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OSCILLATING ABOUT COPLANARITY IN THE 4 BODY PROBLEM.

RICHARD MONTGOMERY

ABSTRACT. For the Newtonian 4-body problem in space we prove that any zero angular momentum bounded solution suffers infinitely many coplanar instants, that is, times at which all 4 bodies lie in the same plane. This result generalizes a known result for collinear instants (“syzygies”) in the zero angular momentum planar 3-body problem, and extends to the $d + 1$ body problem in d -space. The proof, for $d = 3$, starts by identifying the center-of-mass zero configuration space with real 3×3 matrices, the coplanar configurations with matrices whose determinant is zero, and the mass metric with the Frobenius (standard Euclidean) norm. Let S denote the signed distance from a matrix to the hypersurface of matrices with determinant zero. The proof hinges on establishing a harmonic oscillator type ODE for S along solutions. Bounds on inter-body distances then yield an explicit lower bound ω for the frequency of this oscillator, guaranteeing a degeneration within every time interval of length π/ω . The non-negativity of the curvature of oriented shape space (the quotient of configuration space by the rotation group) plays a crucial role in the proof.

1. RESULTS.

Consider the Newtonian 4 body problem in Euclidean 3-space. Typically, the four point masses form the vertices of a tetrahedron. As the masses move about, at isolated instants the tetrahedron which they form might degenerate so that all 4 bodies lie on a single plane. Must such co-planar instants always occur?

A solution is called *bounded* if the interparticle distances r_{ab} between the four masses $m_a, a = 1, 2, 3, 4$ are bounded for all time in the solution’s domain of definition.

Theorem 1.1. *For the 4 body problem in 3-space, any bounded zero angular momentum solution defined on an infinite time interval suffers infinitely many coplanar instants.*

Theorem 1.1 follows directly from the finite time interval oscillation results of Theorems 1.2 and 1.3 and Proposition 1.1 below, results which hold for the $d + 1$ -body problem in d -dimensional Euclidean space. These results generalize the result [12] for the case $d = 2$ of the planar three-body problem .

Write $q_a \in \mathbb{R}^d, a = 1, \dots, d + 1$ for the positions of the bodies. Typically, at each instant the q_a form the vertices of a $d + 1$ -simplex, meaning that their convex hull has nonzero d -dimensional volume. At special instants this volume may vanish by virtue of all bodies instantaneously lying on some affine hyperplane. We call these *degeneration instants*. Write $r_{ab} = |q_a - q_b|$ for the distances between bodies, $M = \Sigma m_a$ for the total mass and G for the universal gravitational constant. G is

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included to get our units straight: GM/r_{ab}^3 has the units of $1/(\text{time})^2$, the units of a frequency squared.

Theorem 1.2. *Consider any zero angular momentum solution to the standard attractive ($1/r$ potential) Newton's equations for $d+1$ bodies in d -dimensional. Suppose that along this solution the inter-body distances satisfy the bound*

$$(1) \quad r_{ab} \leq c$$

Then, within every time interval of size $\frac{1}{\pi}(\frac{c^3}{GM})^{1/2}$, this solution has a degeneration instant.

REMARK. Theorem 1.2 represents a quantitative improvement of the syzygy estimates found earlier in the case $d = 2$ described above.

NECESSITY OF ZERO ANGULAR MOMENTUM IN EVEN DIMENSIONS. The regular simplex is a central configuration in all dimensions d . If the dimension d is even, say $d = 2k$, then one can uniformly rotate the simplex in a way consistent with a splitting of \mathbb{R}^d into k two-planes to get a relative equilibrium solution to the $d + 1$ -body problem in \mathbb{R}^d which has nonzero angular momentum and never degenerates. These even-dimensional analogues of the Lagrange rotating equilateral triangle illustrate that for even dimensions d the hypothesis that the angular momentum be zero is necessary in theorem 1.2.

GENERAL TWO-BODY TYPE POTENTIALS.. There is nothing special about the Newtonian $1/r$ potential in theorem 1.2. It is enough to have a sum of pair potentials of the form

$$(2) \quad V(q) = G \sum_{a \neq b} m_a m_b f_{ab}(r_{ab}(q))$$

where the individual two-body potentials f_{ab} are attractive. Specifically, assuming

$$(3) \quad f'_{ab}(r) > 0, f''_{ab}(r) < 0, \text{ for } r > 0; \lim_{r \rightarrow \infty} \frac{f'_{ab}(r)}{r} = 0.$$

is enough. Examples include the standard Newtonian 3-dimensional gravitational potential $f_{ab}(r) = -1/r$ and the power law potentials $f_{ab}(r) = -k_{ab}/r^\alpha$ for positive exponent α and positive constants k_{ab} . (We choose the units so that f_{ab} has units $1/(\text{length})$.) Hypothesis (3) guarantees that the functions $f'_{ab}(r)/r$ are positive and strictly monotone decreasing so that for each $c > 0$ and pair ab we have that $r_{ab} \leq c \implies \frac{f'_{ab}(r_{ab})}{r_{ab}} \geq \delta_{ab} := \frac{f'_{ab}(c)}{c}$. Taking δ to be the minimum of these δ_{ab} over all pairs we get

$$(4) \quad r_{ab} \leq c \text{ for all pairs } ab \implies \frac{1}{r_{ab}} f'_{ab}(r_{ab}) \geq \delta > 0 \text{ for all pairs } ab.$$

Then, we have

Theorem 1.3. *Consider the zero angular momentum Newton's equations for $N = d + 1$ bodies moving in Euclidean d -dimensional space under the influence of the attractive potential (2) whose 2-body potentials satisfy hypothesis (3). Suppose that along such a solution all its inter-body distances r_{ab} satisfy the bound $r_{ab} \leq c$. Then, in every time interval of size $(GM\delta)^{-1/2}/\pi$, this solution has a degeneration instant. Here δ is as in implication (4) above, and M the total mass.*

We now describe the key ingredients behind these Theorems.

Definition 1.1. Σ is the degeneration locus within configuration space – the set of configurations for which the $d + 1$ masses all lie on a single affine hyperplane.

Σ is a (singular) hypersurface in the full configuration space which cuts it into two disjoint congruent halves, the simplices having positive volume, and those having negative volume. (The sign of the volume depends on the orientation of Euclidean space and the ordering of the masses, which we fix once and for all.) Write $\text{sgn}(\det(q))$ for the sign of the volume, defined for $q \notin \Sigma$. For example, if $d = 3$, then $\text{sgn}(\det(q))$ is the sign of the triple product $(q_2 - q_1) \cdot ((q_3 - q_1) \times (q_4 - q_1))$.

Definition 1.2. *The signed distance $S(q)$ of a configuration q of $d+1$ point masses in \mathbb{R}^d is the distance from q to the degeneration locus relative to the mass inner product (described in subsection 3.1.1), that distance being given a plus sign if the signed volume of q is positive and a minus sign if negative. In symbols:*

$$S(q) = \text{sign}(\det(q)) \text{dist}(q, \Sigma).$$

with $S(q) = 0$ if and only if $q \in \Sigma$.

In Prop. 6.1 below we prove that $|S(q)|$ is the smallest singular value of a $d \times d$ matrix representing q in the center-of-mass frame.

Proposition 1.1. *[Main computation.] If S is smooth along a zero angular momentum solution $q(t)$ to Newton's equations then $S(t) := S(q(t))$ evolves according to*

$$\ddot{S} = -Sg(q, \dot{q}), \text{ with } g > 0 \text{ everywhere.}$$

If, moreover, all interparticle distances r_{ab} satisfy $r_{ab} \leq c$ then $g \geq GM/c^3$ for the Newtonian ($f_{ab}(r) = -1/r$) potential case, and, more generally, $g \geq GM\delta$ for potentials of the form (2), with (3) in force and δ as per (4).

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2. MOTIVATION AND MAIN IDEAS.

Newton's N-body equations in d-space are invariant under the isometry group of the inertial Euclidean space, \mathbb{R}^d , so we can push them down to form a system of ODEs on "shape space", by which we mean the quotient space of the N-body configuration space by the isometry group of \mathbb{R}^d . There are actually two shape spaces, depending on whether or not we allow orientation reversing isometries. In the body of this paper we will work on the "oriented shapes space" which arises from taking the quotient with respect the group $SE(d)$ of orientation-preserving isometries. In appendix B we describe the relationships between these two shape spaces.

We will speak of "downstairs" to mean we are working on the quotient and "upstairs" to mean we are working on the original configuration space. Upstairs, Newton's equations have the form $\ddot{q} = -\nabla V(q)$. Downstairs, the equations have precisely this same form **provided** that the total angular momentum is zero ¹.

¹If the angular momentum is non-zero there are additional 'magnetic' terms" in the equations downstairs, meaning terms linear in velocities, and also additional equations involving 'internal variables" which represent instantaneous rigid body tumbling coupled to dynamics on the shape space, these internal variables lying in co-adjoint orbits for $SO(d)$.

In writing down the downstairs zero-angular momentum Newton's equations, the acceleration \ddot{q} is replaced by the covariant acceleration $\nabla_{\dot{q}}\dot{q}$ where ∇ is the Levi-Civita connection arising out of the the induced *shape metric* downstairs. This shape metric, induced by the flat kinetic energy metric upstairs, is curved.

Robert Littlejohn [8] pointed out to me that the oriented shape space for $d + 1$ bodies in \mathbb{R}^d is homeomorphic to a Euclidean space. This topological fact is well known for the case $d = 2$ of the 3 body problem in the plane where it has proven to be of great utility. For higher d the fact has been known for some time amongst certain statisticians and can be found in [5] and [6]. Although I do not use this fact here, it is this single fact that inspired my faith that something like the theorems in this paper might hold.

Not only is the oriented shape space homeomorphic to a Euclidean space, but it is smooth at most points. The points where it fails to be smooth are those shapes of corank 2 or higher. (The locus of such points has codimension 4.) Here we use the following terminology

Definition 2.1. *The corank of a configuration $q = (q_1, \dots, q_{d+1})$, or of its corresponding shape, is the codimension of the smallest affine subspace in \mathbb{R}^d which contains all $d + 1$ of the vertices q_1, \dots, q_{d+1} . The rank of a configuration is the dimension of this smallest affine subspace.*

We continue to write Σ for the degeneration locus, either upstairs or downstairs. Downstairs, in *oriented* shape space, Σ is a totally geodesic hypersurface, at least at its smooth points. (This is a bit strange since Σ is not totally geodesic upstairs. For example, when $d = 3$, imagine connecting two quadrilaterals which lie in different planes by geodesics, i.e. straight lines between vertices. The resulting curve in configuration space is non-degenerate at most instants.) To see the total geodesy downstairs, select any orientation reversing isometry R , for example, in the case $d = 3$, a reflection about the xy plane. Downstairs R is an isometry of shape space whose fixed point set is precisely Σ . A general theorem in Riemannian geometry now implies that Σ is totally geodesic. In terms of Newton's equations, this 'total geodesy' is basically the assertion that, for example for the case $d = 3$, that a configuration which initially lies in a plane, and all of whose velocities initially lie in that plane, will remain in that plane for all time.

HEURISTICS. Proposition 1.1 asserts that the signed distance S from Σ behaves qualitatively like a one-dimensional harmonic oscillator, oscillating around $S = 0$. The physical intuition behind this phenomenon was pointed out to me by Mark Levi many years ago. The potential is invariant under isometries so descends to a function downstairs. How to interpret this potential downstairs? Write $\Sigma_{ab} \subset \Sigma$ for the binary collision locus $r_{ab} = 0$. One computes that $r_{ab}(s) = \mu_{ab} \text{dist}(s, \Sigma_{ab})$ where $\text{dist}(s, \Sigma_{ab})$ is the distance from s to Σ_{ab} and where $\mu_{ab} = \sqrt{M/m_a m_b}$. Consequently, re-interpreted downstairs, formula (2) for the potential asserts that a point s in shape space is subjected to the force of an attractive potential exerted by the $\binom{d}{2}$ sources Σ_{ab} , all of which lie in the "hyperplane" Σ . So, of course, the shape is always attracted to Σ ! And as long as the shape's "vertical" kinetic energy is not too large, it will always return to cross Σ , oscillating forever back and forth across the attracting 'hyperplane' Σ .

CHOICE OF S VERSUS SIGNED VOLUME. In [12], in proving Theorem 1.1 for the case $d = 2$, I used a function z in place of the S of proposition 1.1. This z was the signed area of the oriented triangle normalized by divided it by the moment of

inertia I that the triangle would have if all masses were assigned the value 1. The obvious generalizations z_d of this z to $d > 2$, namely a normalized signed volume, did not work out. All my attempts at proving a version of proposition 1.1 for such a function in place of S failed. The function z_2 satisfies a kind of monotonicity relation with respect to geodesics orthogonal to Σ which fails for $z_d, d > 2$ and this monotonicity was required to get positivity of g in Proposition 1.1. The need for such a relation led to introducing S . After the fact, one observes that the identity $z = S/\sqrt{I}$ holds for equal masses when $d = 2$, and fails for $d > 2$.)

KEY INGREDIENTS TO THE PROOF. The proof of Proposition 1.1 relies on four key facts .

- Fact 1. S satisfies the Hamilton-Jacobi equation $\|\nabla S\| = 1$ wherever S is smooth. This fact implies that the integral curves of the gradient flow of S are geodesics.
- Fact 2. The shape metric is everywhere non-negatively curved.
- Fact 3. There is a close relationship between the sign of the second fundamental form of distance level sets (the $\{S = t\}$'s) from a totally geodesic submanifold ($\Sigma = \{S = 0\}$) and the sign of the curvature of the ambient space within which the level sets lie. This relation is detailed on p. 34-37 of Gromov [4] and recalled below as proposition 4.1.
- Fact 4. (Theorem 6.1). The singular locus of S has codimension 2. This locus, denoted $Sing(S)$ below, consists of all points at which S is not smooth.

3. SET-UP AND REDUCTION.

The proofs of all of our theorems hinge on Proposition 1.1 which is a computation. We achieve the computation by exploiting the relations between Newton's equations at zero angular momentum as expressed upstairs on the usual configuration space and downstairs on shape space. The process of pushing the equations downstairs is referred to as "reduction". Our reduction procedure is a metric reduction, putting kinetic energy to the fore, as opposed to the oft-used symplectic reduction. The two reduction procedures are formally equivalent but the metric approach makes out computation tractable. In this section we go through the reduction for the case $d = 3$. At the end, in subsection 3.3, we describe the small changes needed for the set-up of reduction for higher d .

Write $M(k, m)$ for the space of $k \times m$ real matrices. The configuration space for the 4 body problem in \mathbb{R}^3 can be naturally identified with the space $M(3, 4)$. To do so, think of the four vectors $q_1, q_2, q_3, q_4 \in \mathbb{R}^3$ which define the positions of the four bodies as column vectors and place them side-by-side to form the 3×4 matrix

$$(5) \quad q = \begin{pmatrix} q_1 & q_2 & q_3 & q_4 \end{pmatrix} \in M(3, 4).$$

The translation subgroup \mathbb{R}^3 acts on $M(3, 4)$ by $q_a \mapsto q_a + b, b \in \mathbb{R}^3, a = 1, 2, 3, 4$, which in matrix terms is

$$(6) \quad q \mapsto q + (b, b, b, b)$$

The quotient of $M(3, 4)$ by this action can be identified with the matrix space $M(3, 3)$. This identification depends on choosing a basis for the 3-dimensional subspace $x_1 + x_2 + x_3 + x_4 = 0$ of the mass label space \mathbb{R}^4 , or, what is the same

thing, a choice of “Jacobi vectors”. The results are independent of this choice ². See Appendix A for details. The rank of a configuration becomes the rank of the representing matrix so that the degeneration locus is

$$\Sigma = \{q \in M(3, 3) : \det(q) = 0\}.$$

3.1. Oriented Shape Space. Rotations of \mathbb{R}^3 act on both $M(3, 4)$ and its translation-quotient $M(3, 3)$ by

$$q \mapsto gq, g \in SO(3).$$

Definition 3.1. *The oriented shape space $Sh = Sh(3, 4)$ is the topological quotient space $M(3, 3)/SO(3)$. The quotient map $M(3, 3) \rightarrow Sh(3, 4)$ will be denoted by π . The projection of a configuration $q \in M(3, 3)$ will be called “the shape” of q .*

REMARK. The group of orientation-preserving isometries of \mathbb{R}^3 , denoted $SE(3)$, is made up of the translations (\mathbb{R}^3) and the rotations ($SO(3)$). We can naturally identify Sh with the quotient $M(3, 4)/SE(3)$, by using reduction in stages: first quotient by translations \mathbb{R}^3 to get $M(3, 3)$ and then by rotations $SO(3)$ to get to Sh . The projection $M(3, 4) \rightarrow Sh$ will also be denoted by π .

Recall that the action of a group G on a set Q is called free if $gq = q \implies g = id$. It is well-known (see for example Prop 4.1.23 of [1]) that the free action of a compact Lie group G on a smooth manifold Q yields a quotient Q/G which is itself a smooth manifold with the quotient map $Q \rightarrow Q/G$ being a smooth projection. The action of $SO(3)$ on $M(3, 3)$ is free on the open dense subset of $M(3, 3)$ consisting of matrices of rank 2 and 3, i.e. on the planar and spatial configurations. Moreover, $M(3, 3)$ is stratified by rank. The rank 3 matrices form an open dense subset whose complement is the singular hypersurface Σ . The rank 2 matrices form the generic points of Σ , the points at which it is smooth. The rank 1 points, i.e. the collinear configurations, have codimension $4 = 2 * 2$ in $M(3, 3)$. See for example, [3] for this computation. There is only one rank 0 matrices, namely the 0 matrix which represents total collision. Hence we get

Proposition 3.1. *Let $U \subset M(3, 3)$ denote the set of rank 2 and 3 matrices, henceforth referred to as “generic configurations”. Then the restriction of $\pi : M(3, 3) \rightarrow Sh(3, 4)$ to U gives the space of rank 2 and 3 shapes within $Sh(3, 4)$ a smooth structure in such a way that this restricted projection is a smooth submersion. Moreover this restricted projection $\pi : U \rightarrow \pi(U)$ gives U the structure of a principal $SO(3)$ bundle. The complement of U has codimension 4 within $M(3, 3)$.*

3.1.1. Newton’s Equations. To write down Newton’s equations for the motion of the 4 bodies, we need the potential and the choice of masses. We have written down the potential (eq (2)). The choice of masses $m_a > 0$ for each body defines an inner product $\langle \cdot, \cdot \rangle$ on $M(3, 4)$ called the “mass metric” or “kinetic energy metric” according to

$$\frac{1}{2} \langle \dot{q}, \dot{q} \rangle = \frac{1}{2} \sum_{a=1}^4 m_a |\dot{q}_a|^2.$$

²Usually the quotient $M(3, 4)/\mathbb{R}^3$ is identified with the codimension 3 linear subspace of $M(3, 4)$ obtained by fixing the center of mass to be zero. There is a *mass independent* way to form the identification of the quotient with $M(3, 3)$ which is more useful for our purposes. This alternative perspective, due to Albouy and Chenciner REF, is reviewed in an appendix. It is equivalent to fixing the center-of-mass, once masses are chosen.

We use the absolute value symbol for the usual norm in our Euclidean inertial \mathbb{R}^3 . When we interpret $\dot{q} \in M(3, 4)$ to represent the velocities of the four bodies then the above expression is the usual expression for the total kinetic energy. Newton's equations can now be written

$$(7) \quad \ddot{q} = -\nabla V(q)$$

where the gradient ∇V is computed using the mass inner product: $dV(q)(\delta q) = \langle \nabla V(q), \delta q \rangle$.

The mass inner product induces an inner product on the translation reduced configuration space $M(3, 3)$ by declaring the projection $M(3, 4) \rightarrow M(3, 3)$ to be a metric projection. Equivalently, we can view $M(3, 3)$ as a subspace of $M(3, 4)$ by fixing the center of mass to be zero, and then take the restricted inner product. (Again, see appendix A.) We can choose a basis for the 1st \mathbb{R}^3 factor of $M(3, 3) = \text{Hom}(\mathbb{R}^3, \mathbb{R}^3)$ such that the inner product becomes

$$\langle q, q \rangle = \text{Tr}(q^t q),$$

namely, the inner product on matrices for which the matrix entries form an orthonormal linear coordinate system. Such a basis corresponds to a choice of *normalized Jacobi vectors*. See Appendix A. So done, Newton's equations have precisely the form, eq (7) when written on $M(3, 3)$.

3.2. Reduced Newton's equations. We push Newton's equations and the kinetic energy metric down to shape space. For this purpose it will be helpful to keep in mind the following generalities.

METRIC PROJECTIONS AND RIEMANNIAN SUBMERSIONS. Whenever we have a metric space M with distance function d_M and an onto map $\pi : M \rightarrow B$ we can *try* to define a metric d_B on B by $d_B(b_1, b_2) = d_M(\pi^{-1}(b_1), \pi^{-1}(b_2))$, or, in English, the distance between points downstairs is the distance between their corresponding fibers upstairs. When this construction works we say that $\pi : M \rightarrow B$ is a metric projection or *submetry*. If $B = M/G$ is the quotient of M by the action of a compact Lie group acting on M by isometries and π is the quotient projection then the construction always works. If, in addition, M is a manifold whose metric d_M comes from a Riemannian metric and if the G -action is free so that the quotient map π is a smooth submersion with smooth B , then the induced distance function d_B also arises as the distance function of a Riemannian metric on M . In this case $\pi : M \rightarrow B$ is a *Riemannian submersion* which has the following infinitesimal meaning. The ‘vertical space’ $V_q \subset T_q M$ through $q \in M$ is defined to be the kernel of $d\pi_q$; equivalently, it is the tangent space at q to the fiber $\pi^{-1}(s) = Gq$ through q . Define the ‘horizontal space’ H_q to be the orthogonal complement to the vertical: $H_q = V_q^\perp$. Then the restriction of $d\pi_q$ to H_q is a linear isomorphism. Declaring this linear isomorphism $H_q \rightarrow T_s B$ to be an isometry induces an inner product on $T_s B$, and this inner product is independent of the point $q \in \pi^{-1}(s)$ since G acts isometrically on M . Distance minimizers between fibers upstairs are geodesics in M orthogonal to the fibers. From this follows the well-known fact that geodesics orthogonal to fibers at one point are orthogonal at every point, and that the geodesics downstairs in B are precisely the projections of horizontal geodesics upstairs.

In this way, starting from the mass metric on $M(3, 4)$ or $M(3, 3)$, we get a metric on $Sh = Sh(3, 4)$ which is Riemannian at the generic shapes (those of rank 2 or 3) and over these points is such that $\pi : M(3, 3) \rightarrow Sh$ is a Riemannian submersion.

To push Newton's equations down to Sh we must understand the *dynamical meaning* of being horizontal in $M(3, 3)$. In [11] (or [15]) I compute that \dot{q} is orthogonal to the $SO(3)$ orbit through q if and only if the total angular momentum $J(q, \dot{q})$ of the pair (q, \dot{q}) is zero. The expression for J as a function on $TM(3, 4) = M(3, 4) \times M(3, 4)$ is $J(q, \dot{q}) = \Sigma m_a q_a \wedge \dot{q}_a$ for $M(3, 4)$, and is the same when restricted to $TM(3, 3)$ viewed as subspace of $TM(3, 4)$. Recall that J is conserved for any potential of the form of eq (2), that is to say $J(q(t), \dot{q}(t)) = J(q(0), \dot{q}(0))$ along solutions $q(t)$ to Newton's equations. Now let ∇ be the Levi-Civita connection for the shape metric. Observe that since the potential is $SE(3)$ invariant it also defines a projection on Sh . We will use the same symbol V for the potential upstairs and downstairs. We have

Lemma 3.1. *Any zero angular momentum solution to Newton's equations passing through generic (i.e. rank 2 and 3) points of $M(3, 3)$ projects to a curve γ in shape space which satisfies*

$$\nabla_{\dot{\gamma}} \dot{\gamma} = -\nabla V(\gamma(t)).$$

Conversely, the horizontal lift of any such solution is a zero-angular momentum solution to Newton's equations upstairs.

Regarding 'horizontal lift' see, again [11] or chapter 13 of [15].

PROOF. This theorem is a general fact, holding for any Hamiltonian of the form kinetic plus potential on any manifold endowed with the smooth free action of a Lie group which keeps both the kinetic (metric) and potentials invariant. For a proof see for example, [15].

The special case when $V = 0$ will be useful below.

Lemma 3.2. *Any zero angular momentum straight line $q + tv$ in $M(3, 3)$ projects to a geodesic in Shape space $Sh(3, 4)$. Conversely, the horizontal lift of any geodesic in $Sh(3, 4)$ is a zero-angular momentum straight line in $M(3, 3)$. The geodesic is parameterized by arc length if and only if $\|v\| = 1$.*

3.3. Set-up for general dimension d . GOING FROM $d = 3$ TO GENERAL d . The configuration space for N bodies in \mathbb{R}^d is the space $M(d, N)$ of $d \times N$ real matrices. Its quotient by the translation group of \mathbb{R}^d forms an $M(d, N - 1)$ once a basis for the hypersurface $x_1 + x_2 + \dots + x_N = 0$ of the mass-label is chosen. Again see Appendix A. In our case of $N = d + 1$ we thus get the translation-reduced configuration space $M(d, d)$ of square matrices. The degeneration locus Σ is given by $\{q : \det(q) = 0\}$. Shape space is $Sh(d, d + 1) = M(d, d)/SO(d) = M(d, d + 1)/SO(d)$. Proposition 3.1 holds with 'rank 2 and 3' replaced by 'rank $d - 1$ and rank d '.

Introducing masses puts an inner product on the mass label space, and so on $M(d, d + 1)$ and on its translation quotient $M(d, d)$. The masses also allow us to identify $M(d, d)$ as a linear subspace, rather than a quotient space, of $M(d, d + 1)$, namely as the subspace of center-of-mass zero configurations. An orthonormal basis for the hypersurface $\Sigma x_i = 0$ is equivalent to a choice of normalized Jacobi vectors and relative to these coordinates the mass-induced inner product structure on $M(d, d)$ is standard : $\langle q, q \rangle = \text{tr}(q^t q) = \Sigma_{i,j} q_{ij}^2$, (This is the inner product whose associated norm is called the Frobenius norm). Relative to this inner product the

rotation group $SO(d)$ acts isometrically by left multiplication and the action leaves the potential invariant and so the metric and the zero-angular momentum Newton's equation push down to the quotient shape space $Sh(d, d+1) = M(d, d)/SO(d)$. The reduction lemmas 3.1 and 3.2 for this reduced dynamics hold as stated, upon replacing '3' by 'd' in the obvious places.

4. THE PROOF OF PROP 1.1: SIGNED DISTANCE AS AN OSCILLATOR.

We proceed to differentiate S along a solution arc which does not pass through any singular point of S . We have

$$\dot{S} = \langle \nabla S, \dot{\gamma} \rangle$$

so that

$$(8) \quad \ddot{S} = \langle \nabla S, \nabla_{\dot{\gamma}} \dot{\gamma} \rangle + \langle \nabla_{\dot{\gamma}} \nabla S, \dot{\gamma} \rangle$$

$$(9) \quad = \langle \nabla S, -\nabla V \rangle + \langle \nabla_{\dot{\gamma}} \nabla S, \dot{\gamma} \rangle$$

We estimate each term of this last equation separately, showing that each term has the form $-Sg$ with $g \geq 0$. We verify that the 'g' for the first term is always positive and satisfies the stated bounds when $r_{ab} \leq c$.

First term, $\langle \nabla S, -\nabla V \rangle$. At smooth points of S , the integral curves of the vector field ∇S are geodesics orthogonal to the level sets of S , and in particular to the level set $S = 0$ which is the degeneration locus Σ . This fact holds true generally for the signed distance function from a hypersurface on any Riemannian manifold, and is closely related to the fact that signed distance satisfies the Hamilton -Jacobi equation: $\|\nabla S\| = 1$.

We proceed in the special case of $d = 3$ for this paragraph, for simplicity. The geodesics in $M(3, 3)$, or in shape space, are the projections of straight lines $q + tv$ in $M(3, 4)$ for which $(q, v) \in M(3, 4) \times M(3, 4)$ has zero total angular momentum and zero total linear momentum. See lemma 3.2 above. (Zero linear momentum arises from working on $M(3, 3) \subset M(3, 4)$ by identifying it with the zero center-of-mass configurations. Alternatively, having zero linear momentum is equivalent to the assertion that the velocity v is orthogonal to the translation action.) The parameter t is arclength provided $\langle v, v \rangle = 1$. Now the smooth points q of the degeneration locus Σ are the planar points. In order for a geodesic to be perpendicular to Σ at such a q we must have that v is perpendicular to all $\delta q \in T_q \Sigma$. By rotating, we may assume that the 4 vertices of q lie in the xy plane which we will denote by " \mathbb{R}^2 ". Then any variation $\delta q = (\delta q_1, \delta q_2, \delta q_3, \delta q_4)$ with $\delta q_a \in \mathbb{R}^2$ represents a planar variation of q and hence a tangent vector to Σ at q . Since

$$\langle \delta q, v \rangle = \sum m_a (\delta q_a) \cdot v_a$$

and since the δq_a are arbitrary vectors in \mathbb{R}^2 , we see that our tangent vector v must have all 4 of its component vectors v_a perpendicular to \mathbb{R}^2 , which is to say, along the z-axis. But then, along our geodesic the squared inter-body distances are

$$(10) \quad r_{ab}^2 = |q_a + tv_a - (q_b + tv_b)|^2 = r_{ab}(0)^2 + t^2 |v_a - v_b|^2$$

where the cross term is zero since q_a, q_b lie in \mathbb{R}^2 while v_a, v_b are orthogonal to \mathbb{R}^2 .

For general d , equation (10) continues to hold for a geodesic orthogonal to the degeneration locus. Indeed, the only real difference between the proof above for $d = 3$ and the proof for $d > 3$ is notational. Now the q_a , representing a point on the degeneration locus, can be taken to all lie in a fixed affine hyperplane of \mathbb{R}^d so

that the variations δq_a , $a = 1, \dots, N = d + 1$ can be taken to be arbitrary vectors tangent to the corresponding linear hyperplane \mathbb{R}^{d-1} . As a consequence the v_a all lie in the one-dimensional orthogonal to this \mathbb{R}^{d-1} and the computation is the same.

Now look at the negative of the potential in the gravitational case:

$$U = -V = G\Sigma \frac{m_a m_b}{r_{ab}}$$

along our geodesic. Each individual term $\frac{m_a m_b}{r_{ab}}$ is *strictly decreasing* or constant in t^2 . Indeed $\frac{d}{dt} \frac{1}{r_{ab}(t)} = -\frac{t|v_{ab}|^2}{r_{ab}(t)^3} = -S \frac{|v_{ab}|^2}{r_{ab}(t)^3}$, since $S = t$ as long as the geodesic is the unique minimizer to the degeneration locus. Summing, we obtain

$$\langle \nabla S, -\nabla V \rangle = \langle \nabla S, \nabla U \rangle = -Sg_1$$

with

$$g_1 = G\Sigma m_a m_b \frac{|v_{ab}|^2}{r_{ab}(t)^3} > 0$$

as desired.

If each r_{ab} is bounded above by c , we have that $g_1 \geq \frac{G}{c^3} \Sigma m_a m_b |v_{ab}|^2$. But, if $\Sigma m_a v_a = 0$, we find that $\|v\|^2 = \Sigma m_a m_b |v_{ab}|^2 / M$ (“Lagrange’s identity”) and since we have that $\|v\|^2 = 1$ (since t is arclength) it follows that $\Sigma m_a m_b |v_{ab}|^2 = M$ which yields $g_1 \geq GM/c^3$, which completes the proof for the gravitational case.

In the case of a general potential satisfying hypothesis (2), (3) we get that $\frac{d}{dt} f_{ab}(r_{ab}) = f'_{ab}(r_{ab}) \frac{d}{dt}(r_{ab}(t)) = f'_{ab}(r_{ab})(tv_{ab}^2)/r_{ab} = S(\frac{f'(r_{ab})}{r_{ab}})(v_{ab})^2$. Summing, we get $\langle \nabla S, -\nabla V \rangle = -Sg_1$ with $g_1 = G\Sigma m_a m_b (\frac{f'(r_{ab})}{r_{ab}})(v_{ab})^2 > 0$. Under the boundedness assumption, eq (4) yields that $\frac{f'(r_{ab})}{r_{ab}} > \delta$ for all pairs a, b and the lower bound for g_1 proceeds exactly as in the previous paragraph.

QED for Term 1.

Second term, $\langle \nabla_v \nabla S, v \rangle$. For a fixed shape p , $p \notin \text{Sing}(S)$

$$v \mapsto Q_p(v, v) := \langle \nabla_v \nabla S, v \rangle, v \in T_p Sh$$

is a quadratic form on the tangent space $T_p Sh$. **We will show that $Q_p(v, v) = -S(p)H_p(v, v)$ where $H_p \geq 0$ is a positive semi-definite quadratic form.** The trick for achieving this inequality is to recognize the quadratic form Q_p as being essentially the second fundamental form of the equidistant hypersurface Σ_t from Σ which passes through p , namely

$$\Sigma_t := \{S = t\}; \text{ where } t = S(p)$$

and then to use a relation between the sign of such second fundamental forms and the sign of the ambient curvature.

Take $v = \nabla S$ in $Q_p(v, v)$. Differentiate the identity $\langle \nabla S, \nabla S \rangle = 1$ with respect to v to see that $\langle \nabla_v \nabla S, v \rangle = 0$, so that $Q_p(v, v) = 0$.

Take $v \perp \nabla S$. Then v is tangent to Σ_t while ∇S is the unit normal N to Σ_t . Recall that second fundamental form to a hypersurface V with unit normal vector field N is the quadratic form $\Pi(v, v) = v \mapsto \langle \nabla_v N, v \rangle$ defined for vectors v tangent to V . It follows that for $Q_p(v, v) = \Pi_p(v, v)$ for $v \perp \nabla S$ is the second fundamental form Π_p of the hypersurface Σ_t at the point $p \in \Sigma_t$. Summarizing:

$$Q_p(v, v) = \begin{cases} 0 & \text{for } v \parallel \nabla S \\ \Pi_p(v, v) & \text{for } v \perp \nabla S \end{cases}$$

We recall some facts about the second fundamental form Π of a hypersurface.

- (1) A hypersurface is *totally geodesic* if and only if $\Pi = 0$.
- (2) Replacing the choice of unit normal N to the hypersurface by its negative $-N$ replaces Π by its negative $-\Pi$.

Our hypersurface Σ is totally geodesic, as mentioned earlier in ‘heuristics’. Indeed, Σ is the fixed point set of an isometric involution $i : Sh \rightarrow Sh$ and fixed point sets of isometric involutions are always totally geodesic. This isometric involution i , called “reflection about Σ ”, is implemented by the nontrivial element of the two-element group $O(d)/SO(d)$. Any orientation reversing orthogonal transformation $R \in O(d)$ realizes this nontrivial element and acts on shape space by sending the shape $s = \pi(q)$ to $i(s) = \pi(Rq)$. Now $i^*S = -S$ from which it follows that $i_*\nabla S = -\nabla S$, and thus, using item (2) above, that $i^*Q = -Q$. It follows that we can write $Q = -SH$ where $i^*H = H$. It remains to show that H is positive semi-definite.

A NECESSARY DETOUR INTO CURVATURES.

Definition 4.1. *A hypersurface is convex relative to the choice of normal N if $\Pi \geq 0$ for this choice of normal, and concave relative to N if $\Pi \leq 0$ for this choice of normal.*

Example 4.1. *The boundary of a convex domain having smooth boundary in Euclidean space is convex in the above sense provided we use the outward pointing normal.*

Let M be a Riemannian manifold and $V \subset M$ a hypersurface in M , together with a choice of unit normal N along the hypersurface. Then close to V we have the family $V_s, -\epsilon < s < \epsilon$ of nearby equidistant hypersurfaces formed by travelling along the geodesics tangent to the unit normal N for a distance s . By flowing along these geodesics we also have diffeomorphisms

$$\phi_s : V \rightarrow V_s.$$

Write Π_0 for the second fundamental form of V relative to N and Π_s for that of the equidistant V_s . Recall that we say that M is “non-negatively curved” if its sectional curvatures are all positive or zero, and “non-positively curved” if all of its sectional curvatures are all negative or zero. The following basic relationship between extrinsic and intrinsic curvature. is found on p. 34-37 of [4].

Proposition 4.1. *(See figures 1). If the ambient curvature of the Riemannian manifold M is non-negative and if the hypersurface $V \subset M$ is concave with respect to the choice of unit normal N for V , then its positive equidistants $V_s, s > 0$ are at least as concave as V : $\phi_s^*\Pi_s \leq \Pi_0 \leq 0$ for $s > 0$.*

If the ambient curvature of M is non-positive and if the hypersurface $V \subset M$ is convex with respect to N , then its positive equidistants $V_s, s > 0$ are at least as convex as V : $\phi_s^\Pi_s \geq \Pi_0 \geq 0$ for $s > 0$.*

END OF PROOF FOR THE 2ND TERM. By the O’neill formula for curvature [17] (see Cor. 1, eq (3), p. 466), the base space B of a Riemannian submersion is non-negatively curved provided its total space Q has zero (or positive) curvature. Applying this to $\pi : M(d, d) \rightarrow Sh$ we get that Sh is a non-negatively curved manifold at all smooth points. (Indeed the sectional curvature of a two-plane in

T_pSh which is spanned by orthonormal vectors $v, w \in T_pSh$ is $\sigma = \frac{3}{4} \|F_p(v, w)\|^2$ where F is curvature of the Riemannian submersion when viewed as a principal $SO(d)$ -bundle. Lemma 2, p.461 of [17] describes the relation between the A occurring there within cor. 1, eq (3), and the curvature.)

By proposition 4.1 and the fact that $\Sigma = \Sigma_0$ is totally geodesic at each smooth point, we have that each $\Sigma_s, s > 0$ is concave relative to ∇S , which is to say that $Q_p \leq 0$ for $S > 0$. It follows that $H \geq 0$ and by symmetry, as above, we have that $Q = -SH$ with H a positive semi-definite form, as desired.

QED for the second term and the proof of proposition 1.1.

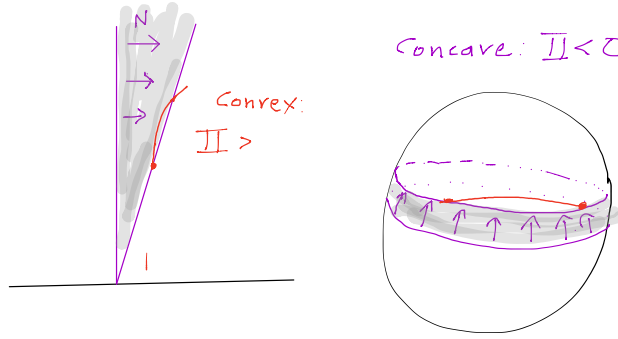


FIGURE 1. The relation between the sign of the sectional curvatures and convexity of equidistant hypersurfaces to a totally geodesic submanifold. The left figure depicts an equidistant from a geodesic in the hyperbolic plane (ambient curvature -1). The right figure pictures an equidistant from a geodesic on the sphere (ambient intrinsic curvature 1). The first equidistant is convex relative to the normal while the second is concave.

5. PROOFS OF THEOREMS

We prove theorems 1.1, 1.2 and 1.3 by strengthening proposition 1.1:

Proposition 5.1. *Regardless of whether or not S is smooth along the zero angular solution γ to Newton's equations, the composition $S \circ \gamma$ is a convex function of t for $S > 0$ and a concave function for $S < 0$. If γ is bounded with bounds $r_{ab} \leq c$ then $S \circ \gamma(t) = 0$ for at least one time t in each time interval of length $\Delta t = \pi(c^3/GM)^{1/2}$, in the Newtonian potential case and length $\pi(GM\delta)^{-1/2}$ for the general potential case as per hypothesis (2), (3) and (4).*

PROOF OF PROPOSITION 5.1 .

We first consider the case when S is smooth along γ , treating the general case as a limit of the smooth case.

If S is smooth along γ then proposition 1.1 asserts that $\ddot{S} = -Sg$ with $g > 0$ and smooth. The convex/concave properties of $S \circ \gamma$ follow immediately. In case the bounds on the r_{ab} are in force then we know that $g \geq \omega^2 = GM/\delta$ with δ as per hypothesis (2), (3) and (4) in the case of general two-body potential and $\delta = 1/c^3$ in

the particular case of the Newtonian potential. Compare our differential equation for S to the oscillator equation $\ddot{S} = -S\omega^2$. The solutions of the later, being $S = A\sin(\omega(t - t_0))$, have successive zeros t_0, t_1, \dots spaced regularly at increments of length π/ω . By the Sturm comparison theorem, between any two of these zeros lies a zero of our S . Since $1/\omega = \sqrt{\delta/GM}$ this yields the result for the smooth case.

For the general case, it will suffice to know that set of points at which S fails to be smooth has codimension 2. We call this the singular set of S and denote it by $Sing(S)$. This assertion regarding the codimension of $Sing(S)$ is theorem 6.1 of the next section.

Assuming the validity of this codimension theorem 6.1, let γ be a zero angular momentum solution to Newton's equations. Then, by using the smooth dependence of solutions on initial conditions, we can find a family of solutions γ_ϵ in $M(d, d)$ which avoids $Sing(S)$ and converges in the uniform (C^0) topology (or even C^k topology for any k) to γ on compact time intervals as $\epsilon \rightarrow 0$. By lemma 1, each $S \circ \gamma_\epsilon$ is convex wherever it is positive and concave wherever negative. The properties of being convex or concave are closed in the C^0 -topology, i.e. the uniform limit of convex functions is convex. Since $S \circ \gamma_\epsilon \rightarrow S \circ \gamma$ in the C^0 topology our result for the convexity / concavity of $S \circ \gamma$ follows. We proceed to the boundedness implications. If the original γ satisfies $r_{ab}(\gamma(t)) \leq c$, then its approximating curves γ_ϵ almost satisfy this bound, namely, they satisfy $r_{ab}(\gamma_\epsilon(t)) \leq c + o(1)$ as $\epsilon \rightarrow 0$, since they C^0 -converge to γ . Thus, by the preceding paragraph, each $S \circ \gamma_\epsilon$ has a zero in any interval of length $\Delta t = \pi/\omega(\epsilon)$ with $\omega(\epsilon) = GM/\delta(\epsilon)$ and $\delta(\epsilon) = \delta(c + o(1))$ as per eq (4) above. (Explicitly for the Newtonian case, $\delta(c + o(1)) = 1/(c + o(1))^3$.) Since $S \circ \gamma_\epsilon \rightarrow S \circ \gamma$ we see that $S \circ \gamma$ must have a zero in every time interval of length $\pi/\omega = \pi\sqrt{\delta/GM}$.

QED

PROOF OF THEOREMS. Theorems 1.2 and 1.3 follow immediately from proposition 5.1. Theorem 1.1 is the case $d = 3$ of theorem 1.2.

All that remains to do now in the way of proofs is to establish that the codimension of $Sing(S)$ is 2.

6. SINGULAR SET OF THE SIGNED DISTANCE AND THE SINGULAR VALUE DECOMPOSITION.

In this section we compute the codimension of $Sing(S) \subset M(d, d)$ (theorem 6.1).

6.1. Democracy group. SVD. The signed distance function S enjoys a larger symmetry group than Newton's equations. These additional symmetries form the "democracy group" and are crucial to identifying $Sing(S)$.

We saw in subsection 3.3 that the translation-reduced configuration space is the space of square matrices $M(d, d)$, that $\Sigma \subset M(d, d)$ is given by $det(q) = 0$ and that by choosing an appropriate basis for "mass label space" we can insure that the mass inner product agrees with the standard Euclidean inner product so that the norm squared of a matrix is $tr(q^t q) = \sum_{i,j} q_{ij}^2$. By inspection, the action of $O(d) \times O(d)$ on $M(d, d)$ by

$$(11) \quad q \mapsto g_1 q g_2^t, g_1, g_2 \in O(d).$$

is an isometric action which preserves Σ . It follows that $|S(q)|$, the distance from q to Σ is invariant under this group action. The action does not quite preserve our signed distance, since the $O(d)$'s can reverse orientation. Indeed

$$S(g_1 q g_2) = \pm S(q); \pm = \det(g_1) \det(g_2).$$

from which it follows that the action of $SO(d) \times SO(d)$ preserves S .

TERMINOLOGY: DEMOCRACY GROUP. The first, or left $O(d)$ action (g_1 , in eq (11)) is the usual action of rotations and reflections. The second, or right $O(d)$ (g_2 , in eq (11)) is called the ‘‘democracy group’’ since its action on the matrix space corresponds to choosing new basis for the mass label space, so in essence, permutes, or ‘‘democratizes’’ the mass labels.

The Singular Value Decomposition [SVD] from Matrix theory [?] is a normal form theorem for this group action (11). This decomposition asserts that for any $q \in M(d, d)$ there is a *diagonal matrix* x and matrices $g_1, g_2 \in O(d)$ such that

$$(12) \quad q = g_1 x g_2^t, \quad [SVD1]$$

Moreover the g_i can be chosen so as to force every nonzero entry of x to be positive, and the diagonal entries to be listed in descending order, thus:

$$(13) \quad x = \text{diag}(x_1, x_2, \dots, x_d), x_1 \geq x_2 \geq \dots \geq x_d \geq 0. \quad [SVD2]$$

The diagonal x written in this form is unique. Its diagonal entries x_i are called the ‘‘ i th principal values’’ of q . The x_i^2 are the eigenvalues of both of the symmetric operators $q^t q$ and of $q q^t$.

Proposition 6.1. *The distance function $|S(q)|$ of $q \in M(d, d)$ to Σ is equal to x_d above, the d th (smallest) principal value of q .*

We prove this proposition in the next subsection, below.

If we impose the constraint that $(g_1, g_2) \in SO(d) \times SO(d)$ when performing the normal form computations, then we get the following ‘specialized’ version of the SVD called the ‘pseudo-singular value decomposition’ by [6] (see p. 361).

Proposition 6.2 (PsSVD). *Given any $q \in M(d, d)$ there is a pair $(g_1, g_2) \in SO(d) \times SO(d)$ and a unique diagonal $x = \text{diag}(x_1, x_2, \dots, x_d)$ satisfying $x_1 \geq x_2 \geq \dots \geq x_{d-1} \geq |x_d|$ such that*

$$q = g_1 x g_2, g_i \in SO(d).$$

Then

$$S(q) = x_d$$

and $\text{sign}(x_d) = \text{sign}(\det(q)) = \text{sign}(S(q))$.

In words, the signed distance S is the last ‘signed’ singular value of q in the pseudo-singular value decomposition.

PROOF OF PROP 6.2 ASSUMING PROP 6.1. The value of $\det(q)$ cannot be changed by acting on it by $(g_1, g_2) \in SO(d) \times SO(d)$ and is equal to $x_1 x_2 \dots x_d$ if $q = g_1 x g_2^t$ with $x = \text{diag}(x_1, \dots, x_d)$. Now use the SVD for q . If either one of the elements g_i of the SVD for q is in $O(d)$ but not in $SO(d)$ then we can premultiply that element by $\text{diag}(1, 1, \dots, 1, -1) \in O(d)$ to get a new $g_i \in SO(d)$ at the expense of perhaps changing x_d to $-x_d$. Keeping track of the signs of $\det(q)$ and of S yields that $S(q) = x_d$, the last ‘special’ (or ‘signed’) singular value.

QED

Finally, here is the assertion we need to complete all our proofs.

Theorem 6.1. *The signed distance function $S : M(d, d) \rightarrow \mathbb{R}$ is smooth at any point q of $M(d, d)$ whose smallest two principal values are distinct. The complementary set, the singular locus of S , is the set of matrices q whose d th and $d - 1$ st singular values are equal: $x_{d-1} = |x_d|$. This locus is a semi-algebraic set of codimension 2 within $M(d, d)$.*

6.2. A slice. The fact underlying the proofs of the propositions and theorems just stated (theorem 6.1 etc) is that the linear subspace $D \subset M(d, d)$ of diagonal matrices is a *global slice* for our $O(d) \times O(d)$ action (eq (11)) on $M(d, d)$. Recall that the *orbit* of $q \in M(d, d)$ under this action is the set $\{g_1 q g_2^t : g_1, g_2 \in O(d)\} \subset M(d, d)$ and that, from basic manifold theory, the orbit is a smooth submanifold. The assertion that D is a slice for the action means a number of things

- (a) every $O(d) \times O(d)$ orbit intersects D
- (b) the orbit intersects D orthogonally
- (c) the intersection is transverse for generic orbit (i.e generic q)

Assertion (a) follows from the SVD.

Assertion (b) is a computation. Let ξ_1 and ξ_2 be skew symmetric matrices representing elements of the Lie algebra of our $O(d)$'s, understood to represent the derivatives of the g_i along curves passing through $g_i = Id$. Then the tangent space to the orbit through x for $x \in D$ of the orbit consists of all $d \times d$ matrices v of the form

$$(14) \quad v = \xi_1 x - x \xi_2.$$

One sees by direct computation that the diagonal entries of v are all zero, so that $v \perp D$.

Assertion (c) follows by taking “generic” matrix to mean one all of whose principal values are **distinct**, and then making a more detailed computation based on the orbit tangent space equation (14). If we take $\xi_2 = -\xi_1$ in that equation and set $\xi = \xi_1$ then we compute that v is skew-symmetric with entries $(x_i + x_j)\xi_{ij}$ where ξ_{ij} are the entries of ξ . On the other hand, if we take $\xi_2 = \xi_1 = \xi$ in eq(14) we obtain that v is a symmetric matrix with entries $(x_j - x_i)\xi_{ij}$. Now if the x_i are the distinct principal values, we have that $x_i \pm x_j \neq 0$ for all $i \neq j$ and it follows easily from this we can obtain any skew-symmetric matrix as a v as per eq (14), and that we can also obtain any symmetric matrix v which has zeros on its diagonal. Since any matrix at all is the sum of a symmetric and a skew-symmetric matrix we see that the tangent space to the orbit at a generic x consists of all matrices v with zero entries on the diagonal, which comprises the orthogonal complement to D .

PROOF OF PROPOSITION 6.1. As noted just after we introduced the action in eq (11), the distance function $|S(q)|$ is invariant under the $O(d) \times O(d)$ action:

$$|S(g_1 x g_2^t)| = |S(x)|.$$

Now $\det(x) = x_1 x_2 \dots x_d$ so that $\Sigma \cap D = \{x_1 x_2 \dots x_d = 0\}$ is the union of the d coordinate hyperplanes $x_i = 0$. The metric on $M(d, d)$ is Euclidean in the entries, and D is a d -dimensional linear space and in particular totally geodesic: any minimizing geodesic connecting points of D is a line segment within D . This implies that for $x \in D$ the $M(d, d)$ -distance of x to Σ equals the D -distance of x to $\Sigma \cap D$, that is the distance as realized by line segments *within* D . It follows that the problem of computing that distance is a problem in Euclidean geometry.

To solve the problem, let us first fix attention to the case $d = 3$. Observe that x_1, x_2, x_3 are orthonormal linear coordinates on D . The Euclidean distance of (x_1, x_2, x_3) from the plane $x_1 = 0$ is $|x_1|$. Since $\Sigma \cap D$ is the union of the three planes $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$, we have that

$$|S(x_1, x_2, x_3)| = \min_i |x_i|.$$

But this minimum is the 3rd singular value of x , namely x_3 when the diagonal values are listed as per the SVD. The same logic works for general d and yields $S(x_1, \dots, x_d) = \min_i |x_i|$, which is by definition the d th singular value of q . This proves proposition 6.1.

PROOF OF THEOREM 6.1. *The case $d = 2$.* We begin with the case $d = 2$ for simplicity and intuition. The configuration space is $M(2, 2)$. The degeneration locus $\Sigma = \{\det(q) = 0\}$ is a quadratic cone of signature $(2, 2)$ in the vector space $M(2, 2)$. The group $O(2) \times O(2)$ acts isometrically on the matrix space and the diagonal matrices D form a global slice as described above. Write $q = \text{diag}(x, y) \in D$. Then $D \cap \Sigma$ forms the “cross” $xy = 0$. Within the plane the distance function is $S(x, y) = \text{sign}(xy) \min(|x|, |y|)$. See figure 6.2. The non-smooth locus of S is the line $x = y$ and $x = -y$ corresponding to the matrices xI and xJ where I is the identity and $J = \text{diag}(1, -1)$. It now follows from symmetry that $\text{Sing}(S)$ is the union of two two-dimensional conical varieties intersecting at the origin, namely $\mathbb{R}SO(2)I$ and $\mathbb{R}SO(2)J$. Taken together this set is simply $\mathbb{R}O(2)$, since $J \in O(2)$ and $\det(J) = -1$. If the masses are all equal then this singular locus corresponds to the Lagrange points (equilateral triangles) with one cone corresponding to the positively oriented Lagrange configurations (the north pole of the shape sphere) and the other cone to the negatively oriented Lagrange configurations.

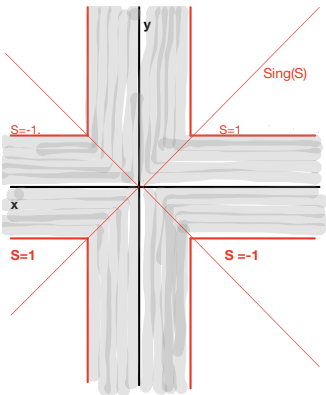


FIGURE 2. Equidistant curves to a cross $xy = 0$ have corners at which the distance function $|S|$ fails to be smooth. This picture models the contours of S restricted to the diagonal slice D for $d = 2$. The thin red diagonal lines indicate $\text{Sing}(S) \cap D$.

The case $d = 3$. The diagonals are still a slice for the $SO(3) \times SO(3)$ action, and S is invariant under this action. It follows that we can understand the singularity set of S by looking at its behaviour on the diagonal matrices $diag(x_1, x_2, x_3)$. First, suppose we are at a point where all $x_i > 0$, and such that $x_1 > x_2 > x_3 > 0$. Then, $S = x_3$ in a neighborhood of our point, which is clearly smooth. As we move from this point towards Σ along a geodesic orthogonal to Σ , the value of $x_3 = S$ steadily decreases until we hit $x_3 = 0$ at which point S continues to decrease, but smoothly. The equality $S = x_3$ continues into the region $x_3 < 0$ as long as $x_1 > x_2 > |x_3|$. This phenomenon is invariant under permutations of the coordinate indices. Indeed, restricted to D , we have that $S(x_1, x_2, x_3) = x_i$ where $|S(x_1, x_2, x_3)| = |x_i| := \min_k |x_k|$. Thus the singular locus of S restricted to D lies on the locus where $|x_i| = |x_j|$ for some $i \neq j$. This locus is the union of 6 planes in D , so has dimension 2, or codimension 1, *within* D . (The singular locus of the restriction of S to D is a bit smaller than the union of these planes, since we do not need that all three principal values are distinct, but only the bottom, two, i.e we only need $x_2 \neq x_3$ if $x_1 \geq x_2 \geq x_3 \geq 0$ are the singular values.)

At first glance, one guesses that since the singular set has codimension 1 within D , then it has overall codimension 1 within $M(3, 3)$. *This logic is wrong.* Points on the singular set of S are *not* generic with respect to the $SO(3) \times SO(3)$ action: their symmetry type jumps. Orbits through points of $Sing(S)$ have dimension 5 or less, not 6 like the dimension of a generic point. (That the orbit through a generic point of D is 6-dimensional is item (b) of ‘slice’ above.) Since S is invariant under our group $SO(3) \times SO(3)$, so is its singular set, $Sing(S)$. Thus the singular set is the union of the orbits through the singular points of the restriction of S to D . $Sing(S) \cap D$ has dimension 2. If the orbit through any point of $Sing(S) \cap D$ has dimension 5 or less then the singular set itself has dimension at most $7 = 2 + 5$. Our space $M(3, 3)$ has dimension 9, which yields the claimed codimension of 2.

It remains to establish that the orbits through points $x \in Sing(S) \cap D$ have dimension 5 or less. The dimension of an orbit of Lie group action is the dimension of the group minus the dimension of the isotropy subgroup of that point. Our group has 6. We show that the isotropy group at such a point x has dimension at least one. Write $x = diag(x_1, \lambda, \lambda)$ for such a singular point. Let $g(t)$ be the rotation about the 1st axis by t radians, and $g(-t)$ its inverse. Clearly $g(t)xg(-t) = x$, establishing that the isotropy group is *at least* one-dimensional, and hence the orbit has dimension $5 = 6 - 1$ *or less*. (A linear algebra computation, following equation (14), shows that this dimension is exactly 5 as long as $x_1 \neq \lambda$, but is unnecessary here since all we need is that the codimension of $Sing(S)$ is at least 2.) In case $x = diag(x_1, \lambda, -\lambda)$ with $S(x_1, \lambda, -\lambda) = \lambda$ so that $|x_1| \geq |\lambda| \geq 0$, use left multiplication by the matrix $g_1 = diag(-1, 1, -1) \in SO(3)$ to replace this x by $x = (-x_1, \lambda, \lambda)$ which lies on the same orbit as the original x but now has the form of the computation just made. Since the orbit is homogeneous its dimension does not depend on where on the orbit we choose to compute dimension, and we arrive again at the fact that its dimension is 5 or less.

The case $d > 3$. The proof is nearly identical to the case $d = 3$. $Sing(S) \cap D$ has codimension 1, being contained in the union of the hyperplanes where $x_i = \pm x_j$. At a generic point of D , which is to say, off of these hyperplanes, the $SO(d) \times SO(d)$ action is “almost free” : the orbit’s dimension equals that of $SO(d) \times SO(d)$, as per item (b) of being a slice above. At a typical point on one of these hyperplanes

the isotropy algebra is again one-dimensional, consisting of rotations of the double eigenvalue plane. (An “atypical” singular point would be one for which the three smallest singular values are all equal and here the isotropy algebra has dimension at least 3.) Hence the *codimension* of $Sing(S)$ is $1 + 1$: 1 for the codimension within D and 1 for the extra continuous symmetry dimension (isotropy) associated to each such double “eigenvalue” diagonal matrix. (Sign discrepancies such as $x_i = -x_j \neq 0$ are at first bothersome, but the trick we used in the previous paragraph of multiplying by an element of $SO(d)$ with ± 1 's to change the entries to $x_i = x_j$ works as before.)

QED

7. DYNAMICAL VISTAS AND OPEN QUESTIONS

PLANAR PRECURSOR.

The planar case of theorem 1.2 or 1.3 asserts that any bounded solution to the planar three-body problem defined on the whole time line will suffer infinitely colinear instants. Colinear instances are also called “syzygies”. Non-collision syzygies come in three flavors, 1, 2, and 3, depending on the mass in the middle. See figure 3. We can thus associate a syzygy sequence to such a solution. *What syzygy sequences are realized?* This question, still largely open, has motivated much work. See for example [10], and also the closely related work in which braids (equivalent to “stutter-reduced” syzygy sequences) rather than syzygy sequences are used for the symbolic encoding [16], [19], [7] and references therein.

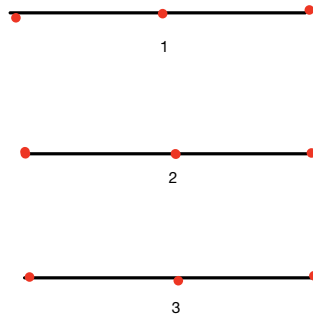


FIGURE 3. The 3 types of generic collinear 3 body shapes.

Theorem 1.1 asserts that that any bounded solution to the spatial four-body problem defined on the whole time line will suffer infinitely coplanar instants. The

generic coplanar configurations divide into 7 types as per FIGURE 4. (We have excluded as “non-generic” configurations for which three of the masses are collinear. Binary collision configurations are thus excluded.) We now have a seven letter

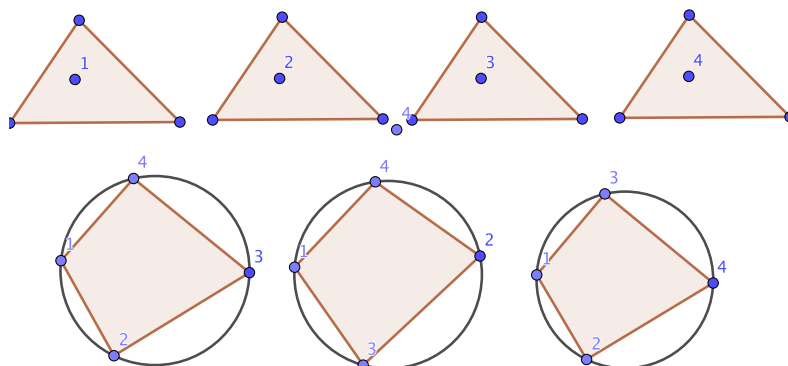


FIGURE 4. The 7 types of generic planar 4 body shapes

alphabet for potential symbol sequences, in analogy with the syzygy sequences of planar three-body dynamics.

OPEN QUESTIONS FOR THE FOUR-BODY PROBLEM IN SPACE.

Q1. Are all possible symbol sequences in this 7-letter alphabet realized by a bounded solution having zero angular momentum?

ENERGY AND ANGULAR MOMENTUM CONSIDERATIONS Bounded solutions for the Newtonian N-body problem necessarily have negative energy. (As soon as $N > 2$ there are negative energy solutions which are unbounded.) Hence the following theorem (see [13]) represents a strengthening of 1.3 for the case $d = 2$. **Theorem:** every zero angular momentum negative energy solution to the planar three-body problem which does not end and begin in triple collision hits the collinear locus infinitely often.

We do not know a single bounded or negative energy solution of the 4-body problem in space which never suffers co-planarities.

MORE QUESTIONS.

Q2. Do there exist negative energy collision-free solutions of the spatial four body problem which are defined over the whole time line and which never suffer coplanar instants ?

Q3. If the answer to Q2 is ‘yes’ then are any of these never-coplanar solutions bounded?

Q4. If the answer to Q2 is ‘yes’ do any of these never-coplanar solutions have zero angular momentum?

In regards to these last two questions, Joseph Gerver has pointed out that there are negative energy collision-free solutions which have no coplanar instants and are defined over a time ray $0 \leq t < \infty$. These solutions have nonzero angular momentum. Take the rotating Lagrange equilateral solution for three of the bodies. Now take the 4th body to be moving away from this triple along the line perpendicular to the plane of the rotating triangle. If the three masses are equal then the situation is symmetric and the 4th body will stay on this orthogonal line as it moves out. The energy of the bound triple can be taken to be sufficiently small so that the overall energy is negative while the 4th escaping body escapes hyperbolically to infinity.

APPENDIX A. DISPOSITIONS: TRANSLATION REDUCTION DONE RIGHT.

We follow the idea of “dispositions” described in Albouy-Chenciner [2] in order to identify the translation quotient of configuration space with the matrix space $M(d, d)$ in a way which is independent of mass choices. We mostly stick to the case $d = 3$. Then we introduce masses and work out how they yield an inner product, choice of Jacobi vectors and the standard mass-dependent embedding of $M(d, d)$ back into configuration space. We strongly recommend [9], pp. 34-40 for a down-to-earth perspective on this subject.

As per eq (5), we view the four position vectors q_a , $a = 1, 2, 3, 4$, describing the instantaneous positions of the four masses as column vectors and place them side by side to form a 3×4 which we can view as a linear operator,

$$q = \begin{pmatrix} q_1 & q_2 & q_3 & q_4 \end{pmatrix} : \mathbb{R}^4 \rightarrow \mathbb{R}^3$$

from the “mass label space” \mathbb{R}^4 to our inertial space of motions \mathbb{R}^3 . Thus the a th mass is located at the point

$$q_a = q(e_a)$$

where e_a , $a = 1, 2, 3, 4$ is the standard basis of \mathbb{R}^4 . Written out in tensor language,

$$q = \sum q_a \otimes \theta^a$$

where θ^a is the basis dual to e_a .

We have identified the full configuration space with

$$M(3, 4) := \text{Hom}(\mathbb{R}^4, \mathbb{R}^3) \cong \mathbb{R}^3 \otimes \mathbb{R}^{4*}.$$

A translation by $b \in \mathbb{R}^3$ acts on the position vectors q_a as per the matrix representation of eq (6) which can be summarized in tensor language as

$$q \mapsto q + b \otimes \Theta \text{ where } \Theta = \sum_a \theta^a$$

since, in matrix terms $\Theta = (1, 1, 1, 1)$ so that $b \otimes \Theta = b(1, 1, 1, 1)$ is the matrix all of whose columns are b . Now $b \otimes \Theta$ is zero on the 3-dimensional linear hyperplane:

$$L := \ker(\Theta) = \{\xi \in \mathbb{R}^4 : \Theta(\xi) := \xi_1 + \xi_2 + \xi_3 + \xi_4 = 0\} \subset \mathbb{R}^4$$

from which it follows that the restriction of q to this subspace is translation invariant.

Definition A.1. *The translation reduction of $q \in M(3, 4)$ is its restriction $q_{red} := q|_L : L \rightarrow \mathbb{R}^3$, to $L \subset \mathbb{R}^4$. The 4-body translation-reduced configuration space is*

$$\text{Hom}(L, \mathbb{R}^3) = \mathbb{R}^3 \otimes L^*$$

and the translation-reduction map is

$$pr : \text{Hom}(\mathbb{R}^4, \mathbb{R}^3) \rightarrow \text{Hom}(L, \mathbb{R}^3); pr(q) = q_{red}.$$

Remark. Following this prescription we have no need of masses in order to identify the quotient of configuration space by translations!

NOTATIONAL WARNING. Albouy-Chenciner [2] use the symbol \mathcal{D}^* for our L , so that their translation-reduced configuration space is $\mathbb{R}^3 \otimes \mathcal{D} = \text{Hom}(\mathcal{D}^*, \mathbb{R}^3)$. Their \mathcal{D} is our $\mathbb{R}^{4^*}/(\mathbb{R}\Theta) = \mathbb{R}^{4^*}/(\mathbb{R}(1, 1, 1, 1))$ which they call the space of dispositions.

DEGENERATION LOCUS. The degeneration locus Σ coincides with those q for which $q_{red} : L \rightarrow \mathbb{R}^3$ has rank less than 3. Indeed, a basis for L is formed by e_{21}, e_{31}, e_{41} where $e_{ij} = e_i - e_j$. For example, $e_{21} = (-1, 1, 0, 0)$. Now $q(e_{ab}) = q_a - q_b$ so relative to this basis, q_{red} is represented by the 3 by 3 matrix

$$\begin{pmatrix} q_2 - q_1 & q_3 - q_1 & q_4 - q_1 \end{pmatrix}$$

whose determinant is 6 times the signed volume of the tetrahedron whose vertices are q_1, q_2, q_3, q_4 .

INTRODUCING MASSES. Once positive masses $m_a > 0$ are chosen, we can form the *mass vector* $\vec{m} = (m_1, m_2, m_3, m_4) \in \mathbb{R}^4$. Please observe that

$$\Theta(\vec{m}) = m_1 + m_2 + m_3 + m_4 := M > 0,$$

demonstrating that mass vector, together with the hyperplane L , spans \mathbb{R}^4 . As a consequence $q \in \text{Hom}(\mathbb{R}^4, \mathbb{R}^3)$ is uniquely determined by its restriction q_{red} to L together with its center of mass :

$$q_{cm} := \frac{1}{\Theta(\vec{m})}q(\vec{m}) = \frac{1}{M}(m_1q_1 + m_2q_2 + m_3q_3 + m_4q_4) \in \mathbb{R}^3.$$

We see in this way that the usual ‘fixing the center of mass to be zero’ way of forming the translation quotient is equivalent to the Albouy method.

MASS INNER PRODUCT, KINETIC ENERGY, AND JACOBI VECTORS. The choice of masses also defines a Euclidean inner product on \mathbb{R}^{4^*} by declaring that $\langle \theta^a, \theta^b \rangle = m_a \delta_{ab}$. In symbols

$$ds_m^2 = \sum m_a (e_a)^2$$

where we are using the fact that the e_a is the dual basis to θ^a .

Let \mathbb{V}^*, \mathbb{W} be real vector spaces. Choosing inner products on each of them induces a canonical inner product on $\mathbb{W} \otimes \mathbb{V}^*$ for which $\langle w \otimes \theta, w \otimes \theta \rangle = \langle w, w \rangle_{\mathbb{W}} \langle \theta, \theta \rangle_{\mathbb{V}^*}$. In our situation, $\mathbb{W} = \mathbb{R}^3$ comes with its standard inner product and we just used the masses to put an inner product on \mathbb{R}^{4^*} , so we now have a mass-dependent inner product on $M(3, 4) = \mathbb{R}^3 \otimes \mathbb{R}^{4^*}$. We call this inner product the ‘‘mass inner product’’ or sometimes the ‘‘kinetic energy metric’’ since it is the inner product for which half of the squared length of a vector is the kinetic energy. Indeed, the four instantaneous velocity vectors of our 4 masses are obtained by differentiating q with respect to time so as to form the velocity matrix $\dot{q} = \sum \dot{q}_a \otimes \theta^a$. We compute

$$\frac{1}{2} \langle \dot{q}, \dot{q} \rangle = \frac{1}{2} \sum m_a |\dot{q}_a|^2 = K(\dot{q})$$

which is the usual expression for the kinetic energy K . Here we have reserved the absolute value sign for the usual norm in our Euclidean \mathbb{R}^3 , as in $|q_1|^2 = q_1 \cdot q_1$.

The mass inner product induces an inner product on our translation-reduced configuration space $\text{Hom}(L, \mathbb{R}^3)$ which is essential for the body of this paper. There are several equivalent ways to arrive at this inner product. We will begin with the isomorphism $\mathbb{R}^{4^*} \rightarrow \mathbb{R}^{4^{**}} = \mathbb{R}^4$ induced by the mass inner product on \mathbb{R}^{4^*} . The

isomorphism sends $\theta^a \mapsto m_a e_a$, $a = 1, 2, 3, 4$ and induces an inner product on \mathbb{R}^4 for which

$$\langle e_a, e_b \rangle = \frac{1}{m_a} \delta_{ab}.$$

The isomorphism sends our basic “translation covector” $\Theta = \Sigma \theta^a \in \mathbb{R}^{4*}$ to the mass vector \vec{m} . Since Θ annihilates L it follows that the mass vector \vec{m} is orthogonal to L . Thus we get the orthogonal decomposition

$$\mathbb{R}^4 = L \oplus \mathbb{R}\vec{m}.$$

We compute $\|\vec{m}\|^2 = \|\Theta\|^2 = \Theta(\vec{m}) = M$, the total mass. Now choose an orthonormal basis E_1, E_2, E_3 for $L \subset \mathbb{R}^4$, and complete it by adding in \vec{m} to form the orthogonal basis E_1, E_2, E_3, \vec{m} for \mathbb{R}^4 . Write the associated dual basis for \mathbb{R}^{4*} as $\omega^1, \omega^2, \omega^3, \Theta/M$. Note that $\omega^i(\vec{m}) = 0$. Then the inner product on L^* can be defined by insisting that the restrictions of the ω^i to L forms an orthonormal basis for L^* . Since $\text{Hom}(L, \mathbb{R}^3) = \mathbb{R}^3 \otimes L^*$ this defines an inner product, as desired.

Definition A.2. *By a choice of Jacobi vectors we mean either an orthogonal basis V_1, V_2, V_3 for L relative to the mass inner product, or the image vectors $q(V_i)$ of this basis under $q \in \text{Hom}(L, \mathbb{R}^3)$. By a choice of normalized Jacobi vectors we mean either an orthonormal basis E_1, E_2, E_3 for L or the image of this basis under $q \in \text{Hom}(L, \mathbb{R}^3)$. These Jacobi vectors are said to form an oriented basis if the orientation they induce agrees with that induced on L by the standard basis for \mathbb{R}^4 together with the mass vector \vec{m} .*

To better understand the induced inner product on $\text{Hom}(L, \mathbb{R}^3)$, observe that given any basis u_a whatsoever for \mathbb{R}^4 , and its corresponding dual basis ω^a for \mathbb{R}^{4*} we can expand out any $q \in M(3, 4)$ as $q = \Sigma q(u_a) \otimes \omega^a$. If the basis is an orthogonal one, i.e. a triple of ‘Jacobi vectors’ as per the definition above, then the terms of this expansion are orthogonal relative to our mass metric, meaning that $\|q\|^2 = \Sigma |q(u_a)|^2 \langle \omega^a, \omega^a \rangle$. Applying these considerations to our orthonormal basis E_1, E_2, E_3 (or ‘normalized Jacobi vectors’) we find that

$$\|q\|^2 = |X_1|^2 + |X_2|^2 + |X_3|^2 + M|q_{cm}|^2, \text{ with } X_i = q(E_i), E_i \text{ orthonormal for } L.$$

Exercise A.1. *With q and E_i as above show that the norm squared of the translation reduction q_{red} of q satisfies*

$$\|q_{red}\|^2 = |X_1|^2 + |X_2|^2 + |X_3|^2, \text{ with } X_i = q_{red}(E_i).$$

We go a bit deeper into the last identity for the inner product after some more examples around Jacobi vectors

Example A.1 (3 bodies in the plane.). *To understand this definition we retreat to the case of 3 bodies in \mathbb{R}^2 where Jacobi vectors are better known. The configuration space is now $\text{Hom}(\mathbb{R}^3, \mathbb{R}^2)$ where \mathbb{R}^3 is the mass label space and \mathbb{R}^2 represents the inertial space in which the bodies move. Then $e_{12} = (1, -1, 0) \in L$ and $q(e_{12}) = q_1 - q_2$, is the standard choice of first Jacobi vector. Take the mass vector to be $\vec{m} = (m_1, m_2, m_3) \in \mathbb{R}^3$ so that the associated mass inner product on the mass label space \mathbb{R}^3 is given by $\langle e_a, e_a \rangle = 1/m_a$, $a = 1, 2, 3$. Then $\langle e_{12}, e_{12} \rangle = \frac{1}{m_1} + \frac{1}{m_2} := \mu_1^2$. If we write $m_{12} = m_1 + m_2$ then $V_2 = (-m_1, -m_2, m_{12}) \in L$. Compute that $\langle V_2, e_{12} \rangle = 0$. Scale V_2 to $U_2 = \frac{1}{m_{12}}(V_2) = (-m_1/m_{12}, -m_2/m_{12}, 1)$. Then $q(U_2) = q_2 - (m_1 q_1 + m_2 q_2)/m_{12}$ is the standard expression for the second Jacobi vector, and $\langle U_2, U_2 \rangle = \frac{1}{m_{12}} + \frac{1}{m_3} = \mu_2^2$ is its normalization factor. The vectors*

$E_1 = \frac{1}{\mu_1}e_{12}$ and $E_2 = \frac{1}{\mu_2}U_2$ are then an orthonormal basis for L , so “normalized Jacobi vectors”. In the standard terminology, the normalized Jacobi vectors are $q(E_1) = Z_1$ and $q(E_2) = Z_2$, and they yield the identity $|q_{red}|^2 = |Z_1|^2 + |Z_2|^2$. After identifying \mathbb{R}^2 with \mathbb{C} , we have $(Z_1, Z_2) \in \mathbb{C}^2$ and this coordinatization of the reduced configuration space as \mathbb{C}^2 is the first step in forming shape space and the shape sphere. See [14]

Example A.2. . [4 bodies in 3-space.] The three vectors

$$J_1 = (1, -1, 0, 0), J_2 = (0, 0, 1, -1), \text{ and } J_3 = (-p_1, -p_2, p_3, p_4)$$

form a basis for $L \subset \mathbb{R}^4$ for any choice of positive p_i with $p_1 + p_2 = p_3 + p_4 = 1$. Now suppose masses m_1, m_2, m_3, m_4 are given and set $m_{12} = m_1 + m_2, m_{34} = m_3 + m_4, p_1 = m_1/m_{12}, p_2 = m_2/m_{12}, p_3 = m_3/m_{34}, p_4 = m_4/m_{34}$. Then J_1, J_2, J_3 are Jacobi vectors – they are mass-orthogonal. Normalizing them we get the corresponding normalized Jacobi vectors $E_i = \sqrt{\mu_i}J_i$ with normalization factors μ_i defined by $\frac{1}{\mu_1} = \frac{1}{m_1} + \frac{1}{m_2}, \frac{1}{\mu_2} = \frac{1}{m_3} + \frac{1}{m_4}$, and $\frac{1}{\mu_3} = \frac{1}{m_{12}} + \frac{1}{m_{34}}$. Then, if we set $\rho_i = q(E_i) \in \mathbb{R}^3$, so that, for example, $\rho_1 = \sqrt{\mu_1}(q_1 - q_2)$, and if $q_{cm} = 0$, then $\|q\|^2 = |\rho_1|^2 + |\rho_2|^2 + |\rho_3|^2$. Compare ([8], eqs (2.5), (2.6), pp 2036).

RETURNING TO THE INNER PRODUCT...

The restriction-to- L map $pr : Hom(\mathbb{R}^4, \mathbb{R}^3) \rightarrow Hom(L, \mathbb{R}^3)$ (see Definition A.1) implements the quotient of $Hom(\mathbb{R}^4, \mathbb{R}^3)$ by the ‘translation subgroup’ $\mathbb{T} = \mathbb{R}^3 \otimes \Theta \subset \mathbb{R}^3 \otimes \mathbb{R}^{4*} = Hom(\mathbb{R}^4, \mathbb{R}^3)$. The choice of masses induces an inner product on $Hom(\mathbb{R}^4, \mathbb{R}^3)$ and thence on $Hom(L, \mathbb{R}^3)$. The masses also induce an inclusion $Hom(L, \mathbb{R}^3) \rightarrow Hom(\mathbb{R}^4, \mathbb{R}^3)$ which is not present without the additional structure of the masses. This inclusion and the inner product on $Hom(L, \mathbb{R}^3)$ are so tightly linked that they are almost the same thing.

We formulate all this in more intrinsic linear algebra terms. Suppose we have an inner product space \mathbb{E} endowed with a subspaces \mathbb{T} . Then \mathbb{E}/\mathbb{T} is naturally endowed with an inner product which makes $\pi : \mathbb{E} \rightarrow \mathbb{E}/\mathbb{T}$ a ‘submetry’ as defined above. Now consider the orthogonal complement $\mathbb{T}^\perp \subset \mathbb{E}$. Then the restriction of π to \mathbb{T}^\perp is an isometry between \mathbb{T}^\perp and \mathbb{E}/\mathbb{T} . Now apply these considerations to the case that $\mathbb{E} = Hom(\mathbb{V}, \mathbb{W})$, $\mathbb{E}/\mathbb{T} = Hom(\mathbb{S}, \mathbb{W})$ where $\mathbb{S} \subset \mathbb{V}$ is a linear subspace of \mathbb{V} and where the projection map π is the map which sends $q : \mathbb{V} \rightarrow \mathbb{W}$ to its restriction $q|_{\mathbb{S}} : \mathbb{S} \rightarrow \mathbb{W}$. Then $\mathbb{T} \subset Hom(\mathbb{V}, \mathbb{W})$ consists of those linear operators which are zero on \mathbb{S} . An inner product on \mathbb{V} and \mathbb{W} induces ones on all these spaces.

Exercise A.2. Show that, continuing with the above terminology, \mathbb{T}^\perp consists of those linear operations $\mathbb{V} \rightarrow \mathbb{W}$ which are identically zero on the orthogonal complement $\mathbb{S}^\perp \subset \mathbb{V}$ to \mathbb{S} and that the induced isometric inclusion $Hom(\mathbb{S}, \mathbb{W}) \rightarrow Hom(\mathbb{V}, \mathbb{W})$ is the map which takes a linear operator $q_{red} : \mathbb{S} \rightarrow \mathbb{W}$ and extends off of \mathbb{S} to obtain a map $\mathbb{V} \rightarrow \mathbb{W}$ equal to q_{red} on \mathbb{S} and identically zero on $\mathbb{S}^\perp \subset \mathbb{V}$.

Applying these considerations to our situation of $L \subset \mathbb{R}^4$ and $q_{red} \in Hom(L, \mathbb{R}^3)$ we see that the mass induced canonical extension $q = i(q_{red})$ of q_{red} is obtained by insisting that $q(\vec{m}) = 0$, since \vec{m} is the normal vector to L relative to the mass metric. But $q(\vec{m}) = Mq_{cm}$. **The mass-induced inclusion $i : Hom(L, \mathbb{R}^3) \rightarrow Hom(\mathbb{R}^4, \mathbb{R}^3)$ of the translation-reduced configuration space into our original configuration space $Hom(\mathbb{R}^4, \mathbb{R}^3)$ is obtained by insisting that the center of mass of the extension $q = i(q_{red})$ is zero.** Thus the process of

using inner products to implement this inclusion amounts to the usual ‘center-of-mass zero’ prescription for performing translation reduction. This inclusion i is an isometry onto its image, and we arrive in this way at a solution to exercise A.1 regarding the inner product on $Hom(L, \mathbb{R}^3)$. Comparing the formulae we see that the prescription for the inner product amounts to the equation that $\|q\|^2 = \|q_{rel}\|^2$ whenever $q_{cm} = 0$.

DEMOCRACY GROUP; LINEAR ISOMETRIES. Let us again suppose that \mathbb{V} and \mathbb{W} are finite-dimensional real inner product spaces. Then $\mathbb{W} \otimes \mathbb{V}^* = Hom(\mathbb{V}, \mathbb{W})$ inherits a canonical inner product for which the orthogonal groups $O(\mathbb{V})$ and $O(\mathbb{W})$ act isometrically according to

$$q \mapsto g_1 \circ q \circ g_2^{-1}, \text{ for } g_1 \in O(\mathbb{W}), g_2 \in O(\mathbb{V}).$$

Specializing to our situation, $\mathbb{V} = L \subset \mathbb{R}^4, \mathbb{W} = \mathbb{R}^3$, so that the 6-dimensional Lie group $O(L) \times O(\mathbb{R}^3) \cong O(3) \times O(3)$ acts isometrically on our translation-reduced configuration space $Hom(L, \mathbb{R}^3)$. The g_1 factor acting by left multiplication is the usual action on a 4-body configuration by rotation. The g_2 action is a bit more mysterious. It is not a symmetry of Newton’s equations.

Definition A.3. *The democracy group action is the action of $O(L)$ on $Hom(L, \mathbb{R}^3)$ by $q \mapsto qg_2^{-1}$.*

To better understand the democracy group action, choose an orthonormal basis for L , which is to say normalized Jacobi vectors E_1, E_2, E_3 . This choice induces an isomorphism $O(L) \cong O(3)$. \mathbb{R}^3 comes with a standard basis u_1, u_2, u_3 so we can just write $O(\mathbb{R}^3) = O(3)$. Relative to these two bases $q_{rel} : L \rightarrow \mathbb{R}^3$ becomes a 3×3 matrix X with entries $X_{ij} = u_i \cdot q(E_j)$. In this way we identify

$$Hom(L, \mathbb{R}^3) = M(3, 3) := \text{the space of all real three-by-three matrices.}$$

The mass metric in these coordinates is simply the standard entry-wise Euclidean structure: $\|q_{rel}\|^2 = tr(X^t X) = \sum_{i,j} X_{ij}^2$. Our $O(L) \times O(\mathbb{R}^3)$ action is simply standard matrix multiplication by $O(3) \times O(3)$:

$$X \mapsto g_1 X g_2^t.$$

The degeneracy locus is

$$\Sigma = \{X \in M(3, 3) : det(X) = 0\}.$$

and is mapped to itself by the $O(3) \times O(3)$ action, since $det(g_1 X g_2^t) = \pm det(X)$.

The idea expressed by the terminology ‘Democracy group’ is that a choice of Jacobi vectors involves selecting out certain masses to play special roles. An element of $O(L)$ changes the basis, i.e the Jacobi vectors, and hence corresponds to choosing different sets of masses for these roles. It permutes the mass labels. Indeed if all the m_a are equal, then the permutation group of the mass labels, i.e. of the basis vectors e_a for the label space \mathbb{R}^4 , forms a subgroup of $O(L)$. We owe the picturesque name ‘democracy group’ to Littlejohn and Reinsch, [8].

APPENDIX B. UNORIENTED VS ORIENTED SHAPE SPACE

We describe the oriented and unoriented shape space and the relation between them. For the general N body problem in \mathbb{R}^d the configuration space is $Hom(\mathbb{R}^N, \mathbb{R}^d)$ and these shape spaces are the quotient spaces of the configuration space by the groups $SE(d)$ and $E(d)$ respectively. The quotient $E(d)/SE(d)$ is the two-element

group (with nontrivial element generated by any orientation reversing isometry) so that we expect that the natural map from oriented to un-oriented shape space will be a 2 : 1 branched cover, branched over the degeneration locus Σ .

We form the quotient spaces in stages. We saw in the previous appendix how the quotient of the configuration space by translations is $Hom(L, \mathbb{R}^d) \cong M(d, d)$ in case $N = d + 1$. It remains to quotient by the linear isometries, i.e. the rotations $SO(d)$ and rotations and reflections $O(d)$. Thus oriented shape space becomes $M(d, d)/SO(d)$ while unoriented shape space becomes $M(d, d)/O(d)$, where $g \in O(d)$ or $SO(d)$ acts on $q \in M(d, d)$ by $q \mapsto gq$.

UNORIENTED SHAPE SPACE. The map $q \mapsto q^t q \in Sym(d)$ realizes the $O(d)$ quotient. Here $Sym(d)$ denotes the space of symmetric $d \times d$ matrices. When we say “realizes the quotient” we mean that for $q', q \in M(d, d)$ there exists a $g \in O(d)$ such that $q' = gq$ if and only if $(q')^t q' = q^t q$. This fact follows from a basic theorem from representation theory. The matrix $q^t q$ is sometimes called the ‘Gram matrix’, being a matrix of inner products of position vectors. Any matrix of form $q^t q$ is positive semi-definite, and any positive semi-definite matrix s can be expressed as $s = q^t q$ for some $q \in M(d, d)$. (Take $q = \sqrt{s} \in Sym(d)$ for example.). These facts prove that un-oriented shape space is the “positive semi-definite cone”: the closed convex cone of positive semi-definite symmetric matrices within $Sym(d)$. The boundary of the cone consists of those non-negative symmetric matrices whose rank is not full and thus corresponds to the shape projection of our degeneration locus Σ .

THE MAP BETWEEN. The map from oriented shape space to shape space maps onto this shape cone, and forms a 2:1 cover branched along Σ . Indeed, an unoriented nondegenerate simplex shape has precisely two oriented representative shapes, one having positive volume, the other having negative volume, while a degenerate shape in $M(d, d)/SO(d)$ has precisely one representative in the unoriented shape space.

A BIT OF TOPOLOGY. Take two identical copies of a closed convex cone with nonempty interior in any finite dimensional real vector space. Glue one copy to the other along the boundary, using the identity map of the boundary as gluing map. One checks without great difficulty that the result is homeomorphic to the original vector space: we have ‘blown up’ or desingularize the boundary. These general considerations may serve to convince the reader that oriented shape space is indeed homeomorphic to the Euclidean space $Sym(d)$

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