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Global practical tracking of a class of nonlinear systems by output feedback☆

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Abstract

In this paper we address the practical tracking problem for a class of nonlinear systems by dynamic output feedback control. Unlike most of the existing results where the unmeasurable states in the nonlinear vector field can only grow linearly, we allow higher-order growth of unmeasurable states. The proposed controller makes the tracking error arbitrarily small and demonstrates nice properties such as robustness to disturbances and universal property to reference signals.

Keywords: Global practical tracking; Output feedback; Higher-order growth

1. Introduction and problem statement

This paper considers the practical tracking problem of the following nonlinear system:

$$\dot{x}_i = x_{i+1} + \phi_i(x, u, d(t)), \quad i = 1, 2, \ldots, n-1,$$

$$\dot{x}_n = u + \phi_n(x, u, d(t)), \quad y = x_1 - y_r(t), \tag{1}$$

where $x = [x_1, \ldots, x_n]^{\mathrm{T}} \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the system state, input and output, respectively. $y_r(t)$ is the reference to be tracked and $d(t) \in \mathbb{R}^s$ represents unknown continuous disturbances. For $i = 1, 2, \ldots, n$, $\phi_i(x, u, d(t))$'s are unknown continuous nonlinear functions of the states, input and disturbances. Similar to the output regulation theory (Huang, 2004; Isidori, 1995) we assume the only measurable signal in system (1) to be the error between the output x_1 and the reference y_r. Therefore, only y is allowed in the design of the controller. Notice that such setting is different from the standard tracking problem where both the reference and its derivatives are assumed to be measurable. There are two reasons to limit the only measurement to be the error signal. Firstly, in some practical control applications, it is inevitable that the error signal is the one to be directly measured. For example, in a missile guidance system, instead of measuring the absolute position of the moving target, i.e. $y_r(t)$, the onboard radar keeps measuring the distance/error between the missile and the target. Secondly, assuming only error signal also makes the actuator design simple, since the controller does not depend on the signal to be tracked explicitly. In this way, the controller is more adaptive to different reference signals.

The global output feedback tracking problem of nonlinear system (1) is virtually unsolvable if the nonlinear vector fields $\phi(\cdot)$'s grow too fast with respect to the unmeasurable states, as shown in (Mazenc et al., 1994). Realizing this difficulty, it is not surprising that most of the existing output feedback results impose restrictive assumptions on the nonlinear vector fields. For example, in Chen and Hunag (2005) and Gong and Lin (2003), the global asymptotic output regulation has been solved for the nonlinear systems in the output feedback form. A unique feature for the output feedback form is that the nonlinearities can only depend on the measurement y. In Chen and Huang (2004), the output regulation problem is solved for a class of nonlinear systems with lower-triangular structure and linear dependency on the measurable states. When the nonlinear systems satisfy global Lipschitz or linear growth type of conditions, the asymptotic tracking can be tackled by

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the methods presented in Gauthier et al. (1992) and Qian and Lin (2002).

To the best of our knowledge, most existing global output regulation/tracking results cannot allow the unmeasurable states to grow faster than linearly. The main contribution of this paper is to solve the tracking problem for the systems with higher-order growing nonlinearities of the unmeasurable states, for instance,

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3 + x_2 \ln(1 + x_2^2),$$
$$\dot{x}_3 = u + d(t)x_2^2 + x_3^{4/3} + x_3^{1/3} + d(t), \tag{2}$$

where $|d(t)| \leqslant 1$ is a bounded disturbance. One particular difficulty imposed by higher-order growing nonlinearities is the lack of observer design tool. So far, many of the global output feedback design methods are fundamentally based on Luenberger type of observer. The linear nature of the observers limits their ability to handle the higher-order growing nonlinearities of the unmeasurable states. Nonlinear observer design methods are proposed in Gauthier et al. (1992) and Krener and Kang (2003), but the observers are locally convergent and therefore are not suitable for the global output tracking problems. Recently, a homogeneous nonlinear observer design is introduced in Qian (2005). This observer is inherently nonlinear and provides the ability to handle higher-order growing unmeasurable states. It has been shown in Qian (2005) that the global stabilization of (1) can be solved by output feedback under suitable growth conditions. In this paper, we extend the result in Qian (2005) to solve the global practical tracking problem. To this end, the following condition is introduced.

Assumption 1. There are constants $\tau \geqslant 0$ and $c \geqslant 0$ such that, for $i = 1, \ldots, n$

$$|\phi_i(\cdot)| \leqslant c(|x_1|^{i\tau+1} + |x_2|^{(i\tau+1)/(\tau+1)} + \cdots + |x_i|^{(i\tau+1)/((i-1)\tau+1)}) + c.$$

Remark 1. Clearly, system (2) satisfies A1 (with $\tau=2$). System (2) exemplifies that A1 covers nonlinear systems with higher-order growing unmeasurable states. It is in sharp contrast to many existing output feedback design methods, where the nonlinear vector field needs to be Lipschitz or linear growth in the unmeasurable states (Gauthier et al., 1992; Qian and Lin, 2002). On the other hand, the counter examples in Mazenc et al. (1994) indicate that, due to the finite escape time phenomenon, the global output feedback stabilization of systems (1) cannot be solved if the nonlinear functions $\phi_i(\cdot)$ grow too fast. From this point of view, A1 is very tight already (see Qian, 2005 for further explanations).

Remark 2. A1 is slightly more general than the assumption imposed in Qian (2005) which contains no constant term. By adding this constant term, we can cover nonlinear systems with both higher-order and lower-order growing unmeasurable states (see x_2^2 and $x_3^{1/3}$ terms in (2)), while the growth condition in Qian (2005) only permits higher-order growing unmeasurable states. In this sense, the result in this paper is a generalization of the result in Qian (2005).

Another difficulty associated with the output tracking problem of system (1) lies in the appearance of the disturbance and uncertainty in the reference. In standard output regulation theory (Isidori, 1995), this difficulty is circumvented by assuming both of the disturbance, $d(t)$, and the reference, $y_r(t)$, be generated by a neutrally stable exosystem. In this paper, we intend to relax such assumptions. In particular, we assume the reference signal $y_r(t)$ satisfying the following condition.

Assumption 2. The reference signal $y_r(t)$ is continuously differentiable. Moreover, there is a known constant $M > 0$, such that $|y_r(t)| + |\dot{y}_r(t)| \leqslant M, \forall t \in [0, \infty)$.

The assumption on the disturbance is imbedded in A1 which basically allows $d(t)$ to be any bounded signal. Note that, such relaxations on the reference and the disturbance do not come free. The price been paid is the solvability to achieve *asymptotic* tracking and *asymptotic* disturbance rejection. For example, in the case of linear systems, the celebrated *internal model principle* (Isidori, 1995) indicates that any regulator that solves the asymptotic tracking problem must incorporate a suitable internal model of the exosystem which generates the disturbance and the reference. In our case, since the disturbance $d(t)$ and the reference $y_r(t)$ are assumed to be unknown and do not belong to any prescribed class of signals, we do not know what kind of exosystems can generate them. The lack of information on the exosystems makes asymptotic tracking extremely difficult. Being aware of aforementioned difficulties, we pursue a less ambitious goal and focus on global practical tracking instead of asymptotic one.

The global practical tracking problem: For any given tolerance $\varepsilon > 0$, design a dynamic output feedback controller u of the form

$$\dot{\xi} = \alpha(\xi, y), \quad \xi \in \mathbb{R}^m, \quad u = u(\xi, y) \tag{3}$$

such that (i) the state of the closed-loop system (1)–(3) is well defined on $t \in [0, \infty)$ and globally bounded; (ii) for any initial condition $(x(0), \xi(0))$, there is a finite time $T > 0$, such that $|y(t)| = |x_1(t) - y_r(t)| \leqslant \varepsilon, \forall t > T$.

In the remainder of this paper, we shall show that the global practical output tracking problem of system (1) can be solved under Assumptions 1 and 2.

2. Review of a stabilization result

In this section, we briefly review a new output feedback stabilization result presented in Polendo and Qian (2006) and Qian (2005). Based on homogeneous theory, the result provides a systematic design tool for the construction of dynamic compensators, and is essential in solving our practical tracking problem.

Consider the linear system

$$\dot{z}_i = z_{i+1}, \quad i = 1, \ldots, n-1, \quad \dot{z}_n = v, \quad y = z_1, \tag{4}$$

where v is the input and y is the output. For system (4), one can easily design a linear observer plus a linear feedback controller to globally stabilize the system. This method has been extended to nonlinear system (1) with linear growth condition on the nonlinear vector field (Qian and Lin, 2002). However, the linear nature of this type of design makes it inapplicable to inherently nonlinear systems. For instance, when the nonlinear vector field has higher-order growth terms such as those satisfying A1, linear dynamic output feedback controller fails to globally stabilize the system. For the output feedback design of inherently nonlinear systems, a genuinely nonlinear observer design method is needed. The recently developed nonlinear homogeneous observer in Qian (2005) provides such a tool to handle inherently nonlinear systems.

According to Qian (2005), one can construct a reduced order homogeneous observer for system (4) as follows:

$$\begin{aligned}
\dot{\eta}_2 &= f_{n+1}(z_1,\eta_2) = -l_1\hat{z}_2,\\
\hat{z}_2 &= \mathrm{sign}(\eta_2 + l_1 z_1)|\eta_2 + l_1 z_1|^{r_2/r_1},\\
\dot{\eta}_k &= f_{n+k-1}(z_1,\eta_2,\ldots,\eta_k) = -l_{k-1}\hat{z}_k,\\
\hat{z}_k &= \mathrm{sign}(\eta_k + l_{k-1}\hat{z}_{k-1})|\eta_k + l_{k-1}\hat{z}_{k-1}|^{r_k/r_{k-1}},
\end{aligned} \tag{5}$$

where $k = 3,\ldots,n$ and $r_i = (i-1)\tau + 1$, $i = 1,\ldots,n$ are the homogeneous dilation and the constants $l_i > 0$, $i = 1,\ldots,n-1$ are observer gains. The sign function is defined as

$$\mathrm{sign}(s) = \begin{cases} 1 & \text{if } s \geqslant 0,\\ -1 & \text{if } s < 0.\end{cases}$$

The controller can be constructed as

$$v = -\mathrm{sign}(\hat{\xi}_n)|\hat{\xi}_n|^{(r_n+\tau)/r_n}\beta_n \tag{6}$$

with $\hat{z}_1 = z_1$ and

$$\begin{aligned}
&\hat{z}_1^* = 0, \quad \hat{\xi}_1 = \hat{z}_1 - \hat{z}_1^*,\\
&\hat{z}_k^* = -\mathrm{sign}(\hat{\xi}_{k-1})|\hat{\xi}_{k-1}|^{r_k/r_{k-1}}\beta_{k-1}, \quad \hat{\xi}_k = \hat{z}_k - \hat{z}_k^*,
\end{aligned} \tag{7}$$

for appropriate controller constants $\beta_k > 0$, $k = 1,\ldots,n$. Denote

$$Z = (z_1, z_2, \ldots, z_n, \eta_2, \ldots, \eta_n)^{\mathrm{T}}, \tag{8}$$

$$F(Z) = (z_2, \ldots, z_n, v, f_{n+1}, \ldots, f_{2n-1})^{\mathrm{T}}. \tag{9}$$

The closed-loop system (4)–(5)–(6) can be rewritten in a compact form $\dot{Z} = F(Z)$. Moreover, it can be verified that $F(Z)$ is homogeneous of degree τ with dilation

$$\Delta = (1, \tau+1, \ldots, (n-1)\tau+1, 1, \ldots, (n-2)\tau+1). \tag{10}$$

Lemma 1 (*Qian, 2005*). *The observer gains $l_i > 0$, $i = 1,\ldots,n-1$ and controller gains $\beta_i > 0$, $i = 1,\ldots,n$ can be recursively determined such that the closed-loop system* (4)–(5)–(6) *admits a Lyapunov function $V(Z)$ with the following properties*:

(1) *V is positive definitive and proper with respect to Z*;
(2) *V is homogeneous of degree $2r_n - \tau$, with dilation* (10);
(3) *the derivative of $V(Z)$ along* (4)–(5)–(6) *satisfies*

$$\dot{V}(Z(t)) = \frac{\partial V}{\partial Z}F(Z) \leqslant -C\|Z\|_{\Delta}^{2r_n}, \quad C > 0, \tag{11}$$

where $\|Z\|_{\Delta} = \sqrt{\sum_{i=1}^{2n-1}|Z_i|^{2/r_i}}$.

Remark 3. The choice of the gains, β_i and l_i, only depends on the homogeneous degree, τ, and the system dimension, n. Once τ and n are given, one can determine β_i and l_i through a recursive manner. Instead of giving the detailed procedure which can be found in Polendo and Qian (2006) and Qian (2005), later we provide an example to illustrate the idea.

Next, we list several useful lemmas.

Lemma 2 (*Hermes, 1991*). *Given a dilation weight $\Delta = (r_1,\ldots,r_n)$, suppose $V_1(x)$ and $V_2(x)$ are homogeneous functions of degree τ_1 and τ_2, respectively. Then $V_1(x)V_2(x)$ is homogeneous with respect to the same dilation Δ. Moreover, the homogeneous degree of $V_1(x)V_2(x)$ is $\tau_1+\tau_2$.*

Lemma 3 (*Hermes, 1991*). *Suppose $V : \mathbb{R}^n \to \mathbb{R}$ is a homogeneous function of degree τ with respect to the dilation weight Δ. Then*: (1) *$\partial V/\partial x_i$ is homogeneous of degree $\tau - r_i$ with r_i being the homogeneous weight of x_i*; (2) *there is a constant $\bar{c}$ such that $V(x) \leqslant \bar{c}\|x\|_{\Delta}^{\tau}$*; (3) *if $V(x)$ is positive definite, $\underline{c}\|x\|_{\Delta}^{\tau} \leqslant V(x)$ for a positive constant $\underline{c}$.*

Lemma 4. *For $x \in \mathbb{R}$, $y \in \mathbb{R}$, $p \geqslant 1$ is a constant, the following inequality holds*: $|x+y|^p \leqslant 2^{p-1}|x^p + y^p|$.

Lemma 5. *Let c, d be positive constants. Given any positive number $\gamma > 0$, the following inequality holds*: $|x|^c|y|^d \leqslant c/(c+d)\gamma|x|^{c+d} + d/(c+d)\gamma^{-c/d}|y|^{c+d}$.

3. Global practical tracking by output feedback

Theorem 1. *Under A1–A2, the global practical output tracking problem of system* (1) *can be solved by a dynamic output feedback controller of the form* (3).

Proof. Define $(e_1, e_2, \ldots, e_n) = (y, x_2, \ldots, x_n)$. Then

$$\begin{aligned}
\dot{e}_i &= e_{i+1} + \hat{\phi}_i(e, u, d(t)), \quad i = 1,\ldots,n-1,\\
\dot{e}_n &= u + \hat{\phi}_n(e, u, d(t)),
\end{aligned} \tag{12}$$

where

$$\hat{\phi}_1(\cdot) = \phi_1(e_1 + y_r(t), e_2, \ldots, e_n, u, d(t)) - \dot{y}_r(t),$$

$$\hat{\phi}_i(\cdot) = \phi_i(e_1 + y_r(t), e_2, \ldots, e_n, u, d(t)), \quad i = 2,\ldots,n.$$

Note that, in the definition of the error signal $(e_1,\ldots,e_n)$, we only change the coordinate of the first state x_1. It is different from the common definition used in solving asymptotic tracking, where the error is defined as the difference between all the states and their steady trajectories. By A1–A2 and Lemma 4, it is readily to show that, for $i = 1, \dots, n$,

$$
\begin{aligned}
|\hat{\phi}_i(\cdot)| &\leqslant c(|e_1 + y_r(t)|^{i\tau+1} + |e_2|^{(i\tau+1)/(\tau+1)} + \cdots \\
&\quad + |e_i|^{(i\tau+1)/((i-1)\tau+1)}) + |\dot{y}_r(t)| + c \\
&\leqslant c(2^{i\tau}|e_1|^{i\tau+1} + |e_2|^{(i\tau+1)/(\tau+1)} \\
&\quad + \cdots + |e_i|^{(i\tau+1)/((i-1)\tau+1)}) \\
&\quad + 2^{i\tau}|y_r(t)|^{i\tau+1} + |\dot{y}_r(t)| + c \\
&\leqslant c_1(|e_1|^{i\tau+1} + \cdots + |e_i|^{(i\tau+1)/((i-1)\tau+1)}) + c_1,
\end{aligned}
\tag{13}
$$

where $c_1 > 0$ is a constant only depending on c, τ (in A1) and M (in A2). Next, introducing the change of coordinates: $z_i = e_i/L^{i-1}$, $i = 1, \dots, n$, and $v = u/L^n$, where $L > 1$ is a scaling constant to be determined later, system (12) is transformed to

$$
\dot{z}_i = L z_{i+1} + \hat{\phi}_i(\cdot)/L^{i-1}, \quad i = 1, 2, \dots, n, \tag{14}
$$

where $z_{n+1} = v$. Following the homogeneous observer and controller design proposed in Qian (2005), we construct a dynamic compensator for (14) as

$$
\begin{aligned}
&\dot{\eta}_k = -L l_{k-1}\hat{z}_k, \\
&\hat{z}_k = \operatorname{sign}(\eta_k + l_{k-1}\hat{z}_{k-1})|\eta_k + l_{k-1}\hat{z}_{k-1}|^{r_k/r_{k-1}}
\end{aligned}
\tag{15}
$$

for $k = 2, \dots, n$, and a controller

$$
u = -L^n \operatorname{sign}(\hat{\xi}_n)|\hat{\xi}_n|^{(r_n+\tau)/r_n}\beta_n, \tag{16}
$$

where $\hat{\xi}_i$ is defined in (7). In (15) and (16), $l_i > 0$, $i = 1, \dots, n-1$ and $\beta_i > 0$, $i = 1, \dots, n$ are constant gains specified as in Lemma 1 and $L > 1$ is a constant to be determined later. Note that the only information used in the construction of (15)–(16) is the measurement $y(t)$. Next, we will determine the gain L such that the global practical output tracking is achieved. By using notations (8) and (9), the closed-loop system (14)–(16) can be written down in a compact form

$$
\dot{Z} = LF(Z) + [\hat{\phi}_1(\cdot), \hat{\phi}_2(\cdot)/L, \dots, \hat{\phi}_n(\cdot)/L^{n-1}, 0, \dots, 0]^{\mathrm{T}}.
$$

By Lemma 1, there exist constants l_i's, β_i's and a Lyapunov function $V(Z)$, such that $\frac{\partial V}{\partial Z}F(Z) \leqslant -C\|Z\|_\Delta^{2r_n}$. Moreover, V is homogeneous of degree $2r_n - \tau$ with dilation (10). Hence, with these choice of l_i, β_i, the derivative of V along the trajectory of (14)–(16) satisfies

$$
\begin{aligned}
\dot{V}(Z) \leqslant &-LC\|Z\|_\Delta^{2r_n} + \frac{\partial V(Z)}{\partial Z}[\hat{\phi}_1(\cdot), \hat{\phi}_2(\cdot)/L, \dots, \\
&\hat{\phi}_n(\cdot)/L^{n-1}, 0, \dots, 0]^{\mathrm{T}}.
\end{aligned}
\tag{17}
$$

From (13) and the fact $L > 1$, it is readily to deduce that

$$
\begin{aligned}
|\hat{\phi}_i(\cdot)/L^{i-1}| &\leqslant c_1(|z_1|^{i\tau+1} + |Lz_2|^{(i\tau+1)/(\tau+1)} + \cdots \\
&\quad + |L^{i-1}z_i|^{(i\tau+1)/((i-1)\tau+1)}) + c_1/L^{i-1} \\
&\leqslant c_1 L^{1-1/((i-1)\tau+1)}(|z_1|^{i\tau+1} + |z_2|^{(i\tau+1)/(\tau+1)} + \cdots \\
&\quad + |z_i|^{(i\tau+1)/((i-1)\tau+1)}) + c_1/L^{i-1}.
\end{aligned}
\tag{18}
$$

By Lemmas 3 and 1, $\partial V/\partial Z_i$ is homogeneous of degree $2r_n - \tau - r_i$ for all i. Hence, from Lemmas 2 and 3, we can find a constant $\rho_i > 0$ such that

$$
\left|\frac{\partial V}{\partial Z_i}\right|(|z_1|^{i\tau+1} + \cdots + |z_i|^{(i\tau+1)/((i-1)\tau+1)}) \leqslant \rho_i\|Z\|_\Delta^{2r_n}. \tag{19}
$$

Substituting (19) into (17) leads to

$$
\begin{aligned}
\dot{V}(Z) \leqslant &-L\left(C - c_1\sum_{i=1}^n \rho_i L^{-1/((i-1)\tau+1)}\right)\|Z\|_\Delta^{2r_n} \\
&+ c_1\sum_{i=1}^n \left|\frac{\partial V(Z)}{\partial Z_i}\right|\frac{1}{L^{i-1}}.
\end{aligned}
\tag{20}
$$

On the other hand, by Lemmas 3 and 5, there are positive constants c_2, c_3, such that for all $1 \leqslant i \leqslant n$,

$$
\begin{aligned}
c_1\left|\frac{\partial V(Z)}{\partial Z_i}\right|\frac{1}{L^{i-1}} &\leqslant c_2\|Z\|_\Delta^{2r_n-\tau-r_i}(L^{-(i-1)/(\tau+r_i)})^{\tau+r_i} \\
&\leqslant \|Z\|_\Delta^{2r_n} + c_3 L^{-(2(i-1)r_n)/(\tau+r_i)}.
\end{aligned}
\tag{21}
$$

Estimations (20) and (21) lead to

$$
\begin{aligned}
\dot{V}(Z) \leqslant &-L\left(C - c_1\sum_{i=1}^n \rho_i L^{-1/((i-1)\tau+1)} - nL^{-1}\right)\|Z\|_\Delta^{2r_n} \\
&+ c_3\sum_{i=1}^n L^{-(2(i-1)r_n)/(\tau+r_i)}.
\end{aligned}
$$

Choose a sufficiently large L, such that

$$
c_1\sum_{i=1}^n \rho_i L^{-1/((i-1)\tau+1)} + nL^{-1} < C/2,
$$

$$
\sum_{i=1}^n L^{-2(i-1)r_n/(\tau+r_i)} < 2.
$$

Then, $\dot{V}(Z) \leqslant -LC/2\|Z\|_\Delta^{2r_n} + 2c_3$. Next, by Lemma 3, there are two positive constants α_1, α_2 such that

$$
\alpha_1\|Z\|_\Delta^{2r_n-\tau} \leqslant V(Z) \leqslant \alpha_2\|Z\|_\Delta^{2r_n-\tau}. \tag{22}
$$

Therefore

$$
\dot{V}(Z) \leqslant -0.5LC\alpha_2^{-2r_n/(2r_n-\tau)}V(Z)^{2r_n/(2r_n-\tau)} + 2c_3. \tag{23}
$$

From (22) and (23) it is not difficult to show that there is a finite time T such that

$$
V(Z) \leqslant \alpha_2(4c_3/(CL))^{(2r_n-\tau)/2r_n} \quad \forall t \geqslant T.
$$

From here it is clear the global practical tracking can be achieved with a sufficiently large L. $\square$

Remark 4. The controller (16) and the observer (15) are constructed only based on the nominal system (4). No precise information of the nonlinearities is needed. It means that the same dynamic controller (15)–(16) can be applied to different nonlinear systems as long as they satisfy A1. This advantage greatly reduces the design complexity normally associated with the dynamic output feedback design. Also note that, there are only three sets of parameters l_i, β_i and L need to be determined. The choice of l_i and β_i only depends on the nominal system (4). Therefore, they can be pre-fixed even for different nonlinear systems. The gain L needs to be assigned as a sufficiently large number to achieve the given tracking accuracy. To calculate the precise value of L could be tedious and most likely conservative. In practice, one can simply choose a large L such that the closed-loop system is stable and keep increasing L until the given tracking accuracy is achieved.

In the following we illustrate the proposed controller on a nonlinear system which describes a particle moving under nonlinear viscous friction. Consider

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u - \mathrm{sign}(x_2)|x_2|^{\alpha}, \quad y = x_1 - \sin(t),$$

where x_1 is the displacement, x_2 is the velocity and u is the control force. The term $\mathrm{sign}(x_2)|x_2|^{\alpha}$ represents nonlinear viscous friction. It is assumed that $1 \leqslant \alpha \leqslant \frac{5}{3}$ but the precise value of α is unknown. The control objective is to force the state x_1 to track the reference $\sin(t)$ using the measurement $y(t)$ only. Note that, although the parameter α can be estimated by experiment, it may not be a constant due to the change of the working environment. Therefore, it is quite desirable to construct a controller not depending on the precise value of α.

A1 can be easily verified with $\tau = 2$. Then, following (15) and (16), we can construct the dynamic output feedback tracking controller as

$$\dot{\eta} = -Ll_1\hat{x}_2, \quad \hat{x}_2 = (\eta + l_1 y)^3,$$
$$u = -L^2\beta_2(\hat{x}_2 + \beta_1 y^3)^{5/3}. \tag{24}$$

The gains β_1, β_2 and l_1 can be determined according to the following three-step procedure (Polendo and Qian, 2006; Qian, 2005).

Step 1: β_1 and β_2 can be found by designing a full state feedback control, $v = -\beta_2(z_1 + \beta_1 z_2^3)^{5/3}$, for the nominal linear system $\dot{z}_1 = z_2, \dot{z}_2 = v$ with the Lyapunov function $V(z) = 0.5z_1^6 + 0.5(z_2 + \beta_1 z_1^3)^2$. Taking derivative of $V(z)$, we have

$$\begin{aligned}\dot{V}(z) &= -3\beta_1 z_1^8 + 3z_1^5\xi_2 + \xi_2(v + 3\beta_1 z_1^2(\xi_2 - \beta_1 z_1^3)) \\ &= -3\beta_1 z_1^8 + 3(1-\beta_1^2)z_1^5\xi_2 + \xi_2 v + 3\beta_1 z_1^2\xi_2^2,\end{aligned}$$

where $\xi_2 = z_2 + \beta_1 z_1^3$. Applying Lemma 5 to the cross terms $z_1^5\xi_2$ and $z_1^2\xi_2^2$, one can find β_1 and β_2 such that

$$\dot{V}(z) \leqslant -c_1 z_1^8 - c_2\xi_2^{8/3}$$

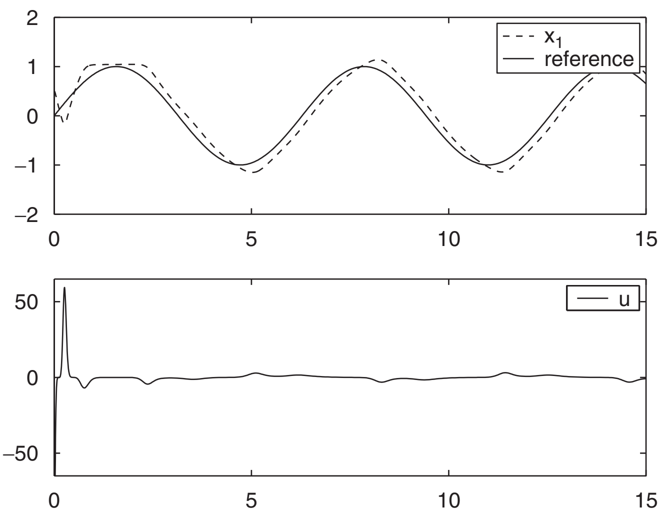

Fig. 1. The trajectory and the control of the closed-loop system with $\beta_1 = 1$, $\beta_2 = 2$, $l_1 = 1$ and $L = 30$.

for some positive constants c_1 and c_2. For instance, one possible choice is $\beta_1 = 1$ and $\beta_2 = 2$.

Step 2: With the controller gains having been fixed, one can now design the dynamic output feedback control for the nominal linear system as

$$\dot{\eta} = -l_1\hat{z}_2, \quad \hat{z}_2 = (\eta + l_1 z_1)^3, \quad v = -\beta_2(\hat{z}_2 + \beta_1 z_1^3)^{5/3}.$$

With this output feedback control, it is easy to show

$$\dot{V}(z) \leqslant -c_1 z_1^8 - c_2\xi_2^{8/3} + \beta_2\xi_2(\xi_2^{5/3} - (\hat{z}_2 + \beta_1 z_1^3)^{5/3}).$$

Step 3: To determine the observer gain l_1, according to Qian (2005), the Lyapunov function

$$U(z) = \int_{(\eta + l_1 z_1)^5}^{z_2^{5/3}} (s^{1/5} - (\eta + l_1 z_1))\,\mathrm{d}s$$

can be employed. Taking derivative of $U(Z)$ and applying Lemmas 4 and 5, one can find an observer gain l_1 such that

$$\mathrm{d}(V(z) + U(z))/\mathrm{d}t \leqslant -k_1 z_1^8 - k_2\xi_2^{8/3} - k_3(z_2 - \hat{z}_2)^{8/3}$$

for some constants k_1, k_2 and k_3.

Once β_1, β_2 and l_1 are fixed, by Theorem 1 the tracking error can be made arbitrarily small with properly chosen L in (24). The value of L can be designed according to Remark 4. In Fig. 1 we plot out the simulation results when $\alpha = 1.5$ and $L = 30$. The steady tracking error is about 0.2.

References

Chen, Z., & Huang, J. (2004). Global robust servomechanism problem for uncertain lower triangular nonlinear systems by output feedback control. In *Proceedings of the 43rd IEEE conference on decision and control* (pp. 4583–4588). Bahamas.

Chen, Z., & Hunag, J. (2005). Global robust output regulation for output feedback systems. *IEEE Transactions on Automatic Control*, *50*, 117–120.

Gauthier, J., Hammouri, H., & Othman, S. (1992). A simple observer for nonlinear systems with applications to bioreactors. *IEEE Transactions on Automatic Control*, *37*, 875–880.

Gong, Q., & Lin, W. (2003). A note on global output regulation of nonlinear systems in the output feedback form. *IEEE Transactions on Automatic Control*, *48*, 1049–1054.

Hermes, H. (1991). *Homogeneous coordinates and continuous asymptotically stabilizing feedback controls*. New York: Dekker, (pp. 249–260).

Huang, J. (2004). *Nonlinear output regulation: Theory and applications*. Philadelphia, USA: SIAM.

Isidori, A. (1995). *Nonlinear control systems*. Berlin: Springer.

Krener, A. J., & Kang, W. (2003). Locally convergent nonlinear observers. *SIAM Journal on Control and Optimization*, *42*(1), 155–177.

Mazenc, F., Praly, L., & Dayawansa, W. (1994). Global stabilization by output feedback: Examples and counterexamples. *Systems Control Letters*, *23*(2), 119–125.

Polendo, J., & Qian, C. (2006). A generalized framework for global output feedback stabilization of inherently nonlinear systems. In S. Tang, & J. Yong (Eds.), *Control of distributed parameter and stochastic systems.* Singapore: World Scientific.

Qian, C. (2005). A homogeneous domination approach for global output feedback stabilization of a class of nonlinear systems. *Proceedings of the ACC* (pp. 4708–4715) Portland, OR. Also under revision in *IEEE Transactions on Automatic Control*.

Qian, C., & Lin, W. (2002). Output feedback control of a class of nonlinear systems: a nonseparation principle paradigm. *IEEE Transactions on Automatic Control*, *47*(10), 1710–1715.