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# EROSION AND OPTIMAL TRANSPORT

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ABSTRACT. We consider the theory of erosion and investigate connections to the theory of optimal transport. The mathematical theory of erosion is based on two coupled nonlinear partial differential equations. The first one describing the water flow can be considered to be an averaged Navier-Stokes equation, and the second one describing the sediment flow is a very degenerate nonlinear parabolic equation. In the first half of this work, we prove existence, uniqueness, and regularity properties of weak solutions to the second model equation describing the sediment flow. This forms the basis to define an optimal transport problem for the movement of sediment; the second half of this work is devoted to this optimal transport problem for the sediment. We solve the optimal transport problem. Furthermore, we demonstrate that the optimal transport problem distinguishes a particular class of solutions to the model equation. The movement of sediment according to the solution of the optimal transport problem is identical to the movement of sediment according to these solutions of the model equation. The physical interpretation is that if the sediment flows according to the model equation on the surface of separable solutions consisting of valleys separated by mountain ridges, that are observed in simulation and in nature, then the sediment is “optimally transported.”

## 1. INTRODUCTION

The continuing evolution of the surface of the earth poses a challenging and fascinating modeling problem. The earth’s surface is composed of many substances: soil, sand, vegetation, different types of rocks. Its surface is further complicated by topography which continues to change over time due to tectonic uplift and earthquakes. Due to the complexity of most landsurfaces and the instability of some it has taken many years to develop models. The theory of fluvial landscape evolution began with geological surveys such as [28] and [21] which were developed into geological models such as [30] and [51]. The investigations performed more recently fall into three groups: (1) empirical investigations of fluid phenomena, (2) computational investigations of discrete models and (3) investigations of continuous model, or partial differential equations, of surface evolution and channelization. The first group includes the field observations [63] on the badlands of the Perth Amboy and the flume [64] and artificial stream [13, 12] experiments that have given deep insight into channelized drainage. The second group has produced remarkable simulations of evolving channel networks; see [79, 80], [31], [74] and [60]. The third group has lead to an increasing understanding of the physical mechanisms that underlie erosion and channel formation; see [66],[68], [50], [72], [48], [46], [49], [69], [35, 36, 37, 38],[7, 8, 9], [34], [65], [76], [11], [26], [6], [67].

The adequacy of mathematical theories of erosion are generally measured by how well they model observable phenomena. This should include: (1) the emergence of channelized drainage patterns from unchanneled surfaces, (2) the development of relatively stable surfaces characterized by branching patterns of ridges and valleys, (3) the decline of the surfaces and the dissipation of the forms, (4) the relationships between evolving surface forms and the flows of water and sediment over the surfaces and (5) the variability of landforms under varying environmental conditions. The theories should be based on sound physical principles and give rise to testable hypotheses. The approach of Smith, Merchant and Birnir [7] is based on Horton’s [30] classification of the problem into three distinct approaches: (a) deterministic modeling continuous in space and time based on conservation principles, (b) stochastic modeling discrete in space and time based on conservation principles and (c) deterministic modeling based on the search for variational principles characterizing self-organizing drainage surfaces in terms of the minimization or maximization of an aggregate quantity. In [7] and [8], a family of two partial differential equations were introduced based on the conservation of water and sediment. These equations describe a transport-limited process [31] in which sediment moves in the same direction in which the surface water flows. The transport-limited case models situations found in badlands and deserts where all the sediment can be transported away if a sufficient quantity of water is available. The detachment-limited case is the other extreme; see [35, 36, 37, 38] and [34] whose work models a situation where the surface is

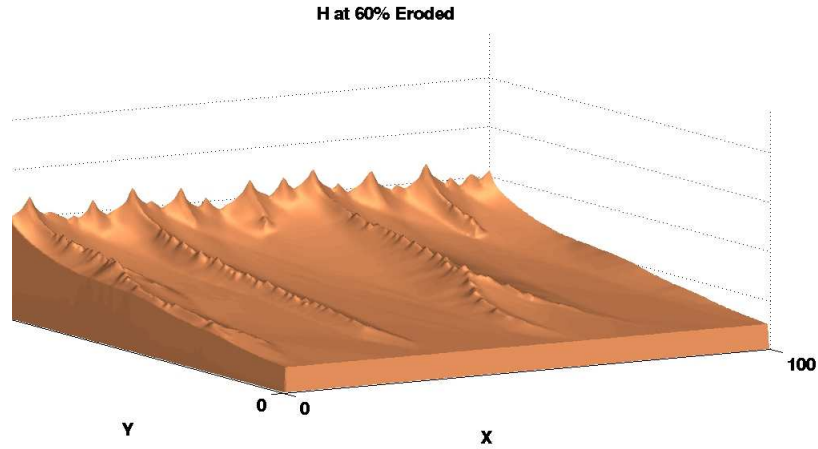


FIGURE 1. A desert landscape consisting of a pattern of valleys separated by mountain ridges, from [10].

covered with rock and one must wait for it to weather before the resulting sediment can be transported away. Different models are required for the latter situation.

In this paper we focus on the equations developed and studied by the first author and his collaborators in [7], [8], [10], [11]. These equations improve the original model in [68] by including a pressure term (in addition to the gravitational and friction terms) that prevents water from accumulating in an unbounded manner in surface concavities; see [48]. They present a representation of the free water surface in a diffusion analogy approximation to the St. Venant equations; see [77]. Although the equations fall into the physical deterministic category (3) and (a) above, in [10] and [11], it has been demonstrated that the equations bridge the deterministic and stochastic theories. In [10] the equations are interpreted as being driven by random influences (noise). Two scaling laws consistent with the theory and observations of [60] and [78] were demonstrated for the equations. Nonlinear systems that are driven by noise have been used to explain the origin of spatial and temporal scaling not only in landscape evolution, but in a wide variety of phenomena [62], [82], [52]. Scaling laws are also related to self organized criticality [2]. The stochastic theory based on our model equations was further developed in [11], in which the landsurfaces were shown to be self-organized critical systems characterized by the spatial and temporal scalings [10]. In [6] the origin of the scaling law for landsurfaces was traced back to the roughness coefficient for turbulent flow in rivers. The fluvial surfaces are formed over time by the meanderings of the rivers over the whole surface. The turbulent flow imparts its roughness to the surface. This gives rise to all the known scaling laws that apply to fluvial landsurfaces and river networks, all of which are determined by Hack's exponent [29]  $h = \frac{1}{1+\chi}$  where  $\chi = 3/4$  is the roughness coefficient for turbulent flow in rivers; see [6].

The analysis of the above nonlinear partial differential equations has so far been mostly numerical, and simulations show a striking time evolution that seems to be similar to the evolution of realistic landscapes. For very general initial data the solutions always seek out a special class of solutions consisting of separable solutions, see [7], consisting of a pattern of mountain ridges and river valleys. A typical such landsurface is illustrated in Figure 1, taken from [10]. One problem is that the model consists of two equations, each of which has a different time scale, one equation for the water flow and the other for the sediment flow. The flow of water over the surface is turbulent, and although the water surface never reaches an equilibrium because

of instabilities that continue to magnify small noise, numerically a statistically stationary water surface is observed to exist. This means that if one runs many simulations with different initial perturbations of the same initial data and takes an ensemble average over these simulations, a statistically stationary water surface emerges. In this paper we will assume that such a statistically stationary water surface exists and use its properties to analyze the partial differential equation describing the sediment flow that is associated with a larger time scale. We will think about the water depth  $h$  as a non-negative quantity, where we have averaged over many rainfall events on a fast scale, assuming that  $h$  only vanishes on a set of measure zero. Namely,  $h$  only vanishes on top of the mountain ridges, and there it vanishes fast enough so that the water and sediment flow over the top of the mountain ridges vanish as well. These hypotheses are observed in simulations and make sense physically. We will then call on the theory of optimal transport to investigate why all initial data eventually approach and stay close to the separable ridges for a long time.

Solving nonlinear equations like those considered here poses a significant challenge. In general, one does not expect smooth solutions. Indeed, based on observations and the stochastic theory, differentiable solutions would be quite surprising! It is then natural to study weak solutions to the model equations. The results in this paper will form the basis for the mathematical theory of the model equations. We study the existence, uniqueness, and regularity of weak solutions. We demonstrate that for Hölder continuous initial data, weak solutions exist, are unique, and are Hölder continuous. This mathematical theory is important to further study both the deterministic and stochastic aspects of landscape evolution. In particular it would provide the basis for the characterization of the noise that is generated in landscape evolution and provides the bridge between the stochastic and deterministic approaches. However, the mathematical theory does not give an immediate interpretation in terms of the physical process and observed phenomena. Moreover, since the solutions are not in closed form, it is challenging to test their accuracy as physical models. Seeking further particulars of the solutions and wishing to demonstrate that they give rise to reasonable models, we were naturally led to a related problem that interestingly provides the connection to an optimality principle.

The theory of optimal transport enjoys a rich and beautiful history; for a careful and thorough exposition, see [75] and [59]. The theory began in 1781 with Monge's simple question [54]: what is the least expensive way to transport mounds of dirt in order to fill holes? Simply put, if one has a collection of mounds of dirt and one also has several holes to fill, assuming the amount of dirt in the mounds is precisely the amount needed to fill the holes, how should one move the dirt from the mounds to the holes using the least amount of work? As a geometer, Monge recognized that the direction of transport should be along straight lines that would be orthogonal to a family of surfaces. Although this proved to be a key observation, the problem remained mostly unsolved for over a century.

In 1938, Kantorovich unknowingly revisited Monge's optimal transport problem. Unfamiliar with Monge's work, Kantorovich was consulted by a laboratory for the solution of a certain optimization problem. This was none other than Monge's optimal transport problem, although it was not until several years after his main results that Kantorovich made the connection with Monge's work. In [41], [42], [43], Kantorovich made the important discovery based on functional analysis that the optimization problem to minimize the work (or "cost") was equivalent to a dual maximization problem. Kantorovich also developed the tools of linear programming and applied them to the problem. In 1975, Kantorovich and Koopmans were awarded the Nobel prize for economics "for their contributions to the theory of optimum allocation of resources." Among Kantorovich's discoveries, he devised a notion of distance between probability measures. This distance is the optimal transport cost from one measure to the other; it is called Kantorovich-Rubinstein or Wasserstein distance. This distance has been used in many branches of mathematics to study spaces of probability measures.

In the 1980s, three distinct areas of mathematics were demonstrated to be intimately related to optimal transport based on the work of [53], [14], and [20]. These connections emphasized that "important information can be gained by a qualitative description of optimal transport" [75]. In the spirit of Monge, Otto [55] introduced a differential point of view to optimal transport theory. This perspective led to a geometric description of the space of probability measures. Differential geometers have used the geometry of optimal transport to define synthetic Ricci curvature to study spaces that do not admit a smooth Riemannian metric [57], [19], and [47]. The geometry of optimal transport is also related to the study of diffusion processes and Bakry-Émery geometry.

In the most naive terms, erosion is nature's process of "moving dirt," so one would expect it to be transported optimally in an appropriate sense. It is thus not unreasonable to expect connections between the theories of

erosion and optimal transport. Erosion takes place at each point of the eroding surface, and the eroded sediment is then transported by the river network to a river or lake at the lower boundary of the region. It turns out that the easiest way of expressing this is in terms of the erosion rate at each point of the surface and the flux of sediment through the boundary of the region. In a period of time this amounts to a layer of sediment being eroded from the surface and transported through the boundary. Whereas the mathematical theory of erosion is still quite young, the theory of optimal transport has been richly developed since its rediscovery by Kantorovich. Demonstrating a meaningful and rigorous connection between these two theories will allow us to exploit the results and tools from optimal transport to further study erosion.

## 2. THE MODEL EQUATIONS

The surfaces generated by erosion are the result of highly nonlinear processes driven by noisy inputs; they are complex and difficult to represent mathematically. We briefly describe the nature and derivation of the family of models and the model equations on which we focus. A detailed description is given in [7] and [10]. The models are based on a conservation principle of water and sediment fluxes over a continuous, erodible surface  $z = z(x, y, t)$ , and on the *advective entrainment and transport of sediment in transport limited conditions* as in [32]. Here  $(x, y) \in \Omega \subset \mathbb{R}^2$ . We assume  $\Omega$  has a piecewise smooth boundary. The equations are

$$\begin{aligned}\frac{\partial h}{\partial t} &= \nabla \cdot (\mathbf{u}_w q_w) + R, \\ \frac{\partial z}{\partial t} &= \nabla \cdot (\mathbf{u}_w q_s),\end{aligned}$$

where  $R$  is the rainfall rate,  $h = h(x, y, t)$  is the depth of the water varying continuously over the landsurface,  $-\mathbf{u}_w = -\nabla H / |\nabla H|$  is a unit vector in the direction of both the water and the advected sediment flows,  $H = H(x, y, t) = z(x, y, t) + h(x, y, t)$  is a free water surface,  $q_w$  represents the flux of water per unit width, and  $q_s$  represents the advected flux of sediment per unit width. The magnitude of water flow is given by a Manning-type equation [22], [69], [71]

$$q_w = h\nu = nh\rho^{\alpha-1}S_H^\beta,$$

in which  $\nu$  is the velocity of water flow averaged over depth,  $n$  is a constant,  $\alpha, \beta > 0$ ,  $S_H = |\nabla H|$  is the slope of the water surface, and  $\rho$  is the hydraulic radius. The magnitude of the sediment flux is

$$q_s = F(q_w)S_H^\delta, \quad \delta > 0,$$

which generalizes the power laws [39] in terms of a monotone increasing function  $F(q_w)$ . Representative scaling units and non dimensional variables are as follows

$$\begin{aligned}H &= [H]\bar{H}, & h &= [h]\bar{h}, & \rho &= [h]\bar{\rho}, \\ x &= [H]\bar{x}, & y &= [H]\bar{y}, & t &= [t]\bar{t}, \\ q_w &= [q_w]\bar{q}_w, & q_s &= [q_s]\bar{q}_s, & R &= [R]\bar{R}, & n &= [n]\bar{n},\end{aligned}$$

where a term  $[\cdot]$  represents a scaling parameter, and a term  $\bar{\cdot}$  represents a non-dimensional quantity. Typical length scales associated with the vertical relief of an elevation surface differ by several orders of magnitude from the typical length scales associated with the depth of flow of surface water. Applying the transformations, one obtains the following relationships between scaling factors,

$$[q_w] = [R][H], \quad [t] = \frac{[H]^2}{[h][R]}, \quad [n][h]^\alpha = [q_w], \quad [F] = [q_s].$$

To reflect the disparity between the rate of water and of sediment, we include the relationship  $[q_s] = [R][h]$ . We arrive at the following non-dimensional form for our general class of models,

$$(2.1) \quad \eta^2 \frac{\partial h}{\partial t} = \nabla \cdot \left[ \frac{\nabla H}{|\nabla H|} h \rho^{\alpha-1} |\nabla H|^\beta \right] + R,$$

$$(2.2) \quad \frac{\partial H}{\partial t} - \eta \frac{\partial h}{\partial t} = \nabla \cdot \left[ \frac{\nabla H}{|\nabla H|} F(h \rho^{\alpha-1} |\nabla H|^\beta) |\nabla H|^\delta \right],$$

in which  $\eta = [h]/[H] = [q_s]/[q_w] = ([F]/n)([R]/[F])^\alpha$  is a dimensionless ‘‘landscape’’ parameter. Typically  $\eta$  is a very small number. It is also possible to derive (2.2) as a special case of the Navier-Stokes equations and the conservation of mass [69], [71].

A simple sub-family of (2.1) and (2.2) are produced by: (1) taking a simple power law of sediment transport  $q_s = kq_w^\gamma |\nabla H|^\delta$  in which  $k$  is a constant, (2) approximating the hydraulic radius  $\rho$  by the depth  $h$  and employing Manning-type exponents  $\alpha = 1.67$  and  $\beta = 0.5$ , and (3) neglecting the terms  $\eta^2 \partial h / \partial t$  and  $\eta \partial h / \partial t$ . The power law of sediment transport implicitly assumes a transport limited erosion process [31] in which sediment transport is dominated by the advection of sediment with flowing water. Such power laws model the erosion and transport of sediment in so-called badland conditions in which there is no vegetation. Omitting the terms  $\eta^2 \partial h / \partial t$  and  $\eta \partial h / \partial t$  is reasonable because at the time scale where significant erosion occurs,  $\eta$  is very small [68], [48]. A realistic value of both  $\gamma$  and  $\delta$  for a range of landsurfaces is 2; see [11] and [8]. With these simplifications, the equations (2.1) and (2.2) become

$$(2.3) \quad -\nabla \cdot \left[ \frac{\nabla H}{|\nabla H|^{1/2}} h^{5/3} \right] = R,$$

and

$$(2.4) \quad \frac{\partial H}{\partial t} = \nabla \cdot \left[ \nabla H |\nabla H|^2 h^{10/3} \right].$$

These equations describe the flow of sediment on a long time scale in the presence of an equilibrium water depth  $h(x, y)$ .

In this paper, we use the same boundary conditions as [7] and [8] to model a ridge defined over a rectangular domain of length  $L$  and width  $W$ ,

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq W, 0 \leq y \leq L\},$$

with boundary conditions

$$(2.5) \quad \begin{aligned} h(0, y) &= 0, \\ h(W, y) &= h_0(y), \\ H(W, y, t) &= 0 \end{aligned}$$

corresponding to a water depth of zero at the top of the ridge and an absorbing body of water at the base of the ridge. While the water surface  $H(0, y, t)$  must be considered to be a free surface at the top of the ridge, it may be viewed as consisting of finitely many smooth curves that are solutions of a nonlinear PDE (the PDE restricted to the boundary) in two variables. These curves are joined in a continuous, but not smooth, moving boundary. The upper boundary is characterized by the additional conditions

$$q_w = q_s = 0,$$

indicating the absence of any flux of water or sediment over this boundary, see [10] for more details. The lateral boundaries are described by periodic boundary conditions

$$(2.6) \quad \begin{aligned} h(x, y + L) &= h(x, y), \\ H(x, y + L, t) &= H(x, y, t) \end{aligned}$$

since  $\Omega$  can be considered to be a (base) section of an extended mountain ridge. For the initial conditions, we will only assume finite regularity of  $H(x, y, 0) = H^0(x, y)$ . For example, in simulations, the initial condition is often given by

$$(2.7) \quad H^0(x, y) = H_o \left( 1 - \frac{1}{W} x \right), \quad 0 \leq y \leq L.$$

This models a linear ridge of height  $H_o$ , uniform in the  $y$  direction and with slope  $-H_o/W$  in the  $x$  direction.

The flux over the lower boundary  $x = W$  is not zero since this is where the sediment is transported by the water into a lake or a river. Let

$$F_\Omega(y) := q_s \frac{\nabla H}{|\nabla H|} (W, y, t) \cdot \hat{n} = |\nabla H|^2 h^{10/3} \nabla H(W, y, t) \cdot \hat{n},$$

where  $\hat{n}$  is the (outward) normal of the boundary. The integral of this flux is negative because the slope  $\nabla H$  increases as  $x$  decreases away from the boundary of the lake or river,

$$(2.8) \quad \int_0^L F_\Omega(y) dy < 0.$$

In case of tectonic uplift the equations above can be modified by adding the uplift (rate) constant  $\mathcal{U}$  as in [61] to the sediment equation so that (2.4) becomes

$$(2.9) \quad \frac{\partial H}{\partial t} = \nabla \cdot \left[ \nabla H |\nabla H|^2 h^{10/3} \right] + \mathcal{U}.$$

Note that if  $H$  is a solution to (2.4), then  $\tilde{H} := H + t\mathcal{U}$  is a solution to (2.9). Moreover,  $\nabla H = \nabla \tilde{H}$ . Since it is easy to pass between solutions of (2.4) and (2.9), for simplicity we study solutions of (2.4).

### 3. SOLUTIONS OF THE MODEL EQUATIONS

The model equations (2.3) and (2.4) are highly nonlinear, and no explicit solution of the initial boundary value problem in the previous section exist. The initial surface is unstable, but in spite of this it is possible to solve the two equations (2.1) and (2.2) numerically with modern numerical methods. The first author and his collaborators did this in [7], [8], [10], [11] and [76]. Thus they gained considerable insight into the properties of the solutions, and the main purpose of this paper is to develop the full nonlinear analysis based on these insights. This will result in a proof of many of the properties that have been observed numerically and enable us to extend the numerical analysis even further. The numerics show that the initial straight slope always evolves into a landsurface similar to the one shown in Figure 1. The simulations were done by seeding the initial linear slope which is unstable with random noise. When the simulation is repeated with a different noise vector, a surface similar to the one in Figure 1 emerges every time. The valleys and the mountain ridges look similar every time, and their number stays the same. Only the location of the ridges and the valleys is unpredictable and changes from one simulation to the next. The important observation is that every initial condition that is a linear ridge with different noise goes to the *same pattern* of separable mountain ridges and valley; see [8]. Moreover, this pattern persists for a long time in the time evolutions and simply decreases in elevation. Thus the separable solutions from [8] can be called a "transient attractor". These solutions attract a large class of initial data and persist for a long time in the dynamics. However, at the very end before the landsurface becomes a flat plain, that constitutes the true attractor, the mountain ridges crumble into small collapsing hills that quickly disappear. These two types of solutions, separable ridges and collapsing hills, will play a role in the discussion of the optimal transport below.

The time scale of the water flow in equation (2.1) is much shorter than that of the sediment flow in equation (2.2). Thus, naively one might hope for an equilibrium water depth on the longer time scale of the sediment flow. This hope is expressed by the equations (2.3) and (2.4) where the time derivative in the water flow is set to zero. The numerical evidence is that the water flow is turbulent and that there is no equilibrium water depth. However, when many simulations are performed and an ensemble average over these simulations is taken, a statistically stationary equilibrium water depth emerges. Based on this numerical evidence we will assume in this paper that a statistically stationary water depth exists and make assumptions on it based on the numerical evidence. We will use this statistically stationary (average) water depth  $h$  to study the sediment flow as described by the second equation (2.4).

The fluvial landsurface described by  $z$  is experimentally not smooth [8], so it is natural to study weak solutions. In this paper, we focus on weak solutions to (2.4); weak solutions to (2.1) giving rise to statistically stationary solutions of (2.3) will be addressed in a later investigation, but in this work we will ignore (2.3). It is then natural to assume that the water depth function  $h$  is given, does not depend on time and satisfies

$$(3.1) \quad h \geq 0, \quad h \in \mathcal{C}^d(\Omega), \quad 0 < d < 1.$$

The water depth  $h$  is shown on the top part of Figure 2 and the gradient of the slope of the water surface  $H$  is shown on the bottom part. The darkest color indicates zero slope and it is clear that the surface consists of three mountain ridges separating three valleys (we use periodic boundary conditions in  $y$ ). The top of the mountain ridges makes a piecewise smooth curve with some straight segments and in addition to the boundary conditions at the top of the surface, we must also impose boundary conditions on the top of the mountain ridges. Physically, this is straight-forward, these are simply the same boundary conditions that we imposed at the top of the initial ridge in the previous section. As for the the upper boundary in Section 2, the water surface on the top of the ridges must be considered to be a free boundary.  $H$  is a solution of a PDE in two variables on top of the ridges and there it is finite but only piecewise smooth. In addition,

$$(3.2) \quad q_w = q_s = 0,$$



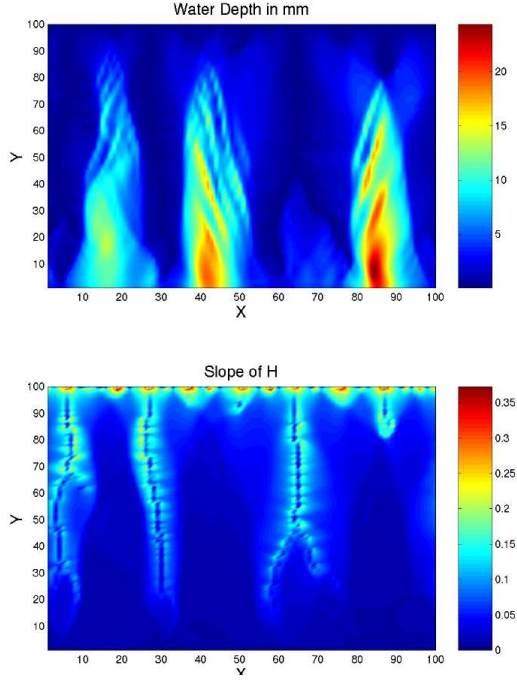


FIGURE 2. The water depth (top) and slope of the water surface (bottom) over the desert landsurface in Figure 1, from [10].

or there is no water or sediment flow over the top of the mountain ridges either. These conditions are verified by the top part of Figure 2. At the top of the mountain ridges the water depth is going to zero. This is indicated by a very dark blue in contrast to the yellow and red at the bottoms of the valleys where the water accumulates. It also makes sense that this statistically stationary water depth, that is averaged over many rainfall events, does not vanish at any point on the surface except at the top of the mountain ridges. Moreover, Figure 2 (bottom) shows that the top of the ridge, being a demarcation of the watersheds of two valleys, is a piecewise linear line. Thus  $h$  only vanishes on this set of measure zero on the surface, the lines that mark the top of the mountain ridges and extend from the upper boundary to the interior of the region, see Figure 2 (bottom). The boundary conditions (3.2) hold on these lines. In some cases the demarcation lines between two watersheds may become a fractal line. We exclude this case for the ease of the computations but the fractal line can be included as a limit of piecewise linear curves.

**Definition 3.3.** *Given  $h$  satisfying (3.1) and (3.2), a weak solution  $H$  of (2.4) is an element of  $\mathcal{D}'(\Omega)$  which is weakly differentiable with respect to  $x, y$ , strongly differentiable with respect to  $t$  for almost every  $(x, y) \in \Omega$ , and satisfies*

$$\int_{\Omega} f \frac{\partial H}{\partial t} d\mathbf{x} = - \int_{\Omega} \langle \nabla f, \nabla H \rangle |\nabla H|^2 h^{10/3} d\mathbf{x}, \quad \text{for all } f \in \mathcal{C}_0^\infty(\Omega).$$

Above,  $\nabla H = \left( \frac{\partial H}{\partial x}, \frac{\partial H}{\partial y} \right)$ , and  $d\mathbf{x}$  is the standard Lebesgue measure on  $\mathbb{R}^2$ .

### 3.1. A priori bounds.

**Lemma 1.** *Assume  $H(x, y, t)|_{t=0} \in L^2(\Omega)$ ,  $h$  is given and satisfies (3.1), and  $H$  is a weak solution of (2.4). Then  $H(x, y, t) \in L^2(\Omega)$  for all  $t > 0$ , and moreover, for  $0 \leq t \leq \tau$ ,*

$$\int_{\Omega} H^2(x, y, \tau) d\mathbf{x} \leq \int_{\Omega} H^2(x, y, t) d\mathbf{x}.$$



*Proof.* We multiply both sides of (2.4) by  $H$  and integrate over  $\Omega$ ,

$$\frac{d}{dt} \|H\|_2^2 = 2 \int_{\Omega} \frac{\partial H}{\partial t} H d\mathbf{x} = -2 \int_{\Omega} |\nabla H|^4 h^{10/3} d\mathbf{x} \leq 0.$$

The second equality follows from integration by parts, the boundary conditions (2.5), (2.6) and (3.2). The inequality follows since the water depth  $h \geq 0$ .  $\square$

*Remark 1.* In the preceding result and throughout this section and the next we will integrate by parts as though we were dealing with smooth solutions. Since  $C^\infty$  is dense in  $W^{1,4}(\Omega)$ , this can be done for a sequence approximating the solutions, and then the appropriate limit taken. The details are straightforward.

**Lemma 2.** *Let  $h$  be given to satisfy (3.1). If  $H$  is a weak solution to (2.4), then*

$$K := \int_{\Omega} \frac{|\nabla H|^4}{4} h^{10/3} d\mathbf{x}$$

*is decreasing in time.*

*Proof.* We compute the functional derivative of  $K$  using integration by parts and the boundary conditions,

$$\dot{K} = \int_{\Omega} \left( |\nabla H|^2 \nabla H h^{10/3} \right) \nabla \dot{H} d\mathbf{x} = - \int_{\Omega} \nabla \cdot \left( |\nabla H|^2 \nabla H h^{10/3} \right) \dot{H} d\mathbf{x},$$

so that

$$D_H K = -\nabla \cdot \left( |\nabla H|^2 \nabla H h^{10/3} \right).$$

The flow is defined by

$$\frac{\partial H}{\partial t} = -D_H K.$$

Consequently,

$$\frac{\partial K}{\partial t} = \int_{\Omega} D_H K \frac{\partial H}{\partial t} d\mathbf{x} = \int_{\Omega} -|D_H K|^2 d\mathbf{x} \leq 0.$$

$\square$

Based on the a priori bounds, we will prove the existence and regularity of solutions to (2.4).

**3.2. Existence and uniqueness.** We start with a technical lemma about the compact embedding of  $W^{1,4}$  in  $L^2$ .

**Lemma 3.**  *$W^{1,4}(\Omega, hdx)$  with the boundary conditions (2.5), (2.6) and (3.2) is compactly embedded into  $L^2(\Omega, dx)$ , and into the space of Hölder continuous functions with index  $\frac{1}{2}$ , with the same boundary conditions.*

*Proof.* Consider the domain  $\Omega$  shown in the bottom part on Figure 2. The domain is a rectangle but with internal boundaries consisting of ridge lines that extend to the upper boundary. These lines are piecewise smooth, in fact piecewise linear, so that the internal domain with the internal boundaries is still ‘‘cone shaped,’’ and can be split into finitely many subregions whose boundary has the strong local Lipschitz property. Thus the usual embedding theorems hold on  $\Omega$ ; see Adams [1]. The value of the water surface  $H$  is a solution of a PDE in two variables on the upper and lower and internal boundaries by the boundary conditions (2.5) and (3.2). On the lateral boundaries periodic boundary conditions (2.6) hold. The issue is whether the vanishing of the weight  $h$  at the upper and internal boundaries can permit function  $H \in W^{1,4}(\Omega, hdx)$  that do not lie in  $W^{1,4}(\Omega, dx)$ . The gradients  $\nabla H$  can have a jump discontinuity at the internal boundaries, however these jumps must be finite because the value of  $H$  is finite on these boundaries. Thus since  $h$  is positive in the interior of  $\Omega$ , and the functions  $H \in W^{1,4}(\Omega, hdx)$  are limits of functions that are continuous at the boundary, with these boundary conditions, the spaces  $W^{1,4}(\Omega, hdx)$  and  $W^{1,4}(\Omega, dx)$  are isomorphic, and the measures in these two spaces are absolutely continuous with respect to each other. Thus  $W^{1,4}(\Omega, hdx)$  and  $W^{1,4}(\Omega, dx)$  are equivalent as Banach spaces. It is well known that the latter space is compactly embedded into  $L^2(\Omega, dx)$ , see the comment above, and lies in the space of Hölder continuous functions with index  $\frac{1}{2}$ .  $\square$

Now consider the PDE (2.4) for the sediment flow,

$$\frac{\partial H}{\partial t} = \nabla \cdot \left[ \nabla H |\nabla H|^2 h^{10/3} \right]$$

By hypothesis,  $H^0 \in W^{1,4}(\Omega)$ , and  $h$  satisfies (3.1) so we compute that

$$\nabla \cdot \left[ \nabla H |\nabla H|^2 h^{10/3} \right]$$

is a distribution on smooth functions with compact support in  $\Omega$  with the internal and external boundaries. The proof of this statement illustrates how the boundary conditions are used. We multiply the equation on both sides with a smooth compactly supported function  $f \in C_0^\infty(\Omega)$  and integrate over  $\Omega$ . Now for the right hand side of the equation an integration by parts gives,

$$\int_{\Omega} f \nabla \cdot \nabla H |\nabla H|^2 h^{10/3} d\mathbf{x} = - \int_{\Omega} \langle \nabla f, \nabla H \rangle |\nabla H|^2 h^{10/3} d\mathbf{x} + \int_{\partial\Omega} f \nabla H \cdot \hat{n} |\nabla H|^2 h^{10/3} dy.$$

We write the boundary terms as

$$f \frac{\nabla H \cdot \hat{n}}{|\nabla H|} |\nabla H|^3 h^{10/3} |_{\partial\Omega} = f \mathbf{u} \cdot \hat{\mathbf{n}} q_s |_{x=0} + f \mathbf{u} \cdot \hat{\mathbf{n}} q_s |_{x=W} + \sum_{j=1}^N [\mathbf{u}]_{RL_j} f q_s = 0$$

Here  $\mathbf{u} = \frac{\nabla H}{|\nabla H|}$  is the unit vector in the direction of the gradient of the water surface,  $q_s = |\nabla H|^3 h^{10/3}$  is the sediment flow, we have summed the internal boundary over  $N$  ridge lines, and  $[\mathbf{u}]_{RL_j}$  denotes the jump in the unit vector across the  $j$ th ridge line. Notice that the boundary terms on the lateral boundaries  $y = 0$  and  $y = L$  cancel due to the periodic boundary conditions (2.6). At the lower boundary,  $\mathbf{u}$  and  $q_s$  are both finite and  $f$  vanishes, thus these boundary terms are zero. At the upper boundary, the slope  $\nabla H$  is finite, and  $q_s$  vanishes, so these terms also vanish by (2.5). At the ridge lines, the jumps  $[\mathbf{u}]_{RL_j}$  are finite, and both  $f$  and  $q_s$  vanish, the latter by the boundary conditions (3.2). This forces all the boundary terms across the ridges to vanish.

Thus for such a function  $f$ , we can apply the Cauchy Schwarz inequality to

$$\left| \int_{\Omega} \langle \nabla f, \nabla H \rangle |\nabla H|^2 h^{10/3} d\mathbf{x} \right| \leq \sqrt{\int_{\Omega} |\langle \nabla f, \nabla H \rangle|^2 h^{10/3} d\mathbf{x}} \sqrt{\int_{\Omega} |\nabla H|^4 h^{10/3} d\mathbf{x}},$$

which by the pointwise Schwarz inequality gives

$$\leq \sqrt{\int_{\Omega} |\nabla f|^2 |\nabla H|^2 h^{10/3} d\mathbf{x}} \sqrt{\int_{\Omega} |\nabla H|^4 h^{10/3} d\mathbf{x}},$$

which by a final application of the Cauchy Schwarz inequality satisfies

$$\leq \left( \int_{\Omega} |\nabla f|^4 h^{10/3} d\mathbf{x} \right)^{1/4} \left( \int_{\Omega} |\nabla H|^4 h^{10/3} d\mathbf{x} \right)^{3/4}.$$

Since  $h$  is continuous (3.1), and  $H \in W^{1,4}(\Omega)$ , it follows that  $\nabla \cdot [\nabla H |\nabla H|^2 h^{10/3}]$  is an element of  $\mathcal{D}'(\Omega)$ .

**Theorem 1.** *Let  $H|_{t=0} \in W^{1,4}(\Omega)$ , and  $h$  satisfying (3.1) be given. Then there exists a weak solution to (2.4) with initial data given by  $H$ ; moreover, at each time  $t$ , the solution is Hölder continuous on  $\Omega$  with index  $1/2$ .*

*Proof.* Now we integrate the Definition 3.3 of the weak solution with respect to  $t$ ,

$$(3.4) \quad \int_{\Omega} f H d\mathbf{x} = \int_{\Omega} f H^0 d\mathbf{x} - \int_0^t \int_{\Omega} \langle \nabla f, \nabla H \rangle |\nabla H|^2 h^{10/3} d\mathbf{x} ds$$

A standard iteration of this equation produces a sequence of Picard iterates  $\{H_k\}$  whose norm is bounded by the norm of the initial condition. This sequence lies in  $C^1([0, T]; W^{1,4}(\Omega)) \subset L^2([0, T]; W^{1,4}(\Omega))$ . Since by Lemmas 1 and 2 the sequence is bounded in  $L^2([0, T]; W^{1,4}(\Omega))$ , by the weak sequential compactness of this reflexive Banach space we can subtract a subsequence also denoted  $\{H_k\}$  that converges weakly in  $L^2([0, T]; W^{1,4}(\Omega))$ . We choose test functions  $f$  which are independent of  $t$  to show that the limit satisfies (3.4) so it also is a weak solution according to Definition 3.3. In fact since the  $W^{1,4}(\Omega)$  norm of the limit is bounded by the initial conditions, by Lemma 2, the weak solution exists for all time. By the Sobolev embedding theorem,  $W^{1,4}(\Omega)$  embeds into  $\mathcal{C}^{0,1/2}(\Omega)$ , so the weak solutions lies in the space of Hölder continuous functions of index  $1/2$ .  $\square$

Next, we demonstrate uniqueness of solutions.

**Theorem 2.** *Assume  $F$  and  $H$  are weak solutions to (2.4) with respect to the same height function  $h$ , and assume that  $W^{1,4}(\Omega) \ni F|_{t=0} = H|_{t=0}$ . Then  $F \equiv H$ .*

*Proof.* We compute the derivative of the  $L^2$  norm of  $H - F$  with respect to time.

$$\frac{d}{dt} \|H - F\|_2^2 = 2 \int_{\Omega} \nabla \cdot \left( (\nabla H |\nabla H|^2 - \nabla F |\nabla F|^2) h^{10/3} \right) (H - F) dx dy.$$

Integrating by parts and using the boundary conditions gives

$$\frac{d}{dt} \|H - F\|_2^2 = -2 \int_{\Omega} [|\nabla H|^4 + |\nabla F|^4 - \langle \nabla H, \nabla F \rangle (|\nabla H|^2 + |\nabla F|^2)] h^{10/3} dx dy.$$

By the pointwise Schwarz inequality applied to  $\langle \nabla H, \nabla F \rangle$ ,

$$(3.5) \quad \frac{d}{dt} \|H - F\|_2^2 \leq -2 \int_{\Omega} [|\nabla H|^4 + |\nabla F|^4 - |\nabla H| |\nabla F| (|\nabla H|^2 + |\nabla F|^2)] h^{10/3} dx dy.$$

It is a straightforward exercise to show that for any  $a, b \geq 0$ ,

$$a^4 + b^4 - ab(a^2 + b^2) \geq 0.$$

Consequently, the integrand in the right side of (6.8) is non-negative almost everywhere on  $\Omega$ , which shows that

$$\frac{d}{dt} \|H - F\|_2^2 \leq 0.$$

Since  $H$  and  $F$  have the same initial data,  $\|H - F\|_2 = 0$  for  $t = 0$ , which implies  $\|H - F\|_2 \equiv 0$  for all  $t \geq 0$ . This implies  $H = F$  almost everywhere for  $t \geq 0$ . By Theorem 1,  $H$  and  $F$  are continuous, hence  $H \equiv F$ .  $\square$

#### 4. OPTIMAL TRANSPORT PROBLEMS

We recall the general setup of optimal transport problems. Let  $\mu$  and  $\nu$  be non-negative Radon measures with (respectively) compact supports  $U, V \subset \mathbb{R}^n$  satisfying,

$$(4.1) \quad \int_U d\mu = \int_V d\nu.$$

A map  $s : U \rightarrow V$  pushes  $\mu$  onto  $\nu$ , and we write  $s_{\#}(\mu) = \nu$  if  $s$  is Borel measurable and for any Borel set  $E \subset V$ ,

$$(4.2) \quad \int_{s^{-1}(E)} d\mu = \int_E d\nu.$$

Associated to the optimal transport problem is a cost function which is typically given by

$$(4.3) \quad C(s) := \int_U c(\mathbf{x}, s(\mathbf{x})) d\mu(\mathbf{x}), \quad c(\mathbf{x}, \mathbf{y}) := \frac{|\mathbf{x} - \mathbf{y}|^p}{p},$$

where  $p \geq 1$  is fixed. Monge's original problem, with  $p = 1$ , is in fact more difficult than the problem with  $p > 1$ ; in this work, we investigate the case  $p = 1$ . A general optimal transport problem is,

$$(4.4) \quad \text{Does there exist } s : U \rightarrow V \text{ which minimizes } C \text{ with } s_{\#}(\mu) = \nu?$$

If it exists, such a map  $s$  is called an ‘‘optimal mass reallocation plan,’’ or an ‘‘optimal mass transport plan.’’

In the context of erosion, we pose the following natural question.

$$(4.5) \quad \text{Is sediment ‘‘optimally transported’’ according to (2.4)?}$$

We must first determine an appropriate mathematical formulation. Some immediate difficulties arise. Monge's problem does not depend on time; erosion does. Moreover, the mass of the sediment is not preserved over time since it flows out of the region  $\Omega$ . We will see in the following arguments that these difficulties are in fact easily overcome.

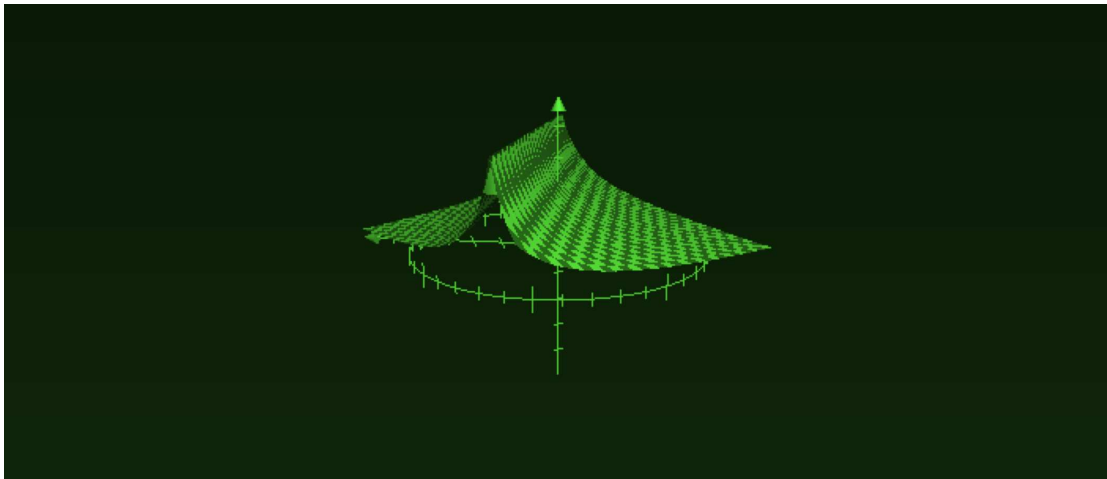


FIGURE 3. A mountain ridge.

## 5. MOUNTAINS AND RIDGES

Simulations and observations of real landsurface shapes that retain their form for a long time but decrease in elevation are studied extensively in [10] and [11]. In [8], separable solutions of the equations (2.1) and (2.2) were discovered; these solutions exhibit the same behavior as the simulations and observations in [10] and [11]. The separable solutions that are of interest to us have the general form

$$(5.1) \quad h(x, y, t) = h(x, y), \quad H(x, y, t) = H_o(x, y)T(t)$$

where  $T(t)$  is a function of time. The following solutions were found by the first author and studied in great detail in [9] in the one dimensional case.

**Lemma 4.** *Let  $a, b, h_1, c, d$ , and  $H_1$  be constants, and assume  $T = T(t)$  is a function that depends only on time. Define*

$$(5.2) \quad \begin{aligned} h_o(x, y) &= h_1(H_1^{1/c} + a(x - x_0) + b(y - y_0))^d \\ H_o(x, y) &= (H_1^{1/c} + a(x - x_0) + b(y - y_0))^c \\ H(x, y, t) &= H_o(x, y)T(t), \end{aligned}$$

Then there exists a function  $u$  such that

$$\frac{\nabla H}{|\nabla H|} = \nabla u$$

*Proof.* The necessary and sufficient condition for the existence of a function  $u$  which satisfies  $\nabla u = \frac{\nabla H}{|\nabla H|}$ , is

$$\nabla \times \frac{\nabla H}{|\nabla H|} = 0,$$

which is equivalent to the following condition on the partial derivatives of  $H$

$$(5.3) \quad H_{xy}(H_x^2 - H_y^2) = H_x H_y (H_{xx} - H_{yy}).$$

The rest of the proof is a computation verifying this last condition.  $\square$

*Remark 2.* When  $a$  and  $b$  have opposite signs, the functions  $h$ ,  $H_o$ , and  $H$  defined in the preceding lemma are called *mountain ridges*, and when  $a$  and  $b$  are both positive, the functions are called *mountains*. Lemma 4 shows that the separable solutions, that are observed both numerically and empirically, satisfy the condition (6.5) that we will impose in Theorem 3.

For the mountain and mountain ridges, if we let

$$(5.4) \quad T(t) = \frac{1}{\sqrt{1 + 2rt}},$$

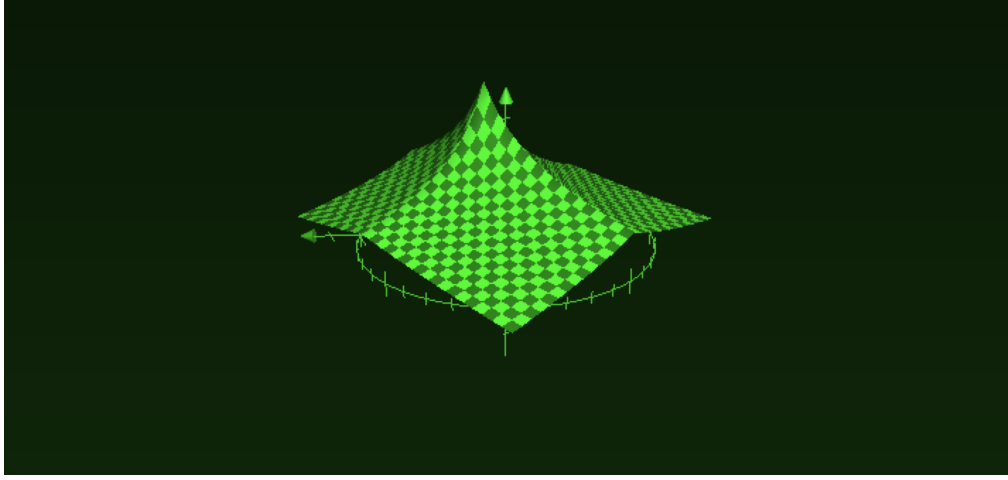


FIGURE 4. A mountain.

then  $h(x, y)$  and  $H(x, y, t)$  satisfy (2.4) if the exponents  $c$  and  $d$  satisfy a certain relationship. We compute

$$\frac{\partial H}{\partial t} = -rT^3(H_1^{1/c} + a(x - x_0) + b(y - y_0))^c.$$

and

$$\nabla \cdot [\nabla H |\nabla H|^2 h^{10/3}] = T^3 h_1^{10/3} c^3 (a^2 + b^2)^2 (3c - 3 + 10d/3) z^{3c-4+10d/3},$$

where for the sake of notation we have let

$$z := H_1^{1/c} + a(x - x_0) + b(y - y_0).$$

Then,  $H$  and  $h$  satisfy (2.4) if and only if

$$-rT^3 z^c = T^3 h_1^{10/3} c^3 (a^2 + b^2)^2 (3c - 3 + 10d/3) z^{3c-4+10d/3}.$$

This simplifies to

$$-r = h_1^{10/3} c^3 (a^2 + b^2)^2 (3c - 3 + 10d/3) z^{2c-4+10d/3}.$$

Since  $z$  is not constant, whereas  $r$ ,  $h_1$ ,  $c$ ,  $a$ ,  $b$ , and  $d$  are, this equation can only be satisfied if either

$$(5.5) \quad 3c - 3 + \frac{10d}{3} = 0 \iff c = 1 - \frac{10d}{9},$$

or

$$(5.6) \quad 2c - 4 + \frac{10d}{3} = 0 \iff c = 2 - \frac{5d}{3}.$$

The first condition (5.5) implies  $r = 0$ , so that  $T$  is constant, and there is no erosion. The second condition (5.6), on the other hand, turns out to be more interesting. This condition implies

$$(5.7) \quad r = -h_1^{10/3} c^3 (a^2 + b^2)^2 (3c - 3 + 10d/3).$$

The constant  $r$  is also related to the flux and the initial volume of sediment,

$$(5.8) \quad r = -c_r \frac{F_0}{V_0}, \quad F_0 = \int_0^L \nabla H |\nabla H|^2 h^{10/3}(W, y, 0) \cdot \hat{n} dy, \quad V_0 = \int_{\Omega} H(x, y, 0) dx,$$

where  $F_0$  is the integration of the initial flux and  $V_0$  is the initial volume of the sediment, and  $c_r > 0$  is a constant. Since the integral of the flux is negative,

$$r > 0.$$

Moreover, since  $h_1 > 0$ , the positivity of  $r$  implies that  $c$  and  $d$  must lie within a certain range. In particular, we have the following.

**Lemma 5.** *Let*

$$h(x, y) = h_o(x, y) = \begin{cases} h_1(H_1^{-2} + a(x - x_0) + b(y - y_0))^{3/2}, & y - y_0 < -\frac{a(x-x_0)}{b} \\ h_1(H_1^{-2} - a(x - x_0) - b(y - y_0))^{3/2}, & y - y_0 > -\frac{a(x-x_0)}{b} \end{cases}$$

where  $a$  and  $b$  have opposite signs, and  $T$  is defined by (5.4). Then the mountain ridge, see Figure 3,  $H(x, y, t) =$

$$H_o(x, y)T(t) = \begin{cases} (H_1^{-2} + a(x - x_0) + b(y - y_0))^{-1/2}T(t), & y - y_0 < -\frac{a(x-x_0)}{b} \\ (H_1^{-2} - a(x - x_0) - b(y - y_0))^{-1/2}T(t), & y - y_0 > -\frac{a(x-x_0)}{b} \end{cases}$$

is a weak solution of (2.4). Let

$$h(x, y) = h_o(x, y) = h_1(H_1^{-2} + a|x - x_0| + b|y - y_0|)^{3/2}$$

where  $a$  and  $b$  are both positive. Then the mountain, see Figure 4,

$$H(x, y, t) = H_o(x, y)T(t) = (H_1^{-2} + a|x - x_0| + b|y - y_0|)^{-1/2}T(t)$$

is a weak solution of (2.4).

*Proof.* For the mountain and mountain ridge functions, we verify that the exponents  $c = -1/2$  and  $d = 3/2$  satisfy (5.6). These are not the only values of the exponents permitted by (5.6) but they are consistent with numerically observed scaling of  $\nabla H$  in [10]. The rest of the proof consists of showing that we can glue together several ridges to form a pattern of valleys and ridges, similar to Figure 1, satisfying the boundary conditions (3.2) and using the abundance of water (depth) at the bottom of the valleys, see Figure 2, to impose similar boundary conditions there. For simplicity of the exposition we consider a uniform ridge directed along the  $x$  axis and given by the formula

$$h(x, y) = (H^{-2} \pm y)^{3/2}, \quad H_o(x, y) = (H^{-2} \pm y)^{-1/2},$$

the general case is similar. We multiply the PDE (2.4) with a smooth function  $f$  and integrate over the domain  $\Omega$  with the internal boundary along the  $x$  axis. This gives

$$\int_{\Omega} f \frac{\partial H}{\partial t} dx = \int_{\Omega} f \nabla \cdot (\nabla H |\nabla H|^2 h^{10/3}) dx = - \int_{\Omega} \nabla f \cdot (\nabla H |\nabla H|^2 h^{10/3}) dx + \int_{\partial\Omega} f \nabla H \cdot \hat{n} |\nabla H|^2 h^{10/3} dy$$

by the divergence theorem. As in Section 3, we write the boundary terms as

$$f \frac{\nabla H \cdot \hat{n}}{|\nabla H|} |\nabla H|^3 h^{10/3} |_{\partial\Omega} = f \mathbf{u} \cdot \hat{\mathbf{n}} q_s |_{x=0} + f \mathbf{u} \cdot \hat{\mathbf{n}} q_s |_{x=W} + \sum_{j=1}^N [\mathbf{u}]_{RL_j} f q_s = 0$$

The periodic boundary conditions at the lateral boundaries  $y = 0$  and  $y = L$  make the corresponding boundary terms vanish. At the upper boundary  $q_s = 0$ , so these terms vanish, and the mountains ridges are sliced of  $f$  by another decaying function before they reach the lower boundary, so there  $q_s$  is finite and  $f = 0$  makes the corresponding boundary terms vanish as well. The only issue is the jump  $[\mathbf{u}]_{y=0} = (0, 2)$  in the unit normal  $\mathbf{u} = \frac{\nabla H}{|\nabla H|}$  to the water surface at the ridge  $y = 0$ . Notice that  $\nabla H = \pm(0, H^3/2)$  where  $H$  is the height of the ridge. But this jump is being multiplied by  $q_s$  the sediment flow over the ridge and  $q_s = 0$  by the boundary conditions (3.2). Thus all the boundary terms vanish. This shows that a single ridge is a weak solution of the PDE (2.4).

The proof for several ridges is similar but then one has extra boundary conditions where the ridges are glued together at the bottom of the valleys. This boundary forms a straight river channel. Along the boundary it makes sense to impose the same boundary conditions (3.2) or to restrict the sediment flow to be along but not across the river. In other words  $h \neq 0$  but  $\frac{\partial H}{\partial y} = 0$  at the center of the rivers. Then again we get  $[\mathbf{u}]_{y=a/2} f q_s$ , at the center  $y = a/2$  of the rivers where the two ridges are joined and since  $q_s = 0$  these boundary terms vanish.  $a$  is the distance between the ridges, it is given by the formula  $a = W^{3/4}$ , see [11]. Consequently, several ridges joined together with the boundary condition (3.2) are a weak solution of (2.4).

The mountains can be considered to be the intersection of two ridges. Thus the proof is the same for them if we apply the boundary condition (3.2) to the mountain top and the four lines of intersection. The boundary conditions then says that the sediment does not flow around the crests along which the four convex faces of the mountain in Figure 4 are joined, again this is physically reasonable.  $\square$

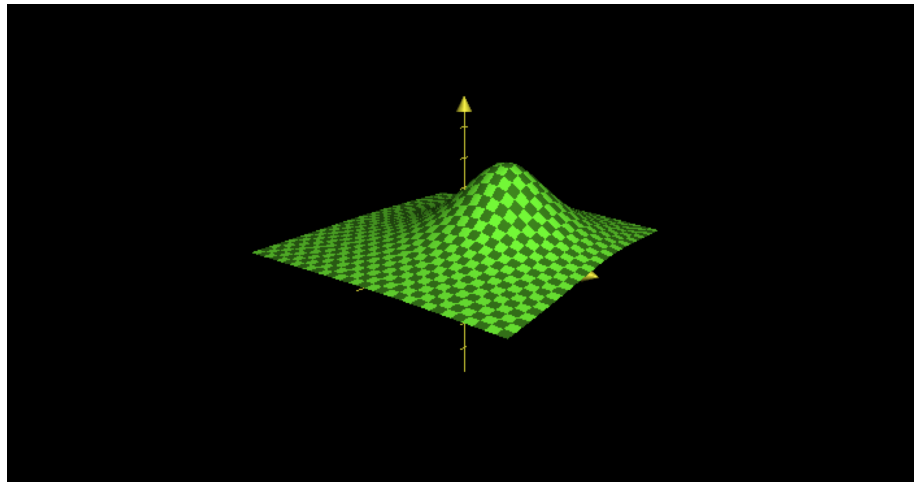


FIGURE 5. A collapsing hill.

*Remark 3.* The mapping  $x - x_o$  to  $-(x - x_o)$  and  $(y - y_o)$  to  $-(y - y_o)$  also produces weak solutions of the initial value problem in §2. These are just the ridges reflected about the line  $y - y_o = x - x_o$ . However, the solutions obtained by mapping  $H$  to  $-H$  are not reachable from the initial conditions. They constitute canyons (negative ridges) that only would appear after all the sediment is gone, and holes (negative mountains) that are not observed in nature.

There also exist solutions of (2.4) which correspond to the Barenblatt solution [4], [58] of the porous medium equation; see [44]. Define the *collapsing hills*, see Figure 5,

$$\begin{aligned} h(x, y, t) &= h_1 [a + b(1 + rt)^\beta ((x - x_0)^2 + (y - y_0)^2)]^d (1 + rt)^{-(3+6\beta)/10} \\ H(x, y, t) &= [a + b(1 + rt)^\beta ((x - x_0)^2 + (y - y_0)^2)]^c, \end{aligned}$$

where  $\beta$  and  $r$  are constants. Again we may assume without loss of generality that  $(x_0, y_0) = (0, 0)$ . Then, the collapsing hills function satisfies  $H(x, y, t) = H(y, x, t)$  which immediately implies (5.3). By a calculation similar to that for the Barenblatt solution of the porous medium equation [4], [58], if  $\beta$ ,  $c$ , and  $d$  are chosen to satisfy certain conditions, then the collapsing hills are a strong solution of (2.4). In particular, let

$$h(x, y, t) = h_1 [a + b(1 + rt)^\beta ((x - x_0)^2 + (y - y_0)^2)]^{9/5} (1 + rt)^{-(3+6\beta)/10}$$

where  $r = 54ab^2/\beta$  and  $0 < \beta < 1/4$ . Then the corresponding *collapsing hill*

$$(5.9) \quad H(x, y, t) = [a + b(1 + rt)^\beta ((x - x_0)^2 + (y - y_0)^2)]^{-3/2},$$

is a strong solution of (2.4).

We are most interested in the mountain ridges, because they are observed both empirically and in simulations for significant time intervals; see [10] and [11]. The empirically observed mountain ridges are in fact more complicated than the ridges modeled by our mountain ridge functions. The observed mountain ridges are actually chains of pieces or slices defined by these ridge functions and linked together. The ridge lines form piecewise linear crests, see Figure 2 (bottom). In the limit of such chains of convex pieces the top of the mountain ridge can even form a fractal curve; see [10] and [11] for figures of simulations of such ridges. The mountains are only observed for much shorter times in the simulations. They occur at the boundary and then usually for relatively short times. Only mountains that anchor stable mountain ridges at the boundary persist for long times. As time becomes large, it is observed that all solutions tend toward these separable solutions, that is a pattern of valleys separated by convex mountain ridges; see [10] and [11]. This is what we recognize as “the landscape.” Based on our results and the related work of Otto [55], we expect that all solutions of (2.4) tend toward these separable mountain ridges; further discussion of this is postponed to §7. The collapsing hills are only observed briefly at the very end of simulations when the surface quickly collapses to a flat plain.



## 6. AN OPTIMAL TRANSPORT PROBLEM FOR THE FLOW OF SEDIMENT

We consider an “instantaneous optimal transport problem” for the sediment similar to the equation used to model sand cone dynamics in [24] §11. The equation considered in that work is

$$\begin{cases} f - u_t \in I_\infty[u] & (t > 0) \\ u = 0 & (t = 0) \end{cases}$$

where  $I_\infty[u]$  is a certain functional defined in [24] (9.16) and (9.17). The physical interpretation of such an instantaneous optimal transport problem is that at each moment in time, the mass  $d\mu^+ = f^+(\cdot, t)dx$  is instantly and optimally transported downhill by the potential  $u(\cdot, t)$  into the mass  $d\mu^- = u_t(\cdot, t)dy$ . In other words, the height function of the sandpile is also the potential generating the optimal transport problem  $u_t dx \mapsto f^+ dy$ . To study the local behavior of the flow of sediment under erosion, it is then natural to introduce a similar instantaneous optimal transport problem.

By the divergence theorem and the boundary condition (2.8)

$$(6.1) \quad \bar{F}_\Omega := \int_\Omega \frac{\partial H}{\partial t} d\mathbf{x} = \int_0^L \nabla H |\nabla H|^2 h^{10/3}(W, y, t) \cdot \hat{n} dy < 0.$$

Physically, this means that the sediment is flowing out of the region  $\Omega$  into the lake or river which meets the  $\{x = W\}$  boundary of  $\Omega$ . We formulate the optimal transport problem using the sediment flux instead of the mass. The problem then becomes an optimal transport problem of the sediment fluxes. This is however equivalent to the optimal transport problem of the masses transported by the sediment fluxes in a small time interval as will be illustrated below.

Define the measures  $\mu$  and  $\nu$  with support on  $\Omega$ ,

$$(6.2) \quad d\mu := -\frac{\partial H}{\partial t}(\mathbf{x}, t)d\mathbf{x} =: f^+(\mathbf{x})d\mathbf{x}, \quad d\nu := -F d\mathbf{x} =: f^-(\mathbf{x})d\mathbf{x}.$$

where

$$F := \bar{F}_\Omega / |\Omega|,$$

and  $|\Omega|$  denotes the area of  $\Omega$ . The density  $F$  is constant on  $\Omega$  but this is the result of averaging the non-constant line density on the boundary in (6.1) and spreading it uniformly over  $\Omega$ . We want to know if this formulation of the optimal transport amounts to nature taking mounds of dirt (mountains) and dumping them in the ocean. To see this we rewrite the balance equation (4.1) as

$$\int_\Omega -\frac{\partial H}{\partial t} d\mathbf{x} = -\int_0^L \nabla H |\nabla H|^2 h^{10/3}(W, y, t) \cdot \hat{n} dy.$$

If we integrate this equality over a small time interval, we get

$$\int_\Omega (H_0(x, y) - H(x, y, t)) d\mathbf{x} = -\int_0^t \bar{F}_\Omega dt.$$

Thus the dirt removed from the surface equals the cumulative flux that exited the lower boundary in the time interval  $[0, t]$ . Here we have formulated the problem in terms of a area density being transported to a line density. However, it is more convenient to be able to integrate over the same domain on both sides of (4.1) and therefore we spread the transported sediment again uniformly over  $\Omega$  in (6.2) for convenience of the exposition.

We make the natural assumption that the landsurface is eroding: that its height is decreasing

$$(6.3) \quad \frac{\partial H}{\partial t} \leq 0 \quad \text{a. e. on } \Omega.$$

Under these assumptions, the measures are non-negative. The physical interpretation of the mass reallocation problem (4.4) for  $\mu \mapsto \nu$ , is that at time  $t_0$  the sediment is instantly and optimally transported. In other words the sediment flux  $-d\nu := F d\mathbf{x}$  is equal to the rate of decrease in the height of the water surface  $-d\mu := \frac{\partial H}{\partial t}(\mathbf{x}, t)d\mathbf{x}$ . We aim to show that if this transport implemented by the sediment flow, is in the direction of the negative surface gradient  $-\nabla H$ , then it is in fact optimal. It is interesting to note that in order to demonstrate a meaningful relationship between optimal mass reallocation and sediment flowing according to (2.4) we must use Monge’s original cost functional (4.3); this is also the case for sand cone dynamics [24]. However, this is a more difficult problem than setting

$$C(s) := \int_U c(\mathbf{x}, s(\mathbf{x})) d\mu, \quad c \text{ is a strictly convex function.}$$

References for the latter variation of Monge's problem include [15], [27], and [59]. Our result is based on the ideas and methods of [25] and its generalizations in [16] and [73].

**Theorem 3.** *Assume  $H(\mathbf{x}, t)|_{t=0} \in W^{1,4}(\Omega)$ , and that for a given function  $h$  satisfying (3.1),  $H$  is a weak solution of (2.4). Assume (6.1) and (6.3) are satisfied at time  $t$ . Let  $\mu$  and  $\nu$  be the measures supported on  $\Omega$  and defined by (6.2). Then, there exists an optimal mass reallocation plan  $s : \Omega \rightarrow \Omega$ , which solves (4.4), and there exists a function  $u$  so that  $s$  and  $u$  satisfy the equation*

$$(6.4) \quad \frac{s(\mathbf{x}) - \mathbf{x}}{|s(\mathbf{x}) - \mathbf{x}|} = -\nabla u.$$

Moreover, if  $H$  satisfies

$$(6.5) \quad H_{xy}(H_x^2 - H_y^2) = H_x H_y (H_{xx} - H_{yy}) \quad \text{and} \quad \nabla H \neq 0, \quad \text{almost everywhere in } \Omega,$$

then

$$(6.6) \quad \nabla u = \frac{\nabla H}{|\nabla H|}$$

is uniquely defined at all points where  $\nabla H$  is defined and nonzero. In this case the sediment flow implements the optimal transport.

*Proof.* Since  $H$  is a weak solution of (2.4),  $f^\pm \in \mathcal{L}^1(\Omega)$ . By definition of  $\mu$  and  $\nu$  and (6.1), the mass balancing condition

$$\int_{\Omega} d\mu = \int_{\Omega} d\nu$$

is satisfied. Namely,

$$\int_{\Omega} (f^+ - f^-) d\mathbf{x} = - \int_0^L \nabla H |\nabla H|^2 h^{10/3} \cdot \hat{n} dy + \int_{\Omega} \frac{F_{\Omega}}{|\Omega|} d\mathbf{x} = -F_{\Omega} + F_{\Omega} = 0.$$

Moreover, the measures are by hypothesis non-negative and absolutely continuous with respect to Lebesgue measure

$$d\mu, d\nu \ll d\mathbf{x}.$$

The existence of the optimal mass reallocation plan  $s$  and a function  $u$  so that  $s$  and  $u$  satisfy (6.4) is well known; see for example [75], [73], and [24]. This proves the first statement in the theorem. Demonstrating (6.6) under the assumption (6.5) will require a bit more work.

The main idea in the proof of the optimal transport is to carefully analyze Kantorovich's dual maximization problem, namely to maximize

$$\mathcal{K}[u, v] := \int_{\Omega} u(\mathbf{x}) d\mu(\mathbf{x}) + \int_{\Omega} v(\mathbf{x}) d\nu(\mathbf{x})$$

subject to the constraint

$$u(\mathbf{x}) + v(\mathbf{y}) \leq c(\mathbf{x}, \mathbf{y}) \quad \text{for } \mathbf{x}, \mathbf{y} \in \Omega.$$

Since we are working with Monge's original cost function,  $c(x, y) = |x - y|$ , by [24] Lemma 9.1 we may assume that

$$u = -v.$$

In fact, [24] requires additional regularity on  $f^\pm$ , but this is not necessary as demonstrated in [73]. The constraint may then be reformulated to

$$(6.7) \quad |v(\mathbf{x}) - v(\mathbf{y})| \leq |\mathbf{x} - \mathbf{y}| \quad \text{almost everywhere on } \Omega.$$

With this simplification, the dual problem is to maximize

$$\mathcal{K}(v) := \int_{\Omega} v(\mathbf{x}) (f^+ - f^-) d\mathbf{x},$$

subject to the Lipschitz constraint (6.7).

Our first step is to reduce the optimal transport problem over  $\Omega$  to an optimal transport problem over

$$\Omega' := \{\mathbf{x} \in \Omega : \nabla H \text{ is defined and nonzero}\}.$$

By the hypothesis (6.5),  $\nabla H$  is defined and nonzero almost everywhere on  $\Omega$  so

$$\int_{\Omega-\Omega'} v(f^+ - f^-) d\mathbf{x} = 0.$$

Consequently,  $v$  maximizes  $\mathcal{K}$  over  $\Omega$  if and only if  $v$  maximizes  $\mathcal{K}$  over  $\Omega'$ . Our next step is to let

$$s(\mathbf{x}) := \mathbf{x} \text{ for } \mathbf{x} \in \Omega - \Omega'.$$

Then

$$\int_{\Omega-\Omega'} c(s(\mathbf{x}), \mathbf{x}) d\mu = 0,$$

so we have indeed reduced the problem to finding an optimal mass reallocation plan over  $\Omega'$ . By (6.6), there exists  $u : \Omega \rightarrow \Omega$  such that

$$(6.8) \quad \nabla u = \frac{\nabla H}{|\nabla H|} \quad \text{almost everywhere on } \Omega.$$

Clearly then  $u$  satisfies the Lipschitz constraint (6.7).

We then compute using: integration by parts, the definition of  $f^\pm$ , the boundary conditions, and (2.4),

$$\int_{\Omega} v(f^+ - f^-) d\mathbf{x} = \int_{\Omega} \langle \nabla v, \nabla H \rangle |\nabla H|^2 h^{10/3} d\mathbf{x}.$$

By the (pointwise) Schwarz inequality,

$$(6.9) \quad |\langle \nabla v, \nabla H \rangle| \leq |\nabla v| |\nabla H|,$$

with equality if and only if  $\nabla v$  is a scalar multiple of  $\nabla H$  so that  $\nabla v = c \nabla H$ . The only scalar multiples consistent with the Lipschitz constraint are  $c = \pm \frac{1}{|\nabla H|}$ . Thus, for any test function  $v$  satisfying the Lipschitz constraint,

$$\int_{\Omega} v(f^+ - f^-) d\mathbf{x} \leq \int_{\Omega} |\nabla H|^4 h^{10/3} d\mathbf{x} = \int_{\Omega} u(f^+ - f^-) d\mathbf{x},$$

where  $u$  is defined to satisfy (6.8). So, we may conclude that the maximizer of  $\mathcal{K}$  is achieved by  $u$  which satisfies (6.8). By [73] Theorem 3.1, there exists an optimal mass reallocation plan  $s$  such that

$$s(\mathbf{x}) = \mathbf{x} \text{ on } \Omega - \Omega',$$

and

$$\frac{s(\mathbf{x}) - \mathbf{x}}{|s(\mathbf{x}) - \mathbf{x}|} = -\nabla u = -\frac{\nabla H}{|\nabla H|}, \quad \text{almost everywhere on } \Omega'.$$

Finally, the uniqueness follows from Theorem 2.  $\square$

By Theorem 1,  $\nabla H$  is defined almost everywhere on  $\Omega$ . The physical interpretation of  $\nabla H(\mathbf{x}, t) = 0$  is that the point  $\mathbf{x}$  lies at the top of a mountain; such points empirically form a set of measure zero. Since the sediment flows in the direction of  $-\nabla H$ , our result shows that the direction of the sediment flow according to the solution of (2.4) is identical to the direction of the instantaneous optimal mass reallocation plan almost everywhere on  $\Omega$ . In other words the direction of the sediment flows according to (2.4) is optimal or when the landsurface evolves towards the separable ridges, in Section 5, then the sediment flow becomes optimal.

## 7. GRADIENT FLOWS AND LONG TIME ASYMPTOTICS

In this paper, we have focused on the local properties of the optimal mass reallocation plan and its relationship to the local properties of the sediment flow. This is related to the elegant works of Otto [55] and McCann [44]. The first of these works concerns the porous medium equation

$$(7.1) \quad \frac{\partial \rho}{\partial t} = \nabla^2 \rho^m,$$

where  $\rho \geq 0$  is a time dependent density function on  $\mathbb{R}^n$ , and  $m \geq 1$ . When  $m > 1$ , this represents so-called ‘‘slow diffusion;’’  $m < 1$  is called fast diffusion. In [55], the exponent satisfies  $m \geq 1 - \frac{1}{n}$  and  $m > \frac{n}{n+2}$ . In an appropriate weak setting, similar to ours, the Cauchy problem for (7.1) is well posed. Then, (7.1) defines an evolution of densities on  $\mathbb{R}^n$ . Although it was previously known that this semi group has the structure of a

gradient flow, in [55] it was demonstrated to be a gradient flow for a certain functional  $E$  on the space  $M$  of non-negative probability measures on  $\mathbb{R}^n$ . Expressing the porous medium equation as the gradient flow

$$\frac{d}{dt}E(\rho) = -g_\rho \left( \frac{d\rho}{dt}, \frac{d\rho}{dt} \right),$$

separates the energetics and kinetics: the energetics are represented by the functional  $E$  on the state space  $M$  while the kinetics endow the state space with Riemannian geometry via the metric tensor  $g$ . This state space  $M$  naturally carries the Wasserstein distance. The main results of [55] demonstrate that the density gradient flow converges, at a certain rate made explicit in the paper, to the Barenblatt solution, which minimizes the energy functional. This is equivalently described on the state space: the gradient flow tends towards the optimal measure. Thus, [55] establishes a connection between the space of probability measures equipped with the Wasserstein metric and the long time behavior of solutions to the porous medium equation.

The setting in [55] does not immediately apply to our problem. The structure seems to be similar but the boundary conditions are different and theory in [55] has to be adapted to our boundary conditions. The boundary conditions also change the asymptotics and make them different from [44]. In [55] and [44], the Barenblatt solution plays the main role, but in our case the collapsing hill (5.9), that is the analog of the Barenblatt solution, is not the main actor in the asymptotics. Instead that role is played by the mountain ridges in Lemma 4. Nevertheless the structure in [55] appears adaptable to our case, and one should be able to use the Wasserstein metric to describe how our general solutions approach the optimal metric, given by the mountain ridges, as time tends to infinity. Even more intriguing is the question of whether the stochastic approach [11] can be formulated on the space  $M$  where the probability measures and the Wasserstein metric live? These questions will be the subject of future work.

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