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Authors
Gong, Qi
Ross, I Michael
Kang, Wei
et al.

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Connections between the covector mapping theorem and convergence of pseudospectral methods for optimal control

Qi Gong · I. Michael Ross · Wei Kang · Fariba Fahroo

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Abstract In recent years, many practical nonlinear optimal control problems have been solved by pseudospectral (PS) methods. In particular, the Legendre PS method offers a Covector Mapping Theorem that blurs the distinction between traditional direct and indirect methods for optimal control. In an effort to better understand the PS approach for solving control problems, we present consistency results for nonlinear optimal control problems with mixed state and control constraints. A set of sufficient conditions is proved under which a solution of the discretized optimal control problem converges to the continuous solution. Convergence of the primal variables does not necessarily imply the convergence of the duals. This leads to a clarification of the Covector Mapping Theorem in its relationship to the convergence properties of PS methods and its connections to constraint qualifications. Conditions for the convergence of the duals are described and illustrated. An application of the ideas to the

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Q. Gong (✉)
Department of Electrical & Computer Engineering, University of Texas at San Antonio,
San Antonio, TX 78249-1644, USA
e-mail: qi.gong@utsa.edu

I.M. Ross
Department of Mechanical and Astronautical Engineering, Naval Postgraduate School, Monterey,
CA 93943, USA
e-mail: imross@nps.edu

W. Kang · F. Fahroo
Department of Applied Mathematics, Naval Postgraduate School, Monterey, CA 93943, USA

W. Kang
e-mail: wkang@nps.edu

F. Fahroo
e-mail: ffahroo@nps.edu

optimal attitude control of NPSAT1, a highly nonlinear spacecraft, shows that the method performs well for real-world problems.

Keywords Optimal control · Pseudospectral · Nonlinear systems

1 Introduction

A fundamental problem in autonomous systems engineering and theory is the computation of constrained nonlinear optimal controls. Since the 1960s, many computational methods have been proposed towards the goal of providing robust and accurate algorithms for solving these problems. These methods are frequently classified under two categories: indirect and direct methods [3]. Historically, many early numerical methods [6, 35] were indirect methods; that is, methods based on finding solutions that satisfy a set of necessary optimality conditions resulting from Pontryagin's Minimum Principle. Implementations of indirect methods have been successfully applied to many real-world problems that include control of flexible structures, launch vehicle trajectory design and low-thrust orbit transfer [1, 3, 5, 6]. Nonetheless, indirect methods suffer from many drawbacks [3, 6]. For instance, the boundary value problem resulting from the necessary conditions is extremely sensitive to initial guesses as a result of the symplectic structure of the Hamiltonian system [6]. In addition, these necessary conditions must be explicitly derived—a labor-intensive process for complicated problems that requires an in-depth knowledge in optimal control theory. Over the last decade, direct methods, that is, methods based on approximations to the original (primal) problem, have gained wide popularity [3, 8, 11–13, 15, 21, 30, 31, 47] as a result of significant progress in theory, large-scale computation and robustness of the approach. In simple terms, the essential idea of this method is to discretize the optimal control problem and solve the resulting large-scale finite-dimensional optimization problem. The simplicity of this approach belies a wide range of deep theoretical issues (see [30]) that lie at the intersection of approximation theory, control theory and optimization. Even though these issues are yet to be satisfactorily addressed and dealt with, a wide variety of industrial-strength optimal control problems have been solved by this approach [2, 3, 23, 25, 29, 44, 49].

Despite the practical successes noted above, results on the convergence of direct methods, particularly, higher-order direct method are far from satisfactory. Only recently has significant progress been made on such problems [9, 11, 12, 21, 28, 52]. These studies show that convergence theorems are quite difficult to prove, and validating computational results by checking the assumptions of these theorems are even harder [22]. In order to overcome suspicions about the extremality of a computed solution, a two-tier method has long been advocated [34] as a means for validating or refining a given solution. In these methods, a direct solution is used as starting point for an indirect method leading to a potential refinement or validation of the computed solution. In recent years, a far simpler technique has been proposed and vigorously exploited [16, 17, 25, 27, 28, 38, 48, 49] in several different forms that can all be encapsulated under the notion of a Covector Mapping Principle [40, 41]. Although some of these ideas go as far back as the early 1990s (see [15, 54]), many key constructs were formulated only around the year 2000 [21, 43]. In this approach, a direct connection between the discrete-time multipliers and the discretized costates associated with the boundary value problem is sought as a means to commute discretization with dualization. The ensuing Covector Mapping Theorem then provides a direct connection to the Pontryagin Minimum Principle, which in turn, facilitates verification and validation of the computed solution as if an indirect method was applied. Thus, all the burdens of a two-tier approach are completely circumvented leading to a simple and robust approach to solving practical optimal control problems. Verification and validation of the computed solution is particularly important in solving industrial-strength problems where safety and robustness are crucial for a successful implementation. In 2000, Hager [21] showed that his transformed adjoint system was in fact crucial for a proper convergence analysis of RK methods. By exploring the discrepancies between the state and costate discretizations, Hager designed new RK methods for control applications that are distinct from the ones developed by Butcher. Thus, what has emerged in recent years is a close juxtaposition of theory, computation and practice enabled through the development and application of covector mapping theorems.

In this paper we focus on pseudospectral (PS) methods. PS methods were largely developed in the 1970s for solving partial differential equations arising in fluid dynamics and meteorology [7], and quickly became "one of the big three technologies for the numerical solution of PDEs" [51]. During the 1990s, PS methods were introduced for solving optimal control problems; and since then, have gained considerable attention [13, 16, 25, 26, 29, 38, 47, 49, 55, 56], particularly in solving aerospace control problems. Examples range from lunar guidance [25], magnetic control [56], orbit transfers [49], tether libration control [55], ascent guidance [29] and a host of other problems. As a result of its considerable success, NASA adopted the Legendre PS method as a problem solving option for their OTIS software package [33]. Results for a test suite of problems are discussed in [39]. In addition, the commercially available software package, DIDO [42], exclusively uses PS methods for solving optimal control problems.

Similar to Hager's research on RK methods, a Covector Mapping Theorem was developed in [16] for the Legendre PS method. In order to address certain discrepancies in the solution, a set of "closure conditions" were identified in [48] to map the Karush-Kuhn-Tucker (KKT) multipliers associated with the discretized optimal control problem to the dual variables associated with the continuous-time optimal control problem. Unlike Hager's RK method which imposes additional conditions on the primal problem (i.e. coefficients of the integration scheme), the conditions of [48] impose constraints on both the primal and dual variables. In the absence of a convergence theorem, this procedure requires solving a difficult primal-dual mixed complementarity problem (MCP). In this paper, we eliminate the need to solve the MCP by strengthening earlier results and weakening prior assumptions. This is done by first establishing a pair of existence and convergence results for the primal problem. These results extend our prior work [20], which was limited to feedback linearizable systems. For general nonlinear systems (i.e. systems not feedback linearizable), we show that the discrete dynamics must be relaxed to guarantee feasibility. Then, we prove that for constrained optimal control problems, the primal solution of the discretized optimal control problem converges to the solution of the continuous optimal control problem under conditions that may be computationally verified. Thus, the difficult primal-dual MCP required for costate evaluation may be replaced by simpler NLP techniques and solvers. More importantly, we demonstrate why the convergence of the primal variables does not necessarily imply the convergence of the KKT multipliers to the continuous costate. The necessity of the closure conditions in [48] is clarified as a means of ensuring the correct selection of the sequence of multipliers that converge to the costates. A simple but illustrative example is introduced to tie these ideas to constraint qualifications. Finally, the theoretical results are demonstrated for the practical optimal control of NPSAT1, a magnetically controlled spacecraft being built at the Naval Postgraduate School and scheduled to be launched in 2007.

This paper is organized as follows: in Sect. 2, we briefly present the PS discretization method for constrained nonlinear optimal control problems. Sections 3 and 4 contain the results regarding existence and convergence of the discretized primal variables. Section 5 focuses on the existence and convergence of the dual variables. Finally, in Sect. 6 we apply the PS method to solve the near-minimum-time minimum-energy control problem for the NPSAT1 spacecraft.

Throughout the paper we make extensive use of Sobolev spaces, $W^{m,p}$, that consist of all functions, $\xi : [-1, 1] \to \mathbb{R}^n$ whose jth distributional derivative, $\xi^{(j)}$, lies in L^p for all $0 \le j \le m$ with the norm,

$$\|\xi\|_{W^{m,p}} = \sum_{j=0}^{m} \|\xi^{(j)}\|_{L^p}$$

where $\|\xi\|_{L^p}$ denotes the usual Lebesgue norm,

$$\|\xi\|_{L^p} = \left(\int_{-1}^{1} \|\xi(t)\|^p dt \right)^{1/p}$$

and $\|\xi(t)\|$ denotes any finite dimensional norm in $\mathbb{R}^n$. By a minor abuse of notation, we sometimes use $\xi(t)$ to mean both the function, ξ, and the value of ξ at time t. For a generic vector $v \in \mathbb{R}^n$, we use $\|v\|_\infty$ to denote the maximum element of v. For notational ease, we suppress the dependence of $W^{m,p}$ on the domain and range of vector-valued functions.

2 The problem and its discretization

Consider the following optimal control problem:

Problem *B:* Determine the state-control function pair, $t \mapsto (x, u) \in \mathbb{R}^{N_x} \times \mathbb{R}^{N_u}$, that minimizes the cost functional

$$J[x(\cdot), u(\cdot)] = \int_{-1}^{1} F(x(t), u(t))\, dt + E(x(-1), x(1))$$

subject to the dynamics,

$$\dot{x}(t) = f(x(t), u(t)) \tag{1}$$

endpoint conditions

$$e(x(-1), x(1)) = 0 \tag{2}$$

and mixed state-control path constraints

$$h(x(t), u(t)) \leq 0. \tag{3}$$

It is assumed that $F : \mathbb{R}^{N_x} \times \mathbb{R}^{N_u} \to \mathbb{R}$, $E : \mathbb{R}^{N_x} \times \mathbb{R}^{N_x} \to \mathbb{R}$, $f : \mathbb{R}^{N_x} \times \mathbb{R}^{N_u} \to \mathbb{R}^{N_x}$, $e : \mathbb{R}^{N_x} \times \mathbb{R}^{N_x} \to \mathbb{R}^{N_e}$, and $h : \mathbb{R}^{N_x} \times \mathbb{R}^{N_u} \to \mathbb{R}^{N_h}$, are continuously differentiable with respect to their arguments and their gradients are Lipschitz continuous over the domain. In order to properly apply the first-order necessary conditions for state constrained problems, particularly, mixed state and control constraints, it is necessary to define appropriate constraint qualifications. For the purpose of brevity, we do not describe these conditions and implicity assume that the problem satisfies such technical conditions which are well documented in [24], [37] and [53]. In addition to these implicit assumptions, we also assume that an optimal solution $(x^*(\cdot), u^*(\cdot))$ exists with the optimal state, $x^*(\cdot) \in W^{m,\infty}$, $m \geq 2$ and optimal control, $u^*(\cdot) \in C^0[-1, 1]$.

Remark 1 If $x^*(\cdot) \in C^1$ and $[-1, 1] \ni t \to \dot{x}^*(t)$ has a bounded derivative everywhere except for finitely many points, then it is clear that $x^*(\cdot) \in W^{2,\infty}$. From Sobolev's Imbedding Theorems [7], any function $x^*(\cdot) \in W^{m,\infty}$, $m \geq 2$ must have continuous $(m-1)$th order classical derivatives (on $[-1, 1]$). Therefore, the condition, $x^*(\cdot) \in W^{m,\infty}$, $m \geq 2$, requires that the optimal state trajectory, $x^*(\cdot)$, be at least continuously differentiable, which in turn requires that the optimal control trajectory, $u^*(\cdot)$, be continuous. The NPSAT1 problem discussed in Sect. 6 illustrates the satisfaction of these regularity conditions.

Remark 2 From the practical point of view, it is desirable to identify conditions on the problem data that guarantee the required regularity on the optimal solution, i.e., $x^*(\cdot) \in W^{m,\infty}$, $m \geq 2$ and $u^*(\cdot) \in C^0[-1, 1]$. It is well-recognized that such regularity theorems are one of the most difficult results to obtain in optimal control [53]. One of the best-known results is Hager's regularity theorem [10, 53] which imposes substantial requirements on the problem data such as affine dynamics, convexity etc. In recognizing the difficulty of developing regularity theorems, we simply assume certain smoothness conditions keeping in mind that many practical problems (e.g. the NPSAT1 problem discussed in Sect. 6) indeed satisfy our assumptions.

Remark 3 In [28], we have provided convergence results for discontinuous $u^*(\cdot)$. Our proof is limited to assuming $u^*(\cdot)$ is a piecewise C^1 with finitely many discontinuities, and the dynamic constraint is in feedback linearizable form.

In the Legendre pseudospectral approximation of Problem B, the basic idea is to approximate $t \mapsto x(t)$ by Nth order polynomials, $t \mapsto x^N(t)$, based on Lagrange interpolation at the Legendre-Gauss-Lobatto (LGL) quadrature nodes, i.e.

$$x(t) \approx x^N(t) = \sum_{k=0}^{N} x^N(t_k)\phi_k(t),$$

where t_k are LGL nodes defined as,

$$\begin{aligned} &t_0 = -1, \quad t_N = 1, \\ &t_k, \quad \text{for } k = 1, 2, \dots, N-1, \text{ are the roots of } \dot{L}_N(t) \end{aligned}$$

where $\dot{L}_N(t)$ is the derivative of the Nth order Legendre polynomial $L_N(t)$. The Lagrange interpolating polynomials, $\phi_k(t)$, are given by [7]

$$\phi_k(t) = \frac{1}{N(N+1)L_N(t_k)} \frac{(t^2-1)\dot{L}_N(t)}{t - t_k}. \tag{4}$$

It is readily verifiable that $\phi_k(t_j) = 1$, if $k = j$ and $\phi_k(t_j) = 0$, if $k \neq j$. The derivative of the ith state $x_i(t)$ at the LGL node t_k is approximated by

$$\dot{x}_i(t_k) \approx \dot{x}_i^N(t_k) = \sum_{j=0}^{N} D_{kj} x_i^N(t_j), \quad i = 1, 2, \dots, N_x$$

where the $(N+1) \times (N+1)$ differentiation matrix D is defined by

$$D_{ik} = \begin{cases} \frac{L_N(t_i)}{L_N(t_k)} \frac{1}{t_i - t_k}, & \text{if } i \neq k; \\ -\frac{N(N+1)}{4}, & \text{if } i = k = 0; \\ \frac{N(N+1)}{4}, & \text{if } i = k = N; \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Let

$$\bar{x}_k = x^N(t_k), \quad k = 0, 1, \dots, N.$$

In a standard PS method for control, the continuous differential equation is approximated by the following nonlinear algebraic equations

$$\sum_{i=0}^{N} \bar{x}_i D_{ki} - f(\bar{x}_k, \bar{u}_k) = 0, \quad k = 0, 1, \dots, N \tag{6}$$

where $\bar{u}_k$ is taken to be analogous to $\bar{x}_k$. This discretization is used in [13, 14, 16, 48] for optimal control problems. *As will be apparent later, it is not necessary to assume $t \mapsto u(t)$ to be approximated by a polynomial.* This is a sharp distinction from previous PS methods. It is also worth observing (6) is imposed at all points and not merely at the interior points. As will be apparent shortly, this implies that a feasible solution to (6) may not exist; hence, to guarantee feasibility of the discretization, we propose the following relaxation,

$$\left\| \sum_{i=0}^{N} \bar{x}_i D_{ki} - f(\bar{x}_k, \bar{u}_k) \right\| \leq (N-1)^{\frac{3}{2}-m}, \quad k = 0, 1, \ldots, N. \tag{7}$$

Deferring a development of this relaxation scheme, note that when N tends to infinity, the difference between conditions (6) and (7) vanishes, since m, by assumption, is greater than or equal to 2. Throughout the paper, we use the "bar" notation to denote discretized variables. Note that the subscript in $\bar{x}_k$ denotes an evaluation of the approximate state, $x^N(t) \in \mathbb{R}^{N_x}$, at the node t_k whereas $x_k(t)$ denotes the kth component of the exact state. The endpoint conditions and constraints are approximated in a similar fashion

$$\|e(\bar{x}_0, \bar{x}_N)\|_\infty \leq (N-1)^{\frac{3}{2}-m}, \tag{8}$$

$$h(\bar{x}_k, \bar{u}_k) \leq (N-1)^{\frac{3}{2}-m} \cdot \mathbf{1}, \quad k = 0, 1, \ldots, N \tag{9}$$

where $\mathbf{1}$ denotes $[1, \ldots, 1]^T$.

Remark 4 The right hand side in (8) and (9) can be set to $(N-r)^{-m+a}$, provided $1 < a < 2$. For simplicity, we choose $a = \frac{3}{2}$.

Remark 5 Although we do not directly use his results, the relaxations in (7–9) are similar in spirit to Polak's theory of consistent approximations [36].

Finally, the cost functional $J[x(\cdot), u(\cdot)]$ is approximated by the Gauss-Lobatto integration rule,

$$J[x(\cdot), u(\cdot)] \approx \bar{J}^N(\bar{X}, \bar{U}) = \sum_{k=0}^{N} F(\bar{x}_k, \bar{u}_k) w_k + E(\bar{x}_0, \bar{x}_N)$$

where w_k are the LGL weights given by

$$w_k = \frac{2}{N(N+1)} \frac{1}{[L_N(t_k)]^2}, \quad k = 0, 1, \ldots, N$$

and $\bar{X} = [\bar{x}_0, \ldots, \bar{x}_N]$, $\bar{U} = [\bar{u}_0, \ldots, \bar{u}_N]$.

Since practical solutions are bounded, we add the following constraints,

$$\{\bar{x}_k \in \mathbb{X}, \ \bar{u}_k \in \mathbb{U}, \ k = 0, 1, \ldots, N\}$$

where $\mathbb{X}$ and $\mathbb{U}$ are two compact sets representing the search region and containing the continuous optimal solution $[-1, 1] \ni t \to (x^*(t), u^*(t))$. Thus, the optimal control Problem B is approximated to a nonlinear programming problem with $\bar{J}^N$ as the objective function and (7), (8) and (9) as constraints; this is summarized as:

Problem B^N*:* Find $\bar{x}_k \in \mathbb{X}$ and $\bar{u}_k \in \mathbb{U}$, $k = 0, 1, \ldots, N$, that minimize

$$\bar{J}^N(\bar{X}, \bar{U}) = \sum_{k=0}^{N} F(\bar{x}_k, \bar{u}_k) w_k + E(\bar{x}_0, \bar{x}_N) \tag{10}$$

subject to

$$\left\| \sum_{i=0}^{N} \bar{x}_i D_{ki} - f(\bar{x}_k, \bar{u}_k) \right\|_\infty \le (N-1)^{\frac{3}{2}-m}, \tag{11}$$

$$\|e(\bar{x}_0, \bar{x}_N)\|_\infty \le (N-1)^{\frac{3}{2}-m}, \tag{12}$$

$$h(\bar{x}_k, \bar{u}_k) \le (N-1)^{\frac{3}{2}-m} \cdot \mathbf{1}. \tag{13}$$

3 Feasibility of Problem B^N

In the case of Eulerian discretizations, for any given initial condition and control discretization, the states are uniquely determined. Hence, there always exists a feasible solution to the discretized dynamic system. For Runge-Kutta methods, a similar property holds if the mesh is sufficiently dense [21]. For pseudospectral methods such an existence result for controlled differential equations is not readily apparent. There are two main difficulties. PS methods are fundamentally different than traditional methods (like Euler or Runge-Kutta) in that they focus on approximating the tangent bundle rather than the differential equation. Since the differential equation is imposed over discrete points, in standard PS methods the boundary conditions are typically handled by not imposing the differential equations over the boundary [4, 51]. This technique cannot be used for controlled differential equations as it implies that the control can take arbitrary values at the boundary. Thus, PS methods for control differ from their standard counterparts in imposing the differential equation at the boundary as well. Although this notion is quite beneficial in extending PS methods to hybrid optimal control problems [44–46], it generates an apparently unfortunate consequence in that the discretized dynamics without relaxation may not have a feasible solution; this is illustrated by the following example.

Example 1 Consider the linear system

$$\begin{aligned} \dot{x}_1 &= x_1 + u, \\ \dot{x}_2 &= x_2 + u. \end{aligned} \tag{14}$$

Its standard PS discretization is

$$D \begin{pmatrix} \bar{x}_{10}^N \\ \vdots \\ \bar{x}_{1N}^N \end{pmatrix} = \begin{pmatrix} \bar{x}_{10}^N \\ \vdots \\ \bar{x}_{1N}^N \end{pmatrix} + \begin{pmatrix} \bar{u}_0^N \\ \vdots \\ \bar{u}_N^N \end{pmatrix},$$

$$D\begin{pmatrix}\bar{x}_{20}^N\\ \vdots\\ \bar{x}_{2N}^N\end{pmatrix}=\begin{pmatrix}\bar{x}_{20}^N\\ \vdots\\ \bar{x}_{2N}^N\end{pmatrix}+\begin{pmatrix}\bar{u}_{0}^N\\ \vdots\\ \bar{u}_{N}^N\end{pmatrix}.$$

Therefore

$$(D-I)\begin{pmatrix}\bar{x}_{10}^N\\ \vdots\\ \bar{x}_{1N}^N\end{pmatrix}=(D-I)\begin{pmatrix}\bar{x}_{20}^N\\ \vdots\\ \bar{x}_{2N}^N\end{pmatrix}.$$

Since D is nilpotent [51], $(D-I)$ is nonsingular. Hence, $(\bar{x}_{10}^N,\ldots,\bar{x}_{1N}^N)=(\bar{x}_{20}^N,\ldots,\bar{x}_{2N}^N)$. Therefore, if the initial condition is such that $\bar{x}_{10}^N\neq\bar{x}_{20}^N$, the discretized dynamics with arbitrary initial conditions has no solution, although a continuous-time solution satisfying (14) always exists for any given initial condition.

In this paper, we propose to relax the equality in (6) to the inequality of (11). In this way, the feasibility of Problem B^N can be guaranteed as proved in Theorem 1 below; but first, we need the following lemma.

Lemma 1 [7] *Given any function* $\xi\in W^{m,\infty}$, $t\in[-1,1]$, *there is a polynomial* $p^N(t)$ *of degree* N *or less, such that*

$$|\xi(t)-p^N(t)|\leq CC_0N^{-m},\quad\forall t\in[-1,1]$$

where C *is a constant independent of* N *and* $C_0=\|\xi\|_{W^{m,\infty}}$. *(*$p^N(t)$ *with the smallest norm* $||\xi(t)-p^N(t)||_{L^\infty}$ *is called the* N*th order best polynomial approximation of* $\xi(t)$ *in the norm of* L^∞.*)*

Proof This is a standard result in approximation theory; see [7]. □

Theorem 1 *Given any feasible solution,* $t\mapsto(x,u)$, *for Problem* B, *suppose* $x(\cdot)\in W^{m,\infty}$ *with* $m\geq 2$. *Then, there exists a positive integer* N_1 *such that, for any* $N>N_1$, *Problem* B^N *has a feasible solution,* $(\bar{x}_k,\bar{u}_k)$. *Furthermore, the feasible solution satisfies* $\bar{u}_k=u(t_k)$ *and*

$$\|x(t_k)-\bar{x}_k\|_\infty\leq L(N-1)^{1-m},\tag{15}$$

for all $k=0,\ldots,N$, *where* t_k *are LGL nodes and* L *is a positive constant independent of* N.

Proof Let $p(t)$ be the $(N-1)$th order best polynomial approximation of $\dot{x}(t)$ in the norm of L^∞. By Lemma 1 there is a constant C_1 independent of N such that

$$\|\dot{x}(t)-p(t)\|_{L^\infty}\leq C_1(N-1)^{1-m}.\tag{16}$$

Define

$$x^N(t)=\int_{-1}^{t}p(\tau)d\tau+x(-1),$$

$$\bar{x}_k = x^N(t_k),$$
$$\bar{u}_k = u(t_k).$$

From (16),

$$\|x(t) - x^N(t)\|_{L^\infty} \le 2C_1(N-1)^{1-m}. \tag{17}$$

It follows that both $x(t_k)$ and $\bar{x}_k$ are contained in some compact set whose boundary is independent of N.

On this compact set, because f is continuously differentiable, it must be Lipschitz continuous. By definition, $t \mapsto x^N(t)$ is a polynomial of degree less than or equal to N. It is known (see [7]) that, for any polynomial of degree less than or equal to N, its derivative at the LGL nodes $t_0, \dots, t_N$ are exactly equal to the value of the polynomial at the nodes multiplied by the differential matrix D, which is defined by (5). Thus we have

$$\sum_{i=0}^{N} \bar{x}_i D_{ki} = \dot{x}^N(t_k).$$

Therefore,

$$\begin{aligned}
\left\| \sum_{i=0}^{N} \bar{x}_i D_{ki} - f(\bar{x}_k, \bar{u}_k) \right\|_\infty &\le \|\dot{x}^N(t_k) - \dot{x}(t_k)\|_\infty + \|\dot{x}(t_k) - f(\bar{x}_k, \bar{u}_k)\|_\infty \\
&= \|p(t_k) - \dot{x}(t_k)\|_\infty + \|f(x(t_k), u(t_k)) - f(\bar{x}_k, \bar{u}_k)\|_\infty \\
&\le C_1(N-1)^{1-m} + C_2\|x(t_k) - x^N(t_k)\|_\infty \\
&\le C_1(1+2C_2)(N-1)^{1-m}
\end{aligned}$$

where C_2 is the Lipschitz constant of f with respect to x. Since there exists a positive integer N_1 such that, for all $N > N_1$,

$$C_1(1+2C_2)(N-1)^{1-m} \le (N-1)^{\frac{3}{2}-m}.$$

It follows that (11) holds for all $N > N_1$.

As for the constraint (13), because h is continuously differentiable, the following estimate holds.

$$\|h(x(t), u(t)) - h(x^N(t), u(t))\|_{L^\infty} \le C_3\|x(t) - x^N(t)\|_{L^\infty} \le 2C_1C_3(N-1)^{1-m}$$

where C_3 is the Lipschitz constant of h with respect to x which is independent of N. Hence

$$h(\bar{x}_k, \bar{u}_k) \le h(x(t_k), u(t_k)) + 2C_1C_3(N-1)^{1-m} \cdot \mathbf{1} \le 2C_1C_3(N-1)^{1-m} \cdot \mathbf{1}.$$

Thus, the constraint (13) holds for all $N \ge N_1$. As for the endpoint condition (12), it can be proved in a similar fashion. Thus, we have constructed a feasible solution $(\bar{x}_k, \bar{u}_k)$ for Problem B^N. Finally, (15) follows directly from (17). □

Remark 6 From the result of Theorem 1, it is easy to see that the feasible set of Problem B^N is nonempty and compact. Therefore, the existence of the optimal solution is guaranteed by the continuity of the cost function, $\bar{J}^N(\cdot)$.

Remark 7 In practice, we use a small number, $\delta_P > 0$ as a feasibility tolerance. Then, Theorem 1 guarantees that for any δ_P howsoever small, (11–13) always has a solution provided a sufficiently large number of nodes are chosen. Although the exact value of N is unknown because m is usually unknown, it is possible to exploit the qualitative information that connects δ_P with N in designing a practically robust spectral algorithm [42].

4 Convergence of the primal variables

With the existence result in hand, we now establish the convergence of the primal variables, (x, u). That is, we will show the existence of a sequence of optimal solutions of Problem B^N converging to an optimal solution of Problem B. The method generalizes the results in [20] and is similar in spirit to Polak's theory of consistent approximations [36]. We indeed show that, under certain conditions, the sequence of finite dimensional nonlinear programming, Problem B^N, consistently approximates the infinite dimensional continuous optimal control Problem B.

Let $(\bar{x}_k^*, \bar{u}_k^*)$, $k = 0, 1, \ldots, N$, be an optimal solution to Problem B^N. Let $x^N(t) \in \mathbb{R}^{N_x}$ be the Nth order interpolating polynomial of $(\bar{x}_0^*, \ldots, \bar{x}_N^*)$ and $u^N(t) \in \mathbb{R}^{N_u}$ be any interpolant of $(\bar{u}_0^*, \ldots, \bar{u}_N^*)$, i.e.

$$x^N(t) = \sum_{k=0}^{N} \bar{x}_k^* \phi_k(t), \qquad u^N(t) = \sum_{k=0}^{N} \bar{u}_k^* \psi_k(t)$$

where $\phi_k(t)$ is the Lagrange interpolating polynomial defined by (4) and $\psi_k(t)$ is any continuous function such that $\psi_k(t_j) = 1$, if $k = j$ and $\psi_k(t_j) = 0$, if $k \neq j$. Note that $u^N(t)$ is not necessarily a polynomial, but an interpolating function. Now consider a sequence of Problems B^N with N increasing from N_1 to infinity. Correspondingly, we get a sequence of discrete optimal solutions $\{(\bar{x}_k^*, \bar{u}_k^*), k = 0, \ldots, N\}_{N=N_1}^{\infty}$ and their interpolating function sequence $\{x^N(t), u^N(t)\}_{N=N_1}^{\infty}$.

Definition 1 A continuous function, $[-1, 1] \ni t \to \rho(t) \in \mathbb{R}^n$ is called a uniform accumulation point of a function sequence, $\{t \mapsto \rho^N(t)\}_{N=0}^{\infty}$, $t \in [-1, 1]$, if there exists a subsequence of $\{t \mapsto \rho^N(t)\}_{N=0}^{\infty}$ that uniformly converges to $t \mapsto \rho(t)$. Similarly, a point $v \in \mathbb{R}^n$ is called an accumulation point of a sequence $\{v^N\}_{N=0}^{\infty}$, if there exists a subsequence of $\{v^N\}_{N=0}^{\infty}$ that converges to v.

Assumption 1 It is assumed that the sequence, $\{(\bar{x}_0^*, \dot{x}^N(\cdot), u^N(\cdot))\}_{N=N_1}^{\infty}$ has a uniform accumulation point, $(x_0^{\infty}, q(\cdot), u^{\infty}(\cdot))$. Moreover, $t \mapsto q(t)$ and $t \mapsto u^{\infty}(t)$ are continuous on $t \in [-1, 1]$.

Lemma 2 [18] *Let t_k, $k = 0, 1, \ldots, N$, be the LGL nodes, and w_k be the LGL weights. Suppose $\xi(t)$ is Riemann integrable; then,*

$$\int_{-1}^{1} \xi(t)dt = \lim_{N\to\infty} \sum_{k=0}^{N} \xi(t_k)w_k.$$

Theorem 2 *Let $\{(\bar{x}_k^*, \bar{u}_k^*), 0 \leq k \leq N\}_{N=N_1}^{\infty}$ be a sequence of optimal solutions to Problem B^N and $\{t \mapsto (x^N(t), u^N(t))\}_{N=N_1}^{\infty}$ be their interpolating function sequence satisfying Assumption 1. Then, $t \mapsto u^{\infty}(t)$ is an optimal control to the original continuous Problem B, and $x^{\infty}(t) = \int_{-1}^{t} q(\tau)d\tau + x_0^{\infty}$ is the corresponding optimal trajectory.*

Proof See Appendix. □

Theorem 2 demonstrates that Problem B^N is indeed a consistent approximation [36] to the continuous optimal control Problem B. In other words, *if the optimal solution of the discrete-time Problem B^N converges as N increases, then the limit point must be an optimal solution of the continuous-time Problem B*. In practical computation, the assumptions in Theorem 2 can be verified up to a large N as illustrated in Sect. 6. In this sense, Theorem 2 provides a certain level of confidence on the optimality of the computed solutions. An additional level of confidence is provided in terms of the Covector Mapping Theorem discussed in the subsequent sections. It should be pointed out that Theorem 2 does not complete convergence analysis. An important question that remains unanswered is the condition under which a uniform accumulation point exists for the sequence of interpolations of the discrete-time optimal solutions $(\dot{x}^N(t), u^N(t))_{N=N_1}^{\infty}$. It is well-recognized that convergence analysis beyond consistent approximation is an important and difficult problem [9, 22]. Progress on Euler and Runge-Kutta discretizations are provided in [11, 21]. Similar results for PS methods are beyond the scope of the present paper.

5 Convergence of the dual variables

As noted in Sect. 1, convergence of dual variables is a critical issue in discrete approximations to optimal control problems for both theory and practice. Furthermore, in the design of optimal feedback control systems, dual variables play a critical role in problem formulation in the construction of the inner loop [50]. Thus, in designing efficient methods, a study of convergence of dual variables takes center stage. In this section, we explore the link between the KKT multipliers and the discrete-time costates and clarify the Covector Mapping Theorem of [48]. Throughout this section, we assume that Assumption 1 holds.

5.1 Necessary conditions for Problems B^N and B

Construct the Lagrangian for Problem B^N as

$$L^N = \bar{J}^N + \sum_{k=0}^{N} \lambda_k^T \left(-\sum_{i=0}^{N} \bar{x}_i D_{ki} + f(\bar{x}_k, \bar{u}_k) \right) + \bar{\nu}^T e(\bar{x}_0, \bar{x}_N) + \sum_{k=0}^{N} \mu_k^T h(\bar{x}_k, \bar{u}_k)$$

where $\lambda_k \in \mathbb{R}^{N_x}$, $\bar{\nu} \in \mathbb{R}^{N_e}$ and $\mu_k \in \mathbb{R}^{N_h}$ are the KKT multipliers associated with Problem B^N. Let, $\delta_P = (N-1)^{\frac{3}{2}-m_x}$; then, a feasible point is called a KKT point if the KKT conditions are approximately satisfied,

$$\left\| \frac{L}{\lambda_k} \right\|_\infty \le \delta_P, \qquad h(\bar{x}_k, \bar{u}_k) \le \delta_P \cdot \mathbf{1}, \qquad \|e(\bar{x}_0, \bar{x}_N)\|_\infty \le \delta_P, \tag{18}$$

$$\left\| \frac{L}{\bar{u}_k} \right\|_\infty \le \delta_D, \qquad \left\| \frac{L}{\bar{x}_k} \right\|_\infty \le \delta_D, \tag{19}$$

$$\|\mu_k \cdot h(\bar{x}_k, \bar{u}_k)\|_\infty \le \delta_D, \qquad \mu_k \ge -\delta_D \cdot \mathbf{1}, \tag{20}$$

where $k = 0, 1, \dots, N$ and $\mathbf{1} = [1, \dots, 1]^T$ with appropriate dimension and δ_D is a dual feasibility tolerance. A proper selection of δ_D will be apparent shortly. Part of the motivation for δ_D comes from the convergence criteria used in solving NLPs; see for example [19]. Motivated by the results of [45] and [48], we use the discrete weights w_k to scale the KKT multipliers as

$$\bar{\lambda}_k = \frac{\lambda_k}{w_k}, \qquad \bar{\mu}_k = \frac{\mu_k}{w_k}.$$

Then, the KKT conditions can be summarized as follows. (For the purpose of brevity, we omit a detailed derivation of an evaluation and subsequent simplification of (18–20); these steps can be found in [48].)

Problem $B^{N\lambda}$*:* Find $(\bar{x}_k^*, \bar{u}_k^*, \bar{\lambda}_k^*, \bar{\mu}_k^*, \bar{\nu}^*)$, $k = 0, 1, \dots, N$, such that

$$\left\| \sum_{i=0}^{N} \bar{x}_i^* D_{ki} - f(\bar{x}_k^*, \bar{u}_k^*) \right\|_\infty \le \delta_P,$$

$$\|e(\bar{x}_0^*, \bar{x}_N^*)\|_\infty \le \delta_P,$$

$$h(\bar{x}_k^*, \bar{u}_k^*) \le \delta_P \cdot \mathbf{1},$$

$$\left\| w_k \left[\sum_{i=0}^{N} \bar{\lambda}_i^* D_{ki} + F_x(\bar{x}_k^*, \bar{u}_k^*) + f_x^T(\bar{x}_k^*, \bar{u}_k^*) \bar{\lambda}_k^* + h_x^T(\bar{x}_k^*, \bar{u}_k^*) \bar{\mu}_k^* \right] + c_k \right\|_\infty \le \delta_D, \tag{21}$$

$$\|w_k [F_u(\bar{x}_k^*, \bar{u}_k^*) + f_u^T(\bar{x}_k^*, \bar{u}_k^*) \bar{\lambda}_k^* + h_u^T(\bar{x}_k^*, \bar{u}_k^*) \bar{\mu}_k^*]\|_\infty \le \delta_D,$$

$$\|w_k \bar{\mu}_k^* \cdot h(\bar{x}_k^*, \bar{u}_k^*)\|_\infty \le \delta_D,$$

$$w_k \bar{\mu}_k^* \ge -\delta_D \cdot \mathbf{1}$$

where $c_i = 0$ for $2 \le i \le N-1$ and

$$c_0 = \bar{\lambda}_0^* + \frac{\partial E}{\partial x_0}(\bar{x}_0^*, \bar{x}_N^*) + \left(\frac{\partial e}{\partial x_0}(\bar{x}_0^*, \bar{x}_N^*)\right)^T \bar{\nu}^*,$$

$$c_N = -\bar{\lambda}_N^* + \frac{\partial E}{\partial x_N}(\bar{x}_0^*, \bar{x}_N^*) + \left(\frac{\partial e}{\partial x_N}(\bar{x}_0^*, \bar{x}_N^*)\right)^T \bar{\nu}^*.$$

The first-order necessary conditions for Problem B are based on the Minimum Principle that uses the D-form of the Lagrangian [24]. Let $\lambda(t)$ be the costate and $\mu(t)$ be the instantaneous KKT multiplier (covector) associated with the Hamiltonian Minimization Condition. Under suitable constraint qualifications [24], the necessary conditions for Problem B can be summarized as:

Problem B^λ*:* If $(x^*(\cdot), u^*(\cdot))$ is the optimal solution to Problem B, then there exist $(\lambda^*(\cdot), \mu^*(\cdot), \nu^*)$ such that

$$\dot{x}^*(t) = f(x^*(t), u^*(t)), \tag{22}$$

$$\dot{\lambda}^*(t) = -F_x(x^*(t), u^*(t)) - f_x^T(x^*(t), u^*(t))\lambda^*(t) - h_x^T(x^*(t), u^*(t))\mu^*(t), \tag{23}$$

$$0 = F_u(x^*(t), u^*(t)) + f_u^T(x^*(t), u^*(t))\lambda^*(t) + h_u^T(x^*(t), u^*(t))\mu^*(t), \tag{24}$$

$$0 = e(x^*(1), x^*(-1)), \tag{25}$$

$$0 \ge h(x^*(t), u^*(t)), \tag{26}$$

$$0 = \mu^*(t)h(x^*(t), u^*(t)), \qquad \mu^*(t) \ge 0, \tag{27}$$

$$\lambda^*(-1) = -E_{x(-1)}(x^*(-1), x^*(1)) - e_{x(-1)}^T(x^*(-1), x^*(1))\nu^*, \tag{28}$$

$$\lambda^*(1) = E_{x(1)}(x^*(-1), x^*(1)) + e_{x(1)}^T(x^*(-1), x^*(1))\nu^*. \tag{29}$$

The discretization of Problem B^λ is denoted as Problem $B^{\lambda N}$ and can be summarized as:

Problem $B^{\lambda N}$*:* Find $\bar{x}_k, \bar{u}_k, \bar{\lambda}_k, \bar{\mu}_k$, $k = 0, 1, \ldots, N$, and $\bar{\nu}_0, \bar{\nu}_N$ such that

$$\left\| \sum_{i=0}^{N} \bar{x}_i D_{ki} - f(\bar{x}_k, \bar{u}_k) \right\|_\infty \le \delta_P,$$

$$\|e(\bar{x}_0, \bar{x}_N)\|_\infty \le \delta_P,$$

$$h(\bar{x}_k, \bar{u}_k) \le \delta_P \cdot \mathbf{1},$$

$$\left\| \sum_{i=0}^{N} \bar{\lambda}_i D_{ki} + F_x(\bar{x}_k, \bar{u}_k) + f_x^T(\bar{x}_k, \bar{u}_k)\bar{\lambda}_k + h_x^T(\bar{x}_k, \bar{u}_k)\bar{\mu}_k \right\|_\infty \le \delta_D,$$

$$\|F_u(\bar{x}_k, \bar{u}_k) + f_u^T(\bar{x}_k, \bar{u}_k)\bar{\lambda}_k + h_u^T(\bar{x}_k, \bar{u}_k)\bar{\mu}_k\|_\infty \le \delta_D,$$

$$\|\bar{\mu}_k \cdot h(\bar{x}_k, \bar{u}_k)\|_\infty \le \delta_D, \quad \bar{\mu}_k \ge -\delta_D \cdot \mathbf{1},$$

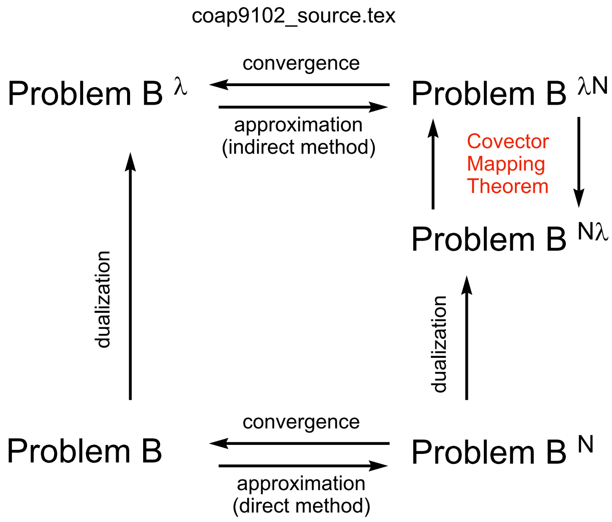

Fig. 1 Schematic for the covector mapping principle

$$\left\| \bar{\lambda}_0 + \frac{\partial E}{\partial x_0}(\bar{x}_0, \bar{x}_N) + \left(\frac{\partial e}{\partial x_0}(\bar{x}_0, \bar{x}_N)\right)^T \bar{\nu} \right\|_\infty \leq \delta_D,$$
$$\left\| \bar{\lambda}_N - \frac{\partial E}{\partial x_N}(\bar{x}_0, \bar{x}_N) - \left(\frac{\partial e}{\partial x_N}(\bar{x}_0, \bar{x}_N)\right)^T \bar{\nu} \right\|_\infty \leq \delta_D.$$

Although they appear to be similar, it is apparent that Problem $B^{\lambda N}$ is not the same as Problem $B^{N\lambda}$. That is, as illustrated in Fig. 1, it is clear that dualization and discretization are not necessarily commutative operations. As noted earlier, a similar observation has been made by Hager on RK methods. Note however, that unlike a Runge-Kutta method, the order and scheme of the discretization is naturally preserved in a PS discretization without any additional conditions. The main points of Fig. 1 are illustrated by the following example (which is a counter example to the widely-held notion that if the primals converge, the KKT multipliers associated with the discretized dynamic constraints converge to the costates).

Example 2 Minimize $J[x(\cdot), u(\cdot)] = x(2)$, subject to

$$\dot{x}(t) = u(t), \quad t \in [0, 2], \tag{30}$$
$$x(0) = 0, \quad u(t) \geq -1. \tag{31}$$

The necessary conditions

$$\begin{aligned} &\dot{\lambda}^*(t) = 0, \quad \lambda^*(2) = 1, \\ &\lambda^*(t) - \mu^*(t) = 0, \\ &\mu^*(t)(-u^*(t) - 1) = 0, \quad \mu^*(t) \geq 0 \end{aligned} \tag{32}$$

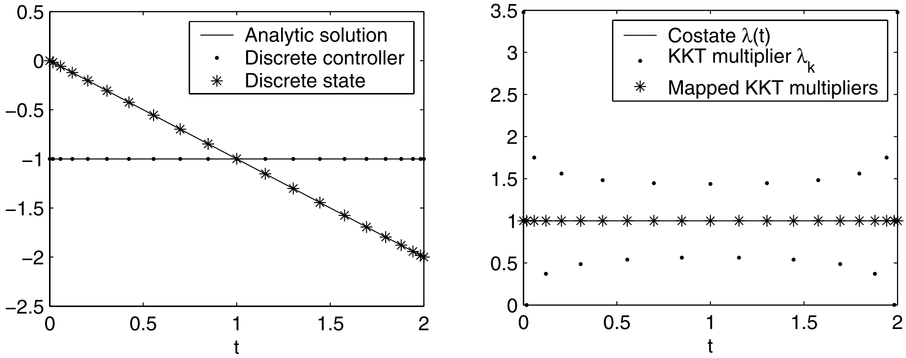

Fig. 2 Discrete PS solution with $N = 20$

uniquely determine the optimal solution as

$$x^*(t) = -t, \quad u^*(t) = -1,$$
$$\lambda^*(t) = 1, \quad \mu^*(t) = 1.$$

A PS solution for 20 nodes is shown in Fig. 2.

The left plot clearly shows that the primal variables $(\bar{x}_k^*, \bar{u}_k^*)$ coincide with the analytic solution $t \mapsto (x^*(t), u^*(t))$. On the other hand, the right plot shows that the weighted KKT multipliers $\bar{\lambda}_k^*$, do not agree with the costate, $\lambda^*(\cdot)$. If the unweighted KKT multiplies are used, the disagreements between the multipliers are even worse (as expected). Clearly, the convergence of the discretized primals does not imply the convergence of the KKT multipliers to the continuous costates. To clarify this point, consider the PS discretization of (30), (31). We ignore the tolerances, δ_P and δ_D, justified by the fact that the optimal continuous-time solutions being polynomials, the discretized problem can be posed exactly without introducing any feasibility problem. Thus, an application of our method yields,

$$\begin{aligned} &\text{Minimize} \quad \bar{J}^N = \bar{x}_N, \\ &\text{subject to} \quad \bar{x}_0 = 0, \quad \bar{u}_k \geq -1, 0 \leq k \leq N \quad \text{and} \\ &\qquad D \begin{pmatrix} \bar{x}_0 \\ \vdots \\ \bar{x}_N \end{pmatrix} = \begin{pmatrix} \bar{u}_0 \\ \vdots \\ \bar{u}_N \end{pmatrix}. \end{aligned} \tag{33}$$

It is easy to show that, for any N, the discretized problem admits a unique globally optimal solution: $\bar{u}_k^* = -1$, $\bar{x}_k^* = -t_k$, $0 \leq k \leq N$, where t_k are the LGL nodes. As a matter of fact, from

$$[w_0, w_1, \ldots, w_N] \cdot D = [-1, 0, \ldots, 0, 1]$$

we have

$$[w_0, w_1, \dots, w_N] \cdot \begin{pmatrix} \bar{u}_0 \\ \vdots \\ \bar{u}_N \end{pmatrix} = [w_0, w_1, \dots, w_N] D \begin{pmatrix} \bar{x}_0 \\ \vdots \\ \bar{x}_N \end{pmatrix} = \bar{x}_N.$$

Therefore,

$$\bar{x}_N = \sum_{k=0}^{N} \bar{u}_k w_k \geq -\sum_{k=0}^{N} w_k = -2$$

and the equality holds if and only if $\bar{u}_k = -1$, for all k. With this result, it is easy to show that $\bar{x}_k = -t_k$. Thus, for any fixed N, the discrete optimal solution, $(\bar{x}_k^*, \bar{u}_k^*) = (-t_k, -1)$, exists and converges to the continuous-time optimal solution. The left plot in Fig. 2 demonstrates this point.

Next, the KKT conditions for the discrete problem are

$$D \begin{pmatrix} \bar{\lambda}_0^* \\ \bar{\lambda}_1^* \\ \vdots \\ \bar{\lambda}_{N-1}^* \\ \bar{\lambda}_N^* \end{pmatrix} = \begin{pmatrix} \bar{\nu} \\ 0 \\ \vdots \\ 0 \\ (\bar{\lambda}_N^* - 1)/w_N \end{pmatrix}, \tag{34}$$

$$\bar{\lambda}_k^* = \bar{\mu}_k^*, \quad 0 \leq k \leq N, \tag{35}$$

$$\bar{\mu}_k^*(-u_k^* - 1) = 0, \quad \bar{\mu}_k^* \geq 0 \tag{36}$$

where $\bar{\nu}$ is the multiplier associated with the initial condition. Since the constraint $\bar{u}_k \geq -1$ is always active at the optimal solution, $\bar{\mu}_k^*$ is undetermined in (36). In addition, (34) has infinitely many solutions since there are $N + 2$ variables, i.e., $(\bar{\lambda}_0^*, \dots, \bar{\lambda}_N^*, \bar{\nu})$, but only $N + 1$ consistent equations. In other words, the KKT multipliers are not unique although the optimal primal solution is unique. It is also straightforward to show that the linear independence constraint qualification is violated in this example but the weaker Mangasarian-Fromovitz constraint qualification [32] holds. Thus, the KKT multipliers exist but are not unique.

From an optimal control perspective [41], there is a loss of information resulting from the act of discretization that is independent of the mesh size. This information loss is different from the well-known notion of hidden convexity [30, 31], since the differential inclusion, $\dot{x} \in \{u : u \geq -1\}$ is convex. The information loss can be restored by supplying the missing information,

$$\bar{\lambda}_N^* = 1$$

to the dual feasibility conditions. This missing condition is obtained simply by comparing (34) with (32). With this additional condition, it is easy to see that the KKT conditions (34–36) admit a unique solution. This is plotted in the right plot of Fig. 2 indicating a perfect match with the costate $t \mapsto \lambda(t)$.

5.2 The augmented KKT conditions

By comparing Problem $B^{\lambda N}$ with Problem $B^{N\lambda}$, it is apparent that the transversality conditions (28), (29) are missing in the KKT conditions. Alternatively, the costate differential equations are not naturally collocated at the boundary points, -1 and 1. By restoring this information loss to the KKT conditions, the KKT multipliers can be mapped to the discretized covectors associated with Problem B^{λ}. More specially, the following conditions are needed in addition to the KKT conditions

$$\left\| -\bar{\lambda}_0^* - \frac{\partial E}{\partial x_0}(\bar{x}_0^*, \bar{x}_N^*) - \left(\frac{\partial e}{\partial x_0}(\bar{x}_0^*, \bar{x}_N^*)\right)^T \bar{\nu}^* \right\|_\infty \leq \delta_D, \tag{37}$$

$$\left\| \bar{\lambda}_N^* - \frac{\partial E}{\partial x_N}(\bar{x}_0^*, \bar{x}_N^*) - \left(\frac{\partial e}{\partial x_N}(\bar{x}_0^*, \bar{x}_N^*)\right)^T \bar{\nu}^* \right\|_\infty \leq \delta_D. \tag{38}$$

These equations generalize the "closure conditions" identified in [48]. They lead to a proof of Theorem 3 which clarifies the Covector Mapping Theorem [48].

Theorem 3 (Covector Mapping Theorem) *Given any feasible solution, $t \mapsto (x(t), u(t), \lambda(t), \nu)$, for Problem B^{λ}, suppose $x(\cdot) \in W^{m_x,\infty}$ and $\lambda(\cdot) \in W^{m_\lambda,\infty}$ with $m_x, m_\lambda \geq 2$. Then, there exists a positive integer N_2 such that, for any $N > N_2$, the augmented KKT conditions, i.e.,* (18–20) *plus* (37), (38), *have a feasible solution with a primal feasibility tolerance of $\delta_P = (N-1)^{\frac{3}{2}-m_x}$ and a dual feasibility tolerance of $\delta_D = (N-1)^{\frac{3}{2}-m}$, where $m = \min\{m_x, m_\lambda\}$.*

The proof of this theorem is very similar to the proof of Theorem 1. The basic idea is to construct a discrete-time solution around the continuous solution of Problem B^{λ}, such that it satisfies both the KKT conditions and the transversality conditions (37), (38). For the purpose of brevity, we skip this proof.

Remark 8 In practice, we often observe the convergence of the primal variables, and as illustrated in Example 2, the KKT multipliers do not converge to the continuous-time covectors. In the absence of Theorem 3, the existence of a solution to the augmented KKT conditions was in doubt. Theorem 3 guarantees the existence of a solution to both the KKT conditions and the augmented KKT conditions. When multiple solutions exist for the KKT multipliers, the closure conditions, (37), (38), act as a selection criterion in picking the correct set of KKT multipliers that map to the continuous-time covectors. In the event the KKT conditions admit a unique solution, the closure conditions do not introduce an infeasibility problem into the augmented KKT conditions.

We now establish a final theorem on the convergence of the sequence of the mapped dual variables. This is done in a manner similar to the analysis of the convergence of the primal variables. Let $(\bar{x}_k^*, \bar{u}_k^*, \bar{\lambda}_k^*, \bar{\mu}_k^*, \bar{\nu}^*)$, $k = 0, 1, \ldots, N$, be a solution to the augmented KKT conditions, i.e., Problem $B^{N\lambda}$ plus the closure conditions (37–(38). Consider a sequence of the augmented KKT conditions with N increasing from N_2 to infinity. Correspondingly we get a sequence of discrete solutions

$\{(\bar{x}_k^*, \bar{u}_k^*, \bar{\lambda}_k^*, \bar{\mu}_k^*, \bar{\nu}^*), k = 0, 1, \ldots, N\}_{N=N_2}^{\infty}$. Furthermore, let $t \mapsto (x^N(t), \lambda^N(t))$ denote Nth order interpolating polynomials of $(\bar{x}_k^*, \bar{\lambda}_k^*)$, and $t \mapsto (u^N(t), \mu^N(t))$ as any interpolating function of $(\bar{u}_k^*, \bar{\mu}_k^*)$, i.e.

$$x^N(t) = \sum_{k=0}^{N} \bar{x}_k^* \phi_k(t), \qquad u^N(t) = \sum_{k=0}^{N} \bar{u}_k^* \psi_k(t),$$

$$\lambda^N(t) = \sum_{k=0}^{N} \bar{\lambda}_k^* \phi_k(t), \qquad \mu^N(t) = \sum_{k=0}^{N} \bar{\mu}_k^* \psi_k(t),$$

where $\phi_k(t)$ is the Lagrange interpolating polynomial defined by (4) and $\psi_k(t)$ is any continuous function such that $\psi_k(t_j) = 1$, if $k = j$ and $\psi_k(t_j) = 0$, if $k \neq j$. For instance, $\psi_k(t)$ can be a linear or spline interpolant.

Assumption 2 It is assumed that the sequence $\{(\bar{x}_0^*, \bar{\lambda}_0^*, \bar{\nu}^*, \dot{x}^N(\cdot), u^N(\cdot), \dot{\lambda}^N(\cdot), \mu^N(\cdot))\}_{N=N_2}^{\infty}$ has a uniform accumulation point $(x_0^{\infty}, \lambda_0^{\infty}, \bar{\nu}^{\infty}, \eta(\cdot), u^{\infty}(\cdot), \rho(\cdot), \mu^{\infty}(\cdot))$, where $t \mapsto (\eta(t), u^{\infty}(t), \rho(t), \mu^{\infty}(t))$ are continuous functions over $t \in [-1, 1]$.

Theorem 4 *Let $\delta_P = (N-1)^{\frac{3}{2}-m_x}$ with $m_x \geq 2$ and $\delta_D = (N-1)^{\frac{3}{2}-m}$ with $m \geq 4$. Let $\{t \mapsto (x^N(t), u^N(t), \lambda^N(t), \mu^N(t))\}_{N=N_2}^{\infty}$ be a sequence of interpolating functions constructed from solutions to the augmented KKT conditions. Suppose Assumption 2 holds. Then the functions $(x^{\infty}(t), u^{\infty}(t), \lambda^{\infty}(t), \mu^{\infty}(t))$ satisfy all the necessary conditions for optimality as indicated by Problem B^{λ}, where*

$$x^{\infty}(t) = \int_{-1}^{t} \eta(\tau) d\tau + x_0^{\infty},$$

$$\lambda^{\infty}(t) = \int_{-1}^{t} \rho(\tau) d\tau + \lambda_0^{\infty}.$$

A sketch of the proof This theorem can be proved in the same manner as Theorem 2. Therefore only a sketch is provided. In the proof of Theorem 2, we already showed that $(x^{\infty}(\cdot), u^{\infty}(\cdot))$ satisfy the state dynamics (22), path constraint (26) and endpoint condition (25). As for the adjoint equation, suppose $t \mapsto (x^{\infty}(t), \lambda^{\infty}(t), u^{\infty}(t), \mu^{\infty}(t))$ does not satisfy (23). Then there is a time instance, $t' \in [-1, 1]$, such that

$$\begin{aligned}&\dot{\lambda}^{\infty}(t') + F_x(x^{\infty}(t'), u^{\infty}(t')) + f_x^T(x^{\infty}(t'), u^{\infty}(t'))\lambda^{\infty}(t') \\ &\quad + h_x^T(x^{\infty}(t'), u^{\infty}(t'))\mu^{\infty}(t') \neq 0.\end{aligned}$$

By assumption, it is easy to show that $\{t \mapsto (x^N(t), \dot{x}^N(t), u^N(t), \lambda^N(t), \dot{\lambda}^N(t), \mu^N(t))\}_{N=N_2}^{\infty}$ has a subsequence that converges uniformly to $\{t \mapsto (x^{\infty}(t), \dot{x}^{\infty}(t), u^{\infty}(t), \lambda^{\infty}(t), \dot{\lambda}(t), \mu^{\infty}(t))\}$. Denote the subsequence as $\{N = M_i\}_{i=1}^{\infty}$, $M_i \in$

$\{1, 2, \ldots\}$ with $\lim_{i\to\infty} M_i = \infty$. Since the LGL nodes, t_k, $k = 0, 1, \ldots, N$, are dense in $[-1, 1]$ as $N \to \infty$ [18], there exists a sequence k^{M_i} satisfying

$$0 < k^{M_i} < M_i \quad \text{and} \quad \lim_{i\to\infty} t_{k^{M_i}} = t'.$$

Thus

$$\begin{aligned}
&\dot{\lambda}^{\infty}(t') + F_x(x^{\infty}(t'), u^{\infty}(t')) + f_x^T(x^{\infty}(t'), u^{\infty}(t'))\lambda^{\infty}(t') \\
&\quad + h_x^T(x^{\infty}(t'), u^{\infty}(t'))\mu^{\infty}(t') \\
&= \lim_{i\to\infty} \{\dot{\lambda}^{M_i}(t_{k^{M_i}}) + F_x(x^{M_i}(t_{k^{M_i}}), u^{M_i}(t_{k^{M_i}})) \\
&\quad + f_x^T(x^{M_i}(t_{k^{M_i}}), u^{M_i}(t_{k^{M_i}}))\lambda^{M_i}(t_{k^{M_i}}) \\
&\quad + h_x^T(x^{M_i}(t_{k^{M_i}}), u^{M_i}(t_{k^{M_i}}))\mu^{M_i}(t_{k^{M_i}})\} \neq 0.
\end{aligned} \tag{39}$$

Because $\lambda^N(\cdot)$ is an Nth order polynomial, we have

$$\dot{\lambda}^{M_i}(t_{k^{M_i}}) = \sum_{j=0}^{M_i} \bar{\lambda}_j^* D_{k^{M_i},j}.$$

On the other hand, from (21) and the fact $k^{M_i} \neq 0, M_i$, we have

$$\begin{aligned}
&\lim_{i\to\infty} \big\| \dot{\lambda}^{M_i}(t_{k^{M_i}}) + F_x(x^{M_i}(t_{k^{M_i}}), u^{M_i}(t_{k^{M_i}})) + f_x^T(x^{M_i}(t_{k^{M_i}}), u^{M_i}(t_{k^{M_i}}))\lambda^{M_i}(t_{k^{M_i}}) \\
&\quad + h_x^T(x^{M_i}(t_{k^{M_i}}), u^{M_i}(t_{k^{M_i}}))\mu^{M_i}(t_{k^{M_i}}) \big\|_{\infty} \\
&\leq \lim_{i\to\infty} \frac{\delta_D}{w_{k^{M_i}}} \leq \lim_{i\to\infty} \frac{M_i(M_i+1)}{2}\delta_D \\
&= \lim_{i\to\infty} \frac{M_i(M_i+1)}{2}(M_i-1)^{\frac{3}{2}-m} = 0
\end{aligned} \tag{40}$$

where the last few equations in (40) follow from the fact that for any N, $w_k \geq \frac{2}{N(N+1)}$, $0 \leq k \leq N$ [18], and the assumption $m \geq 4$. Because (40) contradicts (39), the adjoint equation, (23), holds for all $t \in [-1, 1]$.

Equations (24), (27), (28) and (29) can be proved in a similar way. Note that to prove transversality conditions (28) and (29), the closure conditions (37) and (38) have to be used.

Remark 9 The regularity of the optimal solution required in Theorems 3 and 4 are different. To prove the feasibility of the augmented KKT system, we need $(x^*(\cdot), \lambda^*(\cdot)) \in W^{m,\infty}$ with $m \geq 2$. But to prove the convergence of the costate, we need $(x^*(\cdot), \lambda^*(\cdot)) \in W^{m,\infty}$ with $m \geq 4$. This requirement on higher regularity comes from the fact that in Problem $B^{N\lambda}$ the adjoint equations are naturally weighted by w_k which converges to zero at a quadratic rate as $N \to \infty$; consequently, we need the dual feasibility tolerance δ_D to converge faster than w_k to ensure the convergence

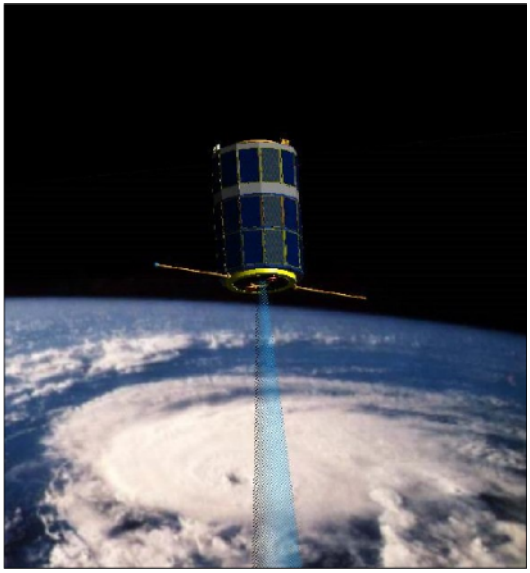

Fig. 3 Artist's view of NPSAT1 in orbit

of the discrete adjoint equations. Therefore, to complete the loop in Fig. 1, a higher regularity of the optimal solution is required.

Remark 10 Theorems 3 and 4 were developed by choosing δ_D to be independent of $k = 0, 1, \ldots, N$. If δ_D is also weighted by w_k, it is possible to reduce the regularity required in Theorem 4 to $m \geq 2$; this would, however, require a substantially longer derivation of the results. In order to maintain the focus of this paper to the main points, we assumed a stronger-than-necessary regularity in Theorem 4.

6 Optimal attitude control of NPSAT1 spacecraft

NPSAT1 is a small satellite being built at the Naval Postgraduate School, and is scheduled to be launched in September 2007. It is currently in an assembly stage. An artist's view of it in orbit is shown in Fig. 3. The spacecraft uses magnetic actuators and a pitch momentum wheel for attitude control. One experiment onboard the NPSAT1 spacecraft is to demonstrate in flight the application of the PS method for time-optimal attitude maneuvers. Choosing the standard quaternion and body rates as the state variables, the dynamical equations of motion for NPSAT1 are given by [17]:

$$\dot{q}_1(t) = \frac{1}{2}[\omega_x(t)q_4(t) - \omega_y(t)q_3(t) + \omega_z(t)q_2(t) + \omega_0 q_3(t)], \tag{41}$$

$$\dot{q}_2(t) = \frac{1}{2}[\omega_x(t)q_3(t) + \omega_y(t)q_4(t) - \omega_z(t)q_1(t) + \omega_0 q_4(t)], \tag{42}$$

$$\dot{q}_3(t) = \frac{1}{2}[-\omega_x(t)q_2(t) + \omega_y(t)q_1(t) + \omega_z(t)q_4(t) - \omega_0 q_1(t)], \tag{43}$$

$$\dot{q}_4(t) = \frac{1}{2}[-\omega_x(t)q_1(t) - \omega_y(t)q_2(t) - \omega_z(t)q_3(t) - \omega_0 q_2(t)], \tag{44}$$

$$\dot{\omega}_x(t) = \frac{I_2 - I_3}{I_1}\left[\omega_y(t)\omega_z(t) - 3\frac{\mu}{r_0^3}C_{23}(q(t))C_{33}(q(t))\right] + \frac{1}{I_1}[B_z(q(t),t)u_2(t) - B_y(q(t),t)u_3(t)], \tag{45}$$

$$\dot{\omega}_y(t) = \frac{I_3 - I_1}{I_2}\left[\omega_x(t)\omega_z(t) - 3\frac{\mu}{r_0^3}C_{13}(q(t))C_{33}(q(t))\right] + \frac{1}{I_2}[B_x(q(t),t)u_3(t) - B_z(q(t),t)u_1(t)], \tag{46}$$

$$\dot{\omega}_z(t) = \frac{I_1 - I_2}{I_3}\left[\omega_x(t)\omega_y(t) - 3\frac{\mu}{r_0^3}C_{13}(q(t))C_{23}(q(t))\right] + \frac{1}{I_3}[B_y(q(t),t)u_1(t) - B_x(q(t),t)u_2(t)] \tag{47}$$

where ω_0 is angular velocity of the orbit with respect to the inertial frame; (I_1, I_2, I_3) are the principal moments of inertia of NPSAT1; $\mu = 3.98601 \times 10^{14}$ m^3/s^2 is Earth's gravitational constant; r_0 is the distance from the mass center of NPSAT1 to the center of the Earth; $C_{ij}(q)$ denote the quaternion-parameterized ijth element of the matrix,

$$C(q) = \begin{bmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1q_2 + q_3q_4) & 2(q_1q_3 - q_2q_4) \\ 2(q_1q_2 - q_3q_4) & q_2^2 - q_1^2 - q_3^2 + q_4^2 & 2(q_2q_3 + q_1q_4) \\ 2(q_1q_3 + q_2q_4) & 2(q_2q_3 - q_1q_4) & q_3^2 - q_1^2 - q_2^2 + q_4^2 \end{bmatrix} \in SO(3).$$

$(B_x(q,t), B_y(q,t), B_z(q,t))$ are the components of the Earth's magnetic field in the body frame,

$$[B_x(q,t), B_y(q,t), B_z(q,t)]^T = C(q)[B_1(t), B_2(t), B_3(t)]^T. \tag{48}$$

$(B_1(t), B_2(t), B_3(t))$ are the time-varying components of the Earth's magnetic field in the orbit frame [17]. The controls, $(u_1, u_2, u_3) \in \mathbb{R}^3$, are the dipole moments on NPSAT1 that are bounded by the maximum available dipole moment, $|u_i| \leq 30$ A m^2, $i = 1, 2, 3$. Clearly, the dynamics of NPSAT1 are quite complex with substantial nonlinearities. Note also that the system is not autonomous. Furthermore, that the quaternions must lie on S^3 is given by the state variable constraint, $q_1^2(t) + q_2^2(t) + q_3^2(t) + q_4^2(t) = 1$. Thus, the NPSAT1 control system contains both state and control constraints.

A benchmark optimal control problem for the NPSAT1 spacecraft is a minimum time slew for a horizon-to-horizon scan. A solution to this problem is expected to demonstrate the agility of NPSAT1; however, it is well-known that minimum-time control problems typically take a substantial amount of energy. Hence, an alternative benchmark control problem for the NPSAT1 is a minimum-energy control problem with the final time fixed near the minimum time. This problem is expected to demonstrate the agility of NPSAT1 while consuming the least amount of electrical energy.

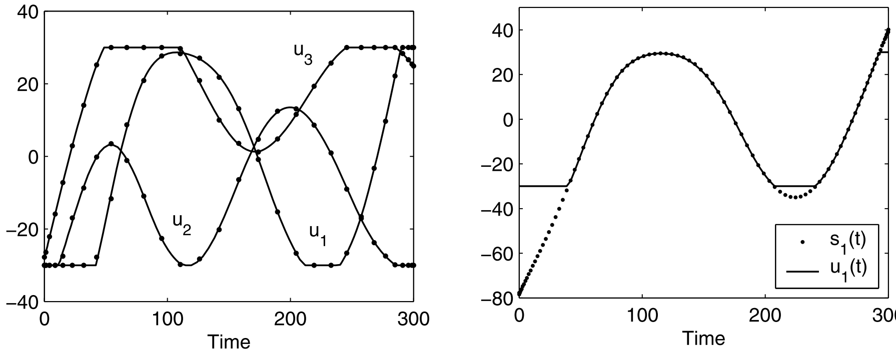

Fig. 4 The *left plot* shows the controls for 30 and 100 nodes. The *solid lines* are generated from a 100-node solution with linear interpolation. The *dotted lines* are discrete optimal controls for 30 nodes. The *right figure* shows the switching function, $t \mapsto s_1(t)$, and the corresponding control, $t \mapsto u_1(t)$

The electrical power is given by the square of the current flow through the magnetic coils; hence, the quadratic cost functional,

$$J[x(\cdot), u(\cdot)] = \int_{t_0}^{t_f} u_1^2(t) + u_2^2(t) + u_3^2(t)\, dt$$

directly measures energy. A benchmark set of endpoint conditions for NPSAT1 are given by [17],

$$\begin{aligned}
&[t_0, t_f] = [0, 300],\\
&[q(t_0), \omega(t_0)] = [0, 0, 0, 1, 0, -0.0011, 0], \qquad (49)\\
&[q(t_f), \omega(t_f)] = [\sin(\phi/2), 0, 0, \cos(\phi/2), 0, 7.725 \times 10^{-4}, 7.725 \times 10^{-4}]
\end{aligned}$$

where $\phi = 135°$ is the principal rotation angle. All of the following solutions to this problem were obtained by way of DIDO [42], and using the NPSAT1 model parameters: $(I_1, I_2, I_3) = (5, 5.1, 2)$ kg m^2; $\omega_0 = 0.00108$ rad/s; $r_0 = 6938$ km and $i = 35.4°$.

A candidate control solution to the optimal control problem is shown in the left plot of Fig. 4 for 30 and 100 nodes. From the negligible difference between the 30-node solution and the 100-node solution, it is apparent that the discrete optimal solution has indeed converged. Furthermore, despite the fact that the NPSAT1 system is highly nonlinear, it is clear that a mere 30-node solution is quite adequate for practical purposes. As a matter of fact, the states obtained from a numerical (RK4/5) propagation of the 30-node discrete-time optimal controller is shown in Fig. 5 along with the 30-node state solution. Clearly, the discrete optimal states match the propagated trajectory quite accurately, which numerically demonstrates the feasibility and accuracy of the 30-node discrete optimal controller.

To further investigate the optimality of the computed solution, we apply Pontryagin's Minimum Principle. From the Hamiltonian Minimization Condition, it is quite

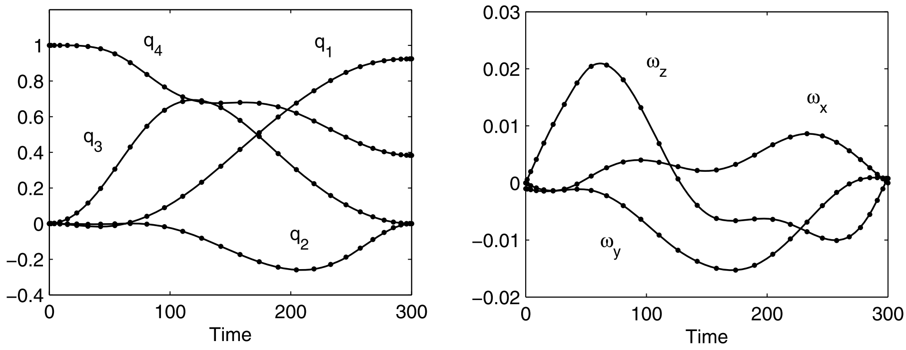

Fig. 5 Quaternions and angular velocities. *Dots* represent the discrete optimal trajectory from a 30-node solution. *Solid lines* are the state trajectories generated by propagating the 30-node discrete optimal controller

straightforward to show that a candidate optimal controller must satisfy,

$$u_1(t) = \begin{cases} -30; & \text{if } s_1(t) \le -30, \\ s_1(t); & \text{if } -30 < s_1(t) < 30, \\ 30; & \text{if } s_1(t) \ge 30 \end{cases}$$

where $s_1(t) = -\frac{1}{2}(\frac{\lambda_7(t)}{I_3} B_y(q,t) - \frac{\lambda_6(t)}{I_2} B_z(q,t))$. Obviously, a check on the optimality (extremality) of the solution via the Minimum Principle requires a computation of the costate trajectory, $t \mapsto \lambda(t)$. We compute these costates by way of the Covector Mapping Theorem. A plot of the control, $t \mapsto u_1(t)$, and its switching function, $t \mapsto s_1(t)$, for N=100 is shown in the right plot of Fig. 4. It is clear that the costates obtained from the Covector Mapping Theorem together with the candidate optimal control, $t \mapsto u_1(t)$, satisfy the Hamiltonian Minimization Condition. In other words, the Covector Mapping Theorem facilitates quick checks on the optimality of the computed solution without the trials and tribulations of generating and solving two-point-boundary-value problems. Tests on other controls and their switching functions reveal the same conclusion. We omit these plots for the purposes of brevity.

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Appendix

Proof of Theorem 2 By assumption, there is a subsequence $N_i \in 1, 2, \dots,$ with $\lim_{i\to\infty} N_i = \infty$ such that

$$\lim_{i\to\infty} (\dot{x}^{N_i}(t), u^{N_i}(t)) = (q(t), u^{\infty}(t))$$

uniform in t. It is easy to show (under Assumption 1)

$$\lim_{i\to\infty} x^{N_i}(t) = x^{\infty}(t) \tag{50}$$

uniformly on $t \in [-1, 1]$. The remaining part of the proof is broken into three steps. First, we show that $(x^{\infty}(t), u^{\infty}(t))$ is a feasible solution to Problem B. Then, we prove the convergence of the cost function $\bar{J}^{N_i}(\bar{X}^*, \bar{U}^*)$ to the continuous cost function $J(x^{\infty}(\cdot), u^{\infty}(\cdot))$, and finally show that $(x^{\infty}(t), u^{\infty})$ is indeed an optimal solution of Problem B.

Step 1 To prove that $(x^{\infty}(t), u^{\infty}(t))$ is a feasible solution to Problem B, we first need to show that $(x^{\infty}(t), u^{\infty}(t))$ satisfies the state equation (1). By the contradiction argument, suppose $(x^{\infty}(t), u^{\infty}(t))$ is not a solution of the differential equation (1). Then there is a time $t' \in [-1, 1]$ so that

$$\dot{x}^{\infty}(t') - f(x^{\infty}(t'), u^{\infty}(t')) \neq 0.$$

Since the LGL nodes t_k are dense with $N \to \infty$ [18], there exists a sequence k^{N_i} satisfying

$$0 < k^{N_i} < N_i \quad \text{and} \quad \lim_{i\to\infty} t_{k^{N_i}} = t'.$$

Thus

$$\dot{x}^{\infty}(t') - f(x^{\infty}(t'), u^{\infty}(t')) = \lim_{i\to\infty} (\dot{x}^{N_i}(t_{k^{N_i}}) - f(x^{N_i}(t_{k^{N_i}}), u^{N_i}(t_{k^{N_i}})) \neq 0. \tag{51}$$

Because $x^N(t)$ is a Nth order polynomial, we have

$$\dot{x}^{N_i}(t_{k^{N_i}}) = \sum_{j=0}^{N_i} \bar{x}_j^* D_{k^{N_i} j}.$$

Thus from (11) and the fact that $(x^N(t), u^N(t))$ are the interpolating functions of $\{(\bar{x}_k^*, \bar{u}_k^*), 0 \le k \le N\}$, the following holds

$$\lim_{i\to\infty} (\dot{x}^{N_i}(t_{k^{N_i}}) - f(x^{N_i}(t_{k^{N_i}}), u^{N_i}(t_{k^{N_i}})) = \lim_{i\to\infty} (N_i - 1)^{\frac{3}{2}-m} = 0. \tag{52}$$

This contradicts (51); therefore, $(x^{\infty}(t), u^{\infty}(t))$ must be a solution of the differential equation (1).

The path constraint can be proved by the same contradiction argument. As for the end-point condition $e(x^{\infty}(-1), x^{\infty}(1)) = 0$, it follows directly from the convergence property, since

$$e(x^{\infty}(-1), x^{\infty}(1)) = \lim_{i\to\infty} e(x^{N_i}(-1), x^{N_i}(1)) = \lim_{i\to\infty} e(\bar{x}_0^*, \bar{x}_{N_i}^*) = 0. \tag{53}$$

Step 2 In this step, we will show that

$$\lim_{i\to\infty} \bar{J}^{N_i}(\bar{X}^*, \bar{U}^*) = J(x^\infty(\cdot), u^\infty(\cdot)),$$

where

$$\bar{J}^{N_i}(\bar{X}^*, \bar{U}^*) = E(\bar{x}_0^*, \bar{x}_{N_i}^*) + \sum_{k=0}^{N_i} F(\bar{x}_k^*, \bar{u}_k^*)w_k,$$

$$J(x^\infty(\cdot), u^\infty(\cdot)) = E(x^\infty(-1), x^\infty(1)) + \int_{-1}^{1} F(x^\infty(t), u^\infty(t))dt.$$

Since $(x^{N_i}(t), u^{N_i}(t))$ converge to $(x^\infty(t), u^\infty(t))$ uniformly, we have,

$$\lim_{i\to\infty} \|x^{N_i}(t_k) - x^\infty(t_k)\|_\infty = \lim_{i\to\infty} \|\bar{x}_k^* - x^\infty(t_k)\|_\infty = 0, \tag{54}$$

$$\lim_{i\to\infty} \|u^{N_i}(t_k) - u^\infty(t_k)\|_\infty = \lim_{i\to\infty} \|\bar{u}_k^* - u^\infty(t_k)\|_\infty = 0 \tag{55}$$

uniformly in k. Therefore, by the fact that $F(x, u)$ is continuously differentiable, there exists a constant $M > 0$ independent of N_i, such that

$$\|F(x^\infty(t_k), u^\infty(t_k)) - F(\bar{x}_k^*, \bar{u}_k^*)\|_\infty \le M(\|x^\infty(t_k) - \bar{x}_k^*\|_\infty + \|u^\infty(t_k) - \bar{u}_k^*\|_\infty)$$

for all $0 \le k \le N_i$. Furthermore, $F(x^\infty(t), u^\infty(t))$ is continuous in t. Thus, by Lemma 2, we have

$$\int_{-1}^{1} F(x^\infty(t), u^\infty(t))dt = \lim_{i\to\infty} \sum_{k=0}^{N_i} F(x^\infty(t_k), u^\infty(t_k))w_k.$$

Therefore,

$$\begin{aligned}
&\int_{-1}^{1} F(x^\infty(t), u^\infty(t))dt \\
&\quad = \lim_{i\to\infty} \left(\sum_{k=0}^{N_i} F(\bar{x}_k^*, \bar{u}_k^*)w_k + \sum_{k=0}^{N_i} [F(x^\infty(t_k), u^\infty(t_k)) - F(\bar{x}_k^*, \bar{u}_k^*)]w_k \right).
\end{aligned}$$

From the uniform convergence of (54) and (55) and the property of w_k, $\sum_{k=0}^{N} w_k = 2$, we know that

$$\begin{aligned}
&\lim_{i\to\infty} \left\| \sum_{k=0}^{N_i} (F(x^\infty(t_k), u^\infty(t_k)) - F(\bar{x}_k^*, \bar{u}_k^*))w_k \right\|_\infty \\
&\quad \le \lim_{i\to\infty} M \sum_{k=0}^{N_i} (\|x^\infty(t_k) - \bar{x}_k^*\|_\infty + \|u^\infty(t_k) - \bar{u}_k^*\|_\infty)w_k = 0.
\end{aligned}$$

Thus,

$$\int_{-1}^{1} F(x^{\infty}(t), u^{\infty}(t))dt = \lim_{i\to\infty} \sum_{k=0}^{N_i} F(\bar{x}_k^*, \bar{u}_k^*)w_k. \tag{56}$$

It is obvious that

$$\lim_{i\to\infty} E(\bar{x}_0^*, \bar{x}_{N_i}^*) = E(x^{\infty}(-1), x^{\infty}(1)). \tag{57}$$

Thus the limit in (6) follows from (56) and (57).

Step 3 Denote $(x^*(t), u^*(t))$ as any optimal solution of Problem B with the property that $x^*(t) \in W^{m,\infty}$, $m \geq 2$ (the optimal solution may not be unique). According to Theorem 1, there exists a sequence of feasible solutions, $(\tilde{x}_k^N, \tilde{u}_k^N)_{N=N_1}^{\infty}$, of Problem B^N that converges uniformly to $(x^*(t), u^*(t))$. Now, from (6) and the optimality of $(x^*(t), u^*(t))$ and $(\bar{x}_k^*, \bar{u}_k^*)$, we have

$$J(x^*(\cdot), u^*(\cdot)) \leq J(x^{\infty}(\cdot), u^{\infty}(\cdot)) = \lim_{i\to\infty} \bar{J}^{N_i}(\bar{X}^*, \bar{U}^*) \leq \lim_{i\to\infty} \bar{J}^{N_i}(\tilde{X}, \tilde{U}). \tag{58}$$

By using the same arguments as in Step 2, it is straightforward to show that

$$J(x^*(\cdot), u^*(\cdot)) = \lim_{i\to\infty} \bar{J}^{N_i}(\tilde{X}, \tilde{U}), \tag{59}$$

since $(\tilde{x}_k^N, \tilde{u}_k^N)_{N=N_1}^{\infty}$ converge uniformly to $(x^*(t), u^*(t))$. Equations (58) and (59) imply that

$$J(x^*(\cdot), u^*(\cdot)) = J(x^{\infty}(\cdot), u^{\infty}(\cdot)).$$

This is equivalent to saying that $(x^{\infty}(t), u^{\infty}(t))$ is a feasible solution that achieves the optimal cost. Therefore, $(x^{\infty}(t), u^{\infty}(t))$ is an optimal solution to the optimal control Problem B. Thus, the conclusions of Theorem 2 follow. □

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