UC Berkeley
UC Berkeley Previously Published Works

Title
Spin glasses and Stein's method

Permalink
https://escholarship.org/uc/item/3sx7j7kx

Journal
Probability Theory and Related Fields, 148(3)

ISSN
1432-2064

Author
Chatterjee, Sourav

Publication Date
2010-11-01

DOI
10.1007/s00440-009-0240-8

Peer reviewed

Spin glasses and Stein's method

Sourav Chatterjee

Abstract   We introduce some applications of Stein's method in the high temperature analysis of spin glasses. Stein's method allows the direct analysis of the Gibbs measure without having to create a cavity. Another advantage is that it gives limit theorems with total variation error bounds, although the bounds can be suboptimal. A surprising byproduct of our analysis is a relatively transparent explanation of the Thouless–Anderson–Palmer system of equations. Along the way, we develop Stein's method for mixtures of two Gaussian densities.

Keywords   Spin glass · Sherrington–Kirkpatrick model · TAP equations · High temperature solution · Stein's method

Mathematics Subject Classification (2000)   60K35 · 82B44

1 Introduction and results

1.1 The Sherrington–Kirkpatrick model

Let N be a positive integer and let $\Sigma_N = \{-1, 1\}^N$. A typical element of $\sigma = (\sigma_1, \ldots, \sigma_N) \in \Sigma_N$ is called a 'configuration' of spins. Let $g = (g_{ij})_{1 \leq i < j \leq N}$ be a

S. Chatterjee (✉)
Department of Statistics, University of California at Berkeley,
367 Evans Hall #3860, Berkeley, CA 94720-3860, USA
e-mail: sourav@stat.berkeley.edu
URL: http://www.stat.berkeley.edu/~sourav

collection of independent standard Gaussian random variables, called the 'disorder' in our context. Given a realization of g, fix any $\beta > 0$ and $h \in \mathbb{R}$ and define a probability distribution G_N on Σ_N as

$$G_N(\sigma) = Z_N^{-1} \exp\left(\frac{\beta}{\sqrt{N}} \sum_{1 \le i < j \le N} g_{ij} \sigma_i \sigma_j + h \sum_{1 \le i \le N} \sigma_i \right).$$

Here $Z_N = Z_N(\beta, h)$ is the normalizing constant (partition function), and G_N is the 'Gibbs measure'. What we have just defined is the well-known Sherrington–Kirkpatrick (SK) model of spin glasses [25]. The parameter β is called the 'inverse temperature' of the model, and h is called the 'external magnetic field', following the conventions of Talagrand [29]. Physicists would replace the h with βh, but the definitions are mathematically equivalent.

Configurations chosen independently from the Gibbs measure (i.e., given the disorder) are denoted by σ^1, σ^2, etc. These are called 'replicas' in physics. If $\sigma^1, \ldots, \sigma^k$ are replicas and f is a function on Σ_N^k, then as usual we define

$$\langle f(\sigma^1, \ldots, \sigma^k) \rangle := \sum_{\sigma^1, \ldots, \sigma^k} f(\sigma^1, \ldots, \sigma^k) G_N(\sigma^1) \cdots G_N(\sigma^k).$$

In the terminology of disordered systems, $\langle f(\sigma) \rangle$ is known as the quenched average of $f(\sigma)$.

The 'overlap' between a pair of replicas σ^1 and σ^2, chosen independently from the Gibbs measure, is defined as

$$R_{12} := \frac{1}{N} \sum_{i=1}^{N} \sigma_i^1 \sigma_i^2. \tag{1}$$

The 'high temperature phase' of the SK model corresponds to the set of (β, h) for which there is a number $q = q(\beta, h) < 1$ such that the overlap R_{12} is approximately equal to q with high probability under the Gibbs measure. This can be made precise in various ways, and the form that will be most suitable for us in this article is:

$$\mathbb{E}\langle (R_{12} - q)^4 \rangle \le \frac{C(\beta, h)}{N^2}, \tag{2}$$

where $C(\beta, h)$ is a constant that depends only on β and h. (At this point, let us declare that throughout this paper, statements like "$T \le C(\beta, h)$" stands for "the term T can be bounded by a constant that depends only on β and h".) It is known (see e.g. [29, p. 72]) that the constant q must satisfy

$$q = \mathbb{E} \tanh^2(\beta z \sqrt{q} + h), \tag{3}$$

where z is a standard Gaussian random variable.

It is not very difficult to show that a consequence of the concentration of the overlap is that small collections of spins become approximately independent under the Gibbs measure (see [29, Theorem 2.4.10]). However, they are not identically distributed

unless $h = 0$. One important objective of the theory of spin glasses is to find ways to compute the marginal distributions of the spins. A way to do this is via the Thouless–Anderson–Palmer (TAP) equations, which we study later in this article.

The high temperature phase of the SK model under zero external field was studied rigorously by Aizenman et al. [1]. A more systematic and powerful approach via stochastic calculus was developed by Comets and Neveu [13] and extended by Tindel [32]. The high temperature phase for $h \neq 0$ was rigorously investigated by Fröhlich and Zegarliński [14] and more extensively by Shcherbina [24] and Talagrand [28]. An extremely thorough rigorous treatment of the high temperature phase with many new results appeared in Chapter 2 of Talagrand's book [29]. An important result, shown in [29, Theorem 2.5.1], is that there exists a constant $\beta_0 > 0$ such that (2) holds whenever $\beta \leq \beta_0$. In this manuscript, this is only result we borrow from the existing theory of spin glasses. We did not attempt to prove this via Stein's method.

As of now, even the low temperature phase is somewhat mathematically tractable, following the deep contributions of Guerra [18], Guerra and Toninelli [17], Talagrand [30], and Panchenko [21]. The recent paper of Comets et al. [12] connecting the SK model and its lattice counterpart is also of interest. For a review of the extensive but mostly unrigorous developments in the theoretical physics literature, let us refer to the classic text of Mézard et al. [20].

1.2 The TAP equations

Since the spins can take only two values, the quenched distribution of the spin at site i is completely described by the quantity $\langle \sigma_i \rangle$. One of the main approaches (as outlined in [20]) to understanding the high temperature phase of the SK model is to understand the quantities $\langle \sigma_1 \rangle, \ldots, \langle \sigma_N \rangle$ via the Thouless–Anderson–Palmer system of equations:

$$\langle \sigma_i \rangle \approx \tanh \left(\frac{\beta}{\sqrt{N}} \sum_{j \neq i} g_{ij} \langle \sigma_j \rangle + h - \beta^2 (1 - q) \langle \sigma_i \rangle \right), \quad i = 1, \ldots, N. \quad (4)$$

Here $\approx$ means, vaguely, 'approximately equal with high probability'. Physicists usually write exact equalities in such cases.

This self-consistent system of equations has a unique solution with high probability if β is small. It was physically argued by Thouless et al. [31] that the quantities $\langle \sigma_1 \rangle, \ldots, \langle \sigma_N \rangle$ must satisfy these equations 'in the large N limit' at any temperature and external field. The first rigorous proof of the validity of the TAP equations in the high temperature phase (where (2) holds) appeared twenty-six years after the publication of the physics paper, in Talagrand's book [29, Theorem 2.4.20]. However, Talagrand's theorem in [29] does not show that *all N* equations hold simultaneously with high probability. This has been proved more recently (Talagrand, private communication), and is going to appear in the forthcoming edition of [29].

Talagrand's proof is based on a remarkable rigorous formulation of the cavity method, which involves studying the system after 'removing the last spin'.

This procedure is known as 'creating a cavity'. Now, if one wants to study the Gibbs measure directly, without having to resort to the essentially inductive process of creating a cavity, is there a way to proceed? This is a key focus in this paper. Let us begin by outlining the approach for the case of the TAP equations and understanding how they arise.

1.3 The Onsager correction term

The first step is to observe that the conditional expectation of σ_i given $(\sigma_j)_{j \neq i}$ under the Gibbs measure is simply $\tanh(\beta \ell_i + h)$, where ℓ_i is the *local field* at site i, defined as

$$\ell_i = \ell_i(g, \sigma) := \frac{1}{\sqrt{N}} \sum_{j \neq i} g_{ij} \sigma_j. \tag{5}$$

The proof of this is quite trivial, following directly from the form of the Gibbs measure. It follows that

$$\langle \sigma_i \rangle = \langle \tanh(\beta \ell_i + h) \rangle, \quad i = 1, \ldots, N. \tag{6}$$

Thus, if we could understand the distribution of the local fields under the Gibbs measure, the problem of computing $\langle \sigma_i \rangle$ would be solved. This motivates the study of the limiting behavior of the local fields.

Incidentally, the naïve mean field heuristic would dictate that the 'average can be moved inside the tanh', and

$$\langle \sigma_i \rangle = \langle \tanh(\beta \ell_i + h) \rangle \overset{?}{\approx} \tanh(\beta \langle \ell_i \rangle + h)$$

$$= \tanh\left(\frac{\beta}{\sqrt{N}} \sum_{j \neq i} g_{ij} \langle \sigma_j \rangle + h \right).$$

However, the naïve heuristic does not work for the SK model. This is not surprising since the local fields are unlikely to be concentrated. The famous discovery of Thouless et al. [31] is that the average can still be moved inside the tanh, but only after adding what has come to be known as the *Onsager correction term*, $-\beta^2(1-q)\langle \sigma_i \rangle$. As stated before, this gives the TAP equations

$$\langle \sigma_i \rangle \approx \tanh\left(\frac{\beta}{\sqrt{N}} \sum_{j \neq i} g_{ij} \langle \sigma_j \rangle + h - \beta^2(1-q)\langle \sigma_i \rangle \right), \quad i = 1, \ldots, N.$$

In view of the approximate independence of the spins under the Gibbs measure at high temperature, it seems natural to surmise that the local fields would be approximately Gaussian. Surprisingly, this is not the case. Rather, the explanation for the Onsager correction is hidden in a property of convex combinations of pairs of Gaussian distributions.

1.4 Onsager correction and mixture Gaussians

For any $\mu \in \mathbb{R}$, $\sigma > 0$, let $N(\mu, \sigma^2)$ denote the Gaussian distribution with mean μ and variance σ^2, with density function

$$\phi_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}. \tag{7}$$

The mixture (i.e. convex combination) of two Gaussian densities has a curious connection with the tanh function. Suppose X is a random variable following the mixture density $p\phi_{\mu_1,\sigma^2} + (1-p)\phi_{\mu_2,\sigma^2}$. Suppose $\mu_1 > \mu_2$, and let

$$a = \frac{\mu_1 - \mu_2}{2\sigma^2}, \quad b = \frac{1}{2}\log\frac{p}{1-p} - \frac{\mu_1^2 - \mu_2^2}{4\sigma^2}. \tag{8}$$

Then a simple computation gives

$$\mathbb{E}\tanh(aX + b) = \tanh(a\mathbb{E}(X) + b - (2p-1)a^2\sigma^2). \tag{9}$$

That is, the 'expectation can be moved inside the tanh', after incurring the quadratic 'correction term' $-(2p-1)a^2\sigma^2$. (The proof of this identity is sketched in the Sect. 1.8. With other values of a and b, the expectation can still be moved inside, but the correction term will no longer have a simple form.) The similarity with the Onsager correction is more than superficial: in fact, it turns out that the distribution of the local fields under the Gibbs measure can be approximated by a mixture of two Gaussian densities, and the correction term in the above equation indeed corresponds to the Onsager correction term in the TAP equations.

1.5 Limit law for the local fields

The precise result about the limiting distribution of the local fields can be described in the following way. For each $1 \le i \le N$, let

$$r_i = r_i(g) := \frac{1}{\sqrt{N}}\sum_{j \ne i} g_{ij}\langle\sigma_j\rangle - \beta(1-q)\langle\sigma_i\rangle, \tag{10}$$

$$p_i := \frac{e^{\beta r_i + h}}{e^{\beta r_i + h} + e^{-\beta r_i - h}}, \tag{11}$$

and let ν_i be the random probability measure on $\mathbb{R}$ with the mixture Gaussian density

$$p_i\phi_{r_i+\beta(1-q),1-q} + (1-p_i)\phi_{r_i-\beta(1-q),1-q}.$$

Then the distribution of the local field ℓ_i [defined in (5)] under the Gibbs measure is close to ν_i, in the sense that the difference between the two (random) measures converges in probability to the zero measure. A more quantitative result is as follows.

Theorem 1.1 *Suppose β and h are such that* (2) *is satisfied for some $q < 1$. Let $v_1, \ldots, v_N$ be defined as above. Then for any bounded measurable $u : \mathbb{R} \to \mathbb{R}$ and any $1 \le i \le N$, we have*

$$\mathbb{E}\left(\langle u(\ell_i)\rangle - \int_{\mathbb{R}} u(x)v_i(dx)\right)^2 \le \frac{C(\beta, h)\|u\|_{\infty}^2}{\sqrt{N}},$$

where $C(\beta, h)$ is a constant depending only on β and h.

Taking $u(x) = \tanh(\beta x + h)$ and using the connection (9) between mixture Gaussian distributions and the Onsager correction, we can now readily prove the TAP equations.

Corollary 1.2 *If* (2) *is satisfied, then for each $1 \le i \le N$ we have*

$$\mathbb{E}\left(\langle\sigma_i\rangle - \tanh\left(\frac{\beta}{\sqrt{N}}\sum_{j\ne i}g_{ij}\langle\sigma_j\rangle + h - \beta^2(1-q)\langle\sigma_i\rangle\right)\right)^2 \le \frac{C(\beta, h)}{\sqrt{N}}.$$

Proof By (6), we know that $\langle\sigma_i\rangle = \langle\tanh(\beta\ell_i + h)\rangle$. Now, for the mixture Gaussian density v_i, a simple computation shows that $a = \beta$ and $b = h$, where a and b are defined as in (8). Taking $u(x) = \tanh(\beta x + h)$ in Theorem 1.1 and using the property (9) of mixture Gaussian densities, it is not difficult to verify that we get the stated result.   □

Note that Talagrand's version of Corollary 1.2 [29, Theorem 2.4.20] has an error bound of order $1/N$, and so our result is suboptimal. However, we are not entirely certain whether Theorem 1.1 itself is suboptimal (although it probably is), because improving the $1/\sqrt{N}$ bound in the proof seems to require some kind of smoothness for the function u, which we are not assuming.

1.6 An explanation of the mixture Gaussianity

Since we know the conditional distribution of σ_i given ℓ_i, and we know the marginal laws of ℓ_i and σ_i via Theorem 1.1 and Corollary 1.2, it is possible to compute the conditional law of ℓ_i given σ_i by Bayes' rule. It turns out that given $\sigma_i = 1$, the law of ℓ_i is approximately Gaussian with mean $r_i + \beta(1 - q)$ and variance $1 - q$, and given $\sigma_i = -1$, the law of ℓ_i is approximately Gaussian with mean $r_i - \beta(1 - q)$ and variance $1 - q$. Thus, the marginal distribution of ℓ_i under the Gibbs measure is approximately a convex combination of these two distributions.

1.7 Stein's method

We prove Theorem 1.1 using our version of the classical probabilistic tool developed by Stein [26,27]. Incidentally, it is also possible to prove it using standard techniques

from the cavity method as developed by Talagrand. However, as we shall see below, one advantage of Stein's method, besides the total variation error bounds, is that it allows us to 'discover' the result before proving it. Let us give a brief primer on Stein's method below.

Suppose we want to show that a random variable X has approximately the same distribution as some other random variable Z. The basic idea behind Stein's method of distributional approximation [26,27] is as follows.

1. Identify a "Stein characterizing operator" T for Z, which has the defining property that for any function f belonging to a fixed large class of functions, $\mathbb{E}Tf(Z) = 0$. For instance, if Z is a standard Gaussian random variable, then $Tf(x) := f'(x) - xf(x)$ is a characterizing operator, acting on all locally absolutely continuous f.
2. Take a function u and find f such that $Tf(x) = u(x) - \mathbb{E}u(Z)$. Relate the smoothness properties of f to those of u.
3. By the definition of f it follows that $|\mathbb{E}u(X) - \mathbb{E}u(Z)| = |\mathbb{E}(Tf(X))|$. Compute a bound on $|\mathbb{E}(Tf(X))|$ by whatever means possible.

The procedure for normal approximation can be simply described as follows: if we want to show that a random variable X is approximately standard Gaussian, Stein's method demands that we show $\mathbb{E}(f'(X) - Xf(X)) \approx 0$ for every f belonging to a large class of functions.

Although the raw version of the method as stated above may seem like a trivial reduction, the replacement of $u(x) - \mathbb{E}u(Z)$ by $Tf(x)$ often gives a high degree of maneuverability in practice. While steps 1 and 2 have to be carried out exactly once for every distribution of Z, the execution of step 3 depends heavily on the problem at hand. A number of techniques for carrying out this step are available in the literature, e.g. exchangeable pairs [27], diffusion generators [4], dependency graphs [2,3], size bias couplings [16], zero bias couplings [15], couplings for Poisson approximation [5,11], specialized procedures like [19,22,23], and some recent advances [7–10]. Incidentally, Stein's method was applied to solve a problem in the interface of statistics and spin glasses in [6].

1.8 Stein's method for mixture Gaussians

For any $a, b, \mu \in \mathbb{R}$ and $\sigma^2 > 0$, let $M(a, b, \mu, \sigma^2)$ be the probability distribution on $\mathbb{R}$ with density function

$$\psi_{a,b,\mu,\sigma^2}(x) = Z_{a,b,\mu,\sigma^2}^{-1} \cosh(ax + b)e^{-(x-\mu)^2/2\sigma^2}. \tag{12}$$

where the normalizing constant is given by

$$Z_{a,b,\mu,\sigma^2} = \sqrt{2\pi}\sigma \cosh(a\mu + b)e^{\frac{1}{2}a^2\sigma^2}.$$

A simple verification shows that $M(a, b, \mu, \sigma^2)$ is in fact a mixture of two Gaussian distributions:

$$\psi_{a,b,\mu,\sigma^2}(x) = p\phi_{\mu+a\sigma^2,\sigma^2}(x) + (1-p)\phi_{\mu-a\sigma^2,\sigma^2}(x), \tag{13}$$

where ϕ stands for the Gaussian density function (7) and

$$p = \frac{e^{a\mu+b}}{e^{a\mu+b} + e^{-a\mu-b}}.$$

Thus, the distributions representable as $M(a, b, \mu, \sigma^2)$ are exactly the distributions arising as mixtures of two Gaussian densities. Note that in particular, $M(0, 0, \mu, \sigma^2)$ is just the Gaussian distribution with mean μ and variance σ^2. An interesting fact about this class of distributions, required for the proof of Corollary 1.2, is that

$$\int_{\mathbb{R}} \tanh(ax + b)\psi_{a,b,\mu,\sigma^2}(x)dx = \tanh(a\mu + b). \tag{14}$$

The computation can be easily done using the convenient representation (12). Note that this is exactly the relation (9). Again, using (12) it is not difficult to verify that the operator

$$Tf(x) = f'(x) - \left(\frac{x - \mu}{\sigma^2} - a\tanh(ax + b)\right) f(x)$$

is a Stein characterizing operator for $M(a, b, \mu, \sigma^2)$. Roughly, this means that to show that a random variable W approximately follows the distribution $M(a, b, \mu, \sigma^2)$, we have to show that for all f,

$$\mathbb{E}\left(f'(W) - \left(\frac{W - \mu}{\sigma^2} - a\tanh(aW + b)\right) f(W)\right) \approx 0.$$

To develop Stein's method for this class of distributions, we have to solve

$$Tf(x) = u(x) - \int_{\mathbb{R}} u(t)\psi_{a,b,\mu,\sigma^2}dt \tag{15}$$

for arbitrary $u : \mathbb{R} \to \mathbb{R}$, and relate bounds on f and its derivatives to properties of u.

Now note that by (13), the measure ν_i in Theorem 1.1 can be alternatively written as $M(\beta, h, r_i, 1-q)$. The randomness of r_i adds an extra level of complexity: We also have to analyze the dependence of the function f in (15) on the parameter μ. So we should start with $f(x, \mu)$ rather than $f(x)$. The bounds required for Stein's method are summarized in the following lemma.

Lemma 1.3 *Fix $a, b \in \mathbb{R}$ and $\sigma^2 > 0$, and a bounded measurable function $u : \mathbb{R} \to \mathbb{R}$. Then there exists an absolutely continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ such that for all $x, \mu \in \mathbb{R}$,*

$$\frac{\partial f}{\partial x}(x, \mu) - \left(\frac{x - \mu}{\sigma^2} - a\tanh(ax + b)\right) f(x, \mu) = u(x) - \int_{\mathbb{R}} u(t)\psi_{a,b,\mu,\sigma^2}(t)dt.$$

Moreover, we can find a solution f such that for some constant $C(a, \sigma)$ depending only on a and σ we have that $|f|$, $|\frac{\partial f}{\partial x}|$, and $|\frac{\partial f}{\partial \mu}|$ are all uniformly bounded by $C(a, \sigma)\|u\|_\infty$.

Note that the case $a = b = 0$ covers the case of the pure Gaussian distributions. The proof of this lemma, which is quite elementary but tedious, is relegated to the end of the manuscript.

1.9 How to apply Stein's method

To see how Stein's method can be used in the SK model, let us sketch a very simple example: the unconditional (i.e. average over the disorder) distribution of the local field at site 1 when $\beta < 1$ and $h = 0$. This is a special case of Theorem 1.1. Since $R_{1,2}$ concentrates around zero in this regime (see e.g. [1], or [29, Chapter 2]), we have $q = q(\beta, 0) = 0$. Also, by symmetry, $\langle \sigma_i \rangle \equiv 0$ for each i. Therefore the measure ν_i is actually a nonrandom probability measure, namely, the mixture Gaussian density $\frac{1}{2}\phi_{\beta,1} + \frac{1}{2}\phi_{-\beta,1}$. Clearly, the nonrandomness of the limiting distribution hugely simplifies our goal. Let us now see how we can prove via Stein's method that this is the limiting distribution of the local fields.

Recall that $\ell_1 = \frac{1}{\sqrt{N}} \sum_{j=2}^{N} g_{1j}\sigma_j$. Fix a smooth function f. For each $j = 2, \ldots, N$, let

$$h_j = \frac{1}{\sqrt{N}} \langle \sigma_j f(\ell_1) \rangle.$$

Then, we have

$$\sum_{j=2}^{N} g_{1j} h_j = \langle \ell_1 f(\ell_1) \rangle. \tag{16}$$

On the other hand, an easy computation gives

$$\frac{\partial h_j}{\partial g_{1j}} = \frac{\langle f'(\ell_1) \rangle + \beta \langle \sigma_1 f(\ell_1) \rangle - \beta \langle \sigma_j f(\ell_1) \rangle \langle \sigma_1 \sigma_j \rangle}{N}.$$

Note that ℓ_1 does not depend on σ_1, and the conditional expectation of σ_1 given $\sigma_2, \ldots, \sigma_N$ is $\tanh(\beta \ell_1)$. Thus,

$$\langle \sigma_1 f(\ell_1) \rangle = \langle \tanh(\beta \ell_1) f(\ell_1) \rangle.$$

Again, it follows from the high temperature condition (2) for $\beta < 1$ and $h = 0$ that for $2 \le j \le N$,

$$\langle \sigma_1 \sigma_j \rangle \approx \langle \sigma_1 \rangle \langle \sigma_j \rangle = 0.$$

Combining, we see that

$$\sum_{j=2}^{N} \frac{\partial h_j}{\partial g_{1j}} \approx \langle f'(\ell_1) \rangle + \beta \langle \tanh(\beta \ell_1) f(\ell_1) \rangle. \tag{17}$$

Now, using integration by parts for Gaussian random variables, we get

$$\mathbb{E}\left(\sum_{j=2}^{N} g_{1j} h_j\right) = \mathbb{E}\left(\sum_{j=2}^{N} \frac{\partial h_j}{\partial g_{1j}}\right).$$

In view of (16) and (17), this is equivalent to

$$\mathbb{E}\left\langle \ell_1 f(\ell_1) - f'(\ell_1) - \beta \tanh(\beta \ell_1) f(\ell_1) \right\rangle \approx 0. \tag{18}$$

As noted in Sect. 1.8,

$$Tf(x) = xf(x) - f'(x) - \beta \tanh(\beta x) f(x)$$

is a Stein characterizing operator for the mixture Gaussian density $\frac{1}{2}\phi_{\beta,1} + \frac{1}{2}\phi_{-\beta,1}$. Note that this procedure 'discovers' that the (averaged) limiting distribution of ℓ_1 is the above Gaussian mixture.

1.10 Quenched distributions and the approximation lemma

In the above example, we sketched a derivation of the limiting unconditional (i.e. averaged over disorder) distribution for the local field at site 1, essentially using Gaussian integration by parts. To prove the result for the quenched distribution, it does not suffice to show (18), but rather, we have to show

$$\left\langle \ell_1 f(\ell_1) - f'(\ell_1) - \beta \tanh(\beta \ell_1) f(\ell_1) \right\rangle \approx 0 \quad \text{with high probability.}$$

In other words, we have to show

$$\sum_{j=2}^{N} g_{1j} h_j \approx \sum_{j=2}^{N} \frac{\partial h_j}{\partial g_{1j}} \quad \text{with high probability.}$$

This is a recurring issue whenever we have to prove a quenched CLT. The following result, which we call the 'approximation lemma', becomes our main tool. The proof of the lemma is so short that we present it right away.

Lemma 1.4 *Suppose* $g = (g_1, \ldots, g_n)$ *is a collection of independent standard Gaussian random variables, and* $h_1, \ldots, h_n$ *are absolutely continuous functions of* g.

Assume that h_i are elements of the Sobolev space $H^{1,2}$ with respect to the Gaussian measure on $\mathbb{R}$. Then

$$\mathbb{E}\left(\sum_{j=1}^{n} g_j h_j - \sum_{j=1}^{n} \frac{\partial h_j}{\partial g_j}\right)^2 = \sum_{j=1}^{n} \mathbb{E}h_j^2 + \sum_{j,k=1}^{n} \mathbb{E}\left(\frac{\partial h_j}{\partial g_k} \frac{\partial h_k}{\partial g_j}\right).$$

Proof By taking convolutions with smooth kernels, we can assume that $h_1, \ldots, h_n$ are twice continuously differentiable. Let

$$h = \sum_{j=1}^{n}\left(g_j h_j - \frac{\partial h_j}{\partial g_j}\right).$$

Then

$$\mathbb{E}h^2 = \sum_{j=1}^{n} \mathbb{E}\left(g_j h_j h - \frac{\partial h_j}{\partial g_j} h\right).$$

Integration-by-parts gives

$$\mathbb{E}(g_j h_j h) = \mathbb{E}\left(\frac{\partial h_j}{\partial g_j} h + h_j \frac{\partial h}{\partial g_j}\right).$$

Thus,

$$\mathbb{E}h^2 = \sum_{j=1}^{n} \mathbb{E}\left(h_j \frac{\partial h}{\partial g_j}\right).$$

Now

$$\frac{\partial h}{\partial g_j} = h_j + \sum_{k=1}^{n}\left(g_k \frac{\partial h_k}{\partial g_j} - \frac{\partial^2 h_k}{\partial g_j \partial g_k}\right).$$

Therefore,

$$\mathbb{E}h^2 = \sum_{j=1}^{n} \mathbb{E}h_j^2 + \sum_{j,k=1}^{n} \mathbb{E}\left(g_k h_j \frac{\partial h_k}{\partial g_j} - h_j \frac{\partial^2 h_k}{\partial g_j \partial g_k}\right).$$

Again, using integration-by-parts, we see that

$$\mathbb{E}\left(g_k h_j \frac{\partial h_k}{\partial g_j}\right) = \mathbb{E}\left(\frac{\partial h_j}{\partial g_k} \frac{\partial h_k}{\partial g_j} + h_j \frac{\partial^2 h_j}{\partial g_j \partial g_k}\right).$$

This completes the proof.   □

1.11 Other results

The following theorems are some further examples of CLTs for the SK model that can be proved via Stein's method. In all cases, we obtain total variation error bounds. Although the bounds are probably suboptimal, this is the only method available that can give such bounds.

1.11.1 The cavity field

Suppose $g_1, \ldots, g_N$ are i.i.d. standard Gaussian random variables, independent of the disorder $(g_{ij})_{i<j\leq N}$. The 'cavity field' ℓ is defined as

$$\ell = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} g_i \sigma_i. \tag{19}$$

The name 'cavity field' comes from the role played by ℓ in the cavity method for solving the SK model in the high temperature regime.

Note that the quenched average of ℓ is

$$\langle \ell \rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} g_i \langle \sigma_i \rangle.$$

The following result states that under the Gibbs measure, ℓ is approximately Gaussian with mean $\langle \ell \rangle$ and variance $1 - q$. The original proof of this result, without the error bound, can be found in Talagrand [29, p. 87].

Theorem 1.5 *Suppose the high temperature condition* (2) *is satisfied and* $u : \mathbb{R} \to \mathbb{R}$ *is a bounded measurable function. Then*

$$\mathbb{E}\left(\langle u(\ell) \rangle - \int_{\mathbb{R}} u(t) \phi_{\langle \ell \rangle, 1-q}(t) dt \right)^2 \leq \frac{C(\beta, h) \|u\|_\infty^2}{\sqrt{N}},$$

where ϕ *is the Gaussian density defined in* (7).

1.11.2 The Hamiltonian

Our next limit theorem is about the quantity

$$H := \frac{1}{N} \sum_{i<j\leq N} g_{ij} \sigma_i \sigma_j - \frac{\sqrt{N}\beta}{2}. \tag{20}$$

Note that this is just a linear transformation of the interaction term in the hamiltonian. We show that in the regime $\beta < 1$, $h = 0$, the quenched law of H is asymptotically

Gaussian with mean 0 and variance $1/2$. The case of general β and h, even in the high temperature phase, seems to be much harder, and is currently under investigation.

Theorem 1.6 *Let H be defined as above. Suppose $\beta < 1$, $h = 0$, and $u : \mathbb{R} \to \mathbb{R}$ is a bounded measurable function. Then*

$$\mathbb{E} \left(\langle u(H) \rangle - \int_{\mathbb{R}} u(t)\phi_{0,1/2}(t)dt \right)^2 \leq \frac{C(\beta)\|u\|_\infty^2}{N}.$$

This gives a total variation error bound of order $1/\sqrt{N}$ in the central limit theorem for H.

Again, this was originally proved in Comets and Neveu [13, Proposition 5.2], albeit without an error bound.

1.11.3 Quenched average of the spin at a site

It is natural to ask about the limiting distribution of the random variables $(\langle \sigma_i \rangle)_{1 \leq i \leq N}$. Although the joint distribution has no simple description, Talagrand proved that for any fixed k, the collection $(\langle \sigma_i \rangle)_{1 \leq i \leq k}$ converges in law to $(\tanh(\beta z_i \sqrt{q} + h))_{1 \leq i \leq k}$, where $z_1, \ldots, z_k$ are independent standard Gaussian random variables (Theorem 2.4.12 in [29]).

Note that the term inside tanh in Corollary 1.2 is simply $\beta r_i + h$, with r_i defined in (10). Hence, to compute the asymptotic distribution of $\langle \sigma_i \rangle$, it suffices to find out the limit law of r_i. The following result shows that r_i is asymptotically Gaussian with mean 0 and variance q, and gives a total variation error bound.

Theorem 1.7 *Suppose (2) holds for some $q > 0$, and r_i is defined as in (10). Let z be a standard Gaussian random variable. Then for any bounded measurable $u : \mathbb{R} \to \mathbb{R}$,*

$$\left| \mathbb{E}u(r_i) - \mathbb{E}u(z\sqrt{q}) \right| \leq \frac{C(\beta, h)\|u\|_\infty}{N^{1/4}}.$$

Note that by Corollary 1.2, this shows that $\langle \sigma_i \rangle$ is asymptotically distributed as $\tanh(\beta z\sqrt{q} + h)$.

The rest of the paper is organized as follows. Since the complete proofs involve some heavy computations, we give brief sketches of the proofs in the next section. The details are in Sect. 3. Section 3 also contains a development of Stein's method for mixture Gaussian densities.

2 Proof outlines

In this section, we sketch the proofs of the theorem from Sect. 1 in the order of difficulty.

Sketch of the proof of Theorem 1.5 Recall the definition (19) of the cavity field ℓ and let $r = \langle \ell \rangle$. Take any smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ and for each j, let

$$h_j := \frac{1}{\sqrt{N}} \left\langle (\sigma_j - \langle \sigma_j \rangle) f(\ell, r) \right\rangle.$$

Then

$$\sum_{j=1}^{N} g_j h_j = \langle (\ell - r) f(\ell, r) \rangle.$$

A careful calculation shows that under (2), the approximation lemma can be applied, and

$$\sum_{j=1}^{N} \frac{\partial h_j}{\partial g_j} \approx (1 - q) \left\langle \frac{\partial f}{\partial x} (\ell, r) \right\rangle.$$

Combining, we get that for any smooth $f : \mathbb{R}^2 \to \mathbb{R}$,

$$\left\langle \frac{\ell - r}{1 - q} f(\ell, r) - \frac{\partial f}{\partial x} (\ell, r) \right\rangle \approx 0.$$

This shows that the law of ℓ under the Gibbs measure approximately satisfies the characterizing equation for the Gaussian law with mean r and variance $1 - q$. The proof can now be completed by standard techniques from Stein's method.

Sketch of the proof of Theorem 1.6 Recall the definition (20) of the centered hamiltonian H, and take any smooth function $f : \mathbb{R} \to \mathbb{R}$. For each $i < j \le N$ let

$$h_{ij} = \frac{1}{N} \langle \sigma_i \sigma_j f(H) \rangle.$$

Then

$$\sum_{i<j\le N} g_{ij} h_{ij} = \left\langle \left(H + \frac{\sqrt{N}\beta}{2} \right) f(H) \right\rangle.$$

In the regime $\beta < 1$, $h = 0$, it is known that $R_{12} = O(N^{-1/2})$. Using this fact and some calculations, it follows that the approximation lemma can be applied to the collection $(g_{ij}, h_{ij})_{i<j\le N}$, and also that

$$\sum_{i<j} \frac{\partial h_{ij}}{\partial g_{ij}} \approx \frac{1}{2} \langle f'(H) \rangle + \frac{\sqrt{N}\beta}{2} \langle f(H) \rangle.$$

This shows that for any smooth f,

$$\left\langle H f(H) - \tfrac{1}{2} f'(H) \right\rangle \approx 0,$$

and Stein's method does the rest.

Sketch of the proof of Theorem 1.1 Recall the definitions (5) and (10) of ℓ_i and r_i. It suffices to prove the theorem for $i = 1$. Take any smooth $f : \mathbb{R}^2 \to \mathbb{R}$, and for each $2 \leq j \leq N$, let

$$h_j(g) = \frac{1}{\sqrt{N}} \left\langle (\sigma_j - \langle \sigma_j \rangle) f(\ell_1, r_1) \right\rangle.$$

Then

$$\sum_{j=2}^{N} g_{1j} h_j = \left\langle (\ell_1 - r_1) f(\ell_1, r_1) \right\rangle.$$

Now h_j depends not only on $(g_{1j})_{2 \leq j \leq N}$ but also on $(g_{ij})_{2 \leq i < j \leq N}$. However, we can condition on $(g_{ij})_{2 \leq i < j \leq N}$ and then apply the approximation lemma to show that

$$\sum_{j=2}^{N} g_{1j} h_j \approx \sum_{j=2}^{N} \frac{\partial h_j}{\partial g_{1j}}.$$

In a number of steps, one can show that under (2),

$$\sum_{j=2}^{N} \frac{\partial h_j}{\partial g_{1j}} \approx (1 - q) \left\langle \beta \tanh(\beta \ell_1 + h) f(\ell_1, r_1) + \frac{\partial f}{\partial x}(\ell_1, r_1) \right\rangle.$$

Combining it follows that for any smooth $f : \mathbb{R}^2 \to \mathbb{R}$,

$$\left\langle \left(\frac{\ell_1 - r_1}{1 - q} - \beta \tanh(\beta \ell_1 + h) \right) f(\ell_1, r_1) - \frac{\partial f}{\partial x}(\ell_1, r_1) \right\rangle \approx 0.$$

It turns out that an exact equality in the above equation characterizes the distribution ν_1 from Theorem 1.1. The proof can now be completed by Stein's method.

Sketch of the proof of Theorem 1.7 Again, it suffices to prove the theorem just for r_1. By a series of steps involving integration by parts and applications of the high temperature condition (2), one can show that for any smooth $f : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}(r_1 f(r_1)) \approx q \mathbb{E}\left(f'(r_1) \langle \eta_1 \rangle \right), \tag{21}$$

where

$$\eta_1 = 1 + \frac{\beta \sigma_1}{\sqrt{N}} \sum_{j=2}^{N} g_{1j} \left(\sigma_j - \langle \sigma_j \rangle \right) - \beta^2 (1 - q) \left(1 - \langle \sigma_1 \rangle \sigma_1 \right),$$

Now let

$$h_j = \frac{\beta \langle (\sigma_j - \langle \sigma_j \rangle) \sigma_1 \rangle}{\sqrt{N}}.$$

Clearly,

$$\langle \eta_1 \rangle = 1 + \sum_{j=2}^{N} g_{1j} h_j - \beta^2 (1 - q) \left(1 - \langle \sigma_1 \rangle^2 \right). \tag{22}$$

A series of steps using the high temperature condition (2) give

$$\sum_{j=2}^{N} \frac{\partial h_j}{\partial g_{1j}} \approx \beta^2 (1 - q) \left(1 - \langle \sigma_1 \rangle^2 \right).$$

Applications of (2) also imply that the approximation lemma can be used in this case to deduce that

$$\sum_{j=2}^{N} g_{1j} h_j \approx \sum_{j=2}^{N} \frac{\partial h_j}{\partial g_{1j}},$$

and therefore the last two terms in (22) approximately cancel each other out, leaving us with the conclusion that $\langle \eta_1 \rangle \approx 1$. Combining with (21), we see that for any smooth $f : \mathbb{R} \to \mathbb{R}$, $\mathbb{E}(r_1 f(r_1)) \approx q \mathbb{E}(f'(r_1))$. The proof is now completed by Stein's method.

3 Complete proofs

3.1 Some estimates

Applying Lemma 1.4 in our problems require a substantial amount of computation. The purpose of this subsection is to organize the computations into a friendly and accessible system.

In this subsection and the rest of the manuscript, we switch to the convention that $C(\beta, h)$ denotes any constant that depends only on β and h. In particular, the value of $C(\beta, h)$ may change from line to line.

Let us first recall our conventions. Configurations chosen independently given the disorder are denoted by σ^1, σ^2, etc. The overlap between σ^1 and σ^2 is defined as

$$R_{12} = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^1 \sigma_i^2.$$

Recall that we have a number q, depending on β and h, such that

$$\mathbb{E}\langle (R_{12} - q)^4 \rangle \leq \frac{C(\beta, h)}{N^2}. \tag{23}$$

Let us begin our computations with the following straightforward formula: For any function $v = v(g, \sigma)$ of the disorder g and the configuration σ, and any i, j, we have

$$\frac{\partial \langle v \rangle}{\partial g_{ij}} = \left\langle \frac{\partial v}{\partial g_{ij}} \right\rangle + \frac{\beta}{\sqrt{N}} \langle v \left(\sigma_i \sigma_j - \langle \sigma_i \sigma_j \rangle \right) \rangle$$

$$= \left\langle \frac{\partial v}{\partial g_{ij}} \right\rangle + \frac{\beta}{\sqrt{N}} \langle (v - \langle v \rangle) \sigma_i \sigma_j \rangle. \tag{24}$$

For each j, let

$$\dot{\sigma}_j = \dot{\sigma}_j(g, \sigma) = \sigma_j - \langle \sigma_j \rangle. \tag{25}$$

Then by (24),

$$\frac{\partial \dot{\sigma}_j}{\partial g_{kl}} = -\frac{\partial \langle \sigma_j \rangle}{\partial g_{kl}} = -\frac{\beta}{\sqrt{N}} \langle \dot{\sigma}_j \sigma_k \sigma_l \rangle. \tag{26}$$

Now let $v(g, \sigma)$ be a bounded function of g and σ. Then

$$\frac{1}{N} \sum_{j=1}^{N} \mathbb{E}\langle \dot{\sigma}_j v \rangle^2 = \frac{1}{N} \sum_{j=1}^{N} \mathbb{E} \left\langle \left(\sigma_j^1 - \langle \sigma_j \rangle \right) \left(\sigma_j^2 - \langle \sigma_j \rangle \right) v(g, \sigma^1) v(g, \sigma^2) \right\rangle$$

$$= \mathbb{E} \left\langle (R_{12} - R_{14} - R_{23} + R_{34}) \, v(g, \sigma^1) v(g, \sigma^2) \right\rangle.$$

From this and the inequality (23), we get

$$\frac{1}{N} \sum_{j=1}^{N} \mathbb{E}\langle \dot{\sigma}_j v \rangle^2 \leq \frac{C(\beta, h) \left(\mathbb{E}\langle v^4 \rangle \right)^{1/2}}{\sqrt{N}}. \tag{27}$$

Next, note that

$$\frac{1}{N} \sum_{j=1}^{N} \langle \dot{\sigma}_j \sigma_j v \rangle = \frac{1}{N} \sum_{j=1}^{N} \left(\langle v(g,\sigma) \rangle - \langle \sigma_j^1 \sigma_j^2 v(g,\sigma^1) \rangle \right)$$
$$= \left\langle (1 - R_{12}) v(g,\sigma^1) \right\rangle.$$

Thus, we have

$$\mathbb{E} \left(\frac{1}{N} \sum_{j=1}^{N} \langle \dot{\sigma}_j \sigma_j v \rangle - (1-q) \langle v \rangle \right)^2 \leq \frac{C(\beta, h) \left(\mathbb{E} \langle v^4 \rangle \right)^{1/2}}{N}. \tag{28}$$

If w is another function of g and σ, then

$$\frac{1}{N} \sum_{j=1}^{N} \langle \sigma_j v \rangle \langle \sigma_j w \rangle = \langle R_{12} v(g,\sigma^1) w(g,\sigma^2) \rangle.$$

Thus, we have

$$\mathbb{E} \left(\frac{1}{N} \sum_{j=1}^{N} \langle \sigma_j v \rangle \langle \sigma_j w \rangle - q \langle v \rangle \langle w \rangle \right)^2 \leq \frac{C(\beta, h) \left(\mathbb{E} \langle v^4 \rangle \langle w^4 \rangle \right)^{1/2}}{N}. \tag{29}$$

Next, note that

$$\frac{1}{N} \sum_{j=1}^{N} \langle \dot{\sigma}_j v \rangle \langle \sigma_j w \rangle = \frac{1}{N} \sum_{j=1}^{N} \left(\langle \sigma_j v \rangle \langle \sigma_j w \rangle - \langle \sigma_j \rangle \langle v \rangle \langle \sigma_j w \rangle \right)$$
$$= \left\langle (R_{12} - R_{13}) v(g,\sigma^1) w(g,\sigma^2) \right\rangle.$$

This shows that

$$\frac{1}{N} \sum_{j=1}^{N} \mathbb{E} \left(\langle \dot{\sigma}_j v \rangle \langle \sigma_j w \rangle \right) \leq \frac{C(\beta, h) \left(\mathbb{E} \langle v^2 \rangle \langle w^2 \rangle \right)^{1/2}}{\sqrt{N}}, \tag{30}$$

and moreover

$$\mathbb{E} \left(\frac{1}{N} \sum_{j=1}^{N} \langle \dot{\sigma}_j v \rangle \langle \sigma_j w \rangle \right)^2 \leq \frac{C(\beta, h) \left(\mathbb{E} \langle v^4 \rangle \langle w^4 \rangle \right)^{1/2}}{N}. \tag{31}$$

The inequality (30) readily implies the following important lemma.

Lemma 3.1 *Let $v_1, \ldots, v_N, w_1, \ldots, w_N$ be arbitrary functions of g and σ. Then, we have*

$$\mathbb{E}\left(\frac{1}{N^2} \sum_{j,k=1}^{N} \langle \dot{\sigma}_j v_k \rangle \langle w_k \sigma_j \rangle\right) \leq \frac{C(\beta, h)}{N^{3/2}} \sum_{k=1}^{N} \left(\mathbb{E}\left\langle v_k^2 \right\rangle \left\langle w_k^2 \right\rangle\right)^{1/2}.$$

The above result is generally used as follows. Given functions $f_2, \ldots, f_N$ of the disorder g, we find v_j and w_j such that

$$\frac{\partial f_j}{\partial g_{1k}} = \frac{\langle \dot{\sigma}_j v_k \rangle}{N} = \frac{\langle w_j \sigma_k \rangle}{N},$$

and apply the bound from Lemma 3.1 to extract information from Lemma 1.4.

The next lemma is necessary for bounding the moments of functions of (g, σ) that arise when we try to apply the inequalities derived above.

Lemma 3.2 *Let $b_1(\sigma), \ldots, b_m(\sigma)$ be arbitrary functions of σ, taking values in the interval $[-1, 1]$. Then for any $k \geq 1$ and any distinct collection of indices $2 \leq j_1, \ldots, j_k \leq N$, we have*

$$\mathbb{E}\left(g_{1j_1} g_{1j_2} \cdots g_{1j_k} \langle b_1 \rangle \langle b_2 \rangle \cdots \langle b_m \rangle\right) \leq \frac{C(m, k)\beta^k}{N^{k/2}},$$

where $C(m, k)$ is a constant depending only on m and k.

Proof Let us use induction on k. For $k = 1$, observe that by integration-by-parts and (24), we have

$$\mathbb{E}\left(g_{1j_1} \langle b_1 \rangle \langle b_2 \rangle \cdots \langle b_m \rangle\right) = \sum_{l=1}^{m} \mathbb{E}\left(\frac{\partial \langle b_l \rangle}{\partial g_{1j_1}} \prod_{l' \neq l} \langle b_{l'} \rangle\right)$$

$$= \frac{\beta}{\sqrt{N}} \sum_{l=1}^{m} \mathbb{E}\left(\langle (b_l - \langle b_l \rangle) \sigma_1 \sigma_{j_1} \rangle \prod_{l' \neq l} \langle b_{l'} \rangle\right) \leq \frac{C(m)\beta}{\sqrt{N}}.$$

Now assume that the result is true up to $k - 1$ (and any m). Again, using integration-by-parts and (24), we have

$$\mathbb{E}\left(g_{1j_1} g_{1j_2} \cdots g_{1j_k} \langle b_1 \rangle \langle b_2 \rangle \cdots \langle b_m \rangle\right) \leq \frac{\beta}{\sqrt{N}} \sum_{l=1}^{m} \mathbb{E}\left(g_{1j_2} \cdots g_{1j_k} \frac{\partial \langle b_l \rangle}{\partial g_{1j_1}} \prod_{l' \neq l} \langle b_{l'} \rangle\right)$$

$$= \frac{\beta}{\sqrt{N}} \sum_{l=1}^{m} \mathbb{E}\left(g_{1j_2} \cdots g_{1j_k} \langle (b_l - \langle b_l \rangle) \sigma_1 \sigma_{j_1} \rangle \prod_{l' \neq l} \langle b_{l'} \rangle\right).$$

A straightforward application of the induction hypothesis for $k - 1$ completes the proof. □

The following function will appear several times in the sequel.

$$\dot{\ell}_1 = \dot{\ell}_1(g, \sigma) = \frac{1}{\sqrt{N}} \sum_{j=2}^{N} g_{1j} \dot{\sigma}_j. \tag{32}$$

Take any $k \geq 1$. A simple application of Lemma 3.2 to each term in the expansion of $\dot{\ell}_1^k$ shows that

$$\mathbb{E}(\dot{\ell}_1^k) \leq C(\beta, k). \tag{33}$$

The important thing is that the bound does not depend on N.

3.2 Proof of Theorem 1.5

We will continue using the notation introduced in the previous subsections. Let us briefly recall the setting. Suppose $g_1, \ldots, g_N$ are i.i.d. standard Gaussian random variables, independent of $(g_{ij})_{i<j\leq N}$. The cavity field is defined as

$$\ell := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} g_j \sigma_j.$$

Our objective is to show that under the Gibbs measure, ℓ is approximately distributed as a Gaussian random variable with mean

$$r := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} g_j \langle \sigma_j \rangle$$

and variance $1 - q$.

Take any bounded measurable function $u : \mathbb{R} \to [-1, 1]$ and suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a solution to

$$\frac{\partial f}{\partial x}(x, y) - \frac{x - y}{1 - q} f(x, y) = u(x) - \int_{\mathbb{R}} u(t) \phi_{y, 1-q}(t) dt.$$

For simplicity, we let f_1 and f_2 denote $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. From Lemma 1.3 it follows that $|f|, |f_1|$ and $|f_2|$ are uniformly bounded by $C(\beta, h)$. For each j, let

$$h_j := \frac{1}{\sqrt{N}} \langle \dot{\sigma}_j f(\ell, r) \rangle.$$

Then

$$\sum_{j=1}^{N} g_j h_j = \langle (\ell - r) f(\ell, r) \rangle . \tag{34}$$

In the rest of the proof, we will simply write f, f_1 and f_2 instead of $f(\ell, r)$, etc. Note that for any j, k,

$$\frac{\partial h_j}{\partial g_k} = \frac{1}{N} \langle \dot{\sigma}_j \sigma_k f_1 \rangle + \frac{1}{N} \langle \dot{\sigma}_j f_2 \rangle \langle \sigma_k \rangle .$$

Thus, putting

$$v_k = \sigma_k f_1 + f_2 \langle \sigma_k \rangle \quad \text{and} \quad w_j = \dot{\sigma}_j f_1 + \langle \dot{\sigma}_j f_2 \rangle ,$$

we see that

$$\frac{\partial h_j}{\partial g_k} = \frac{\langle \dot{\sigma}_j v_k \rangle}{N} = \frac{\langle w_j \sigma_k \rangle}{N} .$$

Hence by Lemma 3.1, we have

$$\sum_{j,k=1}^{N} \mathbb{E} \left(\frac{\partial h_j}{\partial g_k} \frac{\partial h_k}{\partial g_j} \right) \le \frac{C(\beta, h)}{\sqrt{N}} .$$

Again, from (27) we have

$$\sum_{j=1}^{N} \mathbb{E}(h_j^2) \le \frac{C(\beta, h)}{\sqrt{N}} .$$

Combining and applying Lemma 1.4, we get

$$\mathbb{E} \left(\sum_{j=1}^{N} g_j h_j - \sum_{j=1}^{N} \frac{\partial h_j}{\partial g_j} \right)^2 \le \frac{C(\beta, h)}{\sqrt{N}} . \tag{35}$$

Again, note that

$$\sum_{j=1}^{N} \frac{\partial h_j}{\partial g_j} = \frac{1}{N} \sum_{j=1}^{N} \left(\langle \dot{\sigma}_j \sigma_j f_1 \rangle + \langle \dot{\sigma}_j f_2 \rangle \langle \sigma_j \rangle \right) .$$

By (28) and (31), this gives

$$\mathbb{E} \left(\sum_{j=1}^{N} \frac{\partial h_j}{\partial g_j} - (1 - q) \langle f_1 \rangle \right)^2 \le \frac{C(\beta, h)}{N} . \tag{36}$$

Combining (34)–(36), we finally get

$$\mathbb{E}\left(\langle u(\ell)\rangle - \int_{\mathbb{R}} u(t)\phi_{r,1-q}(t)dt\right)^2 = \frac{1}{(1-q)^2}\mathbb{E}\,\langle(1-q)f_1 - (\ell - r)f\rangle^2$$

$$\leq \frac{C(\beta, h)}{\sqrt{N}}.$$

This completes the proof of Theorem 1.5.

3.3 Proof of Theorem 1.6

Recall that the centered hamiltonian H was defined as

$$H := \frac{1}{N}\sum_{i<j\leq N} g_{ij}\sigma_j\sigma_j - \frac{\sqrt{N}\beta}{2}.$$

Take any $u : \mathbb{R} \to [-1, 1]$, and let f be a solution to

$$f'(x) - 2xf(x) = u(x) - \int_{\mathbb{R}} u(t)\phi_{0,1/2}(t)dt.$$

Again by Lemma 1.3, $|f|$ and $|f'|$ are uniformly bounded by $C(\beta, h)$. For each i, j, let

$$h_{ij} = \frac{1}{N}\langle\sigma_i\sigma_j f(H)\rangle.$$

In the following, we will write f and f' for $f(H(\sigma))$ and $f'(H(\sigma))$ for notational convenience. When we have expressions involving multiple replicas, f will stand for $f(H(\sigma^1))$. We have

$$\frac{\partial h_{ij}}{\partial g_{kl}} = \frac{1}{N^2}\langle\sigma_i\sigma_j\sigma_k\sigma_l f'\rangle + \frac{\beta}{N^{3/2}}\left(\langle\sigma_i\sigma_j\sigma_k\sigma_l f\rangle - \langle\sigma_i\sigma_j f\rangle\langle\sigma_k\sigma_l\rangle\right). \qquad (37)$$

Using identities like

$$\frac{1}{N^4}\sum_{i,j,k,l}\langle\sigma_i\sigma_j\sigma_k\sigma_l f\rangle^2 = \left\langle R_{12}^4 f\right\rangle,$$

$$\frac{1}{N^4}\sum_{i,j,k,l}\langle\sigma_i\sigma_j\sigma_k\sigma_l f\rangle\langle\sigma_i\sigma_j f\rangle\langle\sigma_k\sigma_l\rangle = \left\langle R_{12}^2 R_{13}^2 f\right\rangle, \quad \text{etc.,}$$

we get

$$\sum_{i<j,\ k<l} \mathbb{E}\left(\frac{\partial h_{ij}}{\partial g_{kl}} \frac{\partial h_{kl}}{\partial g_{ij}}\right) \le C(\beta) N \mathbb{E}\left\langle R_{12}^4 \right\rangle \le \frac{C(\beta)}{N}.$$

Similarly,

$$\sum_{i<j} \mathbb{E}(h_{ij}^2) \le C(\beta) \mathbb{E}\left\langle R_{12}^2 \right\rangle \le \frac{C(\beta)}{N}.$$

Another similar verification starting from the formula (37) shows that

$$\sum_{i<j} \frac{\partial h_{ij}}{\partial g_{ij}} = \frac{1}{2}\langle f'\rangle + \frac{\sqrt{N}\beta}{2}\langle f\rangle + \mathcal{R},$$

where $\mathcal{R}$ is a remainder term satisfying

$$\mathbb{E}(\mathcal{R}^2) \le \frac{C(\beta)}{N}.$$

The proof is now completed by applying Lemma 1.4.

3.4 Proof of Theorem 1.1

It suffices to prove the result for $i = 1$. Note that ν_1 is simply the probability distribution $M(\beta, h, r_1, 1 - q)$. Without loss of generality, we can assume that $\|u\|_\infty \le 1$. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a solution of the differential equation

$$\frac{\partial f}{\partial x}(x, y) - \left(\frac{x - y}{1 - q} - \beta \tanh(\beta x + h)\right) f(x, y) = u(x) - \int_{\mathbb{R}} u(t)\psi_{\beta, h, y, 1-q}(t)dt. \quad (38)$$

By Lemma 1.3, such an f exists and moreover, we can guarantee that $|f|$, $|\frac{\partial f}{\partial x}|$, and $|\frac{\partial f}{\partial y}|$ are all bounded by $C(\beta, h)$. As before, to lighten notation, we let f_1 and f_2 denote the two partial derivatives of f. We have to prove that for any i,

$$\mathbb{E}\left(\langle u(\ell_i)\rangle - \int_{\mathbb{R}} u(x)\psi_{\beta, h, r_i, 1-q}dx\right)^2 \le \frac{C(\beta, h)\|u\|_\infty^2}{\sqrt{N}},$$

where

$$r_i = r_i(g) := \frac{1}{\sqrt{N}} \sum_{j\neq i} g_{ij}\langle\sigma_j\rangle - \beta(1 - q)\langle\sigma_i\rangle.$$

By the definition and properties of f, this is clearly equivalent to proving that

$$\mathbb{E}\left\langle f_1(\ell_i, r_i) - \left(\frac{\ell_i - r_i}{1-q} - \beta \tanh(\beta \ell_i + h)\right) f(\ell_i, r_i)\right\rangle^2 \leq \frac{C(\beta, h)}{\sqrt{N}},$$

and this is what we aim to prove in the next few pages. Note that it suffices to fix $i = 1$. Recall that we defined

$$\dot{\sigma}_j := \sigma_j - \langle \sigma_j \rangle.$$

For each $j \geq 2$, let

$$h_j(g) = \frac{1}{\sqrt{N}} \langle \dot{\sigma}_j f(\ell_1, r_1) \rangle,$$

where recall that

$$\ell_1 = \ell_1(g, \sigma) = \frac{1}{\sqrt{N}} \sum_{j=2}^{N} g_{1j} \sigma_j$$

and

$$r_1 = r_1(g) = \frac{1}{\sqrt{N}} \sum_{j=2}^{N} g_{1j} \langle \sigma_j \rangle - \beta(1-q) \langle \sigma_1 \rangle.$$

In what follows, the random variable $f(\ell_1, r_1)$ is simply denoted by f to lighten notation. The distinction between the random variable f and the function f should be clear from the context. Similar remarks apply to f_1 and f_2 also.

Now for any $j, k \geq 2$, simple applications of (24) and (26) gives

$$\begin{aligned}
\frac{\partial h_j}{\partial g_{1k}} &= \frac{1}{\sqrt{N}} \left\langle \frac{\partial \dot{\sigma}_j}{\partial g_{1k}} f \right\rangle + \frac{1}{\sqrt{N}} \left\langle \dot{\sigma}_j f_1 \frac{\partial \ell_1}{\partial g_{1k}} \right\rangle + \frac{1}{\sqrt{N}} \langle \dot{\sigma}_j f_2 \rangle \frac{\partial r_1}{\partial g_{1k}} \\
&\quad + \frac{\beta}{N} \langle (\dot{\sigma}_j f)(\sigma_1 \sigma_k - \langle \sigma_1 \sigma_k \rangle) \rangle \\
&= -\frac{\beta}{N} \langle \dot{\sigma}_j \sigma_1 \sigma_k \rangle \langle f \rangle + \frac{1}{N} \langle \dot{\sigma}_j f_1 \sigma_k \rangle + \frac{1}{\sqrt{N}} \langle \dot{\sigma}_j f_2 \rangle \frac{\partial r_1}{\partial g_{1k}} \\
&\quad + \frac{\beta}{N} \langle (\dot{\sigma}_j f)(\sigma_1 \sigma_k - \langle \sigma_1 \sigma_k \rangle) \rangle.
\end{aligned} \tag{39}$$

A further use of (24) and (26) gives

$$\frac{\partial r_1}{\partial g_{1k}} = \frac{\langle \sigma_k \rangle}{\sqrt{N}} + \frac{\beta}{N} \sum_{l \geq 2} g_{1l} \langle \dot{\sigma}_l \sigma_1 \sigma_k \rangle - \frac{\beta^2(1-q)}{\sqrt{N}} \langle \dot{\sigma}_1 \sigma_1 \sigma_k \rangle.$$

Recalling the definition (32) of $\dot{\ell}_1$ and putting

$$\eta_1 := 1 + \beta \dot{\ell}_1 \sigma_1 - \beta^2 (1 - q) \dot{\sigma}_1 \sigma_1, \tag{40}$$

we see that

$$\frac{\partial r_1}{\partial g_{1k}} = \frac{\langle \eta_1 \sigma_k \rangle}{\sqrt{N}}. \tag{41}$$

Thus, putting

$$v_k = -\beta \sigma_1 \sigma_k \langle f \rangle + f_1 \sigma_k + f_2 \langle \eta_1 \sigma_k \rangle + \beta f \sigma_1 \sigma_k - \beta f \langle \sigma_1 \sigma_k \rangle, \quad \text{and}$$
$$w_j = -\beta \dot{\sigma}_j \sigma_1 \langle f \rangle + \dot{\sigma}_j f_1 + \langle \dot{\sigma}_j f_2 \rangle \eta_1 + \beta \dot{\sigma}_j f \sigma_1 - \beta \langle \dot{\sigma}_j f \rangle \sigma_1,$$

and organizing the terms in (39), we get

$$\frac{\partial h_j}{\partial g_{1k}} = \frac{\langle \dot{\sigma}_j v_k \rangle}{N} = \frac{\langle w_j \sigma_k \rangle}{N}.$$

Since f, f_1, and f_2 are uniformly bounded by $C(\beta, h)$ and $\mathbb{E} \langle \dot{\ell}_1^4 \rangle \leq C(\beta, h)$ by (33), an application of Lemma 3.1 gives

$$\mathbb{E} \left(\sum_{j,k=2}^{N} \frac{\partial h_j}{\partial g_{1k}} \frac{\partial h_k}{\partial g_{1j}} \right) = \mathbb{E} \left(\frac{1}{N^2} \sum_{j,k=2}^{N} \langle \dot{\sigma}_j v_k \rangle \langle w_k \sigma_j \rangle \right) \leq \frac{C(\beta, h)}{\sqrt{N}}. \tag{42}$$

Again, since f is bounded by $C(\beta, h)$, we can use (27) to get

$$\sum_{j=2}^{N} \mathbb{E}(h_j^2) \leq \frac{C(\beta, h)}{\sqrt{N}}. \tag{43}$$

Applying Lemma 1.4, using the bounds (43) and (42) obtained above, we finally get

$$\mathbb{E} \left(\sum_{j=2}^{N} g_{1j} h_j - \sum_{j=2}^{N} \frac{\partial h_j}{\partial g_{1j}} \right)^2 = \sum_{j=2}^{N} \mathbb{E}(h_j^2) + \sum_{j,k=2}^{N} \mathbb{E} \left(\frac{\partial h_j}{\partial g_{1k}} \frac{\partial h_k}{\partial g_{1j}} \right) \leq \frac{C(\beta, h)}{\sqrt{N}}. \tag{44}$$

Note that although h_j is a function of the whole collection $(g_{jk})_{1 \leq j < k \leq N}$, we can first condition on $(g_{jk})_{2 \leq j < k \leq N}$ and apply Lemma 1.4, and then take the unconditional expectation to get the first line in (44).

Now let us define

$$\gamma_1 := -\beta \sigma_1 \langle f \rangle + f_1 + \beta \sigma_1 f.$$

Then from the expressions (39) and (41) we see that

$$\sum_{j=2}^{N} \frac{\partial h_j}{\partial g_{1j}} = \frac{1}{N} \sum_{j=2}^{N} \left(\langle \dot{\sigma}_j \sigma_j \gamma_1 \rangle + \langle \dot{\sigma}_j f_2 \rangle \langle \eta_1 \sigma_j \rangle - \beta \langle \dot{\sigma}_j f \rangle \langle \sigma_1 \sigma_j \rangle \right)$$

Applying (28) for the first term and (31) for the second and third terms, we have

$$\mathbb{E} \left(\sum_{j=2}^{N} \frac{\partial h_j}{\partial g_{1j}} - (1-q) \langle \gamma_1 \rangle \right)^2 \leq \frac{C(\beta, h)}{N}. \tag{45}$$

Note the most crucial point in the derivation of (44) and (45) is that by Lemma 1.3, the bounds on f and its derivatives depend only on (β, h). The parameter μ in the mixture Gaussian distribution, which equals r_1 in this proof, does not actually behave as a fixed parameter because we have defined f as a function of two variables, one of which is μ. This is why we need to have f defined on $\mathbb{R}^2$ instead of $\mathbb{R}^1$.

Now $f = f(\ell_1, r_1)$ does not depend on σ_1. Also recall that under the Gibbs measure, the conditional expectation of σ_1 given $\sigma_2, \ldots, \sigma_N$ is simply $\tanh(\beta \ell_1 + h)$. Thus,

$$\langle \sigma_1 f \rangle = \langle \tanh(\beta \ell_1 + h) f \rangle.$$

Using the above identity to compute $\langle \gamma_1 \rangle$, we see that

$$\sum_{j=2}^{N} g_{1j} h_j - (1-q) \langle \gamma_1 \rangle = \langle (\ell_1 - r_1 - \beta(1-q) \tanh(\beta \ell_1 + h)) f \rangle - (1-q) \langle f_1 \rangle.$$

Combining (44) and (45), and dividing by $1 - q$ throughout, we get

$$\mathbb{E} \left\langle f_1 - \left(\frac{\ell_1 - r_1}{1-q} - \beta \tanh(\beta \ell_1 + h) \right) f \right\rangle^2 \leq \frac{C(\beta, h)}{\sqrt{N}}.$$

But by the definition (38) of f,

$$f_1(\ell_1, r_1) - \left(\frac{\ell_1 - r_1}{1-q} - \beta \tanh(\beta \ell_1 + h) \right) f(\ell_1, r_1)$$

$$= u(\ell_1) - \int_{\mathbb{R}} u(x) \psi_{\beta, h, r_1, 1-q}(x) dx.$$

This completes the proof of Theorem 1.1.

3.5 Proof of Theorem 1.7

Recall the definitions

$$r_1 := \frac{1}{\sqrt{N}} \sum_{j=2}^{N} g_{1j} \langle \sigma_j \rangle - \beta(1-q) \langle \sigma_1 \rangle$$

and

$$\eta_1 := 1 + \beta \dot{\ell}_1 \sigma_1 - \beta^2 (1-q) \dot{\sigma}_1 \sigma_1.$$

Let $u : \mathbb{R} \to [-1, 1]$ be a measurable map. By Lemma 1.3, there exists an absolutely continuous function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) - \frac{x}{q} f(x) = u(x) - \mathbb{E}u(z\sqrt{q}),$$

where z is a standard Gaussian random variable, and moreover $|f|$ and $|f'|$ can be uniformly bounded by $C(\beta, h)$. From the definition (10) of r_1, we see that

$$r_1 f(r_1) = \frac{1}{\sqrt{N}} \sum_{j=2}^{N} g_{1j} \langle \sigma_j \rangle f(r_1) - \beta(1-q) \langle \sigma_1 \rangle f(r_1). \tag{46}$$

Now, by integration-by-parts and the identities (26) and (41), we have

$$\mathbb{E}\left(g_{1j} \langle \sigma_j \rangle f(r_1)\right) = \mathbb{E}\left(\frac{\partial \langle \sigma_j \rangle}{\partial g_{1j}} f(r_1) + \langle \sigma_j \rangle f'(r_1) \frac{\partial r_1}{\partial g_{1j}}\right)$$

$$= \frac{\beta}{\sqrt{N}} \mathbb{E}\left(\langle \dot{\sigma}_j \sigma_1 \sigma_j \rangle f(r_1)\right) + \frac{1}{\sqrt{N}} \mathbb{E}\left(\langle \sigma_j \rangle \langle \eta_1 \sigma_j \rangle f'(r_1)\right).$$

Thus, we have

$$\mathbb{E}\left(\frac{1}{\sqrt{N}} \sum_{j=2}^{N} g_{1j} \langle \sigma_j \rangle f(r_1)\right) = \mathbb{E}\left(\beta f(r_1) \frac{\sum_{j=2}^{N} \langle \dot{\sigma}_j \sigma_1 \sigma_j \rangle}{N}\right)$$

$$+ \mathbb{E}\left(f'(r_1) \frac{\sum_{j=2}^{N} \langle \sigma_j \rangle \langle \eta_1 \sigma_j \rangle}{N}\right). \tag{47}$$

By (28) and the bound on $|f|$, we have

$$\left|\mathbb{E}\left(\beta f(r_1) \frac{\sum_{j=2}^{N} \langle \dot{\sigma}_j \sigma_1 \sigma_j \rangle}{N}\right) - \beta(1-q)\mathbb{E}\left(f(r_1) \langle \sigma_1 \rangle\right)\right| \leq \frac{C(\beta, h)}{\sqrt{N}}. \tag{48}$$

Similarly, by (29) and the bound on $|f'|$,

$$\left| \mathbb{E} \left(f'(r_1) \frac{\sum_{j=2}^{N} \langle \sigma_j \rangle \langle \eta_1 \sigma_j \rangle}{N} \right) - q \mathbb{E} \left(f'(r_1) \langle \eta_1 \rangle \right) \right| \le \frac{C(\beta, h)}{\sqrt{N}}. \tag{49}$$

Combining (46)–(49), we get

$$\left| \mathbb{E}(r_1 f(r_1)) - q \mathbb{E} \left(f'(r_1) \langle \eta_1 \rangle \right) \right| \le \frac{C(\beta, h)}{\sqrt{N}}.$$

Since f' is bounded and

$$\mathbb{E}u(r_1) - \mathbb{E}u(z\sqrt{q}) = q^{-1} \mathbb{E}(q f'(r_1) - r_1 f(r_1)),$$

the proof will be complete if we can show that

$$\mathbb{E} \left(\langle \eta_1 \rangle - 1 \right)^2 \le \frac{C(\beta, h)}{\sqrt{N}}. \tag{50}$$

The rest of the proof is devoted to proving (50). From the definition (40) of η_1, we get

$$\langle \eta_1 \rangle = 1 + \frac{\beta}{\sqrt{N}} \sum_{j=2}^{N} g_{1j} \langle \dot{\sigma}_j \sigma_1 \rangle - \beta^2 (1-q) \langle \dot{\sigma}_1 \sigma_1 \rangle.$$

Now let

$$h_j = \frac{\beta \langle \dot{\sigma}_j \sigma_1 \rangle}{\sqrt{N}},$$

so that

$$\langle \eta_1 \rangle = 1 + \sum_{j=2}^{N} g_{1j} h_j - \beta^2 (1-q) \left(1 - \langle \sigma_1 \rangle^2 \right).$$

Our intention is to apply Lemma 1.4 to show that the second and the third terms approximately cancel each other out. First, note that by Eqs. (24) and (26),

$$\frac{\partial h_j}{\partial g_{1k}} = \frac{\beta^2}{N} \left(- \langle \dot{\sigma}_j \sigma_1 \sigma_k \rangle \langle \sigma_1 \rangle + \langle \dot{\sigma}_j \sigma_k \rangle - \langle \dot{\sigma}_j \sigma_1 \rangle \langle \sigma_1 \sigma_k \rangle \right). \tag{51}$$

Thus, if we put

$$v_k = -\sigma_1 \sigma_k \langle \sigma_1 \rangle + \sigma_k - \sigma_1 \langle \sigma_1 \sigma_k \rangle, \quad \text{and}$$
$$w_j = -\dot{\sigma}_j \sigma_1 \langle \sigma_1 \rangle + \dot{\sigma}_j - \langle \dot{\sigma}_j \sigma_1 \rangle \sigma_1,$$

then

$$\frac{\partial h_j}{\partial g_{1k}} = \frac{\beta^2 \langle \dot{\sigma}_j v_k \rangle}{N} = \frac{\beta^2 \langle w_j \sigma_k \rangle}{N}.$$

From Lemma 3.1 it follows that

$$\sum_{j,k=2}^{N} \mathbb{E}\left(\frac{\partial h_j}{\partial g_{1k}} \frac{\partial h_k}{\partial g_{1j}}\right) \le \frac{C(\beta, h)}{\sqrt{N}}.$$

Again, by (27) we get

$$\sum_{j=2}^{N} \mathbb{E}h_j^2 = \frac{\beta^2}{N} \sum_{j=2}^{N} \mathbb{E}\langle \dot{\sigma}_j \sigma_1 \rangle^2 \le \frac{C(\beta, h)}{\sqrt{N}}.$$

Using the last two bounds in Lemma 1.4, we have

$$\mathbb{E}\left(\sum_{j=2}^{N} g_{1j} h_j - \sum_{j=2}^{N} \frac{\partial h_j}{\partial g_{1j}}\right)^2 \le \frac{C(\beta, h)}{\sqrt{N}}. \tag{52}$$

Now from (51) we see that

$$\sum_{j=2}^{N} \frac{\partial h_j}{\partial g_{1j}} = -\frac{\beta^2 \langle \sigma_1 \rangle}{N} \sum_{j=2}^{N} \langle \dot{\sigma}_j \sigma_1 \sigma_j \rangle + \frac{\beta^2}{N} \sum_{j=2}^{N} \langle \dot{\sigma}_j \sigma_j \rangle - \frac{\beta^2}{N} \sum_{j=2}^{N} \langle \dot{\sigma}_j \sigma_1 \rangle \langle \sigma_1 \sigma_j \rangle.$$

Applying (28) to the first two terms and (30) to the third term, we get

$$\mathbb{E}\left(\sum_{j=2}^{N} \frac{\partial h_j}{\partial g_{1j}} + \beta^2 (1-q) \langle \sigma_1 \rangle^2 - \beta^2 (1-q)\right)^2 \le \frac{C(\beta, h)}{N}. \tag{53}$$

Combining (52) and (53), we have

$$\mathbb{E}\left(\langle \eta_1 \rangle - 1\right)^2 = \mathbb{E}\left(\sum_{j=2}^{N} g_{1j} h_j - \beta^2 (1-q)\left(1 - \langle \sigma_1 \rangle^2\right)\right)^2 \le \frac{C(\beta, h)}{\sqrt{N}}.$$

This proves (50) and hence completes the proof of Theorem 1.7.

3.6 Proof of Lemma 1.3

For each $x, \mu \in \mathbb{R}$, let

$$\rho(x, \mu) = \cosh(ax + b)e^{-(x-\mu)^2/2\sigma^2}$$

and

$$r(\mu) = \int_{-\infty}^{\infty} u(x)\psi_{a,b,\mu,\sigma^2}(x)dx = \frac{\int_{-\infty}^{\infty} u(x)\rho(x, \mu)dx}{\int_{-\infty}^{\infty} \rho(x, \mu)dx}.$$

Let

$$f(x, \mu) = \frac{1}{\rho(x, \mu)} \int_{-\infty}^{x} \rho(t, \mu)(u(t) - r(\mu))dt$$

$$= -\frac{1}{\rho(x, \mu)} \int_{x}^{\infty} \rho(t, \mu)(u(t) - r(\mu))dt.$$

For ease of notation, let us write $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial \mu}$ as f_1 and f_2. Multiplying by $\rho(x, \mu)$ on both sides and differentiating w.r.t. x, and finally dividing everything by $\rho(x, \mu)$, we get

$$f_1(x, \mu) - \left(\frac{x - \mu}{\sigma^2} - a\tanh(ax + b)\right)f(x, \mu) = u(x) - r(\mu). \tag{54}$$

This proves the first assertion of the lemma. Now observe that for any $x \geq \mu$,

$$\int_{x}^{\infty} \exp\left(at + b - \frac{(t - \mu)^2}{2\sigma^2}\right)dt = e^{b+a\mu+\frac{1}{2}a^2\sigma^2} \int_{x}^{\infty} \exp\left(-\frac{(t - \mu - a\sigma^2)^2}{2\sigma^2}\right)dt$$

$$= e^{b+a\mu+\frac{1}{2}a^2\sigma^2} \int_{\frac{x-\mu-a\sigma^2}{\sigma}}^{\infty} e^{-\frac{1}{2}y^2}\sigma dy. \tag{55}$$

Let

$$C_1(a, \sigma) = \sigma \sup_{z \geq -a\sigma} e^{\frac{1}{2}z^2} \int_{z}^{\infty} e^{-\frac{1}{2}y^2}dy.$$

It is easy to verify by elementary calculus that $C_1(a, \sigma)$ is finite. Since $x \geq \mu$, we now get from (55) that

$$\int_x^\infty \exp\left(at + b - \frac{(t-\mu)^2}{2\sigma^2}\right) dt \leq C_1(a, \sigma) e^{b+a\mu+\frac{1}{2}a^2\sigma^2} \exp\left(-\frac{(x-\mu-a\sigma^2)^2}{2\sigma^2}\right)$$

$$= C_1(a, \sigma) \exp\left(ax + b - \frac{(x-\mu)^2}{2\sigma^2}\right).$$

Similarly, we have

$$\int_x^\infty \exp\left(-at - b - \frac{(t-\mu)^2}{2\sigma^2}\right) dt \leq C_1(-a, \sigma) \exp\left(-ax - b - \frac{(x-\mu)^2}{2\sigma^2}\right).$$

Combining, and putting $C_2(a, \sigma) = \max\{C_1(a, \sigma), C_1(-a, \sigma)\}$, we get that for $x \geq \mu$,

$$\int_x^\infty \rho(t, \mu) dt \leq C_2(a, \sigma) \rho(x, \mu). \tag{56}$$

Similarly, if $x < \mu$, we have

$$\int_{-\infty}^x \rho(t, \mu) dt \leq C_3(a, \sigma) \rho(x, \mu)$$

for some other constant $C_3(a, \sigma)$. From the definition of f, we can now deduce that

$$\|f\|_\infty \leq C_4(a, \sigma) \|u\|_\infty,$$

where $C_4 = 2 \max\{C_2, C_3\}$. Next, note that for $x \geq \mu$,

$$\int_x^\infty |t - \mu| \exp\left(at + b - \frac{(t-\mu)^2}{2\sigma^2}\right) dt = e^{b+a\mu+\frac{1}{2}a^2\sigma^2} \int_x^\infty (t-\mu) \exp\left(-\frac{(t-\mu-a\sigma^2)^2}{2\sigma^2}\right) dt$$

$$= e^{b+a\mu+\frac{1}{2}a^2\sigma^2} \int_{\frac{x-\mu-a\sigma^2}{\sigma}}^\infty (\sigma y + a\sigma^2) e^{-\frac{1}{2}y^2} \sigma \, dy.$$

Putting

$$C_5(a, \sigma) = \sigma \sup_{z \geq -a\sigma} e^{\frac{1}{2}z^2} \int_z^\infty (\sigma y + a\sigma^2) e^{-\frac{1}{2}y^2} dy = \sigma^2 + a\sigma^2 C_1(a, \sigma),$$

we get

$$\int_x^\infty |t - \mu| \exp\left(at + b - \frac{(t-\mu)^2}{2\sigma^2}\right) dt \le C_5(a,\sigma) \exp\left(ax + b - \frac{(x-\mu)^2}{2\sigma^2}\right).$$

Proceeding as before, this leads to

$$\int_x^\infty |t - \mu| \rho(t,\mu) dt \le C_6(a,\sigma)\rho(x,\mu), \tag{57}$$

and a similar bound for the integral from $-\infty$ to x in the case $x < \mu$. An easy byproduct of these inequalities is the bound

$$\sup_{x \in \mathbb{R}} |(x - \mu)f(x,\mu)| \le C_7(a,\sigma)\|u\|_\infty. \tag{58}$$

Using this and (54), and the previous deduction that $\|f\|_\infty \le C_4(a,\sigma)\|u\|_\infty$, it follows that

$$|f_1(x,\mu)| \le C_8(a,\sigma)\|u\|_\infty.$$

Now note that

$$\frac{1}{\rho}\frac{\partial \rho}{\partial \mu} = \frac{x - \mu}{\sigma^2}.$$

Thus, for $x \ge \mu$, we have

$$f_2(x,\mu) = -\frac{x-\mu}{\sigma^2}f(x,\mu) - \frac{1}{\rho(x,\mu)}\int_x^\infty \frac{t-\mu}{\sigma^2}\rho(t,\mu)(u(t) - r(\mu))dt$$

$$+ \frac{r'(\mu)}{\rho(x,\mu)}\int_x^\infty \rho(t,\mu)dt.$$

Thus from (56)–(58) it follows that

$$|f_2(x,\mu)| \le C_9(a,\sigma)\|u\|_\infty + C_2(a,\sigma)|r'(\mu)|.$$

A simple computation gives

$$r'(\mu) = \frac{\int_{-\infty}^\infty (u(t) - r(\mu))\rho(t,\mu)\frac{t-\mu}{\sigma^2}dt}{\int_{-\infty}^\infty \rho(t,\mu)dt}.$$

Thus,

$$|r'(\mu)| \leq \frac{2\|u\|_\infty}{\sigma^2} \int\limits_{-\infty}^{\infty} |t - \mu| \psi_{a,b,\mu,\sigma^2}(t) dt.$$

From the representation (13) it follows that

$$\int\limits_{-\infty}^{\infty} |t - \mu| \psi_{a,b,\mu,\sigma^2}(t) dt \leq C_{10}(a, \sigma).$$

Thus, for $x \geq \mu$, $|f_2(x, \mu)| \leq C_{11}(a, \sigma)\|u\|_\infty$. The bound for $x < \mu$ follows similarly.

References

1. Aizenman, M., Lebowitz, J.L., Ruelle, D.: Some rigorous results on the Sherrington–Kirkpatrick spin glass model. Comm. Math. Phys. **112**(1), 3–20 (1987)
2. Arratia, R., Goldstein, L., Gordon, L.: Poisson approximation and the Chen-Stein method. Stat. Sci. **5**(4), 403–434 (1992)
3. Baldi, P., Rinott, Y., Stein, C.: A normal approximation for the number of local maxima of a random function on a graph. In: Anderson, T.W., Athreya, K.B., Iglehart, D.L. (eds.) Probability, Statistics and Mathematics, Papers in Honor of Samuel Karlin, pp. 59–81. Academic Press, New York (1989)
4. Barbour, A.D.: Stein's method for diffusion approximations. Probab. Theory Relat. Fields **84**(3), 297–322 (1990)
5. Barbour, A.D., Holst, L., Janson, S.: Poisson approximation. Oxford Studies in Probability, vol. 2. The Clarendon Press, Oxford University Press, New York (1992)
6. Chatterjee, S.: Estimation in spin glasses: a first step. Ann. Stat. **35**(5), 1931–1946 (2007)
7. Chatterjee, S.: A new method of normal approximation. Ann. Probab. **36**(4), 1584–1610 (2008)
8. Chatterjee, S.: Fluctuations of eigenvalues and second order Poincaré inequalities. Probab. Theory Relat. Fields **143**, 1–40 (2009)
9. Chatterjee, S., Fulman, J., Röllin, A.: Exponential approximation by exchangeable pairs and spectral graph theory. Submitted. Available at http://arxiv.org/abs/math/0605552 (2008)
10. Chatterjee, S., Meckes, E.: Multivariate normal approximation using exchangeable pairs. ALEA **4**, 257–283 (2008)
11. Chen, L.H.Y.: Poisson approximation for dependent trials. Ann. Probab. **3**(3), 534–545 (1975)
12. Comets, F., Guerra, F., Toninelli, F.L.: The Ising–Sherrington–Kirpatrick model in a magnetic field at high temperature. J. Stat. Phys. **120**(1–2), 147–165 (2005)
13. Comets, F., Neveu, J.: The Sherrington–Kirkpatrick model of spin glasses and stochastic calculus: the high temperature case. Comm. Math. Phys. **166**(3), 549–564 (1995)
14. Fröhlich, J., Zegarliński, B.: Some comments on the Sherrington–Kirkpatrick model of spin glasses. Comm. Math. Phys. **112**(4), 553–566 (1987)
15. Goldstein, L., Reinert, G.: Stein's method and the zero bias transformation with application to simple random sampling. Ann. Appl. Probab. **7**(4), 935–952 (1997)

16. Goldstein, L., Rinott, Y.: On multivariate normal approximations by Stein's method and size bias couplings. J. Appl. Probab. **33**, 1–17 (1996)
17. Guerra, F., Toninelli, F.L.: The thermodynamic limit in mean field spin glass models. Comm. Math. Phys. **230**, 71–79 (2002)
18. Guerra, F.: Broken replica symmetry bounds in the mean field spin glass model. Comm. Math. Phys. **233**, 1–12 (2003)
19. Fulman, J.: Stein's method and non-reversible Markov chains. Stein's method: expository lectures and applications. IMS Lecture Notes Monogr. Ser., vol. 46, pp. 69–77. IMS, Beachwood, OH (2004)
20. Mézard, M., Parisi, G., Virasoro, M.A.: Spin glass theory and beyond. World Scientific Lecture Notes in Physics, vol. 9. World Scientific Publishing Co., Inc., Teaneck, NJ (1987)
21. Panchenko, D.: A question about the Parisi functional. Elect. Comm. Probab. (2009, to appear)
22. Rinott, Y., Rotar, V.: A multivariate CLT for local dependence with $n^{-1/2} \log n$ rate and applications to multivariate graph related statistics. J. Multivariate Anal. **56**(2), 333–350 (1996)
23. Rinott, Y., Rotar, V.: On coupling constructions and rates in the CLT for dependent summands with applications to the antivoter model and weighted U-statistics. Ann. Appl. Probab. **7**(4), 1080–1105 (1997)
24. Shcherbina, M.V.: On the replica-symmetric solution for the Sherrington–Kirkpatrick model. Helv. Phys. Acta **70**, 838–853 (1997)
25. Sherrington, D., Kirkpatrick, S.: Solvable model of a spin glass. Phys. Rev. Lett. **35**, 1792–1796 (1975)
26. Stein, C.: A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In: Proc. of the Sixth Berkeley Symp. on Math. Statist. and Probab., Vol. II: Probability theory, pp. 583–602 (1972)
27. Stein, C.: Approximate computation of expectations. IMS Lecture Notes—Monograph Series, vol. 7 (1986)
28. Talagrand, M.: The Sherrington–Kirkpatrick model: a challenge to math- ematicians. Probab. Theory Relat. Fields **110**, 109–176 (1998)
29. Talagrand, M.: Spin glasses: a challenge for mathematicians. Cavity and Mean Field Models. Springer, Berlin (2003)
30. Talagrand, M.: The Parisi formula. Ann. Math. (2) **163**(1), 221–263 (2006)
31. Thouless, D.J., Anderson, P.W., Palmer, R.G.: Solution of 'solvable model of a spin glass'. Philos. Mag. **35**, 593 (1977)
32. Tindel, S.: On the stochastic calculus method for spins systems. Ann. Probab. **33**(2), 561–581 (2005)