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THE LONG-TIME BEHAVIOR OF 3 DIMENSIONAL RICCI FLOW ON CERTAIN TOPOLOGIES

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ABSTRACT. In this paper we analyze the long-time behavior of 3 dimensional Ricci flow with surgery. We prove that under the topological condition that the initial manifold only has non-aspherical or hyperbolic components in its geometric decomposition, there are only finitely many surgeries and the curvature is bounded by Ct^{-1} for large t. This proves a conjecture of Perelman for this class of initial topologies.

The proof of this fact illustrates the fundamental ideas that are used in the subsequent papers of the author.

CONTENTS

1. INTRODUCTION

In this paper we analyze the long-time behavior of the Ricci flow with surgery on certain 3 dimensional manifolds. Our main result will be the following theorem, which we will present more precisely at the end of the introduction:

Let (M, g) be a closed 3 dimensional Riemannian manifold that fulfills the purely topological condition that all components of its geometric decomposition are hyperbolic or non-aspherical.
Then there is a Ricci flow that has only finitely many surgeries and whose initial metric is g. This Ricci flow with surgery either goes extinct in finite time or exists for all positive times. Moreover, the

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*Riemannian curvature in this flow is bounded everywhere by Ct^{-1}
for large t.*

The Ricci flow with surgery has been used by Perelman to solve the Poincaré and Geometrization Conjecture ([23], [24], [25]). More precisely, given any initial metric on a closed 3-manifold, Perelman managed to construct a solution for the Ricci flow with surgery on a maximal time interval and showed that the surgery times do not accumulate. This means that on every finite time interval there are only a finite number of surgery times. Furthermore, he could prove that if the given manifold is a homotopy sphere (or more generally a connected sum of prime, non-aspherical manifolds), then the Ricci flow goes extinct in finite time. This implies that the initial manifold is a sphere if it is simply connected and hence establishes the Poincaré Conjecture. On the other hand, if the Ricci flow continues to exist, he could show that the manifold decomposes into a thick part, which approaches a hyperbolic geometry, and an thin part, which becomes arbitrarily collapsed on local scales. Based on this collapsing, it is then possible to show that the thin part can be decomposed into geometric pieces ([27], [21], [15]) and hereby proving the Geometrization Conjecture.

Observe that although the Ricci flow with surgery was used to solve such difficult problems, some of its basic properties are still unknown, because they surprisingly turned out to be irrelevant in the end. For example, it was only conjectured by Perelman that in the long-time existent case there are finitely many surgeries, i.e. that after some time the flow can be continued by a conventional smooth, non-singular Ricci flow defined up to time infinity. Furthermore, it is still unknown whether and in what way the Ricci flow exhibits the the full geometric decomposition of the manifold.

In [16], [17] and [18], Lott and Lott-Sesum could give a description of the long-time behavior of certain Ricci flows on manifolds that consist of a single component in their geometric decomposition. However, they needed to make additional curvature and diameter or symmetry assumptions.

In this paper, we only have to impose a topological condition on the initial manifold. Using the language developed in section 2 our precise result reads:

Theorem 1.1. *Given a surgery model $(M_{\text{stan}}, g_{\text{stan}}, D_{\text{stan}})$, there is a continuous function $\delta : [0, \infty) \to \mathbb{R}_+$ such that:*

Let $\mathcal{M}$ be a Ricci flow with surgery with normalized initial conditions and $\delta(t)$-precise cutoff (see section 2 for more details) such that $\mathcal{M}(0)$ satisfies the following topological condition:

$\mathcal{M}(0) \approx M_1 \# \ldots \# M_m$ is a connected sum of closed 3-manifolds M_i. Each M_i is either spherical, diffeomorphic to $S^2 \times S^1$ or its torus decomposition only consists of hyperbolic pieces (i.e. we can find collections of pairwise disjoint, incompressible, embedded tori $T_{i,1}, \ldots, T_{i,m_i} \subset M_i$ such that the connected components of $M \setminus (T_{i,1} \cup \ldots \cup T_{i,m_i})$ carry complete finite volume hyperbolic metrics).

Then $\mathcal{M}$ has only finitely many surgeries and there are constants $T, C < \infty$ such that $|\text{Rm}_t| < Ct^{-1}$ on $\mathcal{M}(t)$ for all $t \geq T$.

We like to point out that this curvature estimate only used to be known to hold on the thick part. Hence, our theorem contributes towards a better understanding of the geometry of the thin part. Observe that the result implies that the rescaled metrics $t^{-1}g(t)$ have uniformly bounded curvature for large t. Such solutions are said to be of type III and have been subject of study by Hamilton ([11]).

We give a short outline of the proof: The thin part of the manifold is locally collapsed along S^1, T^2 or S^2 fibers. We will show that there are certain "good" areas where the fibers are either diffeomorphic to S^1 or T^2 and incompressible in the manifold. Hence, if we pass to the universal cover, these areas will become non-collapsed on a local scale. We can then use a modification of Perelman's Theorem [24, 7.3] to deduce a curvature bound on the scale $\sqrt{t}$. By looking closer at the decomposition arising from the collapse, we can argue that if not all areas of the thin part are good, there must be some good area that is collapsed along incompressible S^1-fibers over a 2-dimensional space. Hence, by the conclusion above, this collapse takes place at scale $\sqrt{t}$. Next, we establish the existence of minimal annuli that intersect every fiber of this fibration and whose area goes to zero compared to the scale $\sqrt{t}$. This will then give us a contradiction implying that the thin part only consists of good areas and hence the curvature is controlled everywhere.

The paper is organized as follows: In section 2 we clarify the concepts behind Ricci flows with surgery. We are keeping the definitions here as general as possible so that they match or follow from existent literature on the subject. Section 3 recapitulates known existence results for Ricci flows with surgery. In section 4 we quote Perelman's important long-time curvature estimate and generalize it to the universal cover. We then explain the known geometric results arising from the long-time analysis in sections 5 and 6. In section 7 we analyze the behavior of the collapse when passing to the universal cover and in section 8 we prove bounds for the evolution of minimal spheres and annuli in Ricci flow. Finally, the proof of the main theorem can be found in section 9.

2. Definition of Ricci flows with surgery

In this section, we give a precise definition of the Ricci flows with surgery that we are going to analyze. We will mainly use the language developed in [1] here. We first define Ricci flows with surgery in a very broad sense

Definition 2.1 (Ricci flow with surgery)**.** Consider a time interval $I \subset \mathbb{R}$. Let $T^1 < T^2 < \ldots$ be times of the interior of I which form a possibly infinite, but discrete subset of $\mathbb{R}$ and divide I into the intervals

$$I^1 = I \cap (-\infty, T^1), \quad I^2 = [T^1, T^2), \quad I^3 = [T^2, T^3), \quad \ldots$$

and $I^{k+1} = I \cap [T^k, \infty)$ if there are only finitely many T^i's and T^k is the last such time and $I^1 = I$ if there are no such times. Consider Ricci flows $(M^1 \times I^1, g_t^1), (M^2 \times I^2, g_t^2), \ldots$ on 3-manifolds $M^1, M^2, \ldots$. Let $\Omega^i \subset M^i$ be open sets on which the metric g_t^i converges smoothly as $t \nearrow T^i$ to some Riemannian metric $g_{T^i}^i$ on Ω^i and let

$$U_-^i \subset \Omega^i \qquad \text{and} \qquad U_+^i \subset M^{i+1}$$

be open subsets such that there are isometries

$$\Phi^i : (U_-^i, g_{T^i}^i) \longrightarrow (U_+^i, g_{T^i}^{i+1}), \qquad (\Phi^i)^* g_{T^i}^{i+1}|_{U_+^i} = g_{T^i}^i|_{U_-^i}.$$

We assume that we never have $U_-^i = \Omega^i = M^i$ and $U_+^i = M^{i+1}$ and we assume that every component of M^{i+1} contains a point of U_+^i. Then, we call $\mathcal{M} = ((T^i), (M^i \times I^i, g_t^i), (\Omega^i), (U_\pm^i), (\Phi^i))$ a *Ricci flow with surgery on the time interval* I and $T^1, T^2, \ldots$ *surgery times*.

If $t \in I^i$, then $(\mathcal{M}(t), g(t)) = (M^i \times \{t\}, g_t^i)$ is called the *time t-slice of $\mathcal{M}$*. The points in $\mathcal{M}(T^i) \setminus U_+^i \times \{T^i\}$ are called *surgery points*. For $t = T^i$, we define the *(presurgery) time T^{i-}-slice* to be $(\mathcal{M}(T^{i-}), g(T^{i-})) = (\Omega^i \times \{T^i\}, g_{T^i}^i)$. The points $\Omega^i \times \{T^i\} \setminus U_-^i \times \{T^i\}$ are called *presurgery points*. We will call a point that is not a presurgery point a *non-presurgery point*.

If $\mathcal{M}$ has no surgery points, then we call $\mathcal{M}$ *non-singular* and write $\mathcal{M} = M \times I$.

We will often view $\mathcal{M}$ in the *space-time picture*, i.e. we imagine $\mathcal{M}$ as a topological space $\bigcup_{t \in I} \mathcal{M}(t) = \bigcup_i M^i \times I^i$ where the components in the latter union are glued together via the diffeomorphisms Φ^i. The following vocabulary will prove to be useful when dealing with Ricci flows with surgery:

Definition 2.2 (Ricci flow with surgery, points in space-time). For $(x, t) \in \mathcal{M}$, consider a spatially constant line in $\mathcal{M}$ that starts in (x, t) and goes forward or backward in time for some time $\Delta t \in \mathbb{R}$ and that doesn't hit any surgery points, except possibly at its endpoints. When crossing surgery times, we can continue the line via the isometries Φ^i. We denote the endpoint of this line by $(x, t + \Delta t) \in \mathcal{M}$. Observe that this point is only defined if there are no surgery points between (x, t) and $(x, t + \Delta t)$. We say that a point $(x, t) \in \mathcal{M}$ *survives until time $t + \Delta t$* if the point $(x, t + \Delta t) \in \mathcal{M}$ is well-defined.

Observe that this notion also makes sense, if $(x, t^-) \in \mathcal{M}$ is a presurgery point and $\Delta t \leq 0$.

Using this definition, we can define parabolic neighborhoods in $\mathcal{M}$.

Definition 2.3 (Ricci flow with surgery, parabolic neighborhoods). Let $(x, t) \in \mathcal{M}$ (presurgery points are allowed, in this case we have to replace t by t^-), $r \geq 0$ and $\Delta t \in \mathbb{R}$. Consider the ball $B = B(x, t, r) \subset \mathcal{M}(t)$. If (x, t^-) is a presurgery point, we have to look at $B(x, t^-, r) \subset \mathcal{M}(t^-)$. For each $(x', t) \in B$ consider the union $I_{x',t}^{\Delta t}$ of all points $(x', t + t') \in \mathcal{M}$ that are well-defined in the sense of Definition 2.2 for $t' \in [0, \Delta t]$ resp. $t' \in [\Delta t, 0]$. We say that $I_{x',t}^{\Delta t}$ is *non-singular* if $(x', t + \Delta t) \in I_{x',t}^{\Delta t}$. Define the *parabolic neighborhood* $P(x, t, r, \Delta t) = \bigcup_{x' \in B} I_{x',t}^{\Delta t}$. We call $P(x, t, r, \Delta t)$ *non-singular* if all the $I_{x',t}^{\Delta t}$ are non-singular.

We will now characterize three important approximate local geometries that we will have to deal with very often: ε-necks, strong ε-necks and (ε, E)-caps. The notions below also make sense for presurgery times.

Definition 2.4 (Ricci flow with surgery, ε-necks). Let $\varepsilon > 0$ and consider a Riemannian manifold (M, g). We call an open subset $U \subset M$ an ε-neck, if there is a diffeomorphism $\Phi : S^2 \times (-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}) \rightarrow U$ such that there is a $\lambda > 0$ with $\|\lambda^{-2}\Phi^*g(t) - g_{S^2\times\mathbb{R}}\|_{C^{[\varepsilon^{-1}]}} < \varepsilon$ where $g_{S^2\times\mathbb{R}}$ is the standard metric on $S^2 \times (-\frac{1}{\varepsilon}, \frac{1}{\varepsilon})$ of constant scalar curvature 2.

We say that $x \in \mathcal{M}(t)$ is a *center* of U if $x \in \Phi(S^2 \times \{0\})$ for such a Φ. If $\mathcal{M}$ is a Ricci flow with surgery and $(x, t) \in \mathcal{M}$, then we say that (x, t) is a *center of an ε-neck* if (x, t) is a center of an ε-neck in $\mathcal{M}(t)$.

Definition 2.5 (Ricci flow with surgery, strong ε-necks). Let $\varepsilon > 0$ and consider a Ricci flow with surgery $\mathcal{M}$ and a time t_2. Consider a subset $U \subset \mathcal{M}(t_2)$ and assume that all points of U survive until some time $t_1 < t_2$. Then the subset $U \times [t_1, t_2] \subset \mathcal{M}$ is called a *strong ε-neck* if there is a factor $\lambda > 0$ such that after parabolically rescaling by λ^{-1}, the flow on $U \times [t_1, t_2]$ is ε-close to the standard flow on some time-interval $[-a, 0]$ for $a \geq 1$. By this we mean $a = \lambda^{-2}(t_2 - t_1) \geq 1$ and there is a diffeomorphism $\Phi : S^2 \times (-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}) \rightarrow U$ such that

$$\|\lambda^{-2}\Phi^*g(\lambda^2 t + t_2) - g_{S^2\times\mathbb{R}}(t)\|_{C^{[\varepsilon^{-1}]}(S^2\times(-\frac{1}{\varepsilon},\frac{1}{\varepsilon})\times[-a,0])} < \varepsilon.$$

Here $(g_{S^2\times\mathbb{R}}(t))_{t\in(-\infty,0]}$ is the standard Ricci flow on $S^2 \times \mathbb{R}$ that has scalar curvature 2 at time 0 and $\lambda^{-2}\Phi^*g(\lambda^2 t + t_2)$ denotes the pull-back of the parabolically rescaled flow on $U \times [t_1, t_2]$.

A point $(x, t_2) \in U \times \{t_2\}$ is called a *center of $U \times [t_1, t_2]$* if $(x, t_2) \in \Phi(S^2 \times \{0\} \times \{t_2\})$ for such a Φ.

Definition 2.6 (Ricci flow with surgery, (ε, E)-caps). Let $\varepsilon, E > 0$ and consider a Riemannian manifold (M, g) and an open subset $U \subset M$. Suppose that $(\operatorname{diam} U)^2 |\mathrm{Rm}|(y) < E^2$ for any $y \in U$ and $E^{-2}|\mathrm{Rm}|(y_1) \leq |\mathrm{Rm}|(y_2) \leq E^2|\mathrm{Rm}|(y_1)$ for any $y_1, y_2 \in U$. Furthermore, assume that U is either diffeomorphic to $\mathbb{B}^3$ or $\mathbb{R}P^3 \setminus \overline{\mathbb{B}}^3$ and that there is a compact set $K \subset U$ such that $U \setminus K$ is an ε-neck.

Then U is called an (ε, E)-cap. If $x \in K$ for such a K, then we say that x is a *center of U*.

Analogously as in Definition 2.4, we define (ε, E)-caps in Ricci flows with surgery.

With these concepts at hand we can now give an exact description of the surgery process that will underlie the Ricci flows with surgeries which we are going to analyze. The author has chosen the phrasing so that it includes the outcomes of the constructions presented in [24], [14], [20], [2] and [1].

We will first need to fix a geometry that models the metric which we will endow the filling 3-balls with after each surgery.

Definition 2.7 (surgery model). Consider $M_{\mathrm{stan}} = \mathbb{R}^3$ with its natural $SO(3)$-action and let g_{stan} be a complete metric on M_{stan} such that

(1) g_{stan} is $SO(3)$-invariant,
(2) g_{stan} has non-negative sectional curvature,
(3) for any sequence $x_n \in M_{\text{stan}}$ with $\text{dist}(0, x_n) \to \infty$, the pointed Riemann-ian manifolds $(M_{\text{stan}}, g_{\text{stan}}, x_n)$ smoothly converge to the standard $S^2 \times \mathbb{R}$ of scalar curvature $R = 2$.

For every $r > 0$, we denote the r-ball around 0 by $M_{\text{stan}}(r)$. Let $D_{\text{stan}} > 0$ be a positive number. Then we call $(M_{\text{stan}}, g_{\text{stan}}, D_{\text{stan}})$ a *surgery model*.

Definition 2.8 (φ-positive curvature)**.** We say that a Riemannian metric g on a manifold M has φ-*positive curvature* for $\varphi > 0$ if for every point $p \in M$ there is an $X > 0$ such that $\sec_p \geq -X$ and

$$\text{scal}_p \geq -\tfrac{3}{2}\varphi \qquad \text{and} \qquad \text{scal}_p \geq 2X\left(\log(2X) - \log \varphi - 3\right).$$

Observe that by [11] this condition is improved by Ricci flow in the following sense: If $(M, (g_t)_{t \in [t_0, t_1]})$ is a Ricci flow with $t_0 > 0$ and g_{t_0} is t_0^{-1}-positive, then g_t is t^{-1}-positive for all $t \in [t_0, t_1]$.

Definition 2.9 (Ricci flow with surgery, $\delta(t)$-precise cutoff)**.** Let $\mathcal{M}$ be a Ricci flow with surgery defined on some time interval $[0, T)$, let $(M_{\text{stan}}, g_{\text{stan}}, D_{\text{stan}})$ be a surgery model and let $\delta : [0, \infty) \to \mathbb{R}_+$ be a function. We say that $\mathcal{M}$ is *performed by $\delta(t)$-precise cutoff (using the surgery model $(M_{\text{stan}}, g_{\text{stan}}, D_{\text{stan}})$)* if

(1) For all t the metric $g(t)$ (and $g(t^-)$ if t is a surgery time) has t^{-1}-positive curvature.
(2) For every surgery time T^i, the subset $\mathcal{M}(T^i) \setminus U_+^i$ is a disjoint union $D_1^i \cup \ldots \cup D_{m_i}^i$ of smoothly embedded 3-disks.
(3) For every such D_j^i there is an embedding
$$\Phi_j^i : M_{\text{stan}}(\delta^{-1}(T^i)) \longrightarrow \mathcal{M}(T^i)$$
such that $D_j^i \subset \Phi_j^i(M_{\text{stan}}(D_{\text{stan}}))$ and such that for all $j = 1, \ldots, m_i$ the images $\Phi_j^i(M_{\text{stan}}(\delta^{-1}(T^i)))$ are pairwise disjoint and there are constants $0 < \lambda_j^i \leq \delta(T^i)$ such that
$$\left\| g_{\text{stan}} - (\lambda_j^i)^{-2}(\Phi_j^i)^* g(T^i) \right\|_{C^{[\delta^{-1}(T^i)]}(M_{\text{stan}}(\delta^{-1}(T^i)))} < \delta(T^i).$$
(4) For every such D_j^i, the points on the boundary of U_-^i in $\mathcal{M}(T^{i-})$ corresponding to ∂D_j^i are centers of strong $\delta(T^i)$-necks.
(5) For every D_j^i for which the boundary component of ∂U_+^i corresponding to the sphere ∂D_j^i bounds a 3-disk component $(D')_j^i$ of $M^i \setminus U_-^i$ (i.e. a "trivial surgery", see below), the following holds: For every $\chi > 0$, there is some $t_\chi < T^i$ such that for all $t \in (t_\chi, T^i)$ there is a $(1 + \chi)$-Lipschitz map $\xi : (D')_j^i \to D_j^i$ which corresponds to the identity on the boundary.
(6) For every surgery time T^i, the components of $\mathcal{M}(T^{i-}) \setminus U_-^i$ are either diffeomorphic to $S^2 \times I$, D^3, $\mathbb{R}P^3 \setminus B^3$, a spherical space form, $S^1 \times S^2$ or $\mathbb{R}P^3 \# \mathbb{R}P^3$.

We will speak of each D_j^i as *a surgery* and if D_j^i satisfies the property described in (5), we call it a *trivial surgery*.

Observe that we have phrased the Definition so that if $\mathcal{M}$ is a Ricci flow with surgery that is performed by $\delta(t)$-precise cutoff, then it is also performed by $\delta'(t)$-precise cutoff whenever $\delta'(t) \geq \delta(t)$ for all t. Note also that trivial surgeries don't change the topology of the respective component at which they are performed.

3. Existence of Ricci flows with surgery

Ricci flows with surgery and precise cutoff as introduced in Definition 2.9 can indeed be constructed from any given initial metric. We will make this more precise below. To simplify things, we restrict the geometries which we want to consider as initial conditions.

Definition 3.1 (Normalized initial conditions)**.** We say that a Riemannian 3-manifold (M, g) is *normalized* if

(1) M is compact and orientable,
(2) $|\mathrm{Rm}| < 1$ everywhere and
(3) $\mathrm{vol}\, B(x, 1) > \frac{\omega_3}{2}$ for all $x \in M$ where ω_3 is the volume of a standard Euclidean 3-ball.

We say that a Ricci flow with surgery $\mathcal{M}$ has *normalized initial conditions*, if $\mathcal{M}(0)$ is normalized.

Obviously, any Riemannian metric on a compact and orientable 3-manifold can be rescaled to be normalized. Moreover, recall

Definition 3.2 (κ-noncollapsedness)**.** Let $\mathcal{M}$ be a Ricci flow with surgery, $(x, t) \in \mathcal{M}$ (possibly a presurgery point) and $\kappa, \rho > 0$. We say that $\mathcal{M}$ is κ-noncollapsed in (x, t) on scales less than ρ if $\mathrm{vol}_t\, B(x, t, r) \geq \kappa r^3$ for all $0 < r < \rho$ for which

(1) the ball $B(x, t, r)$ is relatively compact in $\mathcal{M}(t)$,
(2) the parabolic neighborhood $P(x, t, r, -r^2)$ is nonsingular and
(3) $|\mathrm{Rm}| < r^{-2}$ on $P(x, t, r, -r^2)$.

In order to construct a Ricci flow with surgery, we need the following characterization of regions of high curvature (see [24, 5.1], [14, sec 77], [20, Propositions 16.1, 17.1], [2, sec 5.3, Propositions B, C], [1, Theorem 7.5.1]). The power of the this proposition lies in the fact that none of the parameters depends on the number or the preciseness of the preceding surgeries. Hence, it provides a tool to perform surgeries in a controlled way.

Proposition 3.3 (Canonical Neighborhood Theorem, Ricci flows with surgery)**.** *There are constants $C_0 < \infty$ and $\kappa_0 > 0$ and for every surgery model $(M_{\mathrm{stan}}, g_{\mathrm{stan}}, D_{\mathrm{stan}})$ and every $\varepsilon > 0$ there are a constant $E < \infty$ and continuous positive functions $r, \delta, \kappa : [0, \infty) \to \mathbb{R}_+$ such that the following holds:*

Let $\mathcal{M}$ be a Ricci flow with surgery on some time interval $[0, T)$ which has normalized initial conditions and which is performed by $\delta(t)$-precise cutoff. Then

(a) *At every time $t \in [0, T)$ the flow $\mathcal{M}$ is $\kappa(t)$-noncollapsed on scale less than $\sqrt{t}$.*

(b) *If $(x, t) \in \mathcal{M}$ is a non-presurgery point with $R(x, t) \geq r^{-2}(t)$, then*

*(1) (x, t) is either the center of a strong ε-neck or an (ε, E)-cap $U \subset$
$\mathcal{M}(t)$.*

*If $U \approx \mathbb{R}P^3 \setminus \overline{B}^3$, then there is a time $t_1 < t$ such that all points on
U survive until time t_1 and such that the flow on $U \times [t_1, t]$ lifted to
its double cover contains strong ε-necks for which either lift of (x, t)
is a center.*

(2) $|\nabla| \operatorname{Rm}|^{-1/2}|(x, t) < C_0$ and $|\partial_t| \operatorname{Rm}|^{-1}(x, t)| < C_0$,

(3) $\mathcal{M}$ is κ_0-noncollapsed in (x, t),

*or property (2) holds and the time-t sectional curvatures on the component
of $\mathcal{M}(t)$ in which x lies are positive and E-pinched, i.e. they lie in an
interval of the form $(\lambda, E\lambda)$ for some $\lambda > 0$ (hence that component of
$\mathcal{M}(t)$ is diffeomorphic to a spherical space form).*

Note that, for future purposes, we have added a more detailed description
of the local geometry in the case in which the (ε, E)-cap U is diffeomorphic
to $\mathbb{R}P^3 \setminus \overline{B}^3$. This additional assertion follows from the popular proofs of the
Canonical Neighborhood Theorem, along with the fact that κ-solutions, which
serve as models for the geometry around points of high curvature, are *isometric*
to a quotient of round $S^2 \times \mathbb{R}$ if they are diffeomorphic to $\mathbb{R}P^3 \setminus \overline{B}^3$.

Using Proposition 3.3, it is possible to give an existence result for Ricci flows
with surgery. For a proof see again the sources indicated above.

Proposition 3.4. *Given a surgery model $(M_{\text{stan}}, g_{\text{stan}}, D_{\text{stan}})$, there is a contin-
uous function $\delta : [0, \infty) \to \mathbb{R}_+$ such that if $\delta' : [0, \infty) \to \mathbb{R}_+$ is a continuous
function with $\delta'(t) \leq \delta(t)$ for all $t \in [0, \infty)$ and (M, g) is a normalized Riemann-
ian manifold, then there is a Ricci flow with surgery $\mathcal{M}$ defined for times $[0, \infty)$
such that $\mathcal{M}(0) = (M, g)$ and which is performed by $\delta'(t)$-precise cutoff. (Observe
that we can possibly have $\mathcal{M}(t) = \emptyset$ for large t.)*

*Moreover, if $\mathcal{M}$ is a Ricci flow with surgery on some time interval $[0, T)$ that
has normalized initial conditions and that is performed by $\delta(t)$-precise cutoff, then
$\mathcal{M}$ can be extended to a Ricci flow on the time interval $[0, \infty)$ that has $\delta'(t)$-precise
cutoff on the time interval $[T, \infty)$.*

We point out that the parameters $\delta(t)$ and ε in Proposition 3.3 and $\delta(t)$ in
Proposition 3.4 depend on the choice of the surgery model.

> ***From now on we will fix a surgery model*** $(M_{\text{stan}}, g_{\text{stan}}, D_{\text{stan}})$
> ***for the rest of this paper and we will not mention this
> dependence anymore.***

4. Perelman's longtime analysis result

Consider a Ricci flow with surgery $\mathcal{M}$. For any non-presurgery point $(x, t) \in
\mathcal{M}$, we define

$$\rho(x, t) = \max\{r > 0 \ : \ \sec_t \geq -r^{-2} \ \text{ on } \ B(x, t, r)\}.$$

If $\sec_t \geq 0$ on $\mathcal{M}(t)$, then we set $\rho(x, t) = \infty$. For any $r_0 > 0$, we moreover define

$$\rho_{r_0}(x, t) = \min\{\rho(x, t), r_0\}.$$

The following Proposition is a consequence of [24, 6.3, 6.8, 7.3]:

Proposition 4.1. *There is a continuous positive function $\delta : [0, \infty) \to \mathbb{R}_+$ such that for every $w > 0$ there are constants $\overline{\rho}(w), \overline{r}(w) > 0$ and $T = T(w), K = K(w) < \infty$ such that:*

Let $\mathcal{M}$ be a Ricci flow with surgery on the time interval $[0, \infty)$ with normalized initial conditions that is performed by $\delta(t)$-precise cutoff. Let $t > T$ and $x \in \mathcal{M}(t)$. Then

(a) *If $0 < r \le \min\{\rho(x,t), \overline{r}\sqrt{t}\}$ and $\mathrm{vol}_t B(x,t,r) \ge wr^3$, then $|\mathrm{Rm}| < Kr^{-2}$ on $B(x,t,r)$.*

(b) *If $\mathrm{vol}_t B(x,t,\rho(x,t)) \ge w\rho^3(x,t)$, then $\rho(x,t) > \overline{\rho}\sqrt{t}$ and $|\mathrm{Rm}| < Kt^{-1}$ on $B(x,t,\overline{\rho}\sqrt{t})$.*

Note that assertion (a) is only a direct consequence of [24, 6.3, 6.8] in the case in which $r \ge \theta^{-1}(w)h$, where $\theta(w)$ is a universal positive constant that only depends on w and h is the maximal cutoff radius on the time interval $[\frac{1}{2}t, t]$. In the case in which $r < \theta^{-1}(w)h$ it is possible to conclude $r \ll r(t)$, for an appropriate choice of $\delta(t)$, where $r(t)$ is the canonical neighborhood scale from Proposition 3.3. Then the desired bound follows from the fact that regions of high curvature below the canonical neighborhood scale look sufficiently "neck-like" and are hence sufficiently collapsed. We omit the proof in this case since both Proposition 4.1(a) and its consequence, Proposition 4.2(a), are only stated for completeness and neither statement will be used in the rest of the paper. Assertion (b) of Proposition 4.1, which will be important for us later on, is indeed a direct consequence of [24, 7.3].

We can generalize this Proposition by passing to the universal cover: Consider a non-presurgery point $(x,t) \in \mathcal{M}$ and $r > 0$. Lift $x \in \mathcal{M}(t)$ to the universal cover $\widetilde{\mathcal{M}}(t)$ of $\mathcal{M}(t)$ to obtain $\widetilde{x}$. Then we call $\mathrm{vol}_t \widetilde{B}(\widetilde{x}, t, r)$ the volume of the r-ball around $\widetilde{x}$ in $\widetilde{\mathcal{M}}(t)$. Obviously, $\mathrm{vol}_t \widetilde{B}(\widetilde{x}, t, r) \ge \mathrm{vol}_t B(x, t, r)$.

We can now state the following more general Proposition, which will be crucial for the proof of Theorem 1.1:

Proposition 4.2. *Under the same assumptions as in Proposition 4.1, we have:*

(a) *If $0 < r \le \min\{\rho(x,t), \overline{r}\sqrt{t}\}$ and $\mathrm{vol}_t \widetilde{B}(\widetilde{x}, t, r) \ge wr^3$, then $|\mathrm{Rm}| < Kr^{-2}$ on $B(x,t,r)$.*

(b) *If $\mathrm{vol}_t \widetilde{B}(\widetilde{x}, t, \rho(x,t)) \ge w\rho^3(x,t)$, then $\rho(x,t) > \overline{\rho}\sqrt{t}$ and $|\mathrm{Rm}| < Kt^{-1}$ on $B(x,t,\overline{\rho}\sqrt{t})$.*

Proof. We first need to define the universal covering flow $\widetilde{\mathcal{M}}$ of $\mathcal{M}$. Recall that $\mathcal{M} = ((T^i), (M^i \times I^i, g^i), (\Omega^i), (U^i_\pm), (\Phi^i))$ where each g^i is a Ricci flow on the closed 3-manifold M^i defined for times I^i. We can lift each of these flows to the universal cover $\widetilde{M^i}$ of M^i. Its lift $\widetilde{g}^i$ still satisfies the Ricci flow equation. Moreover, all its time slices are complete Riemannian metrics and we have bounded curvature on compact subintervals of I^i. Denote by $\widetilde{\Omega}^i$ the preimage of Ω^i under the universal covering projection for each i.

We will now assemble the flows $(\widetilde{M}^i \times I^i, \widetilde{g}_t^i)$ to a Ricci flow with surgery. Observe first that for every i, the subset $U_-^i \subset M^i$ is bounded by pairwise disjoint, embedded 2-spheres. So for every point $p \in U_-^i$, the natural map $\pi_1(U_-^i, p) \to \pi_1(M^i, p)$ is an injection. Now let $\widetilde{U}_+^i \subset \widetilde{M}^{i+1}$ be the preimage of U_+^i under the universal covering projection. The complement of this subset is still a collection of pairwise disjoint, embedded 3-disks and hence each component of $\widetilde{U}_+^i$ is simply connected. Via $(\Phi^i)^{-1} : U_+^i \to U_-^i$ there is a covering map $\widetilde{U}_+^i \to U_+^i \to U_-^i \subset M^i$. Since every component of $\widetilde{U}_+^i$ is simply-connected, we find a lift $\phi^i : \widetilde{U}_+^i \to \widetilde{M}^i$. Using the fact that $U_-^i \to M^i$ is π_1-injective, we conclude that ϕ^i is injective. Denote by $\widetilde{U}_-^i \subset \widetilde{M}^i$ the image of ϕ^i and let $\widetilde{\Phi}^i : \widetilde{U}_-^i \to \widetilde{U}_+^i$ be its inverse. Then $\widetilde{\mathcal{M}} = ((T^i), (\widetilde{M}^i \times I^i, \widetilde{g}_t^i), (\widetilde{\Omega}^i), (\widetilde{U}_\pm^i), (\widetilde{\Phi}^i))$ is a Ricci flow with surgery that satisfies all properties of Definition 2.9 ("$\delta(t)$-precise cutoff") except for (6) if we allow countably many disks $D_1^i, D_2^i, \ldots$ at each surgery time T^i in property (2).

We now argue that the proof of Proposition 4.1, as presented in [24], can still be carried out in the universal covering flow $\widetilde{\mathcal{M}}$. For this, observe first that the time slices of $\widetilde{\mathcal{M}}$ might be non-compact, but they are still complete and the curvature is bounded on compact time intervals away from surgery times. So all arguments in the proof of Proposition 4.1 that involve picking points in $\mathcal{M}$ at which certain geometric quantities are maximal, can still be carried out in the universal covering flow $\widetilde{\mathcal{M}}$. Also all arguments in this proof that make use of the existence of minimizing $\mathcal{L}$-geodesics, stay valid in $\widetilde{\mathcal{M}}$.

Secondly, note that at several points the proof of Proposition 4.1 uses the assertions of the Canonical Neighborhood Theorem, Proposition 3.3. Since these assertions are satisfied for $\mathcal{M}$, and they are stable under taking covers, they must also hold for the universal covering flow $\widetilde{\mathcal{M}}$.

Finally, let us recapitulate the proof of Proposition 4.1 as presented in Perelman's paper [24]. The first ingredient in Perelman's proof of Proposition 4.1 is [24, 6.3(c)], which is a bounded curvature at bounded distance result. The methods used in the proof of this result involve point-picking, $\mathcal{L}$-geometry and the maximum principle applied to functions whose support is contained in balls of a definite radius. All these methods can still be carried out in $\widetilde{\mathcal{M}}$. The second and third ingredients are [24, 6.5 and 6.6]. These Lemmas are statements about smooth Ricci flows that are defined on a parabolic neighborhood or about smooth Riemannian balls and hence don't have to be modified for our purposes. Eventually, in [24, 6.7], these three ingredients are combined to prove Proposition 4.2(a). The arguments used in this proof only involve point-picking and an open-closed argument and can hence also be carried out in $\widetilde{\mathcal{M}}$. Proposition 4.2(b) is a direct consequence of part (a) and the t^{-1}-positivity of the curvature, see [24, 7.3]. $\square$

5. The thick-thin decomposition

We now describe how in the longtime picture Ricci flows with surgery decompose the manifold into a thick and a thin part. In this process, the thick

part approaches a hyperbolic metric while the thin part collapses on local scales. Compare this Proposition with [24, 7.3] and [14, Proposition 90.1].

Proposition 5.1. *There is a function $\delta : [0, \infty) \to \mathbb{R}_+$ such that given a Ricci flow with surgery and $\delta(t)$-precise cutoff $\mathcal{M}$ with normalized initial conditions defined on the interval $[0, \infty)$, we can find a constant $T_0 < \infty$, a function $w : [T_0, \infty) \to \mathbb{R}_+$ with $w(t) \to 0$ as $t \to \infty$ and a collection of orientable finite volume hyperbolic manifolds $(H_1', g_{\mathrm{hyp},1}), \ldots, (H_k', g_{\mathrm{hyp},k})$ such that:*
There are finitely many embedded 2-tori $T_{1,t}, \ldots, T_{m,t} \subset \mathcal{M}(t)$ for $t \in [T_0, \infty)$ which move by isotopies and don't hit any surgery points and which separate $\mathcal{M}(t)$ into two (possibly empty) closed subsets $\mathcal{M}_{\mathrm{thick}}(t), \mathcal{M}_{\mathrm{thin}}(t) \subset \mathcal{M}(t)$ such that

(a) $\mathcal{M}_{\mathrm{thick}}(t)$ does not contain surgery points for all $t \in [T_0, \infty)$.

(b) The $T_{i,t}$ are incompressible in $\mathcal{M}(t)$ and $t^{-1/2} \operatorname{diam}_t T_{i,t} < w(t)$.

(c) The topology of $\mathcal{M}_{\mathrm{thick}}(t)$ stays constant in t and $\mathcal{M}_{\mathrm{thick}}(t)$ is a disjoint union of components $H_{1,t}, \ldots, H_{k,t} \subset \mathcal{M}_{\mathrm{thick}}(t)$ such that the interior of each $H_{i,t}$ is diffeomorphic to H_i'.

(d) We can find an embedded cross-sectional torus $T_{j,t}'$ in each cusp of the H_i' which moves by isotopies such that the following holds: Chop off the ends of the H_i' along the $T_{j,t}'$ and call the remaining open manifolds $H_{i,t}''$. Then each $H_{i,t}''$ contains a $w^{-1}(t)$-tubular neighborhood of the thick part[1] of H_i' and there are smooth families of diffeomorphisms $\Psi_{i,t} : H_{i,t}'' \to H_i$ which become closer and closer to being isometries, i.e.

$$\left\| \tfrac{1}{4t} \Psi_{i,t}^* g(t) - g_{\mathrm{hyp},i} \right\|_{C^{[w^{-1}(t)]}(H_{i,t}'')} < w(t)$$

and which move slower and slower in time, i.e.

$$\sup_{H_{i,t}''} t^{1/2} |\partial_t \Psi_{i,t}| < w(t)$$

for all $t \in [T_0, \infty)$ and $i = 1, \ldots, k$.

(e) A large neighborhood of the part $\mathcal{M}_{\mathrm{thin}}(t)$ is better and better collapsed, i.e. for every $t \geq T_0$ and $x \in \mathcal{M}(t)$ with

$$\operatorname{dist}_t(x, \mathcal{M}_{\mathrm{thin}}(t)) < w^{-1}(t)\sqrt{t}$$

we have

$$\operatorname{vol}_t B(x, t, \rho_{\sqrt{t}}(x,t)) < w(t)\rho_{\sqrt{t}}^3(x,t).$$

6. Analysis of the thin part

Based on property (e) of Proposition 5.1 we can analyze the thin part $\mathcal{M}_{\mathrm{thin}}(t)$ for large t and recover its graph structure geometrically. The following result, Proposition 6.1, follows from the work of Morgan and Tian ([21]). We have altered its phrasing to include more geometric information. We explain below

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[1]On the hyperbolic manifolds H_i' the *thick part* denotes the part in which the injectivity radius is larger than the Margulis constant.

where to find each of the following conclusions in their paper. Similar results can also be found in [15], [3], [6] and [8].

We first summarize the content of Proposition 6.1. Consider a Riemannian 3-manifold (M, g) with boundary. We will impose assumptions on (M, g) that are satisfied by the rescaled metric on the thin part $(\mathcal{M}_{\text{thin}}(t), t^{-1}g(t))$. The first assumption is that (M, g) is locally collapsed at scale $\rho_1(x)$, i.e. for some small $w_0 > 0$ and for all $x \in M$ for which $B(x, \rho_1(x)) \subset \text{Int } M$ we have

$$\text{vol } B(x, \rho_1(x)) < w_0 \rho_1^3(x).$$

Secondly, we assume that the curvature of (M, g) is bounded if we pass to smaller scales on which (M, g) is non-collapsed. In a few words, this means that for any point $x \in M$ and every $r \ll \rho_1(x)$ for which $B(x, r) \subset \text{Int } M$ and

$$\text{vol } B(x, r) > wr^3$$

for some $w > w_0$, we have $|\text{Rm}_t| < K(w)r^{-2}$ on $B(x, r)$. Thirdly, we impose geometric conditions on collar neighborhoods of the boundary components of (M, g), which are natural to the setting of Proposition 5.1.

The conclusions of Proposition 6.1 help us understand both the global topological structure of the collapse on (M, g) as well as its approximate local geometry. Before explaining these conclusions, it is helpful to first consider the case in which (M, g) is collapsed with a *global* lower bound on the sectional curvature. In this case (M, g) is collapsed to either a point, a 1-dimensional or a 2-dimensional space. The following examples illustrate different collapsing behaviors in this setting:

(0) In the case in which (M, g) is collapsed to a point, M has to be closed and we speak of a *total collapse*. Examples for such a behavior would be a small 3-sphere, a small 3-torus or a small nilmanifold.

(1) A collapse to a 1-dimensional space generically occurs along 2-dimensional fibers, which can be either spheres or tori. For example, the Cartesian products $S^2 \times \mathbb{R}$ (collapse along spheres) and $T^2 \times \mathbb{R}$ (collapse along tori) with very small first factor are each collapsed to a line. The $\mathbb{Z}_2$ quotients of these examples, $\mathbb{R}P^2 \widetilde{\times} \mathbb{R}$ and Klein $\widetilde{\times} \mathbb{R}$, are each collapsed along spheres or tori to a ray. Note that in these examples, M is only fibered by spheres or tori on a generic subset, away from an embedded $\mathbb{R}P^2$ or Klein bottle, where the fibration degenerates.

Such a fibration by spheres or tori does not always degenerate along an embedded hypersurface, as the next example illustrates: Consider a 2-dimensional, rotationally symmetric surface of positive curvature that has only one end and that is asymptotic to a thin cylinder. The Cartesian product of this surface with a small S^1-factor is collapsed along tori to a ray. These tori are products of concentric circles around the tip of the surface with the S^1-factor, and they degenerate to a circle over the tip of the surface. Note that in this example M is diffeomorphic to an open solid torus $S^1 \times B^2$. In a similar way we can construct metrics on B^3 that are collapsed to a ray along 2-spheres which degenerate to a point.
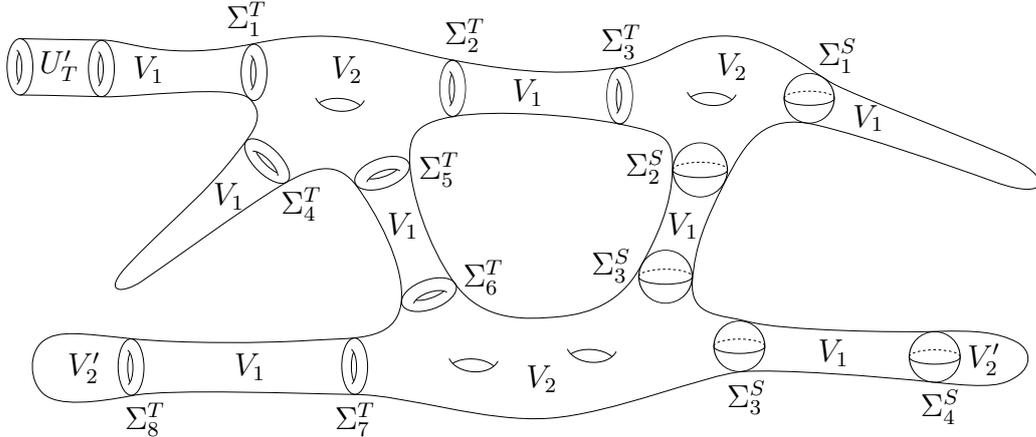
FIGURE 1. A decomposition of M into V_1, V_2 and V_2' along embedded 2-tori $\Sigma_1^T, \ldots, \Sigma_8^T$ and embedded 2-spheres $\Sigma_1^S, \ldots, \Sigma_4^S$. In our proposition, we impose geometric conditions on the collar neighborhood U_T' around the boundary torus of M.

(2) A collapse to a 2-dimensional space generically occurs along S^1-fibers. Basic examples for such a collapse would be Cartesian products $S^1 \times \mathbb{R}^2$, with small S^1-factor, or $S^1 \times \Sigma$, where the S^1-factor is small and Σ is a surface whose curvature is bounded from below. More generally, we can construct collapsing metrics on S^1-fibrations over such surfaces.

Note that similarly as in the previous case, M might only be fibered by S^1-fibers on a generic subset of M. For example if (M, g) is the quotient of $S^1 \times \mathbb{R}^2$ by a cyclic subgroup that acts as non-trivial rotations around 0 on $\mathbb{R}^2$ and as rotations on S^1, then M only possesses such a fibration away from the quotient of $S^1 \times \{0\}$, which is a singular fiber. In this example (M, g) is collapsed to a cone and the tip of this cone corresponds to the quotient of $S^1 \times \{0\}$.

We point out another example in which the fibration on M is degenerate. Consider again a thin 2-dimensional, rotationally symmetric surface of positive curvature that is asymptotic to a thin cylinder and take a Cartesian product with $\mathbb{R}$. This space is collapsed along S^1-fibers to a half plane. The S^1-fibers correspond to concentric circles on the surface. Hence the S^1-fibration only exists away from an embedded line or an embedded solid cylinder.

In the setting of Proposition 6.1, (M, g) is only locally collapsed at scale $\rho_1(x)$ around every point $x \in M$. So the examples given above for the case of the global collapse now only serve as models for these local collapses. One of the main difficulties in the proof of Proposition 6.1 is to understand the transition between those different models, which possibly describe the metric at different scales, and to patch together the induced topological structures on their overlaps.

We now outline the statement of this proposition. If M is locally collapsed to a point, then M must be closed and the collapse must be global. This case is very well understood. So assume that M is not collapsed to a point. In this case we decompose M into three subsets V_1, V_2 and V_2' (see Figure 1). The subset V_1 roughly consists of all points around which we observe a local collapse to a 1-dimensional space, i.e. V_1 is the set of those points whose local models are for example mentioned in (1) of the preceding list. On the subset V_2 we observe local collapses to 2-dimensional spaces and the geometry is locally modeled, for example, by spaces mentioned in (2) of the preceding list. The subset V_2' has the following properties: On a neighborhood around each point $x \in V_2'$ we observe a local collapse to a half-open interval at scale $\rho_1(x)$, but there is a scale $r \ll \rho_1(x)$ at which we observe a collapse to a 2-dimensional space. The difference between V_2 and V_2' will not be important for us in this paper. The subsets V_1, V_2, V_2' are separated from one another by embedded 2-tori denoted by Σ_i^T and embedded 2-spheres denoted by Σ_i^S.

The decomposition of M into the subsets V_1, V_2, V_2' depends on a parameter $\mu > 0$ that governs how well (M, g) is approximated by these local models. Note that, even after fixing μ, this decomposition is still not unique. For example, if around some point $x \in M$ the manifold (M, g) looks like $S^1 \times S^1 \times \mathbb{R}$, with very small first and barely small enough second factor, then it is not clear whether we observe a collapse along S^1-fibers or along 2-tori. Therefore, x could potentially belong to V_1, V_2 or V_2'. On the other hand, this ambiguity allows us to smoothly pass from one collapsing model to another.

The topology of the components of V_1 and V_2' is very restricted and can be classified easily. In order to understand the topology and local geometry of V_2, we decompose V_2 into three subsets $V_{2,\mathrm{reg}}, V_{2,\mathrm{cone}}$ and $V_{2,\partial}$ (see Figure 2). Roughly speaking, $V_{2,\mathrm{reg}}$ is the set of all points where the collapse is modeled on the example $S^1 \times \mathbb{R}^2$ from (2) of the preceding list. Hence this subset admits an S^1-fibration. The set $V_{2,\mathrm{cone}}$ consists of approximately all points whose local model is a finite quotient of $S^1 \times B^2$ as described in (2) of the above list. Around the points of this subset, the manifold is collapsed to a cone. Note that since a cone is regular away from its tip, the components of $V_{2,\mathrm{cone}}$ have bounded diameter and are adjacent to $V_{2,\mathrm{reg}}$. It can moreover be shown that the components of $V_{2,\mathrm{cone}}$ are diffeomorphic to a solid torus $D^2 \times S^1$ and hence bounded by 2-tori, which we will denote by Ξ_i^T.

The set $V_{2,\partial}$ consists of all points whose neighborhoods are collapsed towards a 2-dimensional space with boundary. An example for a local model around such points would be the one involving the surface that is asymptotic to a thin cylinder in (2) of the preceding list (recall that this model is collapsed to a half plane). It is possible to choose $V_{2,\partial}$ such that its components are either diffeomorphic to a solid cylinder $D^2 \times I$ or a solid torus $D^2 \times S^1$ in such a way that the boundary circles of the D^2-factors correspond to the S^1-fibers of $V_{2,\mathrm{reg}}$. The components that are diffeomorphic to a solid torus are bounded by 2-tori, which we denote by Ξ_i^O. Each component that is diffeomorphic to a solid cylinder is positioned within
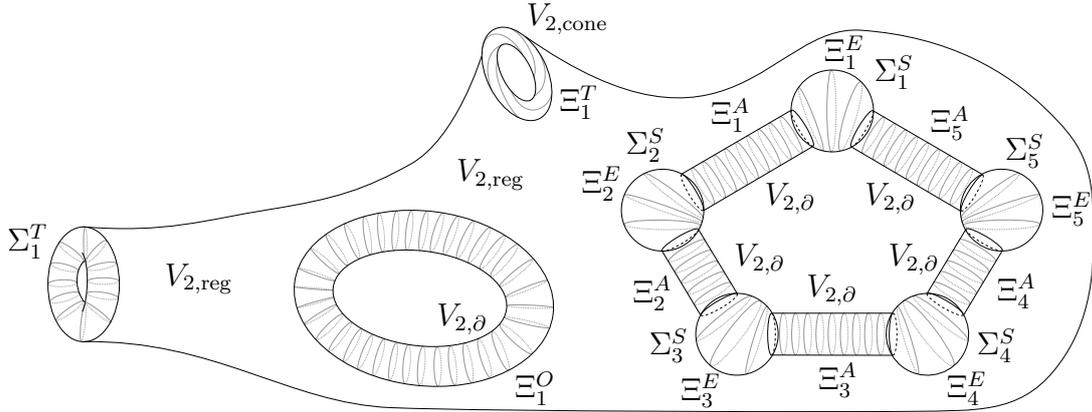
FIGURE 2. An example for a component of V_2. This component has 6 boundary components: one toroidal component Σ_1^T and 5 spherical components $\Sigma_1^S, \ldots, \Sigma_5^S$. The spherical boundary components are connected by 5 components of $V_{2,\partial}$, which are diffeomorphic to solid cylinders $D^2 \times I$. Their annular boundary parts are denoted by $\Xi_1^A, \ldots, \Xi_5^A$. The boundary circles of each of these annuli lie in the Σ_i^S and bound equatorial annuli $\Xi_1^E \subset \Sigma_1^S, \ldots, \Xi_5^E \subset \Sigma_5^S$ within these boundary spheres. The subset $V_{2,\partial}$ also contains a component that is diffeomorphic to a solid torus and bounded by a 2-torus Ξ_1^O. The subset $V_{2,\text{cone}}$ consists of a single solid torus which is bounded by a 2-torus Ξ_1^T. The closure of the complement of $V_{2,\text{cone}} \cup V_{2,\partial}$ is denoted by $V_{2,\text{reg}}$ and carries an S^1-fibration. Thin grey circles illustrate the behavior of this fibration on the boundary components of $V_{2,\text{reg}}$.

V_2 in such a way that its two diskal boundary parts are contained in spherical boundary components Σ_i^S of V_2. That means that if we denote their annular boundary components by Ξ_i^A, then the boundary circles of each Ξ_i^A lie in the spherical boundary components of V_2. Each spherical boundary component $\Sigma_{i'}^S$ of V_2 contains exactly two diskal boundary parts of components of $V_{2,\partial}$ or, in other words, two boundary circles of the annuli Ξ_i^A. These two boundary circles bound an annulus within $\Sigma_{i'}^S$, denoted by Ξ_i^E. The S^1-fibration on $V_{2,\text{reg}}$ restricts to the standard S^1-fibration of this annulus. Summarizing our discussion, we conclude that the boundary of $V_{2,\text{reg}}$ consists of the 2-tori Σ_i^T that are contained in V_2, the 2-tori Ξ_i^T and Ξ_i^O and the union of the annuli Ξ_i^A and Ξ_i^E.

   We now state the precise result in Proposition 6.1. The structure of this proposition is as follows: After stating the assumptions (i)-(iii), we explain what happens in the case in which the manifold is collapsed to a point. If this case does not occur, then the proposition asserts the decomposition of M into subsets V_1, V_2, V_2'. The topological structure of this decomposition is explained in assertions (a1)-(a4). Next, we explain the decomposition of V_2 into $V_{2,\text{reg}}, V_{2,\text{cone}}, V_{2,\partial}$ and list its

topological properties in (b1)-(b4). Finally, in (c1)-(c4), we describe the local collapsing behavior in the different subsets $V_1, V_2, V_{2,\text{reg}}$ and $V_{2,\text{cone}}$.

Proposition 6.1. *For every two continuous functions $\overline{r}, K : (0,1) \to (0,\infty)$ and every $\mu > 0$ there are constants $w_0 = w_0(\mu, \overline{r}, K) > 0$, $0 < s(\mu, \overline{r}, K) < \frac{1}{10}$ and $a(\mu) > 0$, monotonically increasing in μ, such that:*
  Let (M, g) be a compact manifold with boundary such that:

  (i) *Each component T of ∂M is an incompressible torus and for each such T there is a closed subset $U_T' \subset M$ that is diffeomorphic to $T^2 \times I$ such that $T \subset \partial U_T'$ and such that the boundary components of U_T' have distance of at least 2. Moreover, there is a fibration $p_T : U_T' \to I$ such that the T^2-fiber through every $x \in U_T'$ has diameter $< w_0 \rho_1(x)$.*
  (ii) *For all $x \in M$ for which $B(x, \rho_1(x)) \subset \text{Int } M$ we have*

  $$\text{vol } B(x, \rho_1(x)) < w_0 \rho_1^3(x).$$

  (iii) *For all $w \in (w_0, 1)$, $r < \overline{r}(w)$ and $x \in M$ we have: if $B(x, r) \subset \text{Int } M$ and $\text{vol } B(x, r) > wr^3$ and $r < \rho(x)$, then $|\text{Rm}|, r|\nabla \text{Rm}|, r^2|\nabla^2 \text{Rm}| < K(w)r^{-2}$ on $B(x, r)$.*

Then either M is closed and diffeomorphic to an infra-nilmanifold or a manifold that also carries a metric of non-negative sectional curvature, and $\text{diam } M < \mu \rho_1(x)$ for all $x \in M$, or the following holds:
  There are finitely many embedded 2-tori Σ_i^T and 2-spheres $\Sigma_i^S \subset M$, which are pairwise disjoint and disjoint from ∂M, as well as closed subsets $V_1, V_2, V_2' \subset M$ such that (see Figure 1 for an illustration)

  (a1) *$M = V_1 \cup V_2 \cup V_2'$, the interiors of the sets V_1, V_2 and V_2' are pairwise disjoint and $\partial V_1 \cup \partial V_2 \cup \partial V_2' = \partial M \cup \bigcup_i \Sigma_i^T \cup \bigcup_i \Sigma_i^S$. Obviously, no two components of the same set share a common boundary component.*
  (a2) *$\partial V_1 = \partial M \cup \bigcup_i \Sigma_i^T \cup \bigcup_i \Sigma_i^S$. In particular, $V_2 \cap V_2' = \emptyset$ and $V_2 \cup V_2'$ is disjoint from ∂M.*
  (a3) *V_1 consists of components diffeomorphic to one of the following manifolds:*

  $$T^2 \times I, \ S^2 \times I, \ \text{Klein}^2 \widetilde{\times} I, \ \mathbb{R}P^2 \widetilde{\times} I, \ D^2 \times S^1, \ D^3,$$

  a T^2 bundle over S^1, $S^2 \times S^1$ or the union of two (possibly different) components listed above along their T^2- or S^2-boundary.
  (a4) *Every component of V_2' has exactly one boundary component and this component borders V_1 on the other side. Moreover, every component of V_2' is diffeomorphic to*

  $$D^2 \times S^1, \ D^3, \ L(p,q) \setminus B^3, \ \text{Klein}^2 \widetilde{\times} I.$$

We can further characterize the components of V_2 (see Figure 2 for an illustration): In V_2 we find embedded 2-tori Ξ_i^T and Ξ_i^O which are pairwise disjoint and disjoint from the boundary ∂V_2. Furthermore, there are embedded closed 2-annuli $\Xi_i^A \subset V_2$ whose interior is disjoint from the Ξ_i^T, Ξ_i^O and ∂V_2 and whose boundary components lie in the components of ∂V_2 that are spheres. Each spherical component of ∂V_2 contains exactly two such boundary components, which separate

the sphere into two (polar) disks and one (equatorial) annulus Ξ_i^E. We also find closed subsets $V_{2,\mathrm{reg}}, V_{2,\mathrm{cone}}, V_{2,\partial} \subset V_2$ such that

(b1) $V_{2,\mathrm{reg}} \cup V_{2,\mathrm{cone}} \cup V_{2,\partial} = V_2$ and the interiors of these subsets are pairwise disjoint. Moreover, $\partial V_{2,\mathrm{reg}}$ is the union of $\bigcup_i \Xi_i^T \cup \bigcup \Xi_i^O \cup \bigcup_i \Xi_i^A \bigcup_i \Xi_i^E$ and the components of ∂V_2 which are diffeomorphic to tori.

(b2) $V_{2,\mathrm{reg}}$ carries an S^1-fibration which is compatible with its boundary components and all its annular regions.

(b3) The components of $V_{2,\mathrm{cone}}$ are diffeomorphic to solid tori ($\approx D^2 \times S^1$) and bounded by the Ξ_i^T such that the fibers of $V_{2,\mathrm{reg}}$ on their boundaries are not nullhomotopic inside $V_{2,\mathrm{cone}}$.

(b4) The components of $V_{2,\partial}$ are either solid tori ($\approx D^2 \times S^1$) and bounded by the Ξ_i^O such that the S^1-fibers of $V_{2,\mathrm{reg}}$ on the Ξ_i^O are nullhomotopic inside the $V_{2,\partial}$ or they are solid cylinders ($\approx D^2 \times I$) such that their two diskal boundary components are polar disks on ∂V_2 and their annular boundary component is one of the Ξ_i^A. Every polar disk and every Ξ_i^A bounds such a component on exactly one side.

We now explain the geometric properties of this decomposition:

(c1) For every $x \in V_1$, the ball $(B(x, \rho_1(x)), \rho_1^{-2}(x)g, x)$ is μ-close (in the Gromov-Hausdorff sense) to a 1-dimensional interval $(J, g_{\mathrm{eucl}}, \overline{x})$ or an S^1 of length $> a(\mu)$.

Consider the case in which x lies in a component of V_1 that is diffeomorphic to $T^2 \times I$. Then $(J, \overline{x})$ can be chosen to be $((-b, 1), 0)$ for some $0 \leq b \leq 1$, depending on how close x is to ∂M. If $y \in B(x, \rho_1(x))$ is a point that is at least $\frac{1}{25}$ away from the endpoints of J via this identification, then we can find an open subset U with $B(y, \frac{1}{50}\rho_1(x)) \subset U \subset B(y, \frac{1}{25}\rho_1(x))$, a subinterval $J' \subset (-b, 1)$ and a map $p : (U, \rho_1^{-2}(x)g) \rightarrow (J', g_{\mathrm{eucl}})$ such that

(α) p is 1-Lipschitz and its differential has an eigenvalue $> 1 - \mu$ everywhere,

(β) U is diffeomorphic to $S^2 \times J'$ or $T^2 \times J'$ such that p is the projection map onto the interval,

(γ) the fibers of p have diameter at most μ with respect to the metric $\rho_1^{-2}(x)g$.

(c2) For every $x \in V_2$, the ball $(B(x, \rho_1(x)), \rho_1^{-2}(x)g, x)$ is μ-close to a 2-dimensional pointed Alexandrov space $(X, \overline{x})$ of area $> a$.

(c3) For every $x \in V_{2,\mathrm{reg}}$, the ball $(B(x, s\rho_1(x)), s^{-2}\rho_1^{-2}(x)g, x)$ is μ-close to a standard 2-dimensional Euclidean ball $(B = B_1(0), g_{\mathrm{eucl}}, \overline{x} = 0)$.

Moreover, there is an open subset U with $B(x, \frac{1}{2}s\rho_1(x)) \subset U \subset B(x, s\rho_1(x))$, a smooth map $p : U \rightarrow \mathbb{R}^2$ such that:

(α) there are vector fields X_1, X_2 on U such that $dp(X_i) = \frac{\partial}{\partial x_i}$ and X_1, X_2 are almost orthonomal, i.e. $|\langle X_i, X_j \rangle - \delta_{ij}| < \mu$ for all $i, j = 1, 2$,

(β) U is diffeomorphic to $B^2 \times S^1$ such that $p : U \rightarrow p(U)$ corresponds to the projection onto B^2 and the S^1-fibers of p are isotopic in U to the fiber of the fibration on $V_{2,\mathrm{reg}}$ that passes through x.

(γ) *the fibers of p as well as the fibers of $V_{2,\mathrm{reg}}$, belonging to the fibration mentioned in assertion (b2), that are contained in U, have diameter at most μ and both families of fibers enclose an angle $< \mu$ with each other whenever they intersect.*

(c4) *For every $x \in V_{2,\mathrm{cone}}$, the ball $B(x, \frac{1}{10}\rho_1(x))$ covers the component of $V_{2,\mathrm{cone}}$ in which x lies.*

Proof. Our proposition is a consequence of the arguments used for the proof of Theorem 0.2 in Morgan-Tian ([21]). In the following, we will point out the intermediate steps in this proof that imply the assertions of our proposition and we will explain how some of its arguments have to be modified slightly to fit our assumptions.

First note that our proposition and Theorem 0.2 use different philosophies. Our proposition asserts that there is a small $w_0 > 0$ with the property that *every* "w_0-collapsed" manifold (M, g) satisfies the desired topological and geometric assertions while Theorem 0.2 claims that whenever we have a sequence of manifolds (M_n, g_n) that are "w_n-collapsed" with $\lim_{n \to \infty} w_n = 0$, then these assertions hold for sufficiently large n. These two philosophies are equivalent, similarly as the ε-δ-criterion for continuity is in general equivalent to the sequence criterion. Under this equivalence, assumption (ii) of our proposition implies assumption 1. of Theorem 0.2, which reads

> *1. For each point $x \in M_n$ there exists a radius $\rho = \rho_n(x)$ such that the ball $B_{g_n}(x, \rho)$ has volume at most $w_n \rho^3$ and all the sectional curvatures of the restriction of g_n to this ball are all at least $-\rho^{-2}$.*

Except for the higher derivative bounds, which are not really needed in the proof of Theorem 0.2, assumption (iii) of our proposition implies assumption 3. of Theorem 0.2, which reads

> *3. For every $w' > 0$ there exist $\overline{r} = \overline{r}(w') > 0$ and constants $K_m = K_m(w') < \infty$ for $m = 0, 1, 2, \ldots$, such that for all n sufficiently large, and any $0 < r \leq \overline{r}$, if the ball $B_{g_n}(x, r)$ has volume at least $w' r^3$ and sectional curvatures at least $-r^{-2}$, then the curvature and its mth-order covariant derivatives at x, $m = 1, 2, \ldots$, are bounded by $K_0 r^{-2}$ and $K_m r^{-m-2}$, respectively.*

Lastly, assumption (i) of our proposition translates to the following condition (we use "[...]" to indicate repetition):

> *Each component T of ∂M_n is an incompressible torus [...] such that the T^2-fiber through every $x \in U_T'$ has diameter $< w_n \rho_1(x)$.*

This condition does not imply assumption 2. of Theorem 0.2:

> *2. Each component of the boundary of M_n is an incompressible torus of diameter at most w_n and with a topologically trivial collar containing all points withinin distance 1 of the boundary on which the sectional curvatures are between $-5/16$ and $-3/16$.*

It will become evident later, however, that either condition is sufficient for our purposes.

Next, Morgan and Tian make the following simplifying assumption:

> *Assumption 1. For each n, no connected, closed component of M_n admits a Riemannian metric of non-negative sectional curvature.*

In our proposition we don't want to make this assumption. So we have to find alternative arguments whenever this assumption is used. Assumption 1 is essentially used at two places in the proof of Theorem 0.2. Firstly, it is used to rule out certain topologies in the description of the geometric decomposition of (M, g). This issue can be resolved by adding these topologies to the list of possible topologies, e.g. in assertion (a3). Secondly, it is used in the proof of [21, Lemma 1.5] to show that the function $\rho_n(x)$ can be rechosen to be sufficiently regular and $\leq \frac{1}{2} \operatorname{diam} M$. The regularity assumption is automatically satisfied by our choice $\rho_1(x) = \min\{\rho(x), 1\}$ and by any multiple $\lambda\rho_1(x)$ for $0 < \lambda \leq 1$. We will now argue that, nevertheless, we can add the simplifying assumption that

$$\rho_1(x) \leq \max\{C, \mu^{-1}\} \operatorname{diam} M \qquad \text{for all} \qquad x \in M$$

for some universal constant $C < \infty$. This bound will be enough for our purposes, because we can choose $\lambda = \frac{1}{2}\min\{C^{-1}, \mu\}$ to ensure that the function $\rho_n(x)$ in [21] is bounded by $\frac{1}{2}\operatorname{diam} M$.

So assume for the moment that $\rho_1(x) > \max\{C, \mu^{-1}\} \operatorname{diam} M$ for some $x \in M$ and some constant $C < \infty$ that we will determine later. Since $M \subset B(x, \rho_1(x))$, this inequality holds for all $x \in M$ and it implies

$$\operatorname{diam} M < \min\{C^{-1}, \mu\}\rho_1(x) \leq \mu\rho_1(x).$$

By condition (i), M must be closed. It now follows from [9, Corollary 0.13] that we can choose C uniformly such that the lower sectional curvature bound of $-\rho_1^{-2}(x)$ on (M, g) together with the diameter bound imply that M either supports a metric of non-negative sectional curvature or is infranil. This implies that the assertion in the paragraph immediately after condition (iii) is satisfied and we are done. So we may assume from now on that $\rho_1(x) \leq \max\{C, \mu^{-1}\} \operatorname{diam} M$ for all $x \in M$ or, equivalently, that the function $\rho_n(x)$ in [21], being equal to $\lambda\rho_1(x)$, is bounded from above by $\frac{1}{2}\operatorname{diam} M$.

Next, we have to construct the sets V_1, V_2, V_2' as well as $V_{2,\text{reg}}, V_{2,\text{cone}}, V_{2,\partial}$. These sets will arise from the construction of the sets $V_{n,1}$ and $V_{n,2}$ in [21]. Note that the construction of $V_{n,1}$ and $V_{n,2}$ is carried out in several steps. In the following we provide an overview over this construction and point out the necessary changes for the proof of our proposition.

In [21, Proposition 5.2], Morgan-Tian define $X_{n,1} \subset M_n$ to be the set of all points at which (M_n, g_n) is locally collapsed to an open interval. The statement of Proposition 5.2 is that $X_{n,1}$ can be extended to a subset $X_{n,1} \subset U_{n,1} \subset M_n$ such that the components of $U_{n,1}$ are diffeomorphic to $S^2 \times (0, 1)$, $T^2 \times (0, 1)$ or a 2-torus bundle over the circle and such that the ends of $U_{n,1}$ are geometrically controlled. It follows from the proof of this proposition, that all points $x \in U_{n,1}$ satisfy the geometric characterization of assertion (c1) in our proposition. Note that in our setting, due to the lack of Assumption 1, we have to include 2-sphere bundles over the circle to the list of possible topologies of $U_{n,1}$.

Next, Morgan-Tian analyze the components of $A \subset M_n \setminus U_{n,1}$. In [21, Lemma 5.3] they conclude that for each such A there are three possibilities:

(1) (M_n, g_n) is locally collapsed in A to a 2-dimensional space of area bounded from below.

(2) (M_n, g_n) is globally collapsed in A to a half-open interval such that one of its endpoints corresponds to a point in A. In this case A is diffeomorphic to $T^2 \times I$ and adjacent to the boundary of M_n or A is diffeomorphic to $D^2 \times S^1$, $\text{Klein}^2 \widetilde{\times} I$, D^2 or $\mathbb{R}P^2 \widetilde{\times} I$.

(3) A is "a component which is close to an interval but which expands to be close to a standard 2-dimensional ball" (compare with [21, Definition 5.4]). This means roughly that after decreasing $\rho_n(x)$ by a small factor, A satisfies the conditions of (1).

At this point we need to recall that in our setting we are using a different characterization of the metric around the boundary of M. So we have to be careful with arguments that involve points close to the boundary of M. It can however be shown that, for sufficiently small w_0, every component A that is adjacent to ∂M satisfies (2). Using their conclusion, Morgan-Tian define $U'_{n,1}$ to be the union of $U_{n,1}$ with all such components A that satisfy (2). Note that, again all points $x \in U'_{n,1}$ satisfy the geometric characterization of assertion (c1) in our proposition, since the important part of this assertion involves points y that are sufficiently far away from the endpoints of the interval towards which we observe the local collapse. We will later choose V_1 to be a subset of $U'_{n,1}$. This will then establish assertion (c1).

After constructing $U'_{n,1}$, Morgan-Tian remove a small bit of each open end of $U'_{n,1}$ and call the new (closed) subset $W_{n,2}$ and the closure of its complement $W_{n,1}$ (see [21, subsec 5.3]). The reason for doing this is that this way the ends of $W_{n,2}$ are equipped with fibrations by 2-tori or 2-spheres that are compatible with the boundary components of $W_{n,2}$ and the fibrations of the adjacent components of $W_{n,1}$. For every component $A \subset M_n \setminus U'_{n,1}$, Morgan-Tian denote the corresponding component of $W_{n,2}$ by $\widehat{A} \supset A$. Note that the change between A and $\widehat{A}$ is generally negligible. So if A belongs to case (1) in the preceding list, then we will still interpret $\widehat{A}$ to be locally collapsed to a 2-dimensional space; analogously for case (3). We will later choose the subset $V_2 \subset M$ such that for each of its components $\mathcal{C} \subset V_2$ there is some component A from case (1) such that $A \subset \mathcal{C} \subset \widehat{A}$. The same is true for V'_2, with case (3) instead of case (1). Hence, using the arguments in [21, subsec 5.3], assertion (c2) follows.

Next, Morgan-Tian analyze the geometry of the subset $W_{n,2}$ in [21, subsec 5.4]. In order to do this, they use the following intuition: Around every point $x \in W_{n,2}$ the Riemannian manifold (M_n, g_n) is locally collapsed to some 2-dimensional Alexandrov space (X, d), which depends on x. Every point $y \in X$ satisfies one of the following characterizations, which depend on certain parameters (compare with [21, Theorem 3.22]):

(1) y is regular, i.e. after enlarging (X, d) by some uniform factor, the geometry around y is close to a 2-dimensional Euclidean ball.

(2) y is conical, i.e. after rescaling, the geometry around y looks like a subset of a 2-dimensional cone.

(3) y is close to a regular boundary point, i.e. the local geometry around y is close to a half plane.

(4) y is close to a corner, i.e. the local geometry around y is close to a 2-dimensional sector.

Based on this classification of the points of the spaces that $W_{n,2}$ is locally collapsed to, Morgan-Tian derive an induced classification of the points of $W_{n,2}$. As a result, they obtain a covering of $W_{n,2}$ by subsets $U_{2,\text{generic}}$ (case (1)), finitely many "ε'-solid tori near interior cone points" (case (2)), U_{cyl}, being the union of "ε'-solid cylinders near flat 2-dimensional boundary points" (case (3)) and finitely many "3-balls near 2-dimensional boundary corners" (case (4)). The subset $U_{2,\text{generic}}$ carries an S^1-fibration along which the collapse occurs. For the exact statements see [21, Lemmas 5.7, 5.9]. Finally, Morgan-Tian define the subsets $W'_{n,1}, W'_{n,2}$ by removing the "3-balls near 2-dimensional boundary corners" from $W_{n,2}$ and adding their closures to $W_{n,1}$.

Eventually, in [21, subsec 5.5] Morgan-Tian construct the subset $V_{n,1}$. In this construction, they first slightly deform the boundary between $W'_{n,1}$ and $W'_{n,2}$ such that the S^1-fibration on $W'_{n,2} \cap U_{2,\text{generic}}$ is compatible with each boundary component. After redefining $W'_{n,1}$ in that way, they set $V_{n,1} := W'_{n,1}$. For our proposition, we define $V_1 \subset M$ to be the union of this new subset $W'_{n,1}$ minus the components that were added as deformations of "3-balls near 2-dimensional boundary corners" when we passed from $W_{n,1}$ to $W'_{n,1}$. We define the subsets V_2, V'_2 to be the unions of components in the closure of $M \setminus V_1$ depending on whether the corresponding component A belonged to case (1) or (3) in the list before the previous list. The surfaces Σ_i^T and Σ_i^S are defined to be the boundary components of $V_2 \cup V'_2$. Assertions (a1)-(a3) follow immediately. The topology of the components of V'_2, as asserted in (a4), can be deduced using a better lower bound on the sectional curvature at the local scale. We omit the proof of this assertion, since it has only been stated for completeness. For our purposes it will just be important that each component of V'_2 has only one boundary component.

It remains to construct the subsets $V_{2,\text{reg}}, V_{2,\text{cone}}$ and $V_{2,\partial}$. For this we look at the construction of $V_{2,n}$ in [21]. The subset $V_{2,n}$ arises from $W'_{2,n}$ by removing deformations of certain "ε'-solid tori near interior cone points" and "ε'-solid cylinders near flat 2-dimensional boundary points". For our proposition we denote by $V_{2,\text{cone}}$ the union of all these deformations of "ε'-solid cylinders near flat 2-dimensional boundary points" within V_2 and by $V_{2,\partial}$ the union of all these deformations of "ε'-solid cylinders near flat 2-dimensional boundary points" together with the deformations of "3-balls near 2-dimensional boundary corners". So the components of $V_{2,\text{cone}}$ are solid tori; we denote their boundaries by Ξ_i^T. Note that for each deformed "3-balls near 2-dimensional boundary corners" and every

spherical boundary component of V_2 there are exactly two diskal boundary components of deformed "ε'-solid cylinders near flat 2-dimensional boundary points" that are contained in the boundary of this deformed 3-ball or spherical boundary component. So the deformations of the "ε'-solid cylinders near flat 2-dimensional boundary points" and the "3-balls near 2-dimensional boundary corners" form chains, which may or may not close up. Chains that do not close up are homeomorphic to solid cylinders $\approx D^2 \times I$ whose diskal boundary components are contained in spherical boundary components of V_2. After smoothing out the corners equivariantly with respect to the adjacent S^1-fibration, the boundaries of these solid cylinders are smooth annuli; we denote them by Ξ_i^A. Note that $\partial \Xi_i^A \subset \partial V_2$ and every spherical boundary component of V_2 contains exactly two circles of $\bigcup_i \partial \Xi_i^A$, which enclose an annulus $\Xi_i^E \subset \partial V_2$. A chain that does close up is homeomorphic to a solid torus $\approx D^2 \times S^1$ and after smoothing equivariantly, its boundary 2-torus is denoted by Ξ_i^O. This establishes assertions (b1)-(b4).

Assertions (c4) follows from the construction process and assertion (c3) follows from the construction process together with the statement and proof of [21, Proposition 4.4].

Finally, we make a remark on the choice of the parameters μ, w_0, s and $a(\mu)$. The constants in [21] that determine the preciseness of the collapse or the closeness with respect to the Gromov-Hausdorff distance are mainly assumed to be fixed during the construction process of the subsets $V_{1,n}$ and $V_{2,n}$. This is due to the fact that the purpose of [21] was to establish a purely topological theorem. Our proposition, however, also contains a geometric characterization of the collapse, as presented in assertions (c1)-(c4). These assertions involve a degree of preciseness μ, which can be chosen arbitrarily in the beginning of our proposition. Our geometric characterization is more or less a byproduct of the proof in [21] and the Lemmas and Propositions asserting the desired geometric statements, which can mainly be found in section 4 of [21], do allow the choice of arbitrarily small preciseness parameters. Allowing these parameters to depend on μ will however entail a μ-dependence of the collapsing degree w_0 and the lower bound a on the diameters or areas of 1 or 2-dimensional collapsing models. The constant s, which also depends on μ, roughly characterizes at which scale we can distinguish local models of 2-dimensional Alexandrov spaces with preciseness μ. Due to this dependence and the dependence of the area bound, s also depends on μ. $\square$

7. Further geometric properties of the thin part

In this section we will identify parts in the decomposition of Proposition 6.1 that become non-collapsed when we pass to the universal cover.

Lemma 7.1. *There are constants $\mu_0, w_1 > 0$, where w_1 only depends on $s(\varepsilon, \mu_0, \overline{r}, K)$, such that: Consider the situation of Proposition 6.1 and assume $\mu \leq \mu_0$. Assume moreover that (M, g) can be extended to a complete Riemannian manifold (M_0, g_0) without boundary. Let $x \in M$ and consider one of the following cases:*

  (i) $x \in \mathcal{C}$ where $\mathcal{C}$ is a component of V_2 with the property that the S^1-fiber of $\mathcal{C} \cap V_{2,\mathrm{reg}}$ has infinite order in $\pi_1(M_0)$ or

(ii) $x \in \mathcal{C}$ where $\mathcal{C}$ is a component of V_1 that is diffeomorphic to $T^2 \times I$, is not adjacent to any component of V_2' and whose cross-sectional tori are incompressible in M_0.

Then $\operatorname{vol} \widetilde{B}(\widetilde{x}, r) \geq w_1 r^3$ for all $r \leq \rho_1(x)$ where $\widetilde{B}(\widetilde{x}, r)$ denotes the r-ball around a lift $\widetilde{x}$ of x in the universal cover $\widetilde{M_0}$ of M_0.

We remark that the Lemma stays true if in (ii) we consider all components of V_1 whose generic fibers are incompressible tori.

The proof of the Lemma uses comparison geometry. For any three points x_0, x_1, x_1 in a metric space (X, d) we can construct a triangle $\triangle \overline{x}_0 \overline{x}_1 \overline{x}_2 \subset \mathbb{H}^2$ in the hyperbolic plane with the property that $\operatorname{dist}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for all $i, j = 0, 1, 2$. Its angles do not depend on the choice of the $\overline{x}_i$ and are called *comparison angles*. We will write $\widetilde{\sphericalangle} x_1 x_0 x_2 := \sphericalangle \overline{x}_1 \overline{x}_i \overline{x}_2$. Note that this construction can be carried out in any model space of constant curvature, but in this paper we will only be interested in the model space of constant curvature -1. Using this notion, we define the notion of strainers as follows:

Definition 7.2 ((m, δ)-strainer)**.** Let $\delta > 0$ and $m \geq 1$. A $2m$-tuple $(a_1, b_1, \ldots, a_m, b_m)$ of points in a metric space (X, d) is called an (m, δ)-*strainer around a point* $x \in X$ if

$$\widetilde{\sphericalangle} a_i x b_j, \ \widetilde{\sphericalangle} a_i x a_j, \ \widetilde{\sphericalangle} b_i x b_j > \tfrac{\pi}{2} - \delta \qquad \text{for all} \quad i \neq j, \quad i, j = 1, \ldots, m$$

and

$$\widetilde{\sphericalangle} a_i x b_i > \pi - \delta \qquad \text{for all} \quad i = 1, \ldots, m.$$

The strainer is said to have *size* r if $d(x, a_i) = d(x, b_i) = r$ for all $i = 1, \ldots, m$ or *size* $> r$ if $d(x, a_i), d(x, b_i) > r$ for all $i = 1, \ldots, m$.

We will also need the following

Definition 7.3 ($(m + \tfrac{1}{2}, \delta)$-strainer)**.** Let $\delta > 0$ and $m \geq 1$. A $2m + 1$-tuple $(a_1, b_1, \ldots, a_m, b_m, a_{m+1})$ of points in a metric space (X, d) is called an $(m + \tfrac{1}{2}, \delta)$-*strainer around a point* $x \in X$ if

$$\begin{aligned}
\widetilde{\sphericalangle} a_i x b_j &> \tfrac{\pi}{2} - \delta &\text{for all} \quad & i \neq j, \quad i = 1, \ldots, m+1, \quad j = 1, \ldots, m, \\
\widetilde{\sphericalangle} a_i x a_j &> \tfrac{\pi}{2} - \delta &\text{for all} \quad & i \neq j, \ i, j = 1, \ldots, m+1, \\
\widetilde{\sphericalangle} b_i x b_j &> \tfrac{\pi}{2} - \delta &\text{for all} \quad & i \neq j, \ i, j = 1, \ldots, m
\end{aligned}$$

and

$$\widetilde{\sphericalangle} a_i x b_i > \pi - \delta \qquad \text{for all} \quad i = 1, \ldots, m.$$

The strainer is said to have *size* r if $d(x, a_i) = d(x, b_i) = r$ for all $i = 1, \ldots, m$ or $m + 1$ or *size* $> r$ if $d(x, a_i), d(x, b_i) > r$ for all $i = 1, \ldots, m$ or $m + 1$.

Proof. By volume comparison, it suffices to prove the desired inequality only for $r = \rho_1(x)$.

Consider first case (i): Since the fibers on $\mathcal{C} \cap V_{2,\mathrm{reg}}$ are non-contractible, we conclude that $\mathcal{C}$ is disjoint from $V_{2,\partial}$. So either $x \in V_{2,\mathrm{reg}}$ or $x \in V_{2,\mathrm{cone}}$. In the second case we can apply Proposition 6.1(c4) and find an $x' \in B(x, \tfrac{1}{10}\rho_1(x)) \cap V_{2,\mathrm{reg}}$

and $\rho_1(x') > \frac{1}{2}\rho_1(x)$. Let $\widetilde{x}$ be a lift of x in the universal cover $\pi : \widetilde{M_0} \to M_0$. Since $\widetilde{B}(\widetilde{x}', \frac{1}{2}\rho_1(x)) \subset \widetilde{B}(\widetilde{x}, \rho_1(x))$, for some lift $\widetilde{x}'$ of x', we can replace x by x'. So assume without loss of generality that $x \in V_{2,\text{reg}}$.

Consider now the map $p : U \to \mathbb{R}^2$ and the metric $g' = s^{-2}\rho_1^{-2}(x)g_0$ on M_0. For the rest of the proof of case (i) we will only work with the metric g' on M_0 as opposed to g, and we will bound the g'-volume of a 1-ball in the universal cover from below by a universal constant. Observe that the sectional curvatures of the metric g' are bounded from below by -1 on this ball. In the following we will denote by $\delta_k(\mu_0)$ a positive constant that depends on $\mu_0 > 0$ and that goes to zero as μ_0 goes to zero. We will then later choose μ_0 small enough so that all constants δ_k are sufficiently small. In the following paragraphs we carry out concepts that can also be found in [4] or [5].

By the properties of x, we can find a $(2, \delta_1(\mu_0))$-strainer (a_1, b_1, a_2, b_2) of size $\frac{1}{2}$ around x (here $\delta_1(\mu_0)$ is a suitable constant as mentioned above). Recall that this entails that $\text{dist}(a_i, x) = \text{dist}(b_i, x) = \frac{1}{2}$ for all $i = 1, 2$. In the universal cover $\widetilde{M_0}$, we can now choose lifts $\widetilde{x}, \widetilde{a}_i, \widetilde{b}_i$ such that $\text{dist}(\widetilde{a}_i, \widetilde{x}) = \text{dist}(a_i, x) = \frac{1}{2}$ and $\text{dist}(\widetilde{b}_i, \widetilde{x}) = \text{dist}(b_i, x) = \frac{1}{2}$. Since the universal covering map is 1-Lipschitz, we obtain furthermore $\text{dist}(\widetilde{a}_i, \widetilde{b}_j) \geq \text{dist}(a_i, b_j)$, $\text{dist}(\widetilde{a}_1, \widetilde{a}_2) \geq \text{dist}(a_1, a_2)$ and $\text{dist}(\widetilde{b}_1, \widetilde{b}_2) \geq \text{dist}(b_1, b_2)$. So all the comparison angles in the universal cover are at least as large as those on M_0 and hence we conclude that $(\widetilde{a}_1, \widetilde{b}_1, \widetilde{a}_2, \widetilde{b}_2)$ is a $(2, \delta_1(\mu_0))$-strainer around $\widetilde{x}$ of size $\frac{1}{2}$.

Next, we extend this strainer to a $2\frac{1}{2}$-strainer around $\widetilde{x}$. To do this, observe that by the property of the map p there is a sequence $\widetilde{x}_n$ of lifts of x in $\widetilde{M_0}$ which is unbounded and whose consecutive distance is at most $2\mu_0$. So for sufficiently small μ_0, we can find an n such that with $\widetilde{y} = \widetilde{x}_n$ we have

$$|\text{dist}(\widetilde{x}, \widetilde{y}) - 2\sqrt{\mu_0}| \leq 2\mu_0.$$

Note that $\widetilde{x}$ and $\widetilde{y}$ both project to x under the universal covering projection $\pi : \widetilde{M_0} \to M_0$. It follows that for $i = 1, 2$

$$\text{dist}(\widetilde{y}, \widetilde{a}_i) \geq \text{dist}(x, a_i) = \frac{1}{2} \qquad \text{and} \qquad \text{dist}(\widetilde{y}, \widetilde{b}_i) \geq \frac{1}{2}.$$

So in the triangle $\triangle\widetilde{y}\widetilde{x}\widetilde{a}_i$, the segment $|\widetilde{y}\widetilde{a}_i|$ is the longest, which means that it must be opposite to the largest comparison angle, i.e.

$$\widetilde{\sphericalangle}\widetilde{a}_i\widetilde{x}\widetilde{y} \geq \widetilde{\sphericalangle}\widetilde{x}\widetilde{y}\widetilde{a}_i.$$

Since $\text{dist}(\widetilde{x}, \widetilde{y}) \to 0$ as $\mu_0 \to 0$, we find using hyperbolic trigonometry that

(7.1) $$\widetilde{\sphericalangle}\widetilde{a}_i\widetilde{x}\widetilde{y} + \widetilde{\sphericalangle}\widetilde{x}\widetilde{y}\widetilde{a}_i + \widetilde{\sphericalangle}\widetilde{y}\widetilde{a}_i\widetilde{x} > \pi - \delta_2(\mu_0)$$

and

(7.2) $$\widetilde{\sphericalangle}\widetilde{y}\widetilde{a}_i\widetilde{x} < \delta_2(\mu_0).$$

The last three inequalities imply

$$2\widetilde{\sphericalangle}\widetilde{a}_i\widetilde{x}\widetilde{y} > \pi - 2\delta_2(\mu_0).$$

The same is true with $\widetilde{a}_i$ replaced by $\widetilde{b}_i$. So

$$(7.3) \qquad \widetilde{\sphericalangle} \widetilde{a}_i \widetilde{x} \widetilde{y} > \tfrac{\pi}{2} - \delta_2(\mu_0) \qquad \text{and} \qquad \widetilde{\sphericalangle} \widetilde{b}_i \widetilde{x} \widetilde{y} > \tfrac{\pi}{2} - \delta_2(\mu_0).$$

Hence $(\widetilde{a}_1, \widetilde{b}_1, \widetilde{a}_2, \widetilde{b}_2, \widetilde{y})$ is a $(2\tfrac{1}{2}, \delta(\mu_0))$-strainer around $\widetilde{x}$.

Since $|\operatorname{dist}(\widetilde{y}, \widetilde{a}_i) - \operatorname{dist}(\widetilde{x}, \widetilde{a}_i)| < 2\sqrt{\mu_0} + 2\mu_0$ and $|\operatorname{dist}(\widetilde{y}, \widetilde{b}_i) - \operatorname{dist}(\widetilde{x}, \widetilde{b}_i)| < 2\sqrt{\mu_0} + 2\mu_0$, we conclude that $(\widetilde{a}_1, \widetilde{b}_1, \widetilde{a}_2, \widetilde{b}_2)$ is a $(2, \delta_3(\mu_0))$-strainer around $\widetilde{y}$ of size $\geq \tfrac{1}{2} - 2\sqrt{\mu_0} - 2\mu_0$. We now show that, symmetrically, $(\widetilde{a}_1, \widetilde{b}_1, \widetilde{a}_2, \widetilde{b}_2, \widetilde{x})$ is a $2\tfrac{1}{2}$-strainer around $\widetilde{y}$ of arbitrarily good precision: By comparison geometry

$$\widetilde{\sphericalangle} \widetilde{a}_i \widetilde{x} \widetilde{y} + \widetilde{\sphericalangle} \widetilde{b}_i \widetilde{x} \widetilde{y} + \widetilde{\sphericalangle} \widetilde{a}_i \widetilde{x} \widetilde{b}_i \leq 2\pi.$$

Together with (7.3) and the strainer inequality at $\widetilde{x}$, this yields

$$\widetilde{\sphericalangle} \widetilde{a}_i \widetilde{x} \widetilde{y} < \tfrac{\pi}{2} + \delta_1(\mu_0) + \delta_2(\mu_0).$$

Combining this bound with (7.1) and (7.2) yields

$$\widetilde{\sphericalangle} \widetilde{x} \widetilde{y} \widetilde{a}_i > \tfrac{\pi}{2} - \delta_1(\mu_0) - 3\delta_2(\mu_0) = \tfrac{\pi}{2} - \delta_3(\mu_0).$$

The same estimate holds for $\widetilde{\sphericalangle} \widetilde{x} \widetilde{y} \widetilde{b}_i$.

Let $\widetilde{m}$ be the midpoint of a minimizing segment joining $\widetilde{x}$ and $\widetilde{y}$. We will now show that $(\widetilde{a}_1, \widetilde{b}_1, \widetilde{a}_2, \widetilde{b}_2, \widetilde{y}, \widetilde{x})$ is a 3-strainer around $\widetilde{m}$ of arbitrarily good precision. Since the distances of $\widetilde{a}_i$ and $\widetilde{b}_i$ to $\widetilde{m}$ differ from the distances to $\widetilde{x}$ by at most $\sqrt{\mu_0} + \mu_0$, we can conclude that $(\widetilde{a}_1, \widetilde{b}_1, \widetilde{a}_2, \widetilde{b}_2)$ is a $(2, \delta_4(\mu_0))$-strainer of size $\geq \tfrac{1}{2} - \sqrt{\mu_0} - \mu_0$ around $\widetilde{m}$. It remains to bound comparison angles involving the points $\widetilde{x}, \widetilde{y}$: By the monotonicity of comparison angles, we have

$$\widetilde{\sphericalangle} \widetilde{m} \widetilde{x} \widetilde{a}_i \geq \widetilde{\sphericalangle} \widetilde{y} \widetilde{x} \widetilde{a}_i > \tfrac{\pi}{2} - \delta_2(\mu_0) \qquad \text{and} \qquad \widetilde{\sphericalangle} \widetilde{m} \widetilde{x} \widetilde{b}_i \geq \widetilde{\sphericalangle} \widetilde{y} \widetilde{x} \widetilde{b}_i > \tfrac{\pi}{2} - \delta_2(\mu_0).$$

Now, if we apply the same argument as in the last paragraph, replacing $\widetilde{y}$ by $\widetilde{m}$, we obtain $\widetilde{\sphericalangle} \widetilde{x} \widetilde{m} \widetilde{a}_i, \widetilde{\sphericalangle} \widetilde{x} \widetilde{m} \widetilde{b}_i > \tfrac{\pi}{2} - \delta_4(\mu_0)$. For analogous estimates on the opposing angles, we then interchange the roles of $\widetilde{x}$ and $\widetilde{y}$. Finally, $\widetilde{\sphericalangle} \widetilde{x} \widetilde{m} \widetilde{y} = \pi$ is trivially true.

Set $\widetilde{a}_3 = \widetilde{y}$ and $\widetilde{b}_3 = \widetilde{x}$. We have shown that $(\widetilde{a}_1, \widetilde{b}_1, \widetilde{a}_2, \widetilde{b}_2, \widetilde{a}_3, \widetilde{b}_3)$ is a $(3, \delta_5(\mu))$-strainer around $\widetilde{m}$ of size $\geq \sqrt{\mu_0} - \mu_0 > \tfrac{1}{2}\sqrt{\mu_0}$ (for $\mu_0 < \tfrac{1}{4}$) for a suitable $\delta_5(\mu_0)$. We will now use this fact to estimate the volume of the $\lambda\sqrt{\mu_0}$-ball around $\widetilde{m}$ from below for sufficiently small λ and μ_0. We follow here the ideas of the proof of [4, Theorem 10.8.18]. Define the function

$$f : \widetilde{B}(\widetilde{m}, \lambda\sqrt{\mu_0}) \longrightarrow \mathbb{R}^3 \qquad z \longmapsto (\operatorname{dist}(\widetilde{a}_1, z) - \operatorname{dist}(\widetilde{a}_1, \widetilde{m}),$$
$$\operatorname{dist}(\widetilde{a}_2, z) - \operatorname{dist}(\widetilde{a}_2, \widetilde{m}), \operatorname{dist}(\widetilde{a}_3, z) - \operatorname{dist}(\widetilde{a}_3, \widetilde{m})).$$

We will show that f is 100-bilipschitz for sufficiently small μ_0 and λ. Obviously, f is 3-Lipschitz, so it remains to establish the lower bound $\tfrac{1}{100}$. Assume that this was false, i.e. that there are $z_1, z_2 \in \widetilde{B}(\widetilde{m}, \lambda\sqrt{\mu_0})$ with $\operatorname{dist}(z_1, z_2) > 100|f(z_1) - f(z_2)|$. Then for all $i = 1, 2, 3$

$$(7.4) \qquad \operatorname{dist}(z_1, z_2) > 100|\operatorname{dist}(a_i, z_1) - \operatorname{dist}(a_i, z_2)|.$$

By the previous conclusions and the fact that comparison angles can be computed in terms of the distance function, we find that given any $\delta > 0$, we can choose $\lambda > 0$ and $\mu_0 > 0$ sufficiently small, to ensure that $(\widetilde{a}_1, \widetilde{b}_1, \widetilde{a}_2, \widetilde{b}_2, \widetilde{a}_3, \widetilde{b}_3)$ is a $(3, \delta)$-strainer around z_1 and around z_2. Now look at the comparison triangle corresponding to the points $z_1, z_2, \widetilde{a}_i$. By (7.4), it is almost isosceles and hence by elementary hyperbolic trigonometry we conclude for λ sufficiently small

$$\tfrac{9}{10}\tfrac{\pi}{2} < \widetilde{\sphericalangle} z_2 z_1 \widetilde{a}_i, \ \widetilde{\sphericalangle} z_1 z_2 \widetilde{a}_i < \tfrac{11}{10}\tfrac{\pi}{2}.$$

Using comparison geometry

$$\widetilde{\sphericalangle} z_1 z_2 \widetilde{b}_i \leq 2\pi - \widetilde{\sphericalangle} \widetilde{a}_i z_2 \widetilde{b}_i - \widetilde{\sphericalangle} z_1 z_2 \widetilde{a}_i < \tfrac{11}{10}\tfrac{\pi}{2} + \delta.$$

For λ sufficiently small, we obtain furthermore by hyperbolic trigonometry

$$\widetilde{\sphericalangle} \widetilde{b}_i z_1 z_2 + \widetilde{\sphericalangle} z_1 z_2 \widetilde{b}_i + \widetilde{\sphericalangle} z_2 \widetilde{b}_i z_1 > \pi - \delta \qquad \text{and} \qquad \widetilde{\sphericalangle} z_2 \widetilde{b}_i z_1 < \delta.$$

So

$$\widetilde{\sphericalangle} \widetilde{b}_i z_1 z_2 > \tfrac{9}{10}\tfrac{\pi}{2} - 3\delta.$$

Analogously, we obtain

$$\widetilde{\sphericalangle} \widetilde{a}_i z_1 z_2 > \tfrac{9}{10}\tfrac{\pi}{2} - 3\delta.$$

Now join z_1 with $\widetilde{a}_1, \widetilde{b}_1, \widetilde{a}_2, \widetilde{b}_2, \widetilde{a}_3$ by minimizing geodesics. By comparison geometry, these geodesics enclose angles of at least $\tfrac{\pi}{2} - \delta$ or $\pi - \delta$, depending on geodesics, between each other. So their unit direction vectors approximate the negative and positive directions of an orthonormal basis. By the same argument, the minimizing geodesic that connects z_1 with z_2 however encloses an angle of at least $\tfrac{9}{10}\tfrac{\pi}{2} - 3\delta$ with each of these geodesics. For sufficiently small δ this contradicts the fact that the tangent space at z_1 is 3-dimensional. So f is indeed 100-bilipschitz for sufficiently small λ and μ_0.

From the bilipschitz property we can conclude that

$$\operatorname{vol} \widetilde{B}(\widetilde{m}, \lambda \sqrt{\mu_0}) > c(\lambda \sqrt{\mu_0})^3$$

for some universal $c > 0$. Fixing $\mu_0 < \tfrac{1}{4}$ and $\lambda < 1$ such that the argument above can be carried out, we obtain

$$\operatorname{vol} \widetilde{B}(\widetilde{x}, 1) > \operatorname{vol} \widetilde{B}(\widetilde{m}, \lambda \sqrt{\mu_0}) > c(\lambda \sqrt{\mu_0})^3 = c' > 0.$$

By rescaling, this implies the desired inequality for the metric g_0.

Now consider case (ii): By Proposition 6.1 we know that $(B(x, \rho_1(x)), \rho_1^{-2}(x)g_0, x)$ is μ-close to $((-b, 1), g_{\mathrm{eucl}}, 0)$ where $0 \leq b \leq 1$. Let $y \in B(x, \rho_1(x))$ be a point that is at least $\tfrac{1}{25}$ away from the endpoints of $(-b, 1)$ with respect to this closeness and choose $B(y, \tfrac{1}{50}\rho_1(x)) \subset U \subset B(y, \tfrac{1}{25}\rho_1(x))$ and $p : U \to J' \subset (-b, 1)$ as in Proposition 6.1(c1).

Choose $q \in \pi_1(M_0)$ corresponding to a nontrivial simple loop in one of the cross-sectional tori and denote by $\widehat{M}_0$ the covering of M_0 corresponding to the cyclic subgroup generated by q, i.e. if we also denote by q the deck-transformation of $\widetilde{M}_0$ corresponding to q, then $\widehat{M}_0 = \widetilde{M}_0/q$. So we have a tower of coverings $\widetilde{M}_0 \to \widehat{M}_0 \to M_0$.

Consider first the rescaled metric $g' = \rho_1^{-2}(x)g_0$. Using the same arguments as in case (i), we can construct a $(1, \delta_1(\mu_0))$ strainer (a_1, b_1) around y on M_0 of size $\frac{1}{50}$ for a suitable $\delta_1(\mu_0)$. Furthermore, using the covering $\widehat{M_0} \to M_0$, we can find a point $\widehat{m} \in \widehat{M_0}$ within $\sqrt{\mu_0}$-distance away from a lift $\widehat{y}$ of y and a $(2, \delta_2(\mu_0))$ strainer $(\widehat{a}_1, \widehat{b}_1, \widehat{a}_2, \widehat{b}_2)$ around $\widehat{m}$ of size $\geq \sqrt{\mu_0}$. Connect the points $\widehat{a}_i$ and $\widehat{b}_i$ with $\widehat{m}$ by minimizing geodesics and choose points $\widehat{a}'_i$ and $\widehat{b}'_i$ of distance $\sqrt{\mu_0}$ from $\widehat{m}$. By monotonicity of comparison angles, $(\widehat{a}'_1, \widehat{b}'_1, \widehat{a}'_2, \widehat{b}'_2)$ is a $(2, \delta_2(\mu_0))$-strainer of size $\sqrt{\mu_0}$.

Let $g'' = \frac{1}{4}\mu_0^{-1}g'$. Then $(\widehat{a}'_1, \widehat{b}'_1, \widehat{a}'_2, \widehat{b}'_2)$ has size $\frac{1}{2}$ with respect to g''. Using this strainer, the metric g'' and the covering $\widetilde{M_0} \to \widehat{M_0}$, we can apply the same argument from case (i) again and obtain a $(3, \delta_3(\mu_0))$ strainer $(\widetilde{a}_1, \widetilde{b}_1, \widetilde{a}_2, \widetilde{b}_2, \widetilde{a}_3, \widetilde{b}_3)$ around a point $\widetilde{m}' \in \widetilde{M_0}$ which is $\sqrt{\mu_0}$-close to a lift $\widetilde{m}$ of $\widehat{m}$ in $\widetilde{M_0}$.

As in case (i), for a sufficiently small μ_0 we can deduce a lower volume bound $\mathrm{vol}_{g''} \widetilde{B}(\widetilde{m}', 1) > c'$. With respect to g', the point $\widetilde{m}'$ is within a distance of $\mu_0 + \sqrt{\mu_0}$ of a lift $\widetilde{x}$ of $\widehat{x}$. Hence

$$\mathrm{vol}_{g'} \widetilde{B}(\widetilde{x}, 1) > \mathrm{vol}_{g'} \widetilde{B}(\widetilde{y}, \tfrac{1}{50}) > \mathrm{vol}_{g'} \widetilde{B}(\widetilde{m}', 2\sqrt{\mu_0}) > c'(2\sqrt{\mu_0})^3 = c'' > 0.$$

The desired inequality follows by rescaling. $\square$

Definition 7.4. We call a component $\mathcal{C}$ of V_2 resp. V_1 *good* if it suffices the conditions in assumption (i) or (ii) of Lemma 7.1.

8. EVOLUTION OF AREAS OF MINIMAL SURFACES

Lemma 8.1. *Let $\mathcal{M}$ be a Ricci flow with surgery and precise cutoff, defined on the time interval $[T_1, T_2]$ ($T_1 > 0$), assume that the surgeries are all trivial and that $\pi_2(\mathcal{M}(t)) \neq 0$ for all $t \in [T_1, T_2]$. For every time $t \in [T_1, T_2]$ denote by $A(t)$ the infimum of the areas of all homotopically nontrivial, immersed 2-spheres. Then for all $t \in [T_1, T_2]$ we have $A(t) > 0$ and the quantity*

$$t^{1/4}\big(t^{-1}A(t) + 16\pi\big)$$

is monotonically non-increasing in t.

Proof. Compare also with [20, Lemma 18.10 and 18.11]. Let $t_0 \in [T_1, T_2)$. By [26] and [10] or [22], there is a noncontractible, conformal, minimal immersion $f : S^2 \to \mathcal{M}(t_0)$ with $\mathrm{area}_{S^2} f^*(g(t_0)) = A(t_0)$. Call $\Sigma = f(S^2) \subset \mathcal{M}(t)$. We can estimate the infinitesimal change of its area while we vary the metric in positive time direction (and keep f constant!). In the case in which Σ is an $\mathbb{R}P^2$, we count the area twice. Using the fact that the interior sectional curvatures are not larger

than the ambient ones as well as Gauß-Bonnet, we conclude:

$$\frac{d}{dt^+}\Big|_{t=t_0} \text{area}_t(\Sigma) = -\int_\Sigma \text{tr}_{t_0}(\text{Ric}_{t_0}\,|_{T\Sigma})d\text{vol}_{t_0}$$

$$= -\frac{1}{2}\int_\Sigma R_{t_0} d\text{vol}_{t_0} - \int_\Sigma \sec_{t_0}^{\mathcal{M}(t_0)}(T\Sigma)d\text{vol}_{t_0} \leq \frac{3}{4t_0}\,\text{area}_{t_0}(\Sigma) - \int_\Sigma \sec^\Sigma d\text{vol}_{t_0}$$

$$\leq \frac{3}{4t_0}\,\text{area}_{t_0}(\Sigma) - 2\pi\chi(\Sigma) = \frac{3}{4t_0}A(t_0) - 4\pi.$$

Here, $\sec_{t_0}^{\mathcal{M}(t_0)}(T\Sigma)$ denotes the ambient sectional curvature of $\mathcal{M}(t_0)$ tangential to Σ and $\sec_{t_0}^\Sigma$ denotes the interior sectional curvature of Σ. We conclude from this calculation that $\frac{d}{dt^+}|_{t=t_0}(t^{1/4}(t^{-1}A(t) + 16\pi)) \leq 0$ in the barrier sense and hence, the quantity $t^{1/4}(t^{-1}A(t) + 16\pi)$ is monotonically non-increasing in t away from the singular times.

We will now show that $A(t)$ is lower semi-continuous. We can restrict ourselves to the case in which t_0 is a surgery time. Let $t_k \nearrow t_0$ be a sequence converging to t_0 and choose minimal 2-spheres $\Sigma_k \subset \mathcal{M}(t_k)$ with $\text{area}_{t_k}\Sigma_k = A(t_k)$. By property (5) of Definition 2.9, we find diffeomorphisms $\xi_k : \mathcal{M}(t_k) \to \mathcal{M}(t_0)$ which are $(1 + \chi_k)$-Lipschitz for $\chi_k \to 0$. So $A(t_0) \leq \liminf_{k\to\infty}(1 + \chi_k)^2 A(t_k) = \liminf_{k\to\infty} A(t_k)$. This proves the desired result. $\qquad\square$

Lemma 8.2. *Let $\mathcal{M}$ be a Ricci flow with surgery and precise cutoff, defined on the time interval $[T_1, \infty)$ $(T_1 \geq 0)$ and assume that the surgeries are all trivial. Let $\gamma_{1,t}, \gamma_{2,t} \subset \mathcal{M}(t)$ be two families of smoothly embedded noncontractible loops which are homotopic to each other and move by isotopies for all $t \in [T_1, \infty)$. For every $t \in [T_1, \infty)$ let $A(t)$ be the infimum over the areas of all smooth homotopies $S^1 \times I \to \mathcal{M}(t)$ connecting $\gamma_{1,t}$ with $\gamma_{2,t}$.*

Assume that for the geodesic curvatures we have the bound $\kappa(\gamma_{1,t}), \kappa(\gamma_{2,t}) < Ct^{-1/2}$ for all $t \in [T_1, \infty)$ and assume that the normalized lengths $t^{-1/2}\ell(\gamma_{1,t})$, $t^{-1/2}\ell(\gamma_{2,t})$ converge to 0 as $t \to \infty$. Moreover, assume that the velocity by which the given loops move, is bounded in the appropriate rescaling, i.e. $|\partial_t\gamma_{1,t}|$, $|\partial_t\gamma_{2,t}| < Ct^{-1/2}$ for all $t \in [T_1, \infty)$.

Then $t^{-1}A(t) \to 0$ as $t \to \infty$.

Proof. Let $t_0 \in [T_1, \infty)$. By [19], we can find an area minimizing homotopy between γ_{1,t_0} and γ_{2,t_0}. More precisely, there is an $0 < r < 1$ such that if we denote by $A_{r,1} = \overline{B}_1(0) \setminus B_r(0) \subset \mathbb{C}$ the closed $(r, 1)$-annulus, then we can find a continuous map $f : A_{r,1} \to \mathcal{M}(t_0)$ with the following properties: f restricted to the boundary components of $A_{r,1}$ are parameterizations of γ_{1,t_0} and γ_{2,t_0}. Moreover, f is smooth, conformal and harmonic on the interior of $A_{r,1}$ and we have $A(t_0) = \text{area}\, f^*(g(t_0))$. By [13], f is even smooth up to the boundary.

Analogously to the proof of Lemma 8.1, we can compute the infinitesimal change of the area of f as we vary the metric only:

$$\frac{d}{dt}\Big|_{t=t_0} \operatorname{area} f^*(g(t)) = -\int_{A_{r,1}} \operatorname{tr} f^*(\operatorname{Ric}_{t_0}^{\mathcal{M}(t_0)})$$

$$\leq \frac{3}{4t_0} A(t_0) - \int_{A_{r,1}} \sec^{\mathcal{M}(t_0)}(df) d\operatorname{vol}_{f^*(g(t_0))},$$

where $\sec^{\mathcal{M}(t_0)}(df)$ denotes the sectional curvature in the normalized tangential direction of f. Observe that the last integrand is a continuous function on $A_{r,1}$ since the volume form vanishes wherever this tangential sectional curvature is not defined.

In order to avoid issues arising from possible branch points (especially on the boundary of $A_{r,1}$), we employ the following trick (compare with [25]): Let $\varepsilon > 0$ be a small constant and consider the flat cylinder $(N_\varepsilon = S^1 \times [\log r, 0]), \varepsilon(g_{S^1} + g_{\text{eucl}}))$ of size ε. Then $h_\varepsilon : A_{r,1} \to N_\varepsilon, z \mapsto (\log|z|, z|z|^{-1})$ is a conformal and harmonic diffeomorphism. We conclude that the map $f_\varepsilon = (f, h_\varepsilon) : A_{r,1} \to \mathcal{M}(t_0) \times N_\varepsilon$ is a conformal and harmonic *embedding*. Denote its image by $\Sigma_\varepsilon = f_\varepsilon(A_{r,1})$. Since the sectional curvatures on the target manifold $\mathcal{M}(t_0) \times N_\varepsilon$ arise from pulling back the sectional curvatures on $\mathcal{M}(t_0)$ via the projection onto the first factor, we have

$$\lim_{\varepsilon \to 0} \int_{\Sigma_\varepsilon} \sec^{\mathcal{M}(t_0) \times N_\varepsilon}(T\Sigma_\varepsilon) d\operatorname{vol}_{t_0} = \int_{A_{r,1}} \sec^{\mathcal{M}(t_0)}(df) d\operatorname{vol}_{f^*(g(t_0))}.$$

We can now proceed as in the proof of Lemma 8.1, using the fact that the interior sectional curvatures of Σ_ε are not larger than the corresponding ambient ones as well as the Theorem of Gauß-Bonnet:

$$\int_{\Sigma_\varepsilon} \sec^{\mathcal{M}(t_0) \times N_\varepsilon}(T\Sigma_\varepsilon) d\operatorname{vol}_{t_0} \geq \int_{\Sigma_\varepsilon} \sec^{\Sigma_\varepsilon}(T\Sigma_\varepsilon) d\operatorname{vol}_{t_0} = 2\pi\chi(\Sigma_\varepsilon) + \int_{\partial\Sigma_\varepsilon} \kappa_{\partial\Sigma_\varepsilon}^{\Sigma_\varepsilon} ds_{t_0}.$$

In our case $\chi(\Sigma_\varepsilon) = 0$. We now estimate the last integral. Let $\gamma_{i,t_0,\varepsilon} : S^1(l_{i,t_0,\varepsilon}) \to \partial\Sigma_\varepsilon$ be unit-speed parameterizations of the boundary of Σ_ε ($i = 1, 2$). Denote by $\gamma_{i,t_0,\varepsilon}^{\mathcal{M}(t_0)}(s)$ and $\gamma_{i,t_0,\varepsilon}^{N_\varepsilon}(s)$ their component functions in $\mathcal{M}(t_0)$ and N_ε, respectively. Furthermore, let $\nu_{i,t_0,\varepsilon}(s)$ be the outward-pointing unit-normal field along $\gamma_{i,t_0,\varepsilon}(s)$ which is tangent to Σ_ε. As before, denote by $\nu_{i,t_0,\varepsilon}^{\mathcal{M}(t_0)}$ and $\nu_{i,t_0,\varepsilon}^{N_\varepsilon}$ the components in the direction of $\mathcal{M}(t_0)$ and N_ε, respectively. Note that

$$0 = \left\langle \gamma_{i,t_0,\varepsilon}'(s), \nu_{i,t_0,\varepsilon}(s) \right\rangle = \left\langle \left(\gamma_{i,t_0,\varepsilon}^{\mathcal{M}(t_0)}\right)'(s), \nu_{i,t_0,\varepsilon}^{\mathcal{M}(t_0)}(s) \right\rangle + \left\langle \left(\gamma_{i,t_0,\varepsilon}^{N_\varepsilon}\right)'(s), \nu_{i,t_0,\varepsilon}^{N_\varepsilon}(s) \right\rangle$$

Since h_ε is conformal, the first summand on the right hand side is a non-negative multiple of the second summand. So both summands cannot have opposite signs and hence

$$(8.1) \qquad\qquad \left\langle \left(\gamma_{i,t_0,\varepsilon}^{\mathcal{M}(t_0)}\right)'(s), \nu_{i,t_0,\varepsilon}^{\mathcal{M}(t_0)}(s) \right\rangle = 0.$$

Thus since the boundary of N_ε is geodesic and since the $\gamma_{i,t_0,\varepsilon}^{\mathcal{M}(t_0)}(s)$ parameterize the loops γ_{i,t_0} whose geodesic curvatures are bounded by $Ct_0^{-1/2}$, we can compute

$$-\int_{\partial\Sigma_\varepsilon} \kappa_{\partial\Sigma_\varepsilon}^{\Sigma_\varepsilon} ds_{t_0} = -\sum_{i=1,2}\int_0^{l_{i,t_0,\varepsilon}} \left\langle \frac{D}{ds}\left(\frac{d}{ds}\gamma_{i,t_0,\varepsilon}^{\mathcal{M}(t_0)}(s)\right), \nu_{i,t_0,\varepsilon}^{\mathcal{M}(t_0)}(s)\right\rangle ds$$

$$\leq \sum_{i=1,2}\int_0^{l_{i,t_0,\varepsilon}} Ct_0^{-1/2}\left|\left(\gamma_{i,t_0,\varepsilon}^{\mathcal{M}(t_0)}\right)'(s)\right|^2\left|\nu_{i,t_0,\varepsilon}^{\mathcal{M}(t_0)}(s)\right| ds$$

$$\leq Ct_0^{-1/2}\sum_{i=1,2}\int_0^{l_{i,t_0,\varepsilon}} \left|\left(\gamma_{i,t_0,\varepsilon}^{\mathcal{M}(t_0)}\right)'(s)\right| ds = Ct_0^{-1/2}\left(\ell(\gamma_{1,t_0})+\ell(\gamma_{2,t_0})\right).$$

Note that in the first inequality, the terms involving the second derivative of $\gamma_{i,t_0,\varepsilon}^{\mathcal{M}(t_0)}(s)$ in tangential direction to $\gamma_{i,t_0,\varepsilon}$ vanish due to (8.1). Putting everything together and passing to the limit $\varepsilon \to 0$, we hence obtain

$$\frac{d}{dt}\Big|_{t=t_0} \operatorname{area} f^*(g(t)) \leq \frac{3}{4t_0}A(t_0) + Ct_0^{-1/2}\left(\ell(\gamma_{1,t_0})+\ell(\gamma_{2,t_0})\right).$$

In order to bound the derivative of $A(t)$ in the barrier sense, we have to account for the fact that the boundary curves move by isotopies. The maximal additional infinitesimal increase is then

$$\ell(\gamma_{1,t_0})\sup_{\gamma_{1,t_0}}|\partial_t\gamma_{1,t_0}| + \ell(\gamma_{2,t_0})\sup_{\gamma_{2,t_0}}|\partial_t\gamma_{2,t_0}| \leq Ct_0^{-1/2}\left(\ell(\gamma_{1,t_0})+\ell(\gamma_{2,t_0})\right).$$

So in the barrier sense

$$\frac{d}{dt^+}\Big|_{t=t_0} A(t) \leq \frac{3}{4t_0}A(t_0) + 2Ct_0^{-1/2}\left(\ell(\gamma_{1,t_0})+\ell(\gamma_{2,t_0})\right).$$

Thus

$$(8.2) \qquad \frac{d}{dt^+}\left(t^{-1}A(t)\right) \leq -\frac{1}{t}\left(\tfrac{1}{4}\left(t^{-1}A(t)\right) - 2Ct^{-1/2}\left(\ell(\gamma_{1,t})+\ell(\gamma_{2,t})\right)\right).$$

Analogously as in the proof of Lemma 8.1, we conclude that $A(t)$ is lower semi-continuous. So since the last summand in (8.2) goes to 0 for $t \to \infty$, we conclude that for every $a > 0$ there is some time t_1 such that whenever $t \geq t_1$ and $t^{-1}A(t) \geq a$, then $\frac{d}{dt^+}(t^{-1}A(t)) < -\frac{1}{8}t^{-1}a$. Since t^{-1} is not integrable, this implies that $t^{-1}A(t) < a$ for large t. It follows that $t^{-1}A(t) \to 0$ as $t \to \infty$. $\qquad\square$

9. Proof of Theorem 1.1

In order to finish the proof of the main theorem, we will need the following topological statement, which will help us to ensure that minimal annuli pass through certain thin parts of the manifold.

Lemma 9.1. *Let M be a smooth closed 3-manifold and $U_1, \ldots, U_m \subset M$ pairwise disjoint embedded copies of $T^2 \times I$ such that the components of $M'' = M \setminus (U_1 \cup \ldots \cup U_m)$ are hyperbolic (i.e. they carry hyperbolic metrics of finite volume). Let $\sigma_1, \sigma_2 : S^1 \to \partial U_1$ be two non-contractible loops that lie in different boundary components of U_1 and that are freely homotopic to each other within U_1.*

Then the image of every homotopy $f : S^1 \times I \to M$ between σ_1 and σ_2 has to intersect every loop $\gamma \subset U_1$ which is not homotopic to a multiple of σ_1 or σ_2 in U_1.

Proof. Assume that some $\gamma \subset U_1$ does not intersect $f(S^1 \times I)$. By a perturbation argument, we can assume that f is also transverse to the boundaries of all the U_i. So $f^{-1}(\partial U_1 \cup \ldots \cup \partial U_m)$ is a collection of disjoint circles $C_1, \ldots, C_k \subset S^1 \times I$. If one of these circles is contractible in $S^1 \times I$, then pick an innermost contractible circle C_j. It bounds a disk D_j. The image of its interior has to be contained in one of the components of M'' or in one of the U_i. In either case, this implies that $f|_{C_j}$ is homotopically trivial in the corresponding boundary torus and hence, we can replace f by a transverse f' which intersects the boundaries of the U_i in one circle fewer and whose image still does not meet γ.

So after a finite number of reduction steps, we can assume that all the C_j are non-contractible in $S^1 \times I$, which implies that they cut this annulus into $k-1$ nested topological annuli. Assume that one of these annuli is bounded by loops C_{j_1} and C_{j_2} such that the image of C_{j_1} is contained in ∂U_1, but the image of C_{j_2} is contained in some ∂U_i with $i \neq 1$. This means that two cuspidal homotopy classes of M'' that correspond to different cusps are conjugate to each other. However, this is impossible by elementary hyperbolic geometry.

It follows that the images of all C_j must be contained in ∂U_1. By elementary hyperbolic geometry again, we conclude that $f|_{C_j}$ is homotopic to σ_1 or σ_2 in ∂U_1. So if we restrict f to a certain sub-annulus, we obtain a homotopy $f'' : S^1 \times I \to U_1 \approx T^2 \times I$ between loops in each boundary torus which are each homotopic to σ_1 or σ_2 in ∂U_1.

By a simple intersection number argument, the image of f'' has to intersect γ. $\qquad \square$

We now prove that after some large time, all time slices are irreducible and all surgeries are trivial (see also [20, Proposition 18.9]).

Proposition 9.2. *Let $\mathcal{M}$ be a Ricci flow with surgery and precise cutoff, defined on the time interval $[T, \infty)$ $(T \geq 0)$. Then there is some $T_1 \in [T, \infty)$ such all surgeries on $[T_1, \infty)$ are trivial. Moreover, the following holds: For each $t \geq 0$ let $\mathcal{M}'(t)$ be the union of all components of $\mathcal{M}(t)$ that are not diffeomorphic to spherical space forms. Then $\mathcal{M}'(t)$ is irreducible and $\mathcal{M}'(t) \approx \mathcal{M}'(T_1)$ for all $t \geq T_1$.*

Finally, if there is a time $T^* \geq T_1$ such that there are no surgeries on $\mathcal{M}'$ past time T^*, then $\mathcal{M}$ is non-singular past time T^* and there is some time $T^{**} \geq T^*$ such that $\mathcal{M}'(t) \approx \mathcal{M}(t)$ for all $t \geq T^{**}$.

Proof. By definition of Ricci flows with surgery, for any two times $t_2 > t_1 \geq T$, the topological manifold $\mathcal{M}(t_1)$ can be obtained from $\mathcal{M}(t_2)$ by possibly adding spherical space forms or copies of $S^2 \times S^1$ to the components of $\mathcal{M}(t_2)$ and then performing connected sums between some components. So by the existence and uniqueness of the prime decomposition (see e.g. [12, Theorem 1.5]), there are only finitely many times when the topology of $\mathcal{M}'(t)$ can change. This implies

that there is some $T_1 \in [T, \infty)$ such all surgeries on $[T_1, \infty)$ are trivial and hence the time slices $\mathcal{M}'(t)$ are diffeomorphic to each other for all $t \in [T_1, \infty)$.

Assume that $\mathcal{M}'(T_1)$ was not irreducible. Then by [12, Proposition 3.10] and the solution of the Poincaré Conjecture, we find that $\pi_2(N) \neq 0$ for some component N of $\mathcal{M}'(t)$. We can now use Lemma 8.1 and conclude that $t^{-1}A(t)$ goes to zero in finite time. This is a contradiction to the fact that $A(t) > 0$.

In order to establish the last part of the Proposition, note that by assumption the complement $\mathcal{M} \setminus \mathcal{M}'$ restricted to the time-interval $[T^*, \infty)$ is a Ricci flow with surgery which is performed by precise cutoff. So by finite time extinction of spherical components (see [25], [7]), we find that $\mathcal{M}(t) \setminus \mathcal{M}'(t) = \emptyset$ for large t. $\qquad \square$

Proof of Theorem 1.1. First, by Proposition 9.2 and the assumption of the Theorem we conclude that for all $t \geq T_1$ the (topological) manifold $\mathcal{M}'(t)$ consists of components that are irreducible and only contain hyperbolic pieces in their torus decomposition. Moreover, all surgeries on $[T_1, \infty)$ are trivial.

Next, we apply Proposition 5.1 (here we need to assume that $\mathcal{M}$ is performed by sufficiently precise cutoff). This yields, amongst others, a time $T_2 > T_1$, a splitting $\mathcal{M}(t) = \mathcal{M}_{\text{thick}}(t) \cup \mathcal{M}_{\text{thin}}(t)$ for all $t \in [T_2, \infty)$ and a function $w : [T_2, \infty) \to \mathbb{R}_+$ with $w(t) \to 0$ as $t \to \infty$. Set $\mathcal{M}'_{\text{thick}}(t) = \mathcal{M}'(t) \cap \mathcal{M}_{\text{thick}}(t)$ and $\mathcal{M}'_{\text{thin}}(t) = \mathcal{M}'(t) \cap \mathcal{M}_{\text{thin}}(t)$. The interiors of the components of $\mathcal{M}'_{\text{thick}}(t)$ are diffeomorphic to the hyperbolic manifolds $H'_1, \ldots, H'_k$ and $\mathcal{M}'_{\text{thin}}(t)$ satisfies the collapsing condition described in Proposition 5.1(e). Moreover, the components of $\mathcal{M}'_{\text{thick}}(t)$ and $\mathcal{M}'_{\text{thin}}(t)$ are separated by embedded, incompressible tori $T_{1,t}, \ldots, T_{m,t} \subset \mathcal{M}'(t)$.

Choose $\mu = \mu_0$ from Lemma 7.1 and then $w_0 = w_0(\mu, \overline{r}, K)$ from Proposition 6.1, where $\overline{r}$ and K are the functions from Proposition 4.1 (in order to apply this Proposition, we again have to assume that the surgeries of $\mathcal{M}$ are performed by sufficiently precise cutoff). We can find some time $T_3 > T_2$ such that $w(t) < w_0$ for all $t \in [T_3, \infty)$ and hence Proposition 6.1 can be applied to $\mathcal{M}'_{\text{thin}}(t)$ for $\mu = \mu_0$, which gives us a decomposition of the thin part.

We now need to prove that for all $t \in [T_3, \infty)$, all components of $\mathcal{M}'_{\text{thin}}(t)$ are diffeomorphic to $T^2 \times I$: By [21, Theorem 0.2] (observe that this Theorem is a direct consequence of Proposition 6.1), we can choose additional embedded, incompressible tori $T'_{1,t}, \ldots, T'_{m',t} \subset \mathcal{M}'_{\text{thin}}(t)$ that cut $\mathcal{M}'_{\text{thin}}(t)$ into Seifert pieces. Using the uniqueness of the torus decomposition (see [12, Theorem 1.9]) and the topological assumption on $\mathcal{M}'(t)$, we conclude that a subset $\mathcal{T} \subset \mathcal{T}_0 = \{T_{1,t}, \ldots, T_{m,t}, T'_{1,t}, \ldots, T'_{m',t}\}$ cuts $\mathcal{M}'(t)$ into pieces that are hyperbolic. Let $H \subset \mathcal{M}'(t) \setminus \mathcal{T}$ be such a hyperbolic piece and consider a torus $T \in \mathcal{T}_0 \setminus \mathcal{T}$ that is contained in H. Since hyperbolic manifolds are atoroidal, there is a boundary torus $T'' \in \mathcal{T}$ of H such that T and T'' bound an embedded copy of $T^2 \times I$. We conclude that the tori of $\mathcal{T}_0$ that are contained in H cut H into pieces which are diffeomorphic to $T^2 \times I$ except for one piece which is diffeomorphic to H. Since H cannot carry a Seifert structure, this piece cannot be contained in $\mathcal{M}'_{\text{thin}}(t)$.

So $\mathcal{M}'_{\text{thin}}(t) \setminus \mathcal{T}_0$ is a disjoint union of copies of $T^2 \times I$. Piecing these together, implies that all components of $\mathcal{M}'_{\text{thin}}(t)$ are diffeomorphic to $T^2 \times I$.

Having established the topological description, we will now bound the geometry of the thin part using a minimal surface argument. In order to do that, we choose smooth isotopies of loops $\sigma'^1_{1,t}, \sigma'^2_{1,t}, \ldots, \sigma'^1_{m,t}, \sigma'^2_{m,t} : S^1 \to H'_1 \cup \ldots \cup H'_k$ in the model hyperbolic manifolds, defined for times $t \in [T_3, \infty)$ such that there is a function $\varepsilon : [T_3, \infty) \to \mathbb{R}_+$ with $\varepsilon(t) \to 0$ as $t \to \infty$ and:

(1) The lengths of the loops go to zero: $\ell(\sigma'^j_{i,t}) < \varepsilon(t)$ for all $t \in [T_3, \infty)$ and their geodesic curvature is everywhere equal to 1.

(2) For all t, the loops $\sigma'^j_{i,t}$ are contained in $H''_{1,t} \cup \ldots \cup H''_{k,t} \subset H'_1 \cup \ldots \cup H'_k$ (compare with Proposition 5.1(d)).

(3) The velocity by which the loops move, is bounded appropriately: $|\partial_t \sigma'^j_{i,t}| < t^{-1}$.

(4) For every hyperbolic cusp $N' \subset H'_1 \cup \ldots \cup H'_k$, consider the torus $T_{i,t} \subset \mathcal{M}'(t)$ that borders the corresponding almost hyperbolic cusp $N' \subset \mathcal{M}(t)$. Then $\sigma'^1_{i,t}, \sigma'^2_{i,t}$ are contained in N' and for all $t \in [T_3, \infty)$ and represent two nondivisible and linearly independent homotopy classes in $\pi_1(N') \cong \pi_1(T^2 \times I) \cong \mathbb{Z}^2$.

(5) Let $\sigma^j_{i,t} : S^1 \to \mathcal{M}'(t)$ be the loops corresponding to the $\sigma'^j_{i,t}$ under the diffeomorphisms $\Psi_{l,t} : H''_{l,t} \to H_{l,t}$, i.e. $\sigma^j_{i,t} = \Psi_{l,t} \circ \sigma'^j_{i,t}$ for the appropriate l (see Proposition 5.1(d)). We now demand that for every component $\mathcal{C} \subset \mathcal{M}'_{\text{thin}}(t)$ the following is true: let $N_1, N_2 \subset H_1 \cup \ldots \cup H_k$ be the two cusps that are adjacent to $\mathcal{C}$ and let $\sigma^1_{i_1,t}, \sigma^2_{i_1,t}$ be the loops in N_1 and $\sigma^1_{i_2,t}, \sigma^2_{i_2,t}$ the loops in N_2. Then $\sigma^1_{i_1,t}$ and $\sigma^1_{i_2,t}$ as well as $\sigma^2_{i_1,t}$ and $\sigma^2_{i_2,t}$ are freely homotopic in $N_1 \cup \mathcal{C} \cup N_2$.

It is clear that we can find such $\sigma^j_{i,t}$, e.g. by choosing the loops as geodesics of horospherical tori in the cusps, $d(t)$-far away from the thick part, where $d(t)$ is an interpolation of $\min\{w^{-1}(t), \log t\}$.

For each time $t \in [T_3, \infty)$ and component $\mathcal{C} \subset \mathcal{M}_{\text{thin}}(t)$ denote by $A_{\mathcal{C},j}(t)$ the infimum over the areas of all smooth homotopies $S^1 \times I \to \mathcal{M}(t)$ connecting $\sigma^j_{i_1,t}$ and $\sigma^j_{i_2,t}$ from property (5). By Lemma 8.2 and conditions (1)-(3) above, we conclude that $t^{-1} A_{\mathcal{C},j}(t) \to 0$ as $t \to \infty$. So there are time-dependent homotopies $f^j_{\mathcal{C},t} : S^1 \times I \to \mathcal{M}(t)$ such that

$$(9.1) \qquad\qquad t^{-1} \operatorname{area}_t f^j_{\mathcal{C},t} \longrightarrow 0 \qquad \text{as} \qquad t \longrightarrow \infty$$

for all components $\mathcal{C}$ of $\mathcal{M}_{\text{thin}}(t)$ and $j = 1, 2$. (Note that the components $\mathcal{C}$ change in time. However, the combinatorics of the thick-thin decomposition stay the same on $[T_3, \infty)$.)

Now look at the decomposition of a component $\mathcal{C} \subset \mathcal{M}'_{\text{thin}}(t)$ into sets V_1, V_2, V'_2 as given in Proposition 6.1 (applied to the metric $t^{-1} g(t)$). The two boundary tori of $\mathcal{C}$ have to border components of V_1. So either $\mathcal{C} = V_1$ or the boundary components of $\mathcal{C}$ border components $\mathcal{C}_1, \mathcal{C}_2 \subset V_1$ which are diffeomorphic to $T^2 \times I$ (see conclusion (a3)). In the second case, there is a component $\mathcal{C}_3$ of V_2 or V'_2

adjacent to $\mathcal{C}_1$. Since components of V_2' have only one boundary component and $\mathcal{C} \neq \mathcal{C}_1 \cup \mathcal{C}_3$, we must have $\mathcal{C}_3 \subset V_2$. The generic S^1-fibers of $\mathcal{C}_3$ are homotopic to a nontrivial curve in the boundary torus of $\mathcal{C}_1$ adjacent to $\mathcal{C}_3$. This torus is isotopic to one of the $T_{i,t}$ that are incompressible in $\mathcal{M}'(t)$ (see Proposition 5.1(b)). So the generic S^1-fibers of $\mathcal{C}_3$ generate an infinite cyclic subgroup in the fundamental group of the corresponding component of $\mathcal{M}'(t)$.

Hence, we can apply Lemma 7.1 and obtain that for any $x \in \mathcal{C}$ (if $\mathcal{C} = V_1$) or for any $x \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ (if $\mathcal{C} \neq V_1$), we have $\mathrm{vol}_t \widetilde{B}(\widetilde{x}, \rho_{\sqrt{t}}(x,t)) \geq w_1 \rho_{\sqrt{t}}^3(x,t)$ in $\widetilde{\mathcal{M}}(t)$. We can now use Proposition 4.2, to deduce that there is some $T_4 \in [T_3, \infty)$ and constants $\overline{\rho} > 0$ and $K < \infty$ such that for all $t \in [T_4, \infty)$ we have $\rho(x,t) > \overline{\rho}\sqrt{t}$ and $|\mathrm{Rm}| < Kt^{-1}$ on $\mathcal{C}$ (if $\mathcal{C} = V_1$) resp. $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ (if $\mathcal{C} \neq V_1$).

Assume that the second case occurs for some $t \in [T_4, \infty)$. Let $x \in C_3$. Then by Proposition 6.1(c3), we can find an open set U with $B(x,t,\frac{1}{2}s\rho_{\sqrt{t}}(x,t)) \subset U \subset B(x,t,s\rho_{\sqrt{t}}(x,t))$ and a 2-Lipschitz map $p : U \to \mathbb{R}^2$ whose image must contain $B(0,\frac{1}{4}s\rho_{\sqrt{t}}(x,t)) \subset \mathbb{R}^2$ and whose fibers are homotopic to the fibers on $\mathcal{C}_3$ and hence non-contractible in $\mathcal{M}(t)$. So by Lemma 9.1 applied twice, we conclude that each fiber of p has to intersect the images of one of the homotopies $f_{\mathcal{C},t}^1, f_{\mathcal{C},t}^2$. This implies that

$$\mathrm{area}_t\, f_{\mathcal{C},t}^1 + \mathrm{area}_t\, f_{\mathcal{C},t}^2 > cs^2 \rho_{\sqrt{t}}^2(x,t) > cs^2 \overline{\rho}^2 t$$

for some universal $c > 0$. If t is sufficiently large, this however contradicts (9.1).

Hence we conclude that there is some $T_5 \in [T_4, \infty)$ such that for all $t \in [T_5, \infty)$, we have $\mathcal{M}'_{\mathrm{thin}}(t) = V_1$ and $|\mathrm{Rm}| < Kt^{-1}$ on $\mathcal{M}'_{\mathrm{thin}}(t)$. The curvature bound on $\mathcal{M}_{\mathrm{thick}}(t)$ follows directly from Proposition 5.1(d). By Definition 2.9(3), surgeries can only appear when the curvature is comparable to $\delta^{-2}(t)$, where $\delta(t)$ is the preciseness parameter. So if we assume that $\mathcal{M}$ is performed by sufficiently precise cutoff, then there cannot be any surgeries for large t on $\mathcal{M}'$. The Theorem now follows using the last part of Proposition 9.2. $\square$

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