Title
Reinventing Realistic Mathematics Education at Berkeley—Emergence and Development of a Course for Pre-service Teachers

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Chapter 14
Reinventing Realistic Mathematics Education at Berkeley—Emergence and Development of a Course for Pre-service Teachers

Dor Abrahamson, Betina Zolkower and Elisa Stone

Abstract A central principle of Realistic Mathematics Education (RME) is that learners experience guided opportunities to reconstruct cultural practices and artefacts in the course of attempting to solve engaging problems using emerging resources as structuring tools. The same principle plays out at the meta level, across ages, geography, and functions, where instructors experience opportunities to reinvent RME as they adapt its principles to satisfy specific design constraints and local needs. This chapter recounts a collaborative effort to create at the Graduate School of Education, University of California, Berkeley, graduate and undergraduate courses for pre-service mathematics teachers that incorporates tenets of RME, while accommodating to prescribed and emerging constraints of local contexts, such as stipulation of federal funding, as well as the collective histories and prior schooling experiences of pre-service teachers, most of whom are encountering this didactical approach for the first time.

Keywords Course development · Mathematics education · Pre-service teachers · Problem solving · Undergraduate

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14.1 Reinventing Realistic Mathematics Education at Tel Aviv University: Dor's Story

The story begins in the fall of 1992 when I, Dor Abrahamson enrolled as a graduate student in the cognitive psychology master's programme at Tel Aviv University. Having served as a 'big brother' in Jerusalem, taught enrichment classes in the periphery, and enjoyed some circumscribed adventures in designing instructional devices, I arrived with a deep humanistic conviction that children's prospects could be greater than what educational systems offer. I was astonished by the epistemic abyss between students' natural perceptual sensitivities to the phenomenal world and their confusion over mathematical propositions that model the very same phenomena.

Mathematics learning, I believed, should always begin from situated sensorimotor experiences. There should be some engaging activity that gives rise to a surprising problem; and through tackling this problem with available resources, the student should arrive at new insights. Throughout, the teacher facilitates this activity by highlighting relevant elements of the situation, providing these resources that productively problematise the child's judgments, shaping the child's reflection on the experience, and supporting a formulation of the insight in cultural structures that mediate mathematical practices.

During the second year of my studies, I chose to focus my thesis on the early development of multiplicative concepts. In particular, I was looking to evaluate empirically an activity I had previously created for students to ground the concept of fractions in proportional judgement of geometrically similar images. The activity involved an elongated wooden contraption with a stretchable rubber ruler that enabled students to measure the heights of vertically oriented parallel elements in pictures they had judged as 'the same.' For example, by stretching the ruler we find that Danny and Snowman standing side by side are 2 and 3 units tall, respectively, whether in the small, middle-sised, or large prints (see Fig. 14.1).

Yet, if we do not stretch the ruler and keep the small units from the small picture on the left to measure the heights in the other pictures, then the heights of the three pictures would measure at 2 and 3, 4 and 6, and 6 and 9. In particular, the difference between Danny and Snowman's heights then measures as 1, 2, and 3 units, respectively. I was interested in understanding whether or not children's tacit perceptual expectations ('same difference') might cause cognitive conflict with the normative quantification routines ('different differences'!) and if, somehow, resolving this conflict may support students' articulated understanding of proportional equivalence as an entry into the multiplicative conceptual field.

My case studies had suggested that, indeed, children naturally expect pictorial identity to imply uniform measures: "If it is the same picture, then the heights should be the same, too." They were invariably befuddled when I pointed out that the absolute differences between these measured heights are different across the images: "If these are the same pictures, then how can the difference be different?" And yet, then they would reason: "The bigger the picture, the bigger the difference!" This insight, which much later I would learn to theorize either as a hypostatic abstraction,
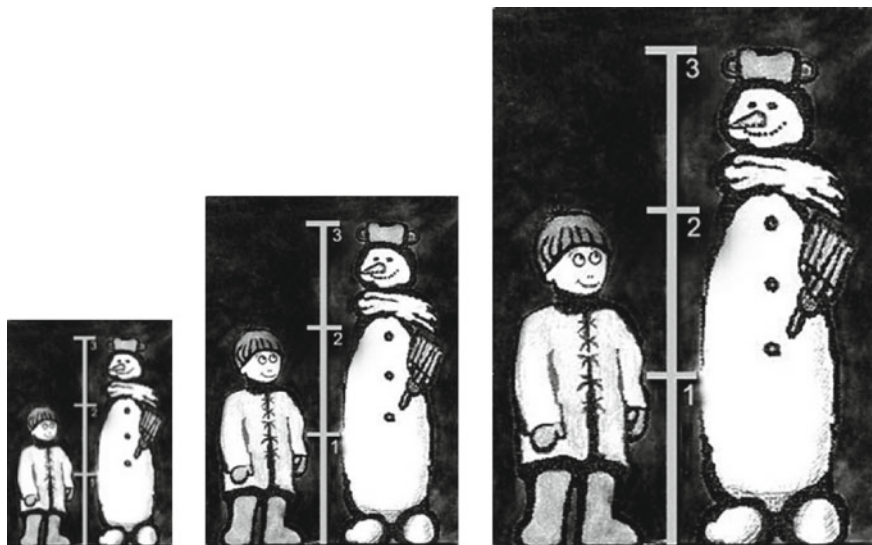
Fig. 14.1 A 'photograph' of Danny and Snowman is printed in three different sizes; they always measure as 2 and 3 units tall, respectively, but the absolute size of the measure unit changes across the prints

reflective abstraction, or abductive inference…, would then impel the children to further experimentation with larger pictures and finally to validation, to their great gratification as well as to mine (Abrahamson, 2002, 2012).

Still back in the 1990s at Tel Aviv University, I sought to develop my thesis by grounding my observations in the literature of the discipline. And yet that did not prove to be too simple. The epistemic climate of cognitive psychology, at least the climate of the leading experimental journals that populated our library, appeared uninviting of a perspective on mathematics learning grounded in tacit perceptual capacity. My quest brought me to the pinnacle of Tel Aviv University—the Education Library that resided on the very top floor of the Sackler Faculty of Medicine. There, amid stunning Mediterranean sunsets, I found it: a paper by Van den Brink and Streefland (1979) that was about to change everything.

Van den Brink and Streefland discuss a conversation between Coen (8;0) and his father about a poster showing a man and a whale. The child realised something was wrong with the ratio between the man and the whale that had been exaggerated for effect. Coen reasoned about this error by citing an image he had seen elsewhere. That child was taken very seriously by the authors of the paper. It is precisely these didactical materials and these forms of reasoning, they argued, that we should be recruiting so as to promote and support meaningful mathematical learning. Finally, I had found my 'sensei' in Realistic Mathematics Education (RME).

Fast-forward a decade or so, this chapter recounts the story of an international effort, a mixed relay to carry the RME torch from Utrecht across the Atlantic, by

way of Brooklyn, to the University of California Berkeley (UC Berkeley). Along the way, this torch was carried by different athletes and took many forms, and yet we have all been attempting to keep the essential flame alive. We describe how, at each station westward, local objectives and contingencies shaped the specific materialisations of the RME didactics. In particular, we discuss the development of courses for preparing pre-service mathematics teachers to design and facilitate problem-based instruction inspired by RME. In the next section, Betina Zolkower, the second author, will narrate her ongoing life-long investment in RME dissemination by way of Southern Argentina, the Netherlands, and Brooklyn, New York. A chance meeting with Abrahamson, the first author, instigated the appropriation of Zolkower's methods for teaching mathematics in middle school courses to Berkeley, where it framed a graduate-level course on cognition catering to pre-service mathematics teachers. At the behest of Stone, the third author, and again with Zolkower's support, this course was re-redesigned into an undergraduate course for pre-service mathematics teachers, part of Berkeley's Cal Teach initiative. A successful staple of Cal Teach, since then this course is taught annually by Abrahamson in collaboration with doctoral students from the Graduate School of Education.

This chapter is not offered as a theoretical piece, neither should it count remotely as experimental. Instead, we present a brief case study—a biography of sorts—with a modest scope of generalisation, with the hope that this story might encourage our fellow practitioners that Realistic Mathematics Education can and should be reinvented in diverse guises. Fiat lux!

14.2   Meanwhile, in New York City: Betina's Story

14.2.1   At the Graduate Center of City University of New York

In 1987, I, Betina Zolkower, enrolled in the PhD Sociology Program at the Graduate Center of City University of New York. As a graduate student in that programme I acquired analytical tools for inquiring into the mechanisms through which, notwithstanding cycles or reforms and counter-reforms, mathematics education continues to perpetuate social inequality by providing uneven opportunities to different socio-economic status groups to acquire that form of academic capital that constitutes successful performance in school mathematics. My thesis included fieldwork in 4th and 5th grade classrooms in Spanish Harlem attended by recently migrated, Spanish-speaking children, many of them from Mexico. There I witnessed first-hand the effects of camouflaging the ubiquitous, stereotypical word problems into supposedly culturally relevant story problems. Rather than contributing meaning and purpose to classroom activities, these micro-narratives added an extra layer of noise to students' efforts at deciphering the underlying mathematical structure of those problems.

14.2.2 Mathematics in the City: Learning and Practicing Realistic Mathematics Education

> Word problems are rather unappealing, dressed up problems in which the context is merely window dressing for the mathematics put in there. One context can be changed for another without substantially altering the problem […]. The aim of RME, by contrast, is to place oneself in the context and learn to think within it […]. (Van den Heuvel-Panhuizen, 1996, p. 20)

The experience of witnessing first-hand the negative effects of story and word problems on English language learners directed me towards RME, an approach premised on the view of mathematics as a human activity that consists of mathematising subject matter from reality, including mathematics itself (Freudenthal, 1991), with the aims of searching for generality, certainty, exactness, and brevity (Gravemeijer & Terwel, 2000), and mathematics teaching/learning as guided reinvention: mathematics is best learned when students are guided to reinvent mathematising by organising or structuring problematic situations embedded in realistic contexts and situations using mathematical tools (Freudenthal, 1991). In other words, as they organise mathematically those situations, students are guided to reconstruct their initial, situated material/mental activity, by verbalising, symbolising, and diagramming the relationships found therein. In Dutch, 'zich realiseren' means to imagine; thus, in this broader sense, a situation is realistic insofar as it appears to the learner as reasonable or imaginable (Freudenthal, 1991; Van den Heuvel-Panhuizen, 1996).

During the final stages of my doctoral studies, I joined Mathematics in the City, a teacher enhancement project funded by the National Science Foundation and directed by Catherine T. Fosnot in collaboration with two faculty members from the Freudenthal Institute, Willem Uittenbogaard and Maarten Dolk. Participation in this project apprenticed me into mathematics learning, teaching, and teacher education from the perspective of RME. In 1996, I visited the Freudenthal Institute to attend a summer institute on the textbook series (in preparation) *Mathematics in Context* (National Center for Research in Mathematical Sciences Education [NCRMSE] & Freudenthal Institute, 1997–1998). This experience as well as the ongoing mentorship of Uittenbogaard, with whom we collaborated in co-teaching lessons and co-facilitating workshops and seminars, helped me appropriate Freudenthal's ideas. I learned how to plan and enact mini-lessons to strengthen students' mental arithmetic skills; acquired an eye for finding mathematisable matter in the world and using it as raw material for instructional design; developed flexibility in using a variety of structuring models (e.g., open and double number line, ratio table, bar model, notebook notation, combination charts); and appreciated the value of models as level-raising tools, the central role of students' constructions and productions in teaching/learning processes, and the paramount function of teacher-guided interaction in expanding students' potential for making and exchanging mathematical meanings.

14.2.3   At Brooklyn College

Since the fall of 2000, as a Brooklyn College faculty member, I have been teaching initial and advanced methods courses as well as the capstone action-research course for graduate students in the 5–9 Grade and 7–12 Grade programmes. In the capstone course, my students, many of them beginning teachers, formulate researchable questions related to the teaching and learning of a specific mathematics topic, analyse relevant units of the textbook series *Mathematics in Context* (National Center for Research in Mathematical Sciences Education [NCRMSE] & Freudenthal Institute, 1997–1998), review literature on that topic (including seminal work by RME specialists) and, in light of all of the above, design and carry out a teaching experiment to address those questions. Among the master's theses I have directed are:

– Mathematising and didactising dissection puzzles
– Connecting geometry, measurement, estimation, and ratio and proportion through problems involving large numbers
– Exploring whole-class share and discussion formats that maximise opportunities for students to exchange mathematical ideas
– Teaching students to use diagrams as tools for solving non-routine problems
– The number line as a tool for solving linear equations
– Using geometric contexts to teach algebra.

  All of the above show evidence of my students' creative appropriation of RME ideas.

A central principle of RME is the pivotal function of interaction in guiding students to reinvent mathematical objects, ideas, tools, and strategies, hence the need to prepare teachers to guide such exchanges in manners that support reinvention. With that in mind, my courses include activities that focus on the multiple intersections between language and mathematics. Worth highlighting among those is the interpretative analysis of whole-group conversations conducted by highly experienced and effective teachers. Treating the transcribed conversations as multi-semiotic texts (Halliday, 1994; O'Halloran, 2000), our interpretative analysis centres on the teachers' choices of grammar and vocabulary and the effect of these choices on expanding students' mathematical meaning potential. Among the problems we studied, which found their way into the Berkeley courses, are 'What do you mean by relationship? (de Freitas & Zolkower, 2009), 'Chunking necklaces' (Zolkower & de Freitas, 2010), 'Numbers on a triangle' (Zolkower & Shreyar, 2007), 'Ways to go' (Zolkower & de Freitas, 2012), and 'Marching ants' (Zolkower, Shreyar, & Pérez, 2015).

14.3   Reinventing Algebra Brick by Brick: A Graduate Level Pre-service Mathematics Teaching Course

In May of 2008, following a chance meeting at a Spencer reception during the annual meeting of the American Educational Research Association in New York City, Zolkower and Abrahamson began collaborating on an RME-inspired research project, Paradigmatic Didactical–Mathematical Problematic Situations. In this section, we will revisit the construct of these situations, which evolved as our means for importing Zolkower's RME course from Brooklyn College to University of California Berkeley. We will introduce the 'Brick pyramid' problem as well as our graduate students' work on it to exemplify its mathematical and didactical–mathematical potential.

14.3.1   Paradigmatic Didactical–Mathematical Problematic Situations

The project involved co-designing and evaluating a course in mathematical cognition, learning, and teaching. As we define them, paradigmatic didactical–mathematical problematic situations are activities evoked as contexts for collaborative inquiry into the practices of mathematics, mathematics learning, and mathematics teaching. Our experimental course builds upon and contributes to a body of work on rich problems as contexts for teaching and learning to teach mathematics. Included in this growing domain are: realistic modelling problems (Verschaffel & De Corte, 1997); emergent modelling problems (Gravemeijer, 1999; Van den Heuvel-Panhuizen, 2003); problems that yield multiple solutions (Silver, Ghousseini, Gosen, Charamboulous, & Font Strawhun, 2005; Zolkower & Shreyar 2007); model-eliciting tasks (Lesh, Middleton, Caylor, & Gupta, 2008); substantial learning environments (Wittmann, 1995, 2002); open-ended problems (Cifarelli & Cai, 2005); spiral tasks (Fried & Amit, 2005); and example-generating problems (Watson & Mason, 2005).

Our common interest in paradigmatic didactical–mathematical problematic situations sparked from noticing the potential of these activities, which we both had been using independently, sporadically, and anecdotally, to engage classroom practitioners as well as researchers-in-training in reflective inquiry into a panoply of cognitive, social, technological, and other aspects of mathematics teaching and learning. Consequently, we designed and implemented a semester-long course based on guided study and classroom try-outs of paradigmatic didactical–mathematical problematic situations. Our experimental course is entirely organised and driven by them. Paraphrasing Turkle and Papert's (1991, p. 117) proverbial call to "put logic on tap, not on top," we place mathematics instruction theories on tap rather than on top. That is, our problematic situations serve as scenarios for the targeted mathematical–didactical ideas to emerge out of participants' guided engagement in those situations.

Abrahamson first taught the course in the Fall Semester of 2008. Our research consisted of investigating the effect of this course on participants' mathematical understandings and didactical-mathematical abilities and disposition. Our data included rich documentation from both the college sessions and the field-placement classrooms, where student–teachers tried out the same or similar problems (Zolkower & Abrahamson, 2009). Central to the paper is a particular RME-style problem, the 'Brick Pyramid' problem, which we introduce and discuss below.

14.3.2   The 'Brick Pyramid' Problem

Figure 14.2 shows the 'Brick Pyramid' problem. Students are first presented a picture of bricks configured in the shape of a triangle. The bricks' numerical contents are bound to each other by the following rule: in each brick triad, the number within the top brick is the sum of the numbers in the two contiguous bricks directly below it. The task is to solve the puzzle-like problem by filling in the missing numbers. The brick pyramid discloses the top number and three more, thus creating an implicit system of constraints that emerges as determining a single solution for each additional input inserted, resulting in a structure with surprising mathematical relationships. The instruction given to the students is: "Fill in the brick pyramid. How many solutions did you find? Could there be more? Explain."

Although this problem can be solved using formal algebraic tools, non-algebraic and proto-algebraic (informal) methods can be used as well. Working with positive integers and 0, one possible algebraic treatment begins by assigning to the bottom-leftmost brick the variable x and then stepping upwards, sideways, and downwards, abiding with the addition rule for each triad, until all the bricks have been filled (see Fig. 14.3).

This process indicates that the range of possible values for x is 0–6 (see in Fig. 14.3 the expression "$6 - x$" for the bottom-rightmost brick), thereby giving seven solutions with integers. Interestingly enough, the rightmost brick on the second row is the only one where x cancels out, resulting in the constant value of 24, which is a puzzling phenomenon that may merit an investigation of its own. For example, several
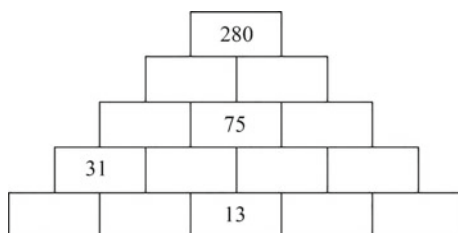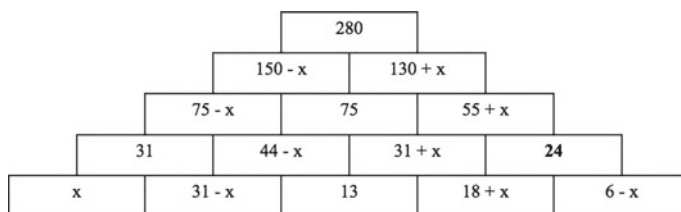

Fig. 14.2 'Brick Pyramid' problem

		280		
	$150 - x$		$130 + x$	
	$75 - x$	75	$55 + x$	
31	$44 - x$	$31 + x$	**24**	
x	$31 - x$	13	$18 + x$	$6 - x$

Fig. 14.3 Algebraic treatment of the brick pyramid as a space shaped by numerical constraints

questions can come up: Why 24? How is this number, 24, related to the four given constraints (280, 75, 31, and 13) and their respective locations on the pyramid?

Four graduate students (all names are pseudonyms) participated in the study: Justin, Emily, and Nora, who are all three in their first year in a master's programme, and Zoran, who is in his second year in a doctoral programme. Zolkower, the designer of this course, assisted Abrahamson, remotely, in facilitating this course. Occasionally, Zolkower participated through video-conferencing. The focus lesson took place during the fifth week of instruction of a Fall Semester. During the fifth week, course participants were just beginning to become involved in school placement observations and other fieldwork-related assignments.

14.3.3 Reinventing Algebra by Thinking Aloud Together About the Brick Pyramid and Beyond

In the subsection below, we analyse and discuss selected excerpts from a whole-group conversation about the brick pyramid in the course. This text, made up of 432 turns (changes of speaker), consists of a series of exchanges whereby participants engaged in thinking aloud together (Shreyar, Zolkower, & Pérez, 2010; Zolkower & Shreyar, 2007; Zolkower, Shreyar, & Pérez, 2015) about the problem at hand. Borrowing from Christie (2002), we parsed the conversation into episodes. These are:

   I. Opening: framing the problem
  II. Solving the problem: thinking aloud together, scribbling, speaking, diagramming, and gesturing
 III. Comparing and contrasting approaches and moving forwards to 'more algebraic' approaches
 IV. Reflecting back on the experience as reinventing algebra while moving forwards towards algebraisation of the situation
  V. Closure: considering the problem as a potential classroom activity while also discussing it from the point of view of instructional design (for this latter part readers are referred to Zolkower & Abrahamson, 2009).

Fig. 14.4 Coding system for labelling the cells
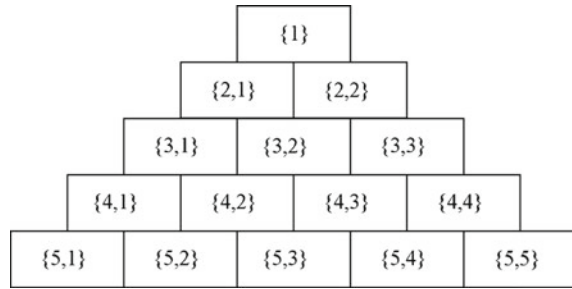
Below we describe the overt activities in this text, that is, what a non-omniscient 'fly-on-the-wall' observer, who is engaged in the solution process, would witness. We begin with Episode II, when students began to think aloud together. In the interest of brevity and clarity, we shall use the following coding system to refer to each of the fifteen cells in the brick pyramid (see Fig. 14.4).

14.3.3.1   Episode II: Solving the Problem

Emily sets off the discussion by going up to the board and presenting her solution procedure (see Fig. 14.5).

Explaining her work, Emily says the following:

> …What I decided to do was, pick a value, put it somewhere in here [indicates bottom row], build off of that. And I figured I'd pick a value that was under one of these given, permanent numbers. So, I put a number here that's less than 31. [indicates {5,2}] Any number I wanted. [enters 17 into {5,2}] And I went with 17 because… I don't know why….

In order to begin familiarizing herself with the problem, Emily inserts a value, 17, which she selects somewhat arbitrarily, into an empty cell in the bottom row {5,2}, and then works that value so as to fill the entire pyramid according to the addition rule. As it turns out, assigning that particular value to that cell-variable is not permissible in this system. Yet, due to an arithmetic error, $45 - 13 = 22$, this violation appears to go unnoticed.

Next, Justin replaces Emily at the board and presents his solution (see Fig. 14.6). Justin, possibly building off Emily's work, is already more systematic than her, in that he orients toward searching for the range of the solutions, and so his choices of input numbers are governed by an attempt to determine one limit of this range.

As he solves, he articulates emergent insights:

> Okay, so, 75 here, 31 here, 13 here, 280 here, oops, right. [enters values as he utters them] And so, <what> I did was, I looked at each box and thought, what are the limits bounding the number in each box? [emphasises boxes with given values] So, I guess I kind of like started up at the top, and I was like, well these have to be 75 or more, [indicates {2,1}, {2,2}] right, and then…. [Nora points to the 'less than' constraints in Justin's approach].
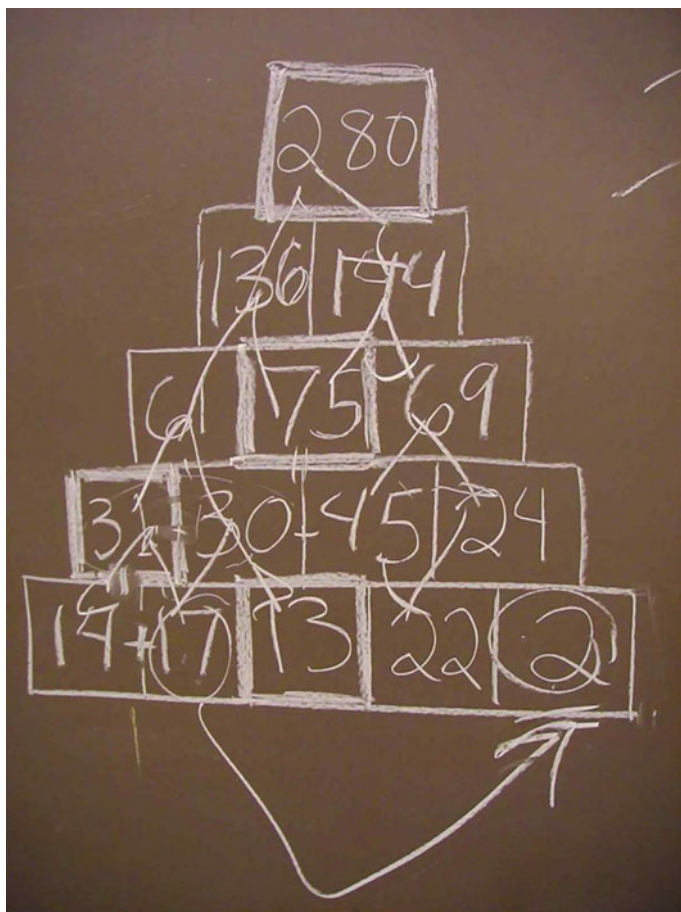
Fig. 14.5 Emily uses numbers and moves from the bottom up; note diagonal lines connecting between cells to express algebraic relations

Dor highlights the double-using issue, which affects the way one distributes the values throughout the pyramid. Namely, Justin begins to realise that the pyramid can be viewed as a system, rather than a collection of local overlapping three-cell triadic structures; because each sum is constituted by addends below it, recursively, there appears to be some overall systematic set of relations governing the distribution of addends. However, Justin does not yet specify the nature of this system of relations.

Nora, commenting on Justin's work, adds that in order to determine the range of values, one would need to find the other limit, too. Dor comments on embodied constructions of addition that, he believes, may be implicitly biasing the solution procedure. First, he suggests a view of addition as 'adding *up*,' for example, a privileging of an upward-adding construction of the problem, at the expense of attending to the equally important downward constraints. Second, he portrays addition as 'using up

**Fig. 14.6** Justin uses 'numbers +' or 'greater than' constraints and moves from the top down

resources'—once a value is used in one sum, e.g., in conjunction with the cell to the right, it must be used again for the cell on the left, and this re-use might be violating the grounding multi-modal dynamical images that tacitly underlie the sense of addition. Emily responds that, indeed, the sum of the numbers on the bottom row is not equal to the number up top, so that her implicit model of the situation seems to have shifted.

Nora thinks aloud:

So, if we did it in this square [indicates {5,4}], we would end up, maybe we would only have to do, like, a fourth of 72, or something. Because then, by the time it gets up to the top, then you'll have the full 72 that you want. But then I don't know if that takes into consideration… I don't know exactly how this ** but I think <it would fit it> somehow if you did some fraction of 72, then, you know, it's gonna be multiplied here, there's gonna be… two of them up here, and then that'll count as one, and that'll count as two, so that's three. [indicates boxes above {5,4}] If we had some number here, then we'd have that number, then twice that number, then three times that number, and then…

…

If we had some number here [enters *a* into {5,2}], then… [draws next row] okay, well, kind of ignoring the numbers that are already in there, once you go up here, so, well, it's like what you were saying, we feel like okay, if the *a* is here, then it's already taken care of, we don't need to worry about it again, but actually it's gonna – you have to count it again in this box and in this box. [enters *a* into {4,1}, {4,2}] And then, when you come to the next level…
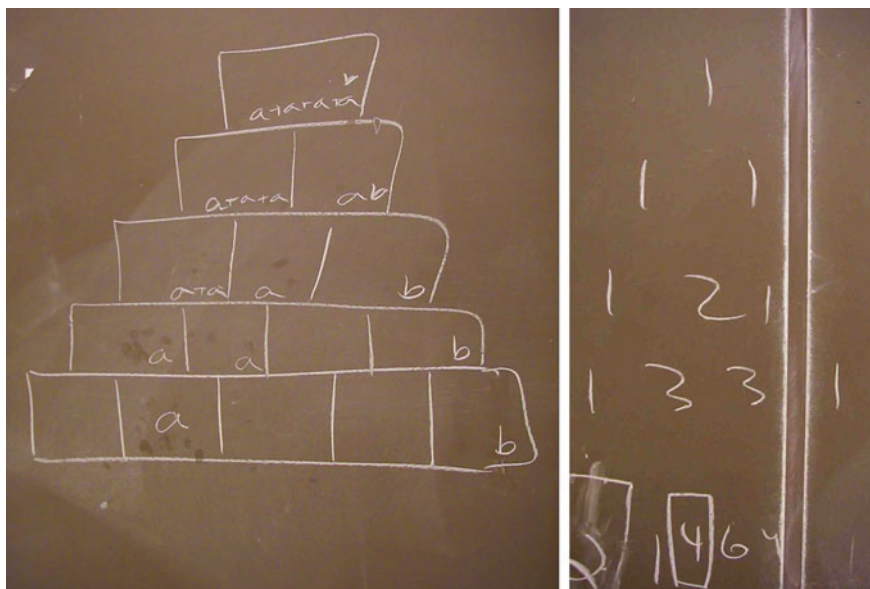
**Fig. 14.7** Nora enters *a*, then *b*; this links to Pascal's triangle

Nora thus explores how—given the repeated use of addends—a sum that is written fairly high in the pyramid could be viewed as constituted by the addends below it. She recognises that the relation is not a direct distribution of the sum into equal parts, but initially she cannot explain just how this distribution works or could be represented to support this inquiry. A significant move forwards in the group's collective inquiry is when Nora uses algebraic symbols to explain how a numerical value on the bottom 'ripples' up (see Fig. 14.7). She distinguishes between values in the centre of the bottom row (a) and those on the extremities (b), in terms of how much they contribute to the upward accumulating sums.

Up to this point, participants have each made unique contributions to the collaborative problem-solving process. One can discern a progression from Emily's first hesitant exploration of a single solution through Justin's analysis of the pyramid as an emerging system of constraints to Nora's introduction of algebraic symbols in an attempt to spell out the spread of upward addend contributions of a single number on the bottom row, depending on whether it is in a central or extreme brick, and how two such 'deltas' (the spreading contribution tributaries) mingle in an addend confluence. The instructor's insistence that the work be done up at the board had two related results. On the one hand, students were initially diffident to share half-baked ideas. On the other hand, these ideas, like Nora's *a* and *b* addends, appear to have spread and mingled upwards, receiving at each level the input and reformulation of additional minds on tap.
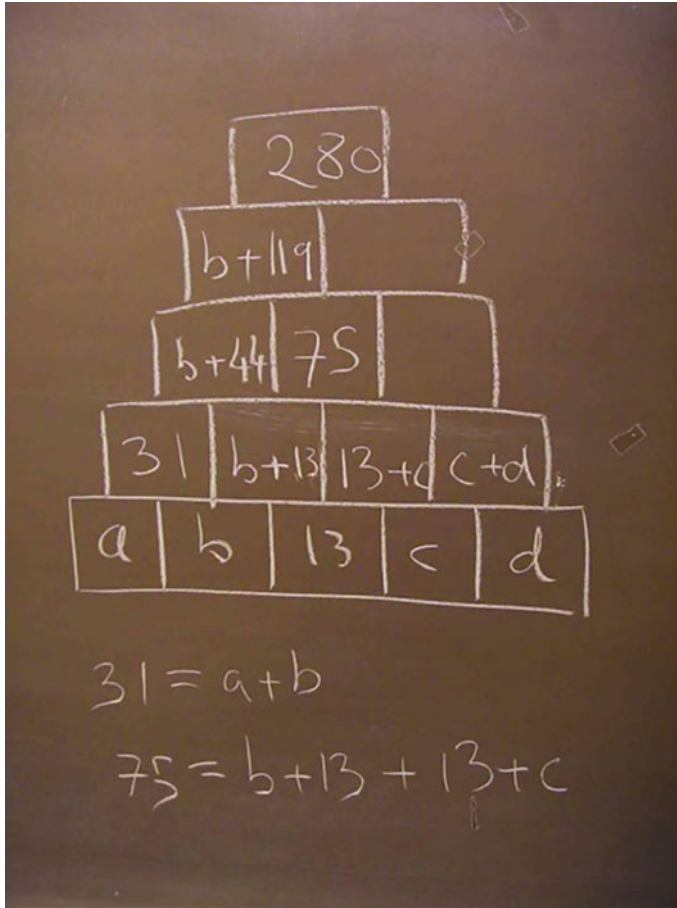
Fig. 14.8 Zoran's brick pyramid; he enters *a*, *b*, *c*, and *d*

14.3.3.2   Episode III: Comparing and Contrasting

Zoran elaborates on the work of Emily, Justin, Nora, and Dor, and he, too, recognises the inadequacy of trial-and-error techniques to cope with the load of arithmetical constraints imposed by the pyramid system. He acknowledges the value of the algebraic system introduced by Nora, to support this inquiry. Zoran introduces additional variables (*a* through *d*), in order to articulate general relations among the fifteen cells of this particular pyramid (see Fig. 14.8).

Zoran continues discussing the distribution issue, talking in terms of 'sets': "With this we have this set here, and this set here… this set is only going up to 31…." Zoran concludes that it would be unreasonable to attempt the problem "purely arithmetically," thereby calling for an algebraic approach.
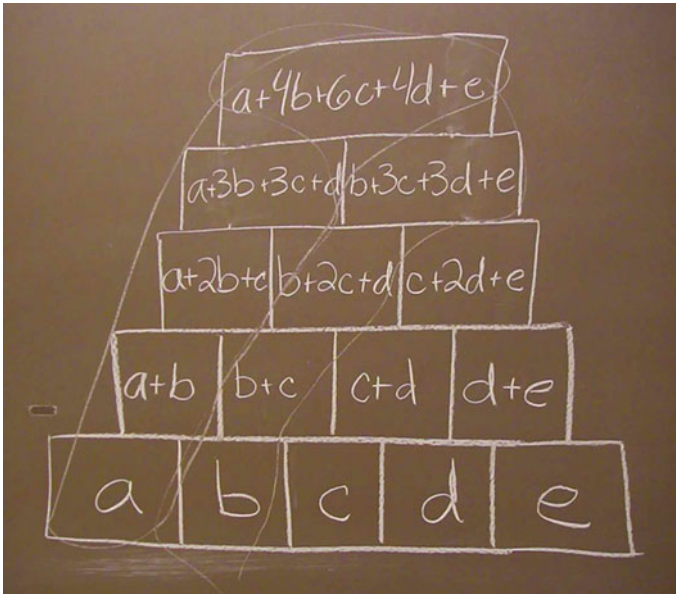
Fig. 14.9 Emily again enters *a*, *b*, *c*, *d*, *e* in the bottom row, and the group notes an unexpected association with Pascal's triangle, albeit in inverted form

Emily, possibly building on Nora and Zoran's suggestion, expands the number of symbols to five (*a* through *e*) and uses these 'variables' to demonstrate attributes of the pyramid family, regardless of the given numbers. In particular, Emily develops the idea that Pascal's triangle can be viewed as lodged upside-down in the pyramid (Fig. 14.9).

Emily says:

> Say you just have your numbers down here: *a*, *b*, *c*, *d*, *e*. [enters letters, as she utters them, into the bottom row, in that order] Okay, there's one of each of those, I'm just saying they're… distinct, or they could be <one> , I don't know, um, they're variables. So, to get here, you do $a + b$, and this is $b + c$…

> …. this is sort of becoming the, uh, Pascal's triangle in reverse. And this one would be $b + 2c + d$. [enters value into {3,2}] This would be $c + 2d + e$. [enters value into {3,3}] And up here, I need to start making the bricks bigger. And now, we have $a + 3b + 3c + d$, I think. [enters sum into {2,1}] Someone correct me if I'm wrong. And then $b + 3c + 3d + e$. [enters sum into {2,2}] And for the last one… Make it a little bigger…! [expands first box borders] Okay. $a + 4b + 6c + 4d + e$. [enters sum into {1}] And then, <see> you have 1, 4, 6, 4, 1, just like this guy. [Indicates Pascal's triangle fragment on the board] And this is just like coefficients.
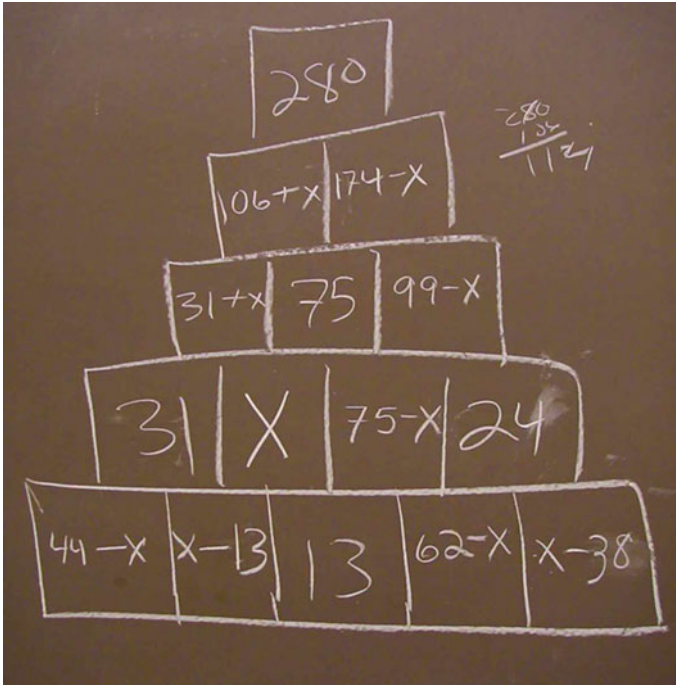
Fig. 14.10 Nora again, now working with a single variable

14.3.3.3 Episode IV: Reflecting

The instructor introduces the following constructs, as they apply to the joint mathematising effort under way: 'sprouting' of algebraic notation from speech and gesture (Radford, 2003); 'distributed cognition' (Hutchins, 1995); 'cognitive artefacts' that amplify reasoning (Norman, 1991); 'working memory' (Baddeley & Hitch, 1974); and 'hypostatic abstraction' (Peirce, 1931–1958). This mini lecture may or may not have had any direct impact on the collaborative solution process. In any case, Nora—possibly also inspired by Emily and Zoran's attempts—then suggests working with a single variable x and expresses all of the numerical values using this variable (see Fig. 14.10).

Nora explains:

> So, I picked this box right here. Just because it was touching so many other things, I figured I could get a lot of information out of it. [places x in {4, 2}] So this one is $31+x$, and then this is $106+x$, and then this is $75-x$, and this is, let's see, $x-13$, and then….

As we have shown in the section above, the brick pyramid allows prospective (and in-service) teachers to encounter central issues in the learning and teaching of algebra, including: (1) symbol sense, informal sense-making, and formal algebra (Arcavi, 1994); (2) the relationship between language proficiency and algebraic learning

(MacGregor & Price, 1999); (3) the role of realistic situations in developing algebra tools (Van Reeuwijk, 1995; Van Reeuwijk & Wijers, 1997); (4) the challenges in guiding the transition from informal to formal algebra (Nathan & Koellner, 2007; Stacey & MacGregor, 2000; Swafford & Langrall, 2000); and (5) the complex interplay among algebraic thinking, algebraic generalisation, and algebraic symbolisation (Linchevski, 1995; Radford, 2003, 2006; Van Ameron, 2003; Zazkis & Liljedahl, 2002).

In what sense is the Brick Pyramid is a paradigmatic problematic situation? In his homage commentary on the life of Hans Freudenthal, Goffree (1993) likens paradigmatic mathematical problems to scientific benchmark discoveries, following which the scientists' extant paradigm shifts and the world is forever seen in a new way. Participants in our graduate course are mathematically literate adults, according to normative standards, such that the algebraic machinery is quite at their fingertips, ready for application. The question is whether, when, and how this application is triggered. This paradigmatic didactical–mathematical problematic situation presents a scenario that calls for mathematising but does not cue or furnish, in an explicit manner, a specific toolbox, let alone a particular tool. Implicit in this principle is the belief that didactical approaches that deny students the opportunity to search for appropriate tools to structure or organise problematic situations have limited effects in developing in them a genuine disposition towards mathematising. The Brick Pyramid is not just a problem-solving tool. It is also and primarily a thinking device (Lotman, 1988). Lyrically speaking, the brick pyramid—once a curious inscribed structure on a worksheet—becomes colourful, animated, mobilised, and enmeshed in multiple intersecting dimensions of mathematical thinking, thereby functioning as a model for thinking about the very meaning and purpose of algebra as a situated human activity.

One might comment that developing paradigmatic didactical–mathematical problematic situations was little more than reinventing RME. We would proudly concede. Our goal of building a graduate-level practicum on mathematics education posed for us the problem of designing sets of interconnected vehicles and experiences around them as means for facilitating encounters between prospective mathematics teachers and Freudenthal's didactical vision. In the course of seeking such architecture, the paradigmatic didactical–mathematical problematic situations emerged in our discourse—first as 'models of,' then as our 'models for'—specifying what we view as opportunities for teachers' productive engagement in thinking about students' thinking. Examining the *Standards for Mathematics Education* (Goffree & Dolk, 1995) published by the Netherlands Institute for Curriculum Development (SLO) and the Freudenthal Institute, we are both heartened to find correlates with our own insights and awed to learn how far we have yet to go. In that volume, eighteen standards are framed as spotlights: "When all eighteen spotlights are 'on,' the entire educational process becomes brightly illuminated" (Goffree & Dolk, 1995, p. 11).

Moving forwards, new problems of practice emerged for us. In particular, we were commissioned to develop an undergraduate course for pre-service mathematics teachers. The next section describes the Education 130 course (hereafter called EDUC 130)—its constitution and credo as well as a set of didactical heuristics that

have evolved as our means of cultivating college cohorts of RME disciples from graduates of the mainstream schooling in the United States.

14.4 An Undergraduate Course for Pre-service Mathematics Teachers

University courses are complex cultural phenomena. They are born at the nexus of institutional needs and individual convictions, evolve in negotiation with emerging constraints from multiple and shifting stakeholders, and become blueprints for communities of practice—emblems of departmental ethos and praxis. Semester after semester, courses, like rivers, function as structures and schemes that host and shape the trajectory of their contained water, and in turn are shaped by these waters. Similar to the Heraclitean river, you can never quite step twice into the same course. In this section, we explain the origins and tributaries of the EDUC 130 course, its credo as it appears in the course syllabus, and the didactical heuristics that evolved over time.

The EDUC 130 course Knowing & Learning at UC Berkeley was designed and implemented as part of Cal Teach, a new undergraduate teacher education programme set within the College of Letters & Sciences, aimed at addressing the critical shortage of mathematics and science teachers in California. In creating EDUC 130, faculty members of the Graduate School of Education were attempting to build a new form of community—a haven safe from what we viewed as less productive aspects of mainstream education, a space where pre-service mathematics teachers could experience what it might be like to centre classroom instruction around guided collaborative activities of framing and solving rich, accessible, open-ended problems. We were attempting to forge a new form of identity for pre-service mathematics teachers as activist careerists—reflective practitioners, intuitive action researchers, keen observers who keep experimenting with their resources and process. In creating EDUC 130 we were thus asking what it might take to administer a course where such a person could grow who would become our ideal teacher: inquisitive, reflective, and knowledgeable of both subject matter and its didactics.

Regardless of the particularities in one's image for the ideal teacher, this image becomes the 'product' that the course should 'deliver.' As such, the process of building EDUC 130 was akin to that of solving a design problem. The question was: What should pre-service mathematics teachers' experience in our course be so that they become our ideal teachers? We therefore created a 'course credo' that specifies our views on why and how teachers should become, in addition to educators, also designers, researchers, and 'hackers' (fluent in current technological developments). We placed the credo in our course syllabus.

Surely, though, we never expected this credo to effect any change. Such an expectation would defeat the very assertion within the credo itself that hermetic definitions rarely suffice as a form of instruction. We posted the text in our syllabus and let it be. At the time, we had our course resources and a collective, unarticulated profes-

sional know-how for making these resources work in our course. With time, this unarticulated know-how would become articulated in the form of a set of didactical heuristics, as we now explain. Still, the credo speaks to a vision. What we needed was a means by which to achieve this vision. Enter our 'didactical heuristics,' which speak to how we might go about realising the credo.

Didactical heuristics are classroom routines for effective participation in the course as well as in future practice as high-school teachers. These didactical heuristics capture recurring framings, emphases, imagery, and metaphors that we use so as to position and project the pre-service mathematics teachers into a new 'figured world' of reform-oriented teachers (see Ma & Singer-Gabella, 2011). We view these didactical heuristics as capturing our 'tricks of the trade,' design solutions for introducing pre-service mathematics teachers to RME.

In developing EDUC 130, we never intended to develop a set of didactical heuristics. Rather, these notions, which instantiate much of the course credo, sprouted into our practice as useful means of explicating our instructional philosophy and methodology, first to ourselves, then to each other, and finally to our colleagues. The didactical heuristics are structures that emerged as our invented means of solving a problem. The set of heuristics constitute EDUC 130's collective models for solving a problem of practice faced by college instructors, that is, the problem of bootstrapping a community of practice (see Visintainer, Little, & Abrahamson, 2011).

We submit that these didactical heuristics constituted our emergent structures for enabling and enhancing the solving of situated problems of practice. They were our 'models of' teaching a RME course for pre-service mathematics teachers. They were models that we were hence able to carry forth, document, and propagate as 'models for,' just as we are doing in this chapter.

14.5 Reflection: Reinventing Reinvention

> Mathematics is said to have, for example, disciplinary value in habituating the pupil to accuracy of statement and closeness of reasoning; it has utilitarian value in giving command of the arts of calculation involved in trade and the arts; culture value in its enlargement of the imagination in dealing with the most general relations of things; even religious value in its concept of the infinite and allied ideas. But clearly mathematics does not accomplish such results, because it is endowed with miraculous potencies called values; it has these values if and when it accomplishes these results, and not otherwise. (Dewey, 1944, p. 245)

The proverbial human capacity to solve problems—a vital skill that is apparently so desirable yet so rare these days that it has been rebranded in the U.S. educational reform discourse as a 21st century skill—has been the vision of Realistic Mathematics Education from its very incipience. Freudenthal makes the following bold statement that connects the teaching of mathematics to youth ethos, mores, identity, and agency in extra-mathematical realms of life.

> Our cultural assets are too dangerous to be offered the youth as ready-made material. The instruction we provide should create the opportunity for youth to acquire the cultural heritage

by their own activity. They should learn that the self-reliance they claim elsewhere extends to their own role in the learning process. (Freudenthal, 1971, p. 415)

What a vision! Mathematics education could fulfil the virtuous role of fostering youth independence and empowerment. To make this vision a reality, college instructors must cultivate a generation of pre-service mathematics teachers who are prepared to offer high-school students experiences by which they can recognise their own capacity to reason deeply as a means of solving problems. A teacher education course guided by RME principles could thus possibly prepare high school students for life.

We have described the evolution of our EDUC 130 course for pre-service mathematics teachers. The course is founded on the idea of preparing pre-service mathematics teachers by running them through the same problem-solving experiences their own prospective high-school students will undergo. Once they become teachers, EDUC 130 course participants are then to emulate our classroom practices 'one step down,' with them now as instructors and their high-school students solving and reflecting on engaging problems.

Building a course is certainly a form of problem solving. And yet how does one know whether and when the problem has been solved? We cannot quite look up the correct answers at the end of the book! Still, a set of guiding principles may serve as a book of sorts by which to evaluate our progress. It is in this sense that RME has served us as a standard by which to evaluate the course design, its implementation, and its effects. And in so doing we have developed our own set of principles, the didactical heuristics, that evolved as our practical means of apprenticing pre-service mathematics teachers into RME. Yet we do not offer these cultural assets as ready-made materials. Rather, we steer our college students to reinvent the didactical heuristics through guided reflection on their problem-solving experiences. In a sense, the didactical heuristics constitute the emergent structures we have cultivated through and for our own guided reinvention of RME at Berkeley.

In designing and promoting this course, we are conscious of valuing depth of experiencing over breadth in coverage. When a college course is planned so as to cover an entire methodology textbook, with its complementary readings, it seems rather certain that one would get through the syllabus by the end of the course. However, when mathematical content, didactical subject matter knowledge, and assigned readings are positioned as ancillary to the 'actual' learning, one may not be as certain to cover it all (and perhaps, even more troublesome, the course is unlikely to be instructor-proof so as to enable standardisation). Yet, as a trade-off, there is the perceived opportunity that participants would have rich experiences of learning 'what really matters.' And what is it that matters, really? Learning to teach mathematics involves reflective back-and-forth movements between classroom practice and instructional theory; mathematical content and didactical form; observing one's learning processes and observing those of others (Freudenthal, 1991). Our RME-inspired course materials, activities, credo, and heuristics, in synergic unison with our grounding of college-classroom discussions in actual school-classroom experiences, engage future teachers and mathematics education researchers in working on

and thinking through mathematics problems and mathematics learning and teaching problems and, in so doing, contribute to the all-cherished goal of making mathematics meaningful, relevant, and accessible for every student in every classroom.

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