Title
Fixed points of local actions of nilpotent Lie groups on surfaces

Journal
Ergodic Theory and Dynamical Systems, 37(4)

ISSN
0143-3857

Author
HIRSCH, MORRIS W

Publication Date
2017-06-01

DOI
10.1017/etds.2015.73

Peer reviewed

Fixed points of local action of nilpotent Lie groups on surfaces

Morris W. Hirsch[*]
Mathematics Department
University of Wisconsin at Madison
University of California at Berkeley

February 24, 2015

Abstract

Let G be a connected nilpotent Lie group with a continuous local action on a real surface M, which might be noncompact or have nonempty boundary ∂M. The action need not be smooth. Let φ be the local flow on M induced by the action of some one-parameter subgroup. Assume K is a compact set of fixed points of φ and U is a neighborhood of K containing no other fixed points.

Theorem: If the Dold fixed-point index of $\varphi_t|U$ is nonzero for sufficiently small $t > 0$, then $\mathsf{Fix}(G) \cap K \neq \varnothing$.

Contents

[*]I thank Joseph Plante for helpful advice.

1 Introduction

Poincaré [34] published a seminal result on surface dynamics in 1885, extended to higher dimensions by Hopf [18] in 1925, and generalized to maps in compact polyhedra and topological manifolds by Lefschetz [23] in 1926. We will use the following version:

Theorem (Poincaré-Hopf-Lefschetz)**.** *Every flow on a compact manifold N of nonzero Euler characteristic $\chi(N)$ has a fixed point.*

In his pioneering 1964 paper E. Lima [24] generalized this result to actions of connected abelian Lie groups on compact surfaces. This was extended to nilpotent groups in 1986 by J. Plante [33]:

Theorem (Plante [33])**.** *Every action of a connected nilpotent Lie group on a compact surface of nonzero Euler characteristic has a fixed point.*

Our main result is Theorem 1.1, an extension of Plante's Theorem to local actions of nilpotent Lie groups on arbitrary surfaces.

Throughout this article G denotes a connected Lie group of positive dimension having unit element e_G. The Lie algebra algebra $\mathfrak{g}$ of G is identified with the set of homomorphisms $\mathbb{R} \to G$.

A *local action* (G, M, α) of G on M assigns to each $g \in G$ a local homeomorphism

$$\alpha(g)\colon\ \mathcal{D}\alpha(g) \approx \mathcal{R}\alpha(g)$$

between open subsets of M, satisfying certain conditions (see Section 2). We may abuse notation by omitting "α" and writing g for $\alpha(g)$.

Define the *fixed-point sets*

$$\begin{aligned} \mathsf{Fix}(g)\ &:= \{x \in \mathcal{D}g\colon\ g(x) = x\}, \\ \mathsf{Fix}(G) &:= \bigcap_{g\in G} \mathsf{Fix}(g) \end{aligned}$$

The local action is *effective* if

$$\mathsf{Fix}(g) = \mathcal{D}g \implies g = e_G.$$

For each $X \in \mathfrak{g}$, composing the homomorphism $X\colon\ \mathbb{R} \to G$ with the maps $\alpha(g)$ produces a local action $(\mathbb{R}, M, X^\alpha)$, called a *local flow*, defined by

$$X^\alpha(t)\colon\ p \mapsto \alpha(X(t))(p), \qquad (p \in \mathcal{D}\,\alpha(X(t))).$$

Define

$$\mathsf{Fix}(X) := \bigcap_{t\in\mathbb{R}} \mathsf{Fix}(\alpha(X(t))).$$

The following terms are made precise in Section 3. Assume (G, M, α) is given. A *block* for X (or an X*-block*) is a compact, relatively open set $K \subset \mathsf{Fix}(X)$. An open neighborhood $U \subset M$ of K is *isolating* for X if its closure $\overline{U}$ is compact and $\mathsf{Fix}(X) \cap \overline{U} = K$.

The *index* $\mathsf{i}_K(X)$ is an integer, defined in Section 3 as Dold's fixed-point index of $\alpha_t^X|U$ for any isolating neighborhood U of K and any sufficiently small $t > 0$. This index is completely determined by M, X and K. The X-block K is *essential* if $\mathsf{i}_K(X) \neq 0$.

Theorem 1.1. *Let G be a connected nilpotent Lie group with an effective local action on a surface M, and let $X \in \mathfrak{g}$ be arbitrary. Then $\mathsf{Fix}(G)$ meets every essential X-block.*

This reduces to Plante's Theorem when M is compact, but even in that case it gives further information:

Corollary 1.2. *Let G, M and X be as in Theorem 1.1,*

(i) *If $\Gamma \subset M$ is a compact attractor for X and $\chi(\Gamma) \neq 0$, then $\mathsf{Fix}(G) \cap \Gamma \neq \varnothing$.*

(ii) *If X has ν essential blocks, then $\mathsf{Fix}(G)$ has ν components.*

Discussion

Theorem 1.1 is inspired by a remarkable result of C. Bonatti [6]:

Theorem. *Assume $\dim(M) \leq 4$, $\partial M = \varnothing$, and X, Y are commuting analytic vector fields on M. If K is an essential block for the local flow generated by X, then $\mathsf{Z}(Y) \cap K \neq \varnothing$.*[1]

When $\partial M = \varnothing$ and X is a C^1 vector field on M, our definition of the index of X in K extends Bonatti's, which runs as follows. If $U \subset M$ is an isolating neighborhood of K then $\mathsf{i}_K(X)$ equals the intersection number of the images in the tangent bundle of M of $X|U$ and the trivial vector field on U.

Plante's theorem does not extend to Lie groups that are solvable, or even supersoluble,[2] by Lima [24], Plante [33].

Related work on the dynamics of Lie group actions can be found in the papers [2, 7, 10, 11, 12, 13, 15, 16, 17, 19, 20, 25, 26, 28, 32, 36, 35, 39, 40], listed in the References.

[1] "The demonstration of this result involves a beautiful and quite difficult local study of the set of zeros of X, as an analytic Y-invariant set. Of course, analyticity is an essential tool in this study, and the validity of this type of result in the smooth case remains an open and apparently hard question." —P. Molino [27]

[2] Supersoluble: All eigenvalues in the adjoint representation are real. This implies solvable.

Open questions: Are there smooth commuting vector fields on the compact 3-ball that point inward at the boundary and have no common fixed points?

Does every compact surface support a smooth fixed-point free action by a solvable Lie group?

Is there an example of a connected nilpotent Lie group acting without fixed-point on a compact 4-manifold of nonzero characteristic?

Does Bonatti's Theorem generalize to three or more vector fields, or to manifolds of dimenison five or greater?

Terminology

The empty set is denoted by $\varnothing$. The sets of integers, natural numbers and positive natural numbers are respectively $\mathbb{Z}$, $\mathbb{N} := \{0, 1, 2, \dots\}$ and $\mathbb{N}_+$. The set of real numbers is $\mathbb{R}$.

The closure of a set S in a topological space X is $\overline{S}$, and its frontier is $\mathsf{Fr}(S) := \overline{S} \cap \overline{X \setminus S}$.

Maps are continuous unless the contrary is indicated. The domain of f is $\mathcal{D}f$ and its range is $\mathcal{R}f := f(\mathcal{D}f)$. We may write $f \cdot x$ for $f(x)$, with the tacit assumption that $x \in \mathcal{D}f$.

Every ordered pair (C, C') of oriented circles (Jordan curves) in an oriented surface M has an intersection number $C\#C' \in \mathbb{Z}$. It depends only on the homology classes $[C], [C'] \in H_1(M)$ represented by C and C', and is denoted also by $[C]\,\#\,[C']$.

A connected oriented surface M has finite genus $\mathsf{g}(M) := k \in \mathbb{N}_+$ if k is the largest integer such that M contains k mutually disjoint circles. An equivalent condition is that M contains k, but not $k + 1$, mutually disjoint sets, each of which is the union of two oriented circles having positive intersection number. If no such k exists we set $\mathsf{g}(M) = \infty$. The genus of nonorientable connected surface is defined as the genus of its orientable double covering space. The genus of a disconnected disconnected surface is the sum of the genera of its components.

M has genus 0 iff the intersection pairing $H_1(M) \times H_1(M) \to \mathbb{R}$ is trivial. A compact surface has genus 0 iff it embeds in the unit sphere $\mathbf{S}^2$ or the projective plane $\mathbf{P}^2$.

2 Local actions

If $f\colon A \to B$ denotes a map, its domain is $\mathcal{D}f := A$ and its range is $\mathcal{R}f := f(A)$.

Let g, f denote maps. Regardless of their domains and ranges, the composition $g \circ f$ is defined as the map $g \circ f : x \mapsto g(f(x))$ whose domain, perhaps empty, is $f^{-1}(\mathcal{D}g)$.

The associative law holds for these compositions: The maps $(h \circ g) \circ f$ and $h \circ (g \circ f)$ have the same domain

$$D := \{x \in \mathcal{D}f\colon\ f(x) \in \mathcal{D}g, \quad g(h(x)) \in \mathcal{D}f\},$$

and

$$x \in D \implies (h \circ g)(f(x)) = h((g \circ f)(x)).$$

A *local homeomorphism* f on a topological space Q is a homeomorphism between open subsets of Q. The set of these homeomorphisms is denoted by $\mathsf{LH}(Q)$.

Definition 2.1. A *local action* of G on a manifold M is a triple (G, M, α), where $\alpha\colon\ G \to \mathsf{LH}(M)$ is a function having the following properties:

- The set $\Omega(\alpha) := \{(g, p) \in G \times M\colon\ p \in \mathcal{D}\alpha(g)\}$ is an open neighborhood of $\{e_G\} \times M$.
- The *evaluation map*

  $$\Omega(\alpha) \to M, \qquad (g, p) \mapsto \alpha(g) \cdot p$$

  is continuous.
- $\alpha(e_G)$ is the identity map of M.
- The maps $\alpha(fg) \circ \alpha(h)$ and $\alpha(f) \circ \alpha(gh)$ agree on the intersection of their domains.
- $\alpha(g^{-1}) = \alpha(g)^{-1}$.

Notation of α may be omitted.

The *orbit* of p is the set

$$G \cdot p := \{g \cdot p\colon\ g \in G,\ p \in \mathcal{D}g\}.$$

A set $S \subset M$ is *invariant* if it contains the orbits of all its elements.

If $\Omega(\alpha) = G \times M$ the local action is a *global action*. When G is simply connected and M is compact, every local action extends to a unique global action.

A local action is often specified by a Lie algebra homomorphism $\theta\colon\ \mathfrak{g} \to \mathfrak{v}^\infty(M)$. The trajectories of the local flow X^α are the integral curves of $\theta(X)$. See Palais [31, Th. II.11], Varadarajan [41, Th. 2.16.6].)

The proof of the next result is left to the reader:

Proposition 2.2. *If $H \subset G$ is a normal subgroup,* $\mathsf{Fix}(H)$ *is invariant under G.*

Local flows

Definition 2.3. A *local flow* on a space *mcalX* is a local action $(\mathbb{R}, \mathcal{S}, \varphi)$, sometimes referred to simply as φ. If $(t, p) \in \mathbb{R} \times M$ we may denote $\varphi(t)(p$ by $\varphi_t(p)$ or $t \cdot p$.

If (G, M, α) denotes a local action, each one-parameter subgroup $X \in \mathfrak{g}$ determines a local flow X^α on M defined by

$$t \cdot p := \alpha(X(t)) \cdot p, \qquad (t \in \mathbb{R}, \quad p \in \mathcal{D}\alpha(X(t))).$$

In the rest of this section α denotes a local flow on a surface M.

For each $p \in M$ there is a unique maximal open interval $J_p \subset \mathbb{R}$, containing 0, on which the *trajectory* of p,

$$\alpha^p\colon\ J_p \to M, \quad t \mapsto \alpha_t(t),$$

is defined. The orbit of p is $\alpha^p(J_p)$, usually denoted by $O(p)$. The restrictions of the trajectory of p to the subintervals on which $t \geq 0$ (respectively, $t \leq 0$) are the forward and backward *semitrajectories* of p. Their images are the forward and backward *semiorbits* $O_+(p)$ and $O_-(p)$.

A set is *invariant* if it contains the orbits of its members.

If $p = t \cdot p$ for some $t > 0$ then p and its orbit are *periodic*. If p is not a fixed point it is *cyclic* and its orbit is a *cycle*. The *period* of a cyclic point is the smallest $t > 0$ such that $\alpha^t(p) = p$.

An *orbit arc* $A \subset M$ is an arc of the form $\alpha^p(J)$ for some (nondegenerate) interval $J \subset \mathbb{R}$.

If $x \neq y \in O_+(x)$ we write $x \prec y$. In this case there is a smallest $s > 0$ such that $s \cdot x = y$, and we define orbit arcs

$$[xy] := \{t \cdot x\colon\ t \in [0, s]\},$$
$$(xy] := \{t \cdot x\colon\ t \in (0, s]\},$$

and so forth.

Definition 2.4. A *flowbox* with domain $S \subset M$ and chart h is a homeomorphism

$$h = (h_1, h_2)\colon\ S \approx I \times J \subset \mathbb{R} \times \mathbb{R}$$

such that

- I and J are intervals,
- h_2 is constant on each orbit arc in S,
- the map $h_1\colon\ S \to J$ converts the action of α on orbit into translation:

  If $s > 0$ and $t \cdot p \in S$ for all $t \in [0, s]$, and $q = \alpha_r(p)$, then $h_1(q) = h_1(p) + r$.

Whitney [42] proved that every nonfixed point lies in the interior of a flowbox.

An open arc $T \subset M$ is a *transversal* if there is a flowbox $h\colon\ S \hookrightarrow \mathbb{R}^2$ such that $T \subset S$ and $h_2|T$ is injective.

Minimal sets of local flows

A point p is *recurrent* if there is a sequence $\{t_i\}$ in $\mathbb{R}$ such that

$$\lim_{i\to\infty} t_i \cdot p = p, \quad |t_i| \to \pm\infty.$$

A recurrent point that is not periodic is *exceptional*.

A *minimal set* is a nonempty, compact invariant set $\mathbf{m}$ satisfying the following equivalent conditions:

- No proper subset of $\mathbf{m}$ is compact and invariant.
- Every orbit in $\mathbf{m}$ is dense in $\mathbf{m}$.
- Every point of $\mathbf{m}$ is exceptionally recurrent.

Periodic orbits (including fixed points) are minimal sets; all others are *exceptional minimal sets*.

The proof of the following result is left to the reader:

Lemma 2.5. *Let* $\mathbf{m} \subset M$ *be a minimal set for the local flow* α *on* M.

(a) $\mathbf{m}$ *is connected.*

(b) *Let* $\pi\colon \tilde{M} \to M$ *be a finite-sheeted covering space of* M. *Let* $\tilde{\alpha}$ *be the local flow on* $\tilde{M}$ *such that* $\pi \circ \tilde{\alpha}_t = \alpha_t \circ \pi$. *There is a minimal set* $\tilde{\mathbf{m}}$ *for* $\tilde{\alpha}$ *such that* $\pi(\tilde{\mathbf{m}}) = \mathbf{m}$. ∎

The next two results are adapted from results of Lima [24] and Plante [33].

Theorem 2.6. *If* $H_1(M)$ *has finite rank, every exceptional minimal set has a neighborhood not meeting any cycle.*

Question: Can the assumption of finite rank be dropped?

Proof. We assume M is orientable, otherwise passing to an oriented finite-sheeted covering space. We also assume $\partial M = \varnothing$, because exceptional minimal sets cannot meet ∂M.

If $B \subset M$ is an oriented topological circle, its homology class is denoted by $[B] \in H_1(M)$.

Let $\{C_j\}$ be an infinite sequence of cycles in M and $\{p_j\}$ a sequence such that

$$p_j \in C_j, \quad \lim_{j\to\infty} p_j = p \in M. \tag{1}$$

As $\mathbf{m}$ is exceptional,

$$C_j \cap \mathbf{m} = \varnothing, \quad (j \in \mathbb{N}_+). \tag{2}$$

We will prove $p \notin \mathbf{m}$.

Each C_j is a 1-manifold having a natural orientation, specified by a generator $\omega_j \in H_1(C_j) \cong \mathbb{Z}$. Let $[C_j] \in H_1(M)$ denote the image of ω_j under the homology homomorphism of the inclusion $C_j \hookrightarrow M$. Let $H \subset H_1(M)$ be the free abelian subgroup generated by the homology classes ω_j. Since $H_1(M)$ has finite rank there exists $n \geq 1$ such that

$$H \text{ is generated by } \{[C_1], \dots, [C_n]\}. \tag{3}$$

By (2) there are arbitrarily small precompact flowbox neighborhood S of p satisfying:

$$S \subset M \setminus (C_1 \cup \cdots \cup C_n). \tag{4}$$

Let $T \subset S$ be a transversal through p. Then T is a *cross-section* for the local flow (Whitney [42]): There exists $\delta > 0$ and a homeomorphism

$$F\colon\ (-\delta, \delta) \times T \approx S, \qquad F(s, y) = s \cdot y. \tag{5}$$

As $\overline{O}_+(p)$ is compact, p is recurrent but not periodic, and T is a transversal, it follows that $\overline{O}_+(p) \cap T$ is a Cantor set. It follows that there are points $x, y, q \in V$ such that:

(a) $x, y, q \in O_+(p) \cap T$,

(b) $x \prec y \prec q$,

(c) $(xq) \cap T = \{y\}$.

Therefore Equation (4) implies:

$$[xy] \cap (C_1 \cup \cdots \cup C_n) = \varnothing. \tag{6}$$

Let $L \subset T$ be the compact arc such that $\partial L = \{x, y\}$. Then $q \in L \setminus \partial L$, and the forward semitrajectory of x meets L first at x, next at y, and next at q. The compact arcs $[xy]$ and L meet only at their common boundary $\{x, y\}$, therefore the set

$$\Sigma := [xy] \cup L \tag{7}$$

is a circle containing q.

Note that $C_j \cap [xy] = \varnothing$ because C_j is a periodic orbit, $[xy]$ lies in the nonperiodic orbit $O(p)$, and distinct orbits are disjoint. Therefore

$$C_j \cap \Sigma = C_j \cap L, \qquad (j \in \mathbb{N}_+), \tag{8}$$

hence from Equation (6):

$$C_i \cap \Sigma = \varnothing, \qquad (i = 1, \dots, n). \tag{9}$$

Assume *per contra*: $p \in \mathbf{m}$. Choose $r \geq 0$ such that $q = r \cdot p$. Set $q_j := r \cdot p_j$. By Equation (1):

$$q_j \in C_j, \quad \lim_{j\to\infty} q_j = q. \tag{10}$$

Therefore there exists a smallest $k > n$ such that

$$C_k \cap L \neq \emptyset. \tag{11}$$

Equation (5) shows there are orientations for the surface M such that the oriented curves C_k and L have local intersection number +1 at each intersection point. Consequently Equation (11) implies:

$$[C_k] \# [\Sigma] > 0. \tag{12}$$

Equation (6) shows there are integers a_i such that

$$[C_k] = \sum_{i=1}^{m} a_i [C_i].$$

Therefore

$$\begin{aligned} [C_k] \# [\Sigma] &= \sum_{i=1}^{n} a_i [C_i] \# [\Sigma], \\ &= 0 \ \text{ by Equation (9)}. \end{aligned}$$

This contradicts Equation (12). ∎

Theorem 2.7.

(a) *If M contains $r \geq 1$ exceptional minimal sets,* $\mathsf{g}(M) \geq r$.

(b) *Every exceptional minimal set has a neighborhood containing no other minimal set.*

Proof. (a) By Lima [24, Lemma 4], $\mathsf{g}(M) \geq 2r - 1 \geq 1$.[3] Assertion (a) follows because distinct minimal sets are contained in disjoint surfaces of genus ≥ 1, and the genus function is additive on unions of disjoint surfaces.[4]

(b) An exceptional minimal set **m** has a neighborhood containing no fixed point. By (a) **m** has a neighborhood meeting no other minimal set, and by Theorem 2.6 **m** has a neighborhood meeting no cycle. The intersection of these neighborhoods proves the conclusion. ∎

---

[3] This also follows from Seibert & Tulley [37]: *Every recurrent point for a local flow in an arbitrary subset of $\mathbb{R}^2$ is periodic.* These authors credit this elegant result to an obscure 1936 work by H. Bohr & W. Fenchel [5].

[4] Plante [33, Lemma 2.3] proved for compact M that the number of exceptional minimal sets is bounded by the rank of $H_1(M)$.

3 The fixed-point index

The late A. Dold [8, 9] defined an integer-valued fixed-point index $I(f)$ for a large class of maps f having compact fixed-point sets. We use this to define an index for blocks of fixed points in local flows.

Dold's index is defined for data $f, V, \mathcal{S}$ such that

- V is an open set in a topological space $\mathcal{S}$,
- $f\colon V \to \mathcal{S}$ is continuous and $\mathsf{Fix}(f)$ is compact,
- V is a *Euclidean neighborhood retract* (ENR):

Some open set in a Euclidean space retracts onto a homeomorph of V.[5]

We will use the following properties of $I(f)$:

(D1) $I(f) = I(f|V_0)$ if $V_0 \subset V$ is an open neighborhood of $\mathsf{Fix}(f)$.

(D2) $I(f) = \begin{cases} 0 \text{ if } \mathsf{Fix}(f) = \varnothing, \\ 1 \text{ if } f \text{ is constant.} \end{cases}$

(D3) $I(f) = \sum_i I(f|V_i)$ if V is the union of finitely many disjoint open sets V_i.

(D4) $I(f_0) = I(f_1)$ if there is a homotopy $f_t\colon V \to \mathcal{S}$, $(0 \le t \le 1)$ such that $\bigcup_t \mathsf{Fix}(f_t)$ is compact.

These correspond to (5.5.11), (5.5.12), (5.5.13) and (5.5.15) in Chapter VII of Dold's book [9].

(D5) Assume $\mathcal{S}$ is a manifold, f is C^1, $\mathsf{Fix}(f)$ is a singleton $p \in M \setminus \partial M$, and $\mathsf{Det}\, f'(p) \neq 0$. Then $I(f) = (-1)^\nu$, where ν is the number of distinct eigenvalues λ of $f'(p)$ such that $\lambda > 1$. (See [9, VII.5.17, Ex. 3]).)

(D6) Suppose $\mathcal{S}$ is compact, $V = \mathcal{S}$, and f is homotopic to the identity. Then $I(f) = \chi(\mathcal{S})$. (See [9, VII.6.22].)

Now let $\varphi := \{\varphi_t\}_{t\in\mathbb{R}}$ be a local flow on $\mathcal{S}$.

A compact set $K \subset \mathsf{Fix}(\varphi)$ is a *block* for φ, or a *φ-block*, if it has an open, precompact ENR neighborhood $U \subset \mathcal{S}$ such that $\overline{U} \cap \mathsf{Fix}(\varphi) = K$. Such a U is *isolating for φ*, and for (φ, K).

Proposition 3.1. *If U is isolating for φ, there exists $\tau := \tau(U) > 0$ such that for all $t \in (0, \tau]$:*

(a) $\mathsf{Fix}(\varphi_t) \cap U$ *is compact,*

(b) $I(\varphi_t|U) = I(\varphi_\tau|U)$.

[5] The class of ENRs includes metrizable topological manifolds and their triangulable subsets.

Proof. If (a) fails there exist sequences $\{t_k\}$ in $(0,\infty)$ and $\{p_k\}$ in U such that

$$t_k \to 0, \quad p_k \in \mathsf{Fix}(\varphi_{t_k}), \quad p_k \to q \in \mathsf{Fr}(U).$$

Joint continuity of $(t,x) \mapsto \varphi_t(x)$ implies $q \in \mathsf{Fix}(\varphi) \cap \mathsf{Fr}(U)$, a contradiction because U is isolating for φ. Assertion (b) is a consequence of (a) and (D4). ∎

Proposition 3.2. *If U_1 and U_2 are isolating for (φ, K), there exists $\sigma > 0$ such that for all $t \in (0,\sigma]$:*

$$I(\varphi_t|U_1) = I(\varphi_t|U_2) = I(\varphi_\sigma|U_1 \cap U_2).$$

Proof. Evidently $U_1 \cap U_2$ is isolating for (φ, K). Set

$$\sigma := \min\{\tau(U_1),\ \tau(U_2),\ \tau(U_1 \cap U_2)\} > 0.$$

where $\tau(\cdot)$ is defined as in Proposition 3.1. The conclusion follows from Proposition 3.1 and (D1). ∎

Definition 3.3. Let $K \subset \mathsf{Fix}(\varphi)$ be a φ-block. By Propositions 3.1, 3.2 there is a unique integer $\mu(\phi, K, U)$ having the following property:

$$\textit{If } U \textit{ is isolating for } (\varphi, K) \textit{ and } 0 < t \le \tau(U), \textit{ then } I(\varphi_t|U) = \mu(\phi, K, U). \quad (13)$$

The *index* of φ in U, and at K, is defined as

$$\mathsf{i}(\varphi, U) = \mathsf{i}_K(\varphi) := \mu(\phi, K, U).$$

We call K *essential* for φ provided $\mathsf{i}_K(\varphi) \neq 0$. This implies $K \neq \varnothing$, by (D2).

Now let (G, M, α) be a local action of the Lie group G on a surface M. Every $X \in \mathfrak{g}$ generates a local flow X^α on M, often denoted simply by X. A block K of fixed points for X is called an X-block. Using Equation 13, define the index of X at K, and in U, as

$$\mathsf{i}_K(X) = \mathsf{i}(X, U) := \mathsf{i}(X^\alpha, U).$$

If $\mathsf{i}_K(X) \neq 0$, K is *essential* for X.

Proposition 3.4. *Let $U \subset M$ be isolating for X.*

(a) *The set*

$$\mathfrak{N}(X, U, \mathfrak{g}) := \{Y \in \mathfrak{g} \colon\ U \textit{ is isolating for } Y \textit{ and } \mathsf{i}(Y, U) = \mathsf{i}(X, U)\}$$

is an open neighborhood of X in $\mathfrak{g}$.

(b) *If $\overline{U}$ is a compact invariant manifold then*

$$Y \in \mathfrak{N}(X, U, \mathfrak{g}) \implies \mathsf{i}(Y, U) = \chi(\overline{U}).$$

Proof. *(a)* Compactness of $\mathsf{Fr}(U)$ implies that the set

$$\mathfrak{U} := \{Y \in \mathfrak{g} \colon\ \mathsf{Fix}(Y) \cap \mathsf{Fr}(U) = \varnothing$$

is an open neighborhood of X, and U is isolating for every $Y \in \mathfrak{U}$. If $Y \in \mathfrak{U}$ is sufficiently close to X and $0 \le s \le 1$, then $Y_s := (1-s)X + sY$ also lies in $\mathfrak{U}$, and therefore $\mathsf{i}(Y, U) = \mathsf{i}(X, U)$ by (D4).

(b) Follows from (D6). ∎

4 Dynamics of nilpotent local action

In this section and the next G denotes a connected, nilpotent nontrivial Lie group, M a surface, and (G, M, α) an effective local action. Connectedness of G implies $\mathsf{Fix}(G) = \mathsf{Fix}(\mathfrak{g})$.

The following result is trivial but useful:

Proposition 4.1. *If $p \in M$ and $\mathfrak{u}, \mathfrak{w} \subset \mathfrak{g}_p$ are linear subspaces such that $\mathfrak{u} + \mathfrak{w} = \mathfrak{g}$, then $p \in \mathsf{Fix}(\mathfrak{g})$.* ∎

The *isotropy group* of $p \in M$ is the closed Lie subgroup $G_p \subset G$ generated by

$$\{g \in G\colon\ g \cdot p = p\}.$$

If $S \subset M$ set $G_S := \bigcap_{p\in S} G_p$. The *stabilizer of p* is the Lie algebra of G_p, equal to the algebra

$$\mathfrak{g}_p := \{X \in \mathfrak{g}\colon\ X_p = 0\},$$

and the stabilizer of S is $\mathfrak{g}_S := \bigcap_{p\in S} \mathfrak{g}_p$, which is the Lie algebra of of G_S.

Lemma 4.2. *If $\mathsf{dim}\,(\mathfrak{g}) \geq 2$, every element of $\mathfrak{g}$ lies in an ideal of codimension one.*

Proof. If $\mathsf{dim}\,(\mathfrak{g}) = 2$ the conclusion is trivial because $\mathfrak{g}$ is abelian. Assume inductively: $\mathsf{dim}\,(\mathfrak{g}) = d \geq 3$ and the lemma holds for Lie algebras of lower dimension. Let $Y \in \mathfrak{g}$ be arbitrary. Fix a 1-dimensional central ideal $\mathfrak{j}$ and a surjective Lie algebra homomorphism

$$\pi\colon\ \mathfrak{g} \to \mathfrak{g}/\mathfrak{j}.$$

By the inductive assumption $\pi(Y)$ belongs to a codimension-one ideal $\mathfrak{f} \subset \mathfrak{g}/\mathfrak{j}$, whence Y belongs the codimension-one ideal $\pi^{-1}(\mathfrak{f}) \subset \mathfrak{g}$. ∎

The set $C(\mathfrak{g})$ of codimension-one ideals has a natural structure as a projective variety in the real projective space $\mathbf{P}^{d-1}$, $d = \mathsf{dim}\,(\mathfrak{g})$, and is given the corresponding metrizable topology.

Proposition 4.3 (Plante [33])**.**

(a) *Every component of $C(\mathfrak{g})$ has positive dimension.*

(b) *Every codimension-one subalgebra is an ideal.*

(c) *Assume $\mathsf{dim}\,(G \cdot p) = 1$. Then $\mathfrak{g}_p$ is an ideal of codimension one, and*

$$\mathfrak{g}_x = \mathfrak{g}_p, \quad (x \in G \cdot p).$$

Moreover:

(d) *There is a one-dimensional subgroup $R \subset G$ such that $G \cdot p = R \cdot p$.*

Proof. (a), (b) and (c) are proved in Plante [33, Lemmas 2.5, 2.6]. The isotropy subgroup G_p of p is normal (Proposition 4.3), hence the action of G on the orbit of p factors through the canonical surjection $\pi\colon\ G \to G/G_p$. As the latter group is connected and one-dimensional, it equals $\pi(R)$ with $R \subset G$ a connected one-dimensional subgroup meeting G_p only at e_G. This implies (d).[6] ∎

[6] While the results in [33] cited in this proof refer to a global action on a compact surface, their proofs apply unchanged to local actions on arbitrary surfaces.

Minimal sets for nilpotent local actions

A *minimal set* $\mathbf{m} \subset M$ for the local surface action (G, M, α) is a nonempty, compact, invariant set containing no smaller such set. Equivalently: $\mathbf{m}$ is compact and the orbit of each of its points is dense in $\mathbf{m}$. Compact orbits are minimal sets; all other minimal sets are *exceptional*. A simple topological condition equivalent to exceptionality is: $\mathbf{m}$ is not a point, a circle, or a component of M.

Proposition 4.4. *Let $\mathbf{m} \subset M$ be a one-dimensional minimal set.*

(a) *There is a connected one-dimensional Lie subgroup $R \subset G$ such that $\mathbf{m}$ is a minimal set for the induced local action of R on M.*

(b) *Every $p \in \mathbf{m}$ has the same isotropy group $G_p = G_{\mathbf{m}}$, a codimension-one closed normal subgroup.*

Proof. *(a)* The action of G on $\mathbf{m}$ factors through the canonical surjection $\pi\colon G \to G/G_{\mathbf{m}}$. As the latter group is connected and one-dimensional, it equals $\pi(R)$ with $R \subset G$ a connected one-dimensional subgroup. Every orbit of G in $\mathbf{m}$ is also an orbit of R, implying the conclusion.

(b) Follows from Proposition 4.3(c). ∎

Theorem 4.5. *Let $M_0 \subset M$ be a compact surface.*

(i) *If M_0 contains $r \geq 1$ exceptional minimal sets, $\mathsf{g}(M_0) \geq r$.*

(ii) *Every exceptional minimal set has a neighborhood $M_0 \subset M$ containing no other minimal set.*

Proof. For local flows this is the same as Theorem 2.7, wherefore the general case follows from Proposition 4.4(b). ∎

Proposition 4.6. *If $M_0 \subset M$ is a compact surface containing no fixed point, the union of the circle orbits in M_0 is compact.*[7]

Proof. Let $\Gamma \subset M_0$ denote the union in question. Evidently $\overline{\Gamma}$ is a compact invariant subset of M_0. We need to show that all orbits in the compact invariant set $\mathsf{Fr}(\Gamma) \subset M_0$ are circles. As these orbits are compact, it suffices to show that their closures. which are connected, compact, one-dimensional invariant sets, are circles.

Each component of the compact invariant set $\overline{\Gamma} \cap \partial M$ is a circle orbit contained in ∂M_0. Therefore it suffices to prove:

$$p \in \mathsf{Fr}(\Gamma) \setminus \partial M \implies p \in \Gamma.$$

The orbit closure $\overline{G \cdot p}$, a compact invariant subset of $\mathsf{Fr}(\Gamma) \subset M_0$, contains a minimal set $\mathbf{m}$ but no fixed point, hence $0 \leq \mathsf{dim}\,(\mathbf{m}) \leq \mathsf{dim}\ \mathsf{Fr}(\Gamma) \leq 1$. Consequently $\mathsf{dim}\,(\mathbf{m}) = 1$, whence $\mathbf{m}$ is a circle orbit by Theorem 4.5(ii). This proves:

[7] Cf. Plante [33, Lemma 2.4].

$$\textit{there is a circle orbit } C \subset \overline{G} \cdot p, \tag{14}$$

Fixing C, we note that Proposition 4.4(a) implies C is a cycle for the local flow X^α induced by a 1-parameter subgroup $X \in \mathfrak{g}$. Therefore C lies in the orbit closure of p under X^α.

Fix an open neighborhood $N \subset M_0$ of C, either an annulus or a Möbius band. Consider the dynamics of X^α in N. Apply the Poincaré-Bendixson Theorem (see Hartman [14],[8] also Ciesielski [7], Markley [25]): There is a relatively open neighborhood $N \subset M_0$, homeomorphic to an annulus or a Möbius band, such that for every $x \in N$ one of the following holds:

(a) *x is cyclic for X^α,*

(b) *x is not cyclic for X^α, and one of the semitrajectories of x under X^α spirals towards its limit set, a cycle in N.*

Moreover:

(c) *If (b) holds, it holds for all points in a neighborhood of x.*

Choose a point $x \in (G \cdot p) \cap A$. Since $\overline{\Gamma}$ is invariant under G, every neighborhood of x meets a cycle for X^α. Therefore (c) shows that (a) holds for x, hence it holds for p. ∎

Proposition 4.7. *Let $M_0 \subset M$ be a compact connected surface with empty boundary, containing no fixed point. There is a compact invariant surface $P \subset M_0$ such that:*

(i) *Each component of P is either an annulus or a Möbius band.*

(ii) *$M_0 \setminus P$ contains at most finitely many minimal sets.*

Proof. Let $\Lambda \subset M_0$ be the union of those circle orbits $C \subset M_0$ that are *nonisolated* in M_0, meaning every neighborhood of C in M_0 contains another circle orbit. Λ is compact by Proposition 4.6. An argument based on Poincaré maps shows that each circle orbit $C \subset \Lambda$ lies in a compact invariant surface $P(C) \subset M_0$ satisfying (ii), whose boundary components are circle orbits.

Compactness of Λ implies the existence of $m \geq 1$ nonisolated circle orbits $C_j \subset M_0$ such that

$$\Lambda \subset P(C_1) \cup \cdots \cup P(C_m).$$

Set $P := \bigcup_i P(C_i)$. Induction on m shows that P is a compact invariant surface satisfying (i). Because M_0 is compact and every nonisolated circle orbit in M_0 lies in P, only finitely many circle orbits lie in $M_0 \setminus P$. Since M_0 contains no fixed points, and the number of exceptional minimal sets on M_0 is finite by Theorem 4.5(i), P satisfies (ii). ∎

[8] Hartman's careful proof is stated for differential equations, but uses only local flows.

5 Proof of Theorem 1.1

Recall the hypotheses: M is a surface, G is a nontrivial, connected nilpotent Lie group with Lie algebra $\mathfrak{g}$, (G, M, α) is an effective local action, and $K \subset M$ is an essential block of fixed points for the induced local flow of a one-parameter subgroup $X \in \mathfrak{g}$. The theorem states that $\mathsf{Fix}(G) \cap K \neq \varnothing$.

If $\dim G = 1$ the conclusion is trivial.

Induction hypothesis: $\dim (G) > 1$ *and the conclusion holds for groups of lower dimension.*

Every neighborhood of K in M contains an isolating neighborhood U for (X, K) with $\overline{U}$ a compact surface. Essentiality of K implies $\mathsf{i}(X, U) \neq 0$. To complete the induction we will prove:

$$\mathsf{Fix}(G) \cap \overline{U} \neq \varnothing. \tag{15}$$

Lemma 5.1. *If U contains only finitely many minimal sets, Equation (15) holds.*

Proof. Since $\dim (\mathfrak{g}) > 1$ and $\mathfrak{g}$ is covered by codimension-one ideals (Lemma 4.2), Proposition 3.4 implies there is an infinite sequence $\{Y_j\}$ in $\mathfrak{g}$ converging to X , and a sequence $\{\mathfrak{h}_j\}$ of pairwise distinct, codimension-one ideals such that:

$$Y_j \in \mathfrak{h}_j, \quad U \textit{ is isolating for } Y_j, \quad \mathsf{i}(Y_j, U) \neq 0.$$

As the set $K_j := \mathsf{Fix}(\mathfrak{h}_j) \cap U$ is compact, nonempty by the induction hypothesis, and invariant by Proposition 2.2, there is a minimal set $L_j \subset K_j \subset U$. The hypothesis of the Lemma implies there exist indices i, j such that $\mathfrak{h}_i \neq \mathfrak{h}_j$ and $L_i = L_j$. Equation (15) now follows from Proposition 4.1. ∎

By Proposition 4.7 there is a nonempty, compact, invariant surface $P \subset \overline{U}$ such that:

$$\chi(P) = 0 \text{ and } \overline{U} \setminus P \text{ contains only finitely many minimal sets.} \tag{16}$$

If $\mathsf{Fix}(G) \cap P \neq \varnothing$ for all choices of U and P, Equation (15) holds by compactness of K. Henceforth we assume:

$$\mathsf{Fix}(G) \cap P = \varnothing. \tag{17}$$

Lemma 5.2. *There exists $Z \in \mathfrak{g}$ such that:*

(a) *U is isolating for Z,*

(b) $\mathsf{i}(Z, U) = \mathsf{i}(X, U) \neq 0$*,*

(c) $\mathsf{Fix}(Z) \cap \partial P = \varnothing$.

Proof. Each of the finitely many components C_i of ∂P is a circle orbit because it contains no fixed point (Equation (17)). Proposition 4.3 shows that its stabilizer is an ideal $\mathfrak{h}_i \subset \mathfrak{g}$. Therefore (c) holds for all Z in the dense open set $\mathfrak{g} \setminus \bigcup \mathfrak{h}_i$, while (a) and (b) hold for Z in the nonempty open set $\mathfrak{N}(X, U, \mathfrak{g}))$ defined in Proposition 3.4). Thus the Lemma is satisfied by all Z in the nonempty set $\mathfrak{N}(X, U, \mathfrak{g}) \setminus \bigcup \mathfrak{h}_i$. ∎

Fix Z as in Lemma 5.2, so that

$$\mathsf{i}(Z, U) \neq 0. \tag{18}$$

Both $U \setminus P$ and $P \setminus \partial P$ are isolating for Z because $\mathsf{Fix}(Z) \cap \partial P = \varnothing$, hence

$$\mathsf{i}(Z, U) = \mathsf{i}(Z, U \setminus P) + \mathsf{i}(Z, P \setminus \partial P) \tag{19}$$

by (D3) in Section 3.

Now $\mathsf{i}(Z, P \setminus \partial P) = \chi(P) = 0$ because $\mathsf{Fix}(Z) \cap \partial P = \varnothing$ (Proposition 3.4(b)). Consequently (18) and (19) imply

$$\mathsf{i}(Z, U \setminus P) \neq 0. \tag{20}$$

Equation (16) shows that $U \setminus P$ contains only finitely many minimal sets. Therefore by Equation (20) and Lemma 5.1,

$$\mathsf{Fix}(G) \cap (U \setminus P) \neq \varnothing.$$

As this implies (15), the proof of Theorem 1.1 is complete. ∎

References

[1] M. Belliart, *Actions sans points fixes sur les surfaces compactes,* Math. Zeit. **225** (1997), 453–465

[2] M. Belliart & I. Liousse, *Actions affines sur les surfaces,* Publications IRMA, Universite de Lille, **38** (1996), exposé X

[3] A. Borel, *Groupes linéaires algebriques*, Annals Math. **64** (1956), 20–80

[4] A. Borel, "Linear algebraic groups," notes taken by Hyman Bass. W. A. Benjamin, Inc., New York-Amsterdam, 1969

[5] H. Bohr & W. Fenchel, *Ein Satz über stabile Bewgungen in der Ebene,* Det Kongelige Danske Videnskabernes Selskab, Mathematisk-Fysiske Meddelelser no. 1, 14 (1936). In: "Harald Bohr, Collected Mathematical Works, Vol. II," Danish Math. Soc., Univ. of Copenhagen (1952)

[6] C. Bonatti, *Champs de vecteurs analytiques commutants, en dimension* 3 *ou* 4*: existence de zéros communs,* Bol. Soc. Brasil. Mat. (N.S.) **22** (1992), 215–247

[7] K. Ciesielski, *The Poincaré-Bendixson theorem: from Poincar to the XXIst century,* Cent. Eur. J. Math. **10** (2012), no. 6, 2110-2128

[8] A. Dold, *Fixed point index and fixed point theorem for Euclidean neighborhood retracts,* Topology **4** (1965), 1–8

[9] A. Dold, "Lectures on Algebraic Topology," Die Grundlehren der matematischen Wissenschaften Bd. 52, second edition. Springer-Verlag New York, 1972

[10] D. B. A. Epstein & W. P. Thurston, *Transformation groups and natural bundles*, Proc. London Math. Soc. **38** (1979), 219–236

[11] N. E. Foland, *The Structure of the orbits and their limit sets in continuous flows,* Pacific Journal of Mathematics **13** (1963), 563–570

[12] O. H ájek, *Structure of dynamical systems,* Comment. Math. Univ. Carolinae **6** 1965, 53-72

[13] O. H ájek, "Dynamical systems in the plane." Academic Press, London-New York 1968

[14] P. Hartman, "Ordinary Differential Equations," John Wiley & Sons, New York (1964)

[15] M. W. Hirsch, *Actions of Lie groups and Lie algebras on manifolds,* "A Celebration of the Mathematical Legacy of Raoul Bott," Centre de Recherches Mathématiques, U. de Montréal. Proceedings & Lecture Notes vol. 50 (P. R. Kotiuga, ed.), Amer. Math. Soc. Providence RI (2010), 69-78

[16] M. W. Hirsch, *Smooth actions of Lie groups and Lie algebras on manifolds,* J. Fixed Point Th. App. **10** (2011), 219–232

[17] M. W. Hirsch, *Zero sets of Lie algebras of analytic vector fields on real and complex* 2*-manifolds*, in preparation (2014)

[18] H. Hopf, *Vektorfelder in Mannifaltigkeiten*, Math. Annalen **95** (1925), 340–367

[19] J. Horne, *A locally compact connected group acting on the plane has a closed orbit*, Illinois J. Math. **9** (1965) 644–650

[20] J. Hounie, *Minimal sets of families of vector fields on compact surfaces*, J. Diff. Geom. **16** (1981), 739–744

[21] M. W. Hirsch & A. Weinstein, *Fixed points of analytic actions of supersoluble Lie groups on compact surfaces,* Ergod. Th. Dyn. Sys. **21** (2001), 1783–1787

[22] J. Humphreys, "Linear Algebraic Groups." Springer-Verlag, New York 1975

[23] S. Lefschetz *Intersections and transformations of complexes and manifolds*, Trans. Amer. Math. Soc. **28** (1926), 1–49

[24] E. Lima, *Common singularities of commuting vector fields on* 2*-manifolds*, Comment. Math. Helv. **39** (1964). 97–110

[25] N. Markley, *The Poincar-Bendixson theorem for the Klein bottle,* Trans. Amer. Math. Soc. **135** (1969), 159-165

[26] N. Markley, *On the number of recurrent orbit closures,* Proc. Amer. Math. Soc. **25** (1970) 413-416

[27] P. Molino, Review of Bonatti [6], Math. Reviews MR1179486 (93h:57044) (1993)

[28] P. Molino & F.-J. Turiel, *Une observation sur les actions de* $\mathbf{R}^p$ *sur les variétés compactes de caractéristique non nulle,* Comment. Math. Helv. **61** (1986), 370-375

[29] G. Mostow, *The extensibility of local Lie groups of transformations and groups on surfaces*, Annals of Math. **52** (1950), 606–636.

[30] A. Onishchik & É. Vinberg, "Lie Groups and Algebraic Groups," Springer-Verlag, Berlin 199)

[31] R. S. Palais, "A Global Formulation of the Lie Theory of Transformation Groups." Memoirs Amer. Math. Soc. no. 22. American Mathematical Society, Providence 1957

[32] J. Plante, *Foliations with measure preserving holonomy*, Annals Math. **102** (1975), 327-361.

[33] J. Plante, *Fixed points of Lie group actions on surfaces,* Erg. Th. Dyn. Sys. **6** (1986), 149–161

[34] H. Poincaré, *Sur les courbes définies par une équation différentielle,* J. Math. Pures Appl. **1** (1885), 167–244

[35] A. J. Schwartz, *Generalization of a Poincaré-Bendixson theorem to closed two dimensional manifolds,* Amer. J. Math. **85** (1963), 453-458

[36] R. Sacksteder, *Degeneracy of orbits of* $\mathbb{R}^m$ *acting on a manifold,* Comm. Math. Helv. **41** (1966-67), 1–9

[37] P. Seibert & P. Tulley, *On dynamical systems in the plane,* Arch. Math. (Basel) **18** (1967), 290–292

[38] A. Sommese, *Borel's fixed point theorem for Kaehler manifolds and an application,* Proc. Amer. Math. Soc. **41** (1973), 51–54.

[39] F.-J. Turiel, *An elementary proof of a Lima's theorem for surfaces*, Publ. Mat. **3** (1989), 555–557

[40] F.-J. Turiel, *Analytic actions on compact surfaces and fixed points,* Manuscripta Mathematica **110** (2003) 195–201

[41] V. Varadarajan, "Lie Groups, Lie Algebras, and Their Representations." Springer-Verlag, New York 1976

[42] H. Whitney, *On regular families of curves*, Bull. Amer. Math. Soc. **47** (1941), 145-147