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RATIONAL HOMOTOPY TYPE AND COMPUTABILITY

FEDOR MANIN

Abstract. Given a simplicial pair (X, A), a simplicial complex Y, and a map $f : A \to Y$, does f have an extension to X? We show that for a fixed Y, this question is algorithmically decidable for all X, A, and f if and only if Y has the rational homotopy type of an H-space. As a corollary, many questions related to bundle structures over a finite complex are decidable.

1. Introduction

When can the set of homotopy classes of maps between spaces X and Y be computed? That is, when can this (possibly infinite) set be furnished with a finitely describable and computable structure? A reasonable first requirement is that X and Y should be finite complexes; this ensures that at least the spaces can be represented as computational objects. Moreover, the question of whether this set has more than one element is undecidable for $X = S^1$, as shown by Novikov as early as 1955[1]. Therefore it is also reasonable to require the fundamental group not to play a role; in the present work, Y is always assumed to be simply connected.[2]

We answer this question with the following choice of quantifiers: for what Y and n can the set of homotopy classes $[X, Y]$ be computed for *every* n-dimensional X? Significant partial results in this direction were obtained by E. H. Brown [Brown] and much more recently by Čadek et al. [Č+14a, Č+14c, Č+14b] and Vokřínek [Vok17]. The goal of the present work is to push their program to its logical limit.

To state the precise result, we need to sketch the notion of an H-space, which is defined precisely in §3. Essentially, an H-space is a space equipped with a binary operation which can be more or less "group-like"; if it has good enough properties, this allows us to equip sets of mapping classes to the H-space with a group structure.

The *cohomological dimension* $\mathrm{cd}(X, A)$ of a simplicial or CW pair (X, A) is the least integer d such that for all $n > d$ and every coefficient group π, $H^n(X, A; \pi) = 0$.

Theorem A. *Let Y be a simply connected simplicial complex of finite type and $d \geq 2$. Then the following are equivalent:*

(i) For any simplicial pair (X, A) of cohomological dimension $d+1$ and simplicial map $f : A \to Y$, the existence of a continuous extension of f to X is decidable.

(ii) Y has the rational homotopy type of an H-space through dimension d. That is, there is a map from Y to an H-space (or, equivalently, to a product of Eilenberg–MacLane spaces) which induces isomorphisms on $\pi_n \otimes \mathbb{Q}$ for $n \leq d$.

Moreover, there is a algorithm which, given a simply connected simplicial complex Y, a simplicial pair (X, A) of finite complexes of cohomological dimension d and a simplicial map $f : A \to Y$,

(1) Determines whether the equivalent conditions are satisfied;

[1]This is the triviality problem for group presentations, translated into topological language. This work was extended by Adian and others to show that many other properties of nonabelian group presentations are likewise undecidable.

[2]The results can plausibly be extended to nilpotent spaces.

(2) If they are, outputs the set of homotopy classes rel A of extensions $[X,Y]^f$ in the format of a (perhaps empty) set on which a finitely generated abelian group acts virtually freely and faithfully (that is, with a finite number of orbits each of which has finite stabilizer).

We give a couple remarks about the statement. First of all, it is undecidable whether Y is simply connected; therefore, when given a non-simply connected input, the algorithm cannot detect this and returns nonsense, like previous algorithms of this type.

Secondly, the difference between $d+1$ in the first part of the theorem and d in the second is important: if $\mathrm{cd}(X,A) = d+1$, then we can decide whether $[X,Y]^f$ is nonempty, but there may not be a group with a natural virtually free and faithful action on it. For example, consider $[S^1 \times S^2, S^2]$. This set can be naturally equipped with the structure

$$[S^1 \times S^2, S^2] \cong \bigsqcup_{r \in \mathbb{Z}} \mathbb{Z}/2r\mathbb{Z};$$

as such, it has a surjection from $\mathbb{Z}^2$, but not an action of it.

1.1. **Examples.** The new computability result encompasses several previous results, as well as new important corollaries. Here are some examples of spaces which satisfy condition (ii) of Theorem A:

(a) Any simply connected space with finite homology groups (or, equivalently, finite homotopy groups) in every dimension is rationally equivalent to a point, which is an H-space. The computability of $[X,Y]$ when Y is of this form was already established by Brown [Brown].

(b) Any d-connected space is rationally an H-space through dimension $n = 2d$. Thus we recover the result of Čadek et al. [Č$^+$14c] that $[X,Y]^f$ is computable whenever X is $2d$-dimensional and Y is d-connected. This implies that many "stable" homotopical objects are computable. One example is the group of oriented cobordism classes of n-manifolds, which is isomorphic to the set of maps from S^n to the Thom space of the tautological bundle over $\mathrm{Gr}_n(\mathbb{R}^{2n+1})$.

(c) The sphere S^n for n odd is rationally equivalent to the Eilenberg–MacLane space $K(\mathbb{Z},n)$. Therefore $[X,S^n]^f$ is computable for any finite simplicial pair (X,A) and map $f : A \to S^n$; this is the main result of Vokřínek's paper [Vok17].

(d) Any Lie group or simplicial group Y is an H-space, so if Y is simply connected then $[X,Y]^f$ is computable for any X, A, and f.

(e) Classifying spaces of connected Lie groups also have the rational homotopy type of an H-space [FHT12, Prop. 15.15]. Therefore we have:

    Corollary 1.1. *Let G be a connected Lie group. Then:*
    (i) The set of isomorphism classes of principal G-bundles over a finite complex X is computable.
    (ii) Let (X,A) be a finite CW pair. Then it is decidable whether a given principal G-bundle over A extends over X.

    In particular, given a representation $G \to GL_n(\mathbb{R})$, we can understand the set of vector bundles with a G-structure. This includes real oriented, complex, and symplectic bundles, as well as spin and metaplectic structures on bundles.

(f) More generally, some classifying spaces of topological monoids have the rational homotopy type of an H-space. This includes the classifying space $BG_n = \mathrm{BAut}(S^n)$ for S^n-fibrations [Mil68, Appendix 1] [Smith]; therefore, the set of fibrations $S^n \to E \to X$ over a finite complex X up to fiberwise homotopy equivalence is computable.

Conversely, most sufficiently complicated simply connected spaces do not satisfy condition (ii). The main result of [Č$^+$14b] shows that the extension problem is undecidable for even-dimensional spheres, which are the simplest example. Other examples include complex projective spaces and most Grassmannians and Stiefel manifolds.

1.2. **Proof ideas.** The proof of the main theorem splits naturally into two pieces. Suppose that Y has the rational homotopy type of an H-space through dimension d, but not through dimension $d+1$. We must show that the extension question is undecidable for pairs of cohomological dimension $d+2$. We must also provide an algorithm which computes $[X,Y]^f$ if $\mathrm{cd}(X,A) \leq d$ and decides whether $[X,Y]^f$ is nonempty if $\mathrm{cd}(X,A) = d+1$. Both of these build on work of Čadek, Krčál, Matoušek, Vokřínek, and Wagner, in [Č+14b] and [Č+14c] respectively.

To show undecidability of the extension problem for a given Y, we reduce a form of Hilbert's Tenth Problem to it. Recall that Hilbert asked for an algorithm to determine whether a system of Diophantine equations has a solution. Work of Davis, Putnam, Robinson, and Matiyasevich showed that no such algorithm exists. It turns out that the problem is still undecidable for very restricted classes of systems of quadratic equations; this was used in [Č+14b] to show that the extension problem for maps to S^{2n} is undecidable. We generalize their work: extension problems to a given Y are shown to encode systems of Diophantine equations in which terms are values on vectors of variables of a fixed bilinear form (or sequence of forms) which depends on Y. We show that Hilbert's Tenth Problem restricted to any such subtype is undecidable.

To provide an algorithm, we use the rational H-space structure of the dth Postnikov stage Y_d of Y. In this case, we can build an H-space H of finite type together with rational equivalences

$$H \to Y_d \to H$$

as well as an "H-space action" of H on Y, that is, a map $\mathrm{act} : H \times Y_d \to Y_d$ which satisfies various compatibility properties. These ensure that the set $[X/A, H]$ (where A is mapped to the basepoint) acts via composition with act on $[X, Y_d]^f$. When $\mathrm{cd}(X,A) \leq d$, the obvious map $[X,Y]^f \to [X,Y_d]^f$ is a bijection; when $\mathrm{cd}(X,A) = d+1$, this map is a surjection. This gives the result.

1.3. **Computational complexity.** Unlike Čadek et al. [Č+14c, ČKV17], whose algorithms are polynomial for fixed d, and like Vokřínek [Vok17], we do not give any kind of complexity bound on the run time of the algorithm which computes $[X,Y]^f$. In fact, there are several steps in which the procedure is to iterate until we find a number that works, with no a priori bound on the size of the number, although it is likely possible to bound it in terms of dimension and other parameters such as the cardinality of the torsion subgroups in the homology of Y. There is much space to both optimize the algorithm and discover bounds on the run time.

1.4. **The fiberwise case.** In a paper of Čadek, Krčál, and Vokřínek [ČKV17], the results of [Č+14c] are extended to the *fiberwise* case, that is, to computing the set of homotopy classes of lifting-extensions completing the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ X & \xrightarrow{g} & B, \end{array} \tag{1.2}$$

where X is $2d$-dimensional and the fiber of $Y \xrightarrow{p} B$ is d-connected. Vokřínek [Vok17] also remarks that his results for odd-dimensional spheres extend to the fiberwise case. Is there a corresponding fiberwise generalization for the results of this paper? The naïve hypothesis would be that $[X,Y]^f_p$ is computable whenever the fiber of $Y \xrightarrow{p} B$ is a rational H-space through dimension n. This is false; as demonstrated by the following example, rational homotopy obstructions may still crop up in the interaction between base and fiber.

Example 1.3. Let $B = S^6 \times S^2$ and Y be the total space of the fibration

$$S^7 \to Y \xrightarrow{p_0} B \times (S^3)^2$$

whose Euler class (a.k.a. the k-invariant of the corresponding $K(\mathbb{Z},7)$-bundle) is

$$[S^6 \times S^2] + [(S^3)^2 \times S^2] \in H^8(B \times (S^3)^2).$$

Then the fiber of $p = \pi_1 \circ p_0 : Y \to B$ is the H-space $(S^3)^2 \times S^7$, but the intermediate k-invariant given above has a term which is quadratic in the previous part of the fiber.

Given a system of s polynomial equations each of the form

$$\sum_{1 \le i < j \le r} a_{ij}^{(k)}(x_i y_j - x_j y_i) = b_k,$$

with variables $x_1, \ldots, x_r, y_1, \ldots, y_r$ and coefficients b_k and $a_{ij}^{(k)}$, we form a space X' by taking $\bigvee_r S^3$ and attaching s 6-cells, the kth one via an attaching map whose homotopy class is

$$\sum_{1 \le i < j \le r} a_{ij}^{(k)}[\mathrm{id}_i, \mathrm{id}_j],$$

where id_i is the inclusion map of the ith 3-sphere. We fix a map $f' : X' \to S^6$ which collapses the 3-cells and restricts to a map of degree $-b_k$ on the kth 6-cell. This induces a map $f = f' \times \mathrm{id}$ from $X = X' \times S^2$ to B.

A lift of f to $B \times (S^3)^2$ corresponds to an assignment of the variables x_i and y_i. The existence of a further lift to Y is then equivalent to whether this assignment is a solution to the system of equations above. Since the existence of such a solution is in general undecidable by [Č+14b, Lemma 2.1], so is the existence of a lift of f through p.

The correct fiberwise statement should relate to rational fiberwise H-spaces, as discussed for example in [LS12]. However, some technical difficulties have thus far prevented the author from obtaining such a result.

1.5. **Structure of the paper.** I have tried to make this paper readable to any topologist as well as anyone who is familiar with the work of Čadek et al. Thus §2 and 3 attempt to introduce all the necessary algebraic topology background which is not used in Čadek et al.'s papers: a bit of rational homotopy theory and some results about H-spaces. For the benefit of topologists, I have tried to separate the ideas that go into constructing a structure on mapping class sets from those required to compute this structure. The construction of the group and action in Theorem A is discussed in §4. In §5, we introduce previous results in computational homotopy theory from [Č+14c,ČKV17,FV20], and in §6 we use them to compute the structure we built earlier. Finally, in §7 and 8, we prove the negative direction of Theorem A.

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2. Rational homotopy theory

Rational homotopy theory is a powerful algebraicization of the topology of simply connected topological spaces first introduced by Quillen [Qui69] and Sullivan [Sul77]. The subject is well-developed, and the texts [GM81] and [FHT12] are recommended as a comprehensive reference. This paper requires only a very small portion of the considerable machinery that has been developed, and this short introduction should suffice for the reader who is assumed to be familiar with Postnikov systems and other constructs of basic algebraic topology.

The key topological idea is the construction of rationalized spaces: to any simply connected CW complex X one can functorially (at least up to homotopy) associate a space $X_{(0)}$ whose homology (equivalently, homotopy) groups are $\mathbb{Q}$-vector spaces.[3] There are several ways of constructing such a rationalization, but the most relevant to us is by induction up the Postnikov tower: the rationalization of a point is a point, and then given a Postnikov stage

$$\begin{array}{ccccc} K(\pi_n(X), n) & \longrightarrow & X_n & \longrightarrow & E(\pi_n(X), n+1) \\ & & \downarrow & & \downarrow \\ & & X_{n-1} & \xrightarrow{k_n} & K(\pi_n(X), n+1), \end{array}$$

one replaces it with

$$\begin{array}{ccccc} K(\pi_n(X) \otimes \mathbb{Q}, n) & \longrightarrow & X_{n(0)} & \longrightarrow & E(\pi_n(X) \otimes \mathbb{Q}, n+1) \\ & & \downarrow & & \downarrow \\ & & X_{n-1(0)} & \xrightarrow{k_n \otimes \mathbb{Q}} & K(\pi_n(X) \otimes \mathbb{Q}, n+1). \end{array}$$

This builds $X_{n(0)}$ given $X_{n-1(0)}$, and then $X_{(0)}$ is the homotopy type of the limit of this construction. We say two spaces are *rationally equivalent* if their rationalizations are homotopy equivalent.

The second key fact is that the homotopy category of rationalized spaces of *finite type* (that is, for which all homology groups, or equivalently all homotopy groups, are finite-dimensional vector spaces) is equivalent to several purely algebraic categories. The one most relevant for our purpose is the Sullivan DGA model.

A *differential graded algebra* (DGA) over $\mathbb{Q}$ is a cochain complex of $\mathbb{Q}$-vector spaces equipped with a graded commutative multiplication which satisfies the (graded) Leibniz rule. A familiar example is the algebra of differential forms on a manifold. A key insight of Sullivan was to associate to every space X of finite type a *minimal* DGA $\mathcal{M}_X$ constructed by induction on degree as follows:

- $\mathcal{M}_X(1) = \mathbb{Q}$ with zero differential.
- For $n \geq 2$, the algebra structure is given by
$$\mathcal{M}_X(n) = \mathcal{M}_X(n+1) \otimes \Lambda \operatorname{Hom}(\pi_n(X); \mathbb{Q}),$$
where ΛV denotes the free graded commutative algebra generated by V.
- The differential is given on the elements of $\operatorname{Hom}(\pi_n(X); \mathbb{Q})$ (*indecomposables*) by the dual of the nth k-invariant of X,
$$\operatorname{Hom}(\pi_n(X); \mathbb{Q}) \xrightarrow{k_n^*} H^{n+1}(X; \mathbb{Q}),$$
and extends to the rest of the algebra by the Leibniz rule. Although it is only well-defined up to a coboundary, this definition makes sense because one can show by induction that $H^k(\mathcal{M}_X(n-1))$ is naturally isomorphic to $H^k(X_{n-1}; \mathbb{Q})$, independent of the choices made in defining the differential at previous steps.

  Note that from this definition, it follows that for an indecomposable y of degree n, dy is an element of degree $n+1$ which can be written as a polynomial in the indecomposables of degree $< n$. In particular, it has no linear terms.

The DGA $\mathcal{M}_X$ is the functorial image of $X_{(0)}$ under an equivalence of homotopy categories.

Many topological constructions can thus be translated into algebraic ones. This paper will use the following:

- The Eilenberg–MacLane space $K(\pi, n)$ corresponds to the DGA $\Lambda \operatorname{Hom}(\pi, \mathbb{Q})$ with generators concentrated in dimension n and zero differential.
- Product of spaces corresponds to tensor product of DGAs. In particular:

[3] It's worth pointing out that this fits into a larger family of *localizations* of spaces, another of which is used in the proof of Lemma 3.3.

Proposition 2.1. *The following are equivalent for a space X:*
(a) X is rationally equivalent to a product of Eilenberg–MacLane spaces.
(b) The minimal model of X has zero differential.
(c) The rational Hurewicz map $\pi_*(X) \otimes \mathbb{Q} \to H_*(X;\mathbb{Q})$ is injective.

Finally, we note the following theorem of Sullivan:

Theorem 2.2 (Sullivan's finiteness theorem [Sul77, Theorem 10.2(i)])**.** *Let X be a finite complex and Y a simply connected finite complex. Then the map $[X, Y] \to [X, Y_{(0)}]$ induced by the rationalization functor is finite-to-one.*

Note that this implies that if the map $Y \to Z$ between finite complexes induces a rational equivalence, then the induced map $[X, Y] \to [X, Z]$ is also finite-to-one.

3. H-spaces

A pointed space (H, o) is an H-space if it is equipped with a binary operation $\mathrm{add} : H \times H \to H$ satisfying $\mathrm{add}(x, o) = \mathrm{add}(o, x) = x$ (the basepoint acts as an identity). In addition, an H-space is *homotopy associative* if

$$\mathrm{add} \circ (\mathrm{add}, \mathrm{id}) \simeq \mathrm{add} \circ (\mathrm{id}, \mathrm{add})$$

and *homotopy commutative* if $\mathrm{add} \simeq \mathrm{add} \circ \tau$, where τ is the "twist" map sending $(x, y) \mapsto (y, x)$. We will interchangeably denote our H-space operations (most of which will be homotopy associative and commutative) by the usual binary operator $+$, as in $x + y = \mathrm{add}(x, y)$.

A classic result of Sugawara [Sta70, Theorem 3.4] is that a homotopy associative H-space which is a connected CW complex automatically admits a *homotopy inverse* $x \mapsto -x$ with the expected property $\mathrm{add}(-x, x) = o = \mathrm{add}(x, -x)$.

Examples of H-spaces include topological groups and Eilenberg–MacLane spaces. If H is simply connected, then it is well-known that it has the rational homotopy type of a product of Eilenberg–MacLane spaces. Equivalently, from the Sullivan point of view, H has a minimal model $\mathcal{M}_H$ with zero differential; see [FHT12, §12(a) Example 3] for a proof. On the other hand, a product of H-spaces is clearly an H-space. Therefore we can add "X is rationally equivalent to an H-space" to the list of equivalent conditions in Prop. 2.1. We will generally use the sloppy phrase "X is a rational H-space" to mean the same thing.

It is easy to see that an H-space operation plays nice with the addition on higher homotopy groups. That is:

Proposition 3.1. *Let (H, o, add) be an H-space. Given $f, g : (S^n, *) \to (H, o)$,*

$$[f] + [g] = [\mathrm{add} \circ (f, g)] \in \pi_n(H, o).$$

Another important and easily verified fact is the following:

Proposition 3.2. *If (H, o, add) is a homotopy associative H-space, then for any pointed space $(X, *)$, the set $[X, H]$ forms a group, with the operation given by $[\varphi] \cdot [\psi] = [\mathrm{add} \circ (\varphi, \psi)]$. If H is homotopy commutative, then this group is likewise commutative.*

Moreover, suppose that H is homotopy commutative, and let $A \to X$ be a cofibration (such as the inclusion of a CW subcomplex), and $f : A \to H$ a map with an extension $\tilde{f} : X \to H$. Then the set $[X, H]^f$ of extensions of f forms an abelian group with operation given by

$$[\varphi] + [\psi] = [\varphi + \psi - \tilde{f}].$$

Throughout the paper, we denote the "multiplication by r" map

$$\underbrace{\mathrm{id} + \cdots + \mathrm{id}}_{r \text{ times}} : H \to H$$

by χ_r. The significance of this map is in the following lemmas, which we will repeatedly apply to various obstruction classes:

Lemma 3.3. *Let H be an H-space of finite type, and let $\alpha \in H^n(H)$ be a cohomology class of finite order. Then there is an $r > 0$ such that $\chi_r^*\alpha = 0$.*

In other words, faced with a finite-order obstruction, we can always get rid of it by precomposing with a multiplication map.

Lemma 3.4. *Let H be a simply connected H-space of finite type. Then for every $r > 0$,*

$$\chi_r^*(H^*(H)) \subseteq rH^*(H) + \textit{torsion}.$$

Proof. By Prop. 3.1, χ_r induces multiplication by r on $\pi_n(H)$. Therefore by Prop. 2.1(c), it induces multiplication by r on the indecomposables of the minimal model $\mathcal{M}_H$. Therefore it induces multiplication by some r^k on every class in $H^n(H; \mathbb{Q})$. □

Combining the two lemmas gives us a third:

Lemma 3.5. *Let H be a simply connected H-space of finite type. Then for any $r > 0$ and any $n > 0$, there is an $s > 0$ such that*

$$\chi_s^*(H^n(H)) \subseteq rH^n(H).$$

Proof of Lemma 3.3. I would like to thank Shmuel Weinberger for suggesting this proof.

Let q be the order of α. By Prop. 3.1, for $f : S^k \to H$, $(\chi_q)_*[f] = q[f]$.

Let $H[1/q]$ be the universal cover of the mapping torus of χ_q; this should be thought of as an infinite mapping telescope. By the above, the homotopy groups of $H[1/q]$ are $\mathbb{Z}[1/q]$-modules (the telescope localizes them away from q). This implies, see [Sul05, Thm. 2.1], that the reduced homology and cohomology groups are also $\mathbb{Z}[1/q]$-modules.

Now we would like to show that for some t, $(\chi_q^*)^t\alpha = 0$, so that we can take $r = q^t$. Suppose not, so that $(\chi_q^*)^t\alpha$ is nonzero for every t. Clearly every element in the sequence

$$\alpha, \chi_q^*\alpha, (\chi_q^*)^2\alpha, \ldots$$

has order which divides q; moreover, since there are finitely many such elements, the sequence eventually cycles. Extrapolating this cycle backward gives us a nonzero element of

$$H^n(H[1/q]) = \varprojlim\bigl(\cdots \xrightarrow{\chi_q^*} H^n(H) \xrightarrow{\chi_q^*} H^n(H)\bigr)$$

which likewise has order dividing q. Since the cohomology groups of $H[1/q]$ are $\mathbb{Z}[1/q]$-modules, this is a contradiction. □

Note that this proof does not produce an effective bound on t. This prevents our algorithmic approach from yielding results that are as effective as those of Vokřínek in [Vok17].

We will also require the similar but more involved fact.

Lemma 3.6. *Let H be an H-space of finite type, U a finite complex, and $n > 0$. Let $i_2 : U \to H \times U$ be the obvious inclusion $u \mapsto (*, u)$.*

(i) Suppose that $\alpha \in H^n(H \times U)$ is torsion and $i_2^*\alpha = 0$. Then there is an $r > 0$ such that $(\chi_r, \mathrm{id})^*\alpha = 0$.

(ii) Suppose that H is simply connected and $\alpha \in H^n(H \times U)$ is such that $i_2^*\alpha = 0$. Then for every $r > 0$,

$$(\chi_r, \mathrm{id})^*\alpha \in rH^n(H \times U) + \textit{torsion}.$$

(iii) Suppose that H is simply connected, and consider

$$G = \ker i_2^* \subseteq H^n(H \times U).$$

Then for every $r > 0$ there is an $s > 0$ such that

$$(\chi_s, \mathrm{id})^* G \subseteq r H^n(H \times U).$$

Proof. We use the Künneth formula, which gives a natural short exact sequence

$$0 \to \bigoplus_{k+\ell=n} H^k(H) \otimes H^\ell(U) \to H^n(H \times U) \to \bigoplus_{k+\ell=n+1} \mathrm{Tor}(H^k(H), H^\ell(U)) \to 0.$$

To demonstrate (i), we will first show that there is an r_0 such that $(\chi_{r_0}, \mathrm{id})^*\alpha$ is in the image of $\bigoplus_{k+\ell=n} H^k(H) \otimes H^\ell(U)$. In other words, we show that the projection of $(\chi_{r_0}, \mathrm{id})^*\alpha$ to $\bigoplus_{k+\ell=n+1} \mathrm{Tor}(H^k(H), H^\ell(U))$ is zero. Now, this group is generated by elementary tensors $\eta \otimes \nu$ where $\eta \in H^k(H)$ and $\nu \in H^\ell(U)$ are torsion elements. By Lemma 3.3, for each such elementary tensor, we can pick $r(\eta)$ such that $\chi_{r(\eta)}^* \eta = 0$ and therefore

$$(\chi_{r(\eta)}, \mathrm{id})^*(\eta \otimes \nu) = 0 \in \mathrm{Tor}(H^k(H), H^\ell(U)).$$

We then choose r_0 to be the least common multiple of all the $r(h)$'s.

Now fix a decomposition of each $H^k(H)$ and $H^\ell(U)$ into cyclic factors to write $\chi_{r_0}\alpha$ as a sum of elementary tensors. Since $i_2^*\alpha = 0$, there are no summands of the form $1 \otimes u$; moreover, each summand is itself torsion. For every other elementary tensor $h \otimes u$, we can use Lemma 3.3 (if h is torsion) or Lemma 3.4 (otherwise, since then u is torsion) to find an $s(h, u)$ such that $\chi_{s(h,u)}^* h \otimes u = 0$.

Finally, we can take r to be the product of r_0 with the least common multiple of the $s(h, u)$'s. This completes the proof of (i).

To demonstrate (ii), we only need to apply Lemma 3.4 to $H^k(H)$ for all $0 < k < n$. Finally, (iii) follows from (i) and (ii). □

4. The algebraic structure of $[X, Y]^f$

We start by constructing the desired structure on $[X, Y]^f$ when Y is a rational H-space. From the previous section, such a Y is rationally equivalent to a product of Eilenberg–MacLane spaces. In particular, it is rationally equivalent to $H = \prod_{n=2}^\infty K(\pi_n(Y), n)$, which we give the product H-space structure. We will harness this to prove the following result.

Theorem 4.1. *Suppose that Y is a rational H-space through dimension d, denote by Y_d the dth Postnikov stage of Y, and let $H_d = \prod_{d=2}^\infty K(\pi_n(Y), n)$. Suppose (X, A) is a finite simplicial pair and $f : A \to Y$ a map. Then $[X, Y_d]^f$ admits a virtually free and faithful action by $[X, H_d]^f$ induced by a map $H_d \to Y_d$.*

Before proving this, we see how computing this structure gives the algorithms of Theorem A.

If (X, A) has cohomological dimension $d + 1$, then there is no obstruction to lifting an extension $X \to Y_d$ of f to Y, as the first obstruction lies in $H^{d+2}(X, A; \pi_{d+1}(Y))$. Therefore $[X, Y]^f$ is nonempty if and only if $[X, Y_d]^f$ is nonempty.

If (X, A) has cohomological dimension d, then in addition every such lift is unique: the first obstruction to homotoping two lifts lies in $H^{d+1}(X, A; \pi_{d+1}(Y))$. Therefore $[X, Y]^f \cong [X, Y_d]^f$.

4.1. An H-space action on Y_n.

Denote the nth Postnikov stages of Y and H by Y_n and H_n, respectively, and the H-space zero and multiplication on H_n by o_n and by $+$ or $\mathrm{add}_n : H_n \times H_n \to H_n$. We will inductively construct the following additional data:

(i) Maps $H_n \xrightarrow{u_n} Y_n \xrightarrow{v_n} H_n$ inducing rational equivalences such that $v_n u_n$ is homotopic to the multiplication map χ_{r_n} for some integer r_n.

(ii) A map $\mathrm{act}_n : H_n \times Y_n \to Y_n$ defining an *H-space action*, that is such that $\mathrm{act}_n(o, x) = x$ and the following diagram homotopy commutes:

$$\begin{array}{ccc} H_n \times H_n \times Y_n & \xrightarrow{(\mathrm{add}_n,\mathrm{id})} & H_n \times Y_n \\ \Big\downarrow{\scriptstyle (\mathrm{id},\mathrm{act}_n)} & & \Big\downarrow{\scriptstyle \mathrm{act}_n} \\ H_n \times Y_n & \xrightarrow{\mathrm{act}_n} & Y_n, \end{array} \tag{4.2}$$

which is "induced by u_n" in the sense of the homotopy commutativity of

$$\begin{array}{ccccc} H_n \times H_n & \xrightarrow{(\mathrm{id},u_n)} & H_n \times Y_n & \xrightarrow{(\chi_{r_n},v_n)} & H_n \times H_n \\ \Big\downarrow{\scriptstyle \mathrm{add}_n} & & \Big\downarrow{\scriptstyle \mathrm{act}_n} & & \Big\downarrow{\scriptstyle \mathrm{add}_n} \\ H_n & \xrightarrow{u_n} & Y_n & \xrightarrow{v_n} & H_n. \end{array} \tag{4.3}$$

Note that when we pass to rationalizations, the existence of such a structure is obvious: one takes $u_{n(0)}$ to be the identity, $\mathrm{act}_{n(0)} = \mathrm{add}_{n(0)}$, and $v_{n(0)}$ to be multiplication by r_n.

4.2. **The action of $[X/A, H_d]$ on $[X, Y_d]^f$.** Now suppose that we have constructed the above structure. Then add_d induces the structure of a finitely generated abelian group on the set $[X/A, H_d]$, which we identify with the set of homotopy classes of maps $X \to H_d$ sending A to $o \in H_d$. Moreover, this group acts on $[X, Y_d]^f$ via the action $[\varphi] \cdot [\psi] = [\mathrm{act}_d \circ (\varphi, \psi)]$.

It remains to show that this action is virtually free and faithful. Indeed, notice that pushing this action forward along v_d gives the action of of $[X/A, H_d]$ on $[X, H_d]^{v_d f}$ via $[\varphi] \cdot [\psi] = r_d[\varphi] + [\psi]$, which is clearly virtually free and faithful. This implies that the action on $[X, Y_d]^f$ is virtually free. Moreover, the map $v_d \circ\ : [X, Y_d]^f \to [X, H_d]^{v_d f}$ is finite-to-one by Sullivan's finiteness theorem. Thus the action on $[X, Y_d]^f$ is also virtually faithful.

4.3. **The Postnikov induction.** Now we construct the H-space action. For $n = 1$ all the spaces are points and all the maps are trivial. So suppose we have constructed the maps u_{n-1}, v_{n-1}, and act_{n-1}, and let $k_n : Y_{n-1} \to K(\pi_n(Y), n+1)$ be the nth k-invariant of Y. For the inductive step, it suffices to prove the following lemma:

Lemma 4.4. *There is an integer $q > 0$ such that we can define u_n to be a lift of $u_{n-1}\chi_q$, and construct v_n and a solution $\mathrm{act}_n : H_n \times Y_n \to Y_n$ to the homotopy lifting-extension problem*

$$\begin{array}{ccccccc} H_n \times H_n & & \xrightarrow{\mathrm{add}_n} & & H_n & \xrightarrow{u_n} & Y_n \\ \Big\downarrow{\scriptstyle (\mathrm{id},u_n)} & & & \overset{\mathrm{act}_n}{\nearrow} & & & \Big\downarrow \\ H_n \times Y_n & \xrightarrow{(\chi_q,\mathrm{id})} & H_n \times Y_n & \twoheadrightarrow & H_{n-1} \times Y_{n-1} & \xrightarrow{\mathrm{act}_{n-1}} & Y_{n-1} \end{array} \tag{4.5}$$

so that the desired conditions are satisfied.

Proof. First, since Y is rationally a product, k_n is of finite order, so by Lemma 3.3 there is some q_0 such that $k_n u_{n-1}\chi_q = 0$, and therefore

$$\begin{array}{ccc} H_n & \xrightarrow{\hat{u}} & Y_n \\ \Big\downarrow & & \Big\downarrow \\ H_{n-1} & \xrightarrow{u_{n-1}\chi_{q_0}} & Y_{n-1}; \end{array}$$

is a pullback square. We will define $u_n = \chi_{q_2 q_1}\hat{u}$, with q_1 and q_2 to be determined and $q = q_2 q_1 q_0$.

Now we construct act_n. We will in fact construct a lifting-extension

$$\begin{array}{ccccccc}
H_n \times H_n & \xrightarrow{(\chi_{q_1},\mathrm{id})} & H_n \times H_n & \xrightarrow{\mathrm{add}_n} & H_n & \xrightarrow{\hat{u}} & Y_n \\
\Big\downarrow{\scriptstyle (\mathrm{id},\hat{u})} & & & \overset{\widehat{\mathrm{act}}}{\nearrow} & & & \Big\downarrow \\
H_n \times Y_n & \xrightarrow[(\chi_{q_1 q_0},\mathrm{id})]{} & H_n \times Y_n & \twoheadrightarrow & H_{n-1} \times Y_{n-1} & \xrightarrow{\mathrm{act}_{n-1}} & Y_{n-1}.
\end{array}$$

It is easy to see that then for any $q_2 > 0$, $\mathrm{act}_n = \widehat{\mathrm{act}} \circ (\chi_{q_2}, \mathrm{id})$ satisfies (4.5).

Note that the outer rectangle commutes since we know (4.3) holds in degree $n-1$. Moreover, the obstruction $\mathcal{O} \in H^{n+1}(H_n \times Y_n, H_n \times H_n; \pi_n(Y))$ to finding the lifting-extension is of finite order since $(\mathrm{id}, u_n) : H_n \times H_n \to H_n \times Y_n$ is a rational equivalence. We will show that when q_1 is large enough, this obstruction is zero.

The obstruction group fits into the exact sequence

$$\cdots \to H^n(H_n \times H_n; \pi_n(Y)) \xrightarrow{\delta} H^{n+1}(H_n \times Y_n, H_n \times H_n; \pi_n(Y)) \xrightarrow{\mathrm{rel}^*} H^{n+1}(H_n \times Y_n; \pi_n(Y)) \to \cdots,$$

and so the image $\mathrm{rel}^* \mathcal{O}$ in $H^{n+1}(H_n \times Y_n; \pi_n(Y))$ is torsion. By Lemma 3.6(i), that means that $(\chi_s, \mathrm{id})^*(\mathrm{rel}^* \mathcal{O}) = 0$ for some $s > 0$.

Now we look at the preimage $\alpha \in H^n(H_n \times H_n; \pi_n(Y))$ of $(\chi_s, \mathrm{id})^*\mathcal{O}$. Applying Lemma 3.6(iii), we can find a t such that $\chi_t^*\alpha \in \ker \delta$ and therefore

$$\delta((\chi_t, \mathrm{id})^*\alpha) = (\chi_{st}, \mathrm{id})^*\mathcal{O} = 0.$$

Thus for $q_1 = st$, we can find a map $\widehat{\mathrm{act}}$ completing the diagram.

Now we ensure that (4.2) commutes by picking an appropriate q_2. Note that the diagram

$$\begin{array}{ccc}
H_n \times H_n \times Y_n & \xrightarrow{(\mathrm{add}_n,\mathrm{id})} & H_n \times Y_n \\
\Big\downarrow{\scriptstyle (\mathrm{id},\widehat{\mathrm{act}})} & & \Big\downarrow{\scriptstyle \widehat{\mathrm{act}}} \\
H_n \times Y_n & \xrightarrow{\widehat{\mathrm{act}}} & Y_n
\end{array}$$

commutes up to finite order; namely, the sole obstruction to commutativity is a torsion class in $H^{n+1}(H_n \times H_n \times Y; \pi_n(Y_n))$. Therefore we can again apply Lemma 3.6(i), this time with $H = H_n \times H_n$ and $U = Y_n$, to find a q_2 which makes the obstruction zero. Finally, (4.2) implies that $\mathrm{act}_n|_{\{o\}\times Y_n}$ is homotopic to the identity, so we modify act_n by a homotopy to make it the identity on the nose.

All that remains is to define v_n. But we know that u_n is rationally invertible, and so we can find some v_n such that $v_n u_n$ is multiplication by some r_n. Moreover, for any such v_n, the right square of (4.3) commutes up to finite order. Thus by increasing r_n (that is, replacing v_n by $\chi_{\hat{r}} v_n$ for some $\hat{r} > 0$) we can make it commute up to homotopy. □

5. Building blocks of homotopy-theoretic computation

We now turn to describing the algorithms for performing the computations outlined in the previous two sections. This relies heavily on machinery and results from [Č+14c], [ČKV17], and [FV20] as building blocks. This section is dedicated to explaining these building blocks.

Our spaces are stored as simplicial sets *with effective homology.* Roughly speaking this means a computational black box equipped with:

- Algorithms which output its homology and cohomology in any degree and with respect to any finitely generated coefficient group.

- A way to refer to individual simplices and compute their face and degeneracy operators. This allows us to, for example, represent a function from a finite simplicial complex or simplicial set to a simplicial set with effective homology.

Now we summarize the operations which are known to be computable from previous work.

Theorem 5.1. *(a) Given a finitely generated abelian group π and $n \geq 2$, a model of the Eilenberg–MacLane space $K(\pi, n)$ can be represented as a simplicial set with effective homology and a computable simplicial group operation. Moreover, there are algorithms implementing a chain-level bijection between n-cochains in a finite simplicial complex or simplicial set X with coefficients in π and maps from X to $K(\pi, n)$ [Č+14c, §3.7].*

(b) Given a finite family of simplicial sets with effective homology, there is a way of representing their product as a simplicial set with effective homology [Č+14c, §3.1].

(c) Given a simplicial map $f : X \to Y$ between simplicial sets with effective homology, there is a way of representing the mapping cylinder $M(f)$ as a simplicial set with effective homology. (In [ČKV17] this is remarked to be "very similar to but easier than Prop. 5.11".)

(d) Given a map $p : Y \to B$, we can compute the nth stage of the Moore–Postnikov tower for p, in the form of a sequence of Kan fibrations between simplicial sets with effective homology [ČKV17, Theorem 3.3].

(e) Given a diagram

$$\begin{array}{ccc} A & \longrightarrow & P_n \\ \big\downarrow & \nearrow & \big\downarrow \\ X & \longrightarrow & P_{n-1} \end{array}$$

where $P_n \to P_{n-1}$ is a step in a (Moore–)Postnikov tower as above, there is an algorithm to decide whether a diagonal exists and, if it does, compute one [ČKV17, Prop. 3.7].

(f) Given a fibration $p : Y \to B$ of simply connected simplicial complexes and a map $f : X \to B$, we can compute any finite Moore–Postnikov stage of the pullback of p along f [ČKV17, Addendum 3.4].

(g) Given a diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ X & \xrightarrow{g} & B, \end{array}$$

where A is a subcomplex of a finite complex X and p is a fibration of simply connected complexes of finite type, we can compute whether two maps $u, v : X \to Y$ completing the diagram are homotopic relative to A and over B [FV20, see "Equivariant and Fiberwise Setup"].

(h) Given a diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ X & \xrightarrow{g} & B \end{array}$$

where A is a subcomplex of a finite complex X, Y and B are simply connected, and p has finite homotopy groups, we can compute the (finite and perhaps empty) set $[X, Y]^f_p$ of homotopy classes of maps completing the diagram up to homotopy.

Proof. We prove only the part which is not given a citation in the statement.

Part (h)**.** Let $d = \dim X$. One starts by computing the dth stage of the Moore–Postnikov tower of $p : Y \to B$ using (f). From there, we induct on dimension. At the kth step, we have computed the (finite) set of lifts to the kth stage P_k of the Moore–Postnikov tower. For each such lift, we use (e) to decide whether it lifts to the $(k+1)$st stage, and compute a lift $u : X \to P_{k+1}$ if it does. Then we compute all lifts by computing representatives of each element of $H^{k+1}(X, A; \pi_{k+1}(p))$ and modifying u by each of them. Finally, we use (g) to decide which of the maps we have obtained are duplicates and choose one representative for each homotopy class in $[X, P_{k+1}]^f_p$. We are done after step d since $[X, P_d]^f_p \cong [X, Y]^f_p$. □

6. Computing $[X, Y]^f$

We now explain how to compute the group and action described in §4. We work with a representation of (X, A) as a finite simplicial set and a Postnikov tower for Y, and perform the induction outlined in that section to compute $[X, Y_d]^f$ for a given dimension d. The algorithm verifies that Y is indeed a rational H-space through dimension d; however, it assumes that Y is simply connected and returns nonsense otherwise.

6.1. **Setup.** Let d be such that Y_d is a rational H-space. Since the homotopy groups of Y can be computed, we can use Theorem 5.1(a) and (b) to compute once and for all the space

$$H_d = \prod_{n=2}^{d} K(\pi_n(Y), n),$$

and the binary operation $\mathrm{add}_d : H_d \times H_d \to H_d$ is given by the product of the simplicial group operations on the individual $K(\pi_n(Y), n)$'s. The group of homotopy classes $[X/A, H_d]$ is naturally isomorphic to $\prod_{n=2}^{d} H^n(X, A; \pi_n(Y))$, making this also easy to compute. Finally, given an element of this group expressed as a word in the generators, we can compute a representative map $X \to H_d$, constant on A, by generating the corresponding cochains of each degree on (X, A) and using them to build maps to $K(\pi_n(Y), n)$.

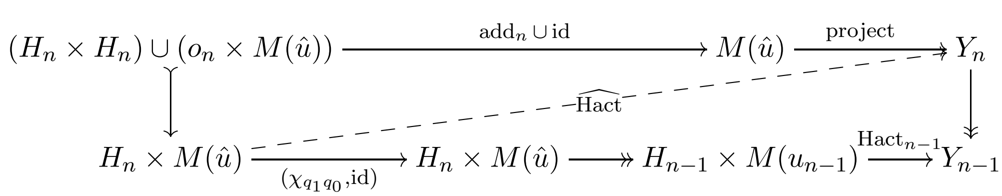
We then initialize the induction which will compute maps u_d, v_d, and act_d and an integer r_d satisfying the conditions of §4. Since $H_1 = Y_1$ is a point, we can set $r_1 = 1$ and u_1, v_1, and act_1 to be the trivial maps.

6.2. **Performing the Postnikov induction.** The induction is performed as outlined in §4.3, although we have to be careful to turn the homotopy lifting and extension problems into genuine ones. Suppose that maps u_{n-1}, v_{n-1}, and act_{n-1} as desired have been constructed, along with a map

$$\mathrm{Hact}_{n-1} : H_n \times M(u_{n-1}) \to Y_{n-1}$$

which restricts to add_{n-1} on $H_{n-1} \times H_{n-1}$ and act_{n-1} on $H_{n-1} \times Y_{n-1}$ (here $M(f)$ refers to the mapping cylinder of f). There are five steps to constructing the maps in the nth step:

1. Find q_0 such that $u_{n-1}\chi_{q_0}$ lifts to a map $\hat{u} : H_n \to Y_n$.
2. Find q_1 such that the diagram



has a lifting-extension along the dotted arrow. Note the modifications to diagram (4.5) which are designed to make it commute on the nose rather than up to homotopy and to make sure that $\widehat{\mathrm{act}}(o, x) = x$. Here $M(f)$ represents the mapping cylinder of the map f.

3. Find q_2 such that $\widehat{\mathrm{Hact}}|_{H_n \times Y_n} \circ (\chi_{q_2}, \mathrm{id})$ makes the diagram (4.2) commute up to homotopy. Now we can set
$$\mathrm{Hact}_n = \widehat{\mathrm{Hact}} \circ (\chi_{q_2}, \mathrm{id}); \qquad \mathrm{act}_n = \mathrm{Hact}_n |_{H_n \times Y_n}; \qquad u_n = \hat{u}\chi_{q_1 q_2}.$$
4. Find q_3 so that the diagram
$$H_{n+1} \rightarrowtail M(u_{n+1}) \dashrightarrow H_{n+1}, \qquad \chi_{q_3} : H_{n+1} \to H_{n+1}$$
can be completed by some $\hat{v}$.
5. Find q_4 so that setting
$$v_{n+1} = \hat{v}\chi_{q_4} \quad \text{and} \quad r_{n+1} = r_k q_0 q_1 q_2 q_3 q_4$$
makes the diagram (4.3) commute.

The first step is done by determining the order of the k-invariant $k_{n+1} \in H^{n+2}(Y_n; \pi_{n+1}(Y))$. If this order is infinite, then Y is not rationally a product of Eilenberg–MacLane spaces, and the algorithm returns failure. Otherwise we compute q_0 by trying various multiples of the order.

The rest of the steps are guaranteed to succeed for some value of q_i, and each of the conditions can be checked using the operations of Theorem 5.1, so this part can be completed by iterating over all possible values until we find one that works.

6.3. **Computing the action.** Let $G = [X/A, H_d]$; we now explain how to compute $[X, Y]^f$ as a set with a virtually free and faithful action by G.

First we must decide whether there is a map $X \to H_d$ extending $v_d f : A \to H_d$. If the set $[X, Y_d]^f$ has an element e, then $v_d f$ has an extension $v_d e$, so if we find that there is no such extension, we return the empty set. Otherwise we compute such an extension ψ_0.

Lemma 6.1. *We can determine whether an extension $\psi_0 : X \to H_d$ of $v_d f$ exists, and compute one if it does.*

Proof. Recall that $H_d = \prod_{n=2}^{d} K(\pi_n(Y), n)$. Write proj_n for the projection to the $K(\pi_n(Y), n)$ factor. Then the extension we desire exists if and only if for each $n < d$, the cohomology class in $H^n(A; \pi_n(Y))$ represented by $\mathrm{proj}_n v_d f$ has a preimage in $H^n(X; \pi_n(Y))$ under the map i^*.

We look for an explicit cocycle $\sigma_n \in C^n(X; \pi_n(Y))$ whose restriction to A is $\mathrm{proj}_n v_d f$. We can compute cycles which generate $H^n(X; \pi_n(Y))$ (because X has effective homology) as well as generators for $\delta C^{n-1}(X; \pi_n(Y))$ (the coboundaries of individual $(n-1)$-simplices in X). Then finding σ_n or showing it does not exist is an integer linear programming problem with the coefficients of these chains as variables.

Now if σ_n exists, then it also determines a map $X \to K(\pi_n(Y), n)$. Taking the product of these maps for all $n \leq d$ gives us our ψ_0. □

We now compute a representative a_N for each coset N of $r_d G \subseteq G$. Since this is a finite-index subgroup of a fully effective abelian group, this can be done algorithmically, for example by trying all words of increasing length in a generating set until a representative of each coset are obtained. For each a_N, we compute a representative map $\varphi_N : X \to H_d$ which is constant on A. Then the finite set
$$S = \{\psi_N = \psi_0 + v_d u_d f_N : N \in G/r_d G\}$$
contains representatives of the cosets of the action of $[X/A, H_d]$ on $[X, H_d]^{v_d f}$ obtained by pushing the action on $[X, Y]^f$ forward along v_d.

Now, for each element of S we apply Theorem 5.1(h) to the square

$$\begin{array}{ccc} A & \xrightarrow{f} & Y_d \\ {\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle v_d} \\ X & \xrightarrow{\psi_N} & H_d \end{array}$$

to compute the finite set of preimages under v_d in $[X, Y_d]^f$. To obtain a set of representatives of each coset for the action of $[X/A, H_d]$ on $[X, Y_d]^f$, we must then eliminate any preimages that are in the same coset. In other words, we must check whether two preimages $\tilde{\psi}$ and $\tilde{\psi}'$ of ψ_N differ by an element of $[X/A, H_d]$; any such element stabilizes $v_d\psi$, and so its order must divide r_d. Since there are finitely many elements whose order divides r_d, we can check for each such element φ in turn whether $[\varphi] \cdot [\tilde{\psi}] \simeq [\tilde{\psi}']$.

Finally, to finish computing $[X, Y_d]^f$ we must compute the finite stabilizer of each coset. This stabilizer is contained in the finite subgroup of $[X/A, H_d]$ of elements whose order divides r_d. Therefore we can again go through all elements of this subgroup and check whether they stabilize our representative.

7. Variants of Hilbert's tenth problem

In [Č+14b], the authors show that the existence of an extension is undecidable by using the undecidability of the existence of solutions to systems of diophantine equations of particular shapes:

Lemma 7.1 (Lemma 2.1 of [Č+14b])**.** *The solvability in the integers of a system of equations of the form*

$$\text{(Q-SYM)} \qquad \sum_{1 \le i < j \le r} a_{ij}^{(q)} x_i x_j = b_q, \qquad q = 1, \ldots, s \quad \textit{or}$$

$$\text{(Q-SKEW)} \qquad \sum_{1 \le i < j \le r} a_{ij}^{(q)} (x_i y_j - x_j y_i) = b_q, \qquad q = 1, \ldots, s$$

for unknowns x_i and (for (Q-SKEW)) y_i, $1 \le i \le r$, is undecidable.

For our purposes, we will need to show the same for systems of one more form, as well as an infinite family generalizing it.

Lemma 7.2. *The solvability in the integers of a system of equations of the form*

$$\text{(Q-DIFF)} \qquad \sum_{i,j=1}^{r} a_{ij}^{(q)} x_i y_j = c_q, \qquad q = 1, \ldots, s$$

for unknowns x_i and y_i, $1 \le i \le r$, is undecidable. More generally, for any (not all zero) family of $m \times n$ matrices $\{B_p\}_{p=1,\ldots,t}$, the solvability in the integers of a system of equations of the form

$$\text{(Q-BLIN}\{B_p\}\text{)} \qquad \sum_{i,j=1}^{r} a_{ij}^{(q)} \vec{u}_i^T B_p \vec{v}_j = c_{pq}, \qquad q = 1, \ldots, s, \qquad p = 1, \ldots, t$$

for unknowns $u_{i1}, \ldots, u_{im}$ and $v_{j1}, \ldots, v_{jn}$, $1 \le j \le r$, is undecidable.

Proof. Systems of the form (Q-DIFF) are a subset of those of the form (Q-SYM). In fact, the proof in [Č+14b] of the undecidability of (Q-SYM) only uses systems of the form (Q-DIFF), and so proves that (Q-DIFF) is undecidable.

To show that (Q-BLIN$\{B_p\}$), for any $\{B_p\}_{p=1,\ldots,t}$ which are not all zero, is undecidable, we show that a system of the form (Q-DIFF) can be simulated with one of the form (Q-BLIN$\{B_p\}$). This proof is closely related to that of the undecidability of (Q-SYM) in [Č+14b].

First, suppose that $r = 1$, so we just have one matrix B. We first show that we replace B wth an invertible square matrix.

Lemma 7.3. *Given an $m \times n$ matrix B, there is a square invertible matrix B' such that for every choice of $\{a_{ij}\}$ and c_q, the system*

$$\sum_{i,j=1}^{r} a_{ij} \vec{u}_i^T B \vec{v}_j = c_q, \qquad q = 1, \ldots, s$$

has a solution if and only if the system

$$\sum_{i,j=1}^{r} a_{ij} (\vec{u}_i')^T B' \vec{v}_j' = c_q, \qquad q = 1, \ldots, s$$

has a solution.

Proof. The rows of B generate a subgroup of $\mathbb{Z}^n$, and by plugging in different $\vec{u}_i$ we can get any vector in that subgroup. So let B'' be an $t \times n$ matrix whose rows are linearly independent vectors generating that subgroup. Then the set of possible values of $(\vec{u}_i')^T B''$ is the same as the set of possible values of $\vec{u}_i^T B$.

Now the columns of B'' generate a subgroup of full rank in $\mathbb{Z}^t$, and by plugging in different $\vec{v}_j$ we can get any vector in that subgroup. So let B' be a $t \times t$ matrix whose columns are linearly independent vectors generating that subgroup. Then the set of possible values of $B' \vec{v}_j'$ is the same as the set of possible values of $B'' \vec{v}_j$. □

Thus we may assume from the start that $m = n$ and $B = (b_{k\ell})$ is invertible. Moreover, by shuffling indices we may assume that b_{11} is nonzero.

Now consider a general system of the form (Q-DIFF). We use it to build a system of the form (Q-BLIN$\{B\}$) with variables

$$\begin{aligned} &u_{i1}, \ldots, u_{in} \text{ and } v_{j1}, \ldots, v_{jn}, && 1 \le i \le r, \\ &z_{k\ell} \text{ and } w_{k\ell}, && 1 \le k, \ell \le n. \end{aligned}$$

Define $n \times n$ matrices $Z = (z_{k\ell})$ and $W = (w_{k\ell})$. Then the equations of our new system are

$$\text{(7.4)} \qquad \begin{cases} \displaystyle\sum_{i,j=1}^{r} a_{ij}^{(q)} \vec{u}_i^T B \vec{v}_j = b_{11} c_q, & q = 1, \ldots, s, \\ \qquad Z^T B W = B, & \\ (\vec{u}_i^T B W)_\ell = 0, & i = 1, \ldots, r, \quad \ell = 2, \ldots, n, \\ (Z^T B \vec{v}_j)_k = 0, & j = 1, \ldots, r, \quad k = 2, \ldots, n. \end{cases}$$

We show that this has a solution if and only if (Q-DIFF) does. It is easy to see that $\{x_i, y_j\}_{1 \le i,j \le r}$ is a solution to (Q-DIFF) if and only if

$$Z = W = I_n, \qquad \vec{u}_i = x_i \vec{e}_1, \qquad \vec{v}_j = y_j \vec{e}_1,$$

where $\vec{e}_1$ is the basis vector $(1, 0, \ldots, 0)$, is a solution to (7.4). In particular, if (Q-DIFF) has a solution, then so does (7.4). Conversely, suppose that we have a solution for (7.4). Since they are integer matrices and B is invertible, Z and W must both have determinant ± 1. Therefore Z^{-1} and W^{-1} are also integer matrices. Then (7.4) also has the solution

$$\vec{u}_i' = Z^{-1} \vec{u}_i, \qquad \vec{v}_j' = W^{-1} \vec{v}_j, \qquad Z' = W' = I_n,$$

which gives us a corresponding solution for (Q-DIFF).

Now we take on the general case. Write $B_p = (b^{(p)}_{k\ell})$; again by reshuffling indices we can assume that $b^{(1)}_{11} \neq 0$. We again use the variables

$$\begin{array}{lll} u_{i1}, \ldots, u_{im} \text{ and } v_{j1}, \ldots, v_{jn}, & 1 \leq i \leq r, & \\ z_{k\ell} \text{ and } w_{k\ell}, & 1 \leq k \leq m, & 1 \leq \ell \leq n \end{array}$$

and the very similar system of equations

$$(7.5) \qquad \begin{cases} \displaystyle\sum_{i,j=1}^{r} a^{(q)}_{ij} \vec{u}_i^T B_p \vec{v}_j = b^{(p)}_{11} c_q, & q = 1, \ldots, s, \quad p = 1, \ldots, t, \\ \qquad Z^T B_p W = B_p, & p = 1, \ldots, t, \\ (\vec{u}_i^T B_p W)_\ell = 0, & p = 1, \ldots, t, \quad i = 1, \ldots, r, \quad \ell = 2, \ldots, n, \\ (Z^T B_p \vec{v}_j)_k = 0, & p = 1, \ldots, t, \quad j = 1, \ldots, r, \quad k = 2, \ldots, m. \end{cases}$$

Once again, (Q-DIFF) has a solution $\{x_i, y_j\}_{1 \leq i,j \leq r}$ if and only if (7.5) has the solution

$$Z = I_m, \qquad W = I_n, \qquad \vec{u}_i = x_i \vec{e}_1, \qquad \vec{v}_j = y_j \vec{e}_1.$$

Conversely, any solution to (7.5) is also a solution to the subsystem consisting of equations involving B_1; by the argument above this can be turned into a solution for (Q-DIFF). ☐

8. Undecidability of extension problems

Theorem 8.1. *Let Y be a simply connected finite complex which is not a rational H-space. Then the problem of deciding, for a finite simplicial pair (X, A) and a map $\varphi : A \to Y$, whether an extension to X exists is undecidable. Moreover,* $\mathrm{cd}(X, A) = d + 1$*, where d is the smallest degree such that Y_d is not a rational H-space.*

Proof. We reduce from the problem (Q-BLIN$\{B_p\}$), for an appropriate set of matrices B_p. For each instance of this problem, we construct a pair (X, A) and map $f : A \to Y$ such that an extension exists if and the instance has a solution.

Fix a minimal model $\mathcal{M}_Y$ for Y and a basis of generators for the indecomposables V_k in each degree k which is dual to a basis for $\pi_k(Y)$/torsion. Since Y is not a rational H-space, there is some least d such that the differential in the minimal model $\mathcal{M}_Y$ is nontrivial. Recall that for a minimal model, each nonzero term in the differential is at least quadratic. For each of the generators η of V_d, $d\eta$ is a polynomial in the lower-degree generators. Denote by *P-degree* the degree of an element of the minimal model as a polynomial in these generators, as opposed to the degree imposed by the grading. Of all the terms in all these polynomials, we choose one with the smallest P-degree and write it as $C\alpha\beta\mu$, where C is a rational coefficient, α and β are elements of V_{d_1} and V_{d_2}, respectively, and μ is some shorter monomial, perhaps 1.

Some of the $d\eta$ may have other terms of the form $\alpha'\beta'\mu$, for various α' and β'. We write

$$d\eta = P_\eta(\vec{\alpha}, \vec{\beta})\mu + \nu_\eta,$$

with ν_η consisting of all the terms which either have higher P-degree or are not multiples of μ.

We note here the connection, first investigated in [AA78], between the differential in the minimal model and higher-order Whitehead products. Given spheres $S^{n_1}, \ldots, S^{n_t}$, their product can be given a cell structure with one cell for each subset of $\{1, \ldots, t\}$. Define their *fat wedge* $\mathbb{V}_{i=1}^t S^{n_i}$ to be this cell structure without the top face. Let $N = -1 + \sum_{i=1}^t n_i$, and let $\tau : S^N \to \mathbb{V}_{i=1}^t S^{n_i}$ be the attaching map of the missing face. By definition, $\alpha \in \pi_N(Y)$ is contained in the r*th-order Whitehead product* $[\alpha_1, \ldots, \alpha_t]$, where $\alpha_i \in \pi_{n_i}(Y)$, if it has a representative which factors through a map

$$S^N \xrightarrow{\tau} \mathbb{V}_{i=1}^t S^{n_i} \xrightarrow{f_\alpha} Y$$

such that $[f_\alpha|_{S^{n_i}}] = \alpha_i$. Note that there are many potential indeterminacies in how higher-dimensional cells are mapped, so $[\alpha_1, \dots, \alpha_t]$ is a set of homotopy classes rather than a unique class.

Some properties of the Whitehead product set $[\alpha_1, \dots, \alpha_t]$ are easy to deduce. It is nonempty if all the $(t-1)$st-order product sets $[\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_t]$ contain zero. Moreover, higher order Whitehead products are multilinear, in the sense that

$$[c\alpha_1, \dots, \alpha_t] \supseteq c[\alpha_1, \dots, \alpha_t],$$

and the factors commute or anticommute as determined by the grading.

The main theorem of [AA78], Theorem 5.4, gives a formula for the pairing between an indecomposable $\eta \in V_n$ and any element of an rth-order Whitehead product set, assuming that every term of $d\eta$ has P-degree at least r. This formula is somewhat complicated, but is r-linear in the pairings between factors of the terms of $d\eta$ and factors of the Whitehead product. In particular, let $\mu = \gamma_1 \cdots \gamma_t$, and let $e_1, \dots, e_t$ be the generators of $\pi_*(Y)$ dual to the γ_i. Then any element f of the Whitehead product set $[a, b, e_1, \dots, e_t]$, for $a \in \pi_{d_1}(X)$ and $b \in \pi_{d_2}(X)$, satisfies

$$\langle \eta, f \rangle = P'_\eta(\langle \alpha_i, a \rangle, \langle \beta_i, b \rangle) \tag{8.2}$$

where P'_η is an integer bilinear form in the two arguments, and α_i and β_i range over generators of V_{d_1} and V_{d_2}, respectively, which occur in terms of η of the form $\alpha_i\beta_i\mu$ (these are the same set if $d_1 = d_2$.)

In general, Whitehead product sets may be empty. However, since the rational homotopy of Y below n is that of a product of Eilenberg–MacLane spaces, for any $a_1, \dots, a_s$ whose degrees add up to $\leq d+1$, there are integers $p_1, \dots, p_s$ such that $[p_1a_1, \dots, p_sa_s]$ is nonempty, and if the degrees add up to $\leq d$ then there are $p_1, \dots, p_s$ such that $0 \in [p_1a_1, \dots, p_sa_s]$. In particular, we can fix integers $p_1, \dots, p_t$ such that $[p_1e_1, \dots, p_te_t]$ contains zero, as well as integers ρ_1 and ρ_2 such that for any $g_1 \in \pi_{d_1}(Y)$ and $g_2 \in \pi_{d_2}(Y)$, $[\rho_1g_1, \rho_2g_2, p_1e_1, \dots, p_te_t]$ is nonempty.

Let $\eta_1, \dots, \eta_r$ be the generators of V_d, $g_1, \dots, g_m$ a generating set for $\rho_1\pi_{d_1}(Y)/\text{torsion}$, $h_1, \dots, h_n$ a generating set for $\rho_2\pi_{d_2}(Y)/\text{torsion}$, and for $p = 1, \dots, r$, let B_p be the matrix which gives P'_{η_p} in terms of those two bases. Now given a system of the form (Q-BLIN$\{B_p\}$), we will build a $(d+1)$-dimensional pair (X, A) and a map $f : A \to Y$ such that the extension problem has a solution if and only if the system does. We define

$$A = \bigvee_{q=1}^{s} S_q^d \vee \bigvee_{i=1}^{t} S^{n_i},$$

where n_i is the degree of e_i, and let $f : A \to Y$ send

- S^{n_i} to Y via a representative of p_ie_i;
- S_q^d to Y via an element whose pairing with η_p is c_{pq}.

Finally, we build X from $A' = A \vee \bigvee_{i=1}^r S_i^{d_1} \vee \bigvee_{j=1}^r S_j^{d_2}$ as follows:

- Add on cells so that for every i and j, X includes the fat wedge $\mathbb{V}(S_i^{d_1}, S_j^{d_2}, S^{n_1}, \dots, S^{n_t})$, and these fat wedges only intersect in A'. Let $\varphi_{ij} : S^d \to X$ be the attaching map of the missing $(d+1)$-cell for the (i,j)th fat wedge.
- Add on spheres $S_i^{d_1\prime}$ together with the mapping cylinder of a map $S_i^{d_1} \to S_i^{d_1\prime}$ of degree ρ_1, and spheres $S_j^{d_2\prime}$ together with the mapping cylinder of a map $S_j^{d_2} \to S_j^{d_2\prime}$ of degree ρ_2.
- Then, for each q, add a $(d+1)$-cell whose boundary is a representative of $\rho([S_q^d] - \sum_{i,j=1}^r a_{ij}^{(q)}[\varphi_{ij}])$, where ρ is the exponent of the torsion part of $\pi_d(Y)$.

It is easy to see that $H_n(X, A) = 0$ for $n > d$.

We claim that (X, A) and f pose the desired extension problem. Indeed, any extension of f to $\tilde{f} : X \to Y$ sends each $S_i^{d_1}$ to an element of $\rho_1\pi_{d_1}(Y)$ and each $S_j^{d_2}$ to an element of $\rho_2\pi_{d_2}(Y)$, as constrained by the mapping cylinders. Now if we write

$$\tilde{f}_*[S_i^{d_1}] = \text{torsion} + \sum_{k=1}^{m} u_k g_k \qquad \text{and} \qquad \tilde{f}_*[S_j^{d_1}] = \text{torsion} + \sum_{\ell=1}^{n} v_\ell h_\ell, \tag{8.3}$$

then the $(d+1)$-cells force, via (8.2), a relationship between the u_k, v_ℓ, and c_{pq} which is exactly (Q-BLIN$\{B_p\}$).

Conversely, given u_k and v_ℓ satisfying (Q-BLIN$\{B_p\}$), there is an extension $\tilde{f} : X \to Y$ satisfying 8.3. To see this, note that there is clearly an extension to the fat wedges and the mapping cylinders. Moreover, under any such extension, $f_*[S_q^d]$ and $\sum_{i,j=1}^r a_{ij}^{(q)} \tilde{f}_*[\varphi_{ij}] \in \pi_d(Y)$ are rationally equivalent; thus when multiplied by ρ they are equal, and the map extends to the $(d+1)$-cells of X. $\square$

References

[AA78] Peter Andrews and Martin Arkowitz, *Sullivan's minimal models and higher order Whitehead products*, Canadian Journal of Mathematics **30** (1978), no. 5, 961–982.

[Brown] Edgar H. Brown, *Finite computability of Postnikov complexes*, Annals of Mathematics (1957), 1–20.

[Č⁺14a] Martin Čadek, Marek Krčál, Jiří Matoušek, Francis Sergeraert, Lukáš Vokřínek, and Uli Wagner, *Computing all maps into a sphere*, J. ACM **61** (2014), no. 3, Art. 17, 44. MR 3215297

[Č⁺14b] Martin Čadek, Marek Krčál, Jiří Matoušek, Lukáš Vokřínek, and Uli Wagner, *Extendability of continuous maps is undecidable*, Discrete & Computational Geometry **51** (2014), no. 1, 24–66.

[Č⁺14c] ———, *Polynomial-time computation of homotopy groups and Postnikov systems in fixed dimension*, SIAM Journal on Computing **43** (2014), no. 5, 1728–1780.

[ČKV17] Martin Čadek, Marek Krčál, and Lukáš Vokřínek, *Algorithmic solvability of the lifting-extension problem*, Discrete & Computational Geometry **57** (2017), no. 4, 915–965.

[FHT12] Yves Félix, Steve Halperin, and Jean-Claude Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics, vol. 205, Springer, 2012.

[FV20] Marek Filakovský and Lukáš Vokřínek, *Are two given maps homotopic? An algorithmic viewpoint*, Foundations of Computational Mathematics **20** (2020), no. 2, 311–330.

[GM81] Phillip A. Griffiths and John W. Morgan, *Rational homotopy theory and differential forms*, Birkhäuser, 1981.

[LS12] Gregory Lupton and Samuel Bruce Smith, *Fibrewise rational H-spaces*, Algebr. Geom. Topol. **12** (2012), no. 3, 1667–1694.

[Mil68] J. Milnor, *On characteristic classes for spherical fibre spaces*, Comment. Math. Helv. **43** (1968), 51–77. MR 226638

[Qui69] Daniel Quillen, *Rational homotopy theory*, Ann. of Math. (2) **90** (1969), 205–295. MR 258031

[Smith] Samuel Bruce Smith, *Rational type of classifying spaces for fibrations*, Groups of homotopy self-equivalences and related topics (Gargnano, 1999), Contemp. Math., vol. 274, Amer. Math. Soc., Providence, RI, 2001, pp. 299–307. MR 1817019

[Sta70] James Stasheff, *H-spaces from a homotopy point of view*, Lecture Notes in Mathematics, Vol. 161, Springer-Verlag, Berlin-New York, 1970. MR 0270372

[Sul77] Dennis Sullivan, *Infinitesimal computations in topology*, Publications Mathématiques de l'IHES **47** (1977), no. 1, 269–331.

[Sul05] Dennis P. Sullivan, *Geometric topology: localization, periodicity and Galois symmetry*, K-Monographs in Mathematics, vol. 8, Springer, 2005. MR 2162361

[Vok17] Lukáš Vokřínek, *Decidability of the extension problem for maps into odd-dimensional spheres*, Discrete & Computational Geometry **57** (2017), no. 1, 1–11.

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