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Authors
Kaufman, A.N.
McDonald, S.W.
Rosengaus, E.

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Lawrence Berkeley Laboratory
UNIVERSITY OF CALIFORNIA

Accelerator & Fusion Research Division

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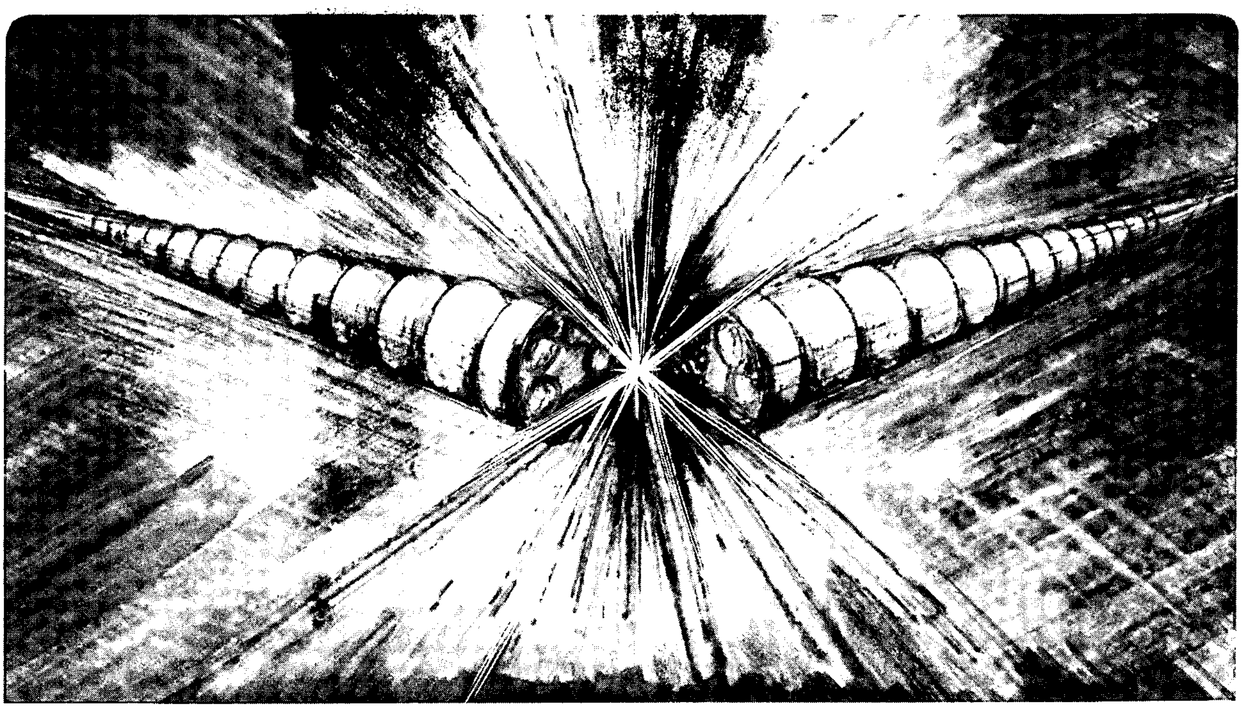

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Allan N. Kaufman, Steven W. McDonald, and Eliezer Rosengaus

Lawrence Berkeley Laboratory and Physics Department,
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ABSTRACT

A coherent wave may be characterized by a single-valued phase function. As the wave propagates, its rays twist and separate, causing its Lagrangian manifold $\underline{k}(\underline{x})$ to develop pleats. Thereby the phase becomes multivalued, and the wave may be termed incoherent. This process is analyzed by studying the local spectral density, which changes from a line spectrum to a continuous spectrum.

The concept of chaos can be applied to classical waves as well as to the quantum solutions of the Schrödinger equation. In this paper, the term "chaos" will refer to the degree of spatial incoherence of the phase of a linear oscillation of a nonuniform, but nonrandom, classical medium.

The motivation for our study arose in the problem of the controlled heating of magnetically confined plasma by electromagnetic radiation. Although the radiation incident on the plasma is typically coherent (from an antenna structure or a wave guide), in the course of propagation through the nonuniform plasma the wave phase may become multivalued, as we show below. The resulting incoherence has important consequences for the deposition of the radiant energy, i.e., the heating of the plasma. A coherent wave typically traps particles, while an incoherent wave causes diffusion. Nonlinear effects (usually undesirable) have a higher threshold for incoherent waves.

The ideas presented here can of course be applied to other classical media, and (with reinterpretation) to the quantum problem. To a large extent these concepts have been developed previously by M. Berry and co-workers [1]. Our contribution to the study was an outgrowth of the thesis research of S. W. McDonald [2], where several alternative descriptions were explored and developed.

In this presentation, we shall consider the field as a scalar $\phi(x)$ satisfying a linear homogeneous integral equation:

$$\int d^4x' \; D(x,x')\phi(x') = 0 \tag{1}$$

This is a simplification of the more correct description [2], wherein the kernel D may depend implicitly on ϕ, and where (1) has a (small) inhomogeneous term, representing sources of the field. Further, the scalar $\phi(x)$ should be replaced by the vector electric field, while the kernel becomes a tensor [2].

In (1), x represents position in space-time, while D is (essentially) the two-point linear response function for the field, which is obtained from the underlying dynamics of the system. The description of the latter may be Hamiltonian (e.g., Vlasov or Klimontovich in the plasma case) or dissipative (e.g. a Fokker-Planck kinetic equation for the particle distribution). For present purposes, it is considered known. Because of causality, D is not symmetric in its arguments $(\underline{x},t)$, $(\underline{x}',t')$. However, we may often neglect the antisymmetric part to lowest order, reconsidering it later (not here) as a perturbation.

Thus, if we replace D by its symmetric part (from this point on), the integral equation (1) for $\phi(x)$ is equivalent to the variational principle $\delta S(\phi) = 0$, where

$$S(\phi) = \int d^4x \int d^4x' \; D(x,x')\phi(x')\phi(x). \qquad (2)$$

It is convenient to introduce the two-point correlation for the field

$$\phi^2(x',x) = \phi(x')\phi(x); \qquad (3)$$

an averaging may be introduced when needed. Thus the action S attains the form of the trace of an inner product:

$$S(\phi) = \int d^4x \int d^4x' \; D(x,x')\phi^2(x',x). \qquad (4)$$

For each of the two-point functions, D and ϕ^2, we introduce the local Fourier transforms:

$$D(k,x) = \int d^4s \; \exp(-ik\cdot s) \; D(x+s/2,x-s/2), \qquad (5)$$

and obtain (ignoring 2π factors)

$$S(\phi) = \int d^4x \int d^4k \; D(k,x)\phi^2(k,x). \qquad (6)$$

(Note that the new functions are denoted by the same symbols as the old; their arguments indicate the representation.)

We now consider fields $\phi(x)$ expressible in eikonal form:

$$\phi(x) = \tilde{\phi}(x)\exp i\theta(x) + c.c., \qquad (7)$$

where the amplitude $\tilde{\phi}$ and the gradient of the phase θ are slowly varying. More explicitly, with $x=(\underline{x},t)$ $k=(\underline{k},\omega)$, we may define the local wave-vector and frequency:

$$\underline{k}(\underline{x},t) = \nabla\theta(\underline{x},t); \quad \omega(\underline{x},t) = -\partial\theta/\partial t. \tag{8}$$

The expression (7) is appropriate for a coherent wave. We desire evolution equations for its amplitude and phase.

We use the "slowly-varying" assumption to evaluate the Wigner function (analog of (5)):

$$\phi^2(k,x) = |\tilde{\phi}|^2(\underline{x},t)\ \delta^3(\underline{k} - \nabla\theta(\underline{x},t))\ \delta(\omega + \partial\theta/\partial t). \tag{9}$$

In this approximation, ϕ^2 is singularly concentrated on a "Lagrangian manifold," a 4-dimensional surface embedded in the 8-dimensional phase space $(k,x) = (\underline{k},\omega;\ \underline{x},t)$. Substitution of (9) into (6) now yields

$$S(\tilde{\phi},\theta) = \int dt \int d^3x\ D(\underline{k} = \nabla\theta,\ \omega = -\partial\theta/\partial t;\ \underline{x},t)\ |\tilde{\phi}|^2(\underline{x},t) \tag{10}$$

This has the form of a Lagrangian variational principle. This approach has previously been utilized by Whitham [3] and by Dewar [4].

The Euler-Lagrange equation for the variation of S with respect to the amplitude yields the Hamilton-Jacobi equation for the phase θ:

$$D(\underline{k} = \nabla\theta,\ \omega = -\partial\theta/\partial t;\ \underline{x},t) = 0. \tag{11}$$

The field $J(\underline{x},t)$ conjugate to the phase $\theta(\underline{x},t)$ is defined in the canonical way:

$$J(\underline{x},t) = \frac{\partial D}{\partial \omega}\ |\tilde{\phi}|^2(\underline{x},t), \tag{12}$$

in terms of the Lagrangian density. Variation of S with respect to $\tilde{\phi}$ yields the standard amplitude-transport equation:

$$\partial J/\partial t + \nabla \cdot (J\ \partial\omega/\partial \underline{k}) = 0. \tag{13}$$

The Poisson structure based on the conjugate fields θ, J has been explored elsewhere [5]. Here we concentrate on the evolution of only the phase θ.

The standard method of solution of (11) is due to Hamilton.

The ray equations:

$$dx/d\tau = -\partial D/\partial k, \qquad dk/d\tau = +\partial D/\partial x,$$
$$dt/d\tau = +\partial D/\partial\omega, \qquad d\omega/d\tau = -\partial D/\partial t, \tag{14}$$

are to be solved, subject to initial conditions, as discussed further below. The solution is expressed as $\underline{k}(\underline{x},t)$, $\omega(\underline{x},t)$; the phase is then

$$\theta(\underline{x},t) - \theta(\underline{x}_0,t_0) = \int(\underline{k}\cdot d\underline{x} - \omega\, dt). \tag{15}$$

The path of integration is arbitrary, since $k(x)$ is curl-free, i.e., $k = d\theta(x)$ is an exact one-form.

For purposes of illustration of the boundary-value problem, we shall consider a monochromatic wave in a time-independent two-dimensional medium:

$$\phi(\underline{x},t) = e^{-i\omega_0 t}\,\tilde{\phi}(x,y)\,e^{i\theta(x,y)} + \text{c.c}; \tag{16a}$$
$$k_x = \partial\theta/\partial x, \qquad k_y = \partial\theta/\partial y; \tag{16b}$$
$$D(k_x,k_y,\omega_0;\ x,y) = 0. \tag{16c}$$

The phase space is 4-dimensional, but we need portray only the three-dimensional (x,y,k_x)-space, since k_y is determined by the dispersion relation (16c). Let the phase be specified on some spatial curve, say $y = 0$; knowing $\theta(x,y=0)$ determines $k_x=\partial\theta/\partial x$ and k_y (from (16c)) on the "boundary" $y=0$. Consider the single-valued curve k_x vs x on the surface $y=0$ (Fig. 1); from each point of that curve, construct the corresponding ray, whose initial conditions are known. The family of rays emanating from the curve generate a smooth surface (the Lagrangian manifold), which represents the desired solution $\underline{k}(\underline{x})$. However, the generic behavior of rays of Hamiltonians with two (or more) degrees of freedom is either twisting about each other ("stable") or exponential separation ("unstable"). In either case, the surface develops pleats, causing the wave-vector field $\underline{k}(\underline{x})$ to become multivalued.

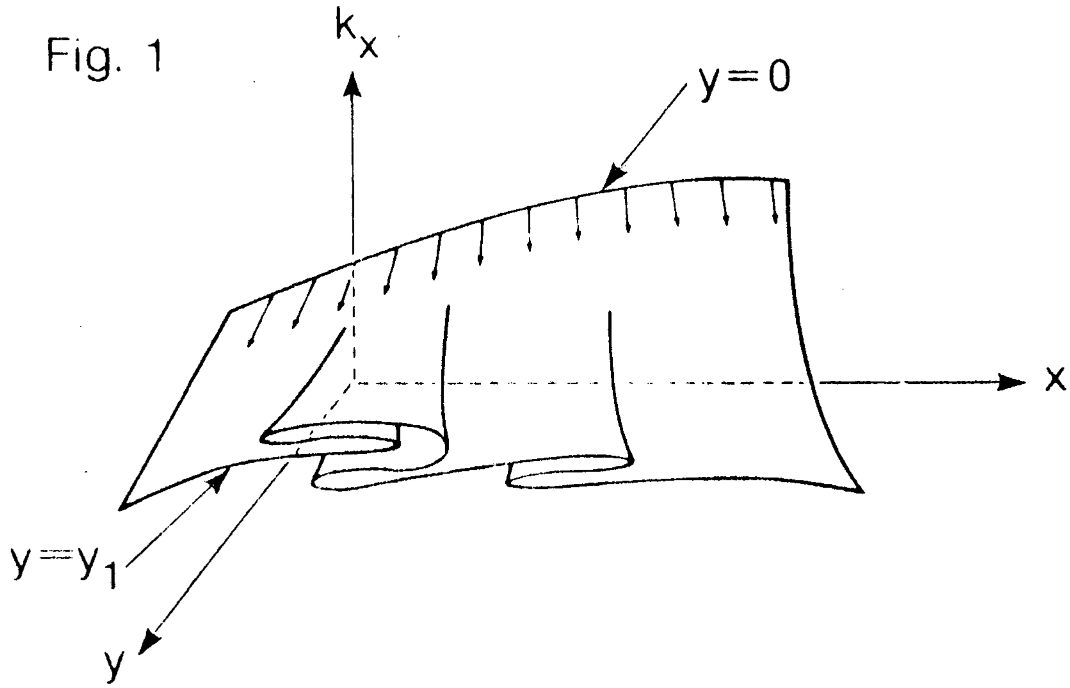

Fig. 1

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When this occurs, we generalize the eikonal representation (16a) to a sum over the several phases at (x,y):

$$\phi(\underline{x},t) = e^{-i\omega_0 t} \sum_j \phi_j(x,y) e^{i\theta_j(x,y)} + \text{c.c.} \tag{17}$$

We wish to study the local spectral density, going beyond the singular approximation (9). Limiting ourselves (again for simplicity) to a spatial field $\phi(\underline{x})$, not necessarily eikonal, we introduce its local Fourier transform:

$$\phi(\underline{k},\underline{x}) = \int_{\underline{s}} e^{-i\underline{k}\cdot\underline{s}}\, \phi(\underline{x}+\underline{s})\, w(\underline{s}), \tag{18}$$

utilizing a window function:

$$w(\underline{s}) = \exp - \underline{s}\,\underline{s} : \underset{\sim}{\sigma}^{-2}/2 \quad , \tag{19}$$

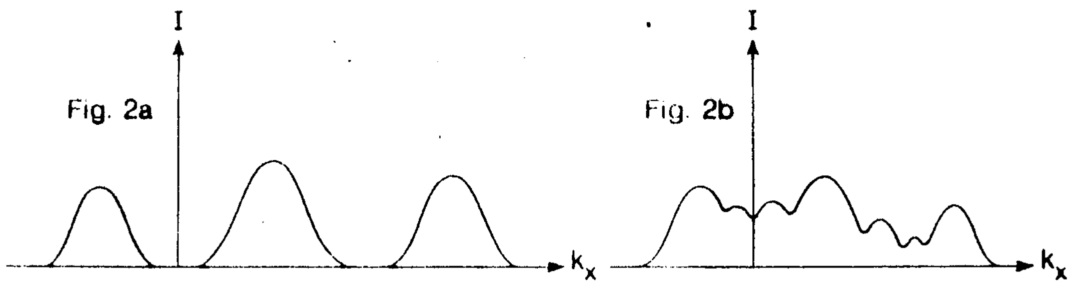
with $\underset{\sim}{\sigma}$ chosen for convenience. We define the local spectral density:

$$I(\underline{k},\underline{x}) = |\phi(\underline{k},\underline{x})|^2; \tag{20}$$

it is the coarse-grained Wigner function, with Gaussian averaging of $\phi^2(\underline{k},\underline{x})$ in both $\underline{k}$ and $\underline{x}$. Its Fourier transform:

$$C(\underline{s},\underline{x}) = \int_{\underline{k}} e^{i\underline{k}\cdot\underline{s}}\, I(\underline{k},\underline{x}) \tag{21}$$

is the coarse-grained field correlation function. When the spectral density changes qualitatively from sharp spectral lines (in $\underline{k}$-space) to a continuous spectrum (Figure 2), then (by Fourier's uncertainty principle) the correlation function changes qualitatively from spatial coherence to spatial incoherence.



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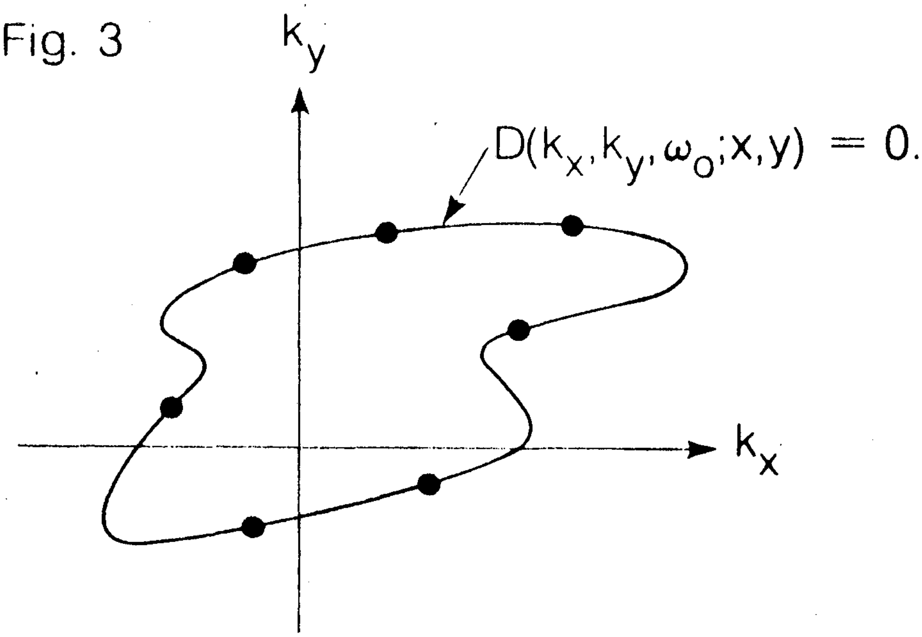
Limiting ourselves now to a single term in the sum (17), we substitute $\phi(\underline{x}) = \tilde{\phi}(\underline{x}) \exp i\theta(\underline{x})$ into (18), expand $\theta(\underline{x}+\underline{s})$ to second order in $\underline{s}$, and ignore the variation of $\tilde{\phi}$. For (20) we then obtain

$$I(\underline{k},\underline{x}) \sim \exp-[\underline{k}-\underline{k}(\underline{x})][\underline{k}-\underline{k}(\underline{x})]:\mathrm{Re}[\underset{\sim}{\sigma}^{-2}-i\nabla\nabla\theta(\underline{x})]^{-1} \quad . \tag{22}$$

Thus the spectral density has a Gaussian spread about the Lagrangian manifold $\underline{k}(\underline{x})$. Since the width of the spectral density still depends on $\underset{\sim}{\sigma}^2$, we minimize it with respect to $\underset{\sim}{\sigma}^2$. The minimizing $\underset{\sim}{\sigma}^2$ is diagonal with respect to the principal axes of $\nabla\nabla\theta(\underline{x})$; its components are $\sigma_\mu^2 = |\theta_{\mu\mu}|^{-1}$, in terms of the diagonal elements $\theta_{\mu\mu}$ of $\nabla\nabla\theta$. Thus a spectral line has a width of order $|\nabla\underline{k}(\underline{x})|^{1/2} \sim (k/L)^{1/2}$, where L is the scale-length for variation of $\underline{k}(\underline{x})$. This is indicated in Figure 2a.

As pleating occurs more spectral lines appear. The several values of $\underline{k}(\underline{x})$ must still satisfy the dispersion relation (11), as indicated on Figure 3. Eventually the lines overlap, as in Figure 2b. The spectrum is then broad, the correlation distance is of the order of the wave length, and the wave may be considered incoherent.

Fig. 3



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