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The merging operation and $(d-i)$-simplicial i-simple d-polytopes

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Abstract. We define a certain merging operation that given two d-polytopes P and Q such that P has a simplex facet and Q has a simple vertex produces a new d-polytope $P \triangleright Q$ with $f_0(P) + f_0(Q) - (d+1)$ vertices. We show that if for some $1 \leqslant i \leqslant d-1$, P and Q are $(d-i)$-simplicial i-simple d-polytopes, then so is $P \triangleright Q$. We then use this operation to construct new families of $(d-i)$-simplicial i-simple d-polytopes. Specifically, we prove that for all $2 \leqslant i \leqslant d-2 \leqslant 6$ with the exception of $(i,d) = (3,8)$ and $(5,8)$, there is an infinite family of $(d-i)$-simplicial i-simple d-polytopes; furthermore, for all $2 \leqslant i \leqslant 4$, there is an infinite family of self-dual i-simplicial i-simple $2i$-polytopes. Finally, we show that for every $d \geqslant 4$, there are $2^{\Omega(N)}$ combinatorial types of $(d-2)$-simplicial 2-simple d-polytopes with at most N vertices.

Keywords. Connected sums, face lattice, face numbers, Gosset–Elte polytopes, self-dual polytopes

Mathematics Subject Classifications. 52B05, 52B11

1. Introduction

A polytope is the convex hull of finitely many points in $\mathbb{R}^d$. For brevity, we refer to d-dimensional polytopes as d-polytopes. While polytopes have been studied since antiquity, many central questions about them remain wide open. In this paper we present progress on one of these questions.

A d-polytope P is called simplicial if every facet of P contains exactly d vertices. Similarly, a d-polytope P is simple, if every vertex of P is in exactly d facets. (Equivalently, P is simple if its dual P^* is simplicial.) Much progress has been made on the study of simplicial and simple

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polytopes, but much less is known about general d-polytopes that are neither simplicial nor simple already when $d = 4$. We refer the reader to [Grü03, Zie95] as excellent books on the theory of polytopes, to [BL81, Sta80] for one of the most celebrated results on the face numbers of simplicial polytopes, and to [Bay87, BZ18, PW06, Zie02, Zie04] for results on general 4-polytopes.

Let $1 \leqslant i \leqslant d-1$. A d-polytope P is called i-simplicial if all of its i-faces are simplices, and it is i-simple if its dual P^* is i-simplicial (equivalently, if every $(d-i-1)$-face of P is contained in exactly $i+1$ facets). In particular, the class of $(d-1)$-simplicial d-polytopes coincides with the class of simplicial d-polytopes, while the class of $(d-1)$-simple d-polytopes is the class of simple d-polytopes. The d-simplex is both simple and simplicial, and it is known that a j-simplicial i-simple d-polytope must be a simplex if $i+j > d$. The question of whether j-simplicial i-simple d-polytopes exist when $i, j > 1$, and especially when $i+j = d$, was raised in the mid-1960s. Such polytopes can be compared to rare combinatorial objects like designs, and the constructions presented in this paper substantially advance our state of knowledge.

Let $2 \leqslant i \leqslant d-2$. While various conjectures (see, for instance [Grü03, Exercise 9.7.7(iii)]) suggest that there should be a large number of $(d-i)$-simplicial i-simple d-polytopes, not many examples are known. The first infinite family of 2-simplicial 2-simple 4-polytopes was constructed by Eppstein, Kuperberg, and Ziegler [EKZ03]. Their approach was generalized by Paffenholz and Ziegler [PZ04] who established the existence of infinite families of $(d-2)$-simplicial 2-simple d-polytopes for all $d \geqslant 4$. Notably, the minimum number of vertices in their d-dimensional construction is $2(d+1)$, realized by $\mathrm{conv}(\Sigma \cup \Sigma^*)$, where Σ is a d-simplex whose $(d-3)$-faces are tangent to the unit sphere $\mathbb{S}^{d-1}$. Additional infinite families of 2-simplicial 2-simple 4-polytopes were constructed by Paffenholz and Werner [PW06]: all their polytopes are elementary (i.e., have $g_2^{\mathrm{toric}} = 0$) and have at least one simplex facet.

As for larger values of i, the d-dimensional demicube with $d \geqslant 4$ (also known as the half-cube) is 3-simplicial $(d-3)$-simple while its dual is $(d-3)$-simplicial 3-simple (see [Grü03, Exercise 4.8.18]). Furthermore, the Gosset–Elte polytopes that arise from Wythoff's construction provide finitely many examples of $(d-i)$-simplicial i-simple d-polytopes for $d \leqslant 8$ and $2 \leqslant i \leqslant d-2$ [Cox63]. These are essentially all known to-date examples of $(d-i)$-simplicial i-simple d-polytopes with $2 \leqslant i \leqslant d-2$. In particular, it is not known whether a 5-simplicial 5-simple 10-polytope exists. In light of this, we further pose the following questions.

Question 1.1.

1. Let $d \geqslant 4$. What is the minimum number of vertices that a non-simplex $(d-2)$-simplicial 2-simple d-polytope can have?

2. Let $d \geqslant 6$ and let $3 \leqslant i \leqslant d/2$. Are there infinite families of $(d-i)$-simplicial i-simple d-polytopes? What is the minimum number of vertices that such a non-simplex polytope can have?

The goal of this paper is to provide new infinite families of $(d-i)$-simplicial i-simple d-polytopes for some values of i and d. To achieve this, we define a certain merging operation that given two d-polytopes P and Q, where P has a simplex facet and Q has a simple vertex, outputs a new d-polytope. This operation is modeled on a familiar notion of connected sums of simplicial polytopes, but designed in a way that preserves the property of being $(d-i)$-simplicial i-simple. Using this operation, we establish the following results:

1. There exist infinite families of $(d-i)$-simplicial i-simple d-polytopes for all pairs (i, d) such that $2 \leqslant i \leqslant d-2 \leqslant 6$ and (i, d) is not $(3, 8)$ or $(5, 8)$; see Theorem 5.1. This partially answers Question 1.1(2) and [Kal97, Problem 19.5.23].

2. There exist infinite families of self-dual i-simplicial i-simple $2i$-polytopes for $2 \leqslant i \leqslant 4$; see Theorem 5.4. This partially answers [Kal97, Problem 19.5.24].

3. For all $d \geqslant 4$, there are $2^{\Omega(N)}$ combinatorial types of $(d-2)$-simplicial 2-simple d-polytopes with at most N vertices; see Theorem 6.13.

To prove the last result, we construct a higher-dimensional analog of the unique 2-simplicial 2-simple 4-polytope with nine vertices. (This 4-polytope is called P_9 in [PW06]; it has the minimum number of vertices among all non-simplex 2-simplicial 2-simple 4-polytopes.) We then apply the merging operation to produce new infinite families of $(d-2)$-simplicial 2-simple d-polytopes.

As for the second result, several examples of (non-simplex) self-dual 2-simplicial 2-simple 4-polytopes were known before, among them polytopes P_9 and P_{10} from [PW06]. In fact, [Paf06] provides a (different) infinite family of self-dual 2-simplicial 2-simple 4-polytopes, that, for instance, includes the 24-cell. An interesting infinite family of self-dual d-polytopes that are neither j-simplicial nor i-simple (for any $d \geqslant 3$ and $j, i > 1$) is the family of multiplexes constructed by Bisztriczky [Bis96].

The outline of the paper is as follows. We review several definitions related to polytopes and face lattices in Section 2. Section 3 serves as a warm-up section where we discuss the minimum number of vertices that a non-simplex 3-simplicial 2-simple 5-polytope can have. In Section 4, we introduce and study the merging operation that applies to pairs of polytopes one of which has a simplex facet and another a simple vertex. This operation has several interesting properties; see, for instance, Theorem 4.6 and Theorem 4.12. Sections 5 and 6 form the most crucial part of this paper: there, we utilize the merging operation and its properties to provide our promised constructions of new $(d-i)$-simplicial i-simple d-polytopes. Specifically, in Section 5.1, we construct infinite families of $(d-i)$-simplicial i-simple d-polytopes for $d \leqslant 8$. In Section 5.2, we construct infinite families of self-dual i-simplicial i-simple $2i$-polytopes for $i \leqslant 4$. In Section 6.1, we revisit the 2-simplicial 2-simple 4-polytopes providing several new constructions. Finally, in Section 6.2, we produce a higher-dimensional analog of P_9 and use it to construct exponentially many (in N) combinatorial types of $(d-2)$-simplicial 2-simple d-polytopes with at most N vertices.

2. Preliminaries

A *polytope* $P \subseteq \mathbb{R}^d$ is the convex hull of a finite set of points in $\mathbb{R}^d$. The *dimension* of P is the dimension of the affine span of P. For brevity, we say that P is a *d-polytope* if P is d-dimensional. In what follows, we always assume that $P \subseteq \mathbb{R}^d$ is a d-polytope.

A hyperplane $H \subseteq \mathbb{R}^d$ is a *supporting hyperplane* of P if P is contained in one of the two closed half-spaces determined by H. A *(proper) face of* P is the intersection of P with any supporting hyperplane of P. A face of a polytope is by itself a polytope. We refer to $(d-1)$-faces of P as *facets* of P, to $(d-2)$-faces as *ridges*, to 1-faces as *edges*, and to 0-faces as *vertices*. We denote by $V(P)$ the vertex set of P. If $V(P)$ consists of $d+1$ affinely independent points, then P is a *d-simplex*; we denote it by σ_d.

The face poset of P, $\mathcal{L}(P)$, is the set of faces of P (including P and $\varnothing$) ordered by inclusion, and two polytopes P and Q have the same *combinatorial type* if $\mathcal{L}(P)$ and $\mathcal{L}(Q)$ are isomorphic. The face poset of P is a lattice. We usually write the maximum element of $\mathcal{L}(P)$ (namely, P) as $\hat{1}$ and the minimum element (namely, $\varnothing$) as $\hat{0}$. For a subset S of $\mathcal{L}(P)$, we let $\vee S$ and $\wedge S$ denote the join and the meet of elements of S, respectively.

By using translation, if necessary, we can always assume that the origin, $\mathbf{0}$, lies in the interior of P. The set

$$P^* = \{y \in \mathbb{R}^d : \ y^t x \leqslant 1, \ \forall x \in P\}$$

is then a polytope called the *dual polytope* of P; see [Zie95, Chapter 2]. The dual construction has the following properties: for every d-polytope $P \subseteq \mathbb{R}^d$ (with $\mathbf{0}$ in the interior of P), $P^{**} = P$ and there are *order-reversing* bijective maps $\phi : \mathcal{L}(P) \to \mathcal{L}(P^*)$ and $\phi : \mathcal{L}(P^*) \to \mathcal{L}(P^{**}) = \mathcal{L}(P)$, which by slight abuse of notation we denote by the same symbol, such that $\phi(\phi(G)) = G$ for all $G \in \mathcal{L}(P) \sqcup \mathcal{L}(P^*)$. If $\mathcal{L}(P)$ is self-dual, that is, if there is an order reversing bijection from $\mathcal{L}(P)$ to itself, then we say that P is a *self-dual* polytope.

Let $1 \leqslant i \leqslant d-1$. A d-polytope P is *i-simplicial* if all of its i-faces are simplices; equivalently, if all of its i-faces have $i+1$ vertices. Similarly, P is *i-simple* if every $(d-i-1)$-face is contained in exactly $i+1$ facets. The class of $(d-1)$-simplicial d-polytopes is known as the class of simplicial d-polytopes, while the class of $(d-1)$-simple d-polytopes is known as the class of simple d-polytopes. In particular, if P is i-simplicial, then the interval $[\hat{0}, \tau]$ is a Boolean lattice for any face τ with $\dim \tau \leqslant i$. Likewise, if P is i-simple, then $[\tau, \hat{1}]$ is Boolean for any face τ with $\dim \tau \geqslant d-i-1$. Hence P is i-simplicial if and only if P^* is $(d-i)$-simple.

If v is a vertex of P, then the *vertex figure of P at v*, denoted P/v, is the polytope obtained by intersecting P with a hyperplane H that has v on one side and all other vertices of P on the other side. The combinatorial type of P/v does not depend on the choice of H. In fact, $\mathcal{L}(P/v)$ is exactly the interval $[v, \hat{1}]$ in $\mathcal{L}(P)$. We say that a vertex v of a d-polytope P is simple if P/v is a simplex, or equivalently, if v belongs to exactly d facets of P.

If P is a simplicial polytope, then the collection of vertex sets of faces of P, including $\varnothing$ but not including P itself, forms an *abstract simplicial complex* ∂P called the *boundary complex* of P. When V is a finite set, we let $\partial\overline{V} := \{\tau \subset V : \tau \neq V\}$ denote the boundary complex of an abstract simplex with vertex set V.

Consider a d-polytope $P \subset \mathbb{R}^d \times \{\mathbf{0}\} \subset \mathbb{R}^d \times \mathbb{R}^{d'}$ and a d'-polytope $Q \subset \{\mathbf{0}\} \times \mathbb{R}^{d'} \subset \mathbb{R}^d \times \mathbb{R}^{d'}$ such that the origin is in the relative interior of both P and Q. The polytope $P \oplus Q := \mathrm{conv}(P \cup Q)$ is called the *free sum* of P and Q. All faces of $P \oplus Q$ are of the form $\mathrm{conv}(F \cup G)$, where $F \neq P$ is a face of P and $G \neq Q$ is a face of Q. Consequently, if P and Q are simplicial polytopes then the boundary complex of $P \oplus Q$ coincides with the *join* of ∂P and ∂Q:

$$\partial(P \oplus Q) = \partial P * \partial Q := \{\sigma \cup \tau : \sigma \in \partial P, \tau \in \partial Q\}.$$

For a d-polytope P, we let $f(P) = (f_0(P), f_1(P), \ldots, f_{d-1}(P))$ be the *f-vector* of P; here $f_i(P)$ denotes the number of i-faces of P. Also, for $0 \leqslant i < j \leqslant d-1$, we let $f_{i,j}(P)$ denote the number of pairs of faces $F_i \subset F_j$ of P such that $\dim F_i = i$ and $\dim F_j = j$.

To conclude this section, we note that for all $0 \leqslant i \leqslant d-1$, $f_i(P) = f_{d-i-1}(P^*)$. This is immediate from the existence of an order-reversing bijection $\phi : \mathcal{L}(P) \to \mathcal{L}(P^*)$.

3. A warm-up: the minimum number of vertices

As mentioned in the introduction, for every $d \geqslant 4$, there exists a $(d-2)$-simplicial 2-simple d-polytope with $2(d+1)$ vertices. Furthermore, for $d = 4$, there is a 2-simplicial 2-simple 4-polytope with only 9 vertices. Are there non-simplex $(d-2)$-simplicial 2-simple d-polytopes with fewer than $2d+2$ vertices for $d > 4$? (Cf. Question 1.1(1).) The goal of this warm-up section is to answer this question for $d = 5$; see Proposition 3.3. To do this, we first establish a criterion that the f-vectors of $(d-i)$-simplicial i-simple d-polytopes (if they exist) must satisfy; cf. [Grü03, Exercise 9.7.7(ii)]. We include the proof for completeness.

Lemma 3.1. *Let $d \geqslant 2$ and $1 \leqslant i \leqslant d-1$. Let P be a $(d-i)$-simplicial d-polytope. Then P is i-simple if and only if $(d-i+1)f_{d-i}(P) = (i+1)f_{d-i-1}(P)$.*

Proof. If P is $(d-i)$-simplicial, then every $(d-i)$-face of P is a simplex; hence, every $(d-i)$-face contains $d-i+1$ faces of dimension $d-i-1$. This means that $f_{d-i-1,d-i}(P) = (d-i+1)f_{d-i}(P)$. On the other hand, a $(d-i-1)$-face of any d-polytope is contained in at least $i+1$ faces of dimension $d-i$. Thus, $f_{d-i-1,d-i}(P) \geqslant (i+1)f_{d-i-1}(P)$, and we conclude that $(d-i+1)f_{d-i}(P) = f_{d-i-1,d-i}(P) \geqslant (i+1)f_{d-i-1}(P)$. Furthermore, equality holds if and only if every $(d-i-1)$-face is in exactly $i+1$ faces of dimension $d-i$ which happens if and only if P is i-simple. $\square$

Corollary 3.2. *For all $i \geqslant 1$, an i-simplicial $2i$-polytope P is i-simple if and only if $f_{i-1}(P) = f_i(P)$.*

Proposition 3.3. *The minimum number of vertices that a non-simplex 3-simplicial 2-simple 5-polytope can have is 12.*

Proof. There exists a 3-simplicial 2-simple 5-polytope with $2(5+1) = 12$ vertices. Thus, we only need to show that there is no non-simplex 3-simplicial 2-simple 5-polytope with fewer than 12 vertices.

It is known (see [PW06]) that every non-simplex 2-simplicial 2-simple 4-polytope has at least 9 vertices, and the only such polytope with 9 vertices is the polytope denoted by P_9 in [PW06]. Since vertex figures of 3-simplicial 2-simple 5-polytopes are 2-simplicial 2-simple, it follows that a non-simplex 3-simplicial 2-simple polytope Q must have at least 10 vertices.

Assume that $f_0(Q) = 10$. Then each vertex figure is either the 4-simplex σ_4 or P_9, and so each vertex of Q has degree 5 or 9. Since Q is not simple, at least one of the vertex figures of Q is P_9. Consider Q^*; it has 10 facets each of which is either σ_4 or P_9. (This is because both σ_4 and P_9 are self-dual.) Now consider a facet F of Q^* that is isomorphic to P_9. It has 7 non-simplex facets (one cross-polytope, also known as an octahedron, and six bipyramids); see

Construction 6.1. Each of these seven 3-faces must lie in F and one additional facet of Q^*, which cannot be a simplex. This shows that Q^* has at least eight facets isomorphic to P_9. Then in Q, at least 8 out of 10 vertices are of degree 9. This implies that all vertices of Q have degree $\geqslant 8$. Consequently, all vertices of Q have degree 9, and so $f_1(Q) = \binom{10}{2} = 45$.

Since Q is 3-simplicial 2-simple, $4f_3(Q) = 3f_2(Q)$ by Lemma 3.1. Furthermore, since Q is 3-simplicial and since the toric h-vector of a 5-polytope is symmetric [Sta87],

$$0 = g_3^{\text{toric}}(Q) = f_2(Q) - 4f_1(Q) + 10f_0(Q) - 20.$$

Finally, by the Euler relation,

$$f_0(Q) - f_1(Q) + f_2(Q) - f_3(Q) + f_4(Q) = 2.$$

This uniquely determines the f-vector of Q: $f(Q) = (10, 45, 100, 75, 12)$. But then we must have $75 = f_3(Q) \leqslant \binom{f_4(Q)}{2} = 66$, which is a contradiction.

Similarly, if $f_0(Q) = 11$, then $f_2(Q) = 4f_1(Q) - 10f_0(Q) + 20 = 4f_1(Q) - 90$, which is not a multiple of 4. On the other hand, $4f_3(Q) = 3f_2(Q)$ still holds, so $f_3(Q)$ is not an integer, which is again a contradiction. ☐

While a 2-simplicial 2-simple 4-polytope with 9 vertices is unique, this is not the case with 3-simplicial 2-simple 5-polytopes with 12 vertices. (For instance, in Section 6 we will see that there is such a polytope with a simplex facet.) For $d \geqslant 6$, Question 1.1(1) remains unsolved. It would be very interesting to shed any light on whether the answer is $2d+2$ or smaller than $2d+2$.

4. The merging operation

Throughout, let $d \geqslant 2$. Recall that a connected sum of two simplicial d-polytopes[1] is a *simplicial* d-polytope. In other words, taking connected sums preserves the property of being $(d-1)$-simplicial 1-simple. Is there an analogous operation that preserves the property of being $(d-i)$-simplicial i-simple for an arbitrary $2 \leqslant i \leqslant d-1$? The goal of this section is to discuss one such operation that can be applied to two d-polytopes as long as one of them has a simplex facet and another one has a simple vertex. The order in which we list the vertices will be important for our construction. Specifically, we write $[a_1, \dots, a_m]$ to denote the polytope $\mathrm{conv}(a_1, \dots, a_m)$ whose vertices are ordered as $a_1, \dots, a_m$. We will mainly use this notation to describe faces of a given polytope. For brevity, we also write the edge $[u, v]$ as uv.

4.1. The definition and basic properties

We start with setting up a few notations, conventions and definitions that will be repeatedly used throughout this section. Let P_1 and P_2 be two d-polytopes such that P_1 has a *simplex facet* $F := [u_1, \dots, u_d]$ and P_2 has a *simple vertex* v whose neighbors are ordered as $u'_1, \dots, u'_d$.

[1] The connected sum of two simplicial polytopes P and Q is defined by gluing them along a common facet whose hyperplane separates P and Q. To guarantee that the result is a polytope we first apply an appropriate projective transformation to P (or Q); see [RG96, Lemma 3.2.4].

We adopt the following notation: for $1 \leqslant j \leqslant d$, let H_j be the facet of P_1 that is adjacent to F along the ridge $G_j := [u_1, \ldots, \widehat{u_j}, \ldots, u_d]$. Similarly, for $1 \leqslant j \leqslant d$, let H'_j be the facet of P_2 that contains all the edges of P_2 incident with v but vu'_j.

By applying a projective transformation to P_1, we may assume that the hyperplanes

$$\mathrm{aff}(F), \mathrm{aff}(H_1), \ldots, \mathrm{aff}(H_d)$$

define a d-simplex Σ that *contains* P_1. (The existence of such a projective transformation follows from the proof of [RG96, Lemma 3.2.4].) Denote the vertex of Σ that does not lie in F by u. By applying the unique affine transformation that maps v to u, and u'_k to u_k for $1 \leqslant k \leqslant d$, we may further assume that the d-simplices $\Sigma' = [v, u'_1, \ldots, u'_d]$ and Σ coincide, and in particular that $P_1 \subseteq \Sigma = \Sigma'$ is a convex subset of P_2.

Finally, let $P'_2 := \mathrm{conv}(V(P_2)\backslash v)$ and $F' := [u'_1, \ldots, u'_d]$ be two subpolytopes of P_2. Note that if P_2 is a d-simplex, then P'_2 is F', and otherwise, F' is a facet of P'_2.

Definition 4.1. Under the above assumptions on P_1 and P_2, define a new d-polytope $P_1 \triangleright P_2$ obtained from P_2 by replacing $\Sigma' = \Sigma$ with P_1. Alternatively, $P_1 \triangleright P_2$ is the union of P_1 and P'_2 where we identify u_k with u'_k for $1 \leqslant k \leqslant d$. (Observe that P_1 and P'_2 share the facet $F = F'$, lie on the opposite sides of F and that their union is a polytope.) The new polytope is called the *merge* of P_1 and P_2 along F and v.

Example 4.2. Consider two polygons P_1 and P_2 whose boundary complexes are cycles

$$(u_1, \ldots, u_n, u_1) \text{ and } (v_0, v_1, \ldots, v_k, v_0).$$

Then the merge of P_1 and P_2 along the edge $F = u_1u_n$ and the vertex v_0 is the polygon whose boundary complex is the cycle $(v_1 = u_1, u_2, \ldots, u_{n-1}, u_n = v_k, v_{k-1}, \ldots, v_2, v_1 = u_1)$. In other words, in dimension two, $P_1 \triangleright P_2$ is exactly the connected sum of P_1 and $P'_2 = \mathrm{conv}(V(P_2)\backslash v_0)$.

Figure 4.1 illustrates how to merge two 3-polytopes.

Remark 4.3. For $d \geqslant 3$, the set of facets of $P_1 \triangleright P_2$ consists of

- old facets: all facets of P_1 with the exception of $F, H_1, \ldots, H_d$, and all facets of P_2 with the exception of $H'_1, \ldots, H'_d$;
- new facets: for each $1 \leqslant j \leqslant d$, H_j and H'_j merge into a single facet $H_j \triangleright H'_j$ where the merge is along $G_j = [u_1, \ldots, \widehat{u_j}, \ldots, u_d]$ and v (with the neighbors of v in H'_j ordered as $u'_1, \ldots, \widehat{u'_j}, \ldots u'_d$).

Remark 4.4. The description of facets of $P_1 \triangleright P_2$ leads to the following observation: the combinatorial type of $P_1 \triangleright P_2$ may depend on the ordering of vertices of F and neighbors of v. That is, letting $F = [u_{\sigma(1)}, \ldots, u_{\sigma(d)}]$ and relabeling the neighbors of v as $v_{\sigma'(1)}, \ldots, v_{\sigma'(d)}$, for some permutations σ, σ' of $[d] := \{1, 2, \ldots, d\}$, may result in a polytope with a different combinatorial type; see Section 6 for examples. This is analogous to the situation with the connected sum of two simplicial polytopes.

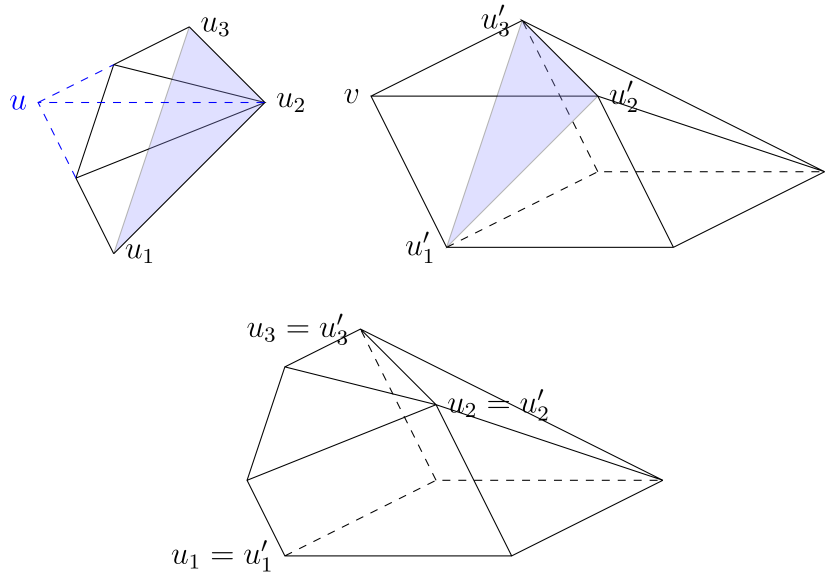

Figure 4.1: $P_1 \subseteq \Sigma$, $P_2 \supseteq \Sigma'$, and $P_1 \triangleright P_2$, where the merge is along $[u_1, u_2, u_3] \cong [u'_1, u'_2, u'_3]$ and v.

It follows from Definition 4.1 that if P_1 is a simplex, then $P_1 \triangleright P_2 = P_2$, and similarly if P_2 is a simplex, then $P_1 \triangleright P_2 = P_1$. In all other cases, F is not a facet of $P_1 \triangleright P_2$ and v is not a vertex of $P_1 \triangleright P_2$. Furthermore, if both P_1 and P_2 are simplicial and P_2 has a simple vertex v, then the merge of P_1 and P_2 along any facet F of P_1 and v is the connected sum of P_1 and $P'_2 = \mathrm{conv}(V(P_2)\backslash v)$.

We summarize this discussion in the following lemma.

Lemma 4.5. *Let $d \geqslant 2$. Let P_1 be a d-polytope with a simplex facet and let P_2 be a d-polytope with a simple vertex. Then $f_0(P_1 \triangleright P_2) = f_0(P_1) + f_0(P_2) - (d+1)$. In particular, $f_0(P_1 \triangleright P_2) \geqslant \max\{f_0(P_1), f_0(P_2)\}$ and equality holds if and only if at least one of P_1 and P_2 is a simplex. In the case that one of P_1 and P_2 is a simplex, $P_1 \triangleright P_2$ is equal to the other polytope.*

The following theorem and corollary explain the significance of the merging operation.

Theorem 4.6. *Let $d \geqslant 2$ and $1 \leqslant i, j \leqslant d-1$, and let P_1 and P_2 be d-polytopes with a simplex facet and a simple vertex, respectively. If P_1 and P_2 are j-simplicial, then so is $P_1 \triangleright P_2$. If P_1 and P_2 are i-simple, then so is $P_1 \triangleright P_2$.*

Proof. We first discuss j-simplicial polytopes. The proof is by induction on d. The statement holds for $j = 1$ for any d (since all polytopes are 1-simplicial). Hence the statement holds for $d = 2$.

Now, assume the statement holds for $d-1$ and any $1 \leqslant j \leqslant d-2$. We prove that the statement holds for d and any $1 \leqslant j \leqslant d-1$. Let P_1 and P_2 be two j-simplicial d-polytopes.

If one of them is a simplex, there is nothing to prove. Also, if $j = d-1$, then $P_1 \triangleright P_2$ is the connected sum of two simplicial polytopes P_1 and P_2', which is $(d-1)$-simplicial.

Thus assume that $2 \leqslant j \leqslant d-2$ and that neither P_1 nor P_2 is a simplex. Let τ be a j-face of $P_1 \triangleright P_2$. Then either τ is a j-face of P_1 or it is a j-face of P_2 or it is a j-face of $H_k \triangleright H_k'$ for some k. In the first two cases, τ is a simplex because P_1 and P_2 are j-simplicial. In the last case, it is a simplex because both H_k and H_k' are j-simplicial, and so τ is a simplex by the induction hypothesis.

We now discuss i-simple polytopes. The proof is again by induction on d. The statement holds for $i = 1$ and any d (since all polytopes are 1-simple). Hence the statement holds for $d = 2$. Now assume the statement holds for $d-1$ and any $2 \leqslant i \leqslant d-2$. Let $2 \leqslant i \leqslant d-1$ and let P_1 and P_2 be two i-simple d-polytopes. To see that $P_1 \triangleright P_2$ is i-simple, let τ be a $(d-i-1)$-face of $P_1 \triangleright P_2$. There are two possible cases.

Case 1: τ is a face of one of $H_k \triangleright H_k'$. Since P_1 and P_2 are i-simple, H_k and H_k' are $(i-1)$-simple $(d-1)$-polytopes. Thus, by the induction hypothesis, $H_k \triangleright H_k'$ is an $(i-1)$-simple $(d-1)$-polytope. Since τ is a face of $H_k \triangleright H_k'$ of dimension $d-i-1 = (d-1)-(i-1)-1$, it follows that there are exactly i facets of $H_k \triangleright H_k'$ (and hence ridges of $P_1 \triangleright P_2$) that contain τ. Each of these i ridges is contained in two facets of $P_1 \triangleright P_2$: $H_k \triangleright H_k'$ and one additional facet. Thus, τ is contained in exactly $i+1$ facets of $P_1 \triangleright P_2$, namely, $H_k \triangleright H_k'$ and the i additional facets just described.

Case 2: τ is not contained in any $H_k \triangleright H_k'$ (for $k = 1, \dots, d$). Then either τ is a face of P_1 not contained in any of $F, H_1, \dots, H_d$, or τ is a face of P_2 that does not contain v and is not contained in any of $H_1', \dots, H_d'$. In the former case, the facets of $P_1 \triangleright P_2$ that contain τ are the facets of P_1 that contain τ and there are $i+1$ of them since P_1 is i-simple. Similarly, in the latter case, the facets of $P_1 \triangleright P_2$ that contain τ are the facets of P_2 that contain τ and there are $i+1$ of them. □

Corollary 4.7. *Let $d \geqslant 2$ and $1 \leqslant i \leqslant d-1$. Let P be a $(d-i)$-simplicial i-simple d-polytope such that (1) P is not a simplex, (2) P has a simplex facet F, and (3) P has a simple vertex v not contained in F. Finally, let $P \triangleright P$ be the merge of P with itself along F and v. Then $P \triangleright P$ is a $(d-i)$-simplicial i-simple d-polytope that has a simplex facet and a simple vertex not contained in that facet; furthermore, $f_0(P \triangleright P) > f_0(P)$. Consequently, there exists an infinite family of $(d-i)$-simplicial i-simple d-polytopes obtained by iterative merging with P.*

Proof. Consider two copies of P: P_1 and P_2. Denote the copy of F in P_j by F_j, and the copy of v in P_j by v_j. Merge P_1 and P_2 along F_1 and v_2. By Theorem 4.6, $P_1 \triangleright P_2$ is $(d-i)$-simplicial and i-simple; it has a simplex facet F_2 and a simple vertex $v_1 \notin F_2$. □

This corollary implies that to find infinitely many $(d-i)$-simplicial i-simple d-polytopes, it suffices to find the "building blocks" — those with simplex facets and simple vertices. Hence we propose the following question that strengthens Question 1.1(2).

Question 4.8. Let $d \geqslant 4$ and $2 \leqslant i \leqslant d-2$. Are there infinite families of $(d-i)$-simplicial i-simple d-polytopes, each of which has a **simplex** facet and a **simple** vertex?

4.2. The face lattice

In this subsection, we assume that P_1 and P_2 are two $(d-i)$-simplicial i-simple d-polytopes that will be merged along a simplex facet $F = [u_1, \ldots, u_d]$ of P_1 and a simple vertex v of P_2. Our goal is to describe the face lattice of $P_1 \triangleright P_2$, $\mathcal{L}(P_1 \triangleright P_2)$. We continue using notation introduced in Section 4.1. The following definitions depend on P_1, P_2 but also on d and i.

Definition 4.9. Consider the following two subposets of $\mathcal{L}(P_1)$ and $\mathcal{L}(P_2)$:

$$\mathcal{L}(P_1)^- := \mathcal{L}(P_1) \backslash \{\sigma : \sigma \subseteq F, \dim \sigma \geqslant d-i\},$$
$$\mathcal{L}(P_2)^- := \mathcal{L}(P_2) \backslash \{\sigma : v \in \sigma, \dim \sigma < d-i\},$$

and let $\mathcal{L}(P_1)^- \sqcup \mathcal{L}(P_2)^-$ be their *disjoint sum*, i.e., the disjoint union of $\mathcal{L}(P_1)^-$ and $\mathcal{L}(P_2)^-$ with the original partial orders on $\mathcal{L}(P_1)^-$ and $\mathcal{L}(P_2)^-$, and no other comparable pairs.

Definition 4.10. Let $\mathcal{L}$ be the following quotient poset of $\mathcal{L}(P_1)^- \sqcup \mathcal{L}(P_2)^-$. As a set, it is $(\mathcal{L}(P_1)^- \sqcup \mathcal{L}(P_2)^-) \,/\, \sim$, where

$$[u_k : k \in S] \sim [u'_k : k \in S] \text{ for all } S \subseteq [d],\ |S| \leqslant d-i,$$

$$\text{and } \cap_{k \in S} H_k \sim \cap_{k \in S} H'_k \text{ for all } S \subseteq [d],\ |S| \leqslant i.$$

The partial order on $\mathcal{L}$ is inherited from $\mathcal{L}(P_1)^- \sqcup \mathcal{L}(P_2)^-$: $[\tau] < [\sigma]$ if there are representatives τ' and σ' of the equivalence classes $[\tau]$ and $[\sigma]$ such that $\tau' < \sigma'$ in $\mathcal{L}(P_1)^- \sqcup \mathcal{L}(P_2)^-$.

The main result of this subsection —Theorem 4.12— asserts that $\mathcal{L}$ is the face lattice of $P_1 \triangleright P_2$. The proof relies on the following lemma.

Lemma 4.11. *Let $S \subseteq [d]$.*

1. *If $|S| \leqslant i$, then $\cap_{k \in S} H_k$ is a $(d-|S|)$-face of P_1 not contained in F, while $\cap_{k \in S} H'_k$ is a $(d-|S|)$-face of P_2 containing v.*
2. *If $|S| \leqslant d-i$, then $[u_k : k \in S]$ is an $(|S|-1)$-face of P_1 and $[u'_k : k \in S]$ is an $(|S|-1)$-face of P_2.*
3. *If H is a facet of P_1 that is not one of $F, H_1, \ldots, H_d$, then H shares with F at most $d-i-1$ vertices, and H does not contain any intersection of the form $\cap_{k \in S} H_k$, for $S \subseteq [d]$, $|S| \leqslant i$. Hence, $\mathcal{L}(H)$ is equal to $[\hat{0}, H]$ computed in both $\mathcal{L}(P_1)^-$ and $\mathcal{L}$.*
4. *If H is a facet of P_2 that does not contain v, then H does not contain any intersection of the form $\cap_{k \in S} H'_k$. Thus $\mathcal{L}(H)$ is equal to $[\hat{0}, H]$ computed in both $\mathcal{L}(P_2)^-$ and $\mathcal{L}$.*

Proof. For part (1), we only need to show that $\cap_{k \in S} H_k$ is $(d-|S|)$-dimensional and that it is not contained in F. Consider $\tau := (\cap_{k \in S} H_k) \cap F = \cap_{k \in S}(H_k \cap F)$. Since F is a $(d-1)$-simplex, τ is a face of P_1 of dimension $d-|S|-1$. Now, since $|S| \leqslant i$, and so $d-|S|-1 \geqslant d-i-1$, the assumption that P_1 is i-simple implies that the interval $[\tau, \hat{1}]$ is a Boolean lattice whose coatoms are H_k, for $k \in S$, and F. This, in turn, implies the desired properties of $\cap_{k \in S} H_k$.

For part (2), since F is a simplex facet of P_1, $[u_k : k \in S]$ must be a simplex $(|S|-1)$-face of P_1. Also, since v is simple, the edges vu'_k for $k \in S$ determine an $|S|$-face of P_2, and this face must be a simplex since P_2 is $(d-i)$-simplicial. Thus $[u'_k : k \in S]$ is an $(|S|-1)$-face of P_2.

For part (3), note that if H contained $d-i$ vertices of F, say, $u_1, \dots, u_{d-i}$, then $[u_1, \dots, u_{d-i}]$ would be a $(d-i-1)$-face of P_1 contained in at least $i+2$ facets, namely, F, $H_{d-i+1}, \dots, H_d$, and H; this is impossible since P is i-simple. Similarly, if H contained, say, the face $H_1 \cap \cdots \cap H_i$, then this $(d-i)$-face would be in at least $i+1$ facets, namely, $H_1, \dots, H_i$, and H, which is again a contradiction.

Part (4) follows from the fact that $v \in \cap_{k \in S} H'_k$ but $v \notin H$, and from the definition of $\mathcal{L}(P_2)^-$ and $\mathcal{L}$. □

Let S be a subset of $[d]$. Note that $\hat{0}_{P_1} = \vee_{k \in \varnothing} u_k \sim \vee_{k \in \varnothing} u'_k = \hat{0}_{P_2}$ is the minimum element of $\mathcal{L}$, while $\hat{1}_{P_1} = \wedge_{k \in \varnothing} H_k \sim \wedge_{k \in \varnothing} H'_k = \hat{1}_{P_2}$ is the maximum element. Furthermore, Lemma 4.11 implies that if $|S| \leqslant d-i$, then $\vee_{k \in S} u_k \in \mathcal{L}(P_1)$ and $\vee_{k \in S} u'_k \in \mathcal{L}(P_2)$ are both elements of $\mathcal{L}(P_1)^- \sqcup \mathcal{L}(P_2)^-$, and that they have the same rank. Similarly, if $|S| \leqslant i$, then $\wedge_{k \in S} H_k$ and $\wedge_{k \in S} H'_k$ both belong to $\mathcal{L}(P_1)^- \sqcup \mathcal{L}(P_2)^-$ and have the same rank there. We are now ready to prove that $\mathcal{L}$ is the face lattice of $P_1 \triangleright P_2$. Specifically, for $S \subseteq [d]$, $|S| \leqslant i$, the class $\wedge_{k \in S} H_k \sim \wedge_{k \in S} H'_k$ in $\mathcal{L}$ represents the face $\cap_{k \in S}(H_k \triangleright H'_k)$ of $P_1 \triangleright P_2$.

Theorem 4.12. *Let $d \geqslant 2$ and $1 \leqslant i \leqslant d-1$. Let P_1 and P_2 be $(d-i)$-simplicial i-simple polytopes such that P_1 has a simplex facet $F = [u_1, \dots, u_d]$ and P_2 has a simple vertex v whose neighbors are $u'_1, \dots, u'_d$. Then $\mathcal{L} = \mathcal{L}(P_1 \triangleright P_2)$.*

Proof. The proof is by induction on d and i. First we consider the case where P_1 and P_2 are both $(d-1)$-simplicial 1-simple d-polytopes. This case splits into two subcases:

1. If P_2 is not a simplex, then $P_1 \triangleright P_2 = P_1 \# P'_2$. The lattice $\mathcal{L}(P_1 \triangleright P_2)$ is obtained from $\mathcal{L}(P_1)$ and $\mathcal{L}(P'_2)$ by removing facets $[u_1, \dots, u_d]$ and $[u'_1, \dots, u'_d]$ and identifying their boundary complexes; this agrees with our definition of $\mathcal{L}(P_1)^- \sqcup \mathcal{L}(P_2)^- / \sim\, = \mathcal{L}$.

2. If P_2 is a simplex, then $P_1 \triangleright P_2$ is P_1. That $\mathcal{L}$ is equal to $\mathcal{L}(P_1)$ in this case, again follows easily from the definition of $\mathcal{L}$.

This discussion completes the proof of the base case $i = 1$ and arbitrary $d \geqslant 2$.

Now assume that the statement holds in dimension $\leqslant d-1$ and consider two $(d-i)$-simplicial i-simple d-polytopes P_1 and P_2, where $i \geqslant 2$. By definition, $\mathcal{L}$ and $\mathcal{L}(P_1 \triangleright P_2)$ have the same coatoms. So it suffices to show that for every facet H of $P_1 \triangleright P_2$, the interval $[\hat{0}, H]$ in $\mathcal{L}$ is equal to $\mathcal{L}(H)$.

First, if H is a facet of P_1 not equal to $F, H_1, \dots, H_d$, or H is a facet of P_2 that does not contain v, then by Lemma 4.11, the interval $[\hat{0}, H]$ in $\mathcal{L}$ is equal to $\mathcal{L}(H)$. For $1 \leqslant k \leqslant d$, both H_k and H'_k are $(d-i)$-simplicial $(i-1)$-simple $(d-1)$-polytopes. In particular,

$$\mathcal{L}(H_k)^- = \mathcal{L}(H_k) \backslash \{\sigma : \sigma \subseteq F \backslash u_k,\ \dim \sigma \geqslant (d-1)-(i-1) = d-i\},$$
$$\mathcal{L}(H'_k)^- = \mathcal{L}(H'_k) \backslash \{\sigma : v \in \sigma,\ u'_k \notin \sigma,\ \dim \sigma < (d-1)-(i-1) = d-i\}.$$

Hence $[0, H_k]$ computed in $\mathcal{L}(P_1)^-$ is $\mathcal{L}(H_k)^-$ and $[0, H'_k]$ computed in $\mathcal{L}(P_2)^-$ is $\mathcal{L}(H'_k)^-$. Then the inductive hypothesis implies that $[\hat{0}, H_k \triangleright H'_k]$ in $\mathcal{L}$ is equal to $\mathcal{L}(H_k \triangleright H'_k)$. This proves that $\mathcal{L} = \mathcal{L}(P_1 \triangleright P_2)$. □

One application of Theorem 4.12 is the following result on the f-numbers of $P_1 \triangleright P_2$.

Corollary 4.13. *Let $d \geqslant 2$ and $1 \leqslant i \leqslant d-1$. Let P_1 and P_2 be $(d-i)$-simplicial i-simple d-polytopes that can be merged along a simplex facet F of P_1 and a simple vertex v of P_2. Then for all $0 \leqslant j \leqslant d-1$, $f_j(P_1 \triangleright P_2) = f_j(P_1) + f_j(P_2) - \binom{d+1}{j+1}$.*

Proof. First assume that $0 \leqslant j \leqslant d-i-1$. By definition of $\mathcal{L}(P_1 \triangleright P_2)$, each j-face of F (i.e., each $(j+1)$-subset of $\{u_1, \dots, u_d\}$), is identified with the corresponding j-face of F' (i.e., the corresponding $(j+1)$-subset of $\{u'_1, \dots, u'_d\}$). In addition, all j-faces of P_2 that contain v (i.e., all $(j+1)$-subsets of $\{v, u'_1, \dots, u'_d\}$ that contain v) are removed from $\mathcal{L}(P_1 \triangleright P_2)$. Hence

$$f_j(P_1 \triangleright P_2) = f_j(P_1) + f_j(P_2) - \binom{d}{j+1} - \binom{d}{j} = f_j(P_1) + f_j(P_2) - \binom{d+1}{j+1}.$$

Similarly, for $d-i \leqslant j \leqslant d-1$, by definition of $\mathcal{L}(P_1 \triangleright P_2)$, all j-faces of P_1 contained in F (i.e., $(j+1)$-subsets of $\{u_1, \dots, u_d\}$) are removed from $\mathcal{L}(P_1 \triangleright P_2)$, while for each $(d-j)$-subset S of $[d]$, the j-face $\cap_{k \in S} H_k$ is identified with the j-face $\cap_{k \in S} H'_k$. Hence $f_j(P_1 \triangleright P_2) = f_j(P_1) + f_j(P_2) - \binom{d}{j+1} - \binom{d}{d-j} = f_j(P_1) + f_j(P_2) - \binom{d+1}{j+1}$. □

5. Applications: part I

5.1. Infinite families of $(d-i)$-simplicial i-simple polytopes for small d

The goal of this section is to answer Question 4.8 in the affirmative for small values of d. Our starting point is the uniform 8-polytope 2_{41} constructed within the symmetry of the E_8 group. (It was first discovered by Gosset and Elte; see also [Cox63, Section 11]). This polytope has 17280 simplex facets and it is 4-simplicial and 4-simple. The polytope 2_{41} gives rise to the following 7-polytopes:

- Each nonsimplex facet of 2_{41} is the 7-polytope 2_{31}. It is 4-simplicial 3-simple and it has 576 simplex facets.
- Each vertex figure of 2_{41} is the 7-demicube.

Recall that the d-*demicube* is defined as follows (see [Grü03, Exercise 4.8.18]). Consider the d-cube $C_d = [0,1]^d$. For each vertex v in C_d whose coordinates have an even number of ones, truncate C_d along the hyperplane that contains all d vertices adjacent to v. The resulting polytope is called the d-demicube; we denote it by Q_d. This polytope has the following properties:

- When $d > 4$, Q_d has exactly 2^{d-1} simplex facets (these are the facets defined by truncating hyperplanes), and $2d$ non-simplex facets (these are the facets obtained by truncating the facets of C_d). Moreover, no two simplex facets are adjacent in Q_d.
- When $d \geqslant 4$, Q_d is 3-simplicial and $(d-3)$-simple.

We are now in a position to prove the main result of this subsection:

Theorem 5.1. *For every element of* $\{(i,d) : 2 \leqslant i \leqslant d-2 \leqslant 6\} \setminus \{(3,8),(5,8)\}$*, there exists an infinite family of* $(d-i)$*-simplicial* i*-simple* d*-polytopes, each of which has a simplex facet and a simple vertex not in that facet.*

Proof. By considering dual polytopes, it suffices to prove the statement for $i \leqslant d/2 \leqslant 4$. The case of $i = 2$ and an arbitrary $d \geqslant 4$ will be discussed in Section 6. For now, we mention that for $i = 2$ and $d = 4$, the result follows by applying Corollary 4.7 to P_9. (For the description of facets of P_9, see Construction 6.1.) Consider the case of $i = 3$ and $d = 6$. Since both Q_6 and Q_6^* are 3-simplicial 3-simple, and since Q_6 has a simplex facet (in fact, 32 of them) and Q_6^* has a simple vertex (in fact, 32 of them), the merge of Q_6 and Q_6^*, $P = Q_6 \triangleright Q_6^*$, is well-defined; furthermore, P has a simplex facet F and a simple vertex v not contained in F. Hence, Corollary 4.7 applies to P and results in a desired infinite family of 3-simplicial 3-simple 6-polytopes. Similarly, in the case of $i = 3$ and $d = 7$, apply Corollary 4.7 to $P = 2_{31} \triangleright Q_7^*$. Finally, in the case of $i = 4$ and $d = 8$, apply Corollary 4.7 to $P = 2_{41} \triangleright 2_{41}^*$. □

The proof of Theorem 5.1 provides the following partial answer to Question 4.8.

Corollary 5.2. *Let* $2 \leqslant i \leqslant 4$*. There exists an infinite family of* i*-simplicial* i*-simple* $2i$*-polytopes, each of which has a simplex facet and a simple vertex not in that facet.*

5.2. Self-dual polytopes

Kalai [Kal97, Problem 19.5.24] asked for which values of i and d there are self-dual i-simplicial d-polytopes other than the d-simplex. For the rest of this section, assume that $d = 2i$ and consider an i-simplicial i-simple $2i$-polytope P with a simplex facet $F = [u_1, \dots, u_{2i}]$. As before, assume that $H_1, \dots, H_d$ are the facets of P adjacent to F, where $H_k \cap F = [u_1, \dots, \widehat{u_k}, \dots, u_d]$. Let $\phi : \mathcal{L}(P) \to \mathcal{L}(P^*)$, $\phi : \mathcal{L}(P^*) \to \mathcal{L}(P)$ be the order-reversing bijections on the face lattices. Then P^* is an i-simplicial i-simple $2i$-polytope with a simple vertex $v := \phi(F)$. The neighbors of v are $u_k' := \phi(H_k)$ for $1 \leqslant k \leqslant d$. Let H_k' be the facet of P^* determined by the edges $vu_1', \dots, \widehat{vu_k'}, \dots, vu_d'$. In other words, $H_k' = (\vee_{j \in [d] \setminus k} u_j') \vee v$, and hence

$$\phi(H_k') = \big(\wedge_{j \in [d] \setminus k} \phi(u_j')\big) \wedge \phi(v) = \big(\wedge_{j \in [d] \setminus k} H_j\big) \wedge F = u_k.$$

The next proposition is our main tool for constructing self-dual i-simplicial i-simple $2i$-polytopes. We follow assumptions and notation introduced in the previous paragraph.

Proposition 5.3. *The merge of* P *and* P^* *along* $F = [u_1, \dots, u_d]$ *and* v *(whose neighbors are ordered as* $u_1', \dots, u_d'$*) is a self-dual polytope.*

Proof. The map $\phi : \mathcal{L}(P) \to \mathcal{L}(P^*), \mathcal{L}(P^*) \to \mathcal{L}(P)$ provides us with an order-reversing involution on $\mathcal{L}(P) \sqcup \mathcal{L}(P^*)$. Since $\phi(H_k) = u_k'$ and $\phi(H_k') = u_k$, it follows that for $S \subseteq [d]$,

$$\phi(\vee_{k \in S} u_k) = \wedge_{k \in S} H_k', \quad \phi(\vee_{k \in S} u_k') = \wedge_{k \in S} H_k. \tag{5.1}$$

In particular, ϕ maps ℓ-faces of F to $(d-\ell-1)$-faces containing v. Since $d = 2i$, it follows that ϕ induces an order-reversing involution on $\mathcal{L}(P)^- \sqcup \mathcal{L}(P^*)^-$. Furthermore, by (5.1), this involution descends to an order-reversing involution on the quotient $\mathcal{L}$ described in Definition 4.10. Thus $\mathcal{L}$ is a self-dual lattice. The result follows since by Theorem 4.12, $\mathcal{L} = \mathcal{L}(P \triangleright P^*)$. □

Theorem 5.4. *For all $2 \leqslant i \leqslant 4$, there exists an infinite family of self-dual i-simplicial $2i$-polytopes.*

Proof. Let $2 \leqslant i \leqslant 4$. By Corollary 5.2, there exists an infinite family of i-simplicial i-simple $2i$-polytopes each of which has a simplex facet. The result follows by applying Proposition 5.3 to this family. □

6. Applications: part II

This section is devoted to $(d-2)$-simplicial 2-simple d-polytopes for all $d \geqslant 4$. We show that for such values of parameters, the answer to Question 4.8 is yes, and, in fact, that for every $d \geqslant 4$, there are $2^{\Omega(N)}$ combinatorial types of $(d-2)$-simplicial 2-simple d-polytopes with at most N vertices, each of which has a simplex facet and a simple vertex. Section 6.1 concentrates on a few constructions for $d = 4$; Section 6.2 treats the general case.

6.1. Revisitng 2-simplicial 2-simple 4-polytopes

By a result of Paffenholz and Werner [PW06], there exist infinite families of 2-simplicial 2-simple 4-polytopes each of which has a simplex facet and a simple vertex. This solves Question 4.8 in the affirmative in dimension $d = 4$.

In this section, we provide alternative (and more symmetric) constructions. We start by revisiting the construction from [PW06] of P_9 — the unique 2-simplicial 2-simple 4-polytope with nine vertices — casting it in a way that will help us construct higher-dimensional analogs of P_9 in Section 6.2. We then provide another construction of a highly symmetric 2-simplicial 2-simple 4-polytope with 18 vertices that appears to be new. The promised infinite families are obtained by merging k copies of P_9 (respectively, P_{18}) for all natural numbers $k \geqslant 2$. The cross-polytope is featured prominently in our constructions, and we often abbreviate it as CP. (The notion of a *point beyond or beneath a facet* is defined in [Grü03, page 78].)

Construction 6.1. To construct P_9, start with a regular 4-simplex $\Sigma := [u'_1, u'_2, u'_3, u'_4, u'_5]$. Now add the vertices u_1, u_2, u_3, v_2 in the following way. (Why we label the vertices in this fashion will become clear in Section 6.2.) For $i = 1, 2, 3$, place u_i in the affine hull of the facet $\Sigma \backslash u'_i$ of Σ so that it is positioned beyond the 2-face $\Sigma \backslash u'_i u'_5$ and so that $[u_1, u_2, u_3, u'_1, u'_2, u'_3]$ is a 3-cross-polytope; cf. Definition 6.8 below. (Hence u_i can be thought of as a perturbation of the barycenter of $[u'_j, u'_k, u'_\ell]$, where $\{i, j, k, \ell\} = [4]$.) Then position v_2 on the intersection of the affine hulls of $[u'_1, u'_4, u_2, u_3]$, $[u'_2, u'_4, u_1, u_3]$, and $[u'_3, u'_4, u_1, u_2]$ (this intersection is a line) and beyond the hyperplane $\mathrm{aff}(u'_4, u_1, u_2, u_3)$; cf. Definitions 6.7 and 6.9. (Thus, v_2 is a special perturbation of the barycenter of $[u_1, u_2, u_3, u'_4]$).

The resulting polytope has nine vertices $\{v_2, u_1, u_2, u_3, u'_1, \ldots, u'_5\}$; it is also convenient to let $v_1 = u'_4$. Figure 6.1 shows part of the Schlegel diagram of $P'_9 = \mathrm{conv}(V(P_9) \backslash u'_5)$. The complete list of facets of P_9 is given as follows (cf. Lemma 6.10):

1. a CP with antipodal facets $[u_1, u_2, u_3]$ and $[u'_1, u'_2, u'_3]$ (colored in blue) and a simplex $[u'_1, u'_2, u'_3, u'_5]$;

2. three bipyramids $[u_1, u'_5, u'_2, u'_3, u'_4]$, $[u_2, u'_5, u'_1, u'_3, u'_4]$, and $[u_3, u'_5, u'_1, u'_2, u'_4]$, where the pairs of suspension vertices are (u_1, u'_5), (u_2, u'_5), and (u_3, u'_5), respectively;

3. three more bipyramids

$$[v_2, u'_1, u_2, u_3, v_1] \text{ (colored in purple)}, [v_2, u'_2, u_1, u_3, v_1], \text{ and } [v_2, u'_3, u_1, u_2, v_1],$$

where the pairs of suspension vertices are (v_2, u'_1), (v_2, u'_2), and (v_2, u'_3), respectively;

4. another simplex $[v_2, u_1, u_2, u_3]$ (colored in orange).

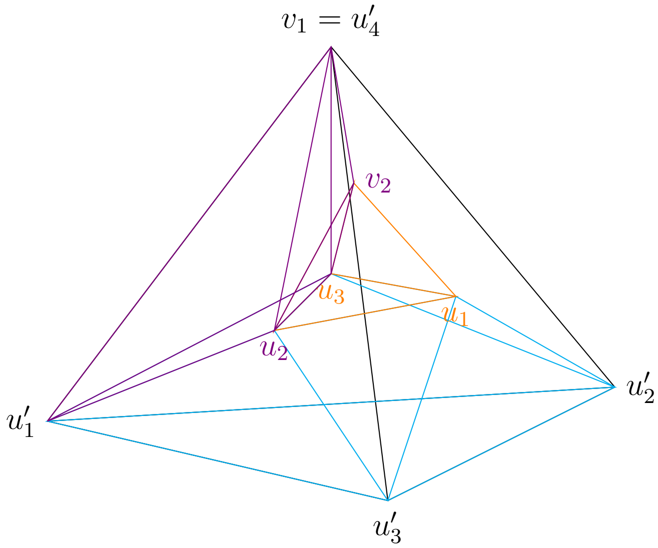

Figure 6.1: Parts of the Schlegel diagrams of P'_9.

The list of facets shows that P_9 is 2-simplicial. The f-vector of P_9 is symmetric, namely, $f(P_9) = (9, 26, 26, 9)$. Thus, by Corollary 3.2, P_9 is also 2-simple. Furthermore, P_9 has two pairs of a simplex facet and a simple vertex not in that facet: $([v_2, u_1, u_2, u_3], u'_5)$ and $([u'_1, u'_2, u'_3, u'_5], v_2)$. Take two copies of P_9, P^l_9 and P^r_9, and consider the merge $P^l_9 \triangleright P^r_9$ along $[v_2, u_1, u_2, u_3]$ from P^l_9 and u'_5 from P^r_9. Since the facets of P_9 containing u'_5 consist of a simplex and three bipyramids, depending on the order in which we list the neighbors of u'_5, the cross-polytopal facet of P^l_9 will either be merged with a 3-simplex or with a bipyramid of P^r_9, resulting in two distinct combinatorial types of 2-simplicial 2-simple 4-polytopes, each of which has a simplex facet and a simple vertex not in that facet. This observation will allow us to construct exponentially many (in the number of vertices) 2-simplicial 2-simple 4-polytopes. We will return to this discussion (and provide many more details) in Section 6.2 after we construct a d-dimensional analog of P_9 for all $d \geqslant 4$; see Theorem 6.13 and Remark 6.14.

How does merging with P_9 affect the f-numbers? Let Q be a 2-simplicial 2-simple 4-polytope that has a simplex facet and a simple vertex not in this facet (for instance, $Q = P_9$).

Then $P_9 \triangleright Q$ and $Q \triangleright P_9$ are both defined and by Corollary 4.13,

$$f(P_9 \triangleright Q) - f(Q) = f(Q \triangleright P_9) - f(Q) = f(P_9) - \left(\binom{5}{1}, \binom{5}{2}, \binom{5}{3}, \binom{5}{4}\right)$$
$$= (9, 26, 26, 9) - (5, 10, 10, 5) = (4, 16, 16, 4).$$

Recall that the toric g_2-number of a 2-simplicial 4-polytope is given by $g_2^{\text{toric}} = f_1 - 4f_0 + 10$ and that any polytope with $g_2^{\text{toric}} = 0$ is called an *elementary* polytope. It then follows that P_9 is an elementary polytope and that $g_2^{\text{toric}}(P_9 \triangleright Q) = g_2^{\text{toric}}(Q \triangleright P_9) = g_2^{\text{toric}}(Q)$. In other words, if Q is also an elementary polytope, then so are $P_9 \triangleright Q$ and $Q \triangleright P_9$. (Elementary polytopes play an important role in the Lower Bound Theorem, see [Kal87].)

It is worth pointing out that if one applies to Q the second construction from [PW06, Section 3.2], the resulting polytope $\mathcal{I}^2(Q)$ has the same f-vector as $f(P_9 \triangleright Q) = f(Q \triangleright P_9)$; see [PW06, Theorem 3.7]. At the same time, both polytopes $P_9 \triangleright Q$ and $Q \triangleright P_9$ are different from $\mathcal{I}^2(Q)$. Indeed, merging with P_9, on the left or on the right, always generates a facet (contributed by the cross-polytopal facet of P_9) that is isomorphic to either CP or the connected sum of CP with another 3-polytope, while in the second construction of [PW06], all new facets are stacked 3-polytopes with either 4, 5, or 6 vertices.

Our next task is to describe another highly-neighborly 2-simplicial 2-simple 4-polytope with a simplex facet and a simple vertex. This polytope has 18 vertices and we denote it by P_{18}.

Construction 6.2. We start with a regular 3-simplex $F = [v_1, v_2, v_3, v_4]$ in $\mathbb{R}^3 \times \{0\}$. Specifically, let

$$v_1 = (0, 0, 0, 0), \quad v_2 = (2, 2, 0, 0), \quad v_3 = (2, 0, 2, 0), \quad v_4 = (0, 2, 2, 0). \tag{6.1}$$

Define $u = (1, 1, 1, h)$ for some $h > 0$. Let $0 < \epsilon \ll 1$. For all distinct $1 \leqslant i, j, k \leqslant 4$, let

$$u_{ji,k} = u_{ij,k} = \frac{1}{2}(v_i + v_j) + \epsilon(u + v_k - v_i - v_j).$$

That is,

$$\begin{aligned}
u_{12,3} &= (1+\epsilon, 1-\epsilon, 3\epsilon, h\epsilon), & u_{12,4} &= (1-\epsilon, 1+\epsilon, 3\epsilon, h\epsilon),\\
u_{13,2} &= (1+\epsilon, 3\epsilon, 1-\epsilon, h\epsilon), & u_{13,4} &= (1-\epsilon, 3\epsilon, 1+\epsilon, h\epsilon),\\
u_{14,2} &= (3\epsilon, 1+\epsilon, 1-\epsilon, h\epsilon), & u_{14,3} &= (3\epsilon, 1-\epsilon, 1+\epsilon, h\epsilon),\\
u_{23,1} &= (2-3\epsilon, 1-\epsilon, 1-\epsilon, h\epsilon), & u_{23,4} &= (2-3\epsilon, 1+\epsilon, 1+\epsilon, h\epsilon),\\
u_{24,1} &= (1-\epsilon, 2-3\epsilon, 1-\epsilon, h\epsilon), & u_{24,3} &= (1+\epsilon, 2-3\epsilon, 1+\epsilon, h\epsilon),\\
u_{34,1} &= (1-\epsilon, 1-\epsilon, 2-3\epsilon, h\epsilon), & u_{34,2} &= (1+\epsilon, 1+\epsilon, 2-3\epsilon, h\epsilon).
\end{aligned}$$

Note that each $u_{ij,k}$ can be viewed as a certain perturbation of the barycenter of $[v_i, v_j]$ that keeps it in the hyperplane defined by $[u, v_i, v_j, v_k]$. Note also that the set of vertices $\{u_{1i,j} : \{i, j\} \in \{2, 3, 4\}\}$ forms a hexagon H_1 that lies in the plane defined by equations $x_1 + x_2 + x_3 = 2 + 3\epsilon, x_4 = h\epsilon$. Similarly, the sets of vertices

$$\{u_{2i,j} : \{i, j\} \subset \{1, 3, 4\}\}, \quad \{u_{3i,j} : \{i, j\} \subset \{1, 2, 4\}\}, \quad \text{and} \quad \{u_{4i,j} : \{i, j\} \subset \{1, 2, 3\}\}$$

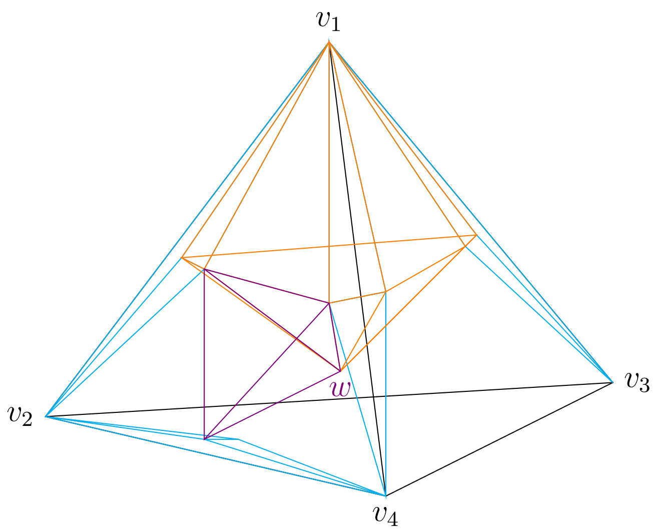

Figure 6.2: Parts of the Schlegel diagrams of P'_{18}.

form hexagons H_2, H_3, H_4 in the planes defined by equations

$$\begin{aligned} &\{x_1 + x_2 - x_3 = 2 - 3\epsilon,\ x_4 = h\epsilon\}, \\ &\{x_1 - x_2 + x_3 = 2 - 3\epsilon,\ x_4 = h\epsilon\}, \\ \text{and} \qquad &\{-x_1 + x_2 + x_3 = 2 - 3\epsilon,\ x_4 = h\epsilon\}, \end{aligned}$$

respectively. It follows that

$$\begin{aligned} \operatorname{aff}(v_1 \cup H_1) &= \{\mathbf{x} \in \mathbb{R}^4 \ :\ -h\epsilon(x_1 + x_2 + x_3) + (2 + 3\epsilon)x_4 = 0\}, \\ \operatorname{aff}(v_2 \cup H_2) &= \{\mathbf{x} \in \mathbb{R}^4 \ :\ h\epsilon(x_1 + x_2 - x_3) + (2 + 3\epsilon)x_4 = 4h\epsilon\}, \\ \operatorname{aff}(v_3 \cup H_3) &= \{\mathbf{x} \in \mathbb{R}^4 \ :\ h\epsilon(x_1 + x_3 - x_2) + (2 + 3\epsilon)x_4 = 4h\epsilon\}, \\ \operatorname{aff}(v_4 \cup H_4) &= \{\mathbf{x} \in \mathbb{R}^4 \ :\ h\epsilon(x_2 + x_3 - x_1) + (2 + 3\epsilon)x_4 = 4h\epsilon\}. \end{aligned}$$

The intersection of these four hyperplanes is the point $(1, 1, 1, \frac{3h\epsilon}{2+3\epsilon})$; we denote it by w.

Define P'_{18} as the convex hull of all 17 vertices $\{w, v_1, \ldots, v_4, u_{ij,k} \ :\ 1 \leqslant i, j, k \leqslant 4\}$. When ϵ is very small, the polytope P'_{18} has the following 19 facets (see Figure 6.2 for part of the Schlegel diagram). We used $\epsilon = 0.05$, $h = 2$ and verified this list with software SAGE.

1. Six simplices of the form $[v_i, v_j, u_{ij,k}, u_{ij,m}]$, where $\{i, j, k, m\} = [4]$. Parts of four of them are shown in blue in Figure 6.2.

2. Four simplices of the form $[u_{ij,k}, u_{ik,j}, u_{jk,i}, w]$, where $1 \leqslant i, j, k \leqslant 4$ are distinct. One such simplex is shown in purple in Figure 6.2.

3. The simplex $[v_1, v_2, v_3, v_4]$.

4. Four polytopes of the form $[v_i, w, u_{ij,k}, u_{ij,m}, u_{ik,j}, u_{ik,m}, u_{im,j}, u_{im,k}]$. Each is the suspension over H_i, with suspension vertices v_i and w. (Here $\{i, j, k, m\} = [4]$.) One such polytope is shown in orange in Figure 6.2.

5. Four cross-polytopes of the form $[v_i, v_j, v_k, u_{ij,k}, u_{ik,j}, u_{jk,i}]$, where $1 \leqslant i, j, k \leqslant 4$ are distinct.

To complete the construction of P_{18}, we apply a projective transformation π to P'_{18} to ensure that the adjacent facets of $G = [v_1, v_2, v_3, v_4]$, i.e., the four cross-polytopes from the last item, intersect at a point w' beyond G. We let $P_{18} = \text{conv}(\pi(P'_{18}) \cup w')$. Then G is not a facet of P_{18} and each facet $[v_i, v_j, v_k, u_{ij,k}, u_{ik,j}, u_{jk,i}]$ is replaced by its connected sum with $[v_i, v_j, v_k, w']$. It can be checked that $f(P_{18}) = (18, 64, 64, 18)$. Since P_{18} is a 2-simplicial 4-polytope that has $f_1 = f_2$, it follows by Corollary 3.2 that P_{18} is also 2-simple. A direct computation shows that $g_2^{\text{toric}}(P_{18}) = 2$. In other words, P_{18} is not elementary.

Observe that P_{18} has a simple vertex w' and many simplex facets not containing w' (see the first item in the list). Thus we can iteratively merge P_{18} with itself and obtain an infinite sequence of 2-simplicial 2-simple 4-polytopes, each having at least one simplex facet and one simple vertex. By Corollary 4.13, any polytope obtained by merging $k \geqslant 1$ copies of P_{18} will have $5+13k$ vertices and $g_2^{\text{toric}} = 2k$. Other families of 2-simplicial 2-simple 4-polytopes where the kth polytope has $g_2^{\text{toric}} = 2k$ (but $f_0 = 10 + 4k$) were constructed in [PZ04, Corollary 4.2].

To close this section, we propose the following problem.

Question 6.3. Is there a sequence of 2-simplicial 2-simple 4-polytopes that approximate the unit ball?

In light of [ANS16, Theorem 3.2], it is natural to conjecture that if such a sequence of 4-polytopes $\{Q_i\}$ exists, then $\lim_{i\to\infty} g_2^{\text{toric}}(Q_i) = \infty$.

6.2. Many $(d-2)$-simplicial 2-simple d-polytopes

In this section we construct a d-dimensional analog of P_9 for all $d \geqslant 4$. We then use this polytope along with Corollary 4.7 to show that there are $2^{\Omega(N)}$ combinatorial types of $(d-2)$-simplicial 2-simple d-polytopes with at most N vertices and an additional property that each of these polytopes has a simplex facet and a simple vertex.

As in Section 6.1, the d- and $(d-1)$-dimensional cross-polytopes are used frequently, and we abbreviate them as CP. To start, we introduce the notion of a pseudo-regular CP and prove some of its properties. Let $\mathbf{0}$ denote the origin of $\mathbb{R}^{d-1}$.

Definition 6.4. Let $G \subset \mathbb{R}^{d-1}$ be a regular $(d-1)$-simplex centered at the origin, let $G^* \subset \mathbb{R}^{d-1}$ be the dual of G, and let $\alpha > 0$ be a real number. Assume also that G is contained in the interior of αG^*, denoted $\text{int}(\alpha G^*)$. A d-cross-polytope is called *pseudo-regular* if it is congruent to $\text{conv}(G \times \{1\} \cup \alpha G^* \times \{-1\})$.

Consider a regular simplex $G = [\mu_1, \dots, \mu_d] \subset \mathbb{R}^{d-1}$ centered at the origin and let $\alpha > 0$. Then $\alpha G^* = [\mu'_1, \dots, \mu'_d] \subset \mathbb{R}^{d-1}$ is also a regular simplex centered at the origin. We label the vertices in such a way that μ'_i is an outer normal vector to the facet $[\mu_1, \dots, \widehat{\mu_i}, \dots, \mu_d]$

of G. By our assumptions on G, this is equivalent to labeling the vertices so that for all $i \in [d]$, $\mu'_i = a\sum_{j\in[d]\setminus i}\mu_j = -a\mu_i$, where a is a positive scalar independent of i.

For a nonempty subset I of $[d]$, let $G_I = [\mu_i : i \in I]$ be a face of G and $G'_I = [\mu'_i : i \in I]$ be a face of αG^*; let $\beta_I = \frac{1}{|I|}\sum_{i\in I}\mu_i$ be the barycenter of G_I and $\beta'_I = \frac{1}{|I|}\sum_{i\in I}\mu'_i$ be the barycenter of G'_I. Since for all $i \in [d]$, $\mu'_i = a\sum_{j\in[d]\setminus i}\mu_j = -a\mu_i$, it follows that for any proper subset I of $[d]$, $\sum_{i\in I}\mu_i = -\frac{1}{a}\sum_{i\in I}\mu'_i = \frac{1}{a}\sum_{j\in[d]\setminus I}\mu'_j$. Thus, β_I is a positive multiple of $\beta'_{[d]\setminus I}$, and so the ray from $\mathbf{0}$ and through β_I coincides with the ray from $\mathbf{0}$ and through $\beta'_{[d]\setminus I}$. Furthermore, since G is regular, the distance from $\mathbf{0}$ to β_I is the same for all k-subsets I of $[d]$; we denote it by ρ_k and note that $\rho_1 > \cdots > \rho_{d-1}$. Similarly, for all k-subsets J of $[d]$, the distance from $\mathbf{0}$ to β'_J is the same number ρ'_k, where $\rho'_1 > \cdots > \rho'_{d-1}$. Finally, since $G \subset \operatorname{int}(\alpha G^*)$, $\rho'_{d-1} > \rho_1$. To summarize,

$$\rho'_1 > \cdots > \rho'_{d-1} > \rho_1 > \cdots > \rho_{d-1}. \tag{6.2}$$

Consider the d-cross-polytope $\mathrm{CP} = \operatorname{conv}(G\times\{1\}\cup\alpha G^*\times\{-1\})$. We label the vertices of CP by $u_j = (\mu_j, 1)$ and $u'_j = (\mu'_j, -1)$ (for $j = 1,\dots,d$), so that $G\times\{1\} = [u_1,\dots,u_d]$ and $\alpha G^*\times\{-1\} = [u'_1,\dots,u'_d]$. For a subset I of $[d]$, we denote the barycenter of $G_I\times\{1\}$ by b_I and the barycenter of and $G'_I\times\{-1\}$ by b'_I. Finally, we let H_I denote the hyperplane in $\mathbb{R}^d$ determined by the following set of d points: $\{u_i : i\in I\}\cup\{u'_j : j\in[d]\setminus I\}$.

Lemma 6.5. *Let $0 \leqslant k \leqslant d$. Then all hyperplanes H_I, where $I \subseteq [d], |I| = k$, intersect the x_d-axis at the same point. When $0 < k < d$, the dth coordinate of this point is > 1.*

Proof. First note that $H_{[d]}$ and $H_\varnothing$ intersect the x_d-axis at $\mathbf{e}_d := (0,\dots,0,1)$ and $-\mathbf{e}_d$, respectively. Now let I be any k-subset of $[d]$, where $1 \leqslant k \leqslant d-1$. Consider the points b_I and $b'_{[d]\setminus I}$. Both of them lie in H_I; hence, so does the line $\ell = \operatorname{aff}(b_I, b'_{[d]\setminus I})$.

We claim that ℓ intersects the x_d-axis. Consequently,

$$H_I \cap x_d\text{-axis} = \ell \cap x_d\text{-axis}.$$

To prove the claim, consider the lines $\operatorname{aff}(\mathbf{e}_d, b_I)$ and $\operatorname{aff}(-\mathbf{e}_d, b'_{[d]\setminus I})$. By discussion following Definition 6.4, these lines are parallel, and thus determine a 2-dimensional plane $\mathcal{L}$. For the rest of the proof, we work in this plane. It contains ℓ and the x_d-axis. Also, since, β_I is a positive multiple of $\beta'_{[d]\setminus I}$, the points b_I and $b'_{[d]\setminus I}$ lie on the same side of the x_d-axis in $\mathcal{L}$. Finally, since the distance from b_I to the x_d-axis is ρ_k, the distance from $b'_{[d]\setminus I}$ to the x_d-axis is ρ'_{d-k}, and $\rho'_{d-k} > \rho_k$, it follows that ℓ and the x_d-axis are not parallel. Hence they intersect and the point of intersection, which we denote by $a_I = (0,\dots,0,c_I)$, satisfies $c_I > 1$. This proves the claim.

To complete the proof of the lemma, it remains to show that c_I depends only on $|I| = k$. Indeed, consider triangles $[a_I, \mathbf{e}_d, b_I]$ and $[a_I, -\mathbf{e}_d, b'_{[d]\setminus I}]$. They are similar; hence,

$$\frac{c_I - 1}{\rho_k} = \frac{\operatorname{dist}(a_I, \mathbf{e}_d)}{\operatorname{dist}(\mathbf{e}_d, b_I)} = \frac{\operatorname{dist}(a_I, -\mathbf{e}_d)}{\operatorname{dist}(-\mathbf{e}_d, b'_{[d]\setminus I})} = \frac{c_I + 1}{\rho'_{d-k}}.$$

Solving this equation yields $c_I = \frac{\rho'_{d-k}+\rho_k}{\rho'_{d-k}-\rho_k}$. The result follows. □

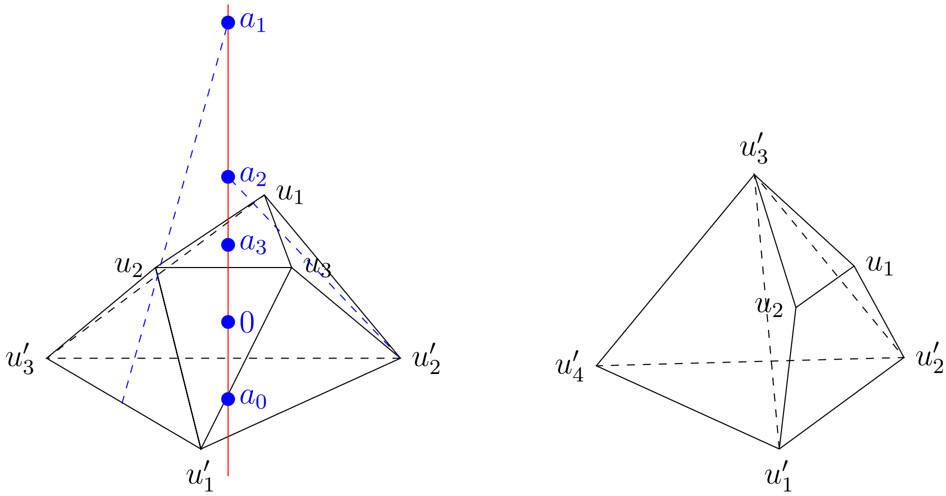

Figure 6.3: Left: a pseudo-regular CP of dimension 3 and the points $\{a_0, \dots, a_3\}$. Right: The polytope $P^{3,1}$.

Let $0 \leqslant k \leqslant d$. In view of Lemma 6.5, we denote by a_k the point of intersection of H_I and the x_d-axis, where I is any subset of $[d]$ of size k, and by $c_k := \frac{\rho'_{d-k}+\rho_k}{\rho'_{d-k}-\rho_k}$ the last coordinate of a_k; see Figure 6.3 for an illustration in dimension 3.

Corollary 6.6. *The heights of points $a_1, \dots, a_d$ satisfy $c_1 > \cdots > c_{d-1} > c_d = 1$. In particular, if q is a point on the x_d-axis that lies strictly between a_{k-1} and a_k, then q is beneath the facet $H_I = [u_i, u'_j : i \in I, j \in [d]\backslash I]$ of the CP if $|I| \leqslant k-1$, and beyond the facet H_I if $|I| \geqslant k$.*

Proof. By equation (6.2), for all $1 \leqslant k \leqslant d-1$, $\rho'_{d-k} - \rho_k > 0$. Hence $c_k = \frac{\rho'_{d-k}+\rho_k}{\rho'_{d-k}-\rho_k} > 1 = c_d$. Furthermore, for $2 \leqslant k \leqslant d-1$,

$$\begin{aligned} c_k - c_{k-1} &= \frac{\rho'_{d-k}+\rho_k}{\rho'_{d-k}-\rho_k} - \frac{\rho'_{d-k+1}+\rho_{k-1}}{\rho'_{d-k+1}-\rho_{k-1}} \\ &= 2\left(\frac{\rho_k}{\rho'_{d-k}-\rho_k} - \frac{\rho_{k-1}}{\rho'_{d-k+1}-\rho_{k-1}}\right) \\ &= 2\left(\frac{1}{\frac{\rho'_{d-k}}{\rho_k}-1} - \frac{1}{\frac{\rho'_{d-k+1}}{\rho_{k-1}}-1}\right) < 0, \end{aligned}$$

where the last step follows from the fact that $\rho'_{d-k} > \rho'_{d-k+1} > \rho_{k-1} > \rho_k$; see eq. (6.2). □

Definition 6.7. Let $\mathrm{CP} = \mathrm{conv}(G \times \{1\} \cup \alpha G^* \times \{-1\})$ be a pseudo-regular d-cross-polytope. The set $\{a_k = \cap_{I \subset [d], |I|=k} H_I : 1 \leqslant k \leqslant d\}$ is called the *sequence of points associated with* CP.

Our construction of a $(d-2)$-simplicial 2-simple polytope starts with a certain d-polytope $P^{d,1}$ described in Definition 6.8 and proceeds by recursively adding to $P^{d,1}$ a total of $d-3$ additional vertices; see Figure 6.3 for an illustration of $P^{3,1}$. As we will see below, one of the facets of $P^{d,1}$ is a pseudo-regular CP (of dimension $d-1$). By a slight abuse of notation, we continue to label the vertices of this facet by $u_1,\ldots,u_{d-1},u'_1,\ldots,u'_{d-1}$.

Definition 6.8. Let $\Sigma = [u'_1,\ldots,u'_{d+1}]$ be a regular d-simplex. Choose an arbitrary $0<\epsilon \ll \operatorname{dist}(u'_1,u'_2)$. For $1\leqslant i\leqslant d-1$, let p_i be the barycenter of the $(d-2)$-face $\Sigma\backslash u'_iu'_{d+1}$, and let $u_i := p_i+\epsilon(p_i-u'_{d+1})$. We define $P^{d,1}$ as $\operatorname{conv}(u'_1,\ldots,u'_{d+1},u_1,\ldots,u_{d-1})$.

Since p_i is the barycenter of the $(d-2)$-face $\Sigma\backslash u'_iu'_{d+1}$, it follows that $[p_1,\ldots,p_{d-1}]$ is a regular $(d-2)$-simplex and $[p_1,\ldots,p_{d-1},u'_1,\ldots,u'_{d-1}]$ is a pseudo-regular $(d-1)$-cross-polytope. By our choice of u_i, $[u_1,\ldots,u_{d-1}]$ is a regular $(d-2)$-simplex obtained from $[p_1,\ldots,p_{d-1}]$ by dilation with factor $(1+\epsilon)$ (where ϵ is small) followed by translation in the direction perpendicular to $\operatorname{aff}(p_1,\ldots,p_{d-1},u'_1,\ldots,u'_{d-1}) = \operatorname{aff}(\Sigma\backslash u'_{d+1})$. In particular, $\operatorname{aff}(u_1,\ldots,u_{d-1})$ is parallel to $\operatorname{aff}(u'_1,\ldots,u'_{d-1})$ and $\mathrm{CP} := [u_1,\ldots,u_{d-1},u'_1,\ldots,u'_{d-1}]$ is also a pseudo-regular $(d-1)$-cross-polytope.

This discussion shows that the polytope $P^{d,1}$ is the union of the simplex Σ and the pyramid with apex u'_d over the cross-polytope CP (glued along the simplex $[u'_1,\ldots,u'_d]$). Furthermore, for each $1\leqslant i\leqslant d-1$, the points $\{u_i,u'_1,\ldots,\widehat{u'_i},\ldots,u'_d,u'_{d+1}\}$ lie in the same hyperplane, and, in this hyperplane, the sets $\operatorname{conv}(u_i,u'_{d+1})$ and $\operatorname{conv}(u'_1,\ldots,\widehat{u'_i},\ldots u'_d)$ intersect in their relative interiors. For $1\leqslant k\leqslant d-1$, let $\mathcal{H}_k$ be the set of facets H of CP with $|H\cap\{u_1,\ldots,u_{d-1}\}|=k$. (Each such H is a $(d-2)$-face of $P^{d,1}$.) Also, let $H^+ := H\cap[u_1,\ldots,u_{d-1}]$ and $H^- := H\cap[u'_1,\ldots,u'_{d-1}]$. Let $v_0 := u'_{d+1}$ and $v_1 := u'_d$. It follows that $P^{d,1}$ has the following facets:

1. The simplex $\Sigma\backslash u'_d$ and the pseudo-regular cross-polytope CP.
2. $d-1$ bipyramids of the form $\operatorname{conv}(H\cup\{v_0,v_1\})$, where $H\in\mathcal{H}_1$; the boundary complex of such facet is $\partial(\overline{V(H^+)\cup v_0}) * \partial(\overline{V(H^-)\cup v_1})$.
3. $2^{d-1}-d$ simplex facets of the form $\operatorname{conv}(H\cup v_1)$, where $H\in\cup_{2\leqslant k\leqslant d-1}\mathcal{H}_k$.

In particular, CP is adjacent to all other facets of $P^{d,1}$.

Since CP is pseudo-regular, by Lemma 6.5, there is a sequence of points associated with CP (lying in $\operatorname{aff}(\mathrm{CP})$): $a_i = \cap_{F\in\mathcal{H}_i}\operatorname{aff}(F)$, $1\leqslant i\leqslant d-1$; see Definition 6.7. The points $\{a_i : 1\leqslant i\leqslant d-1\}$ all lie on the line through the barycenters $b_{[d-1]}$ of $[u_1,\ldots,u_{d-1}]$ and $b'_{[d-1]}$ of $[u'_1,\ldots,u'_{d-1}]$, and, according to Corollary 6.6, they appear on this line in the order $a_1,\ldots,a_{d-2},a_{d-1}$, with a_{d-2} closest to $a_{d-1}=b_{[d-1]}$ and a_1 farthest from $b_{[d-1]}$.

We are now ready for the main definition of this section:

Definition 6.9. Consider the sequence of points $\{a_i : 1\leqslant i\leqslant d-2\}$ associated with the facet $\mathrm{CP}=[u'_1,\ldots,u'_{d-1},u_1,\ldots,u_{d-1}]$ of $P^{d,1}$. Let $v_1=u'_d$. Inductively, for $2\leqslant i\leqslant d-2$, choose a point v_i in the relative interior of the line segment $[a_i,v_{i-1}]$ and let $P^{d,i}=\operatorname{conv}(P^{d,i-1}\cup v_i)$. Finally, let $P^d=P^{d,d-2}$.

The process of adding vertices similar to the one described in Definition 6.9 is illustrated in Figure 6.4, where the vertices are added to the pyramid over a hexagon. (Unfortunately, Definition 6.9 itself is non-vacuous only when $d\geqslant 4$, and as such is hard to illustrate.)

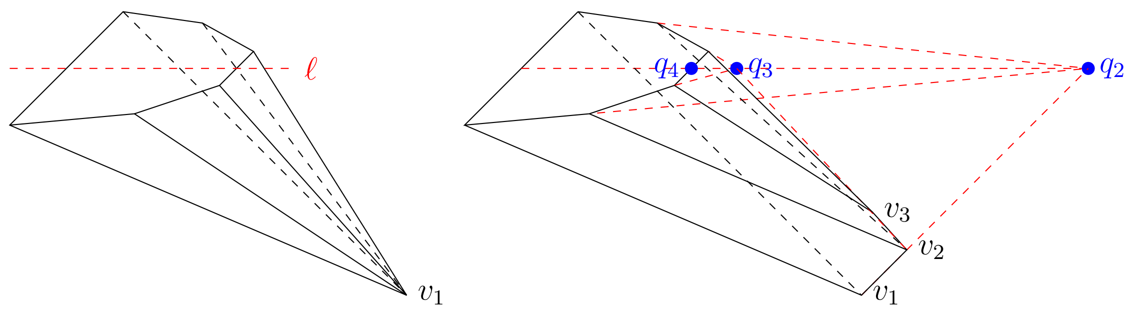

Figure 6.4: Left: The pyramid over a hexagon H symmetric about the line ℓ. Right: A new 3-polytope obtained by adding vertices v_2 and v_3, with v_{i+1} in the interior of the line segment $[q_{i+1}, v_i]$; here q_{i+1} is the intersection of affine spans of the appropriate symmetric edges of H.

Our next goal is to prove that P^d is the promised high-dimensional analog of the 4-polytope P_9; see Theorem 6.11. This requires describing the facets of P^d. We do so by induction, showing that for $2 \leqslant k \leqslant d-2$, the set of facets of $P^{d,k}$ is obtained from that of $P^{d,k-1}$ as follows.

1. For each $H \in \cup_{k+1 \leqslant i \leqslant d-1} \mathcal{H}_i$, the facet $\mathrm{conv}(H \cup v_{k-1})$ of $P^{d,k-1}$ gets replaced with the facet $\mathrm{conv}(H \cup v_k)$.
2. For each $H \in \mathcal{H}_k$, the facet $\mathrm{conv}(H \cup v_{k-1})$ of $P^{d,k-1}$ gets replaced with the facet $\mathrm{conv}(H \cup \{v_{k-1}, v_k\})$ whose boundary complex is $\partial(\overline{V(H^+) \cup v_{k-1}}) * \partial(\overline{V(H^-) \cup v_k})$. There are $\binom{d-1}{k}$ such facets.
3. The rest of the facets of $P^{d,k-1}$ remain unchanged.

In particular, it follows by induction that CP is a facet of $P^{d,k}$ and that it is adjacent to *all* other facets of $P^{d,k}$, and, furthermore, that the collection of facets in item 3 consists of $\Sigma \backslash u'_d$, CP, and for each $1 \leqslant i \leqslant k-1$ and $H \in \mathcal{H}_i$, a facet that contains $H \cup v_i$.

The proof is based on:

Claim 1: *For every $H \in \mathcal{H}_k$, $v_k \in \mathrm{aff}(H \cup v_{k-1})$.* This is because a_k lies on the hyperplane $\mathrm{aff}(H)$, and $v_k \in [a_k, v_{k-1}]$.

Claim 2: *For $i > k$ and $H \in \mathcal{H}_i$, v_k is beyond $\mathrm{conv}(H \cup v_{k-1})$.* Indeed, by Corollary 6.6, in $\mathrm{aff}(\mathrm{CP})$, a_k is beyond H. Hence in $\mathrm{aff}(\mathrm{CP} \cup v_{k-1}) = \mathbb{R}^d$, the point $v_k \in \mathrm{int}[a_k, v_{k-1}]$ is beyond $\mathrm{conv}(H \cup v_{k-1})$.

Claim 3: *v_k is beneath the rest of the facets of $P^{d,k-1}$.* First, as easily seen from the definition of sequences $\{a_j\}$ and $\{v_j\}$, v_k is beneath both $\Sigma \backslash u'_d$ and CP. Thus it only remains to show that if G is a facet of $P^{d,k-1}$ that contains $H \cup v_i$ for some $i < k$ and $H \in \mathcal{H}_i$, then v_k is beneath G. This follows from Corollary 6.6 along with another simple induction on j, where $i+1 \leqslant j \leqslant k$. For the base case, by Corollary 6.6, in $\mathrm{aff}(\mathrm{CP})$, a_{i+1} is beneath H. Hence, in $\mathrm{aff}(\mathrm{CP} \cup v_i) = \mathbb{R}^d$,

a_{i+1} is beneath G. Since v_{i+1} is in the interior of $[v_i, a_{i+1}]$, v_{i+1} is also beneath G. The inductive step is very similar: by the inductive hypothesis, v_j is beneath G and by Corollary 6.6, so is a_{j+1}; hence $v_{j+1} \in [v_j, a_{j+1}]$ is also beneath G. The claim follows.

The above three claims uniquely determine the facets of $P^{d,k}$. Claim 3 implies that the facets of $P^{d,k-1}$ from item 3 in the list are unaffected by adding v_k, and hence remain facets of $P^{d,k}$.

Claim 1 implies that for every $H \in \mathcal{H}_k$, the facet $\operatorname{conv}(H \cup v_{k-1})$ of $P^{d,k-1}$ is replaced by a new facet $\operatorname{conv}(H \cup \{v_k, v_{k-1}\})$. Note that the barycenter b_{H^+} of H^+ lies on the line segment connecting a_k and the barycenter b_{H^-} of H^- (see the proof of Lemma 6.5). Hence, if v_k is an interior point of the line segment $[a_k, v_{k-1}]$, then $[b_{H^+}, v_{k-1}]$ and $[b_{H^-}, v_k]$ intersect at a point p. This implies that $\operatorname{conv}(H^+ \cup v_{k-1}) \cap \operatorname{conv}(H^- \cup v_k) = p$. Thus the boundary complex of $\operatorname{conv}(H \cup \{v_k, v_{k-1}\})$ must be $\partial(\overline{V(H^+) \cup v_{k-1}}) * \partial(\overline{V(H^-) \cup v_k})$. These facets are exactly[2] the facets of $P^{d,k}$ containing $v_{k-1}v_k$.

Finally, the rest of the facets of $P^{d,k}$ are those arising from $H \in \mathcal{H}_i$ for $i > k$. By Claim 2 and the previous paragraph, they must be of the form $\operatorname{conv}(H \cup v_k)$, replacing $\operatorname{conv}(H \cup v_{k-1})$ of $P^{d,k-1}$.

We thus obtain the following result (for convenience we let $v_{d-1} = v_{d-2}$):

Lemma 6.10. *The polytope P^d in Definition 6.9 has $3(d-1)$ vertices and $2^{d-1}+1$ facets. The vertex set of P^d is*

$$\{u_1, \dots, u_{d-1}, u'_1, \dots, u'_{d-1}, u'_d = v_1, u'_{d+1} = v_0, v_2, \dots, v_{d-3}, v_{d-2} = v_{d-1}\}.$$

The set of facets of P^d naturally splits into the following d subfamilies:

1. *$\mathcal{F}_0$ consists of the simplex $[u'_1, \dots, u'_{d-1}, u'_{d+1}]$ and the cross-polytope* CP.
2. *For $1 \leqslant k \leqslant d-1$, $\mathcal{F}_k$ consists of $\binom{d-1}{k}$ polytopes of dimension $d-1$ whose boundary complexes are of the form $\partial(\overline{V(H^+) \cup v_{k-1}}) * \partial(\overline{V(H^-) \cup v_k})$, where $H \in \mathcal{H}_k$. In particular, $\mathcal{F}_{d-1} = \{[u_1, \dots, u_{d-1}, v_{d-2}]\}$.*

Theorem 6.11. *The d-polytope P^d is $(d-2)$-simplicial and 2-simple. It has two pairs of a simplex facet and a simple vertex not in that facet; they are $([u_1, \dots, u_{d-1}, v_{d-2}], u'_{d+1})$ and $([u'_1, \dots, u'_{d-1}, u'_{d+1}], v_{d-2})$.*

Proof. Let $U = \{u_1, \dots, u_{d-1}\}$ and let $U' = \{u'_1, \dots, u'_{d-1}\}$. For $M = \{u_{i_1}, \dots, u_{i_k}\} \subseteq U$, we let $M' := \{u'_{i_1}, \dots, u'_{i_k}\} \subseteq U'$. Also, for brevity, we write u, uv, uvw instead of $\{u\}$, $\{u, v\}$, and $\{u, v, w\}$.

The description of facets in Lemma 6.10 guarantees that P^d is $(d-2)$-simplicial. To show that P^d is also 2-simple, it suffices to check that every $(d-3)$-face τ of P^d is contained in exactly three facets. By examining families $\mathcal{F}_i$, $0 \leqslant i \leqslant d-1$, of Lemma 6.10, we see that there are the following possible cases:

[2] To see this, we invite the reader to compute the link of $v_{k-1}v_k$ in the polytopal complex generated by these facets and check that it is a $(d-3)$-dimensional pseudomanifold (i.e., every ridge is in two facets). Thus it must coincide with the link of $v_{k-1}v_k$ in the boundary of $P^{d,k}$.

1. $u'_{d+1} \in V(\tau)$. In this case, $V(\tau) \subset U' \cup u'_d u'_{d+1}$. If u'_d is also in τ, then τ is contained in three bipyramids from $\mathcal{F}_1$; otherwise, τ is contained in two bipyramids from $\mathcal{F}_1$ and the simplex $[u'_1, \ldots, u'_{d-1}, u'_{d+1}]$ from $\mathcal{F}_0$.

2. $V(\tau) \subset U'$. In this case, τ is contained in the cross-polytope and the simplex from $\mathcal{F}_0$, and one bipyramid from $\mathcal{F}_1$.

3. $V(\tau) = K \cup M'$, where $K \sqcup M \sqcup u_\ell = U$ and $|K| = i$ for some $1 \leqslant \ell \leqslant d-1$ and $1 \leqslant i \leqslant d-2$. Then τ is a face of CP from $\mathcal{F}_0$, of $\partial(\overline{K \cup u_\ell v_i}) * \partial(\overline{M' \cup v_{i+1}})$ from $\mathcal{F}_{i+1}$, and of $\partial(\overline{K \cup v_{i-1}}) * \partial(\overline{M' \cup u'_\ell v_i})$ from $\mathcal{F}_i$.

4. $V(\tau) = K \cup M' \cup v_i$, where $1 \leqslant i \leqslant d-2$ and $K \sqcup M \sqcup u_j u_k = U$ for some $1 \leqslant j < k \leqslant d-1$. There are two cases:

   (a) $|K| = i-1$. Then τ is a face of $\partial(\overline{K \cup u_j u_k v_i}) * \partial(\overline{M' \cup v_{i+1}})$ from $\mathcal{F}_{i+1}$ and of two facets $\partial(\overline{K \cup u_j v_{i-1}}) * \partial(\overline{M' \cup u'_k v_i})$, $\partial(\overline{K \cup u_k v_{i-1}}) * \partial(\overline{M' \cup u'_j v_i})$ from $\mathcal{F}_i$.

   (b) $|K| = i$ (and so, $i < d-2$). Then τ is a face of $\partial(\overline{K \cup v_{i-1}}) * \partial(\overline{M' \cup u'_j u'_k v_i})$ from $\mathcal{F}_i$. and of two facets $\partial(\overline{K \cup u_j v_i}) * \partial(\overline{M' \cup u'_k v_{i+1}})$, $\partial(\overline{K \cup u_k v_i}) * \partial(\overline{M' \cup u'_j v_{i+1}})$ from $\mathcal{F}_{i+1}$.

5. $V(\tau) = K \cup M' \cup v_{i-1} v_i$, where $2 \leqslant i \leqslant d-2$ and $K \sqcup M \sqcup u_j u_k u_\ell = U$ for some $1 \leqslant j < k < \ell \leqslant d-1$. There are two cases:

   (a) $|K| = i-2$. Then τ is contained in three facets from $\mathcal{F}_i$:
$$\begin{aligned} &\partial(\overline{K \cup u_k u_\ell v_{i-1}}) * \partial(\overline{M' \cup u'_j v_i}),\\ &\partial(\overline{K \cup u_j u_\ell v_{i-1}}) * \partial(\overline{M' \cup u'_k v_i}),\\ \text{and}\quad &\partial(\overline{K \cup u_j u_k v_{i-1}}) * \partial(\overline{M' \cup u'_\ell v_i}). \end{aligned}$$

   (b) $|K| = i-1$. Then τ is contained in three facets from $\mathcal{F}_i$:
$$\begin{aligned} &\partial(\overline{K \cup u_\ell v_{i-1}}) * \partial(\overline{M' \cup u'_j u'_k v_i}),\\ &\partial(\overline{K \cup u_j v_{i-1}}) * \partial(\overline{M' \cup u'_k u'_\ell v_i}),\\ \text{and}\quad &\partial(\overline{K \cup u_k v_{i-1}}) * \partial(\overline{M' \cup u'_j u'_\ell v_i}). \end{aligned}$$

The result follows. $\square$

Remark 6.12. It is worth noting that the polytope P^d is d-dimensional and has $3d-3$ vertices. This is the smallest number of vertices that a non-simplex $(d-2)$-simplicial 2-simple d-polytope can have in dimensions $d = 3, 4, 5$ (cf. Proposition 3.3).

As the last theorem of the paper, we show that iteratively merging n copies of P^d from Theorem 6.11 results in exponentially many (w.r.t. the number of vertices) combinatorially distinct $(d-2)$-simplicial 2-simple d-polytopes. Recall from Theorem 6.11 that

- The polytope P^d has two simple vertices u'_{d+1} and v_{d-2}, and two simplex facets $F' := [u'_1, \dots, u'_{d-1}, u'_{d+1}]$ and $F := [u_1, \dots, u_{d-1}, v_{d-2}]$; u'_{d+1} is a vertex of F' but not of F, and v_{d-2} is a vertex of F but not of F'. All other facets containing u'_{d+1} and v_{d-2} are bipyramids.

- The CP facet $[u_1, \dots, u_{d-1}, u'_1, \dots, u'_{d-1}]$ is adjacent to all other facets of P^d.

Let T_1 and T_2 be two copies of P^d with the copy of CP, F, and F' in T_i denoted by CP_i, F_i, and F'_i, respectively, and the copy of u'_{d+1} from T_2 denoted by w. We merge T_1 and T_2 along F_1 and w. Since CP_1 is adjacent to F_1, and since w is in one simplex facet (namely F'_2) and $d-1$ bipyramids, exactly as in the 4-dimensional case, there are two ways to merge leading to two distinct combinatorial types (recall that σ_{d-1} denotes a $(d-1)$-simplex):

- In $T_1 \triangleright T_2$, the facet CP_1 gets merged with the simplex F'_2. The merged facet is then again a CP. Since CP_2 is adjacent to all other facets of T_2, including F'_2, it follows that the polytope $T_1 \triangleright T_2$ has two CP facets and that they are adjacent to each other.

- In $T_1 \triangleright T_2$, the facet CP_1 gets merged with a bipyramid, resulting in a facet of the form $\mathrm{CP}\#\sigma_{d-1}$. In this case, $T_1 \triangleright T_2$ has two "large" facets: $\mathrm{CP}_1\#\sigma_{d-1}$ and CP_2, and they are adjacent to each other; every other facet has at most $d+1$ vertices.

With these observations in hand, we are ready to prove the following.

Theorem 6.13. *There are $2^{\Omega(N)} = 2^{\Omega(k)}$ combinatorially distinct $(d-2)$-simplicial 2-simple d-polytopes with $N = (3d-3) + k(2d-4)$ vertices.*

Proof. Consider $k+1$ copies of P^d, which we denote by $T_1, \dots, T_{k+1}$, with the corresponding copies of the CP facet denoted by CP_i. Each T_i has two pairs of a simplex facet and a simple vertex not in that facet, which in this proof we will denote by (F_i, w_i) and (F'_i, w'_i). Consider all polytopes resulting from $(\cdots((T_1 \triangleright T_2) \triangleright T_3) \cdots) \triangleright T_{k+1}$ by the following rules:

- In the first step, we merge T_1 and T_2 so that the facet CP_1 is merged with a bipyramid. In the ith step where $2 \leqslant i \leqslant k$, we have two choices of whether we merge CP_i with a simplex or with a bipyramid.

- In the ith step, when computing the merge of $(\cdots((T_1 \triangleright T_2) \triangleright T_3) \cdots) \triangleright T_i$ with T_{i+1}, we always merge along F_i and w_{i+1}.

Denote by R_k the polytope obtained in the kth step. In the ith step ($1 \leqslant i < k$), F_{i+1} from T_{i+1} remains untouched and can be used for the $(i+1)$st step. For $1 \leqslant j \leqslant k+1$, we refer to the facet of R_k resulting from CP_j as the jth *special facet*. By remarks above, for each $2 \leqslant j \leqslant k$, the jth special facet is either a CP or a $\mathrm{CP}\#\sigma_{d-1}$; the $(k+1)$st special facet is always a CP while the first special facet is always a $\mathrm{CP}\#\sigma_{d-1}$. Furthermore, for all $1 \leqslant i, j \leqslant k+1$, the ith and jth special facets are adjacent if and only if $|i-j| = 1$.

We show that this procedure produces at least 2^{k-1} pairwise non-isomorphic polytopes. First note that the boundary complexes of all non-special facets of R_k are either simplices, joins of two simplices, or stackings over these, and so a non-special facet can never be isomorphic to CP or CP$\#\sigma_{d-1}$. Associate with R_k its *profile* which is given by the following abstract graph: the nodes represent the facets of the form CP and CP$\#\sigma_{d-1}$, and two such nodes are connected by an edge if the corresponding facets are adjacent; also, label each node with a 0 or 1 depending on whether it represents a facet that is a CP or a CP$\#\sigma_{d-1}$. The resulting profile is then a *path* with $k+1$ nodes labeled by 0's and 1's; one of the endpoints is always labeled by 1 (the node representing the 1st special facet) and the other endpoint is always labeled by 0 (the node representing the $(k+1)$st special facet).

There are 2^{k-1} such $0/1$-paths, and we claim that each of them is a valid profile. Indeed, given such a path, walk along it from the endpoint labeled by 1 to the endpoint labeled by 0 and read the labels of the nodes. The node at distance $i-1$ from the first endpoint corresponds to the special facet coming from T_i and the label of that node simply tells us whether at the ith step we should merge CP$_i$ with a simplex or with a bipyramid. This claim completes the proof since isomorphic polytopes have the same profile. In other words, two polytopes with distinct profiles have different combinatorial types. □

Remark 6.14. When $d=4$, we can further merge R_k with a 2-simplicial 2-simple 4-polytope with 10, 11, or 16 vertices. Such polytopes can be found in [PW06, Section 4.1], where they are denoted by $P_{10}, P_{11}, P_{16} = \mathcal{I}^1(P_{11})$. This allows us to create exponentially many (in N) 2-simplicial 2-simple 4-polytopes with N vertices for all sufficiently large integers N (not just those with $N \equiv 1 \mod 4$). It follows from Corollary 4.13 that all resulting polytopes are elementary. Hence for $d=4$, the number of combinatorially distinct 2-simplicial 2-simple 4-polytopes that are also elementary grows exponentially with the number of vertices. This strengthens [PZ04, Corollary 4.2].

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