

UC San Diego

UC San Diego Previously Published Works

Title

Chemical potential and magnetization of a Coulomb island

Permalink

<https://escholarship.org/uc/item/3s87t04x>

Journal

Physical Review B, 49(19)

ISSN

2469-9950

Authors

Fogler, MM

Levin, EI

Shklovskii, BI

Publication Date

1994-05-15

DOI

10.1103/physrevb.49.13767

Peer reviewed

Chemical potential and magnetization of a Coulomb island

M. M. Fogler, E. I. Levin, and B. I. Shklovskii

Theoretical Physics Institute, School of Physics and Astronomy, University of Minnesota, 116 Church Street Southeast, Minneapolis, Minnesota 55455

(Received 18 August 1993; revised manuscript received 28 October 1993)

We consider variations of the low temperature chemical potential and magnetization of laterally confined two-dimensional interacting electrons in a strong magnetic field. Cases of axially symmetrical (a quantum dot) and strongly elongated (a quantum wire) spatial distributions of electron density are studied. We commonly refer to both types of systems as "islands." The calculations are performed in the mean-field approximation. In this approximation the system consists of alternating regions of compressible and incompressible liquids. We are focusing on islands with typical widths b , which are much larger than the Bohr radius in a semiconductor a_B , and argue that the density profile is formed mainly by electrostatic forces, has a domelike shape, and changes only slightly in the whole range of magnetic fields. We calculated numerically the chemical potential and the magnetization as functions of the magnetic field and found them to be in sharp contrast to the conventional result for noninteracting electrons. We suggest a simple analytic theory adequately explaining these functional dependences. The results are compared with the experimental data.

I. INTRODUCTION

A number of authors have studied the effect of a magnetic field on the energy spectra of quantum dots and quantum wires formed in the two-dimensional electron gas both experimentally^{1,2} and theoretically.²⁻⁵

Regarding variations of physical quantities (the chemical potential, magnetization, etc.) with an increasing magnetic field normal to the plane of the electron gas, two scales of magnetic field can be distinguished. One scale corresponds to a change in a filling factor ν by unity, and the other, a much lesser scale, is related to an increment of the total magnetic flux through the island by a flux quantum. Following Ref. 6 we call the variations on the former scale de Haas-van Alfvén⁷ type (dHvA) and on the latter one Aharonov-Bohm type. The behavior of physical quantities on both scales is oscillatory. The amplitude of large-scale variations is generally much larger than the amplitude of small-scale ones for islands consisting of many electrons. Due to this fact, physical quantities, e.g., the chemical potential, exhibit a smooth trend on the scale of dHvA-type oscillations slightly distorted by small rapid oscillations. These small oscillations were discussed elsewhere^{5,6} and result from the discreteness of either the energy spectrum or electric charge. The goal of this paper is to examine the large-scale variations (dHvA type) of the chemical potential and the magnetization, which take their origin in the grouping of energy levels into discreet Landau levels (LL's). According to our goal, we treat electrons as a continuous medium (charged liquid) in a self-consistent electric field. Spin splitting is ignored, but it can be incorporated without principal difficulties.

In this paper we will limit our studies to the case of smooth potentials varying by an amount much smaller than Landau gap $\hbar\omega_c$ on the scale of the magnetic length.

For these kinds of potentials the energy spectrum of electrons may be approximated by a set of LL's. The energy of k th LL at a given point in space is elevated by $(k + 1/2)\hbar\omega_c$ above the local value of the self-consistent potential. Therefore, LL's are bent and follow the variations of the self-consistent potential.

At zero temperature electrons fill all the energy states up to the level of the chemical potential μ . Hence the total energy of the system and consequently the behavior of the chemical potential and magnetization are defined by the shape of LL's. Let us start from the conventional results for the noninteracting electrons. In this case, the "self-consistent" potential is simply equal to the external one. The classical example of uniform energy LL's brings about the well-known sawtooth behavior of the chemical potential [Fig. 1(a)] and the magnetization [Fig. 1(b)].⁸ The discontinuities in physical quantities, clearly visible in the figures, occur where the electrons start populating the next LL. Indeed, since there is a $\hbar\omega_c$ gap between any adjacent LL's, populating the next LL with a decrease of the field results in jumps in the magnetization and chemical potential equal to $2\mu_B$ (two Bohr magnetons) and $\hbar\omega_c$, respectively.

The case of a nonuniform, in fact, parabolic external potential was studied, e.g., in Refs. 6 and 9. The corresponding plots look different [see Fig. 2(b) and 2(c)]. They consist of connected concave arcs and show no discontinuities. Here, due to the bending of the LL's, while belonging to the same LL, an electron may have different energies at different spatial locations. The next LL begins to fill only when its bottom energy coincides with the highest filled energy state at the preceding level. In this sense there are no energy gaps between the filled parts of adjacent LL's and, consequently, the chemical potential (and magnetization) vary continuously. The filled parts of LL's are determined by the intersections of

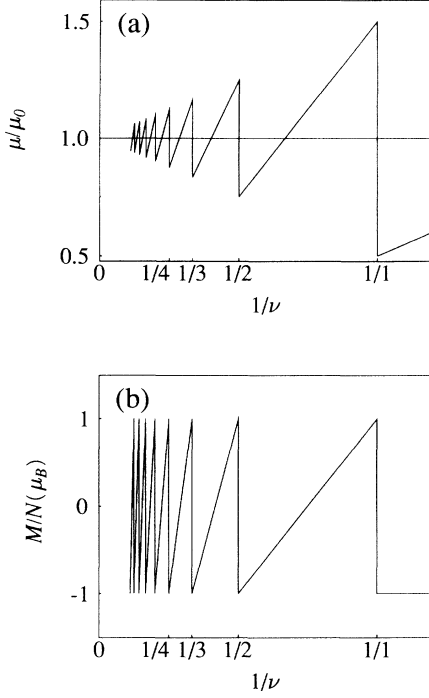


FIG. 1. Variations of the (a) chemical potential and (b) magnetization of free electrons with magnetic field.

bent Landau levels with the level of chemical potential μ [see Fig. 2(a) and Ref. 10]. The exact calculation of, e.g., the chemical potential, requires finding the spatial distribution of the electron density. This distribution is nonuniform but steplike [Fig. 2(a)], i.e., consists of several plateaus.

An island of interacting electrons, however, is a more complicated object. The shape of LL's is now defined by the self-consistent potential, which is not known *a priori* like the external potential. The spatial distribution of electrons is also different. Indeed, the steplike configuration with sharp edges cannot exist for Coulomb interacting electrons, because of the divergences of the electric field at such edges. In the interacting liquid there are necessarily finite-width transitory regions (compressible liquid) (Refs. 11 and 12) between the neighboring plateaus (incompressible liquid); see Fig. 3(a). As a consequence, the functional dependences of the chemical potential and magnetization on magnetic field are expected to be different from the conventional ones. These differences should be the largest for the parameters of the island that emphasize the role of interaction. In our model we describe our systems by a single parameter b/a_B , which is the ratio of the width of the island to the Bohr radius in the semiconductor.

We consider systems with typical sizes b much larger than the Bohr radius a_B . The estimate of the strength of the interaction is as follows: if the typical electron density is n , then the interaction part of the energy per particle is of the order of $E_{\text{int}} \sim e^2 nb$, which is much larger than the magnetic part $E_{\text{mag}} \sim \nu \hbar \omega_c \sim e^2 n a_B$. Thus the role of interaction is crucial.

Let us call the charge density profile providing the

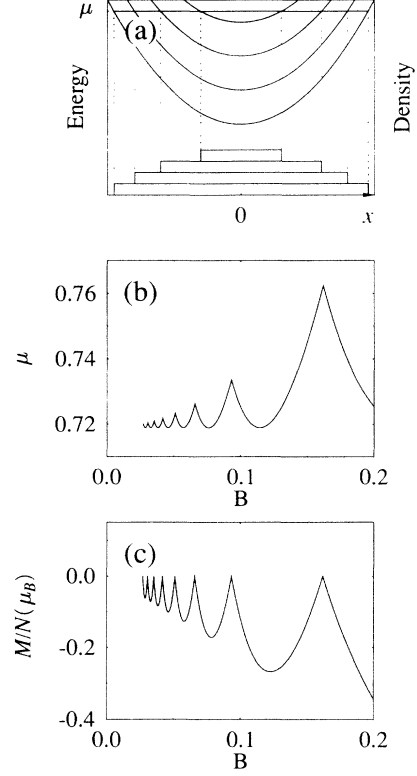


FIG. 2. Structure of the noninteracting 2D electron gas with N electrons confined in a parabolic potential well when a strong magnetic field is applied. Units used in the plots are as follows. The unit for magnetic field B is $\Phi_0 n_0$ and the unit for the chemical potential μ is $e^2 n_0 a_B$. Here Φ_0 is a flux quantum and n_0 is such that $(2\pi/3)n_0 b^2 = N$. The parameter b/a_B is taken to be 0.05. This parameter is related to the oscillator frequency of the potential well by means of Eq. (29). (a) Local energy of Landau levels and the density profile in a cross section through the center of the dot. (b) Chemical potential and (c) magnetization as functions of the magnetic field.

complete electrostatic screening of the external potential throughout the island “electrostatic.” The spatial distribution of the electron density [Fig. 3(a), shaded area] in the magnetic field may deviate only slightly from this smooth “electrostatic” profile [Fig. 3(a), dashed line] because of a considerable amount of work against electrostatic forces needed for this deviation. Hence, for wide islands the incompressible areas, where there is no screening, are small.¹³ Moreover, the plateaus of electron density exist only at moderate magnetic fields. Indeed, these plateaus correspond to integer values of a local filling factor. Therefore, they are absent in very large magnetic fields where the filling does not exceed unity, i.e., the actual density profile conforms to the electrostatic one. In the other limit of zero magnetic field the density profile is also smooth and corresponds to the screening of the external potential with finite screening radius (equal to a_B). Again, if $b \gg a_B$, then deviations from the electrostatic profile are small.

In this paper the actual thermodynamic quantities of

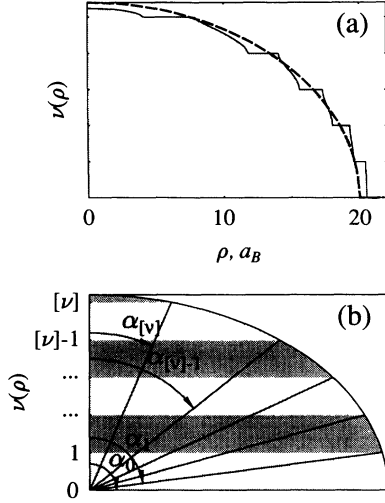


FIG. 3. Structure of a quantum dot in a strong magnetic field. (a) Electron density profile with five Landau levels occupied. The dashed line is the profile of the electrostatic solution. (b) Splitting of the density profile into spherical segments corresponding to different Landau levels used for calculation of the magnetic energy of the dot. The α_k 's are apex angles of the segments.

the interacting system are calculated first in the approximation that the electron liquid has the electrostatic density profile throughout the range of magnetic field of interest. Next we try to account for incompressible areas and consider the effect of the widest one, which is formed in the center of the island when the local value of the filling factor there is close to an integer number (see Ref. 13). To verify the results we have performed a numerical simulation discussed in Sec. II. Comparison with experiments on real Coulomb islands and quantum dots is given in Section IV.

II. QUANTUM DOT

Ignoring small incompressible regions [Fig. 3(a)], we accept, in the first approximation, that the electron density profile remains unchanged in the whole range of magnetic fields and coincides with the one providing perfect screening of the external confining potential. For a smooth confining potential this profile has a domelike shape,¹³ and in a quantum dot's case it is a hemisphere (see below).

Our goal is to calculate the chemical potential and the magnetization as functions of magnetic field, which is accomplished by taking the derivatives of the energy of the system with respect to the number of particles and the field.

The energy of an electron at the k th LL consists of the electrostatic part (energy in the self-consistent potential) and the magnetic part $\hbar\omega_c(k + 1/2)$. Since we assume, in our approximation, that the spatial distribution of electrons does not change with the magnetic field, neither does the electrostatic potential nor the electrostatic part of the energy. The total electrostatic energy

of the dot depends only on the total number of electrons. The derivative of this dependency is equal to the value of the uniform potential created by the electrostatic distribution $e\phi_0$. However, the number of particles at different LL's and, consequently, the magnetic part of the energy changes with the field. To calculate the number of electrons at a particular LL it is convenient to visualize the spatial distribution of electrons in three-dimensional space of n, x, y (electron density and the two spatial coordinates). If n_L is Landau level degeneracy, then the number of electrons in k th LL is proportional to the volume of the slice of the electron liquid density profile $n(x, y)$ cut out by two horizontal planes $n = kn_L$ and $n = (k + 1)n_L$ [see Fig. 3(b)]. This volume and, as a consequence, the variations of physical quantities of the system are determined by the geometrical shape of the electrostatic solution, which we are going to describe now.

We adopt a simplified model of an axially symmetrical confining potential with pure parabolic radial dependence:

$$\phi_{\text{ext}} = C\rho^2. \quad (1)$$

This is a reasonable approximation when distances to the sources of the external potential (gates, surface charges) are larger than the radius of the dot. The electron liquid is confined to a $z = 0$ plane. Above and below the plane there is a semiconductor with a dielectric constant ϵ .

The electric field must be zero at the edge of the electron density to ensure mechanical equilibrium. The problem of the external potential's screening then reduces to the solution of the Laplace equation $\nabla^2\phi(\rho, z) = 0$ with mixed boundary conditions

$$\phi(\rho, 0) = \phi_0 - \phi_{\text{ext}}(\rho), \quad (2)$$

$$\left. \frac{\partial}{\partial z} \phi(\rho, z) \right|_{z \rightarrow \pm 0} = \pm \frac{2\pi e}{\epsilon} n(\rho), \quad (3)$$

$$0 < n(\rho) < +\infty, \quad (4)$$

for $0 < \rho < b$, and

$$n(\rho) = 0, \quad \rho > b. \quad (5)$$

It is convenient to choose C in the form

$$C = \pi^2 en_0 / 4b\epsilon; \quad (6)$$

then the solution may be proven to be a hemisphere (see Ref. 14):

$$n(\rho) = n_0 \sqrt{1 - (\rho/b)^2}, \quad \phi_0 = \pi^2 en_0 b / 2\epsilon. \quad (7)$$

Note that for the external potential given by Eq. (1) the radius of the dot b grows proportionally to the density at the center n_0 . From now on we will avoid explicit mentioning of the dielectric constant ϵ , assuming that the electron charge always enters in the combination e^2/ϵ .

Given the density profile above, we determine the number of electrons at different LL's. With n_L being a degeneracy of both LL spin subbands (twice as large as conventionally defined n_L), we designate by $\nu = n_0/n_L$

the filling factor at the center of the dot. Let us designate by N_k the total number of electrons at LLs $k, k+1, \dots$. Clearly, N_k is the number of electrons occupying areas with filling factor equal or greater than k and is proportional to the volume of a spherical segment of the density profile above the line $n(\rho) = kn_L$ [Fig. 3(b)]. In these notations the total number of electrons N is N_0 . The total magnetic energy may be written as

$$E = \hbar\omega_c \left(\frac{1}{2}N_0 + \sum_{k=1}^{[\nu]} N_k \right). \quad (8)$$

For a hemispherical shape N_k 's are related in a straightforward way to the segments' apex angles α_k 's :

$$\cos \alpha_k = kn_L/n_0, \quad k = 0, 1, \dots, [\nu], \quad \alpha_{[\nu]+1} = 0, \quad (9)$$

where $[\]$ stands for the integer part.

However, bearing in mind a general case of an arbitrary confining potential, we proceed in a slightly different way: we calculate a change in the energy upon adding δN more electrons. If we know exactly how this additional electron liquid arranges itself we will determine changes in N_k 's and substitute them in Eq. (8). It is easy to see that not too close to the edge of the dot the charge is distributed in the same way as on a charged metallic disk. Indeed, the electrostatic potential created by the additional charge density is equal to the change in the electrostatic potential before and after the adding of the new piece of electron density. Both are perfectly flat up to the distance b from the center. Therefore, the electrostatic potential created by the additional charge density is uniform at these distances as for the case of an ideal conductor. The exact density distribution near the edge is unimportant for calculating the magnetic energy because near the edge only the lowest LL is filled. The solution of the problem of the metallic disk is well known:¹⁴

$$\delta n(\rho) = \frac{\delta N}{2\pi b^2} [1 - (\rho/b)^2]^{-1/2}. \quad (10)$$

The change in N_k is

$$\begin{aligned} \delta N_k &= \frac{\delta N}{2\pi b^2} \int_{y_k}^{x_k} 2\pi \rho [1 - (\rho/b)^2]^{-1/2} d\rho \\ &= \delta N \left[\sqrt{1 - (y_k/b)^2} - \sqrt{1 - (x_k/b)^2} \right]. \end{aligned} \quad (11)$$

Here x_k and y_k are the outer and inner radii of the k th slice. If the density distribution is true domelike (without a dip at the center), then all the y_k 's are zeros. Finally, the chemical potential is

$$\begin{aligned} \mu = \frac{\delta E}{\delta N} &= \hbar\omega_c \left\{ \frac{1}{2} + \sum_{k=1}^{[\nu]} \left[\sqrt{1 - (y_k/b)^2} \right. \right. \\ &\quad \left. \left. - \sqrt{1 - (x_k/b)^2} \right] \right\}, \end{aligned} \quad (12)$$

which reduces to

$$\begin{aligned} \mu &= \hbar\omega_c \left\{ \frac{1}{2} + \sum_{k=1}^{[\nu]} \left(1 - \frac{k}{\nu} \right) \right\} \\ &= \hbar\omega_c \left\{ \frac{1}{2} + [\nu] \left(1 - \frac{[\nu] + 1}{2\nu} \right) \right\} \end{aligned} \quad (13)$$

for the parabolic potential.

If the number of electrons fixed, $1/\nu$ is proportional to the magnetic field: $1/\nu = B/n_0\Phi_0$, with $\Phi_0 = \pi\hbar c/e$ being the flux quantum. Then the plot of μ as a function of magnetic field is a set of parabolic arcs standing on top of the line $\mu = \mu_0/2$, where $\mu_0 = \pi e^2 n_0 a_B$ is the chemical potential of a free electron gas with density n_0 ; see Fig. 4(a). Note the difference from the one-particle model [Fig. 2(b)].

We now examine the effects induced by the creation of the incompressible areas. The widest one is formed at the center of the dot [Fig. 5(a)] when the local value of the filling factor ν there is close to an integer number. Following the approach of Ref. 13, the incompressible region may be considered as a region with a redistributed electron density near the top of the hemisphere [Fig. 5(b)]. This redistribution, which is of a quadrupolar character (negative in the middle and positive on the sides), enhances the

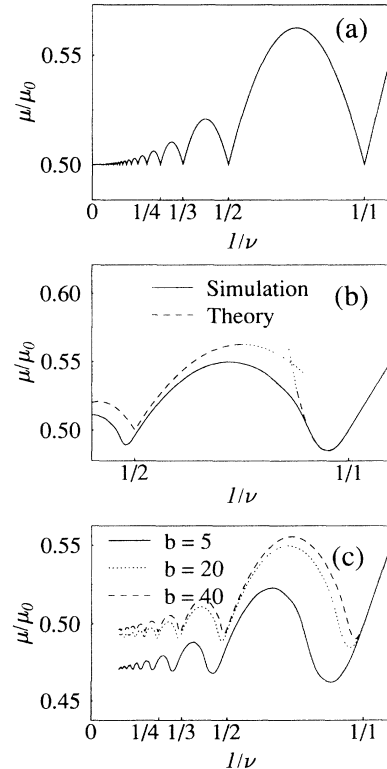


FIG. 4. Chemical potential of a quantum dot as a function of magnetic field. Zero energy at the graphs is $e\phi_0$ and the energy unit for all the graphs is $\mu_0 = \pi e^2 n_0 a_B$. Instead of the magnetic field a proportional quantity $1/\nu$ is used on the x axis. (a) Chemical potential of a quantum dot as given by Eq. (13). (b) Previous picture corrected near integer ν 's and the numerical data for $b = 20a_B$. (c) Numerical data on the chemical potential for $b = 5a_B, 20a_B, 40a_B$.

electrostatic energy of the dot. We essentially repeat the argument of Ref. 13 regarding the two-dimensional (2D) case. We start from an integer filling factor $\nu = k$ and add a small amount of electron density so that $\nu = k + \delta\nu$ and $\delta n_0/n_0 = \delta\nu/\nu$. The redistributed electron density has the form

$$\delta n(\rho) = kn_L - n_0\sqrt{1 - (\rho/b)^2} \approx (kn_L - n_0) - \frac{1}{2}n_0\rho^2/b^2, \quad 0 < \rho < a, \quad (14)$$

where a is the radius of the incompressible area. The quadrupolar region creates a lowering of the electrostatic potential at the center [Fig. 5(c)] of the order of $\delta\phi \sim e\Delta N/a$, where ΔN is the number of electrons in the region where the δn is positive: $\Delta N \sim a^2\delta n_0 \sim a^2\delta n(0) \sim n_0a^4/b^2$. The additional electrostatic energy is

$$\begin{aligned} \delta E_{\text{int}} &= \frac{1}{2} \int_0^a 2\pi\rho\delta n(\rho)e\delta\phi(\rho)d\rho \\ &\sim e\Delta N\delta\phi \sim e^2n_0^2a^7/b^4 \sim e^2n_0^2b^3(\delta n_0/n_0)^{7/2}. \end{aligned} \quad (15)$$

The correction to the chemical potential due to this

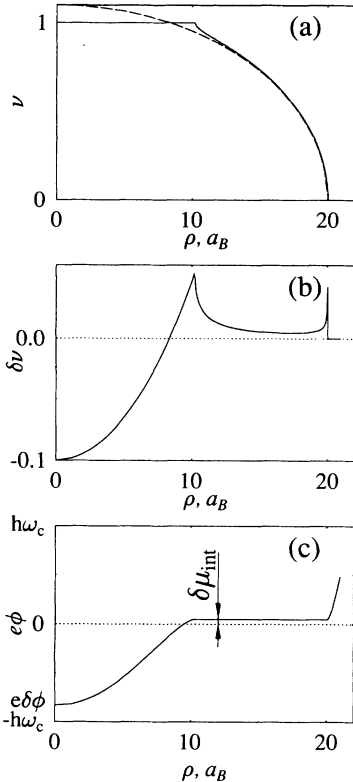


FIG. 5. An incompressible region at the center of the dot. ρ is the radial distance. (a) Electron density profile for $\nu = 1.1$ and $b = 10a_B$ featuring the incompressible region. The dashed line is the ideal electrostatic solution. The deviations in (b) the electron density and (c) the potential from the corresponding quantities of the electrostatic solution. The charge distribution is quadrupolar and creates a potential well. $\delta\mu_{\text{int}}$ represents an increase in the electrochemical potential of the dot due to formation of the incompressible region.

increase in interaction energy $\delta\mu_{\text{int}}$ is a derivative of δE_{int} with respect to $\delta N = \delta(2\pi n_0 b^2/3) = 2\pi b^2\delta n_0$, which is

$$\delta\mu_{\text{int}} = \gamma e^2 n_0 b (\delta\nu/\nu)^{5/2}, \quad (16)$$

where by making use of a method in Ref. 14 the numerical coefficient was found to be $\gamma = 8\sqrt{3}/5$.

On the other hand, the magnetic energy is lower than predicted in our first approximation (of a “frozen” density profile) because none of the electron density is at the topmost LL. This difference is of the order of

$$\delta E_{\text{mag}} = -\hbar\omega_c\Delta N \sim -e^2 n_L a_B n_0 a^4/b^2. \quad (17)$$

The corresponding correction to the chemical potential μ may be proven to be

$$\delta\mu_{\text{mag}} = -\pi e^2 n_0 a_B \delta\nu/\nu^2. \quad (18)$$

The applicability of this argument was verified against a numerical simulation and results are in Fig. 4. For this simulation the dot was divided into a number of narrow concentric rings. The total energy of the dot is taken to be a sum of the energy of Coulomb interaction of the rings, their energy in the external potential, and the magnetic energy as given by Eq. (8). Starting from some initial density distribution over the rings, the algorithm attempts to find the ground state of the system iteratively redistributing small amounts of the electron density among the rings. Eventually the picture of alternative compressible and incompressible regions is reproduced. When the iterations converge all the compressible regions have the same electrochemical potential within the required accuracy. Indeed, although the electrostatic energies of electrons in adjacent compressible regions may differ by $\hbar\omega_c$, their magnetic energies differ by the same amount but with opposite sign as they are at different LL’s. This value of the electrochemical potential is accepted as the chemical potential of the dot. Clearly, when a small amount of electron liquid is added to the system, it is distributed among the compressible regions and the increase in energy per particle is exactly equal to the electrochemical potential of the compressible areas.

Figure 4(c) demonstrates the chemical potential computed numerically for dots of different radii. Comparing Fig. 4(a) with Fig. 4(c) we see that the analytical formula (13) works well for large dots ($b = 20a_B, 40a_B$). Figure 4(b) focuses on the range of filling factors $1 \leq \nu < 2$. The analytical predictions showed by the dashed lines are in agreement with the numerical simulation. The left dashed line is given by Eq. (13); the right dashed line starting from $\nu = 1$ includes the corrections. It may be observed that the effect of the central incompressible region takes place near integer values of ν and consists in changing the convexity of the curves obtained from the “frozen shape” approximation.

We have one comment about the validity interval of Eqs. (16) and (18): the quadrupolar disk breaks into two dipolar areas when the depth of the potential well attains the value of $\hbar\omega_c$ and it is advantageous for electrons to fill its center. The interval of $\delta\nu$ where the quadrupolar area

exists may be found from this condition and the result is similar to that of Ref. 13:

$$\frac{\delta\nu}{\nu} \leq \frac{3}{4} \left(\frac{\pi a_B}{3\nu b} \right)^{2/3}. \quad (19)$$

We anticipate a larger deviation from our theory even with the corrections included when the parameter a_B/b grows. Consider, for instance, the zero field value of the chemical potential. One can see [Fig. 4(c)] that it is less than the predicted value. The discrepancy appears to be linear in a_B/b .

Once the energy of the system is known, it is easy to calculate the magnetization $M = -\partial E/\partial B$ solely in the frozen shape approach as well as its correction:

$$M/N\mu_B = -1 - 2[\nu] + \frac{3[\nu](\nu + 1)}{\nu} - \frac{[\nu]^2(\nu + 1)^2}{\nu^3}, \quad (20)$$

$$\delta M/N\mu_B = -6\delta\mu/\hbar\omega_c. \quad (21)$$

The results for the magnetization and their comparison with the numerical data may be found in Fig. 6, and comments are much the same as on the variations of the chemical potential: our theory well describes wide systems and there is a sharp contrast to the one-particle picture [Figs. 2(b) and 2(c)]. For the noninteracting electrons, the plots consist of concave arcs with cusps

pointing upward. For interacting electrons, the plots are composed of both concave and *convex* arcs, but mainly of the latter. The concave parts are located near integer ν 's; they are small, but they grow as the radius of the dot shrinks.

Our numerical simulations show that for sufficiently small radii of the dot the concave parts oust the convex ones completely and the curves strikingly resemble those in the one-particle limit. The density profile becomes similar to the one for the noninteracting electrons [Fig. 2(a)] as well: it has wide density plateaus and narrow transitory regions. This tendency to recover the results for noninteracting electrons at small values of the parameter b/a_B is expected from our earlier estimate of the relative strength of the interaction and magnetic effects. However, when the magnetic field is strong such that all electrons are at the lowest LL, we still have the electrostatic density profile and the full screening. In this case the electrostatic energy does not depend on the magnetic field, while the magnetic energy is $\hbar\omega_c/2$ per particle. Therefore, in this range of the field, the dependency of the energy and all physical quantities related to it on the magnetic field is universal, e.g., the chemical potential linearly increases with the magnetic field with a universal slope of μ_B and the magnetization is constant and equal to a universal value of $-\mu_B$.

Before we proceed to extend our approach to the case of the quantum wire in the next section, we would like to comment on several approximations made in the model. Throughout the paper we neglect the effects of finite temperature, exchange, and correlations. All of these lead to the finiteness of the screening radius in the compressible regions. For low enough temperatures the screening radius is of the order of magnetic length, i.e., it is much smaller than widths of all compressible and incompressible regions. Hence neither the density profile necessary for calculating the magnetic energy nor the widths of the incompressible regions used for calculating the electrostatic energy corrections are appreciably affected by exchange and correlations (we discuss this also from another point of view in Sec. IV regarding a high-field behavior of the chemical potential).

III. QUANTUM WIRE

We apply the term “quantum wire” to a strongly elongated island isolated from the gate. The requirement of the isolation is essential; otherwise, the chemical potential of the island is defined by the gate voltage and is independent on the magnetic field. For the purpose of finding the electrostatic density profile we treat it as infinite in the y direction. In this approximation the external potential depends only on the x coordinate $e\phi_{\text{ext}}(x, y) = Cx^2$ and we deal with the Laplace equation with boundary conditions similar to Eqs. (2)–(5). The electrochemical potential of the gate is taken to be the energy reference point. Let $C = \pi e^2 n_0/b$. Instead of the hemisphere [Eq. (7)] here we have a semicircle¹⁵

$$n(x) = n_0 \sqrt{1 - (x/b)^2}, \quad (22)$$

$$e\phi_0 = \pi e^2 n_0 b [1 + 2 \ln(4d/b)]/2,$$

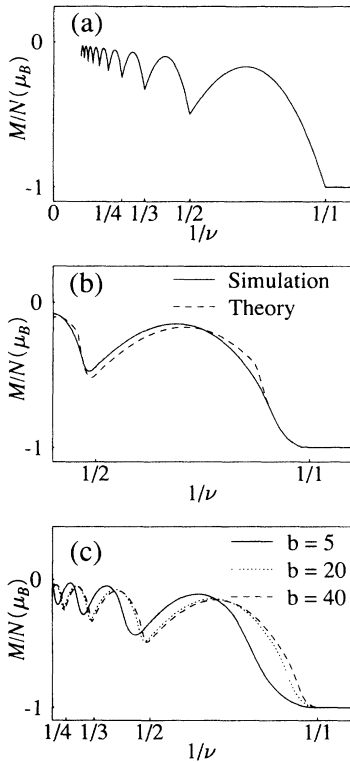


FIG. 6. Magnetization of a quantum dot as a function of magnetic field. Instead of the magnetic field a proportional quantity $1/\nu$ is used on the x axis. (a) Magnetization of a quantum dot as given by Eq. (21). (b) Previous picture corrected near integer ν 's and the numerical data for $b = 20a_B$. (c) Numerical data on the magnetization for $b = 5a_B, 20a_B, 40a_B$.

where $2d \gg b$ is the distance between the gate.

Following the approach of the preceding section for this particular profile, we find that

$$\mu = \frac{2}{\pi} \mu_0 \sum_{k=1}^{[\nu]} \arccos \left(\frac{k}{\nu} \right). \quad (23)$$

The result is graphically represented in Fig. 7(a). It is easy to see that $\mu(\nu = \infty) = 2\mu_0/\pi$. The corrections when an incompressible strip happens to be in the middle of the island are

$$\delta\mu_{\text{int}} = \frac{2}{\pi} \mu_0 (\delta\nu/\nu)^2, \quad (24)$$

$$\delta\mu_{\text{mag}} = -\frac{2\sqrt{2}}{\pi\nu} \mu_0 (\delta\nu/\nu)^{1/2}, \quad (25)$$

$$\frac{\delta\nu}{\nu} \leq \frac{1}{4} \left(\frac{3a_B}{b} \right)^{2/3} \quad (26)$$

and may be understood on the same grounds as in Sec. II. In this case, though, $\Delta N \sim n_0 a^3/b^2$ and $\delta\phi \sim e\Delta N$.

For the magnetization we have

$$M/N\mu_B = -1 - \sum_{k=1}^{[\nu]} \left\{ \arccos \frac{k}{\nu} - \frac{k}{\nu} \left[1 - \left(\frac{k}{\nu} \right)^2 \right]^{1/2} \right\}, \quad (27)$$

$$\delta M/N\mu_B = -4\delta\mu/\hbar\omega_c. \quad (28)$$

The corresponding plots may be found in Fig. 8.

Comparing Fig. 7 with Fig. 4, and Fig. 8 with Fig. 6, one can see that our theory works even better in the quantum wire case than in the case of the quantum dot at the same values of the parameter a_B/b . The graphs depicting the magnetization of the quantum dot and the quantum wire look very similar. As for the chemical potential, in the quantum wire case there is a pronounced drop of the curves towards large fields (small ν 's). It seems plausible that, when the island is neither axially symmetrical nor strongly elongated, but it is something in between these two limits of the quantum wire and the quantum dot cases, one should also get a negative slope. Numerical simulations for nonparabolic potentials show that the behavior of the chemical potential is sensitive to the functional form of the potential as well.

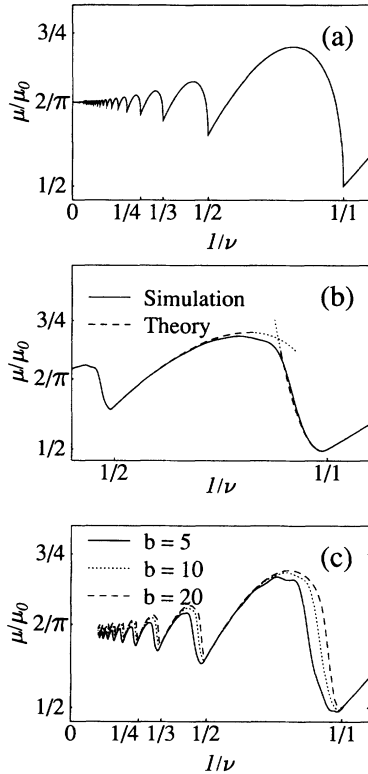


FIG. 7. Chemical potential of a quantum wire as a function of magnetic field. Zero energy at the graphs is $e\phi_0$ and the energy unit for all the graphs is $\mu_0 = \pi e^2 n_0 a_B$. Instead of the magnetic field a proportional quantity $1/\nu$ is used on the x axis. (a) Chemical potential of a quantum wire as given by Eq. (23). (b) Previous picture corrected near integer ν 's and the numerical data for $b = 10a_B$. (c) Numerical data on the chemical potential for $b = 5a_B, 10a_B, 20a_B$.

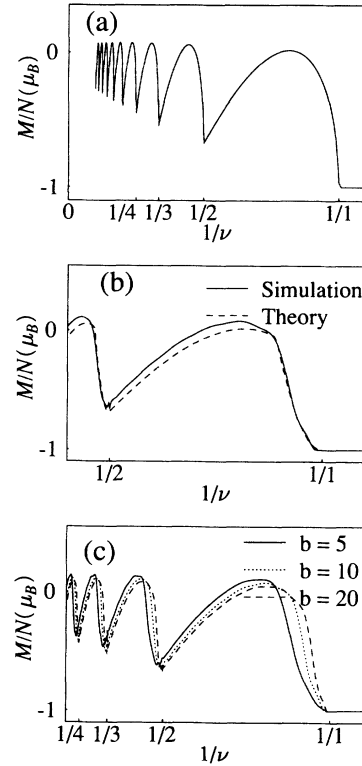


FIG. 8. Magnetization of a quantum wire as a function of magnetic field. Instead of the magnetic field a proportional quantity $1/\nu$ is used on the x axis. (a) Magnetization of a quantum wire as given by Eq. (28). (b) Previous picture corrected near integer ν 's and the numerical data for $b = 10a_B$. (c) Numerical data on the magnetization for $b = 5a_B, 10a_B, 20a_B$.

IV. COMPARISON WITH THE EXPERIMENTAL DATA

We compare our results with experimental data obtained by two different techniques. The first one is transport measurements of a Coulomb island.² One can see in Fig. 2 from their original work that the shape of their curve averaged over small-scale oscillations is reminiscent of the set of parabolas predicted by the frozen shape approximation with an extra uniform tilt, though small. These authors describe their system by the total number of electrons $N = 39$ and the oscillator frequencies in the confining potential $\hbar\omega_x = 3.5$ meV and $\hbar\omega_y = 0.8$ meV with $\hbar\omega_0 = (\hbar\omega_x \hbar\omega_y)^{1/2} = 1.6$ meV. To make the comparison one has to translate these into our choice of parameters by

$$\frac{b}{a_B} = \left[\frac{3\pi}{4} N \left(\frac{e^2/\epsilon a_B}{\hbar\omega_0} \right)^2 \right]^{1/3}. \quad (29)$$

This yields $b/a_B \approx 17.4$ against theirs, ≈ 15 . The amplitude of the variation of the chemical potential, which is about 0.25 meV agrees well with our value ≈ 0.2 meV calculated as

$$\begin{aligned} \Delta\mu &= \pi e^2 n_0 a_B / 16\epsilon = \hbar\omega_c(B_{\max})/12 \\ &= (3N/32)(e^2/\epsilon a_B)/(b/a_B)^2, \end{aligned} \quad (30)$$

where $B_{\max}$ stands for the magnetic field, at which μ is at its maximum.

The authors of Ref. 2 have performed a numerical simulation of their device on the very same footing as we did, but in very restricted range of filling factors (approximately at $1 < \nu < 2$) and have not provided a qualitative interpretation of their data. However, they have taken into account the spin splitting and exchange effects and used an additional constraint: that in each LL they have an integer number of electrons. It is natural to expect that being averaged over small-scale oscillations their data should essentially resemble ours, and this is, indeed, the case.

The largest discrepancy between the experiment and the results of both their and our simulations is the slope of the large field asymptote of the chemical potential. The slope of the experimental curve is approximately four times smaller than the expected value of $\frac{1}{2}\partial(\hbar\omega_c)/\partial B$. Naturally, the question arises if we overlooked the effects of the exchange and correlations. To make a quantitative estimate we can take advantage of the interpolation formula of Fano and Ortolani,¹⁶ who calculated the energy density of the interacting electron liquid as a function of the filling factor in the range $0 < \nu < 1$. We integrated the Fano-Ortolani correction over the area of the dot assuming the electrostatic density profile and found that the absolute value of a negative sign relative correction to the slope of the asymptote does not exceed

$$\frac{0.08}{\sqrt{N}} \frac{b}{a_B}, \quad (31)$$

which is less than 20%. Therefore, exchange and corre-

lations effects are relatively small and do not seem to be the cause of this puzzling discrepancy.

Approximately the same low value for the slope of the large field asymptote can be observed in the other set of experimental data¹ obtained on the basis of capacitance spectroscopy of quantum dots. As for the amplitude of the chemical potential variations, it is surprisingly large and is of the order of 2 meV against ~ 0.3 meV from Eq. (30). Also there is an overall negative slope resembling the quantum wire case, which might be owing to nonparabolicity of the confinement potential. The experimental parameters are $N=30$ and $\hbar\omega_0 = 5.4$ meV, and Eq. (29) seems to give a good estimate for the radius of the dot, which is $7a_B$ vs $9.5a_B$ in the experiment. We do not think that smaller size of the dot in Ref. 1 is responsible for poorer agreement with our predictions, since the parameter b/a_B is large enough in both cases.

V. CONCLUSIONS

In this paper, we have studied how the thermodynamic quantities of a 2D interacting electron liquid confined in a parabolic potential vary with magnetic field. We showed that these variations are defined by the shape of the density profile, which the electron liquid forms. This shape is different for interacting and noninteracting electrons. Noninteracting electrons build up a steplike structure [Fig. 2(a)], while the interacting liquid forms a domelike profile where compressible regions with a smooth varying density dominate [Fig. 3(a)]. The interaction is crucial in wide systems with typical sizes larger than the Bohr radius in a semiconductor and is virtually unimportant for narrow Coulomb islands (unless the field is so strong that filling factor is less than unity).

We suggested an analytical theory explaining the functional form of the variations of the chemical potential and the magnetization of a wide island in two cases: an axially symmetrical island modeling a quantum dot and a strongly elongated island as typical for quantum wires. We verified this analytical theory against the results of the numerical simulations of islands of finite sizes.

The appearances of the curves depicting the magnetization and the chemical potential as functions of the magnetic field are quite distinctive for interacting and noninteracting electrons. In the noninteracting case these are concave arcs with cusps pointing upward, while in the interacting case the curves consist mainly of convex arcs and the cusps point downward. However, the concave parts are also present; they are small for wide islands, but grow when the size of the system decreases. Sufficiently small islands essentially recover the noninteracting limit.

Due to the lack of the experimental data on the magnetization of quantum dots we were able to compare our predictions only regarding the behavior of the chemical potential. The comparison shows both similarity and differences between our theory and the experiment. The most significant discrepancy is the large field asymptote, which is unexpectedly flat in the experiment. However, the proposed theory seems to describe well the shape of the chemical potential's variations.

ACKNOWLEDGMENTS

We are grateful to R. C. Ashoori and D. B. Chklovskii for helpful discussions and to L. I. Glazman and K. A. Matveev for critical reading of the manuscript. One of us (M. F.) is thankful to D. J. Thomson for valuable comments. The work was supported by the NSF under Grant No. DMR 9020587.

-
- ¹ R. C. Ashoori, H. L. Stormer, J. S. Weiner, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, *Phys. Rev. Lett.* **71**, 613 (1993).
² P. L. McEuen, E. B. Foxman, Jari Kinaret, U. Meirav, M. A. Kastner, N. S. Wingreen, and S. J. Wind, *Phys. Rev. B* **45**, 11 419 (1992).
³ A. Kumar, S. E. Laux, and F. Stern, *Phys. Rev. B* **42**, 5166 (1990).
⁴ P. A. Maksym and T. Chakraborty, *Phys. Rev. Lett.* **65**, 108 (1990).
⁵ A. K. Evans, L. I. Glazman, and B. I. Shklovskii, *Phys. Rev. B* **48**, 11 120 (1993).
⁶ Y. Meier, O. Entin-Wohlmann, and Y. Gefen, *Phys. Rev. B* **42**, 8351 (1990).
⁷ N. W. Ashcroft and N. D. Mermin, *Solid State Physics* (Saunders College, Philadelphia, 1976).
⁸ R.E. Peierls, *Z. Phys.* **81**, 186 (1933).
⁹ R. G. Mani and K. v. Klitzing, *Phys. Rev. B* **46**, 9877 (1992).
¹⁰ M. Büttiker, *Phys. Rev. B* **45**, 11 354 (1992).
¹¹ A. M. Chang, *Solid State Commun.* **74**, 871 (1990).
¹² C. W. J. Beenakker, *Phys. Rev. Lett.* **64**, 216 (1990).
¹³ D. B. Chklovskii, K. A. Matveev, and B. I. Shklovskii, *Phys. Rev. B* **47**, 12 605 (1993); D. B. Chklovskii, B. I. Shklovskii, and L. I. Glazman, *ibid.* **46**, 4026 (1992); **46**, 15 606 (E) (1992).
¹⁴ Ian N. Sneddon, *Mixed Boundary Value Problems in Potential Theory* (John Wiley & Sons, Inc., New York, 1966).
¹⁵ I. A. Larkin and V. B. Shikin, *Phys. Lett. A* **151**, 9877 (1992).
¹⁶ G. Fano and F. Ortolani, *Phys. Rev. B* **37**, 8179 (1988).

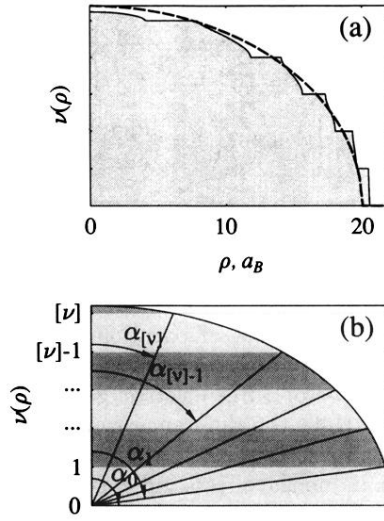


FIG. 3. Structure of a quantum dot in a strong magnetic field. (a) Electron density profile with five Landau levels occupied. The dashed line is the profile of the electrostatic solution. (b) Splitting of the density profile into spherical segments corresponding to different Landau levels used for calculation of the magnetic energy of the dot. The α_k 's are apex angles of the segments.