In particular, if a single empty vertex is completely surrounded by stones of the opposing color, a player can move there only if the move results in the removal of one or more of the adjacent men. (Otherwise, the move creates a group of one stone with no breathing space, which causes it to be removed, i.e., suicide.)

Suppose v is a vacant vertex and that all vertices adjacent to v belong to the same group G of stones. Then v is an *eye* for G . A group of stones with two or more eyes cannot be captured because Rule 2 prevents the opponent from filling one eye while another exists. This principle is fundamental. There are other kinds of impregnable groups too; all such are called *living*.

Figure 1 shows a Black group with two eyes on a 1×3 board. This is the smallest board which can have a group of stones with two eyes.



FIGURE 1. A group of Black stones with two eyes, on the smallest board for which this is possible, 1×3 .

Isolated single vertices can be replaced by larger connected groups of vacant vertices in the above discussion. We call any such group which is “interior” to a group G of stones a *vacuole* for the group G . Note that the “principle of two eyes” does not generalize to a “principle of two vacuoles,” even when one of the vacuoles is an eye. This is because, if a vacuole V of G is large enough, the opponent may be able to build a group H inside which has its own vacuole and which fills all the breathing spaces which V supplied for G (Fig. 1a). In this case, he may or may not be able to prevent the ultimate creation of an eye for G in the vacuole V .



FIGURE 1a. A 1×4 board where the Black group G had one eye and one vacuole. The white group destroys the breathing space provided by the vacuole and G is threatened.

Of immediate practical interest is the question of which vacuoles can be converted into two eyes if (a) the player whose group it is has the move, or (b) his opponent has the move. For instance, the Black player in Figure 2 can make two eyes if it is his turn by moving to the point x . If it is White's



FIGURE 2. If Black moves to X , he makes two eyes. If White moves there, Black cannot make two eyes.



FIGURE 3. If Black moves to either of the X vertices, he makes two eyes. Thus, even if it is White's turn, the group can live.

turn and White moves to x , Black cannot make two eyes. In Figure 3, Black can make two eyes no matter whose turn it is.

3. COMBINATORICS

For us, the combinatorial magnitude of the game means the number of "distinct" plays of the game, i.e., the number of "distinct" branches of the game tree. The word "distinct" has many interpretations, as we shall see.

The number of distinct board configurations, i.e., ways that the pieces can be arranged on the oriented 19×19 board, is usually said to have order of magnitude 3^{361} . The vertices on an oriented board might, for instance, be labeled (i, j) for the vertex in the i th row and j th column where $1 \leq i, j \leq 19$. Then in building a configuration there are three choices for each of the 361 vertices, i.e., vacant, Black, or White, thus, the figure 3^{361} . Any such oriented configuration we call an *arrangement*. Olmstead says "a *configuration* is any arrangement on the whole board of vacant points, Black stones and White stones." We are not sure whether he means what we call an "oriented configuration" or "arrangement," or whether he considers as identical two equivalent unoriented configurations (see space symmetry discussion below).

[*Note:* After this paper was completed, we learned of the profound and significant work of Olmstead and Robinson (1964) and [19]. This has led to a substantial revision of the present paper, particularly in Sections 2, 3, and 4 on topology, combinatorics, and rules.]

But the number of distinct arrangements on an oriented board does not tell us how to continue the play. This leads us to distinguish four levels of increasing information which might be available to an observer (or computer) during play, each of which is necessary for determining how to continue a branch of the game tree. First, there is the (oriented) board arrangement. However, we also need to know who is to move (equivalently, who has just moved). We call the pair (board arrangement, B or W has just moved) a board

position. (Olmstead uses “position” to mean an arrangement in which every group has a breathing space, i.e., any arrangement that could arise after the completion of a turn. The information B or W is not included.) This is a second level of information and is the one used in practice for presenting chess problems.

Yet, in both chess and Go, the game cannot be played out from knowledge of position alone. In chess, for instance, the repeat of a position three times ends the game in a draw (see, e.g. Fine 1969). Thus a list of past positions is required in order to play out a given position. In practice this is not a problem because irreversible transformations (the move or transformation of a pawn, the capture of a piece) occur frequently enough so that players and makers of chess problems generally assume that there is no effect from past positions.

In contrast to chess, repetition of position in small board Go is too common to be ignored. Also, certain vacant squares in a given position may be illegal moves (see Ko rule and cycle rule, discussed below) or may be legal moves, depending on the past history of the game. Thus each vertex may conceivably be in one of four states: Black, White, vacant and legal for the next player to occupy, or vacant and illegal for the next player to occupy. Thus we specify as a third level of information the board position plus the state of each vacant vertex. We call this a *complete position*. It is the minimum information required for the next player to move.

However, even knowledge of the complete position is insufficient to continue the game. To do this we require a fourth level of information, the entire sequence of previous complete positions. We need this both to determine the states of vacant vertices in positions arising more than one move in the future, and in ending and evaluating the game in those cases in which repetitions of position end the game. An inductive argument, using any selection of rules (below), shows that the sequence of previous complete positions may be recovered from the sequence of previous arrangements. We call the (ordered) sequence of previous arrangements a *history*. It uniquely and completely describes the current situation in the game and is sufficient to continue the play unambiguously.

Each board has an associated symmetry group of reflections and rotations (Birkhoff and MacLane 1965, Chapter 6) under which it is carried onto itself. We call these *spatial symmetries*. Each $1 \times N$ board, with $N \geq 2$, has two distinct orientations, depending on which vertex is at “the origin.” A 2×2 board and an $M \times N$ board, with $M, N \geq 2$ and unequal, each has four distinct orientations, corresponding to which vertex is at the origin. For $N \geq 3$, each $N \times N$ board has eight distinct orientations, determined by specifying whether the board is face up or face down and which vertex is at the origin.

A given board arrangement will be carried into n distinct board arrangements under the board’s space symmetry group, where n is a divisor of the

order of the board's space symmetry group. Since this order is 1, 2, 4, or 8 the values of n are limited to those numbers. Thus the set of all board arrangements is divided into space-equivalent subsets of 1, 2, 4, or 8. We shall consider two arrangements as identical under the rules if they are space-equivalent. The resulting reduction in the game tree is helpful in studying the small board game. In practice space symmetry is not useful in the 19×19 game after the initial moves.

Applying space symmetry to the first move we find that, instead of MN distinct subtrees arising after the first (active) Black move on an $M \times N$ board, $M \neq N$, there are $[(M+1)/2][(N+1)/2]$ inequivalent active first moves for Black, where $[x]$ stands for the integer part of x . On a 1×1 board there is no legal active first move because of the suicide rule discussed below. There are $[(N+1)/2][(N+1)/2 + 1]/2$ legal active first moves on an $N \times N$ board if $N \geq 2$. Thus, the number of first moves is reduced by a factor of roughly 2, 4, or 8, depending on the board dimensions.

On the 19×19 board, for instance, there are 55 inequivalent (active) Black first moves, rather than 361, reduction by a factor of 6.6. White has 54 inequivalent replies to the Black move to the center. There are 9 inequivalent non-central Black moves on a ray from the center perpendicular to the edge, and 189 inequivalent White replies to each of them. A Black move to a diagonal is similar. For each of the other 36 Black moves, there are 360 distinct White replies. Thus, there are 16,416 active inequivalent (active) two-move beginnings to the game, rather than 361×360 , reduction by a factor of $95/12$ or 7.9.

On a $1 \times N$ board, $N \geq 4$, there are for N odd $(N^2 - 3)/4$ distinct arrangements after two active moves. The result for N even is $(N+1)(N-2)/2$.

The space symmetry principle used above is well known in ordinary tic-tac-toe. An interesting and subtle variation for three-dimensional tic-tac-toe appears in Silver (1967).

Another principle which reduces the possible number of distinct board arrangements is that, after a turn is completed, and captures have been made, every connected group of a given color must have a "breathing space" (adjacent vacant vertex). In particular, the board will have one or more vacant vertices when a player is about to move. This means, for instance, that on a 19×19 board the 2^{361} arrangements obtained by filling the board with stones can never result from a completed turn. There are also many illegal board arrangements having some board vacancies.

A standard (19×19) game averages roughly 250 choices ("moves") and the players have roughly $361 \times 360 \times \dots \times (361 - 250 + 1)$ sequences of choices, or plays of the game, divided by say 10, for symmetries. For precision of expression, we try in this section to adhere to the (von Neumann-Morgenstern 1944) distinctions of "choice," "move," "play," and "game." Taking

200 as a suitable average number of available choices per move, this yields roughly 200^{250} games. If only half the choices were level at each move, we would have $100^{250} = 10^{500}$ plays of the game to consider. If 20 choices presented themselves at each move as potentially optimal to the best human player, we might have 20^{250} or about 10^{325} potentially optimal plays of the game to consider in the game tree.

To get an upper bound to the number of branches in the game tree, we anticipate some results from the following section on rules. In particular, we note here that we shall adopt rules allowing either player to pass on any turn (Rule 8(b)), forbidding an active move by a player which would repeat a board arrangement after an earlier move by *that player* (Rule 4 and Rule 5(a)), and terminating the game whenever there are two consecutive passes.

Using these rules, any play of the game, i.e., branch of the game tree, is fully described as an ordered sequence of board arrangements, alternately labeled B or W according to whether Black or White moved. Note that if all passes are omitted from the sequence, the branch is still fully described by the reduced sequence. If two arrangements labeled B occur consecutively, we infer that W passed in between.

It might seem from our previous discussion as if there is still ambiguity over whether vacant vertices are permitted or proscribed moves for the next player. However, by induction, and use of the full set of rules (below), it follows that the state of each vacant vertex is determined. At the end of the sequence we infer that two consecutive passes occurred. We know also that distinct branches of the game tree correspond to distinct reduced sequences. Hence the number of reduced sequences is an upper bound for the number of branches of the game tree.

We know further that all arrangements labeled B are distinct board positions. Similarly, all arrangements labeled W are distinct. Since there are no more than 3^{MN} distinct arrangements on an $M \times N$ board, it follows that a reduced sequence, all of whose entries are distinct, has length no more than $k = 2 \cdot 3^{MN}$. Thus there are at most $k(k-1) \dots (k-r+1) \equiv (k)_r$ reduced sequences of length r , hence no more than $(k)_r$ (reduced) plays of the game of length r . Thus the game tree has no more than

$$\sum_{r=1}^k (k)_r = \sum_{r=1}^k k!/r! \leq [k!(e-1)] = [(e-1)(2 \cdot 3^{MN})!]$$

branches. For the 19×19 board this very crude upper bound is roughly $10^{10^{110}}$. On a 1×1 board there is only one branch and the bound is a generous $[6!e] = 1957$. Numerous obvious improvements in the estimate are apparent.

If, for instance, we count the actual number of (non-empty) arrangements a and use a in place of 3^{MN} , we have $[(2a)!(e-1)] + 1$ as an upper bound. For a 1×1 board this gives 2. On a 1×2 board, $a = 2$ and our upper bound

becomes $[4!(e-1)] + 1 = 42$. On a 1×3 board, $a = 10$ compared with $k = 27$, and on a 2×2 board $a = 13$, compared with $k = 81$.

With Rule 5(b) replacing Rule 5(a) as in our earlier work, a reduced sequence of length r can end as before (with two passes), or it can end with the repetition at step $r + 1$ of one of the r previous board arrangements. Thus in the previous argument we may replace $(k)_r$ by $(r + 1)(k)_r$. This readily leads to $[k!(2e - 1)]$ or, better $[2a!(2e - 1)]$, as an upper bound to the number of branches. The preceding estimates can be improved.

Similar calculations for chess show that it is a smaller game than 19×19 Go, no matter which of the preceding bases of comparison is used.

For instance, suppose 70 choices is a fair average length for a play of a chess game and an average of 5 choices per turn would seem potentially optimal to a human expert. Then we would have 5^{70} or about 10^{50} potentially optimal game tree branches to consider.

Now suppose that $N^2/10$ moves are on average potentially optimal to a Go expert playing $N \times N$ Go. Take the average game length as $N^2/2$, which is conservative. Then we have $(N^2/10)^{N^2/2}$ potentially optimal game tree branches to consider.

For $N = 10$ this is 10^{50} , the same as our estimate for chess. Allowing for the fact that Go has some space symmetries and that the men are identical, whereas in chess they vary, it seems plausible to suggest: chess and 11×11 Go are of comparable complexity, and the standard 19×19 Go is much "deeper" than chess.

Good (1965) suggests an interesting way of comparing games of skill, such as Go and chess, which could shed light on the discussion of which is "deeper." Let two players be one "step" apart in skill if the better player has probability $2/3$ of winning. Determine the number of steps from the best player(s) to the poorest. This is the degree of skill or "depth" to be associated with the game.

We could modify this to nearly remove dependence on the population extremes by using as our measure of depth the average number of steps separating two players selected at random from a suitable population. The objections remain that the measure is sensitive to (1) the threshold of skill which determines admission to the population measured and (2) the general social level of skill: as a game is cultivated, its "depth" increases.

4. RULES

In the game as played, the rules seem to be imprecise. We now develop a precise reformulation which agrees with the game as played, and which is suitable for programming.

First we describe how the game is played in practice. Black and White move alternately, beginning with Black. Eventually the board is partitioned into (1) regions surrounded by and “belonging to” Black, (2) regions surrounded by and “belonging to” White, (3) boundary areas in which there is not enough room for either player to capture territory, and (4) rarely, situations called *Seki*. At this point the game ends.

A *Seki* occurs when a move by either player into the area is unfavorable for him. See Figure 4. A region surrounded by White “belongs” to White

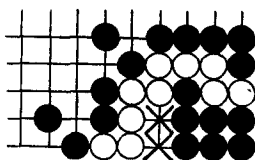


FIGURE 4. *Seki*. If either player moves to an *X* vertex, the other player can capture by playing at the other *X*. Other considerations aside, it is unfavorable for a player to move to an *X* vertex.

if Black does not wish to contest it, i.e., if he believes he has no hope of building a living group in the region or of neutralizing it by *Seki*. Black stones in the region, if any, are removed and counted as White captives. The discussion is similar for regions surrounded by Black.

Note that the decision rule for ending the game is not well defined. It depends on the players' judgment as to whether or not the board has been partitioned into the four types of regions. If the two players disagree about this, one player may pass one or more times while the other one moves. Finally, when neither player wishes to move, the game ends.

The incompleteness and ambiguity in the rules of Go, as practiced, are more subtle and complex than we have so far indicated. A detailed discussion of the difficulties is given by Olmstead and Robinson (1964). For a game-theoretic and computer study of the game it is necessary to unambiguously restate and complete the rules. This is the first objective of our discussion of rules. We adopt what is essentially one of Olmstead's versions. We are guided by the principles that the rules should be as consistent as possible with practice and that, where choice is possible in this framework, elegance and simplicity are desirable.

It is not easy for the computer to decide which groups are dead without playing out the game tree much farther than human players would in an actual game. We shall therefore assume the tree is played out to the end, in order to be able to use the following scoring rule on the machine.

Rule 3 (Scoring). At the end of the game, a group of empty vertices which

have only Black [White] stones adjacent to the group are counted for Black [White], each such vertex counting one point (or "stone"). Each captured Black [White] stone counts one point for White [Black]. The player with the greater number of points wins by the difference in point totals.

In Figure 5, Black gets 4 points for territory. Note that he surrounds this territory even though no vertex is surrounded by a single Black group.



FIGURE 5. Black surrounds four vertices. White has no legal active move and must pass.

Note that draws (i.e., 0 points net for Black) are possible under this rule. We have been told that in international match play there are no draws. If Black wins by 5 or more "stones," he wins and otherwise he loses. Consequently, we see that one number of interest in $M \times N$ Go is the net number $V(M, N)$ of points (conceivably negative) that Black will have under best play by both sides. We call $V(M, N)$ the *value* of the game.

In order to use the computer to work out the game tree for small-board Go, we need a well-defined rule for ending the game. To do this we need to discuss Ko, cycles, and passes.

Rule 4 (Ko). An (active) move which would reproduce the board arrangement as it was after that player's preceding turn, if any, is illegal.

This rule prevents certain endless repetitions (Fig. 6). It also yields subtle

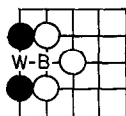


FIGURE 6. By Black and White alternately moving in squares B and W , respectively, the board position would repeat endlessly. The Ko rule forbids this.

sequences of threat and counterthreat in games between better players. This use of "Ko threats," where a player threatens elsewhere before recapturing, is generally considered to greatly enrich the game.

Ko rule 4 alone does not prevent all endless repetition or cycling in the game. In Figure 7, if Black and White move as indicated, the board returns to its previous arrangement in eight moves. To make the game tree finite, we must prevent endless cycling. To do this, we introduce Rule 5 (*Cycles*). Our sources disagree as to the rule in use. The most natural one, and the one we

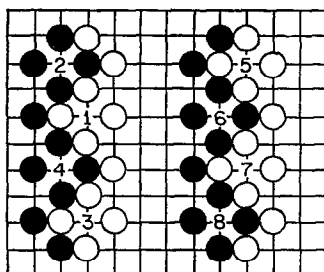


FIGURE 7. A cycle which is completed in 8 moves.

feel is correct, is Rule 5(a) below. It is implied by Rule 3 and the related discussion in Olmstead and Robinson (1964).

Rule 5(a) (Generalized Ko). An active move which would reproduce a board arrangement that occurred an even number of moves earlier is illegal.

The reason for stipulating “active” move will appear when we analyze the 1×3 game. (See notes to Figure 13.) We shall adopt Rule 5(a). Note, that when Rule 5(a) is used, Rule 4 is redundant and may be omitted.

Rule 4 and Rule 5(a) tell us that the permitted future moves in the play of game cannot be determined alone from the current board arrangement and the knowledge of who is to move. The Ko Rule tells us that a player may need to know the board arrangement after his preceding turn to determine whether a certain move is permitted him. The Generalized Ko Rule tells us that a player may need to know prior board arrangements after all his turns to determine whether certain moves are permitted. In general, the result of any previous move may be of importance on a future turn.

When are two partial histories or sequences of (prior) positions “equivalent” for the future play of the game? In Rules 4 and 5(a) we interpret repetition of board arrangement to mean just to within a space equivalence, rather than requiring the identical oriented arrangement to recur. Then suppose that for two branch points of the game tree the following conditions are fulfilled *to within space equivalence*: (i) The current board arrangements are space equivalent and the tempo is the same (i.e., in each case the same player has just moved), i.e., the positions are space equivalent. (ii) Corresponding members of the sequence of past positions are space equivalent. Then two such branch points are called *space equivalent*. Thus we have the:

Space Symmetry Principle. Space equivalent branch points are game equivalent, i.e., give rise to identical subtrees. In particular, they have the same value and the same strategies.

Now it is clear that, by applying a fixed member of the space symmetry group to each arrangement in the partial history leading to a given branch

point, we get space equivalent branch points. This reduces by a factor of 1, 2, 4, or 8 the number of branches in the game tree which have to be analyzed. However, there are in general many other kinds of branches also eliminated by the Space Symmetry Principle. We do not presently know the order of magnitude of this additional reduction or even whether it deserves to be called "substantial."

By analogy with chess and according to Good (1965), instead of Rule 5(a) we should use Rule 5(b) below. Lasker (1960, pages 58–60) also says this is the rule.

Rule 5(b). A move which reproduces a board arrangement that occurred 4 or more even number of moves earlier terminates the game at once in a draw.

Refinements of Rule 5(b) suggest themselves. One might wish to terminate the game as before, but evaluate the position and then award points. Call this *Rule 5(c)*.

Suppose that there has been a change over the cycle in the *capture count*, i.e., the number of stones captured by Black minus the number of stones captured by White. Then sufficiently many repetitions of the cycle lead to an arbitrarily large gain for one player. One might argue that, in such cases, this player should be awarded the win. We note that such a change in capture count over a cycle occurs if and only if one player has passed more than the other over the cycle.

To illustrate this possibility, let M and N be such that the board has at least three vertices, i.e., the board is at least 1×3 or 2×2 . Let Black begin by moving to a corner. Then Black passes while White fills the board, capturing Black's stone on his last move. Black recaptures and the cycle is repeated. Note that over each cycle the change in capture count is $MN - 2$ in favor of Black.

A natural way to end a play of the game tree is by two consecutive passes:

Rule 6(a) (End). The game ends if there are two consecutive passes.

Using Rule 5(a), this is the only way it can end. With Rule 5(b), it can also end with a cycle.

When can a player pass?

Rule 7 (Forced Pass). A player with no legal active move must pass.

On a 1×1 board, for instance, Black and White are forced to pass in turn because suicide is illegal. Rule 6 ends the game and Rule 3 assigns the value 0 to the completed game. (We shall use Rule 6(b) below, rather than Rule

6(a). Anticipating this, we speak of Rule 6. Both rules end the game in the same way. The difference is only in the scoring.)

In the 1×2 game, the sequence beginning with an active Black move, then an active White move (capture), is followed by a forced Black pass (Ko rule) and a forced White pass (suicide rule), which ends the game (Rule 6). See Figure 8. The capture count is indicated below each board. The small dot

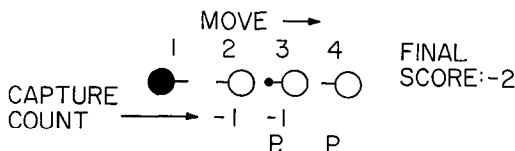


FIGURE 8. Ending the game by two forced passes on the 1×2 board.

on board 3 indicates that Black may not move there on this turn because of Ko. A large *P* indicates a forced pass. The final score for Black is -2 , for White has one prisoner and one point for territory.

To decide whether a subset of one color that is surrounded by another color is dead or alive, we work through the game tree for such situations. This allows us to use the unambiguous Rule 3 for scoring. However, Rule 3 which counts territory for a player only if it is bounded by his stones alone, is very restrictive, compared with the game as played.

In our previous paper (Thorp and Walden 1964) we allowed either player to pass voluntarily at any turn without penalty:

Rule 8(a) (Free Voluntary Pass). Either player may pass his turn at any time without penalty.

In the game as played, no one would pass until the game was ending, at which point both players would generally agree to pass in succession, ending the game. Thus, this voluntary pass rule would seem not to be in conflict with practice.

However, in conjunction with the restrictive scoring Rule 3, Rule 8(a) may lead to quite different scores than in practice. This was pointed out to us by Dr. Joseph A. Schatz of the Sandia Corporation. We illustrate with an example.

Suppose that the simple configuration in Figure 9 has arisen and that it is now White's turn. Most players would terminate the game now by mutual agreement. Black does not wish to move: He cannot move into any vacant vertex belonging to White (suicide rule) and any other Black move reduces Black's own territory and costs him a point.

On the other hand, if White moves into some of the 10 permissible vacant vertices surrounded by Black, he cannot kill the Black group. He also cannot

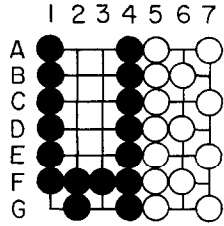


FIGURE 9(a). Black controls 12 vacant vertices, White controls 7. The capture count $C = -5$ and the game would be called a draw. White is to move.

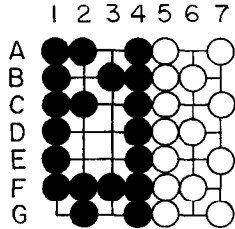


FIGURE 9(b). After the moves $B2$; $A2$, Pass; $C2$, Pass; $B3$, Black controls 9 units of territory, White controls 7, and $C = -4$. White wins by 2 points.

build a living group there, for it is not wide enough to allow an eye. Thus, White stones in this area are dead and each White move there loses 1 point for White. Each move elsewhere reduces White's territory and costs White a point. White also wishes to pass. Thus, both players pass and the game ends.

Now consider the same situation with Rule 3. Suppose White moves to $B2$. The group of ten vertices into which White has moved are no longer counted for Black. If Black wishes to remove the White stone at $B2$, he must move to $A2$, $B3$, and $C2$, with White passing twice. Black has filled in 3 units of territory and has captured one stone. If the game were scored at this point either by Rule 3 or in the customary manner, the result would be the same: Black now loses by 2 points.

We resolved this dilemma by replacing Rule 8(a) with:

Rule 8(b) (Penalized Voluntary Pass). A player may pass his turn at any time. The penalty is one point for any pass, whether voluntary or forced.

This is the form of the pass rule given in Good (1965).

Several careful players to whom we have talked object to Rule 8(b), saying that they have formulated and used a different rule. They divide the game into phase I and phase II. The game begins in phase I and continues until one player feels that the game should end. Rule 8(a) holds in phase I. Next

the game shifts to phase II, in which Rule 8(b) holds. Finally, with two consecutive passes, the game ends and is scored in the customary fashion.

It is the general belief that to pass in the early and middle parts of the game is unwise. Thus, in practice, no one ever passes in phase I, so it is academic as to whether Rule 8(a) or 8(b) is used, if we simulate actual play.

The introduction of voluntary passes throughout the game adds an additional branch to the game tree at each branch point. This increase in the size of the game tree reduces the board size for which the game can be solved by working out the game tree. It seems plausible for large and intermediate boards (but perhaps not for the sometimes pathological very tiny boards) that these extra branches do not affect the values of the game. Thus we may wish to prohibit passing during the early and intermediate stages of the game, by:

Rule 8(c). Select a positive integer $t(M, N)$ for each $M \times N$ board, such that Rule 8(b) holds on and after the $t(M, N)$ turn and voluntary passes are illegal before turn $t(M, N)$.

Presumably $t(M, N)$ would be sufficiently small so that forced passes could not arise before turn $t(M, N)$. For $M, N \geq 3$, probably $t(M, N) = MN/2$ would suffice.

If the game ends with a given board arrangement and capture count, we wish the score to be well defined, as it is supposed to be in practice. But since Rule 8(b) penalizes passes, it might happen that the net change in score due to passes can vary.

This does happen and the variation, if it occurs, is just one point (which is the expectation expressed by Joseph Schatz in our correspondence). To see this, let a_B and a_W be the number of Black and White stones, respectively, which have been played in the game. Since every stone played is either a prisoner or is on the board, we have $a_B = s_B + n_B$, where s_B is the number of Black stones which have been captured and n_B is the number of Black stones on the board. Similarly, $a_W = s_W + n_W$. Thus, a photograph of the board, plus the number of prisoners in each pile, fully specifies a_B and a_W .

We also have that $m_B = p_B + a_B$, where p_B is the number of Black passes, and m_B is the total number of Black moves. Similarly, $m_W = p_W + a_W$. We now distinguish two cases. *Case (E):* If the game ends in an even number of moves, then $m_B = m_W$ so $p_B + a_B = p_W + a_W$ and $p_W - p_B = a_B - a_W$; hence, the net contribution to Black's score from passes is $(s_B - s_W) + (n_B - n_W)$, which can be determined from a photograph of the board plus the number of prisoners in each pile. *Case (O):* If the game ends in an odd number of moves, then $m_B = m_W + 1$ and $p_W - p_B = a_B - a_W - 1$. Again the contribution to Black's score from passes is well determined, but the results in cases (E) and (O) differ by 1 point. To eliminate this disparity, we are led to modify Rule 6(a) for ending the game to read:

Rule 6(b) (End). The game ends if there are two consecutive passes. If the second pass is by Black, then one point is added to Black's total.

This makes the scoring the same in cases (O) and (E).

It is convenient to think of Rule 6(b) as providing for an additional pass by White if the two consecutive passes which end the game do not end on a White move. Thus, Rule 6(b) is equivalent to modifying 6(a) by adding an extra White pass, if necessary, so that the game always ends with a White pass.

As our final set of rules for the use of the computer, we now adopt Rules 1 through 4, 5(a), 6(b), 7, and 8(b).

It appears to us that these rules are equivalent to those for Wei-chi as defined in Olmstead (1964). Replacing Rule 6(b) by Rule 6(a) and using Japanese scoring, as we do for Theorem 4 below, seems to be equivalent to Olmstead's definition of Igo. These differ from his formalization of the game as currently played, which he calls Orthodox Igo. The difference is in the considerable complexities involving Seki. We prefer to avoid this detailed consideration of Seki in this paper in order to greatly simplify our treatment. It also seems to us that the detailed separate consideration of Seki is unsatisfactory. Instead, the rules as given plus the minimax principle applied to the game tree automatically evaluate Seki positions and identify best strategies. If the results are at variance with practice then we suggest the simple set of rules should if possible be revised to achieve an acceptable fit with practice, rather than building a complex elaborate formalization of existing practice.

5. ILLUSTRATIVE GAME TREES FOR TINY BOARDS

We illustrate the rules we have adopted, and some of the alternatives, by discussing the game tree for the tiniest boards.

Figure 10 shows the trivial game tree for 1×1 Go. Note the format of

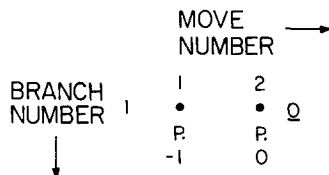


FIGURE 10. The game tree for 1×1 Go. Value of branch to Black follows and is underlined.

the figure. The *P.* below a board indicates a forced pass. The -1 and 0 below the *P.*'s indicate the sum of the capture count $C_B^t = s_W - s_B$ at turn t and the

net gain $P_B^t = p_W - p_B$ to Black from passes. The final value of the branch to Black is the underlined 0 following the branch.

Figure 11 shows the game tree for 1×2 Go. A P indicates a voluntary pass. Trivial though Figures 10 and 11 are, careful commentary will clarify our analysis of larger trees to follow.

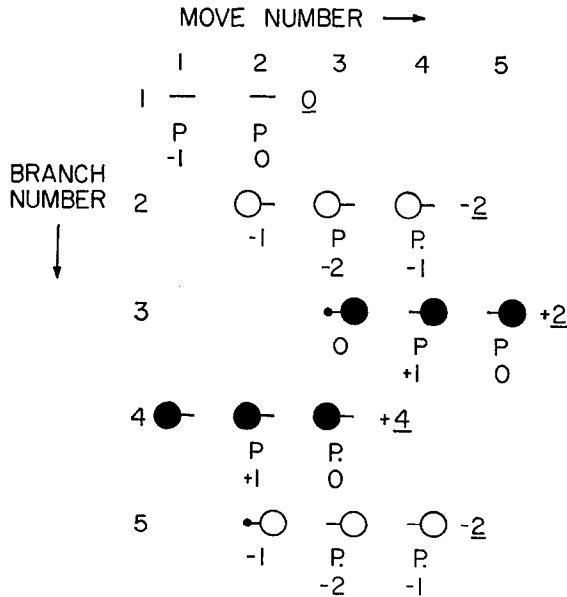
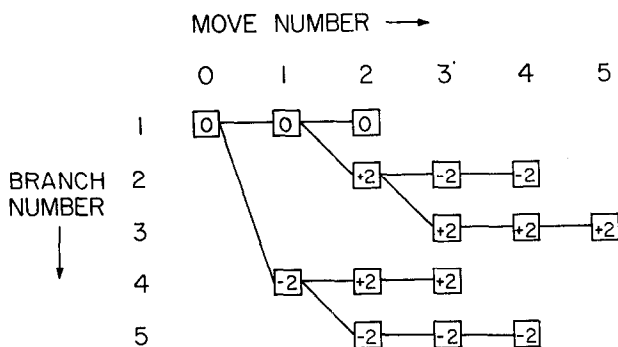


FIGURE 11. The game tree for 1×2 Go. P denotes forced pass and P denotes voluntary pass. In branch 2, a White move 2 to the other vertex is game-equivalent, by spatial symmetry, so we may omit the branches corresponding to 2 and 3 which would otherwise arise. At the end of branch 3, Black gains one point for territory and one point for tempo (Rule 6(b)). Spatial symmetry eliminates two more branches corresponding to branches 4 and 5, arising from a Black move 1 to the other vertex.

First we find the value $V(M, N)$ of the game for these cases. Obviously $V(1, 1)$ is 0. We obtain $V(1, 2)$ by the minimax technique: White on the odd moves will choose the most negative branches, and Black on the even moves will choose the most positive branches. Figure 12 illustrates the minimax calculation for the tree of Figure 11.

Figure 12 shows us that $V(1, 2) = 0$ and that the only branch which arises under best play by both sides is branch 1. We also see the role of spatial symmetry. A spatial equivalent of branches 2 and 3, arising in the same way from branch 1 at move 1, is omitted. It does not affect the minimax calculation. A duplicate of branches 4 and 5, attached to branch 1 at move 0, is omitted and also does not affect the minimax calculation.

FIGURE 12. $V(1,2)$ calculated from Figure 11.

Note that there are just three distinct legal board arrangements which are inequivalent under rotational symmetry. Following the method used earlier for obtaining an upper bound to the number of branches, we had thus obtained the bound of 42. The actual number of branches is of course 5, as Figure 11 shows.

The 1×3 game has 10 board configurations which are distinct under space symmetry. The tree therefore has no more than $[20!(e-1)]$ branches. But this is an impracticably large number of possible branches. It seems prudent therefore to establish at this point a theorem which says, among other things, that we need not analyze in detail the part of the game tree beginning with a Black pass at move 1.

We will see that, if the active first moves for Black all have negative value, then best play by Black is to pass on the first move and the unique best White reply is a pass. Thus, the value of the game and the best strategy will be identical with that in the trivial 1×2 game. We have learned something about the general case by considering the 1×2 game! Note too that this means that we need only have considered the two branches 4 and 5 in the analysis of the 1×2 game given in Figures 11 and 12.

THEOREM 1. (i) For any $M \times N$ board, the value $V(M, N)$ of the game for Black is non-negative, i.e., the game is either a draw or a win for Black under best play. (ii) If every active first move for Black leads to a loss with best play by both sides, then $V(M, N) = 0$, i.e., the game is a draw. In this case, the unique optimal strategy is for both players to pass initially. (iii) In general, when $V(M, N) = 0$, the optimal White strategies after an initial Black pass are exactly those obtained by picking a Black branch at move 1 which gives Black a draw, and imitating optimal Black play on that branch, but one move later, with colors reversed.

Proof. First we show (ii). Suppose the value $V(B)$ for every active Black

first move is negative. Let Black pass on the first move. He gets -1 and White plays. If White passes, Black gets $+1$ for a net of 0 . The game ends on a B, W pass sequence and $V(B) = 0$. If White instead makes an active first move, $V(W) = 1 \times V(B) - 1 = V(B) < 0$, where the first 1 is from the initial Black pass, $V(B)$ is the value of a Black active first move to the same vertex, and the -1 arises as follows:

The subtree S_W of the game tree arising from the active White first move may be obtained by taking the subtree S_B arising from a Black first move to the same vertex and reversing the colors of all the stones. If a branch of S_B ends with the pass sequence B, W , the corresponding branch of S_W ends with the pass sequence W, B . Thus, we must add a W pass by Rule 6(b), reducing the value to White by 1 point.

If, instead, a branch of S_B ends with the pass sequence W, B, W , then the corresponding branch of S_W ends with B, W, B , and the last B is dropped, which again reduces the value of the branch to White by 1 point.

Since every active White move loses for White, and a White pass draws, White's unique best reply to a Black pass on move 1 is to also pass.

This establishes (ii). We note that Corollary 3 below yields a quicker proof: one sees at once that corresponding branches of color-reversed sub-trees S_B and S_W must have equal and opposite values; hence S_B and S_W have equal and opposite values, and the rest follows quickly.

Assertion (i) follows at once from (ii): If every active first move for Black loses, $V(M, N) = 0$ by (ii). If some active Black first move does not lose, then $V(M, N) \geq 0$.

To see assertion (iii), take any Black first move that leads to a draw. Use the argument in (ii) applied to the subtrees S_B and S_W to show that S_W after an initial Black pass gives a draw under best play. Conversely, if after an initial Black pass White has a first move leading to a draw, then the corresponding Black first move leads to a draw. This completes the proof.

We remark that this theorem makes the same assertion as Theorem 1 of Thorp and Walden (1964). However, the rules here are different so it is a different theorem and requires a separate proof.

THEOREM 2. *For any $M \times N$ board, $V(M, N) \leq MN$.*

Proof. Use the interpretation of 6(b) as adding an additional White pass, when necessary, so that the game always ends on a White move (a pass). Then we have $V = A_B - A_W + p_W - p_B + s_W - s_B$, where V is the value of some branch of the game tree, A_B is the area controlled by Black at the end, and A_W is the area controlled by White. Other notation is as in the discussion preceding Rule 6(b).

Referring to that discussion, with the above interpretation we have

$$p_B + s_B + n_B = m_B = m_W = p_W + s_W + n_W$$

so

$$p_W - p_B + s_W - s_B = n_B - n_W$$

from which

$$V = A_B - A_W + n_B - n_W \leq A_B + n_B \leq A + n = MN,$$

where $n = n_B + n_W$ and A is the total number of vacant vertices.

Since the value of each branch of the game tree (outcome of the game) is bounded by MN , $V(M, N) \leq MN$. Note that the bound MN is best possible and is attained by any legal final position in which $n_B > 0$ and $n_W = 0$.

COROLLARY 3 (Chinese Scoring). *If V for a final position is computed as $(A_B - A_W) + (n_B - n_W)$, known as Chinese scoring, the result is exactly the same as if V is computed as $(A_B - A_W) + (p_W - p_B) + (s_W - s_B)$.*

Proof. Given in the course of the proof of the preceding theorem.

Corollary 3 allows us to score the board by observing it directly. We no longer have to keep track of the capture count and the net number of passes. This allows us to omit the number under the boards in tree diagrams like Figure 13. Corollary 3 also shows us at once, without computing the tree in Figure 13, that the only possible values are 3 (one or two Black stones), 0 (board empty or one stone of each color), and -3 (one or two White stones). It follows that $V(1, 3) = 0$ or 3. Branch 1 has the value $+3$ and is an optimal strategy for Black. An analysis like that given in Figure 12 for the 1×2 game shows it is the unique optimal strategy for Black.

In the game as played, $p_W - p_B$ is zero and V is generally computed as $(A_B - A_W) + (s_W - s_B)$, which is known as *Japanese scoring*. Rule 6(b), in effect that the game end on the proper (White) tempo, is not used. Thus in practice Japanese scoring sometimes gives Black one point less than Chinese scoring. This disparity occurs in about half the games that are played.

Figure 13, branch 1, illustrates the difference between Chinese and Japanese scoring as applied in practice. Chinese scoring scores the branch $+3$ whereas in Japanese scoring as used, the branch would end after move 3, with no tempo adjustment, and the score would be $+2$.

Theorem 1 holds for the game with our list of rules, whether Chinese or Japanese scoring is used. We now establish a similar theorem when Rule 6(b) is replaced by Rule 6(a) and Japanese scoring is used. Note first that in practice the complete game tree, such as that in Figure 13, is not used. Rather than work out to the end of a branch, players in practice stop the game earlier and, in effect, agree upon the value determined by the subsequent subtree.

Even so, further consideration will show that the value thus determined

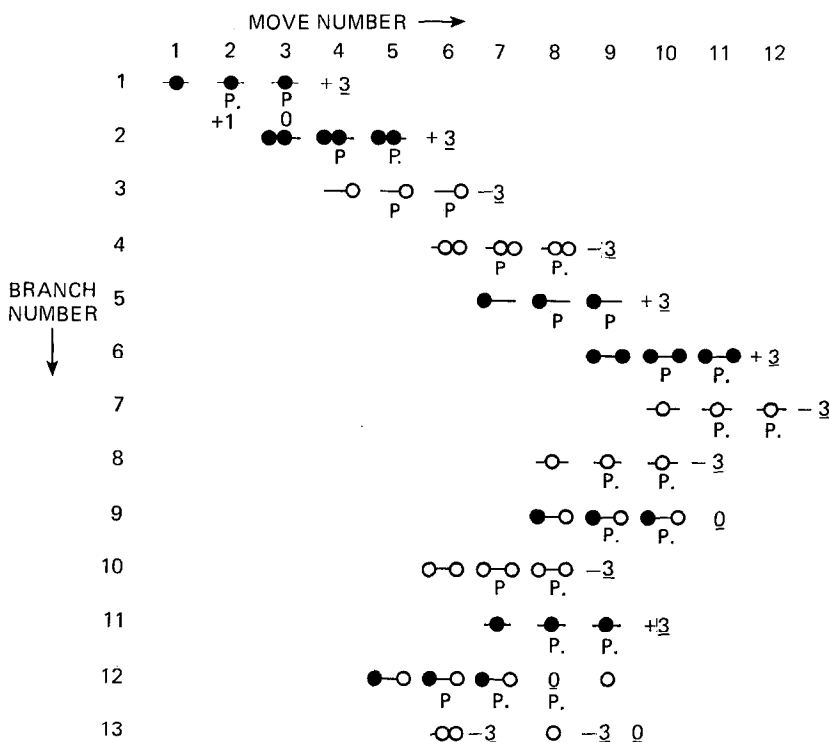


FIGURE 13. A portion of a condensed game tree for 1×3 Go. To complete the tree, first add the subbranches arising from an active Black move 1 to the end vertex. Next note that the section of the tree beginning with a Black pass at move 1 and an active White move 2 now follows by reversing colors, values, and shifting one move later. Adding the branch with passes on moves 1 and 2 completes the tree.

FIGURE 13. *Notes:* Note that (12,7) repeats (12,5). Here the only active move repeats the previous position at (2,3) and a pass repeats a previous position at (12,5). To resolve this, we specify that, if every active move is illegal, a pass is allowable (and forced) even though it returns the board to a previous position. Recall that returning the board to a previous position always means here a position which occurred an *even* number of turns earlier.

Black cannot move to the middle vertex after (3,4), after (5,8), after (11,8), or after (12,6) because of repetition of position. For the same reason, White must pass at (7,12), at (8,10), and at (9,10). The generalized Ko Rule 5(a) plays a role for the first time on the 1×3 board.

(15,3) and (2,3) have different predecessors. Therefore, whereas a Black move to the center was illegal under Rule 5(a) at (3,5), it will not be illegal when it later arises at (18,5) as a consequence of (15,3), and so the subsequent tree differs.

The continuation of (13,6) is the same as branches 4 through 8, but *not* 9, of the continuation of (4,6). This leads to the value -3 after (13,6).

in practice is bounded above by the maximum value of a branch. In Japanese scoring, the maximum possible value of a branch is $MN - 1$. This occurs if and only if the board contains one or more stones, all of which are Black. (The 1×1 board is the lone exception.) Thus $V(M, N) \leq MN - 1$. Repeating the argument in the proof of Theorem 1(ii), we have $V(W) = 1 + V(B) \leq 0$ instead of $V(W) = V(B) < 0$. Arguments like those in the proof of Theorem 1 now yield:

THEOREM 4. *Using Japanese scoring, in the game as played we have: (i) For any $M \times N$ board, the value $V(M, N)$ satisfies $0 \leq V(M, N) \leq MN - 1$. (ii) If every active first move for Black leads to a loss with best play by both sides, then $V(M, N) = 0$.*

We remark that, if Black is not allowed to pass initially, the proofs given for Theorems 1 and 4 are no longer true. The 1×2 board shows the conclusions fail also.

With methods similar to those used in proving Theorems 1 and 4, we have proved:

THEOREM 5. *In K^m tic-tac-toe, with best play the game is always either a draw or a win for the first player.*

By K^m tic-tac-toe we mean the obvious generalization to m dimensions with K "squares" in each "direction."

THEOREM 6. *In the game of chess, if Black has a win under best play then, if White is spotted one additional move, he has a win under best play.*

Chess is believed to be a draw or a win for the first player, White, under best play. If this is true then Theorem 6 is vacuous. However, it is interesting to see that a theorem of this type can be proven for chess.

Theorem 2, crude though it may seem, is a powerful tool in the study of small and intermediate size boards. For instance, if we find a line of play for Black that has the value MN , we know that $V(M, N) = MN$ and that we have an optimal line of play. Thus, we noted at once that branch 1 in Figure 13 is an optimal forced win for Black and that $V(1, 3) = 3$.

Branch 1 is the unique best strategy for Black. Thus, there are no pathological optimal strategies. Things are "as they should be." This encourages us to reduce our goal to finding the value of the game, and an accompanying optimal strategy.

6. EVALUATION OF TINY BOARDS

With the aid of our theorems we solve the 1×4 case at once. Black begins by moving to the center. If White passes, Black passes and Chinese scoring

gives him +4. If White places a stone, Black captures it, White must pass, and Chinese scoring again gives +4.

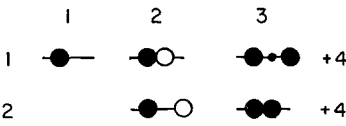


FIGURE 14. Simplified tree for 1×4 Go, showing an optimal Black strategy and proving $V(1,4) = 4$.

Thus, $V(1,4) = +4$ and the strategies described are optimal. This is indicated in Figure 14, which shows only a subtree with an optimal Black strategy and all White responses.

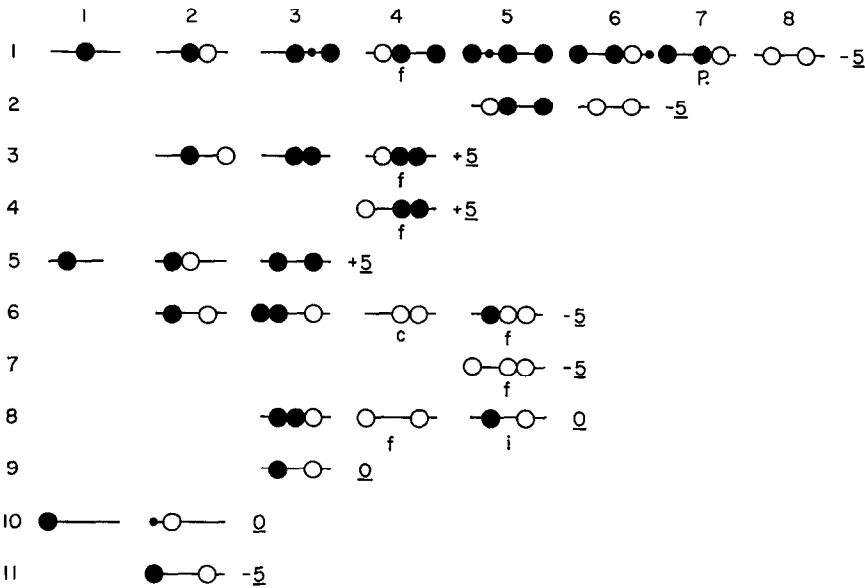


FIGURE 15. Simplified tree for 1×5 Go, showing an optimal strategy and that $V(1,5) = 0$.

Figure 15 is a similar subtree for 1×5 Go. An “f” indicates a forced move, i.e., the alternatives are “obviously” worse. Then “i” reminds us that the move is an illegal continuation of branch 8. We see that 1×5 Go is a draw and that the unique optimal branch is 9.

The analysis in Thorp and Walden (1964) shows that 3×3 Go has the value 9. A modification of this analysis shows that $V(2,4) = 8$. An optimal strategy for Black begins with a move to the center. Black’s second move is to the adjacent center square in the other row, if possible. Otherwise Black

moves to the diagonally opposite center square. Similarly, we can show $V(3, 4) = 12$. We think $V(3, 5) = 15$ will follow similarly but have not verified this yet.

We have not yet determined $V(M, N)$ for other pairs. However, with minor revisions our program should yield $V(M, N)$ and an optimal strategy for $1 \times N$, $N \leq 8$; and $2 \times N$, $N \leq 4$. The 1964 results and the present results to date are summarized in Tables 1 and 2. The dashes denote redundant entries. In Table 2 we expect $V(2, 2) = 0$, $V(2, 3) = 0$, and $V(1, N)$ for $N \geq 5$ to be as conjectured below. We expect to find $V(1, 7) = 1$ and $V(1, 8) = 2$ in Table 1.

TABLE 1

VALUE OF THE GAME USING THORP AND WALDEN (1964) RULES AND RESULTS

		$N \rightarrow$							
		1	2	3	4	5	6	7	8
M $\downarrow$	1	0	0	2	3	0	0	B	B
	2	—	0	0	7				
	3	—	—	8	11	14?			

TABLE 2

VALUE OF THE GAME USING PRESENT RULES AND RESULTS

		$N \rightarrow$							
		1	2	3	4	5	6	7	8
M $\downarrow$	1	0	0	3	4	0	0?	2?	3?
	2	—	0?	0?	8	10?	12?		
	3	—	—	9	12	15?	18?		
	4	—	—	—	0?	20?			
	5	—	—	—	—	25?			
	6	—	—	—	—	—			
	7	—	—	—	—	—	—		

Question marks indicate conjectured values. These have been supported by the experience of several players. We suggest that readers try small board Go as a recreation.

Our experience with the $1 \times N$ game suggests to us the following:

CONJECTURE. For $N = 6k - 1$, $N = 6k$, $V(1, N) = 0$; for $N = 6k + 1$ $V(1, N) = 2$; for $N = 6k + 2$, $6k + 3$, $V(1, N) = 3$; for $N = 6k + 4$, $V(1, N) = 1$, for all $k \geq 1$.

The motivation is our conjecture that an optimal strategy consists of a first Black move to vertex 2, a first White move to vertex $N - 2$, then each player moving toward the center 3 vertices at a time if it is possible to do so

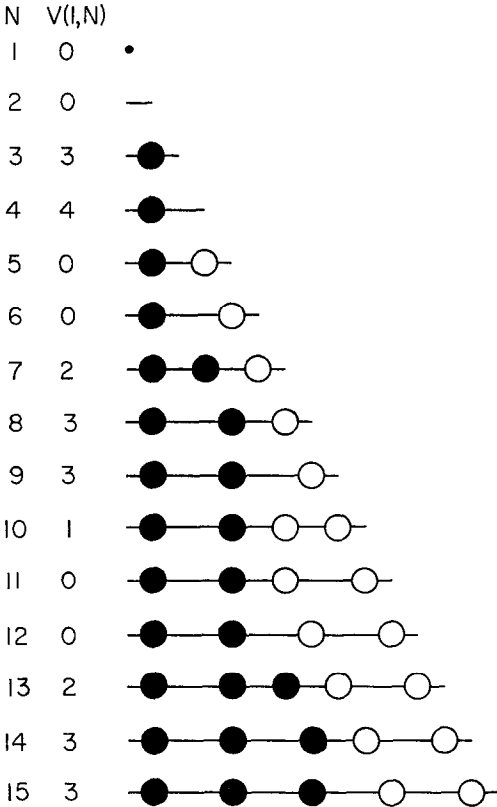


FIGURE 16. Illustrating conjectured optimal strategy and game value for $1 \times N$ Go. Move numbers are given for board 15.

without “touching” the opposing pieces. If not, try two vertices. If this touches the opponent also, then pass. Figure 16 illustrates this.

7. HEURISTICS

Our smallboard analyses and the experience of players on the standard board suggest the following heuristic. Think of each man as exerting a “force” that decreases with “distance.” Further, think of each vertex as having a positive “value” that increases as we move in from the edge, peaking about three or four rows in, then decreasing as we move in further from the edge.

On the square of vertices a fixed distance in from the edge, this increases toward the diagonals and falls off away from them.

Then a heuristic for Black's first move is to maximize $\sum F_i v_i$, where F_i is the force exerted on the i th vertex and v_i is the value of that vertex. In particular, on small $M \times N$ boards such as $3 \leq M, N \leq 7$, M and N odd, we expect a Black move to the center to be so powerful that White has no adequate reply and thus $V(M, N) > 0$.

For M odd and N even, the center contains two points and it is therefore possible for White to safely approach the center more closely on his first move. We expect, therefore, that $V(M, N) > 0$ for $3 \leq M, N \leq 7$ but that the edge is generally less than what we expect by interpolating from the results with M, N both odd.

If M and N are both even, the center has four points and after a Black move to the center, White can move to the diagonally opposite center vertex. This may be enough to give $V(N, N) = 0$, $N = 2, 4, 6$.

Here is heuristic support for the general view that vertices near the corners, and to a lesser extent the edges, are more valuable. On boards big enough so boundary effects disappear, the smallest living group in a corner has 6 stones (Fig. 17(a)). On an edge 8 stones are needed (Fig. 17(b)). The center requires 10 stones (Fig. 17(c)). We know four distinct configurations in the corner, four on the edge, and three in the center.

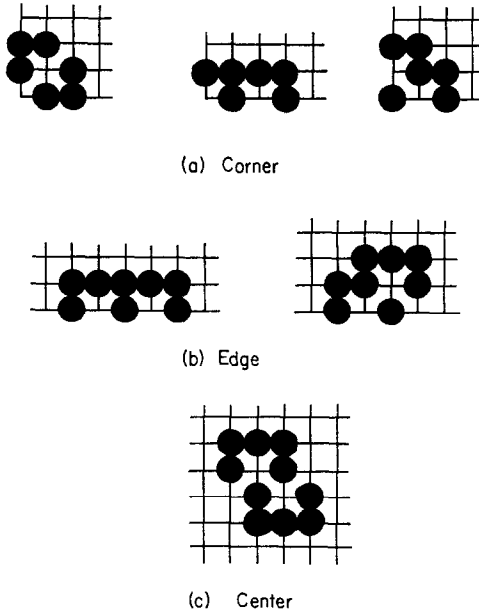


FIGURE 17. Samples of the smallest living groups on "sufficiently large" boards. (a) Corner. (b) Edge. (c) Center.

is a tag word. If $F_{t,k}$ has no meaning, then $H_{t,k} = 0$. If $F_{t,k}$ has meaning, then $H_{t,k} \neq 0$. In particular, $H_{t,k} = j < k$ if $F_{t,k} \sim F_{t,j}$ and otherwise $H_{t,k} = -1$.

Actually, two programs were used but the second is a simple modification of the one we have diagrammed. The reason for two programs stems from the way that draws are handled. If we are seeking a winning strategy for Black,

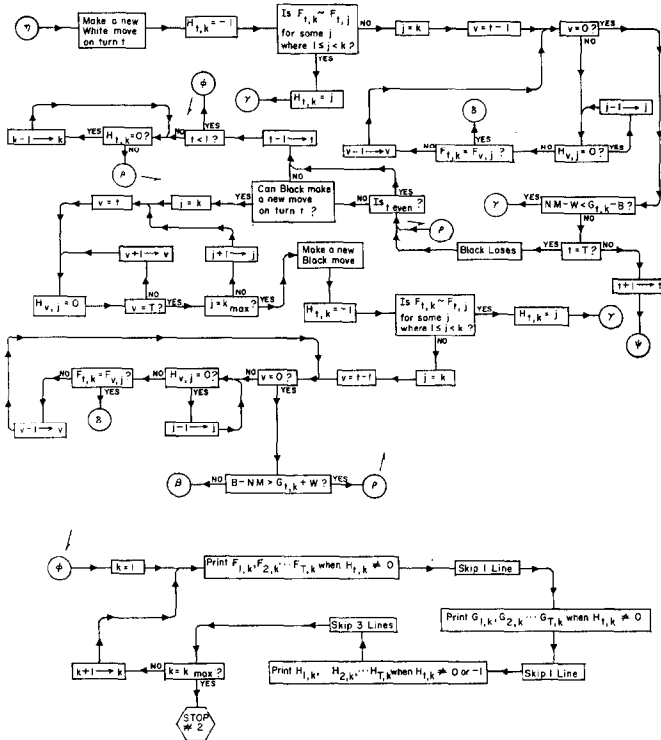


FIGURE 19. Computer program diagram, Part 2.

draws are considered to be losses for Black. This is the version we have diagrammed. If we are seeking a draw strategy for Black, draws are considered to be wins for Black.

We now give a rough description of the way the program works. This should be an aid to following the diagram. The program plays T -truncated Go (Thorp and Walden 1964) beginning at any desired Black turn. A Black turn is played, followed by a White turn, followed by a Black turn, etc. When the game is terminated, if Black wins, the last White play is changed, and, if White wins, the last Black play is changed. If no more moves are possible on the last turn, the play for the player's previous turn is changed. We continue

this process until either White has exhausted all possible plays for a first Black play or all first Black plays have been made. Of course, some of the possible moves have been eliminated by applying the results of sections 5 and 6.

9. POTENTIAL USES OF THE COMPUTER PROGRAM

A man-machine symbiosis should enable us to find $V(M, N)$ or a lower bound for it on other tiny boards. We envision a "good" Go player making the first few Black moves. The machine would for the first few moves ask the player to add Black's move to the current configuration. It would then try all possible White replies and, finally, determine the value of the subtree.

This man-machine combination could also be used in evaluating special situations in actual 19×19 play. Consider, for instance, areas like those in Figure 20. These areas are walled off by a living group and under most

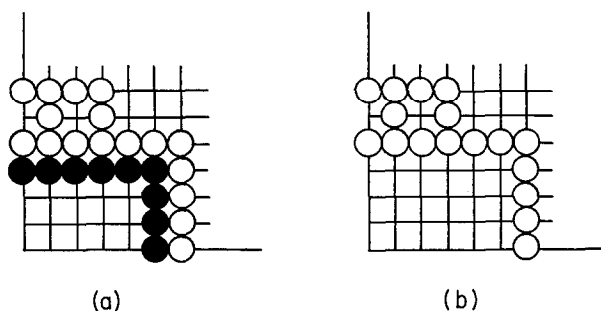


FIGURE 20. A smallboard situation arising from a larger board.

circumstances can be analyzed separately from the rest of the game. With given tempo and best play, what is the value of the subgame played out in these areas? It is probably within the range of our computer program and associated techniques to solve such problems.

Given such an isolated area, one can ask, and probably answer with one program, questions like: Which are the isolated minimal areas in which (given the tempo) Black can build a living group?

Small board Go can serve as a progressive testing ground, as M and N increase, for:

1. Techniques to yield the complete solution.
2. The power of learning programs, evaluation functions, and the like.
3. Human-machine symbiosis.
4. Human-machine confrontation.

10. MISCELLANY

With the Generalized Ko rule, there are complete positions in which no legal active move can be made. But if repetitions of position were instead allowed without restriction, one can show that from any given complete position there is a sequence of moves which leads to any given complete position except perhaps an empty board.

The proof of Theorem 1 suggests a generalization: An arrangement is *color-symmetric* if, when the colors of the stones are interchanged, there is a space symmetry carrying the resulting arrangement onto the original. If it were not for the role of past history, one could usefully generalize Theorem 1 from the special case of an empty board to any color-symmetric board.

For $M \times N$ Go, with M and N both odd, we define "symmetric play by black," or briefly, "symmetric play," as follows: Black moves to the center on his first move. Thereafter, Black attempts to move opposite the center from White, i.e., on every turn, Black tries to move in such a way as to insure that the center of gravity of the configuration of men (each assigned unit mass) is at the center of the board after Black completes his move. If White passes, Black passes. Of course White may eventually move in such a way that Black cannot do this, as when White captures a group of Black men and the loss of the group so changes things that the corresponding group of White men cannot now be captured. After this point we simply assume best play by both sides.

Assuming best play by White, for which M and N does the strategy of symmetric play give Black at least a draw?

The question is of interest in view of the story that master Go players out of courtesy avoided an initial Black move to the center because Black would be assured a win. Also there is the coin game (Pólya 1954, pages 23–24) in which a similar strategy always wins. (Compare Lasker 1960, pages xv, xvi.)

We have shown with examples that, for $3 \leq M \leq N$, $V(M, N) = -MN$. Thus, just as in chess, symmetric play is useless.

For M or N even, the geometric center of the board is not a vertex. In this case, let symmetric play by White mean that, after each Black move, White moves to the corresponding vertex across the center. Once the symmetry is destroyed, each side plays optimally. We have examples to show that $V(M, N) = MN$ if $2 \leq M < N$ or $3 \leq M \leq N$.

The reader interested in variants of Go will find suggestions in Gardner (1962) and Good (1965). We add to the list "hexagonal Go," played on a grid of hexagons, rather than the usual rectangular grid.

Preliminary study of $1 \times n$ toroidal Go, i.e., Go played on a ring with n

locations, suggests that the game is somewhat simpler and that both hand and computer analysis will be somewhat more rewarding. There are, for example, far fewer configurations after (the first) two active moves. When n is odd there are $(n - 1)/2$ and when n is even there are $n/2$.

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