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Discrete Differential Structures on Simplicial Complexes

A dissertation submitted in partial satisfaction of the
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by

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Professor Melvin Leok

2016

The dissertation of John Brogan Moody is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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Chair

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VITA

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ABSTRACT OF THE DISSERTATION

Discrete Differential Structures
on
Simplicial Complexes

by

John Brogan Moody

Doctor of Philosophy in Mathematics

University of California, San Diego, 2016

Professor Michael Holst, Chair

One of the principle concerns of computational mathematics is the discrete representation and approximation of mathematical objects. It is common for classical definitions of mathematical objects to allow for elegant mathematical analysis yet lead to computational models which are either inefficient or unimplementable. One mathematical object that fits this mold is a differentiable manifold. Differentiable manifolds are of increasing interest in modern computational mathematics as more geometrically complex problems are considered. In this dissertation, we propose a computational model for representing compact C^1 differentiable manifolds without boundary and their function spaces. This model is based on a combina-

tion of simplicial complexes and splines. Simplicial complexes are a standard tool for computing in both the pure and applied math settings. Splines are piecewise polynomials relative to some tessellation of a domain of interest whose coefficients have been chosen to enforce differentiability at all points of the domain.

1 Introduction

One of the primary motivations for this dissertation is the standard linear variational problem in Hilbert spaces and its approximation. To precisely state both problems, we define the following:

- W - a Hilbert space with inner product $(\cdot, \cdot)$
- $V \subset W$ - a closed subspace of W.
- V^* - the dual space of V.
- $a(\cdot, \cdot)$ - a bilinear form on V which is both bounded and coercive.

Then the standard linear variation problem is

$$\text{Given } F \in V^*, \text{ find } u \in V \text{ such that } a(u, v) = F(v) \text{ for every } v \in V. \tag{1.1}$$

Unless V is finite dimensional, even this linear problem cannot generally be solved for all choices of a and F, so we must accept an approximation. Formally, we choose a finite-dimensional subspace $V_h \subset V$ and solve the problem in the subspace:

$$\text{Given } F \in V^*, \text{ find } u_h \in V_h \text{ such that } a(u_h, v) = F(v) \text{ for every } v \in V_h. \tag{1.2}$$

The Lax-Milgram Theorem ensures that both problems (1.1) and (1.2) have unique solutions [3]. Céa's Lemma then provides an estimate of the error made in solving the subspace problem (1.2) instead of the full problem (1.1). In particular, if C_1

is the continuity constant, and C_2 is the coercivity constant, i.e.,

$$a(u,v) \leq C_1 \|u\|_V \|v\|_V, \tag{1.3}$$

$$C_2 \|u\|_V \geq a(u,u), \tag{1.4}$$

it can be shown that the following quasi-optimal estimate holds:

$$\|u - u_h\|_V \leq \frac{C_1}{C_2} \min_{v \in V_h} \|u - v\|_V. \tag{1.5}$$

A differential structure is not required for the theory at this point - nor is any geometric information. They are generally required in the definition of the bilinear form $a(\cdot,\cdot)$ as in Poisson's Equation. The following example can be found with more detail in [3].

Example 1. *Let Ω be a bounded open subset of $\mathbb{R}^n$ and consider the boundary value problem:*

$$\begin{aligned} -\Delta u &= f \ \ on\ \Omega, \\ u &= 0 \ \ on\ \partial\Omega. \end{aligned} \tag{1.6}$$

A variational formulation of this problem arises from defining:

$$\begin{aligned} V &:= \{v \in H^1(\Omega) \bigm| v|_{\partial\Omega} = 0\}, \\ a(u,v) &:= \int_\Omega \nabla u \cdot \nabla v \ dx, \\ F(v) &:= (f,v) = \int\limits_\Omega fv \ dx. \end{aligned} \tag{1.7}$$

In this example, not only do the derivatives ∇u and ∇v appear, but so does the inner product on $\mathbb{R}^n$. A more subtle geometric property is also used in order to show that the variational problem has anything at all to do with the boundary value problem. This connection comes from a version of Stokes' Theorem,

$$\begin{aligned} (f, v) &= \int_\Omega (-\Delta u) v \, dx \\ \text{(Stokes' Theorem)} \quad &= \int_\Omega \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} v \frac{\partial u}{\partial \nu} \, ds \\ &= \int_\Omega \nabla u \cdot \nabla v \, dx \\ &= a(u, v) \end{aligned} \tag{1.8}$$

where the boundary term vanishes due to our definition of V. Because of this deep connection, we cannot choose the derivative operators, the metric, and the measure independently. They are intertwined, and must be treated with care if we are to approximate solutions of this equation on manifolds.

The way forward is shown in [1, 10, 4], which we summarize here. For the moment, we shall forget about trying to untangle the geometrically linked objects above, and instead construct a new variational problem. Define a family of Hilbert spaces W^k and closed subspaces $V^k \subset W^k$. Let $V = \bigoplus_{k=1}^{n} V^k$ and let $d : V \to V$ be a graded linear operator such that $d^2 = 0$. Then we have the following sequence, called a Hilbert complex (W, d):

$$\cdots \xrightarrow{d} V^{k-1} \xrightarrow{d} V^k \xrightarrow{d} V^{k+1} \xrightarrow{d} \cdots \tag{1.9}$$

We further restrict to the case where the image of d is closed, in which case the sequence is called a closed Hilbert complex.

The dual complex (W^*, d^*) can also be considered.

$$\cdots \xleftarrow[d^*]{} V^*_{k-1} \xleftarrow[d^*]{} V^*_k \xleftarrow[d^*]{} V^*_{k+1} \xleftarrow[d^*]{} \cdots, \tag{1.10}$$

where $V^* \subset W* = W$ and $\langle d^* u, v\rangle = \langle u, dv\rangle$. We now have access to some useful tools. First, we state the Hodge decomposition. Letting

$$\begin{aligned} \mathfrak{B}^k &= \mathrm{d}\, V^{k-1}, \\ \mathfrak{H}^k &= \ker \mathrm{d}^k \cap \ker \mathrm{d}_k^*, \\ \mathfrak{B}_k^* &= \mathrm{d}\, V_{k+1}^*, \end{aligned} \tag{1.11}$$

we can decompose W^k as

$$W^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{B}_k^*. \tag{1.12}$$

This is known as the Hodge decomposition. Second, we can define the operator $L = \mathrm{d}\,\mathrm{d}^* + \mathrm{d}^*\,\mathrm{d}$, known as the abstract Hodge Laplacian. It can be shown that this is equivalent to the Laplacian appearing in equation (1.6).

We can now state a "mixed" variational problem for $Lu = f$.

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle u, \mathrm{d}\,\tau \rangle &= 0, & \forall \tau \in V^{k-1}, \\ \langle \mathrm{d}\,\sigma, v \rangle + \langle \mathrm{d}\,u, \mathrm{d}\,v \rangle + \langle p, v \rangle &= \langle f, v \rangle\,, & \forall v \in V^k, \\ \langle u, q \rangle &= 0, & \forall q \in \mathfrak{H}^k. \end{aligned} \tag{1.13}$$

This provides an alternative framework for a weak solution of $Lu = f$, but we still require a finite dimensional approximation. However, instead of approximating a single Hilbert space, we must approximate the structure contained in the Hilbert complex. This comparison is done through a morphism of Hilbert complexes. Given the complexes (W, d) and (W_h, d_h), a morphism is a collection of bounded linear maps $\pi_h^k : W^k \to W_h^k$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & V^{k-1} & \xrightarrow{d} & V^k & \xrightarrow{d^k} & V^{k+1} & \xrightarrow{d} & \cdots \\ & & \downarrow{\scriptstyle \pi_h^{k-1}} & & \downarrow{\scriptstyle \pi_h^k} & & \downarrow{\scriptstyle \pi_h^{k+1}} & & \\ \cdots & \longrightarrow & V_h^{k-1} & \xrightarrow{d} & V_h^k & \xrightarrow{d} & V_h^{k+1} & \xrightarrow{d} & \cdots \end{array} \tag{1.14}$$

We may now choose a finite dimensional subcomplex $V_h \subset V$ so that $V_h^k \subset V^k$ is a Hilbert subspace for each k, and the inclusion $i_h : V_h \hookrightarrow V$ is a morphism of Hilbert complexes. The finite dimensional mixed variational problem is then to find $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ which satisfies

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle u, \mathrm{d}\,\tau \rangle &= 0, & \forall \tau \in V_h^{k-1}, \\ \langle \mathrm{d}\,\sigma, v \rangle + \langle \mathrm{d}\,u, \mathrm{d}\,v \rangle + \langle p, v \rangle &= \langle f, v \rangle\,, & \forall v \in V_h^k, \\ \langle u, q \rangle &= 0, & \forall q \in \mathfrak{H}_h^k. \end{aligned} \tag{1.15}$$

If we assume the existence of a Hilbert complex projection π_h, we obtain the following quasi-optimal error estimate, analogous to Céa's Lemma (1.5):

$$\|u - \pi_h u\|_V = \inf_{v \in V_h} \|(I - \pi_h)(u - v)\|_V \leq \|I - \pi_h\|_V \inf_{v \in V_h} \|(u - v)\|_V \tag{1.16}$$

The natural choice would be to consider the orthogonal projection, i_h^*, however, in general this does not commute with the differential, and so is not a morphism of Hilbert complexes.

As shown in [10], it is not necessary that V_h be a subcomplex of V, which is termed a *variational crime*. All that is required is that $i_h : V_h \to V$ be an injective morphism of Hilbert complexes with $\pi_h \circ i_h$ the identity. This framework is then used to generalize the work Dziuk [7] and Demlow [5] who study finite elements on co-dimension one hypersurfaces. This is done by approximating a differentiable manifold M by a family of (piecewise) differentiable manifolds M_h along with (piecewise) diffeomorphisms $\varphi_h : M_h \to M$. We then have the map between function spaces $i_h : M_h \to M$ defined by $[i_h(f)]\,(x) = \big(f \circ \phi_h^{-1}(x)\big)$. The spaces are further endowed with Riemannian metrics g and g_h, respectively. It is not assumed that $g_h = \varphi_h^* g$, so that V_h is not in general a subset of V. However, under appropriate assumptions on g_h and g, it can be shown that i_h is indeed a bounded injective morphism of the de Rham complexes.

In this dissertation we show that there is a class of Riemannian manifolds for which we can further delay committing a variational crime. We construct finite dimensional differentiable manifolds which have a piecewise polynomial function space.

2 Background

In this chapter we review existing theory. In §2.1 we describe the combinatorial structure which forms the scaffolding of the construction. Abstract simplicial complexes allow the discrete representation of a large class of topologies. Basic manifold theory is described in §2.2. These concepts are combined in §2.3, where combinatorial manifolds are presented. Of particular importance to us here is that the topology of all differentiable manifolds can be represented by an abstract simplicial complex. Lastly, in §2.4 we review the space of piecewise polynomial functions on simplicial complexes while setting notation.

2.1 Abstract Simplicial Complexes

An abstract simplicial complex is a combinatorial structure which can be used to represent the connectivity of a simplicial mesh, independent of any geometric information. They are a common tool in algebraic topology, and some general references for the material are [19] [9] and [14]. The abstract simplicial complex associated with Figure 2.1, below, captures that e_3 is directly connected to f_1, f_2, p_1 and p_3. A quick inspection of the figures shows that their connectivity is the same, and is represented by the same abstract simplicial complex

The formal definition of an abstract simplicial complex is as follows.

Definition 1. *Given a vertex set X, an* ***abstract simplicial complex*** $\mathcal{F}$ *of X is a set of non-empty subsets of X with the following property: for every set $f \in \mathcal{F}$, every non-empty subset of f is also a member of $\mathcal{F}$.*

A simple example of an abstract simplicial complex is shown in Figure 2.2.

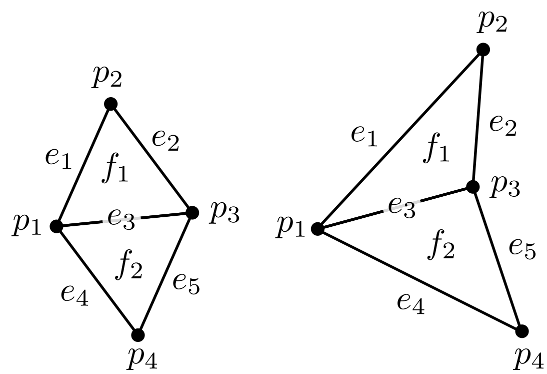


Figure 2.1: Two triangles sharing a common edge, e_3. While the geometry of these meshes is different, the connectivity is the same.

In general we will not explicitly write each element of $\mathcal{F}$ as a set as we have done in that example. Instead, it is convenient to set $p_1 = \{1\}$, $p_2 = \{2\}$ and so on. If e_1 is the edge connecting p_1 and p_2, then $e_1 = p_1 \cup p_2$. In this paper, we will focus mostly on the properties of abstract simplicial meshes endowed with additional structure, and will rarely construct them in this manner.

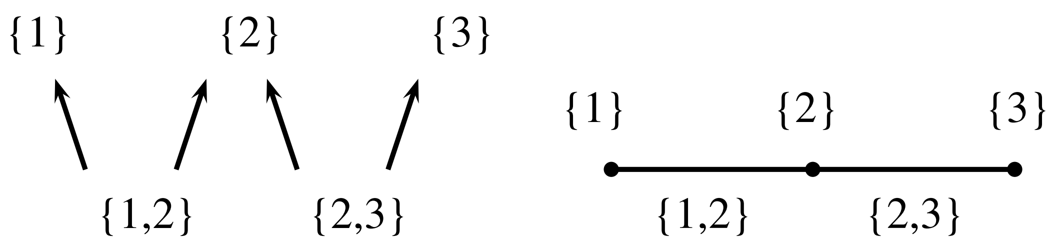


Figure 2.2: The abstract simplicial complex of two line segments sharing a common point. The left image shows the set structure, while the right image shows a geometric representation of the abstract simplicial complex

Continuing the example in Figure 2.1, we can visually represent the abstract simplicial complex associated to it with the diagram in Figure 2.3. In the general two dimensional case, the elements of an abstract simplicial complex represent the faces, edges and vertices of a triangulation. This definition then states that if a face is in a triangulation, then so are the edges and vertices of that face. This definition excludes quadrilateral meshes (see Figure 2.4)

The letters p, e, f and s will generally be reserved for elements of $\mathcal{F}$. We will

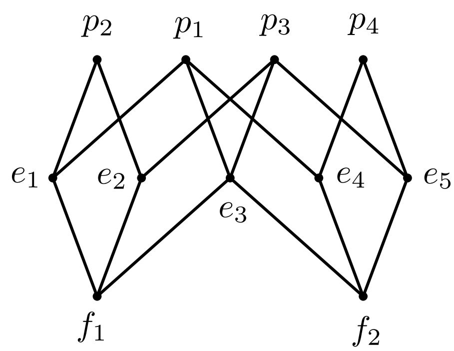


Figure 2.3: The abstract simplicial complex of two triangles sharing a common edge, e_3. Set construction and containment goes downward e.g. $p_1 = \{1\}$, $e_1 = p_1 \cup p_2$ and $p_1 \subseteq e_1 \subseteq f_1$.

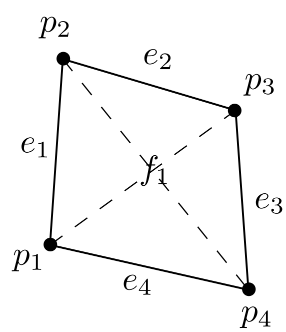


Figure 2.4: The connectivity of a quadrilateral cannot be represented by an abstract simplicial mesh. With $f_1 = p_1 \cup p_2 \cup p_3 \cup p_4$, an abstract simplicial mesh containing f_1 would also contain the edges $p_1 \cup p_3$ and $p_2 \cup p_4$ (indicated by the dashed lines), however, neither of those edges are present in the quadrilateral.

adhere to the convention that $p \subseteq e \subseteq f$, while there will be no such restriction on s. While the definition allows for sets of arbitrary cardinality, we restrict ourselves to the case where X is finite.

2.1.1 Operations on Abstract Simplicial Complexes

Having introduced the concept of abstract simplicial complexes, we now introduce a string of operations useful when dealing with abstract simplicial complexes.

Definition 2 (Dimension)**.** *For $s \in \mathcal{F}$, define $|s|$ to be the number of elements in* *the set* s *and let* $\dim s = |s| - 1$

Definition 3 (k-skeleton)**.** *Given an abstract simplicial complex* $\mathcal{F}$ *and* $k \in \mathbb{N}$, *define the* k*-skeleton of* $\mathcal{F}$ *as:*

$$\mathcal{F}_k = \left\{ s \in \mathcal{F} \;\middle|\; k = \dim s \right\}.$$

Definition 4 (Closure)**.** *For* $f \in \mathcal{F}$*, the closure of* f *is* $\bar{f} = \{s \in \mathcal{F} \mid s \subseteq f\}$*.*

This is closely related to the usual notion of closure, as we will see below. It is a simple exercise to see that $\bar{f}$ is an abstract simplicial complex.

Definition 5 (Star)**.** *For* $e \in \mathcal{F}$*, the star of* e *is* $St(e) = \{s \in \mathcal{F} \mid e \subset s\}$*.*

Definition 6 (Halo)**.** *For* $e \in \mathcal{F}$*, the halo of* e *is* $\breve{e} = \{s \in \mathcal{F} \mid e \subseteq s\}$*.*

Strictly speaking, $\breve{e}$ is not an abstract simplicial complex, since given $f \in \breve{e}$ and $s \subseteq f$, it is not necessarily true that $s \in \breve{e}$. However, it is an abstract simplicial complex in spirit, since by placing the additional restriction on s that $e \subset s$, then $s \in \mathcal{F}$. The literature commonly uses the following definition instead.

The closure and halo are such common operations, that we define an alternate notion which can combine the two:

$$\mathcal{F}_e^f = \text{Halo}(e) \cap Cl(f)$$

With this notation, we could also write:

$$\mathcal{F}_e = \text{Halo}(e)$$

$$\mathcal{F}_f = Cl(f)$$

In general though, except when a more compact notation is needed, we will prefer Halo and Cl.

Definition 7 (Link)**.** *For* $s \in \mathcal{F}$*, the link of* s *is* $Link(s) = Cl \circ St(s) - St \circ Cl(s)$*.*

The following theorem shows that the Halo and the Link are essentially the same thing.

Proposition 1. $Link(s) = \{x \setminus s \mid x \in Halo(s)\}$.

Proof. Let $x \in Link(s)$. Then there exists $y \supseteq s$ such that $x \subseteq y$. Then $s \subset x \cup s \subseteq y \cup s = y$ implies that $x \in \mathcal{F}$ and $x \cup s \in Halo(s)$. Now let $x \in Halo(s) - s$ and let $y = x \cup s$. Then $y \supseteq s$ and $x \subseteq y$ implies $x \in Cl \circ St(s)$. Let $z \subseteq s$, then $x \cap z \subseteq x \cap s = \emptyset$ implies $x \notin St \circ Cl(s)$. Thus $x \in Link(s)$. □

Definition 8 (Cover). *For $e \in \mathcal{F}$, the cover of e is $e^+ = \bigcup_{s \in \breve{e}} s$.*

We will also find that the union $s \cup s^+$ occurs often enough that defining

$$s^\oplus = s \cup s^+$$

is worth the weight of the additional notation.

Proposition 2. *For $e \subset s \in \mathcal{F}$, the following relationships hold:*

1. $s^+ \subset e^+$.
2. $s \cup s^+ \subset e \cup e^+$.
3. $e^+ \cap s = s \setminus e$.
4. $e^\oplus$ *is the disjoint union of* e, $s \setminus e$, s^+ *and* $e^\oplus \setminus s^\oplus$.

Definition 9 (Crown). *For $e \in \mathcal{F}$, the crown of e is $\widetilde{e} = \{f \in \breve{e} \mid s \in \breve{e} \implies s \subseteq f\}$.*

Definition 10 (Vertices). *For $e \in \mathcal{F}$, $\dot{e} = \{f \in \bar{e} \mid s \in \bar{e} \implies f \subseteq s\}$.*

Lastly, We use the following notation for a horizontal "slice" of an abstract simplicial complex.

$$\mathcal{F}_j^k = \left\{s \in \mathcal{F} \mid j \leq \dim s \leq k\right\}$$

2.1.2 Paths in Abstract Simplicial Complexes

The next several definitions allow us to speak of paths in a partial ordering. We restrict to the case where the partial ordering is obtained by set containment on elements from $\mathcal{P}(X)$, however, the concept generalizes easily.

Definition 11. *Let $\mathcal{U}$ be a subset of $\mathcal{P}(X)$, and let $f_1, f_2 \in \mathcal{U}$. We say f_1 and f_2 are adjacent if either $f_1 \subseteq f_2$ or $f_1 \supseteq f_2$ and we write $f_1 \perp f_2$. If f_1 and f_2 are not adjacent, we write $f_1 \parallel f_2$.*

We now define the notion of a path on the partial ordering. to be a sequence of elements of an abstract simplicial complex such that adjacent elements of the sequence are also adjacent in the abstract simplicial complex.

Definition 12. *Let $\mathcal{U}$ be a subset of $\mathcal{P}(X)$, and let $n \in \mathbb{N}$. A path in $\mathcal{U}$ is a function $p : [0..n) \to \mathcal{U}$ such that, for all $i, j \in [0..n)$ with $|i - j| \leq 1$, $p(i) \perp p(j)$.*

We could now define an integer valued metric function on the space of connected abstract simplicial complexes as the smallest length path which connects $s_1, s_2 \in \mathcal{F}$.

Now that we have a notion of path, we describe the spaces where their existence is guaranteed.

Definition 13 (Path Connected)**.** *We say a subset $\mathcal{U}$ of $\mathcal{P}(X)$ (which can be, but is not always, an abstract simplicial complex) is **path connected** if, for any $u_1, u_2 \in \mathcal{U}$, there is a path connecting u_1 and u_2.*

This can be generalized

Definition 14 (k-Connected)**.** *We say that $\mathcal{F}$ is **k-connected** if $\mathcal{F}^+_{k-1}$ is path connected.*

2.1.3 Topology from an Abstract Simplicial Complex

Abstract simplicial complexes are tied closely to topologies. Here we review two methods for constructing a topology associated with an abstract simplicial complex.

Geometric Realization

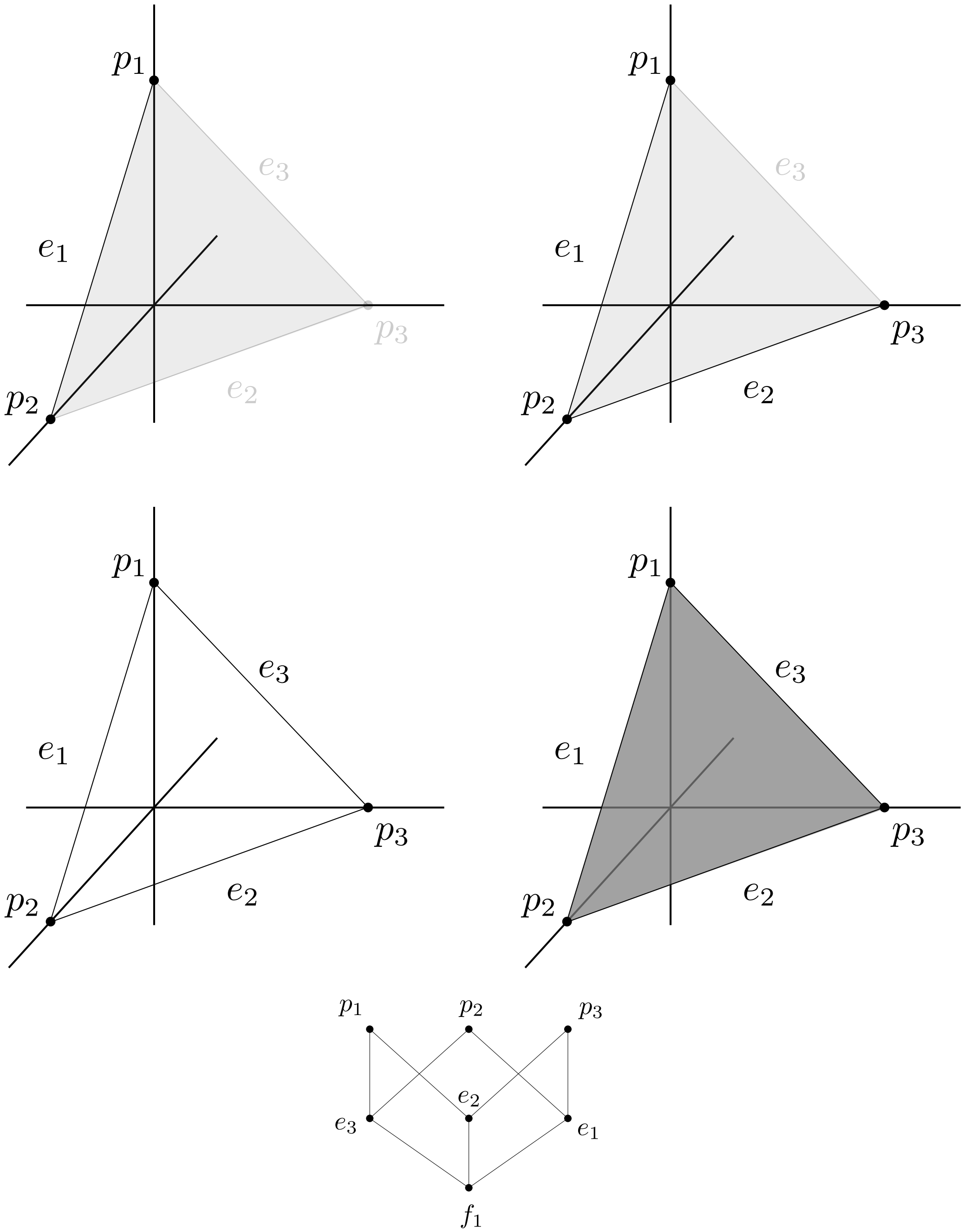


Figure 2.5: Geometric realizations of various components of an abstract simplicial complex.

The first method is slightly easier to work with. In §2.4, we will use it to compactly express continuous piecewise polynomial functions.

Definition 15. *Let $n, N \in \mathbb{N}$ where $n \leq N$. Let $\mathcal{F}$ be an abstract simplicial complex of the set $\{1, \ldots, N\}$. For each $i \in [0..N]$, let $\pi_i : \mathbb{R}^N \to \mathbb{R}$ be the projection onto the i-th coordinate axis. For $f \in \mathcal{F}$, define the* ***geometric realization*** *as:*

$$\Delta_f := \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i=1}^{N} \pi_i(x) = 1, \text{ where } \pi_i(x) > 0, \forall i \in f, \text{ and } \pi_i(x) = 0, \forall i \notin f \right\}$$

This definition allows us to embed each element of an abstract simplicial complex in a euclidean space. We extend this notation to abstract simplicial complexes as:

$$\Delta_{\mathcal{F}} := \bigcup_{f \in \mathcal{F}} \Delta_f$$

An example showing the geometric realizations of a triangle is shown in Figure 2.5

We can now define a topology on $\Delta_{\mathcal{F}}$ by the induced (or subspace) topology from $\mathbb{R}^N$. Since $\mathbb{R}^N$ is a metric topology, $\Delta_{\mathcal{F}}$ is as well.

Conjecture 1. *$\Delta_{\mathcal{F}}$ is path connected if and only if $\mathcal{F}$ is path connected.*

Proof. This should not be difficult, but have no immediate use. □

Gluing

Where the first method constructs a topology by constructing an embedding and using the induced topology, this second method uses the quotient topology to glue simplexes together. The advantage of this approach is that it will allow us to consider discontinuous piecewise polynomial functions. This construction is common in algebraic topology and can be found in Chapter 2 and the appendix of [9]. We briefly review it here.

Let $\mathcal{F}$ be an abstract simplicial complex. For each $s \in \mathcal{F}$, define:

$$\Delta_s = \left\{ x \in \mathbb{R}^s \mid x > 0, \; \sum_{k \in s} x_k = 1 \right\}$$

$$\overline{\Delta}_s = \left\{ x \in \mathbb{R}^s \;\middle|\; x \geq 0,\ \sum_{k \in s} x_k = 1 \right\}$$

And define the barycentric coordinate functions $\pi_k : \Delta_s \to \mathbb{R}$ as $\pi_k(x) = x_k$ for $k \in s$.

Remark: given two sets, X and Y, it is common notation to use X^Y as the set of functions $f : Y \to X$, as above. When using this notation, these functions are referred to as Y-tuples. Further, the function value $f(y)$ is called the yth coordinate of f and is often denoted by f_y. Detailed explanations of this notation with numerous examples can be found in [20].

Define a chain of continuous injections $\{\iota_{s_1}^{s_2} : \overline{\Delta}_{s_1} \to \overline{\Delta}_{s_2}\}_{s_1,s_2 \in \mathcal{F}}$ with the following properties:

1. For each $s_1, s_2 \in \mathcal{F}$, with $s_1 \subset s_2$, then $\pi_k \circ \iota_{s_1}^{s_2} = 0$ whenever $k \in s_2 \setminus s_1$
2. For $s_1, s_2, s_3 \in \mathcal{F}$ with $s_1 \subset s_2 \subset s_3$, then $\iota_{s_1}^{s_3} = \iota_{s_2}^{s_3} \circ \iota_{s_1}^{s_2}$

And consider the disjoint unions $X = \coprod_{s \in \mathcal{F}} \overline{\Delta}_s$ and $X^* = \coprod_{s \in \mathcal{F}} \Delta_s$. The chain of functions $\{\iota_{s_1}^{s_2}\}_{s_1,s_2 \in \mathcal{F}}$ can then be considered as a single *relation*, $\iota : X^* \to X$. We can define the surjective function $p : X \to X^*$, and place the quotient topology on X^* via p (see §22 of [20]).

Quotient topologies are notoriously ill behaved with respect to regularity. In particular, the quotient of a Hausdorff space need not be Hausdorff (the canonical example is the Line with Two Origins, see Example 2). Our goal is to construct differentiable manifolds, and as we will outline in the next section, this lack of being Hausdorff is a serious concern. However, the appendix of [9] shows that this construction (in fact, a more general construction, called a CW complex) yields a space which is not only Hausdorff, but metrizable. We now explore why that is important.

2.2 Manifold Theory

According to [23], the use of geometric ideas in higher dimensions can probably be traced back to at least Lagrange in the late 1700's. Gauss, in particular, in his analysis of $(n-k)$ dimensional surfaces in n dimensional real space very likely had an important impact on Riemann. However, it was Riemann, beginning with his famous *Habilitationsvortrag*, who put us on the trajectory towards the modern day understanding of geometry on manifolds.

2.2.1 Manifolds

The following is a summary of manifold theory from several sources [17, 24, 13].

Definition 16 (Atlas). *An atlas on a topological space M is a set of charts $\Gamma = \{(\gamma_\alpha, U_\alpha)\}_{\alpha \in \mathcal{A}}$ with $\{U_\alpha\}_\alpha$ an open cover of M and $\gamma_\alpha : U_\alpha \to \mathbb{R}^n$ a continuous injection.*

Definition 17 (Topological n-Manifold). *A topological n-manifold is a metrizable space locally homeomorphic to $\mathbb{R}^n$.*

It is interesting to note that a space which is simply locally homeomorphic to $\mathbb{R}^n$ does not, by itself, imply that the space is metrizable. Indeed, it does not imply the space is Hausdorff, as shown in the following example from [20]

Example 2 (Line with Two Origins). *Let X be the union of $\mathbb{R} - \{0\}$ with the set $\{p, q\}$. Take, as a topological basis, any open set of $\mathbb{R}$ which does not contain 0, as well as sets of the form $(-x, 0) \cup \{p\} \cup (0, x)$ and $(-x, 0) \cup \{q\} \cup (0, x)$. Then every neighborhood of p intersects every neighborhood of q, so X is not Hausdorff.*

Adding the additional requirement that a manifold be Hausdorff adds a great deal more structure – every Hausdorff "manifold" is regular [24] – but still does not imply the space is metrizable. The counter example in one dimension is less obvious to construct, details can be found in [24]. However, many desirable properties are equivalent:

Theorem 1 (Spivak [24]). *Let M be a Hausdorff "manifold." The following properties are equivalent.*

1. *Each component of M is σ-compact.*

2. Each component of M is second countable.

3. M is metrizable.

4. M is paracompact.

Definition 18 (Differential Structure). *A C^r differential structure on an n-manifold M is an atlas $\Gamma = \{(\gamma_\alpha, U_\alpha)\}_{\alpha \in \mathcal{A}}$ such that and for any $\alpha, \beta \in \mathcal{A}$*

$$\gamma_\alpha \circ \gamma_\beta^{-1} : \gamma_\beta(U_\alpha \cap U_\beta) \to \gamma_\alpha(U_\alpha \cap U_\beta)$$

is a C^r function.

Definition 19 (Differentiable Manifold). *Let $r \geq 0$. A C^r manifold M is a topological manifold equipped with a C^r differential structure Γ, denoted by (M, Γ).*

a We will use the term differentiable manifold to refer to any C^r manifold with $r > 0$.

Definition 20 (Differentiable Functions on Manifolds). *Let M, N be C^r-manifolds with structures $\{(x_\alpha, U_\alpha)\}_{\alpha \in \mathcal{A}}$ and $\{(y_\beta, V_\beta)\}_{\beta \in \mathcal{B}}$, respectively. A function $f : M \to N$ is C^k if $y_\beta \circ f \circ x_\alpha^{-1}$ is C^k.*

Definition 21 (Induced Differential Structure). *Let N be a topological space, (M, Γ) a differentiable manifold and $f : N \to M$ a homeomorphism, then $f^*\Gamma$, containing the charts $(\gamma_\alpha \circ f, f^{-1}(U_\alpha))$, is called the induced differential structure on N.*

Example 3 (Differential Structures on $\mathbb{R}$). *Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by:*

$$f(x) = \begin{cases} x & x \leq 0 \\ 2x & x > 0 \end{cases} \tag{2.1}$$

The usual differential structure on $\mathbb{R}$ consists of a single chart $\Gamma = \{(\mathrm{id}, \mathbb{R})\}$. With this differential structure, $f : (\mathbb{R}, \Gamma) \to (\mathbb{R}, \Gamma)$ is not a differentiable function. However, f is a homeomorphism so we can consider the induced differential structures $f^*\Gamma = \{(f, \mathbb{R})\}$ and $f^{-1*}\Gamma = \{f^{-1}, \mathbb{R}\}$. Then f is a differentiable function when *using these alternate differential structures such as either* $f : (\mathbb{R}, f^*\Gamma) \to (\mathbb{R}, \Gamma)$ *or* $f : (\mathbb{R}, \Gamma) \to (\mathbb{R}, f^{-1*}\Gamma)$.

Definition 22 (Diffeomorphism)**.** *Let* (M_0, Γ_0) *and* (M_1, Γ_1) *be differentiable manifolds. Then a homeomorphism* $f : M_0 \to M_1$ *is a diffeomorphism if* $f^*\Gamma_1 = \Gamma_0$.

Let $p \in M$, let $\phi : (-\epsilon, \epsilon) \to M$ be a C^1 function with $phi(0) = p$. $\phi_1 = \phi_2$ if, for every $f \in C^r(M)$, $\frac{d}{dt}\Big|_{t=0} (f \circ \phi_1) = \frac{d}{dt}\Big|_{t=0} (f \circ \phi_2)$.

Definition 23 (Tangent Vector)**.** *Let* M *be a* C^r *manifold with chart* $\Gamma = \{(\gamma_\alpha, U_\alpha)\}_{\alpha \in \mathcal{A}}$. *A tangent vector to* M *is an equivalence class of triples*

$$(p, i, v) \in M \times \mathcal{A} \times \mathbb{R}^n$$

under the equivalence relation

$$(p, i, v) = (q, j, w)$$

if and only if $p = q$ *and*

$$\left[D\Big|_{\gamma_i(p)} (\gamma_j \gamma_i^{-1}) \right] v = w$$

Lemma 1. *Let* M *be a* C^r *n-manifold with* $r > 0$. *Then* T_pM *is a vector space with dimension* n.

2.2.2 Fiber Bundles

A fiber bundle is a triple of topological spaces, E, called the total space, B, called the base space, and F, called the fiber, together with a continuous surjective map $\pi : E \to B$, called the projection map. In addition, a local triviality condition must hold. Specifically, for every $b \in B$, there must be an open neighborhood $U \subset B$ such that:

$$\pi^{-1}(U) \cong U \times F.$$

It is common to write a fiber bundle as a 'short exact sequence of spaces':

$$F \longrightarrow E \xrightarrow{\pi} B \ . \tag{2.2}$$

Given two fiber bundles E_1 and E_2 sharing a common base space B, a fiber preserving map is a continuous map $f : E_1 \to E_2$ such that the following diagram commutes.

$$\begin{array}{ccc} E_1 & \xrightarrow{\quad f \quad} & E_2 \\ & {\scriptstyle\pi}\searrow \qquad \swarrow{\scriptstyle\pi} & \\ & B & \end{array} \tag{2.3}$$

This notion can be generalized to a changing base as well. A fiber bundle morphism is a pair of continuous maps $f : E_1 \to E_2$, $g : B_1 \to B_2$ so that the following diagram commutes.

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ B_1 & \xrightarrow{g} & B_2 \end{array} \tag{2.4}$$

Lastly, a section of a fiber bundle is a continuous map $\theta : B \to E$ such that $\pi \circ \theta = \mathrm{id}$. Or, in diagram form:

$$\begin{array}{ccc} E & \xhookrightarrow{\mathrm{id}} & E \\ {\scriptstyle\theta}\uparrow & & \downarrow{\scriptstyle\pi} \\ B & \xhookrightarrow{\mathrm{id}} & B \end{array} \tag{2.5}$$

A standard reference for fiber bundles is [9].

2.2.3 Vector Bundles

An important special case of fiber bundles, and of particular interest to us, is vector bundles. They arise is when F in diagram (2.2) is a vector space. As vector bundles are also fiber bundles, the above notions of morphisms and sections carry over, with one important additional specialization: a vector bundle morphism must act linearly on each fiber.

A vector bundle E' is a sub-bundle of E if the following diagram commutes,

$$\begin{array}{ccc} E' & \hookrightarrow & E \\ & {\scriptstyle\pi'}\searrow & \downarrow{\scriptstyle\pi} \\ & & B \end{array} \tag{2.6}$$

and each fiber of E' is a subspace of the corresponding fiber of E. We can then consider the quotient bundle $\frac{E}{E'}$. That is, for $b \in B$ and $x_1, x_2 \in \pi^{-1}(b)$ we consider $x_1 \sim x_2$ if there exists $w \in \pi'^{-1}(b)$ such that $x_1 - x_2 = w$.

2.2.4 Tensors

Tensors are the universal objects of multi-linear functions. An abstract treatment of them can be found in [6], while a more applied treatment can be found in [2].

Let U and V be vector spaces over a field F. Given $u \in U$ and $v \in V$, we can take the tensor product over F, denoted by $u \otimes_F v$. When the field is implied, the shorter notation of $u \otimes v$ will be used. In this dissertation, the field will either be $\mathbb{R}$ or $\mathbb{R}$-valued functions.

The product has the following properties for $u_1, u_2 \in U$, $v_1, v_2 \in V$ and $a, b \in F$:

1. $(au_1) \otimes v_1 = u_1 \otimes (av_1) = a(u_1 \otimes v_1)$
2. $(u_1 + u_2) \otimes v_1 = u_1 \otimes v_1 + u_2 \otimes v_1$
3. $u_1 \otimes (v_1 + v_2) = u_1 \otimes v_1 + u_1 \otimes v_2$

These products are called simple tensors. The span of F-linear combinations of simple tensors are called tensor products and is denoted by $U \otimes_F V$, or simply $U \otimes V$ when the field is clear by context.

The tensor product is associative. Given an additional vector space W, and $w \in W$ we have:

$$(u \otimes v) \otimes w = u \otimes (v \otimes w) = u \otimes v \otimes w.$$

$$(U \otimes V) \otimes W = U \otimes (V \otimes W) = U \otimes V \otimes W.$$

Repeated tensor products are then well defined.

We will frequently encounter repeated tensor products of V with V^*. $\mathcal{T}_k^l(V)$

Basis of Tensors

Let $\{e_1, \ldots, e_n\}$ and $\{e^1, \ldots, e^n\}$ be dual bases for V and V^* so that we have $e^j e_i = \delta_i^j$. Then the set of simple tensors:

$$\left\{e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes e^{j_1} \otimes \cdots \otimes e^{j_l} \;\middle|\; i_1, \ldots, i_k, j_1, \ldots, j_l \in [1..n]\right\}$$

forms a basis for $\mathcal{T}_k^l(V)$, implying it has dimension n^{k+l}. An element $T \in \mathcal{T}_k^l$ can be represented as

$$T = \sum A_{i_1 \cdots i_k}^{j_1 \cdots j_l}\; e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes e^{j_1} \otimes \cdots \otimes e^{j_l}$$

where $A_{i_1 \cdots i_k}^{j_1 \cdots j_l} \in F$ are the coordinates of T in the basis determined by $\{e_1, \ldots, e_n\}$.

Contractions of Tensors

A contraction C_i^j is the linear map $\mathcal{T}_k^l(V) \to \mathcal{T}_{k-1}^{l-1}(V)$ induced by the following operation on simple tensors:

$$u_1 \otimes \cdots \otimes u_k \otimes v_1 \otimes \cdots \otimes v_l \mapsto u_i v_j \; u_1 \otimes \cdots \otimes \widehat{u_i} \otimes \cdots \otimes u_k \otimes v_1 \otimes \cdots \otimes \widehat{v_j} \otimes \cdots \otimes v_l$$

We will adopt the convention that unless otherwise specified, a contraction without indices will refer to the C_k^l. Tensors will generally be built from left to right. This places a 'LIFO', or 'stack', ordering to the construction and subsequent contraction of tensors.

To represent a contraction of a tensor in a basis.

$$C(T) = \sum \left[\sum_{p=1}^{n} A_{i_1 \cdots i_{k-1} p}^{j_1 \cdots j_{l-1} p}\right] \; e_{i_1} \otimes \cdots \otimes e_{i_{k-1}} \otimes e^{j_1} \otimes \cdots \otimes e^{j_{l-1}}$$

Example 4. *Let $\mathcal{B} = \{e_1, e_2\}$ be the standard basis of $\mathbb{R}^2$ with dual basis $\{e_1, e_2\}$. Then an element of $\mathcal{T}_1^1(\mathbb{R}^2)$ represented in the basis $\mathcal{B}$ has the form:*

$$T = A_1^1\; e_1 \otimes e^1 + A_2^1\; e_1 \otimes e^2 + A_1^2\; e_2 \otimes e^1 + A_2^2\; e_2 \otimes e^2.$$

Or, using matrix notation:

$$T \cong \begin{bmatrix} A_1^1 & A_2^1 \\ A_1^2 & A_2^2 \end{bmatrix} := A$$

The contraction $C(T) = A_1^1 + A_2^2$ *is the trace of the matrix* A.

Now let $X = X^1\ e_1 + X^2\ e_2$ *and consider*

$$C(T \otimes X) = (A_1^1 X^1 + A_2^1 X^2)\ e_1 + (A_1^2 X^1 + A_2^2 X^2)\ e_2.$$

In matrix notation,

$$X \cong \begin{bmatrix} X^1 \\ X^2 \end{bmatrix} := x$$

and

$$C(T \otimes X) \cong Ax.$$

So that tensors are a generalization of matrices, where matrix multiplication is replaced by the tensor product and contraction.

Definition 24. *A (Koszul) connection on a* C^∞ *manifold M is a function* ∇ *which associates a* C^∞ *vector field* $\nabla_X Y$ *to any two* C^∞ *vector fields* X *and* Y, *and which satisfies:*

1. $\nabla_{fX_1+gX_2} Y = f\ \nabla_{X_1} Y + g\ \nabla_{X_2} Y$
2. $\nabla_X(\alpha Y_1 + \beta Y_2) = \alpha \nabla_X Y_1 + \beta \nabla_X Y_2$
3. $\nabla_X(fY) = f \cdot \nabla_X Y + X(f) \cdot Y$

We can extend the connection to operate on tensors of any type with the following:

1. $\nabla_X f =$ X(f)
2. $\nabla_X(\alpha A + \beta B) = \alpha \nabla_X A + \beta \nabla_X B$
3. $\nabla_X(A \otimes B) = \nabla_X A \otimes B + A \otimes \nabla_X B$

4. $\nabla_X \circ C = C \circ \nabla_X$

This induces a linear operator $\nabla : X \to \nabla_X$.

2.3 Combinatorial and Piecewise Linear Manifolds

We now restrict our attention to manifolds which admit a combinatorial structure, that is, to those manifolds which are homeomorphic to Δ_F for some simplicial complex $\mathcal{F}$. It is interesting to note that not all manifolds have this property, details can be found in [11]. However, fortunately for our purposes, all differentiable manifolds have a triangulation [25]. Moreover, all differentiable manifolds have a special type of triangulation called a Brouwer Triangulation.

Let $\mathcal{F}$ be an abstract simplicial complex of some universal set X of size N, and let $\Delta_{\mathcal{F}}$ be its geometric representation in $\mathbb{R}^N$. For $n < N$, consider a rank-n linear map $\gamma : \mathbb{R}^N \to \mathbb{R}^n$ and its restriction to $\Delta_{\mathcal{F}}$, denoted by $\gamma|_{\Delta_{\mathcal{F}}}$.

For an appropriate $\mathcal{F}$ and choice of n and γ, which we will restrict to shortly, this construction produces the normal starting point for a FEM method. $\gamma(\Delta_{\mathcal{F}})$ is the domain Ω and $\pi_i = \lambda_i \circ \gamma$ gives a necessary constraint for the barycentric coordinate functions.

In order for this construction to work, we require two things of $\gamma|_{\Delta_{\mathcal{F}}}$. First, in order for the necessary condition $\pi_i = \lambda_i \circ \gamma$ to become a sufficient one, and thus uniquely define the barycentric coordinate functions, $\gamma|_{\Delta_{\mathcal{F}}}$ must be one-to-one. Specifically, when γ_k is an injection, the barycentric coordinates on the simplex $f \in \mathcal{F}_k$, can be expressed as:

$$\lambda_f^i = \pi_i \circ (\gamma_k|_{\Delta_f})^{-1}$$

Second, in order to speak of derivatives on Ω, the interior of Ω must be non-empty (or lay within the same affine hyperplane of $\mathbb{R}^n$ and have non-empty interior in the restricted topology induced on the hyperplane from $\mathbb{R}^n$). This requires that $\gamma|_{\Delta_{\mathcal{F}}}$ be an open map.

This places subtle restrictions on admissible $\mathcal{F}$ which we will not attempt to answer in the current article. We will instead use existence of these coordinate functions $\gamma|_{\Delta_\mathcal{F}}$ to define the appropriate abstract simplicial complexes we will consider.

Definition 25. *Let X be a finite set of points, with $N = |X|$. Let $\mathcal{F}$ be an abstract simplicial complex of X. For $n \in \mathbb{N}$, we say $\mathcal{F}$ is **n-representable** if there exists a linear map, $\gamma : \mathbb{R}^N \to \mathbb{R}^n$, such that $\gamma|_{\Delta_{\mathcal{F}_{\breve{s}}}}$ is an open injection.*

This definition is more restrictive than we would like. While the construction built so far applies to typical FEM situations, the intent behind this construction is to allow topologies which are not homeomorphic to $\mathbb{R}^n$. For example, there is no appropriate γ for the abstract simplicial complex representing the surface of a tetrahedron.

In order to admit a larger class of abstract simplicial complexes, we make the following definition.

Definition 26. *Let X be a finite set of points, with $N = |X|$. Let $\mathcal{F}$ be an abstract simplicial complex over X. For $n \in \mathbb{N}$, we say $\mathcal{F}$ is **locally n-representable** if for each $s \in \mathcal{F}$, the closure of the halo of s, $\mathrm{Cl} \circ \mathrm{Halo}(s)$, is n-representable.*

Remark: In the theory of differential geometry, $(\gamma_k, \Delta_{\mathcal{F}_k})$ are coordinate charts for the manifold $\Delta_\mathcal{F}$. Indeed, the construction presented thus far is enough to produce C^0-manifolds. However, we are interested in constructing differentiable manifolds. Doing so requires producing an atlas of charts which are mutually C^k, for $k \geq 1$. While we could attempt to create those from the start, doing so requires admitting at least rational tensor polynomials.

Lemma 2. *$\gamma_i \circ (\gamma_j|_{\Delta_f})^{-1}$ is affine.*

Proof. Let $v, u \in \gamma_j(\Delta_f)$. Then there exists $p, q \in \Delta_f$ such that $v = \gamma_j(x)$, $u = \gamma_j(y)$. Let $w = \alpha v + \beta u = \gamma_j(\alpha x + \beta y + z)$, for any $z \in \ker \gamma_j$, $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta = 1$ and $\alpha x + \beta y > 0$. Then $(\gamma_j|_{\Delta_f})^{-1}(w) = \alpha x + \beta y \in \Delta_f$ and:

$$\begin{aligned}
\gamma_i \circ (\gamma_j|_{\Delta_f})^{-1}(\alpha v + \beta u) &= \gamma_i \circ (\gamma_j|_{\Delta_f})^{-1}(w) \\
&= \gamma_i(\alpha x + \beta y) \\
&= \alpha\gamma_i(x) + \beta\gamma_i(y) \\
&= \alpha\gamma_i \circ (\gamma_j|_{\Delta_f})^{-1}(u) + \beta\gamma_i \circ (\gamma_j|_{\Delta_f})^{-1}(v)
\end{aligned}$$

□

Notice that it requires that . This is the smallest (ordered by containment) element of $\{\Delta_{\mathcal{U}} \mid \mathcal{U} \subseteq \mathcal{F}\}$ which contains $s \in \mathcal{F}$.

The following "conjecture" would be wonderful to have.

Conjecture 2. *Let X be a finite set. Let $\mathcal{F}$ be an abstract simplicial complex over X. Then there exists a map $\gamma : \Delta_{\mathcal{F}} \to \mathbb{R}^n$ if and only if [some decidable about $\mathcal{F}$ is true].*

Unfortunately, it is known to be an undecidable problem to determine if a simplicial complex is homeomorphic to a manifold.

2.4 Function Spaces

Given an abstract simplicial complex $\mathcal{F}$, we now construct function spaces on the set $\Delta_{\mathcal{F}}$. We are primarily concerned with polynomial function spaces.

2.4.1 Multi-Index Notation

We use standard multi-index notation, which we review here. Let X be a finite set. A multi-index α is an element of $\mathbb{N}^X$ (the set of functions from X to $\mathbb{N}$).

The two quintessential operations on multi-indexes are

$$|\alpha| = \sum_{x \in X} \alpha(x) \tag{2.7}$$

$$\alpha! = \prod_{x \in X} \alpha(x)! \tag{2.8}$$

where $|\alpha|$ is called the degree of α. Often we will need to refer to spaces of multi-indexes with specific domains and degrees. For this we use the notation:

$$[X]^r = \left\{ \alpha \in \mathbb{N}^X \;\middle|\; |\alpha| = r \right\}. \tag{2.9}$$

We extend this notation to a simplicial complex $\mathcal{F}$ through:

$$[\mathcal{F}]^r = \bigcup_{s \in \mathcal{F}} [s]^r. \tag{2.10}$$

If $X \subset Y$ are finite sets, and $\alpha \in \mathbb{N}^Y$, through an abuse of notation, we say $\alpha \subset X$ whenever $\operatorname{supp} \alpha \subset X$.

2.4.2 Polynomial Spaces

In section §2.1.3, two constructions of a topology on the space $\Delta_{\mathcal{F}}$ were presented. For the case that $\mathcal{F}$ is finite, which is the only case we are considering, these are equivalent. With this topology in place, we can consider the space of continuous real valued functions on $\Delta_{\mathcal{F}}$, denoted by $\mathcal{C}^0(\Delta_{\mathcal{F}})$. We will, however, have need to consider more general functions. Our first generalization will be to consider, instead of real valued functions, functions which take values in topological vector spaces. In general, these will be finite dimensional, so going to topological vector spaces is somewhat overkill as there is a unique topology on finite vector spaces for which the vector space operations are continuous [21]. The second generalization, is that we will find it convenient to consider functions which are not necessarily continuous, which is why the alternate construction of $\Delta_{\mathcal{F}}$ is considered.

We will discuss three bases for polynomials on $\Delta_{\mathcal{F}}$: homogeneous, Bernstein and Lagrange.

Homogeneous Basis

We will consider the homogeneous polynomials of order r as the standard basis. For a given $r \in \mathbb{N}$, simplex $s \in \mathcal{F}$, and coefficients $\{c_\alpha \in V\}_{\alpha \in [s]^r}$ we can consider the homogeneous polynomials on Δ_s as:

$$\phi = \sum_{\alpha \in [s]^r} c_\alpha \pi^\alpha \tag{2.11}$$

Where π_i is the i-th coordinate projection function, or barycentric coordinate function, as defined in §2.1.3.

Extended this to the entire complex $\Delta_{\mathcal{F}}$ is straight forward when we are not enforcing continuity

$$\phi = \sum_{s \in \mathcal{F}} \sum_{\alpha \in [s]^r} c_\alpha \pi^\alpha. \tag{2.12}$$

To describe continuous functions, it is easier to use the geometric realization approach. We begin with polynomials $\phi : \mathbb{R}^N \to V$:

$$\phi = \sum_{|\alpha| = r} c_\alpha \pi^\alpha. \tag{2.13}$$

Here, we are taking the sum over all multi-indexes, α, of $[1..N]$ such that $|\alpha| = r$. We then define $\mathcal{H}^r \Delta_{\mathcal{F}}$ as the functions of $\mathcal{H}^r(\mathbb{R}^N)$ restricted to $\Delta_{\mathcal{F}}$. This is generally an extremely wasteful summation. The following lemmas show that $\pi^\alpha|_{\Delta_{\mathcal{F}}}$ will be zero for a large number of $\alpha \in [1..N]^r$. The terms needed in the sum, and by consequence the dimension of the function space, will therefore be substantially reduced.

Lemma 3. *Let X be a finite set. Let $\mathcal{F}$ be an abstract simplicial complex on X. Let $J \in [X]$ and $f \in \mathcal{F}$. Then $\pi^J|_{\Delta_f} = 0$ if and only if $J \not\subseteq f$*

Proof. Suppose $J \not\subseteq f$ and let $x \in \Delta_f$. Then there exists $j \in J$ such that $j \notin f$.

$$\begin{aligned}\pi^J(x) &= \pi^j(x) \cdot \pi^{J-\{j\}}(x) \\ &= 0 \cdot \pi^{J-\{j\}}(x) \\ &= 0\end{aligned}$$

Now suppose $J \subseteq f$. The case when $J = \emptyset$ is trivial, so assume J is non-empty. Let $g = \operatorname{supp} J$, and $x = \sum\limits_{i \in g} \frac{1}{|g|}\mathbf{e}_i$. Then:

$$\begin{aligned}\pi^J(x) &= \prod_{i \in g} (\pi_i(x))^{J(i)} \\ &= \prod_{i \in g} \left(\frac{1}{|g|}\right)^{J(i)} \\ &= \left(\frac{1}{|g|}\right)^{\left(\sum\limits_{i \in g} J(i)\right)} \\ &= \left(\frac{1}{|g|}\right)^{|J|} \\ &> 0\end{aligned}$$

□

Corollary 1. *Let X be a finite set. Let $\mathcal{F}$ be an abstract simplicial complex on X. Let $J \in [X]$. If* $\operatorname{supp} J \notin \mathcal{F}$ *then* $\pi^J|_{\Delta_\mathcal{F}} = 0$.

We are now able to express $\phi|_{\Delta_\mathcal{F}}$ in reduced form:

$$\phi|_{\Delta_\mathcal{F}} = \sum_{\alpha \in [\mathcal{F}]^r} c_\alpha \pi^\alpha|_{\Delta_\mathcal{F}}.$$

As a more compact notion, we shall drop the $|_{\Delta_\mathcal{F}}$ and write:

$$\phi = \sum_{\alpha \in [\mathcal{F}]^r} c_\alpha \pi^\alpha.$$

The dimension of $\mathcal{H}^r(\mathbb{R}^N)$ is $\binom{r+N-1}{N-1}$, while the dimension of $\mathcal{H}^r(\Delta_\mathcal{F})$ depends on $\mathcal{F}$. Using the notation that $\mathcal{F}_k$ is the k-skeleton of $\mathcal{F}$ we can write a closed form expression for the dimension of $\mathcal{H}^r(\mathcal{F})$:

$$\sum_{k=0}^{n} \binom{r}{k+1} |\mathcal{F}_k|.$$

For piecewise linear functions ($r = 1$), the dimension of both these spaces is $N = |\mathcal{F}_0|$.

The importance of the homogeneous basis function is the simplicity of the evaluation functional at a point $x \in \Delta_{\mathcal{F}}$. It is induced by:

$$\pi^\alpha(x) = \prod_{i \in \alpha} (\pi_i(x))^{\alpha(i)}.$$

$$\phi(x) = \sum_{I \in \mathcal{N}^r(\mathcal{F})} a_I \pi^I(x) = \sum_{I \in \mathcal{N}^r(\mathcal{F})} \prod_{i \in \operatorname{supp} J} (\pi_i(x))^{J(i)} a_I \pi^i(p).$$

Bernstein Basis

We now briefly introduce the Bernstein basis. [15] The Bernstein basis is a term by term rescaling of the standard basis. Specifically, for each $\alpha \in [\mathcal{F}]^r$, define the Bernstein ordinal as

$$B_\alpha = \frac{|\alpha|!}{\alpha!} \tag{2.14}$$

and the corresponding Bernstein basis function as

$$\mathcal{B}^\alpha = B_\alpha \pi^\alpha. \tag{2.15}$$

Bernstein polynomials have a number of nice properties, among them is that they form a partition of unity at every $x \in \Delta_{\mathcal{F}}$. More importantly for us - they greatly simplify the expressions for derivatives,

$$\nabla^k \mathcal{B}^\alpha = \frac{|\alpha|!}{(|\alpha| - k)!} \sum_{\substack{|\alpha_2| = k \\ \alpha_1 + \alpha_2 = \alpha}} \mathcal{B}^{\alpha_1} (d\mathcal{B})^{\alpha_2} \tag{2.16}$$

where we have used $(d\mathcal{B})^\alpha$ to denote the Bernstein polynomial basis of symmetric covariant tensors.

$$(d\mathcal{B})^\alpha = B_\alpha (d\pi)^\alpha$$

The exponent taken on $d\pi$ is the symmetric tensor exponent:

$$(d\pi)^\alpha = \frac{1}{|\alpha|!} \sum_{\sigma \in S^{|\alpha|}} d\pi_{\alpha(\sigma(1))} \otimes \cdots \otimes d\pi_{\alpha(\sigma(|\alpha|))}$$

While this notation is a slight abuse, it allows us to absorb the usual constants which appear when expressing higher order derivatives. One must, however, be on alert when manipulating these equations. For an example showing how it interacts with a function pullback, see equation (3.7). Remark: the author believes this particular notation for the derivative is new, however, it is clear that the literature is at some level aware of this. See [15].

3 Splines on $\mathbb{R}^n$

In this chapter we present a geometric approach to obtain constraints. While not necessary for the main development of the dissertation, it connects the research to existing theory and provides an example from which we will generalize.

3.1 Computing the Derivative

We first present an approach to compute the derivatives of piecewise polynomials on a simplicial complex embedded in a vector space. In this section, we assume that $\mathcal{F}$ is an n-representable manifold (see Definition 25). This implies the existence of a coordinate chart $\gamma : \Delta_{\mathcal{F}} \to \mathbb{R}^n$. We will denote $\Omega_{\mathcal{F}} = \gamma(\Delta_{\mathcal{F}}) \subset \mathbb{R}^n$ and $\Omega_f = \gamma(\Delta_f) \subset \mathbb{R}^n$ for any $f \in \mathcal{F}$.

Our main object of study will be the degree r piecewise polynomial function ϕ on $\Delta_{\mathcal{F}}$:

$$\phi = \sum_{|\alpha|=r} c_\alpha \mathcal{B}^\alpha, \tag{3.1}$$

where B^α is defined in equation (2.15) and $c_\alpha \in \mathbb{R}$.

Using the linearity of ∇^k and combining equations (2.16) and (3.1) yields the following expression for the k-th order derivative of ϕ:

$$\begin{aligned}\nabla^k\phi &= \sum_{|\alpha|=r} c_\alpha \nabla^k \mathcal{B}^\alpha \\ &= \frac{r!}{(r-k)!}\sum_{\substack{|\beta|=k\\|\alpha|=r-k}} c_{\alpha+\beta}\mathcal{B}^\alpha(d\mathcal{B})^\beta \\ &= P^r_k \sum_{\substack{|\beta|=k\\|\alpha|=r-k}} c_{\alpha+\beta}\mathcal{B}^\alpha(d\mathcal{B})^\beta\end{aligned} \tag{3.2}$$

Where we have used the notation $P^r_k = \frac{r!}{(r-k)!}$. From here we will derive an expression for the derivatives of:

$$\phi \circ \gamma^{-1} : \mathbb{R}^n \to \mathbb{R}. \tag{3.3}$$

In equation (3.2), we computed the derivatives of ϕ with respect to $\mathbb{R}^N$ (where N is the number of vertices in the complex $\mathcal{F}$, see Definition 15). To obtain the derivatives of equation (3.3), we shall pull equation (3.2) back through γ^{-1}. The difficulty is that γ is only piecewise affine (and therefore only piecewise differentiable), so we do this pullback simplex by simplex. For simplicity of notation, we introduce the function $\eta_s : \Omega_s \to \Delta_s$ for any $s \in \mathcal{F}_n$ (where $\mathcal{F}_n$ is the n-skeleton of $\mathcal{F}$):

$$\eta_s = (\gamma|^{-1}_{\Delta_s}). \tag{3.4}$$

Further, we will use $\lambda_{s,k} : \mathbb{R}^n \to \mathbb{R}$ to denote the barycentric coordinate (or 'hat' function) of the k-th vertex of the embedded simplicial complex Ω_s; for the multi-index α, λ^α_s is the natural extension to multi-indexes. Using this notation, we can restate our immediate objective as seeking an expression for

$$\eta^*_s(\nabla^k\phi) : \Omega_f \to (T^*\mathbb{R}^n)^k \tag{3.5}$$

in terms of the barycentric coordinates λ_k.

In order to obtain this, we first obtain expressions for $\eta_s^*(\mathcal{B}^\alpha)$ and $\eta_s^*(d\mathcal{B}^\alpha)$

$$\begin{aligned} \eta_s^*(\mathcal{B}^\alpha) &= \mathcal{B}^\alpha \circ \eta_s \\ &= B_\alpha \pi^\alpha \circ (\gamma|_{\Delta_s}^{-1}) \\ &= B_\alpha \lambda_s^\alpha \end{aligned} \tag{3.6}$$

The differentials of these terms behave similarly.

$$\begin{aligned} \eta_s^*(d\mathcal{B}^\alpha) &= B_\alpha \eta_s^*(d\pi^\alpha) \\ &= B_\alpha \eta_s^* \left(\frac{1}{|\alpha|!} \sum_{\sigma \in S^{|\alpha|}} d\pi_{\alpha(\sigma_1)} \otimes \cdots \otimes d\pi_{\alpha(\sigma_{|\alpha|})} \right) \\ &= B_\alpha \frac{1}{|\alpha|!} \sum_{\sigma \in S^{|\alpha|}} d\eta_s^* \pi_{\alpha(\sigma_1)} \otimes \cdots \otimes d\eta_s^* \pi_{\alpha(\sigma_{|\alpha|})} \\ &= B_\alpha \frac{1}{|\alpha|!} \sum_{\sigma \in S^{|\alpha|}} d\lambda_{s,\alpha(\sigma_1)} \otimes \cdots \otimes d\lambda_{s,\alpha(\sigma_{|\alpha|})} \\ &= B_\alpha d\lambda_s^\alpha \end{aligned} \tag{3.7}$$

Combining equations (3.2), (3.6) and (3.7) we obtain an equation for the derivative in the coordinate system γ:

$$\eta_s^*(\nabla^k \phi) = P_k^r \sum_{\substack{|\beta|=k \\ |\alpha|=r-k}} c_{\alpha+\beta} B_\alpha B_\beta \lambda_s^\alpha (d\lambda_s)^\beta \tag{3.8}$$

3.2 Developing Spline Constraints

Next, we derive and analyze conditions on the coefficients c_α of ϕ to enforce differentiability across the "edges" of the complex. More accurately, to enforce the differentiability across the lower dimensional simplices which bind the simplices of the n-skeleton together.

To do this we let $e \subset s \in \mathcal{F}$ be the "edge" across which we enforce differentiability, and restrict equation (3.8) to Δ_e. By Lemma 3 (see page 26), only those terms with $\alpha \subset e$ will be non-zero. This restricted function can then be expressed as:

$$\eta_s^*(\nabla^k \phi)\Big|_{\gamma(\Delta_e)} = P_k^r \sum_{\alpha \in [e]^{r-k}} \left[\sum_{\beta \in [s]^k} c_{\alpha+\beta} B_\alpha B_\beta (d\lambda_s)^\beta \right] \lambda_e^\alpha. \tag{3.9}$$

We view this as a tensor-valued polynomial of degree $r-k$. To achieve k-th order continuity along a simplex $e \in \mathcal{F}$ we must have mutual equality of $\eta_s^*(\nabla^k \phi)|_{\gamma(\Delta_e)}$ for every $s \in \text{Crown}(e)$. As the λ's form a basis for this tensor-valued polynomial, this gives us a finite set of constraints for the coefficients $\{c_\alpha\}_{\alpha \in [\mathcal{F}]^r}$. Formally, for each $e \in \mathcal{F}$, we define an equivalence relation on the elements of $\mathcal{F}_n$:

$$s_1 \sim_e s_2 \iff \eta_{s_1}^*(\nabla^k \phi)\Big|_{\gamma(e)} = \eta_{s_2}^*(\nabla^k \phi)\Big|_{\gamma(e)}. \tag{3.10}$$

The k-th derivative of ϕ is well defined on the simplex e with respect to the coordinate map γ if the set $\text{Crown}(e)$ has exactly one equivalence class.

While we have not yet produced a form of these constraints which are particularly amenable to computation, we can pause here to prove a stronger version of a standard result in spline theory – that obtaining differentiability across all edges gives differentiability at each vertex.

Definition 27. *We say that $\phi \in C^k_=(e)$ if, for all $s_1, s_2 \in \text{Crown}\,(e)$, $s_1 \sim_e s_2$. Also define $C^k(e) = \bigcap_{i=0}^{k} C^i_=(e)$.*

Lemma 4. *Let p be a path of length n in $\text{Halo}\,(e)$ and suppose for any $f \in \text{Halo}\,(e)$ that $s_1 \sim_e s_2$ whenever $s_1, s_2 \in \text{Crown}\,(f)$. Then $s_1 \sim_e s_2$ for any $s_1, s_2 \in \bigcup_{i=1}^{n} \text{Crown}\,(p(i))$.*

Proof. We proceed by induction on the length of p. When p is empty, the statement is vacuously true. When p has length 1, the statement is trivial by assumption of the theorem. Now suppose $|p| = n$. Decompose $p = q_1 \cdot q_2$ where $|q_1| = m_1 > 0$ and $|q_2| = m_2 > 0$ and $n = m_1 + m_2$. Let $f_1 = q_1(m_1 - 1) = p(m_1 - 1)$ and $f_2 = p(m_1) = q_2(0)$. Since p is a path, it follows that $f_1 \perp f_2$, and there exists $s_3 \in \text{Crown}\,(f_1) \cap \text{Crown}\,(f_2)$. By the induction hypothesis, $s_1 \sim_e s_3$ and $s_3 \sim_e s_2$, and so the transitive property gives $s_1 \sim_e s_2$. ☐

Corollary 2. *Let* $e \in \mathcal{F}$ *be such that* Halo(e) *is connected and suppose that for each* $f \in$ Halo(e)*, we have* $\phi \in C^k_{\equiv}(f)$*. Then* $\phi \in C^k_{\equiv}(e)$*.*

Proof. Let $s_1, s_2 \in$ Crown(e). Since Halo(e) is connected, there exists a path from s_1 to s_2. By Lemma 4, $s_1 \sim_e s_2$. □

Theorem 2. *Let* $\mathcal{F}$ *be a an abstract simplicial complex, and suppose that for each* $f \in \mathcal{F}$*, we have either or both of the following conditions:* Halo(f) *is connected or* $\phi \in C^k(f)$*. Then* $\phi \in C^k(\mathcal{F})$*.*

4 Vector Bundles

We describe a discrete method for representing vector bundles over a topology represented by a simplicial complex. While one could use this as a computational model for representing vector bundles, as we will see, there are still obstacles to overcome in order for it to be a practical tool – perhaps the largest being substantial non-uniqueness in representation. The main use of the material in this chapter will be as a general framework for constructing tangent bundles, which are a special case of vector bundles. The main goal will be to develop sufficient conditions for a section of a vector bundle to be continuous. In a later chapter, these conditions will be used to determine when the piecewise defined derivative of a piecewise polynomial is a continuous section of the tangent bundle.

4.1 Motivating Examples

We consider two motivating examples. The first will construct the vector bundle over the circle, and the second will construct vector bundles over the sphere. The first example conveys the basic idea, while the second exposes one of the primary obstacles to discretely representing vector bundles.

4.1.1 One Dimensional Simplicial Complexes

Consider the abstract simplicial complex which represents a circle,

$$\mathcal{F} = 2^{\{0,1,2\}} - \{0, 1, 2\}.$$

We will now construct several one dimensional vector bundles over $\Delta_{\mathcal{F}}$. We will closely follow the construction of $\Delta_{\mathcal{F}}$ from the components $\{\Delta_s\}_{s\in\mathcal{F}}$. In addition to the functions $\iota_{s_1}^{s_2}$ used in the construction of $\Delta_{\mathcal{F}}$, we also require bijective homomorphisms $g_{s_1}^{s_2} : \mathbb{R} \to \mathbb{R}$, in this case (the one dimensional case), multiplication by a non-zero real number.

For each point, 0, 1 or 2 the "linear" sections are just constant maps – the linear structure is trivial. The degrees of freedom are then three real valued constants, one for each point, which we note as c_0^0, c_1^1, c_2^2. The subscript on these degrees of freedom denotes the barycentric coordinate to which it corresponds, and the superscript denotes the simplex – of course, in the case the simplex is a singleton, the subscript is redundant. For each segment, 01, 02 and 12, the linear sections now require two degrees of freedom. We label them $c_0^{01}, c_1^{01}, c_0^{02}, c_2^{02}, c_1^{12}, c_2^{12}$, with the same convention as above. The following constraints are imposed, in order to take into account that these vector spaces have been glued together.

$$\begin{aligned}
g_0^{01} \cdot c_0^0 &= c_0^{01}, \\
g_0^{02} \cdot c_0^0 &= c_0^{02}, \\
g_1^{01} \cdot c_1^1 &= c_1^{01}, \\
g_1^{12} \cdot c_1^1 &= c_1^{12}, \\
g_2^{02} \cdot c_2^2 &= c_2^{02}, \\
g_2^{12} \cdot c_2^2 &= c_2^{12}.
\end{aligned} \tag{4.1}$$

This system can be expressed by the following matrix equation.

$$\begin{bmatrix} g_0^{01} & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & g_1^{01} & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ g_0^{02} & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & g_2^{02} & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & g_1^{02} & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & g_2^{12} & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} c_0^0 \\ c_1^1 \\ c_2^2 \\ c_0^{01} \\ c_1^{01} \\ c_0^{02} \\ c_2^{02} \\ c_1^{12} \\ c_2^{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \qquad (4.2)$$

It is easy to see that this system has full column rank, so that the null space has dimension 3. In fact, the solution of the system can be parameterized by c_0^0, c_1^1 and c_2^2, exactly by equation (4.2), as one would expect.

We now consider two specific choices for the functions $g_{s_1}^{s_2}$. First, consider the effect of setting $g_{s_1}^{s_2} = 1$ for all $s_1 \subset s_2 \in \mathcal{F}$, and let us compute the piecewise linear sections of this bundle. So we must consider the linear functions on each simplex.

Now consider the situation that presents when g_0^{01} is changed to -1. The result is a möbius band.

4.1.2 Two Dimensional Simplicial Complexes

We now move on to the case of constructing vector bundles when the base space is two-dimensional. Let $\mathcal{F}$ be the simplicial complex generated by the sets $\{0,1,2\}$, $\{0,1,3\}$, $\{0,2,3\}$ and $\{1,2,3\}$ so that $\Delta_{\mathcal{F}}$ is topologically the 2-sphere. We focus our attention on the continuity constraints at the point 0, and let us at first proceed naively, by assuming independent choices of the functions $g_{s_1}^{s_2}$. Since we are focusing on the point 0, there are 7 coefficients to track: c_0^0, c_0^{01}, c_0^{02}, c_0^{03}, c_0^{012}, c_0^{013}, c_0^{023}. The constraint system for a continuous vector bundle at the point 0 becomes:

$$\begin{bmatrix} g_0^{01} & -1 & 0 & 0 & 0 & 0 & 0 \\ g_0^{02} & 0 & -1 & 0 & 0 & 0 & 0 \\ g_0^{03} & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & g_{01}^{012} & 0 & 0 & -1 & 0 & 0 \\ 0 & g_{01}^{013} & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & g_{02}^{012} & 0 & -1 & 0 & 0 \\ 0 & 0 & g_{02}^{023} & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & g_{03}^{013} & 0 & -1 & 0 \\ 0 & 0 & 0 & g_{03}^{023} & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} c_0^0 \\ c_0^{01} \\ c_0^{02} \\ c_0^{03} \\ c_0^{012} \\ c_0^{013} \\ c_0^{023} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The situation here is tenuous, for if we do not choose $g_{s_1}^{s_2}$ well, the null space of this matrix can be trivial, forcing all continuous sections of the fiber bundle to be 0 at the point 0. To resolve this, we enforce that $g_{e_1}^f \circ g_v^{e_1} = g_{e_2}^f \circ g_v^{e_2} =: g_v^f$ for any $v \subset e_1 \subset f$ and $v \subset e_2 \subset f$. With this condition, it is straightforward to verify that the null space is spanned by the vector $\begin{bmatrix} 1 & g_0^{01} & g_0^{02} & g_0^{03} & g_0^{012} & g_0^{013} & g_0^{023} \end{bmatrix}$.

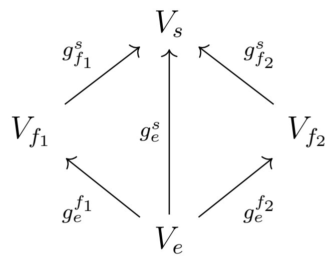
4.2 Covariant Construction

We now examine the first of two methods we will describe for constructing discrete vector bundles on a simplicial complex $\mathcal{F}$. In this first method, which we call the *covariant construction*, the central object is the covariant vector bundle morphism, which we assume for every $e \subset s \in \mathcal{F}$:

$$\begin{array}{ccc} V_e & \xrightarrow{g_e^s} & V_s \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \Delta_e & \xrightarrow{\iota} & \Delta_s \end{array} \tag{4.3}$$

We require that g_e^s be an isomorphism on each fiber. In order to make these discrete transforms, we require some of them to be homogeneous polynomials over Δ_e. Specifically, we require for every maximal $s \in \mathcal{F}$, in the sense that there is no simplex in $\mathcal{F}$ which contains s, that the maps g_e^s be polynomial. Requiring these operator valued polynomials to be isomorphisms on their domain puts nontrivial constraints on the coefficients. A sufficient condition on the coefficients, using the Bernstein polynomial basis, is that the convex space bounded by the coefficients is a subset of invertible linear maps. This has been studied in [12]. Alternatively, one could use some form of geodesic interpolation as employed in [22].

It would of course be far more simple if we just required every g_e^s to be polynomial. However, that would put severe restrictions on the admissible g's, as the following diagram would need to commute:



(4.4)

Worse, this is only the beginning – the commutative diagram would need to mimic the full simplicial complex. So instead, we require that only g_e^s, $g_{f_1}^s$, $g_{f_2}^s$ be polynomial, and define $g_e^{f_1} = (g_{f_1}^s)^{-1} \circ g_e^s$ and $g_e^{f_2} = (g_{f_2}^s)^{-1} \circ g_e^s$. While these are not polynomials, they are still vector bundle injections which are isomorphisms on the fibers.

Now we consider a polynomial section θ over the simplicial complex $\mathcal{F}$. We enforce continuity by requiring that the following diagram commutes for all $e \subset s \in \mathcal{F}$.

$$\begin{array}{ccc} V_e & \xrightarrow{g_e^s} & V_s \\ {\scriptstyle\theta_e}\uparrow & & \uparrow{\scriptstyle\theta_s} \\ \Delta_e & \xrightarrow{\iota} & \Delta_s \end{array} \tag{4.5}$$

We can find independent degrees of freedom by decomposing the section θ into "bump" functions. That is, functions $\mathring{\theta}_s : \Delta_s \to V_s$ such that for any proper subset $e \subset s$, we have $\theta_s \circ \iota_e^s = 0$. We will also make use of an extension operator E_e^s with the following properties:

1. For any $f : \Delta_e \to X$, $E_e^s(f) : \Delta_s \to X$

2. $E_e^s(0) = 0$

3. $E_e^{s_2}(f) \circ \iota_{s_1}^{s_2} = E_{e\cap s_1}^{s_1}(f \circ \iota_{e\cap s_1}^{e})$.

4. For any $f : \Delta_e \to X$ and $g : X \to Y$, $E_e^s(g \circ f) = g \circ E_e^s(f)$

With these definitions, we can use choices of bump functions as the degrees of freedom for polynomial section θ by defining the component of $s \in \mathcal{F}$ as:

$$\theta_s = \sum_{e \subset s} E_e^s(g_e^s \circ \mathring{\theta}_e). \tag{4.6}$$

Which satisfies (4.5) via:

$$\begin{aligned}
\theta_{s_2} \circ \iota_{s_1}^{s_2} &= \sum_{e \subset s_2} E_e^{s_2}(g_e^{s_2} \circ \mathring{\theta}_e) \circ \iota_{s_1}^{s_2} \\
&= \sum_{e \subset s_2} E_{e\cap s_1}^{s_1}(g_e^{s_2} \circ \mathring{\theta}_e \circ \iota_{e\cap s_1}^{e}) \\
&= \sum_{e \subset s_1} E_e^{s_1}(g_e^{s_2} \circ \mathring{\theta}_e) \\
&= \sum_{e \subset s_1} g_{s_1}^{s_2} \circ E_e^{s_1}(g_e^{s_1} \circ \mathring{\theta}_e) \\
&= g_{s_1}^{s_2} \circ \theta_{s_1}.
\end{aligned} \tag{4.7}$$

Notice that, assuming the transition functions g are not constant, but are polynomial, that the degree of the polynomials as we move up the simplicial complex will necessarily increase. The degree of polynomial the user wishes to have at each level, then, determines polynomial degree of each $\mathring{\theta}_s$. That then decides the degrees of freedom in terms of multi-indexes.

4.3 Contravariant Construction

We will now perform a similar analysis to that done in §4.2 when the central diagram of (4.3) is replaced with (4.8).

$$\begin{array}{ccc} V_e & \xleftarrow{g_s^e} & V_s \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\ \Delta_e & \xrightarrow{\iota} & \Delta_s \end{array} \tag{4.8}$$

The maps g are again isomorphisms, homogeneous polynomials when $s \in \mathcal{F}$ is a maximal simplex, and the same comments regarding polynomial isomorphisms as were made in §4.2 apply here as well. It must be pointed out that we are taking some liberties with this diagram. The map g_e^s here is a partial function, defined only on $V_s\big|_e := \pi^{-1} \circ \iota(\Delta_e)$. A more complete diagram is the following.

$$\begin{array}{ccccc} V_e & \xleftarrow{g_s^e} & V_s\big|_e & \hookrightarrow & V_s \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\ \Delta_e & \xrightarrow{\iota} & \Delta_s\big|_e & \hookrightarrow & \Delta_s \end{array} \tag{4.9}$$

We now continue to examine sections of these vector bundles.

$$\begin{array}{ccc} V_e & \xleftarrow{g_e^s} & V_s \\ {\scriptstyle \theta_e}\Big\uparrow & & {\scriptstyle \theta_s}\Big\uparrow \\ \Delta_e & \xrightarrow{\iota} & \Delta_s \end{array} \tag{4.10}$$

The full diagram is:

$$\begin{array}{ccccc} V_e & \xleftarrow{g_s^e} & V_s\big|_e & \hookrightarrow & V_s \\ {\scriptstyle \theta_e}\Big\uparrow & & {\scriptstyle \theta_s}\Big\uparrow & & {\scriptstyle \theta_s}\Big\uparrow \\ \Delta_e & \xrightarrow{\iota} & \Delta_s\big|_e & \hookrightarrow & \Delta_s \end{array} \tag{4.11}$$

By following the diagram, we now are able to fully define θ_e from θ_s (or

$\theta_s\big|_e$), instead of only partially as in the covariant case.

$$\theta_e = g_s^e \circ \theta_s \circ \iota_e^s \tag{4.12}$$

But, perhaps unexpectedly, this feature makes the contravariant system more difficult than the covariant one. This is because, given two simplices which contain $e,\ s_1, s_2 \in \mathcal{F}$, we must satisfy both of the following equations.

$$\begin{aligned} \theta_e &= g_{s_1}^e \circ \theta_{s_1} \circ \iota_e^{s_1}, \\ \theta_e &= g_{s_2}^e \circ \theta_{s_2} \circ \iota_e^{s_2}. \end{aligned} \tag{4.13}$$

Therefore θ_e must be in the range of both $g_{s_1}^e$ and $g_{s_2}^e$. We do not yet have a general method for solving these systems, but present the following example to indicate hope for the development of such a method.

Example 5. *Let $\mathcal{F}$ be the simplicial complex generated by $\{0,1,2\}$ and $\{1,2,3\}$ - two triangles sharing a common edge.*

$$\begin{aligned} g_{012}^{01} &= A_0 x_0 + A_1 x_1 \\ g_{013}^{01} &= B_0 x_0 + B_1 x_1 \end{aligned} \tag{4.14}$$

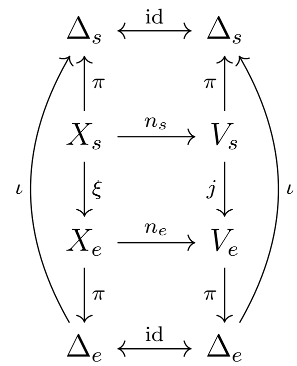
$$\begin{aligned} g_{012}^{01} \circ \theta_{012} \circ \iota_{01}^{012} &= g_{013}^{01} \circ \theta_{013} \circ \iota_{01}^{013} \\ g_{012}^{01} \circ \theta_{012} \circ \iota_{01}^{012} - g_{013}^{01} \circ \theta_{013} \circ \iota_{01}^{013} &= 0 \end{aligned} \tag{4.15}$$

$$\begin{bmatrix} A_0 & 0 & -B_0 & 0 \\ A_1 & A_0 & -B_1 & -B_0 \\ 0 & A_1 & 0 & -B_1 \end{bmatrix} \begin{bmatrix} \theta_{012}^0 \\ \theta_{012}^1 \\ \theta_{013}^0 \\ \theta_{013}^1 \end{bmatrix} = 0 \tag{4.16}$$

This system must have a non-trivial solution, implying the existence of θ_{012} and θ_{013} which solves (4.15)*.*

4.4 Quotient Construction

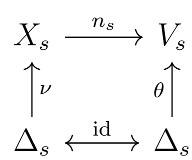
We now present a potentially powerful way to represent a vector bundle through a quotient map. The central object in this case is the following diagram.



(4.17)

In (4.17), n_s, n_e, ξ and j are injections. The map j will be particularly special, as we will assume it is constant over Δ_s. Ideally, the vector spaces $V_s \subset V_e$ will have enough structure that the map j will be a natural choice, as it will be when we make us of this structure in the next chapter. We are taking the same liberties with ξ and j as described in the previous section in that they are only partial functions, defined only on those fibers associated with Δ_e, however, a complete diagram in this case would be overly cumbersome.

Let us now break down the sections of (4.17). Below is the diagram of the bundle over a simplex $s \in \mathcal{F}$.



(4.18)

We will consider the sections of this bundle to be the functions θ_s such that $\pi \circ \theta_s = \text{id}$ with the equivalence relationship that $\theta_1 \sim \theta_2$ if and only if $\theta_1 - \theta_2 = n_s \circ \nu$ for some $\nu : \Delta_s \to X_s$. This is the quotient bundle $\frac{V_s}{n_s(X_s)}$.

The diagram which describes the sections of (4.17) is diagram (4.19), below.

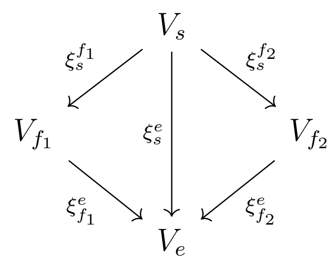
$$\begin{tikzcd}
\Delta_s \arrow[r, leftrightarrow, "\mathrm{id}"] \arrow[d, "\nu_s"] & \Delta_s \arrow[d, "\theta_s"'] \\
X_s \arrow[r, "n_s"] \arrow[d, "\xi"] & V_s \arrow[d, "j"'] \\
X_e \arrow[r, "n_e"] & V_e \\
\Delta_e \arrow[r, leftrightarrow, "\mathrm{id}"] \arrow[u, "\nu_e"] \arrow[uuu, bend left=40, "\iota"] & \Delta_e \arrow[u, "\theta_e"'] \arrow[uuu, bend right=40, "\iota"']
\end{tikzcd} \tag{4.19}$$

Due to the number of components of this diagram, it is convenient to further simplifying by treating the base spaces as implied, as in:

$$\begin{array}{ccc} X_s & \xrightarrow{n_s} & V_s \\ \downarrow{\scriptstyle \xi} & & {\scriptstyle j}\downarrow \\ X_e & \xrightarrow{n_e} & V_e \end{array} \tag{4.20}$$

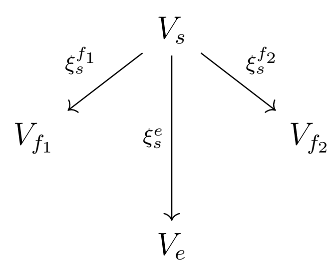
Due to the simplicity of j, we can easily maintain any structure which must be preserved. This is important constructing tangent bundles, as the relationship between vectors in the tangent space and directions in the manifold must be maintained. This would be a primary concern if we used the covariant or contravariant constructions to model a tangent bundle. But with a proper choice of j, the quotient construction makes this a non-issue.

However, this does not come for free. The first obstacle which must be overcome is that the map between quotient spaces induced by j and ξ is not necessarily an isomorphism of the quotient spaces. This issue we overcome in the next chapter by introducing sufficient conditions in the particular instance of tangent bundles, but do not yet have a general set of necessary and sufficient conditions on the transition maps j, ξ, n_s and n_e to guarantee this. The second issue concerns the existence of the maps ξ. This issue is similar to the one described in 4.4. In 4.21, we produce a segment of the complete quotient diagram which illustrates the issue with finding appropriate ξ's.



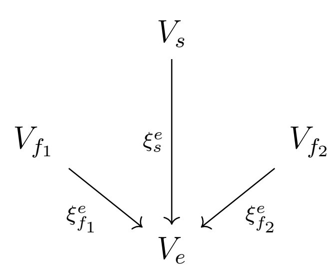
(4.21)

If we were to follow the solution we employed in the covariant and contra-variant constructions, we would define only the functions



(4.22)

and use those functions to complete diagram (4.21). The difficulty here is that each ξ is an injection, not an isomorphism. An injection has a well defined left inverse, but we need a right inverse to complete this diagram. Another possible path to a solution is to instead define:



(4.23)

But this too has a difficulty. Since each ξ in the diagram is a partial function, the diagram can only be completed over Δ_e. Whether an extension to Δ_{f_1} and Δ_{f_2} is unclear.

We now move on to constructing and using discrete differential structures.

5 Tangent Bundles

We use the work of the previous chapter to construct tangent bundles of a simplicial complex. This is then used to construct conditions first for continuous covector fields, and ultimately to construct differentiable functions. Provided we are able to obtain an ample set of differentiable functions, we can then define a C^0 metric on the simplicial complex.

5.1 Tangent Bundles of Simplices

In this section, we describe the tangent bundles of individual simplices. This first definition mirrors the standard definition of the tangent plane at a point of a manifold. For $s \in \mathcal{F}$ and $p \in \Delta_s$, let

$$T_p\Delta_s = \left\{ v \in \mathbb{R}^s \;\middle|\; \sum_{i \in s} v_i = 0 \right\}. \tag{5.1}$$

Continuing with the classical procedure, we then define, for $e \subset s$, $p \in \Delta_e \subset \Delta_s$.

$$\begin{aligned} T_e\Delta_s &= \bigcup_{p \in \Delta_e} T_p\Delta_s \\ &= \left\{ (x, v) \in \mathbb{R}^e \times \mathbb{R}^s \;\middle|\; x \geq 0,\ \sum_{i \in e} x_i = 1,\ \sum_{i \in s} v_i = 0 \right\}, \end{aligned} \tag{5.2}$$

with the convention that $T\Delta_s := T_s\Delta_s$. This is isomorphic to $\Delta_s \times \mathbb{R}^{\dim s}$, and it is that topology we place on $T_e\Delta_s$. This means that the usual notion of differentiability remains the same on each simplex. Of particular importance to us, is that polynomials of the barycentric coordinates are differentiable on Δ_s.

Strictly speaking Δ_s is, as defined, not a manifold, but a manifold with boundary. As we have defined $T\Delta_s$ above, it has vectors which point into the boundary. These are convenient in the construction of the discrete differential structure, but we shall also require the "correct" tangent space - one which restricts the allowed vectors to those that point into the simplex.

$$T_e^+\Delta_s = \left\{ (x, v) \in \mathbb{R}^e \times \mathbb{R}^s \;\middle|\; x > 0,\ v_{s-e} \geq 0,\ \sum_{i\in e} x_i = 1,\ \sum_{i\in s} v_i = 0 \right\}, \tag{5.3}$$

with the notation that $T^+\Delta_s = \bigcup_{e\subset s} T_e^+\Delta_s$

The last group of vector spaces we make use of may seem the most obscure, as it is not the tangent space of a simplex of $\mathcal{F}$. In the following sections, we will describe a structure used to glue together the vector spaces $\{T_e\Delta_s\}_{s\in\mathrm{Halo}(e)}$. But in order to do this, we will need a vector space which contains each of $\{T_e\Delta_s\}_{s\in\mathrm{Halo}(e)}$. That space is:

$$T_e\Delta_{s^\oplus} = \left\{ (x, v) \in \mathbb{R}^e \times \mathbb{R}^{s^\oplus} \;\middle|\; x \geq 0,\ \sum_{i\in e} x_i = 1,\ \sum_{i\in s^\oplus} v_i = 0 \right\} \tag{5.4}$$

With the required vector spaces now defined, we shall now describe an object which will allow us to glue them together in such a way that the positive cones are disjoint yet spanning.

5.2 Positive Simplicial Bases

In the previous section, we described the pieces. In this section, we describe the glue.

Definition 28. *Let $v_1, \ldots, v_n \in \mathbb{R}^m$. The set $V = [v_1, \ldots, v_n]$ is called a* ***positive frame*** *if for every $x \in \mathbb{R}^m$ there exists $\alpha \in (\mathbb{R}^+)^m$ such that $x = \sum_{i=1}^{n} \alpha_i v_i$.*

Definition 29. *Let $V = [v_1, \ldots, v_n]$ be a positive frame and $\mathcal{F}$ an abstract simplicial complex of the set $\{1, \ldots, n\}$. The pair $(V, \mathcal{F})$ is called a* ***positive simplicial***

basis *if for every* $x \in \mathbb{R}^m$ *there exists a unique* $\alpha \in (\mathbb{R}^+)^n$ *with* $\operatorname{supp}\alpha \in \mathcal{F} \cup \varnothing$ *such that* $x = \sum_{i=1}^{n} \alpha_i v_i$.

This conjecture is important because it guarantees the existence of positive simplicial bases we will need. If $\mathcal{F}$ is locally representable, then for each $s \in \mathcal{F}$, $\Delta_{\operatorname{Link} s}$ is homeomorphic to the $(m - |s|)$-sphere. And assuming the conjecture below is true, they admit a positive simplical basis of $\mathbb{R}^{m-\dim s}$). We will need this in the next section.

Conjecture 3. *The abstract simplicial complexes which admit a positive simplicial basis of* $\mathbb{R}^m$ *are homeomorphic to the* $(m-1)$*-sphere.*

Lemma 5. *Let* $m, n, p \in \mathbb{N}$ *and let* $V_1 \in \mathbb{R}^{m,m}$, $V_2 : \mathbb{R}^{n,m}$, $N_1 : \mathbb{R}^{p,m}$ *and* $N_2 : \mathbb{R}^{p,n}$. *Suppose* $\ker \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \operatorname{range} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$ *and* $\ker N_2^T = 0_n$. *Then* $\operatorname{range} V_1 \supset \operatorname{range} V_2$. *Furthermore, if* $\begin{bmatrix} V_1 & V_2 \end{bmatrix}^T$ *is a basis which completes the vector space spanned by* $\begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$, *then* V_1 *is invertible.*

Proof. Since $\ker N^T = 0$, then $\operatorname{range} N_2 = \mathbb{R}^n$. $\ker \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \operatorname{range} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$ implies that for every $y \in \mathbb{R}^n$, there exists $x \in \mathbb{R}^m$ such that $V_1 x + V_2 y = 0$. Thus $\operatorname{range} V_2 \subset \operatorname{range} V_1 = \operatorname{range} \begin{bmatrix} V_1 & V_2 \end{bmatrix}$. If $\begin{bmatrix} V_1 & V_2 \end{bmatrix}$ spans $\mathbb{R}^m$, then $\operatorname{range} V_1 = \mathbb{R}^m$. □

Definition 30. *Let* X, Y *be vector spaces such that* $(V, \mathcal{F}$ *be a positive simplicial basis of a vector space* X, Y *a vector space with* $f : \mathbb{R}^m \to V$. f *is called* ***piecewise linear relative to*** $\mathcal{F}$ *if for every* $s \in \mathcal{F}$:

$$f\left(\sum_{i\in s} \alpha^i v_i\right) = \sum_{i\in s} \alpha^i f(v_i)$$

Lemma 6. *Let* $v_1, \ldots, v_n$ *and* $\mathcal{F}$ *be a positive simplicial basis of* $\mathbb{R}^m$, V *a vector space and* $f : \mathbb{R}^m \to V$ *piecewise linear relative to* $\mathcal{F}$. *Let* N *be a matrix whose* *columns span the null space of* $v_1, \ldots, v_n$*. Further, suppose that:*

$$\begin{bmatrix} f(v_1) & \ldots & f(v_n) \end{bmatrix} N = 0$$

Then f *is linear.*

Proof. Part 1. We show that for every $x_1, x_2 \in \mathbb{R}^m$, $f(-x_1)+f(-x_2)+f(x_1+x_2) = 0$.

We first use the definition of a positive frame to express:

$$-x_1 = \sum_{i \in s_1} \alpha_i^1 v_i \tag{5.5}$$

$$-x_2 = \sum_{i \in s_2} \alpha_i^2 v_i \tag{5.6}$$

$$x_1 + x_2 = \sum_{i \in s_{12}} \alpha_i^{12} v_i \tag{5.7}$$

Since $x_1 + x_2 + (-x_1) + (-x_2) = \sum_{i \in s_1} \alpha_i^1 v_i + \sum_{i \in s_2} \alpha_i^2 v_i + \sum_{i \in s_{12}} \alpha_i^{12} v_i = 0$, this set of coefficients is in the span of N. By assumptions on f, we have:

$$f(-x_1) + f(-x_2) + f(x_1 + x_2) = \sum_{i \in s_1} \alpha_i^1 f(v_i) + \sum_{i \in s_2} \alpha_i^2 f(v_i) + \sum_{i \in s_{12}} \alpha_i^{12} f(v_i) \tag{5.8}$$

$$= 0 \tag{5.9}$$

Part 2. We now repeatedly apply Part 1.

Let $x \in \mathbb{R}^m$. Then by choosing $x_1 = x$ and $x_2 = -x$, by part 1, we have

$$f(-x) = -f(x).$$

This immediately implies that for any $x_1, x_2 \in \mathbb{R}^m$ we have

$$f(x_1 + x_2) = f(x_1) + f(x_2).$$

In addition, for any $\beta \in \mathbb{R}$:

$$f(\beta x) = f(\operatorname{sgn}\beta|\beta|x) \tag{5.10}$$
$$= \operatorname{sgn}\beta f(|\beta|x) \tag{5.11}$$
$$= \operatorname{sgn}\beta|\beta|f(x) \tag{5.12}$$
$$= \beta f(x) \tag{5.13}$$

Thus, for $\beta_1, \beta_2 \in \mathbb{R}$:

$$f(\beta_1 x_1 + \beta_2 x_2) = \beta_1 f(x_1) + \beta_2 f(x_2).$$

□

The converse of this theorem is trivial. This therefore provides necessary and sufficient conditions for a co-vector field f to act linearly on the seams of a mesh, and allows us to represent the basis vectors v_i in the coordinate system used for f - barycentric coordinates. More to the point, we do not need a set of overlapping coordinate systems, we just need N (at each point) to glue disjoint systems together. To further emphasize the importance of this lemma, suppose that ω, v and N are polynomial. Then the expression $w(v)N$ is also polynomial, which means that enforcing the condition that $w(v)N = 0$ is a finite dimensional constraint.

Lemma 7. *Let* $m, n, p \in \mathbb{N}$ *and let* $V_1 \in \mathbb{R}^{m,m}$, $V_2 : \mathbb{R}^{n,m}$, $N_1 : \mathbb{R}^{p,m}$ *and* $N_2 : \mathbb{R}^{p,n}$. *Suppose* $\ker \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \operatorname{range} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$ *and* $\ker N_2^T = 0_n$. *Then* $\operatorname{range} V_1 \supset \operatorname{range} V_2$. *Furthermore, if* $\begin{bmatrix} V_1 & V_2 \end{bmatrix}^T$ *is a basis which completes the vector space spanned by* $\begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$, *then* V_1 *is invertible.*

Proof. Since $\ker N^T = 0$, then $\operatorname{range} N_2 = \mathbb{R}^n$. $\ker \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \operatorname{range} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$ implies that for every $y \in \mathbb{R}^n$, there exists $x \in \mathbb{R}^m$ such that $V_1 x + V_2 y = 0$. Thus range $V_2 \subset$ range $V_1 =$ range $\begin{bmatrix} V_1 & V_2 \end{bmatrix}$. If $\begin{bmatrix} V_1 & V_2 \end{bmatrix}$ spans $\mathbb{R}^m$, then range $V_1 = \mathbb{R}^m$. □

5.3 Discrete Differential Structures

We now pull the previous two sections together. Our aim is to use the positive simplicial basis of §5.2 (more accurately, their dual orthogonal complements) to glue together the tangent spaces of the simplices as described in section §5.1. We will first describe the mathematical structure of a discrete differential structure proposed by this dissertation. This structure will define a subspace of $T_s\Delta_{s^\oplus}$ which is to be considered $T_s\Delta_{\mathcal{F}}$. Once this is defined, we can determine the linear functionals on $T_s\Delta_{s^\oplus}$ which are to be considered $T_s^*\Delta_{\mathcal{F}}$. Moreover, we will produce a linear system the solution of which gives the polynomial elements of $T_s^*\Delta_{\mathcal{F}}$.

5.3.1 Construction

For $s \in \mathcal{F}$, let Y_s be the trivial bundle of $\mathbb{R}^{s^+}$ over Δ_s, and let X_s be the trivial bundle of $\mathbb{R}^{m_s}$ (for $m_s = \dim(s) + |s^+| - n$) over Δ_s, and recall that $T_s\Delta_{s^\oplus}$ is a trivial vector bundle over Δ_s. We now consider two continuous injective vector bundle morphisms

$$\begin{aligned} v_s &: Y_s \to T_s\Delta_{s^\oplus} \\ n_s &: X_s \to Y_s. \end{aligned} \tag{5.14}$$

Continuity implies that for any continuous section $\xi \in Y_s$ we have that $v_s(\xi)$ is a continuous section of $T_s\Delta_{s\cup s^+}$. Since v_s is a vector bundle morphism, the base point is unchanged through the map:

$$\pi \circ v_s = \pi.$$

Furthermore, the structure of the vector space fibers is also preserved by the map v_s - specifically,

$$v_s(\alpha_1\xi_1 + \alpha_2\xi_2) = \alpha_1 v_s(\xi_1) + \alpha_2 v_s(\xi_2)$$

for $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\xi_1, \xi_2 \in Y_s$. These points apply also to n_s.

There are additional properties we require of v_s and n_s. Of v_s, we require that $v_s(e_k) \in T_s^+\Delta_{s\cup k} \subset T_s\Delta_{s^\oplus}$ for each $k \in s^+$. This is to say that v_s acts as a positive diagonal operator on the elements of s^+ (but not on the elements of s). Of n_s, we require that it is a dual simplicial positive basis.

With these definitions in place, we can now construct a discrete differential structure. In order to do so, however, we must first restrict to the case where v_s and n_s are not just continuous but polynomial in the barycentric coordinates of Δ_s. Note that this adds a substantial difficulty to the requirement that the maps be injective at all points. Sufficient conditions for this can be found in [12]. Alternatively, one could use geodesic averages, as done in [22]. In order to produce a robust solution, it is important this be ensured.

A Discrete Differential Structure on a simplicial complex $\mathcal{F}$ is then the pairs (v_s, n_s) for each $s \in \mathcal{F}$. That is, we have the following short chain (not exact) of injective bundle morphisms for each $s \in \mathcal{F}$:

$$X_s \xrightarrow{n_s} Y_s \xrightarrow{v_s} T_s\Delta_{s^\oplus} \tag{5.15}$$

We can therefore decompose $T_s\Delta_{s\cup s^+}$ as:

$$T_s\Delta_{s^\oplus} = \frac{T_s\Delta_{s^\oplus}}{v_s(Y_s)} \oplus \frac{v_s(Y_s)}{(v_s \circ n_s)(X_s)} \oplus (v_s \circ n_s)(X_s), \tag{5.16}$$

And note the dimensions of the decomposition.

$$\begin{aligned} \dim\left(\frac{T_s\Delta_{s^\oplus}}{n_s(Y_s)}\right) &= |s| + |s^+| - 1 - |s^+| = \dim(s) \\ \dim\left(\frac{v_s(Y_s)}{(v_s \circ n_s)(X_s)}\right) &= |s^+| - (\dim(s) + |s^+| - n) = n - \dim(s) \end{aligned} \tag{5.17}$$

This suggests that $\frac{T_s\Delta_{s^\oplus}}{v_s(Y_s)}$ represents the tangent component and $\frac{v_s(Y_s)}{(v_s \circ n_s)(X_s)}$ represents the normal component.

5.3.2 Covector Fields

We consider here two critical aspects of covector fields: linearity and continuity. The first, that the covector field act as a linear function on the tangent space at every point is a requirement by definition. The second, continuity of the field itself, is important to be able to speak of differentiable scalar functions.

Linearity

A covector field in the traditional FEM/FEEC sense, translates into the structures we have developed so far as a linear function of the form:

$$\omega_s : T_s\Delta_s \to \mathbb{R}. \tag{5.18}$$

In §5.3.1, we extended vector fields into the surrounding simplices of $\mathcal{F}$, and so we do the same to covector fields. We therefore consider covectors to be linear functions of the form:

$$\omega_s : T_s\Delta_{s^\oplus} \to \mathbb{R}. \tag{5.19}$$

However, in order for these to be linear on $T_s\Delta_{\mathcal{F}} = \frac{T_s\Delta_{s^\oplus}}{(v_s \circ n_s)(X_s)}$, we enforce that the following sequence be exact:

$$X_s \xrightarrow{v_s \circ n_s} T_s\Delta_{s^\oplus} \xrightarrow{\omega_s} \mathbb{R} \tag{5.20}$$

This gives us the condition,

$$(\omega_s \circ v_s) \circ n_s = 0 \tag{5.21}$$

which is, of course, the equivalent of Lemma 6 in this context.

Continuity

Let us begin by reviewing how continuity of covector fields is defined in the FEM setting, in the notation we have developed. In the FEM setting, we consider only $T_s\Delta_s$ as the tangent space of a simplex. And, given two simplices $e \subset s \in \mathcal{F}$, we have the following bundle morphism.

$$\begin{array}{ccc} T_e\Delta_e & \xrightarrow{d\iota_e^s} & T_s\Delta_s \\ \big\downarrow{\scriptstyle\pi} & & \big\downarrow{\scriptstyle\pi} \\ \Delta_e & \xrightarrow{\iota_e^s} & \Delta_s \end{array} \tag{5.22}$$

The natural condition to enforce, and indeed the condition which is enforced, is:

$$(d\iota_e^s)^* w_s = w_e.$$

This is "trace" continuity as described in [1], and is the result of pulling back the covector from Δ_s to Δ_e through the natural injection.

The situation now has a slight twist, as seen by the bundle morphism diagram:

$$\begin{array}{ccc} T_e\Delta_{e\oplus} & \xleftarrow[j_s^e]{} & T_s\Delta_{s\oplus} \\ \big\downarrow{\scriptstyle\pi} & & \big\downarrow{\scriptstyle\pi} \\ \Delta_e & \xrightarrow{\iota_e^s} & \Delta_s \end{array} \tag{5.23}$$

The inclusion of the bundle is now contravariant. It is tempting to define continuity by simply pulling back the vector from e to s,

$$\omega_s\big|_{\Delta_e} = \omega_e \circ j_s^e. \tag{5.24}$$

The problem with doing so, however, is that this pulled back form may no longer be linear as defined by (5.21). The trouble is not with the form, it is with the differential structure.

The way forward is to add additional constraints to the discrete differential structure. We call a discrete differential structure **continuous** if there exist injective maps ξ and η so that the following diagram commutes:

$$\begin{array}{ccccc} X_s & \xrightarrow{n_s} & Y_s & \xrightarrow{v_s} & T_s\Delta_{s\oplus} \\ \big\downarrow{\scriptstyle\xi} & & \big\downarrow{\scriptstyle\eta} & & \big\downarrow{\scriptstyle j} \\ X_e & \xrightarrow{n_e} & Y_e & \xrightarrow{v_e} & T_e\Delta_{e\oplus} \end{array} \tag{5.25}$$

With this structure in place, we can now see that equation (5.24) yields a linear covector field provided that w_e is linear:

$$\begin{aligned} \omega_s\big|_{\Delta_e} \circ (v_s \circ n_s) &= (\omega_e \circ j_s^e) \circ (v_s \circ n_s) \\ &= \omega_e \circ (j_s^e \circ v_s \circ n_s) \\ &= \omega_e \circ (v_e \circ n_e \circ \xi) \\ &= 0. \end{aligned} \tag{5.26}$$

Note that ξ and η need not be polynomial.

We end this section exploring some implications the structure of n_s, n_e, v_s and v_e have on ξ and η. First, we must have that $(v_e \circ \eta)_{e^\oplus \setminus s^\oplus} = 0$, and since v_e is diagonal on $e^+ \supset e^\oplus \setminus s^\oplus$, we have that $\eta = 0$ on $e^\oplus \setminus s^\oplus$. It immediately follows that $\mathrm{Im}(\xi) \subset \ker(n_e\big|_{e^\oplus \setminus s^\oplus})$. What is perhaps less obvious, is that it is indeed the case that $\mathrm{Im}(\xi) = \ker(n_e\big|_{e^\oplus \setminus s^\oplus})$.

We can see this by considering the following decompositions of X_e.

$$\begin{aligned} X_e &= \ker(n_e\big|_{e^\oplus \setminus s^\oplus}) \oplus \mathrm{Im}(n_e\big|_{e^\oplus \setminus s^\oplus}{}^*) \\ &= \mathrm{Im}(\xi) \oplus \Big[\mathrm{Im}(\xi) \setminus \ker(n_e\big|_{e^\oplus \setminus s^\oplus})\Big] \oplus \mathrm{Im}(n_e\big|_{e^\oplus \setminus s^\oplus}{}^*). \end{aligned} \tag{5.27}$$

As ξ is injective, $\mathrm{Im}(\xi)$ has dimension equivalent to the dimension of X_s, $\dim(s) + |s^+| - n$. Since the rows of n_e have a matroid structure which is dual to $\mathcal{F}$, $n_e\big|_{e^\oplus \setminus s^\oplus}{}^*$ is also injective, so the image has dimension $(\dim(e) + |e^+|) - (\dim(e) + |s^+|)$. This then shows that $\mathrm{Im}(\xi) \setminus \ker(n_e\big|_{e^\oplus \setminus s^\oplus})$ has dimension 0. We conclude that $\mathrm{Im}(\xi) = \ker(n_e\big|_{e^\oplus \setminus s^\oplus})$ and we can complete 5.27 with:

$$X_e = \mathrm{Im}(\xi) \oplus \mathrm{Im}(n_e\big|_{e^\oplus \setminus s^\oplus}{}^*) \tag{5.28}$$

5.3.3 Normal and Tangent Decomposition

In §5.3.1, we noted that the dimensions of the decomposition of $T_s\Delta_{s^\oplus}$ implied by a discrete differential structure,

$$T_s\Delta_{s^\oplus} = \frac{T_s\Delta_{s^\oplus}}{v_s(Y_s)} \oplus \frac{v_s(Y_s)}{(v_s \circ n_s)(X_s)} \oplus (v_s \circ n_s)(X_s), \tag{5.29}$$

suggested that $\frac{T_s\Delta_{s^\oplus}}{v_s(Y_s)}$ represents the tangent component and $\frac{v_s(Y_s)}{(v_s\circ n_s)(X_s)}$ represents the normal component. We now explore that further by looking at the quotient maps induced by j.

Lemma 8 (Tangent Injection)**.** *The map $j : T_s\Delta_{s^\oplus} \to T_e\Delta_{e\cup e^+}$ induces a well defined injection $\phi : \frac{T_e\Delta_{e^\oplus}}{v_e(Y_e)} \to \frac{T_s\Delta_{s^\oplus}}{v_s(Y_s)}$.*

Proof. Let $u_e \in T_e\Delta_{e^\oplus}$, and let $u + v_s(y)$ such that $u \in T_e\Delta_e$. This u must exist because v_s acts diagonally on s^+, so a unique $y \in Y_s$ can always be chosen so that $u + v_s(y)$ is zero on the s^+ coordinates, implying that u is also unique. Then u is in the range of j, so there is a unique $u_s \in T_s\Delta_{s^\oplus}$ such that $u = j(u_s)$ (in fact it is just u_e without the components of $e^\oplus \setminus s^\oplus$, which are all zero). Define $\phi(u_e) = u_s$. Since u_s is unique as above, ϕ is well defined. Now suppose $\phi(u_e) = u_s = v_s(y_s)$. Then $u_e = j(u_s) = (j \circ v_s)(y_s) = (v_e \circ \eta)(y_s) = v_e(\eta(y_s))$. Therefore ϕ is an injection. □

Lemma 9 (Normal Injection)**.** *The map $j : T_s\Delta_{s^\oplus} \to T_e\Delta_{e^\oplus}$ induces a well defined injection $\phi : \frac{v_s(Y_s)}{(v_s\circ n_s)(X_s)} \to \frac{v_e(Y_e)}{(v_e\circ n_e)(X_e)}$.*

Proof. Let $y \in Y_s$ so that $v_s(y)$ is a representative element of $\frac{v_s(Y_s)}{(v_s\circ n_s)(X_s)}$, and define $\phi(v_s(y)) = (j \circ v_s)(y)$. We see that ϕ is well defined since $\phi(v_s \circ n_s) = j \circ v_s \circ n_s = (v_s \circ n_s)(\eta(y))$. Now suppose $\eta(y) = n_e(x_e)$. Since $\eta(y) = 0$ on $e^\oplus \setminus s^\oplus$, we have that $n_e^{e^\oplus \setminus s^\oplus}(x_e) = 0$. This implies that x_e is in the image of ξ, which gives us that $\eta(y) = (n_e \circ \xi)(x_s) = (\eta \circ n_s)(x_s)$, for some $x_s \in X_s$. Therefore ϕ is an injection. □

This shows that even though the bundle morphism in (5.23) is contravariant, the covariance of (5.22) is recovered on the classical (tangent) components of the covector field. Furthermore, notice that the proof of the "Tangent Injection" makes no use of the dimension of X_s, or of ξ being an injection. In fact, it only depends on the right hand side of diagram (5.15). This suggests that if we were to relax those conditions, and allow differential structure which are somehow less regular, we will still maintain the ability to enforce trace continuity. On the other hand, we will lose our ability to have a strict "Normal Injection". Upon reflection, this is an obvious fact, and is the only way this could work.

5.4 Differentiable Scalar Functions

In this section, we combine the notions of continuous discrete differential structures with the differentials of scalar valued functions to produce conditions for continuously differentiable functions. We view scalar valued functions as continuous sections of a trivial one dimensional vector bundle over $\Delta_{\mathcal{F}}$ (see §4.2), and while we restrict attention to this case, generalizing the following analysis to arbitrary covariant vector bundles ought to be natural.

Let $e \in \mathcal{F}$ and let $\theta : \Delta_{\mathcal{F}} \to X$ be a function of the form 4.6. Consider the task of determining the derivative on the interior of Δ_e, denoted by $\mathring{\Delta}_e$. In the direction which is tangent to Δ_e, there is no difficulty, it is simply the derivative of a homogeneous polynomial within a simplex. However, in the direction normal to Δ_s within $\Delta_{\mathcal{F}}$, there is not a single simplex in which to take the derivative, it must be taken across simplices. One nice approach to this problem, similar to the approach taken in §3, is to embed this into a higher dimensional space. There, we took a rather extreme and global approach, and embedded in $\mathbb{R}^N$, where N is the number of vertices. Instead, here we will be as local as possilbe by using the natural embedding of $\Delta_e \subset \Delta_{e^\oplus}$ and natural extension of θ to $\Delta_{e^\oplus}$.

$$\begin{array}{ccc} \Delta_{\mathcal{F}} & & \\ \uparrow & \overset{\theta}{\searrow} & \\ \Delta_s & \longrightarrow & V \\ \downarrow & \underset{E(\theta)}{\nearrow} & \\ \Delta_{e^\oplus} & & \end{array} \tag{5.30}$$

This is where we break with the generalized approach to vector bundles - we are ignoring that there is a transition map between the fibers over each Δ_s.

This extended function $E(\theta)$ then has a natural derivative on $e^\oplus$ which is completely determined by θ, this is a linear function $T_{e^\oplus}\Delta_{e^\oplus} \to \mathbb{R}$, or an element of $T^*_{e^\oplus}\Delta_{e^\oplus}$. By restricting this to $T_e\Delta_{e^\oplus}$, we can apply the techniques of §5.3. We take this restricted map as the derivative of θ at e, denoted by $\nabla\theta\big|_e \in T^*_e\Delta_{e^\oplus}$.

We have already performed this calculation in concrete terms in Chapter 3, but we perform it here again, with this abstraction in mind.

For $s \in \mathcal{F}$ and multi-index α with $\mathrm{supp}(\alpha) = s$:

$$\nabla\mathcal{B}^\alpha = r \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ |\alpha_2|=1}} \mathcal{B}^{\alpha_1}(d\mathcal{B})^{\alpha_2}. \tag{5.31}$$

Restricting this to some $e \in \mathcal{F}$ yields:

$$\nabla\mathcal{B}^\alpha\big|_e = r \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ |\alpha_2|=1}} \mathcal{B}^{\alpha_1}(d\mathcal{B})^{\alpha_2}. \tag{5.32}$$

If we now apply this to the function:

$$\theta = \sum_{\alpha\in[\mathcal{F}]^r} c_\alpha \mathcal{B}^\alpha \tag{5.33}$$

we get the expressions:

$$\nabla\theta = r\sum_{s\in\mathcal{F}}\sum_{\substack{\alpha_1\in(s)^{r-1}\\ \alpha_2\in s^{\oplus}}} c_{\alpha_1+\alpha_2}\mathcal{B}^{\alpha_1}(d\mathcal{B})^{\alpha_2} \tag{5.34}$$

and

$$\begin{aligned}\nabla\theta\Big|_e &= r\sum_{s\subset e}\sum_{\substack{\alpha_1\in(s)^{r-1}\\ \alpha_2\in e^{\oplus}}} c_{\alpha_1+\alpha_2}\mathcal{B}^{\alpha_1}(d\mathcal{B})^{\alpha_2}.\\ &= r\sum_{\substack{\alpha_1\in[e]^{r-1}\\ \alpha_2\in e^{\oplus}}} c_{\alpha_1+\alpha_2}\mathcal{B}^{\alpha_1}(d\mathcal{B})^{\alpha_2}.\end{aligned} \tag{5.35}$$

From here we can apply the material of §5.3. First, it is apparent that $\nabla\theta$ is already *continuous*, though not necessarily *linear*. To get linearity, we must apply (5.21). In order to make this calculation more concrete, we represent $v_e \circ n_e$ as a k-th order matrix valued polynomial:

$$v_e \circ n_e = \sum_{\alpha\in[e]^k} T_\alpha \mathcal{B}^\alpha. \tag{5.36}$$

Then (5.21) becomes:

$$\begin{aligned}w_e \circ v_e \circ n_e &= \nabla\theta\Big|_e \cdot \sum_{\alpha\in[e]^k} T_\alpha\mathcal{B}^\alpha\\ &= r\sum_{\substack{\alpha_1\in[e]^{r-1}\\ \alpha_2\in e^{\oplus}}}\sum_{\alpha_3\in[e]^k}\left[c_{\alpha_1+\alpha_2}(d\mathcal{B})^{\alpha_2}T_{\alpha_3}\right]\mathcal{B}^{\alpha_1}\mathcal{B}^{\alpha_3}\\ &= r\sum_{\alpha\in[e]^{r+k-1}}\sum_{\substack{\alpha_1\in[e]^{r-1}\\ \alpha_2\in e^{\oplus}\\ \alpha_3\in[k]^e\\ \alpha_1+\alpha_3=\alpha}} B_{[\alpha_1,\alpha_3]}\left[c_{\alpha_1+\alpha_2}(d\mathcal{B})^{\alpha_2}T_{\alpha_3}\right]\mathcal{B}^{\alpha}\end{aligned} \tag{5.37}$$

where $B_{[\alpha_1,\alpha_3]} = \frac{B_{\alpha_1}B_{\alpha_3}}{B_{\alpha_1+\alpha_3}}$ – the Bernstein combination ordinals. Forcing this final expression, though terrible in form, to be zero is the constraint which must be satisfied in order for θ to have a derivative which is both continuous and linear with respect to the discrete differential structure represented here by T. More specifically, since $\mathcal{B}^\alpha$ forms a basis for the homogeneous polynomials, we must have, for each $\alpha \in (e)^{r+k-1}$,

$$\sum_{\substack{\alpha_1 \in [e]^{r-1} \\ \alpha_2 \in e^{\oplus} \\ \alpha_3 \in [k]^e \\ \alpha_1 + \alpha_3 = \alpha}} B_{[\alpha_1, \alpha_3]} \left[c_{\alpha_1 + \alpha_2} (d\mathcal{B})^{\alpha_2} T_{\alpha_3} \right] = 0. \tag{5.38}$$

Notice that we leave off those multi-indices which are on the boundary of e, and only take those which have $\operatorname{supp} \alpha = e$, since for any $s \subset e$, multi-indices with $\operatorname{supp} \alpha = s$ will be handled when enforcing linearity at s.

5.5 Remarks

In this chapter, we have proposed a method for representing a discrete differential structure on a simplicial complex, and used that structure to determine conditions for continuously differentiable piecewise polynomials. Yet, there are several obvious future paths to take, including areas which are left unresolved.

1. Existence of ξ. We require that both v_s and n_s be polynomial, but we do not require that of ξ. Indeed, due to the same reasons as presented in (4.4), it seems likely that they *cannot* be polynomial. Their existence in all but the trivial cases (when this commuting is required only over points) is therefore not obvious. Without them, discrete differential structures would exist only on dimensions 1, 2 and 3, but not the coveted 4.

2. Relax the condition that ξ be a complete adjoint to j. In this case, the tangent space may grow as we move down the simplicial tree in dimension. This would allow a way to admit approximate differential structures, and may provide a path forward if it turns out that ξ's cannot exist beyond dimension 3.

3. When performing this calculation numerically, error will be introduced in the form of non-commutativity. It would be valuable to be able to use the structure presented here to analyze the effects of this lack of commutativity.

4. While we focused on simplicial complexes of differentiable manifolds, the structure strongly suggests that generalization to any (compact and finite) simplicial topology is possible.

In the next chapter, we will produce several examples of constructing and using discrete differential structures.

6 Examples

We now present examples of constructing and using discrete differential structures in cases where the theory simplifies enough that it can be worked out by hand. Our first two examples deal with the two dimensional case, where we can completely determine a differential structure. Similar work has been done in two dimensions, recently by [18]. Other work, such as [8], relies upon transition maps of overlapping charts.

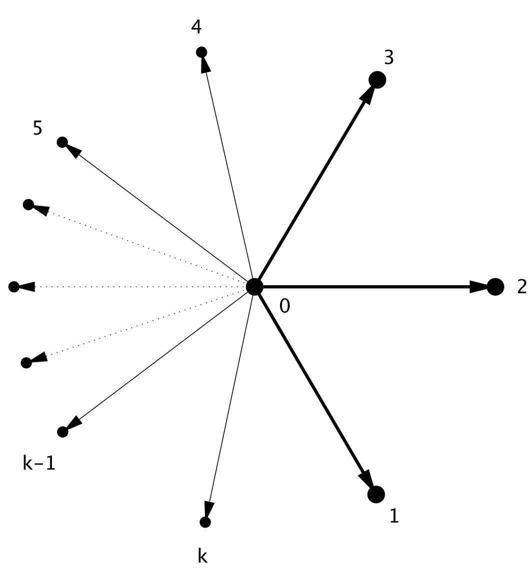


Figure 6.1: The "spokes" of a point 0 in a two dimensional simplicial complex $\mathcal{F}$.

6.1 Construction of a Discrete Differential Structure in Two Dimensions

In two dimensions, the theory simplifies considerably. A two dimensional simplicial complex has three levels, commonly called vertices ($\mathcal{F}_0$), edges ($\mathcal{F}_1$) and faces ($\mathcal{F}_2$). To construct a discrete differential structure, constraints are placed on the vertices and edges. Consistency need only be enforced at the vertices, which is what makes two dimensional analysis so simple. We will first work through the construction of a differential structure on an appropriate two dimensional abstract simplicial complex, then give an explicit example of the 2-sphere.

Let $\mathcal{F}$ be a two dimensional simplicial complex with differential structure (v, n). Without loss of generality, we focus on the point $p = \{0\}$, and the edge $e = \{0, 2\}$ as labeled in Figure 6.1.

$$\begin{array}{ccccc} X_e & \xrightarrow{\mathbb{1}_{13}} & Y_e & \xrightarrow{v_e} & T_e\Delta_{e^\oplus} \\ \downarrow \xi & & \downarrow \eta & & \downarrow j \\ X_p & \xrightarrow[n_p]{} & Y_p & \xrightarrow[\begin{bmatrix} -\mathbb{1}_{p^+}^{\intercal} \\ \mathrm{id}_{p^+} \end{bmatrix}]{} & T_p\Delta_{p^\oplus} \end{array} \tag{6.1}$$

We now determine continuous differentiable structure on this complex. This reduces to completing diagram (6.1), which is diagram (5.25) where we have made the following choices:

1. Instead of $e \subset s$, we have used the labels $p \subset e$.

2. $v_p = \begin{bmatrix} -\mathbb{1}_{p^+}^{\intercal} \\ \mathrm{id}_{p^+} \end{bmatrix}$.

3. $n_e = \mathbb{1}_{13} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

4. n_p is given as part of the problem. This allows the user to control the relative angles the edges meet at the point.

The only terms which are free are ξ, η and v_e.

As discussed in §5.3, ξ must span the null space of $n_p^{p^\oplus \setminus e^\oplus}$. Solving for ξ is routine. Once ξ is obtained, we can find v_e by noting that v_e is diagonal on $e^+ = \{1, 3\}$. That is, we can express v_e by:

$$v_e = \tau_e \oplus \Lambda_e = \begin{bmatrix} \tau_e \\ \Lambda_e \end{bmatrix} \tag{6.2}$$

where Λ_e is a diagonal matrix. Then we have:

$$(n_p\xi)_{1,3} = \Lambda_e n_e = \Lambda_e \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_{e,1} \\ \lambda_{e,2} \end{bmatrix} \tag{6.3}$$

Determining the tangent component, τ_e is not unique. Indeed, if τ_e is a solution, then any solution of $wn_e = 0$ for w, yields the solution $(\tau_e + w)n_e = \tau_e n_e$. So we take the least squares solution for τ_e to:

$$(n_p\xi)_{1,3} = \tau_e n_e \tag{6.4}$$

Once we have v_e, obtaining η follows immediately from chasing the right hand side of diagram 6.1.

This then gives us a consistent differential structure on a two dimensional simplicial complex $\mathcal{F}$. In the next section, we will use this structure to construct piecewise polynomials which are C^1 relative to the differential structure.

6.2 2-Sphere from the Simplicial Complex of a Tetrahedron

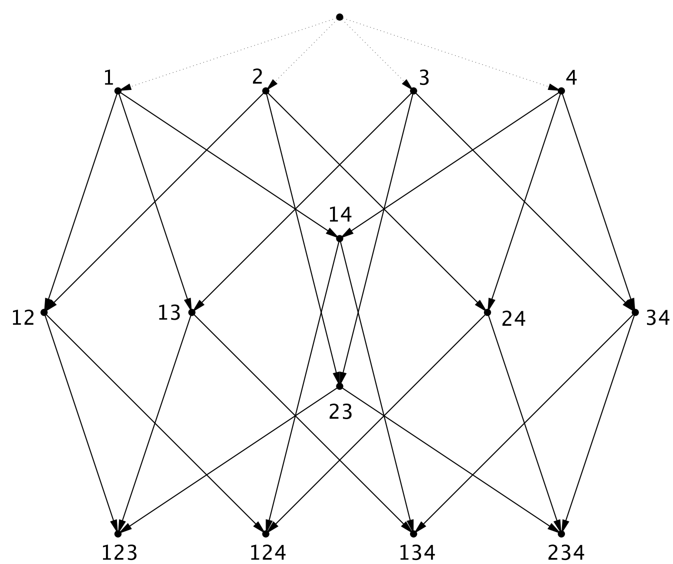


Figure 6.2: The abstract simplicial complex representing the surface of a tetrahedron.

Let $\mathcal{F} = 2^{\{1,2,3,4\}} - \{1, 2, 3, 4\}$ be the surface of a tetrahedron – see Figure 6.2. We construct the discrete differentiable structure from the vertexes up. Each vertex connects 3 edges, and we must connect the 2 dimensional normal vector space of each vertex into the tangent space of the edges. We focus on vertex 1 to show the detailed setup.

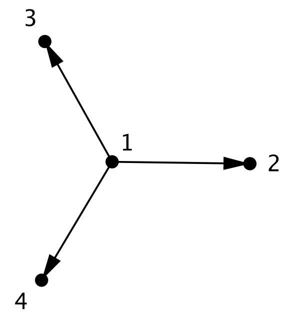


Figure 6.3: The "spokes" of a vertex of a tetrahedron.

Following the previous section, we let

$$v_1 = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The corresponding vectors in the tangent plane of vertex 1, must be 2 dimensional. Therefore there must be a one dimensional linear dependence on these vectors. The symmetric choice is to choose

$$N = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

This connects the tangent spaces of the edges to the "normal" space of the vertexes (normal with respect to the vertex embedded in the manifold). We now need to map the remaining "normal" direction of each edge into the tangent space of the faces. We focusing on edge $\{1, 2\}$, the others working analogously.

To do this, we must choose the "normal" vector fields v_{12}^{123} and v_{12}^{124} (normal here is quoted for two reasons, the first is explained above, the second is that these vectors cannot be normal as there is no metric defined). We must also choose the linear dependence, as we did for the vertex.

Complicating this choice, however, is the relationship to the vertices bounding the edge. In order to choose v_{12}^{123} and v_{12}^{124} in a consistent way, we set up the following. Let $\eta_3 = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix} = 2v_1^{13} - v_1^{12}$, $\eta_4 = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 2 \end{bmatrix} = 2v_1^{14} - v_1^{12}$, $w = \alpha v_1^{12}$ and let $v_{12,1}^{123} = \eta_3 + w$, $v_{12,1}^{124} = \eta_4 + w$. Further, suppose n_3 and n_4 are chosen so that $n_3 v_{12,1}^{123} + n_4 v_{12,1}^{124} = 0$. Then we have

$$\begin{aligned} n_3 v_{12,1}^{123} + n_4 v_{12,1}^{124} &= 0 \\ n_3(\eta_3 + w) + n_4(\eta_4 + w) &= 0 \\ n_3\left(2v_1^{13} - v_1^{12} + \alpha v_1^{12}\right) + n_4\left(2v_1^{14} - v_1^{12} + \alpha v_1^{12}\right) &= 0 \\ 2n_3 v_1^{13} + 2n_4 v_1^{14} + (1-\alpha)(n_3 + n_4)\, v_1^{12} &= 0 \end{aligned} \tag{6.5}$$

Since the only linear combination of v_1^{13}, v_1^{14} and v_1^{12} which can achieve 0 is a multiple of $v_1^{13} + v_1^{14} + v_1^{12}$, we must have that $n_3 = n_4$, which we choose to be 1. Continuing to find α,

$$\begin{aligned} 2v_1^{13} + 2v_1^{14} + 2(1-\alpha)\, v_1^{12} &= 0 \\ -2v_1^{12} + 2(1-\alpha)\, v_1^{12} &= 0 \\ \alpha &= 2 \end{aligned} \tag{6.6}$$

So we find that $v_{12,1}^{123} = \begin{bmatrix} -3 \\ 1 \\ 2 \\ 0 \end{bmatrix}$ and $v_{12,1}^{124} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 2 \end{bmatrix}$. Similarly, we also find that

$$v^{123}_{12,2} = \begin{bmatrix} 1 \\ -3 \\ 2 \\ 0 \end{bmatrix} \text{ and } v^{124}_{12,2} = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix}. \text{ Then } v^{123}_{12} = v^{123}_{12,1}\mathcal{B}_{12,1} + v^{123}_{12,2}\mathcal{B}_{12,2}.$$

6.3 Two Triangles Sharing a Common Edge

In this section, we present an example of constructing an explicit linear system, the solution of which provides coefficients of C^1 splines relative to the discrete differential structure. Let $\mathcal{F}$ be the abstract simplicial complex of two triangles connected by a single common edge. In this example we will find piecewise

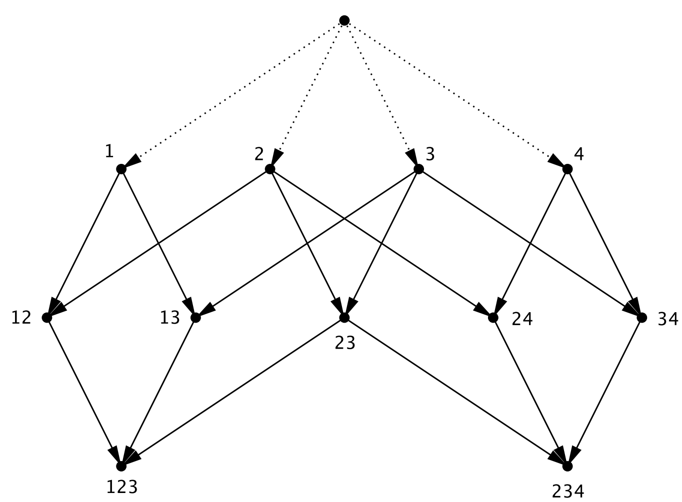


Figure 6.4: The abstract simplicial complex representing two triangles sharing a common edge.

cubic polynomials on $\Delta_{\mathcal{F}}$ which are differentiable with respect to a differential structure not induced by a 'γ' projection function defined above.

$\mathcal{P}_3(\mathcal{F})$ is the space of piecewise cubic polynomials on $\Delta_{\mathcal{F}}$, and the coefficients are indexed by the set $[\mathcal{F}]^3$.

Let us first consider the case when v and w are constant, and we wish to find all piecewise linear, continuous, covariant tensor fields. Taking an expression for all piecewise linear covariant tensor fields:

$$\sum_{s\mathcal{F}} \sum_{\alpha\in s,\beta\in s} c^s_{\alpha,\beta}(\alpha;\beta)_s$$

Applying the tensor continuity constraint, we get the expression:

$$\sum_{\alpha\in e}\left[\sum_{\beta\in f_1} c^{f_1}_{\alpha,\beta}(\alpha;\beta)_{f_1}v + \sum_{\beta\in f_2} c^{f_2}_{\alpha,\beta}(\alpha;\beta)_{f_2}w\right] = 0$$

From which we deduce that each term in the outer most sum must be zero.

This handles the normal constraints, the tangent constraints are handled similarly:

$$\sum_{\beta\in f_1} c^e_{\alpha,\beta} - \sum_{\beta\in f_2} c^e_{\alpha,\beta} = 0$$

Using this we can determine C^1 piecewise quadratic functions.

First we express all functions as:

$$\phi = \sum_{s\in\mathcal{F}} \sum_{\alpha\in s^2} c^s_\alpha(\alpha,\epsilon)$$

Continuity constraints force $c^{s_1}_\alpha = c^{s_2}_\alpha$ for all $s_1, s_2 \in \mathcal{F}$. So we may simplify the expression for functions as:

$$\phi = \sum_{s\in\mathcal{F}} \sum_{\alpha\in s^2} c_\alpha(\alpha,\epsilon)$$

This immediately satisfies all tangent constraints.

Then apply the operator ∇:

$$\nabla\phi = \sum_{s\in\mathcal{F}} \sum_{\alpha\in s,\beta\in s} c_{\alpha+\beta}(\alpha,\beta)$$

And apply the conditions derived above, i.e. for each α:

$$\sum_{\beta \in f_1} c_{\alpha+\beta}(\alpha;\beta)_{f_1} v + \sum_{\beta \in f_2} c_{\alpha+\beta}(\alpha;\beta)_{f_2} w = 0$$

This is a 2 dimensional constraint, and identical to the constraints derived in the splines section.

Now we use the framework to allow the normal vectors to change across the edge. We set v_1, w_1 at one corner, v_2, w_2 at the other, and linearly interpolate between them. The constraint then becomes:

$$\sum_{\alpha \in e} \left[\sum_{\beta \in f_1} c^{f_1}_{\alpha,\beta}(\alpha 1,\beta)_{f_1} v_1 + (\alpha 2,\beta)_{f_1} v_2 + \sum_{\beta \in f_2} c^{f_2}_{\alpha,\beta}(\alpha 1,\beta)_{f_2} w_1 + (\alpha 2,\beta)_{f_2} w_2 \right] = 0$$

Explicitly, we are looking for the null space of the matrix:

$$\begin{bmatrix} v_1^0 & v_1^1 & v_1^2 & 0 & 0 & 0 & w_1^1 & w_1^2 & w_1^3 & 0 & 0 & 0 \\ v_2^0 & v_2^1 & v_2^2 & v_1^0 & v_1^1 & v_1^2 & w_2^1 & w_2^2 & w_2^3 & w_1^1 & w_1^2 & w_1^3 \\ 0 & 0 & 0 & v_2^0 & v_2^1 & v_2^2 & 0 & 0 & 0 & w_2^1 & w_2^2 & w_2^3 \end{bmatrix}$$

The columns of which correspond to the coefficients:

$$\begin{bmatrix} c^{f_1}_{1,0} & c^{f_1}_{1,1} & c^{f_1}_{1,2} & c^{f_1}_{2,0} & c^{f_1}_{2,1} & c^{f_1}_{2,2} & c^{f_2}_{1,1} & c^{f_2}_{1,2} & c^{f_2}_{1,3} & c^{f_2}_{2,1} & c^{f_2}_{2,2} & c^{f_2}_{2,3} \end{bmatrix}$$

.

We can now use this to derive constraints for C^1 quadratic functions under this discrete differentiable structure. Again, explicitly, they are the null space of the matrix:

$$\begin{bmatrix} 0 & v_1^1 + w_1^1 & 0 & 0 & v_1^0 & 0 & v_1^2 + w_1^2 & w_1^3 & 0 \\ 0 & v_2^1 + w_2^1 & v_1^2 + w_1^2 & 0 & v_2^0 & v_1^0 & v_2^2 + v_1^1 + w_2^2 + w_1^1 & w_2^3 & w_1^3 \\ 0 & 0 & v_2^2 + w_2^2 & 0 & 0 & v_2^0 & v_2^1 + w_2^1 & 0 & w_2^3 \end{bmatrix}$$

Corresponding to the coefficients:

$$\begin{bmatrix} c_{00} & c_{11} & c_{22} & c_{33} & c_{01} & c_{02} & c_{12} & c_{13} & c_{23} \end{bmatrix}$$

.

This gives us a 6 dimensional generalized spline space. Note that this is one dimension less than above. Allowing the normal vector to change therefore comes at the cost of a reduced spline dimension.

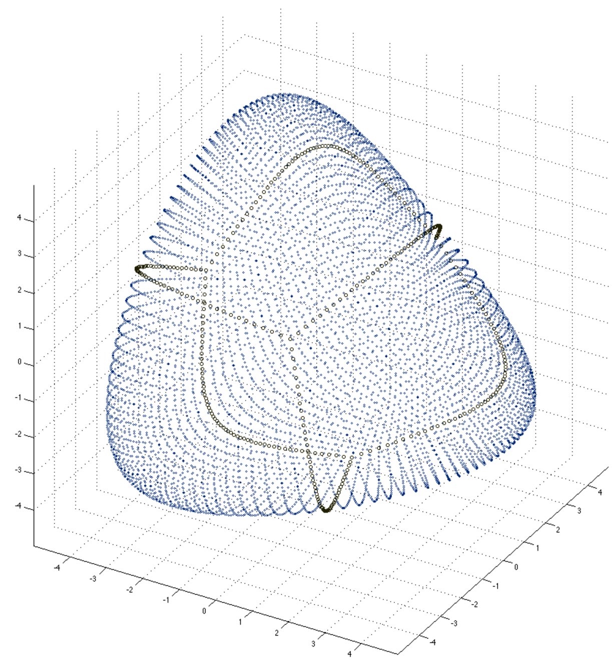


Figure 6.5: A continuously differentiable embedding of the tetrahedron created by functions which are differentiable with respect to the discrete differentiable structure.

Now, we find the piecewise cubic polynomials which are differentiable with respect to this differential structure. Begin with the C^0 piecewise cubic polynomials. These have a degree of freedom for each $\alpha \in [\mathcal{F}]^3$, so we can write a typical element as

$$\phi = \sum_{s \in \mathcal{F}} \sum_{\alpha \in [s]^3} c^s_\alpha \mathcal{B}^\alpha_s \tag{6.7}$$

with the constraint that for every $s_1 \subset s_2 \in \mathcal{F}$ and $\alpha \in [s_1]^3$, that $c^{s_1}_\alpha = c^{s_2}_\alpha$. There are 10 degrees of freedom for each of the 4 2-simplexes, 4 degrees of freedom for each of the 6 1-simplexes, and 1 degree of freedom for each of the 4 0-simplexes, for a total of 68. However, there are 6 constraints for each of the 4 0-simplexes, and 4 constraints for each of the 6 1-simplexes, for a total of 48 constraints. This leaves 20 degrees of freedom. From here we obtain the first derivative, a section of the covector bundle (not necessarily continuous)

$$\nabla\phi = \sum_{s\in\mathcal{F}} \sum_{\substack{\alpha\in[s]^2 \\ \beta\in[s]^1}} c^s_{\alpha+\beta} \mathcal{B}^\alpha_s (d\mathcal{B}_s)^\beta. \tag{6.8}$$

We can enforce continuity by applying the operator (v, N) constructed above. This will yield one constraint at each vertex and two at each edge, for a total of 16 constraints.

6.4 Representing the n-Sphere

Let $W = [0..n]$ be a vertex set. Let $\mathcal{F} = 2^W - W$, the power set of W and remove the set W. This is the minimal simplicial complex representing the n-sphere. The structure of this complex allows us to generate a differential structure with relative ease.

The main fact which makes this complex easy to analyze is that $s^\oplus = W$ for every $s \in \mathcal{F}$. Because of this, $\dim(X_s) = 1$ for every $s \in \mathcal{F}$ and so ξ is just scalar multiplication at every level. Furthermore, for $s \in \mathcal{F}$ it is natural to choose, $n_s = \mathbb{1}_{s^+}$, and

$$v_p = \begin{bmatrix} -\mathbb{1}^*_{p^+} \\ \mathrm{id}_{p^+} \end{bmatrix}. \tag{6.9}$$

The only functions left to find are then v_s and η.

$$\begin{array}{ccccc} X_s & \xrightarrow{\mathbb{1}_{s^+}} & Y_s & \xrightarrow{v_s} & T_s\Delta_{s^\oplus} \\ \downarrow \xi & & \downarrow \eta & & \downarrow \mathrm{id} \\ X_p & \xrightarrow{\mathbb{1}_{p^+}} & Y_p & \xrightarrow{v_p} & T_p\Delta_{p^\oplus} \end{array} \tag{6.10}$$

A solution is:

$$\begin{aligned} \xi &= |s^+| \\ v_s &= \begin{bmatrix} -|p^+|\mathbb{1}_s^* \\ \mathbb{1}_{s^+}^* \\ \mathrm{id}_{s^+} \end{bmatrix} \end{aligned} \tag{6.11}$$

since we have the following expressions of the commutative diagram.

$$\begin{aligned} v_p \circ n_p \circ \xi = \begin{bmatrix} \mathbb{1}_{p^+}^* \\ \mathrm{id}_{p^+} \end{bmatrix} \cdot \mathbb{1}_{p^+} \cdot |s^+| \quad &= \begin{bmatrix} |p^+| \\ \mathbb{1}_{p^+} \end{bmatrix} |s^+| \\ v_s \circ n_s = \begin{bmatrix} -|p^+|\mathbb{1}_s^* \\ \mathbb{1}_{s^+}^* \\ \mathrm{id}_{s^+} \end{bmatrix} \cdot \mathbb{1}_{s^+} \quad &= \begin{bmatrix} |p^+| \\ \mathbb{1}_{p^+} \end{bmatrix} |s^+| \end{aligned} \tag{6.12}$$

This allows us to place a discrete differential structure on spheres of any dimension. An interesting question to investigate would be if the exotic spheres of Milnor [16] can be represented through this structure.

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