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of Finite Groups

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by

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ABSTRACT OF THE DISSERTATION

Cohomological Invariants

of Finite Groups

by

Vivian J. Bailey

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2017

Professor Alexander Sergee Merkurjev, Chair

This dissertation is concerned with calculating the group of unramified Brauer invariants of a finite group over a field of arbitrary characteristic. We present a formula for the group of degree-two cohomological invariants of a finite group G with coefficients in $\mathbb{Q}/\mathbb{Z}(1)$ over a field F of arbitrary characteristic. We then specialize this formula to the case of decomposable invariants and compute the unramified decomposable cohomological invariants with coefficients in $\mathbb{Q}/\mathbb{Z}(1)$ for various finite groups G. When the order of G is prime to the characteristic of F, we obtain a formula for the degree-two unramified normalized decomposable invariants of finite abelian G and certain nonabelian groups G. We additionally compute the degree-two unramified normalized decomposable invariants with coefficients in $\mathbb{Q}/\mathbb{Z}(1)$ in the case that G is cyclic of order p over a field of characteristic p.

The dissertation of Vivian J. Bailey is approved.

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2017

TABLE OF CONTENTS

1 Introduction **1**

2 Preliminaries **4**

2.1 Torsors 4

2.2 Classifying Varieties 6

2.3 The Groups $H^{d+1}(K, \mathbb{Q}/\mathbb{Z}(d))$ 7

2.4 Galois Cohomology 8

2.5 Cohomological Invariants 10

2.6 Quaternion Algebras 12

3 A Formula for Inv$^2(G, \mathbb{Q}/\mathbb{Z}(1))$ **and Inv**$^1(G, \mathbb{Q}/\mathbb{Z})$ **13**

3.1 Formula and Proof 13

3.1.1 Balanced Elements 13

3.1.2 Computing $H^d(U/G, \mathbb{Q}/\mathbb{Z}(d-1))$ for $d = 1, 2$ 19

3.1.3 Proof of Theorem 23

3.2 Evaluation Map for Degree-Two Invariants 26

3.2.1 Crossed Product for Rings 27

3.2.2 Crossed Product for Schemes 27

3.2.3 Describing the Image of ψ 28

4 Unramified Degree One Invariants of a Finite Group and Decomposable Invariants **30**

4.1 An Example 32

4.2 Decomposable Invariants 34

4.3 Cyclic Algebras 37

5 Degree-Two Decomposable Invariants in Characteristic Prime to $|G|$ . . 41

5.1 Unramified Degree-Two Decomposable Invariants of a Finite Cyclic Group . 41

5.2 $\mathrm{Inv}^2(G, \mathbb{Q}/\mathbb{Z}(1))$ for Finite Abelian G in Characteristic Prime to $|G|$ 48

5.3 Quaternion Group Q_8 50

5.4 Dihedral Group D_8 54

5.5 Dihedral Group D_{16} 57

5.6 Dihedral Group D_{2n} for n Odd 63

6 Degree-Two Invariants in Characteristic Dividing $|G|$ 67

6.1 Case for Cyclic G of Prime Order 67

References 74

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CHAPTER 1

Introduction

Let G be a finite group over a field F. Noether's problem asks whether there exists an embedding

$$\rho : G \to \mathbb{GL}_n(F)$$

such that the subfield $F(X_1, \cdots, X_n)^G$ of $F(X_1, \cdots, X_n)$ fixed by the action of G is a purely transcendental extension of F. To show that Noether's problem has a negative answer for a choice of G and F, it is enough to show that the classifying space BG is not stably rational. In the 1980s, D. Saltman [22] demonstrated that there exist finite G such that Noether's problem has a negative solution over an algebraically closed field of characteristic zero, by finding an example of a group G with BG is not stably rational over F. Since then, other results have been found, particularly in the case when F is algebraically closed or has characteristic prime to the order of G. In particular, BG is known to be stably rational over $\mathbb{Q}$ for groups Q_8, $\mathbb{Z}/8\mathbb{Z}$ [9] and D_8 [15], for $G = D_{16}$ over an arbitrary field, and for finite abelian groups of exponent e when the base field F contains a primitive e^{th} root of unity [8].

In order to study the rationality of BG for a finite group G over a field F, we will compute the degree-two normalized cohomological invariants of G with coefficients in $\mathbb{Q}/\mathbb{Z}(1)$.

Definition. Let G be an algebraic group over a field F. A degree d cohomological invariant with coefficients in $\mathbb{Q}/\mathbb{Z}(d-1)$ is a natural transformation

$$a : H^1(-, G) \longrightarrow H^d(-, \mathbb{Q}/\mathbb{Z}(d-1))$$

where $H^1(-, G)$ and $H^d(-, \mathbb{Q}/\mathbb{Z}(d-1))$ considered as functors from the category **Fields**/F

to **Sets**. The group of degree d cohomological invariants with coefficients in $\mathbb{Q}/\mathbb{Z}(d-1)$ is denoted $\mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z}(d-1))$.

There is an embedding

$$H^d(F, \mathbb{Q}/\mathbb{Z}(d-1)) \hookrightarrow \mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z}(d-1))$$

which sends an element $x \in H^d(F, \mathbb{Q}/\mathbb{Z}(d-1))$ to an invariant $a_x \in \mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z}(d-1))$ defined by setting $a_x(w)$ as the image of x in $H^d(F, \mathbb{Q}/\mathbb{Z}(d-1))$ for any $w \in H^1(K, G)$, for any field K/F. Invariants of the form a_x as above are called *constant* invariants. An invariant b is said to be *normalized* if b sends the distinguished point of $H^1(F, G)$ to the identity element of $H^d(F, \mathbb{Q}/\mathbb{Z}(d-1))$. Any element of $\mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z})$ can be written uniquely as the sum of a constant invariant and a normalized invariant.

A cohomological invariant b of G of degree d over a field F is said to be *unramified* if, for all fields K/F and all G-torsors E over K, the element $b(E) \in H^d(K, \mathbb{Q}/\mathbb{Z}(d-1))$ is unramified with respect to all discrete valuatons of K over F.

Our motivation for determining the group of unramified cohomological invariants of a group G comes from the fact that the existence of a nontrivial normalized unramified cohomological invariant of G is enough to show that the classifying space BG is not stably rational. To show that BG is not stably rational over a field F, it suffices to show that there exists a nontrivial unramified element of $H^d(K, \mathbb{Q}/\mathbb{Z}(d-1))$, where K is the field of rational functions of a classifying variety for G over F. However, because $H^d(K, \mathbb{Q}/\mathbb{Z}(d-1))$ is large, we do this in two steps, as the diagram below indicates: we can first determine $\mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z}(d-1))$ and then determine the unramified invariants in this group, yielding the unramified elements of $H^d(K, \mathbb{Q}/\mathbb{Z}(d-1))$.

$$\begin{array}{ccc} H^d(K, \mathbb{Q}/\mathbb{Z}(d-1))_{\mathrm{nr}} & \hookrightarrow & H^d(K, \mathbb{Q}/\mathbb{Z}(d-1)) \\ \| & & \nearrow \\ \mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z}(d-1))_{\mathrm{nr}} & \hookrightarrow & \mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z}(d-1)) \end{array}$$

The construction of the maps in the diagram above will be discussed in more detail in the following chapter.

In this dissertation, we are interested in computing the group of degree-two unramified cohomological invariants of G in the case when G is a finite group over a field F. The first chapter contains background, including a discussion of G-torsors and the classifying spaces BG for an algebraic group G. Next, we present a formula for the degree-two cohomological invariants of a finite group G over an arbitrary field F. Namely, we will show that

$$\mathrm{Inv}^2(G, \mathbb{Q}/\mathbb{Z}(1)) = H^2(G, F^\times) \oplus \mathrm{Br}(F).$$

In the third chapter, we will introduce decomposable invariants and compute the group

$$\mathrm{Inv}^2(G, \mathbb{Q}/\mathbb{Z}(1))^{\mathrm{nr}}_{\mathrm{norm}}$$

of degree-two unramified normalized decomposable invariants of G in the case when G is a finite cyclic group. In the final chapter, we compute $\mathrm{Inv}^2(G, \mathbb{Q}/\mathbb{Z}(1))^{\mathrm{nr}}_{\mathrm{norm}}$ for specific examples of finite G and F, such as the case when G is the dihedral group D_{16} and F has characteristic different from two.

CHAPTER 2

Preliminaries

In this chapter, we review some of the background that will be used later in this dissertation.

2.1 Torsors

Definition. Let G be an algebraic group over a field F. Then a G-torsor over X is the data of a flat, surjective morphism $\pi : P \to X$ of schemes P and X over F with an action of G on P

$$G \times P \to P$$
$$(g, p) \mapsto g \cdot p$$

such that the map

$$P \times_F G \to P \times_X P$$
$$(p, g) \mapsto (p, g \cdot p)$$

is an isomorphism.

Notice that there is a torsor $P = G \times_F X$ with the action $g \cdot (h, x) = (gh, x)$, called the trivial G-torsor over X. When G is a finite group over F and $X = \mathrm{Spec}(K)$ for a field K/F, the definition above implies that a G-torsor P is of the form $\mathrm{Spec}(L)$ for an étale K-algebra L of dimension $|G|$ over K such that G acts simply transitively on $\mathrm{Hom}_{\text{K-Alg}}(L, K_{\mathrm{sep}})$. Such an L is called a Galois G-algebra over K. So when G is finite, the set of G-torsors over a field K can be identified with the set of Galois G-algebras over K.

In general, for an algebraic group G over a field F, there is an isomorphism of pointed sets from the set of G-torsors over a field K/F and the first Galois cohomology set $H^1(K, G)$

which sends the trivial torsor to the distinguished element of $H^1(K, G)$. Here, $H^1(K, G) = H^1(\Gamma_K, G(K_{\mathrm{sep}}))$ where $\Gamma_K = \mathrm{Gal}(K_{\mathrm{sep}}/K)$ and $G(K_{\mathrm{sep}})$ denotes the K_{sep}-points of G. In the case when G is the algebraic group associated to a finite group, we can describe this isomorphism explicitly following Chapter 18 of [16].

Because G is finite, $G(K_{\mathrm{sep}}) \simeq G$. Since the action of Γ_K on G is trivial, we have

$$H^1(K, G) = \mathrm{Hom}_{\mathrm{cts}}(\Gamma_K, G).$$

Given a continuous homomorphism

$$\phi : \Gamma_K \to G,$$

let $\Gamma' \subset \Gamma$ denote the kernel of ϕ, $H \subset G$ be the image of ϕ and set E to be the subfield of K_{sep} fixed by Γ'; i.e. $E = {K_{\mathrm{sep}}}^{\Gamma'}$. Now, let $L = \mathrm{Map}_H(G, E)$ be the set of functions from G to E satisfying $f(hg) = h \cdot f(g)$ where the action is given by the Galois action of H on E.

$$\begin{array}{c} K^{\mathrm{sep}} \\ \Big|\,{\scriptstyle \Gamma'} \\ E \\ \Big|\,{\scriptstyle H} \\ K \end{array}$$

We will see that L has the structure of a G-Galois algebra over K. There is a K-algebra isomorphism

$$\mathrm{Map}_H(G, E) \xrightarrow{\simeq} E^r \tag{2.1}$$

where $r = [G : H]$. To see this, write G as a coproduct of the cosets of H

$$G = \amalg_{1 \leq i \leq r} g_i H$$

and then the isomorphism (2.1) is given by sending $f \in \mathrm{Map}_H(G, E)$ to $(f(g_1), \cdots, f(g_r)) \in E^r$. This shows that the algebra L has dimension $|G|$ over K. Also, there is a G-action on

L given by $(g \cdot f)(g') = f(g'g)$ for all $g, g' \in G$. To see that L is G-Galois, it remains to see that G acts simply transitively on

$$\mathrm{Hom}_{\text{K-Alg}}(L, K_{\text{sep}}),$$

but by [16, Proposition 18.14], this is equivalent to showing that $L^G = K$. However, if $f \in L$ is fixed by G, then $f(g) = f(1)$ for all $g \in G$ and so f is a constant map; and f fixed by H means that the image of f is fixed by the Galois action of H, and so must lie in K. So $L^G = K$.

This construction gives a correspondence

$$H^1(K, G) \to \text{Galois G-Algebras}$$

which sends $\phi \in H^1(K, G)$ to the Galois G-algebra $\mathrm{Map}_{\mathrm{Im}(\phi)}(G, {K_{\text{sep}}}^{\mathrm{Ker}(\phi)})$. A proof that it is a bijection is given in [16].

2.2 Classifying Varieties

As mentioned in the introduction, we are interested ultimately in determining whether the classifying space BG of a finite group G over a field F is stably rational. Recall that a variety W over a field F is called stably rational if there exist integers $n, m \geq 0$ such that $W \times \mathbb{P}^m_F$ is birational to $\mathbb{P}^n_F$. We will need the following definition from [9, Section 5].

Definition. Let G be an algebraic group over a field F. A *generic* G-torsor is a G-torsor $P \to \mathrm{Spec}(K)$ for some finitely generated extension K/F, such that there exists a smooth, irreducible variety X over F whose function field is K, and a G-torsor $Q \to X$ satisfying the following properties:

1. P is the fiber of Q a the generic point $\mathrm{Spec}(K) \to X$

$$\begin{array}{ccc} P & \longrightarrow & Q \\ \downarrow & & \downarrow \\ \mathrm{Spec}(K) & \longrightarrow & X \end{array}$$

2. For every extension k/F where k is infinite, every G-torsor $T \to \mathrm{Spec}(k)$, and every nonempty subvariety U of X there is some $x \in U(k)$ whose fiber Q_x is isomorphic to T.

The G-torsor $Q \to X$ above is called a *versal* G-torsor.

From this definition, we see that G-torsors over K are parametrized by K-points of X. Let V be a faithful representation of G and $U \subset V$ is a G-equivariant subset of V such that $U \to U/G$ is a G-torsor. Then $U \to U/G$ is a versal G-torsor. The variety U/G can be thought of as an approximation of BG. Since the "no-name lemma" [3] says that given two such representations V_1 and V_2 with G-equivariant subsets $U_1 \subset V_1$ and $U_2 \subset V_2$, then U_1/G is stably rational if and only if U_2/G is. So we say that BG is stably rational if some U/G as above is stably rational.

2.3 The Groups $H^{d+1}(K, \mathbb{Q}/\mathbb{Z}(d))$

As mentioned in the introduction, we are interested in computing the degree-two cohomological invariants of a finite group G whose coefficients lie in groups $H^2(K, \mathbb{Q}/\mathbb{Z}(1))$, for fields K/F. We follow the description of the groups $H^{d+1}(K, \mathbb{Q}/\mathbb{Z}(d))$ for $d > 0$ given in [9, Appendix A].

When the characteristic of K is different than p, we define

$$\mathbb{Z}/p^m\mathbb{Z}(d) = (\mu_{p^m})^{\otimes d}$$
$$\mathbb{Q}_p/\mathbb{Z}_p(d) = \lim_{\rightarrow} \mathbb{Z}/p^m\mathbb{Z}(d)$$

and $H^{d+1}(K, \mathbb{Q}/\mathbb{Z}(d))$ is defined as

$$H^{d+1}(K, \mathbb{Q}/\mathbb{Z}(d)) = \amalg_{p \text{ prime}} H^{d+1}(K, \mathbb{Q}_p/\mathbb{Z}_p(d)) = \lim_{\rightarrow} \amalg_{p \text{ prime}} H^{d+1}(K, \mathbb{Z}/p^m\mathbb{Z}(d))$$

In particular,

$$H^1(K, \mathbb{Q}_p/\mathbb{Z}_p(0)) = \mathrm{Hom}_{\mathrm{cts}}(\mathrm{Gal}(K_{\mathrm{sep}}/K), \mathbb{Q}_p/\mathbb{Z}_p)$$
$$H^2(K, \mathbb{Q}_p/\mathbb{Z}_p(1)) = \lim_{\rightarrow} H^2(K, \mu_{p^m}) =_p \mathrm{Br}(K)$$

In the case when K has characteristic p, we follow [9] and define

$$H^{d+1}(K, \mathbb{Q}_p/\mathbb{Z}_p(d)) = H^2(K, K_d(K_{\mathrm{sep}}))\{p\}$$

where $K_d(K_{\mathrm{sep}})$ is the Milnor K-group of K_{sep}. In this case, note that we have

$$H^1(K, \mathbb{Q}_p/\mathbb{Z}_p(0)) = H^2(K, \mathbb{Z})\{p\} = \mathrm{Hom}_{\mathrm{cts}}(\mathrm{Gal}(K_{\mathrm{sep}}/K), \mathbb{Q}_p/\mathbb{Z}_p)$$
$$H^2(K, \mathbb{Q}_p/\mathbb{Z}_p(1)) = H^2(K, K_{\mathrm{sep}}^{\times})\{p\} =_p \mathrm{Br}(K)$$

in degrees one and two.

2.4 Galois Cohomology

Let G be a finite group over a field F, K/F a field extension and let L/K be a Galois G-algebra. Then for any discrete valuation v on K trivial on F, there exist maps on étale cohomology

$$\alpha_v : H^d_{ét}(\mathrm{Spec}(\mathcal{O}_v), \mathbb{Q}/\mathbb{Z}(d-1)) \rightarrow H^d_{ét}(\mathrm{Spec}(K), \mathbb{Q}/\mathbb{Z}(d-1))$$

induced by the inclusion $\mathcal{O}_v \to K$. Since the étale cohomology group

$$H^d_{ét}(\mathrm{Spec}(K), \mathbb{Q}/\mathbb{Z}(d-1))$$

coincides with the Galois Cohomology group $H^d(K, \mathbb{Q}/\mathbb{Z}(d-1))$, α_v gives a map

$$\alpha_v : H^d_{ét}(\mathrm{Spec}(\mathcal{O}_v), \mathbb{Q}/\mathbb{Z}(d-1)) \to H^d(K, \mathbb{Q}/\mathbb{Z}(d-1)).$$

An element $x \in H^d(K, \mathbb{Q}/\mathbb{Z}(d-1))$ is called *unramified* if for every discrete valuation v on K trivial on F, x is in the image of the map α_v.

In the case when the characteristic of F is zero, for any discrete valuation v on K trivial on F, the maps α_v as above fit into an exact sequence

$$\cdots \to H^d_{ét}(\mathrm{Spec}(\mathcal{O}_v), \mathbb{Q}/\mathbb{Z}(d-1)) \stackrel{\alpha_v}{\to} H^d(K, \mathbb{Q}/\mathbb{Z}(d-1)) \stackrel{\partial_v}{\to} H^{d-1}(\kappa_v, \mathbb{Q}/\mathbb{Z}(d-2)) \to \cdots \tag{2.2}$$

where κ_v is the residue field of the valuation v on K. The maps

$$\partial_v : H^d(K, \mathbb{Q}/\mathbb{Z}(d-1)) \to H^{d-1}(\kappa_v, \mathbb{Q}/\mathbb{Z}(d-2))$$

are called residue maps. In this case, an equivalent condition for $x \in H^d(K, \mathbb{Q}/\mathbb{Z}(d-1))$ to be unramified is for x to be in the kernel of the residue map ∂_v for each discrete valuation v on K trivial on F.More generally if F has characteristic p, we define

$$\mathbb{Q}/\mathbb{Z}(d)' = \amalg_{\text{prime } q \neq p} \mathbb{Q}_q/\mathbb{Z}_q(d)$$

and we have maps

$$\partial'_v : H^d(K, \mathbb{Q}/\mathbb{Z}(d-1)') \to H^{d-1}(\kappa_v, \mathbb{Q}/\mathbb{Z}(d-2)')$$

defined outside of characteristic p which fit into the exact sequence below.

$$\cdots \to H^d_{\acute{e}t}(\mathrm{Spec}(\mathcal{O}_v), \mathbb{Q}/\mathbb{Z}(d-1)') \xrightarrow{\alpha_v} H^d(K, \mathbb{Q}/\mathbb{Z}(d-1)') \xrightarrow{\partial_v} H^{d-1}(\kappa_v, \mathbb{Q}/\mathbb{Z}(d-2)') \to \cdots \tag{2.3}$$

In order to show that an element $x \in H^d(K, \mathbb{Q}/\mathbb{Z}(d-1))$ is ramified, it is enough to show that it is not in the kernel of some ∂'_v.

2.5 Cohomological Invariants

As mentioned in the introduction, our main objects of study are cohomological invariants. Let F be a field, let G be a finite group over F, and let C be a $\mathrm{Gal}(F_{\mathrm{sep}}/F)$-module. Then a cohomological invariant with coefficients in C is a natural transformation of functors

$$a : H^1(-, G) \to H^d(-, C)$$

where the Galois Cohomology functors $H^1(-, G)$ and $H^d(-, C)$ are considered as functors from Fields$/F$ to **Sets**. In this case, d is called the degree of a. In this dissertation, we will be primarily interested in the case when C is $\mathbb{Q}/\mathbb{Z}(d-1)$ as described above, for $d = 1, 2$. We write

$$\mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z}(d-1))$$

for the group of cohomological invariants of G of degree d with coefficients in $\mathbb{Q}/\mathbb{Z}(d-1)$. If an invariant $a \in \mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z}(d-1))$ sends the trivial G-torsor in $H^1(F, G)$ to zero in $H^d(F, \mathbb{Q}/\mathbb{Z}(d-1))$, then a is called *normalized*. A cohomological invariant $a \in \mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z}(d-1))$ is called *unramified* if for all fields K/F and for all $L/K \in H^1(K, G)$, $a(L/K) \in H^d(K, \mathbb{Q}/\mathbb{Z}(d-1))$ is unramified. The group of normalized unramified cohomological invariants of G of degree d with coefficients in $\mathbb{Q}/\mathbb{Z}(d-1)$ is denoted

$$\mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z}(d-1))^{\mathrm{norm}}_{\mathrm{nr}}.$$

Let

$$U \to U/G$$

be a versal G-torsor as discussed in section 2.2, and let K/F be function field $F(U/G)$. Then the pullback along the generic point $\mathrm{Spec}(K) \to U/G$ gives

$$\begin{array}{ccc} E & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathrm{Spec}(K) & \longrightarrow & U/G \end{array}$$

a G-torsor $E \to \mathrm{Spec}(K)$. The map

$$f : \mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z}(d-1))^{\mathrm{norm}} \hookrightarrow H^d(K, \mathbb{Q}/\mathbb{Z}(d-1))$$

given by evaluation of an invariant a at the generic torsor in $H^1(F, G)$ gives an injection by [9, Part II, Theorem 3.3]. This map induces the following diagram.

$$\begin{array}{ccc} \mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z}(d-1))^{\mathrm{norm}} & \hookrightarrow & H^d(K, \mathbb{Q}/\mathbb{Z}(d-1)) \\ \uparrow & & \uparrow \\ \mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z}(d-1))^{\mathrm{norm}}_{\mathrm{nr}} & \hookrightarrow & H^d(K, \mathbb{Q}/\mathbb{Z}(d-1))_{\mathrm{nr}} \end{array}$$

To show that BG is not stably rational, it suffices to show that there exists a nontrivial unramified element of $H^d(K, \mathbb{Q}/\mathbb{Z}(d-1))$. However, because $H^d(K, \mathbb{Q}/\mathbb{Z}(d-1))$ is usually a large group, it is useful to examine the smaller subgroup $\mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z}(d-1))$ for unramified elements.

$$\begin{array}{ccccc} H^d(K, \mathbb{Q}/\mathbb{Z}(d-1))_{\mathrm{nr}} & & \hookrightarrow & & H^d(K, \mathbb{Q}/\mathbb{Z}(d-1)) \\ \| & & & \nearrow & \\ \mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z}(d-1))_{\mathrm{nr}} & \hookrightarrow & \mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z}(d-1)) & & \end{array}$$

So the existence of a nontrivial normalized unramified cohomological invariant for G over F of degree d with coefficients in $\mathbb{Q}/\mathbb{Z}(d-1)$ is enough to show that the classifying space BG is not stably rational over F, which motivates our interest in determining the group of unramfied cohomological invariants for various finite G. In this dissertation we consider invariants of degree $d = 1, 2$ with coefficients in $\mathbb{Q}/\mathbb{Z}(d-1)$ and proceed in two steps: first, we compute the invariants $\mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z}(d-1))$, and second, we then determine the subgroup of unramified invariants $\mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z}(d-1))_{\mathrm{nr}} \subset \mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z}(d-1))$.

2.6 Quaternion Algebras

Later in this dissertation when determining $\mathrm{Inv}^2_{\mathrm{dec}}(G, \mathbb{Q}/\mathbb{Z}(1))$ for certain nonabelian groups, we will need criteria for a quaternion algebra over a field F to split. In this discussion we follow [10, Section 1.1] and [16, Section 2].

Definition. Let F be a field. A *quaternion algebra* over F is a central simple algebra of dimension 4 over F. Two examples of quaternion algebras are real Hamiltonians, $(-1,-1)$ over $\mathbb{R}$, and the matrix algebra $M_2(F)$ over a field F, for any field F.

When the characteristic of F is different from 2, a quaternion algebra Q over F has an F-basis $\langle 1, i, j, k\rangle$ and is determined by the relations

$$\begin{aligned} i^2 &= a \\ j^2 &= b \\ ij = -ji &= k \end{aligned} \tag{2.4}$$

for some $a, b \in F^\times$. In this case, we write $Q = (a, b)$. When F has characteristic two, the above relations no longer give a central simple algebra since in this case, $ij = -ji = ji$. However, a quaternion algebra Q still has a basis $\langle 1, u, v, w\rangle$, and satisfies the relations

$$\begin{aligned} u^2 + u &= a \\ v^2 &= b \\ uv = vu + v &= w \end{aligned} \tag{2.5}$$

and in this case we write $Q = [a, b)$.

Definition. Let A be a central simple algebra over a field F. Then A is said to be split if $A \simeq M_n(F)$ for some n. If E/F is a field extension, A is said to split over E if $A \otimes_F E$ is a split E-algebra.

From Proposition 1.1.7 of [10], we have that a quaternion algebra (a, b) over a field F of characteristic different from 2 is split if and only if b is the field norm $N_{F(\sqrt{a})/F}(u)$ for some $u \in F(\sqrt{a})$. When $\mathrm{char}(F) = 2$, $Q = [a, b)$ is split if and only if $b = N_{F(\alpha)/F}(w)$ for some $w \in F(\alpha)$, where α is a root of the equation $x^2 + x = a$.

CHAPTER 3

A Formula for $\mathbf{Inv}^2(G, \mathbb{Q}/\mathbb{Z}(1))$ and $\mathbf{Inv}^1(G, \mathbb{Q}/\mathbb{Z})$

In this chapter, we present formulas for the degree-one and degree-two cohomological invariants of a finite group G over an arbitrary field F with coefficients in $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{Q}/\mathbb{Z}(1)$, respectively.

3.1 Formula and Proof

Our objective is to prove the following theorem.

Theorem. *Let G be a finite group over a field F. Then,*

(a) $Inv^1(G, \mathbb{Q}/\mathbb{Z}(0)) = H^2(G, \mathbb{Z}) \oplus H^1(F, \mathbb{Q}/\mathbb{Z}(0))$. Thus, $Inv^1(G, \mathbb{Q}/\mathbb{Z}(0))_{norm} = H^2(G, \mathbb{Z})$.

(b) $Inv^2(G, \mathbb{Q}/\mathbb{Z}(1)) = H^2(G, F^\times) \oplus H^2(F, \mathbb{Q}/\mathbb{Z}(1))$. Thus, $Inv(G, Br)_{norm} = H^2(G, F^\times)$.

3.1.1 Balanced Elements

Throughout this chapter, let G be a finite group over a field F, and let V be a faithful representation of G over F. Set U to be a G-equivariant open subset of V such that $\mathrm{codim}_V(V - U) \geq 2$ and $U \to U/G$ is a G-torsor. As discussed in the previous chapter,

$$U \to U/G$$

is a classifying G-torsor, and U/G can be thought of as an approximation of the classifying space BG. To prove the above theorem, we first show that for $d = 1, 2$, $\mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z}(d-1))$ can be expressed as a subgroup of $H^d(U/G, \mathbb{Q}/\mathbb{Z}(d-1))$ consisting of the so-called balanced elements. Let p_1 and p_2 denote the projections

$$\begin{array}{ccc} (U \times U) & \xrightarrow{p_1} & U/G \\ {\scriptstyle p_2}\downarrow & & \\ U/G & & \end{array}$$

so that for any functor $\mathcal{F} : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Ab}$, we have the following maps.

$$\begin{array}{ccc} & & \mathcal{F}(U/G) \\ & & \downarrow{\scriptstyle p_1^*} \\ \mathcal{F}(U/G) & \xrightarrow[p_2^*]{} & \mathcal{F}((U \times U)/G) \end{array}$$

The kernel of $p_1 - p_2$ is called the the group of balanced elements of $\mathcal{F}(U/G)$, denoted $\mathcal{F}(U/G)_{\mathrm{bal}}$. We have the following lemma.

Lemma 1. *Let G be a finite group over a field F and let U, V be as above. Then for any functor $\mathcal{F} : \boldsymbol{Sch}^{op} \to \boldsymbol{Ab}$,*

$$Inv(G, \mathcal{F}) = \mathcal{F}(U/G)_{bal}$$

In particular, when $\mathcal{F} = H^d(-, \mathbb{Q}/\mathbb{Z}(d-1))$, we have

$$\mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z}(d-1)) = H^d(U/G, \mathbb{Q}/\mathbb{Z}(d-1))_{\mathrm{bal}}.$$

Proof. According to [2, Theorem 3.4], there is an isomorphism

$$\mathrm{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j)) \simeq H^0_{\mathrm{Zar}}(U/G, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))_{\mathrm{bal}}$$

where $\mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))$ is the Zariski sheaf associated with the presheaf

$$W \mapsto H^n(W, \mathbb{Q}/\mathbb{Z}(j)).$$

Let X be an irreducible smooth variety over F. Note that from the Leray spectral sequence

$$E_2^{p,q} = H^p(X, R^q j_*(\mathbb{Z})) \Longrightarrow H^{p+q}(F(X), \mathbb{Z})$$

we have that $H^1(X, \mathbb{Z}) = 0$ and $H^2(X, \mathbb{Z})$ is torsion, where

$$j : \text{Spec}(F(X)) \longrightarrow X$$

is the inclusion of the generic point. Indeed, the spectral sequence gives the exact sequence

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1 \longrightarrow E_2^{0,1} \longrightarrow E_2^{2,0} \longrightarrow H^2$$

Since $j_*(\mathbb{Z}) = \mathbb{Z}_X$, the first three terms then give the exact sequence

$$0 \longrightarrow H^1(X, (\mathbb{Z})) \longrightarrow H^1(F(X), \mathbb{Z})$$

The Galois cohomology group $H^1 = H^1(F(X), \mathbb{Z}) = 0$, as there are no nontrivial continuous homomorphisms from a profinite group into $\mathbb{Z}$. Thus,

$$H^1(X, \mathbb{Z}) = 0. \tag{3.1}$$

We also then get the exact sequence

$$0 \longrightarrow E_2^{0,1} \longrightarrow E_2^{2,0} \longrightarrow H^2$$

which yields

$$0 \longrightarrow H^0(X, R^1(\mathbb{Z})) \longrightarrow H^2(X, \mathbb{Z}) \longrightarrow H^2(F(X), \mathbb{Z})$$

Since $R^1(\mathbb{Z})$ is the Zariski sheaf associated with $W \mapsto H^1(W, \mathbb{Z})$ and $H^1(W, \mathbb{Z})$ is trivial for each W by the above, we have an injection

$$H^2(X, \mathbb{Z}) \hookrightarrow H^2(F(X), \mathbb{Z}).$$

Since the Galois cohomology group $H^2(F(X), \mathbb{Z})$ is torsion, we have $H^2(X, \mathbb{Z})$ is also torsion.

Now, write $\mathrm{Ch}(X)$ for $H^2(X, \mathbb{Z})$ and $\mathrm{Br}(X)$ for $H^2(X, \mathbb{G}_m)$. Let $\mathcal{C}h$ denote the Zariski sheaf associated to the presheaf $W \mapsto \mathrm{Ch}(W)$ and let $\mathcal{B}r$ denote the Zariski sheaf associated to $W \mapsto \mathrm{Br}(W)$. We will now see that $\mathcal{C}h = \mathcal{H}^1(\mathbb{Q}/\mathbb{Z}(0))$, and that $\mathcal{B}r = \mathcal{H}^2(\mathbb{Q}/\mathbb{Z}(1))$, where $\mathcal{H}^1(\mathbb{Q}/\mathbb{Z}(0))$ and $\mathcal{H}^2(\mathbb{Q}/\mathbb{Z}(1))$ are the Zariski sheaves associated to $W \mapsto H^1(W, \mathbb{Q}/\mathbb{Z}(0))$ and $W \mapsto H^2(W, \mathbb{Q}/\mathbb{Z}(1))$, respectively.

Let $\mathbb{Z}(0)$ and $\mathbb{Q}(0)$ denote the motivic complexes with $\mathbb{Z}$ and $\mathbb{Q}$, respectively, in degree zero, and zero in other degrees. Let $\mathbb{Z}(1) = \mathbb{G}_m[-1]$, and $\mathbb{Q}(1) = \mathbb{Q} \otimes \mathbb{G}_m[-1]$ and let $\mathbb{Q}/\mathbb{Z}(i-1)$ be determined by the exact triangle of motivic complexes for $i = 1, 2$ below, in the derived category of sheaves in étale cohomology.

$$\mathbb{Z}(i-1) \longrightarrow \mathbb{Q} \otimes \mathbb{Z}(i-1) \longrightarrow \mathbb{Q}/\mathbb{Z}(i-1) \longrightarrow \mathbb{Z}(i-1)[1] \tag{3.2}$$

In the case $i = 1$, we have the exact triangle

$$\mathbb{Z}(0) \longrightarrow \mathbb{Q} \otimes \mathbb{Z}(0) \longrightarrow \mathbb{Q}/\mathbb{Z}(0) \longrightarrow \mathbb{Z}(0)[1] \tag{3.3}$$

For each $W \to X$ etale, the exact triangle (3.3) yields the exact sequence of abelian groups

$$H^1(W, \mathbb{Z}) \otimes \mathbb{Q} \longrightarrow H^1(W, \mathbb{Q}/\mathbb{Z}(0)) \longrightarrow H^2(W, \mathbb{Z}) \longrightarrow H^2(W, \mathbb{Z}) \otimes \mathbb{Q}$$

However, $H^1(W, \mathbb{Z}) = 0$ and $H^2(X, \mathbb{Z})$ is torsion by (3.1) above, and so $H^2(W, \mathbb{Z}) \otimes \mathbb{Q} = 0$. Thus $H^1(W, \mathbb{Q}/\mathbb{Z}(0)) \simeq H^2(W, \mathbb{Z})$ for each open W, and so $\mathcal{C}h \simeq \mathcal{H}^1(\mathbb{Q}/\mathbb{Z}(0))$.

In the case $i = 2$, for each $W \to X$ etale, (3.3) gives the exact sequence

$$H^2(W, \mathbb{Z}(1)) \longrightarrow H^2(W, \mathbb{Q}(1)) \longrightarrow H^2(W, \mathbb{Q}/\mathbb{Z}(1)) \longrightarrow H^3(W, \mathbb{Z}(1)) \longrightarrow H^3(W, \mathbb{Q}(1))$$

Since $\mathbb{Z}(1)$ is the complex with $\mathbb{G}_m$ in degree -1 and zero in other degrees, we then have the exact sequence

$$H^1(W, \mathbb{G}_m) \longrightarrow H^1(W, \mathbb{G}_m) \otimes \mathbb{Q} \longrightarrow H^2(W, \mathbb{Q}/\mathbb{Z}(1)) \longrightarrow H^2(W, \mathbb{G}_m) \longrightarrow H^2(W, \mathbb{G}_m) \otimes \mathbb{Q}$$

which we can rewrite as

$$\mathrm{Pic}(W) \longrightarrow \mathrm{Pic}(W) \otimes \mathbb{Q} \longrightarrow H^2(W, \mathbb{Q}/\mathbb{Z}(1)) \longrightarrow \mathrm{Br}(W) \longrightarrow \mathrm{Br}(W) \otimes \mathbb{Q}$$

Since $\mathrm{Br}(W)$ injects into the torsion group $\mathrm{Br}(F(W))$, $\mathrm{Br}(W)$ is torsion, and so $\mathrm{Br}(W) \otimes \mathbb{Q} = 0$. We thus have an exact sequence of presheaves

$$\mathrm{Pic}(-) \otimes \mathbb{Q} \longrightarrow H^2(-, \mathbb{Q}/\mathbb{Z}(1)) \longrightarrow \mathrm{Br}(-) \longrightarrow 0$$

After sheafification, we get an exact sequence of the associated Zariski sheaves. However, since line bundles are locally trivial, the Zariski sheaf associated to the presheaf $W \mapsto \mathrm{Pic}(W)$ is trivial, and so $\mathcal{B}r \simeq \mathcal{H}^2(\mathbb{Q}/\mathbb{Z}(1))$.

Now, we will apply the Leray spectral sequence

$$H^p_{\mathrm{Zar}}(X, R^q\alpha_*(\mathcal{F})) \Longrightarrow H^{p+q}_{et}(X, \mathcal{F}),$$

where α is given by sheafification, to show that $H^0_{\mathrm{Zar}}(X, \mathcal{C}h) = \mathrm{Ch}(X)$ and $H^0_{\mathrm{Zar}}(X, \mathcal{B}r) =$

$\mathrm{Br}(X)$. Thus, we will have $\mathrm{Inv}(G, \mathrm{Ch}) = H^0_{\mathrm{Zar}}(U/G, \mathcal{C}h)_{\mathrm{bal}} = \mathrm{Ch}(U/G)_{\mathrm{bal}}$ and $\mathrm{Inv}(G, \mathrm{Br}) = H^0_{\mathrm{Zar}}(U/G, \mathcal{B}r)_{\mathrm{bal}} = \mathrm{Br}(U/G)_{\mathrm{bal}}$. We treat the two cases together, letting $\mathcal{G}_1 = \mathbb{Z}$, $\mathcal{G}_2 = \mathbb{G}_m$.

Consider the Leray spectral sequence

$$E_2^{p,q} = H^p_{\mathrm{Zar}}(X, R^q\alpha_*(\mathcal{G}_i)) \Longrightarrow H^{p+q}_{et}(X, \mathcal{G}_i),$$

for $i = 1, 2$. Since $\mathrm{Ch}(X) = H^2(X, \mathcal{G}_1)$ and $\mathrm{Br}(X) = H^2(X, \mathcal{G}_2)$, we are interested in terms of total degree $p + q = 2$.

- $R^0\alpha_*(\mathcal{G}_i)$ is the Zariski sheaf associated to the presheaf $W \mapsto H^0_{et}(W, \mathcal{G}_i) = \mathcal{G}_i(W)$, which for $i = 1, 2$, is just the Zariski sheaf $\mathcal{G}_i$. Since we have the flasque, and thus acyclic resolutions of sheaves on X for $\mathcal{G}_i$, $i = 1, 2$

$$0 \longrightarrow \mathbb{Z} = \mathcal{G}_1 \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0 \tag{3.4}$$

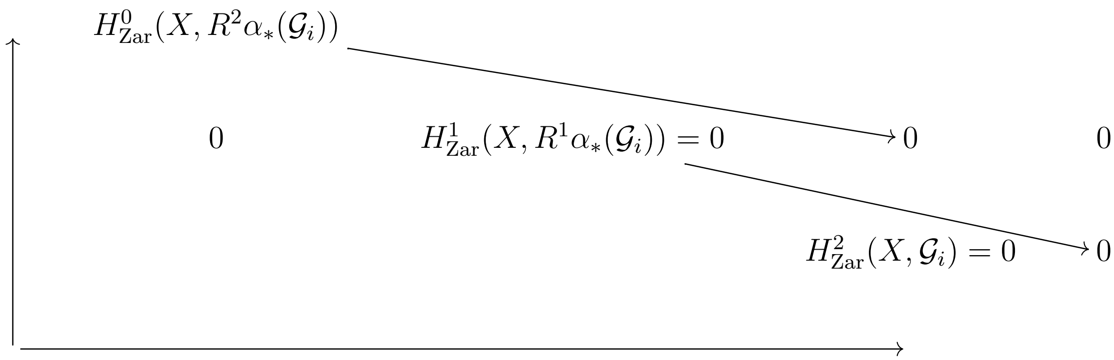
$$0 \to \mathbb{G}_m = \mathcal{G}_2 \to i_*\mathbb{G}_{m,\mathrm{Spec}(K)} \to \mathrm{Div}_X \to 0 \tag{3.5}$$

  $H^j_{\mathrm{Zar}}(X, \mathcal{G}_i) = 0$ for $j \geq 1$. Thus, $E_2^{p,0} = 0$ for $p \geq 1$ in both the above spectral sequences.

- $R^1\alpha_*(\mathcal{G}_i)$ is the Zariski sheaf associated to $W \mapsto H^1_{et}(W, \mathcal{G}_i)$. As in (3.1), each $H^1_{et}(W, \mathcal{G}_1) = 0$, and so $R^1\alpha_*(\mathcal{G}_1) = 0$. $R^1\alpha_*(\mathcal{G}_2)$ is the Zariski sheaf associated with the presheaf $W \mapsto \mathrm{Pic}(W)$. Since line bundles have a local trivialization, the associated sheaf is trivial. So $H^1_{\mathrm{Zar}}(X, R^1\alpha_*(\mathbb{G}_m)) = 0$. In particular, $E_2^{0,1} = E_2^{1,1} = 0$ for both spectral sequences $H^p_{\mathrm{Zar}}(X, R^q\alpha_*(\mathcal{G}_i)) \Longrightarrow H^{p+q}_{et}(X, \mathcal{G}_i)$, $i = 1, 2$.

- $R^2\alpha_*(\mathcal{G}_i)$ is the Zariski sheaf associated with $W \mapsto H^2_{et}(W, \mathcal{G}_i) = \mathcal{F}_i(W)$. $R^2\alpha_*(\mathcal{G}_1) = R^2\alpha_*(\mathbb{Z})$ is denoted $\mathcal{C}h$, and $R^2\alpha_*(\mathcal{G}_2) = R^2\alpha_*(\mathbb{G}_m)$ is denoted $\mathcal{B}r$.

We can thus write the second sheet of the spectral sequences



and we have $R^2\alpha_*(\mathcal{G}_i)(X) = H^1_{et}(X, \mathcal{G}_i) = H^0_{\text{Zar}}(X, R^2\alpha_*(\mathcal{G}_i))$ $i = 1, 2$. Notice that any open subset W of X is also a smooth, irreducible variety over F, and so the sections of $\mathcal{F}_i(-)$ coincide with those of its sheafification $R^2\alpha_*(\mathcal{G}_i)(-)$, and so $\mathcal{F}_i(-) = R^2\alpha_*(\mathcal{G}_i)(-)$. Combining this with our above result,

$$\text{Inv}(G, \mathcal{F}_i) = H^0_{\text{Zar}}(U/G, R^2\alpha_*(\mathcal{G}_i))_{\text{bal}} = \mathcal{F}_i(U/G)_{\text{bal}}.$$

□

3.1.2 Computing $H^d(U/G, \mathbb{Q}/\mathbb{Z}(d-1))$ for $d = 1, 2$

Now we have reduced the problem to computing the balanced elements of $H^d(U/G, \mathbb{Q}/\mathbb{Z}(d-1))$ for $d = 1, 2$. The following lemma gives us relationships between $H^1(U/G, \mathbb{Q}/\mathbb{Z}(0))$ and $H^2(G, \mathbb{Z})$ and $H^2(U/G, \mathbb{Q}/\mathbb{Z}(1))$ and $H^2(G, F^\times)$.

Lemma 2. *Let G be a finite group over a field F, let V be a faithful representation of G over $F$and let U to be a G-equivariant open subset of V such that codim$_V(V - U) \geq 2$ and*

$U \to U/G$ is a G-torsor, as before. Then, there are exact sequences

$$\begin{aligned} 0 \longrightarrow H(G, \mathbb{Z}) \longrightarrow Ch(U/G)_{bal} \longrightarrow Ch(U)_{bal} \\ 0 \longrightarrow H(G, F^{\times}) \longrightarrow Br(U/G)_{bal} \longrightarrow Br(U)_{bal} \end{aligned} \tag{3.6}$$

where for a variety X over F, $Ch(X) = H^2(X, \mathbb{Q}/\mathbb{Z}(0))$ and $Br(X) = H^2(X, \mathbb{Q}/\mathbb{Z}(1))$.

Proof. Since the G-torsor $g : U \longrightarrow U/G$ is a finite Galois covering with Galois group G, we have the Hochschild-Serre spectral sequence

$$H^p(G, H^q(U, \mathcal{F}_i)) \Longrightarrow H^{p+q}(U/G, \mathcal{F}_i)$$

for $\mathcal{F}_1 = \mathbb{Z}$ and $\mathcal{F}_2 = \mathbb{G}_m$.

In the case $i = 1$ we get the exact sequence

$$H^1(U, \mathbb{Z})^G \to H^2(G, \mathbb{Z}(U)) \to \ker(H^2(U/G, \mathbb{Z}) \longrightarrow H^2(U, \mathbb{Z}))$$

Since the codimension of U in $V \simeq \mathbb{A}^n$ is atleast 2, U is connected, and so $\mathbb{Z}(U) = \mathbb{Z}$. Also, $H^1(U, \mathbb{Z}) = 0$ as in (3.1), and so we have the exact sequence

$$0 \longrightarrow H^2(G, \mathbb{Z}) \to H^2(U/G, \mathbb{Z}) \longrightarrow H^2(U, \mathbb{Z}) \tag{3.7}$$

Analogously, for $\mathcal{F}_2 = \mathbb{G}_m$, the Hochschild-Serre spectral sequence yields the exact sequence below.

$$\mathrm{Pic}(U)^G \to H^2(G, F[U]^{\times}) \to \mathrm{Br}(U/(U/G)) \to H^1(G, \mathrm{Pic}(U))$$

Because $V \simeq \mathbb{A}^n$ for some n, and since U is an open subset of V, we have $\mathrm{Pic}(U) \simeq \mathrm{Pic}(V) \simeq 0$. Thus we an exact sequence

$$0 \to H^2(G, F[U]^{\times}) \to \mathrm{Br}(U/G) \longrightarrow \mathrm{Br}(U) \tag{3.8}$$

Since U is regular, the Cartier Divisors are Weil Divisors and we have $\mathrm{Div}_U(U) = \amalg_{x \in U_{(1)}} \mathbb{Z}$. Recall that the codimension of the complement of U in V is at least 2, and so $\amalg_{x \in U^{(1)}} \mathbb{Z} = \amalg_{x \in V^{(1)}} \mathbb{Z}$. So from (3.5) we get the two exact sequences below.

$$\begin{array}{ccccccc} 0 & \longrightarrow & F[U]^\times & \longrightarrow & F(U)^\times & \longrightarrow & \amalg_{x \in U^{(1)}} \mathbb{Z} \\ & & & & \| & & \| \\ 0 & \longrightarrow & F[V]^\times & \longrightarrow & F(V)^\times & \longrightarrow & \amalg_{x \in V^{(1)}} \mathbb{Z} \end{array}$$

Comparing them, we see that $F[U]^\times = F[V]^\times = F^\times$, and so we have the exact sequence:

$$0 \to H^2(G, F^\times) \to \mathrm{Br}(U/G) \longrightarrow \mathrm{Br}(U) \tag{3.9}$$

Now we would like exact sequences similar to (3.7) and (3.9), but which relate the balanced elements to $H^2(G, \mathbb{Z})$ and $H^2(G, F^\times)$, respectively. Let $\mathcal{F}_1 = \mathrm{Ch}(-)$, $\mathcal{F}_2 = \mathrm{Br}(-)$, and let $H_1 = H^2(G, \mathbb{Z})$, $H_2 = H^2(G, F^\times)$. We have reduced the calculation of $\mathrm{Inv}(G, \mathcal{F}_i)$ to that of $\mathcal{F}_i(U/G)_{\mathrm{bal}}$ for $i = 1, 2$. We will now use (3.4) and (3.5) above to obtain exact sequences of balanced elements

$$0 \to H_i \to \mathcal{F}_i(U/G)_{\mathrm{bal}} \longrightarrow \mathcal{F}_i(U)_{\mathrm{bal}} \tag{3.10}$$

for $i = 1, 2$. Because the action of G commutes with the projection, we have the commutative diagram below where p_i, q_i are the natural projections for $i = 1, 2$.

$$\begin{array}{ccc} U^2 & \longrightarrow & U^2/G \\ \downarrow{\scriptstyle q_i} & & \downarrow{\scriptstyle p_i} \\ U & \longrightarrow & U/G \end{array}$$

After applying the contravariant functor $\mathcal{F}_i(-)$, this becomes

$$\begin{array}{ccc} \mathcal{F}_i(U/G) & \longrightarrow & \mathcal{F}_i(U) \\ \Big\downarrow{\scriptstyle p_i^*} & & \Big\downarrow{\scriptstyle q_i^*} \\ \mathcal{F}_i(U^2/G) & \longrightarrow & \mathcal{F}_i(U^2) \end{array}$$

Since the same construction that gave (3.10) can be applied to the G-torsor $g' : U^2 \longrightarrow U^2/G$ we have exact sequences for $i = 1, 2$

$$0 \to H_i \to \mathcal{F}_i(U^2) \longrightarrow \mathcal{F}_i(U^2/G) \tag{3.11}$$

Thus, we the commutative diagrams for $i = 1, 2$

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_i & \longrightarrow & \mathcal{F}_i(U/G) & \longrightarrow & \mathcal{F}_i(U) \\ & & \Big\downarrow{\scriptstyle id} & & \Big\downarrow{\scriptstyle p_i^*} & & \Big\downarrow{\scriptstyle q_i^*} \\ 0 & \longrightarrow & H_i & \longrightarrow & \mathcal{F}_i(U^2/G) & \longrightarrow & \mathcal{F}_i(U^2) \end{array}$$

And so we get the commutative diagram below.

$$\begin{array}{ccccccc} & & & & \mathcal{F}_i(U/G)_{\text{bal}} & \xrightarrow{\phi_0} & \mathcal{F}_i(U)_{\text{bal}} \\ & & & & \Big\downarrow & & \Big\downarrow \\ 0 & \longrightarrow & H & \longrightarrow & \mathcal{F}_i(U/G) & \xrightarrow{\phi} & \mathcal{F}_i(U) \\ & & \Big\downarrow{\scriptstyle 0} & & \Big\downarrow{\scriptstyle p_1^*-p_2^*} & & \Big\downarrow{\scriptstyle q_1^*-q_2^*} \\ 0 & \longrightarrow & H & \longrightarrow & \mathcal{F}_i(U^2/G) & \longrightarrow & \mathcal{F}_i(U^2) \end{array}$$

The exactness of the second row and the commutativity of the right square give us that the image of H_i lies in $\mathcal{F}_i(U/G)_{\text{bal}}$. The sequence

$$0 \longrightarrow H_i \longrightarrow \mathcal{F}_i(U/G)_{\text{bal}} \longrightarrow \mathcal{F}_i(U)_{\text{bal}}$$

is exact, since anything in the kernel of ϕ_0 maps to zero in $\mathcal{F}_i(U)$, and so comes from an element of H. Thus, we have the exact sequences

$$0 \longrightarrow H(G,\mathbb{Z}) \longrightarrow \mathrm{Ch}(U/G)_{\mathrm{bal}} \longrightarrow \mathrm{Ch}(U)_{\mathrm{bal}}$$

$$0 \longrightarrow H(G,F^{\times}) \longrightarrow \mathrm{Br}(U/G)_{\mathrm{bal}} \longrightarrow \mathrm{Br}(U)_{\mathrm{bal}}$$

□

3.1.3 Proof of Theorem

We now apply Lemmas (1) and (2) to prove the theorem.

Proof (of Theorem)**.** First, from a result of B. Kahn we have the following isomorphisms in étale motivic cohomology.

$$H^2(\mathbb{P}^n,\mathbb{Z}(0)) \simeq H^2(F,\mathbb{Z}(0)) \oplus H^0(F,\mathbb{Z}(-1)) \tag{3.12}$$

$$H^3(\mathbb{P}^n,\mathbb{Z}(1)) \simeq H^3(F,\mathbb{Z}(1)) \oplus H^1(F,\mathbb{Z}(0)) \tag{3.13}$$

We will now use (3.12) and (3.13) to show that $\mathrm{Ch}(\mathbb{P}^n) = \mathrm{Ch}(F)$ and that $\mathrm{Br}(\mathbb{P}^n) = \mathrm{Br}(F)$. By definition, $\mathbb{Z}(-1) = \mathbb{Q}/\mathbb{Z}(-1)[-1]$, and so $H^0(F,\mathbb{Z}(-1)) = H^{-1}(F,\mathbb{Q}/\mathbb{Z}(-1)) = 0$. Also, the hypercohomology $H^2(F,\mathbb{Z}(0))$ is the same as $H^2(F,\mathbb{Z})$. Since $H^2(X,\mathbb{Z})$ is torsion, from the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

we see have

$$H^2(X,\mathbb{Z}) \simeq H^1(X,\mathbb{Q}/\mathbb{Z}) \tag{3.14}$$

and so from (3.12) we have

$$H^2(\mathbb{P}^n,\mathbb{Z}(0)) \simeq H^1(X,\mathbb{Q}/\mathbb{Z}) = \mathrm{Ch}(F).$$

On the other hand,

$$H^2(\mathbb{P}^n, \mathbb{Z}(0)) \simeq H^2(\mathbb{P}^n, \mathbb{Z}) \simeq H^1(X, \mathbb{Q}/\mathbb{Z}) = \mathrm{Ch}(\mathbb{P}^n),$$

using (3.14), and so $\mathrm{Ch}(\mathbb{P}^n) \simeq \mathrm{Ch}(F)$. Also, $H^1(F, \mathbb{Z}(0)) \simeq H^1(F, \mathbb{Z}) = 0$ as there are no nontrivial continuous homomorphisms from a profinite group into $\mathbb{Z}$, and we have

$$H^3(F, \mathbb{Z}(1)) = H^3(F, \mathbb{G}_m[-1]) = H^2(F, \mathbb{G}_m) = \mathrm{Br}(F).$$

So from (3.14) we also have $\mathrm{Br}(\mathbb{P}^n) = \mathrm{Br}(F)$.

We need to calculate $\mathcal{F}_i(\mathbb{A}^n)_{\mathrm{bal}}$ for $\mathcal{F}_1 = \mathrm{Ch}$ and $\mathcal{F}_2 = \mathrm{Br}$, which we will do by comparing with $\mathcal{F}_i(\mathbb{P}^n)_{\mathrm{bal}} = \mathcal{F}_i(F)$. Consider again the exact sequence (**??**) given by Prop A.10 in [**BM**]. Taking $X = \mathbb{P}^n, \mathbb{A}^n$ gives the following diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}_i(\mathbb{P}^n) & \longrightarrow & \mathcal{F}_i(F(\mathbb{P}^n)) & \xrightarrow{\amalg \partial_x} & \amalg_{x \in (\mathbb{P}^n)^{(1)}} H_x^{i+1}(\mathbb{P}^n, \mathbb{Q}/\mathbb{Z}(i-1)) \\ & & & & \| & & \\ 0 & \longrightarrow & \mathcal{F}_i(\mathbb{A}^n) & \longrightarrow & \mathcal{F}_i(F(\mathbb{A}^n)) & \xrightarrow{\amalg \partial_x} & \amalg_{x \in (\mathbb{A}^n)^{(1)}} H_x^{i+1}(\mathbb{A}^n, \mathbb{Q}/\mathbb{Z}(i-1)) \end{array}$$

From the diagram above and the equality

$$\amalg_{x \in (\mathbb{P}^n)^{(1)}} H_x^{i+1}(\mathbb{P}^n, \mathbb{Q}/\mathbb{Z}(i-1)) = \amalg_{x \in (\mathbb{A}^n)^{(1)}} H_x^{i+1}(\mathbb{A}^n, \mathbb{Q}/\mathbb{Z}(1)) \amalg H_\infty^{i+1}(\mathbb{P}^n, \mathbb{Q}/\mathbb{Z}(i-1))$$

where ∞ denotes the point at infinity in $\mathbb{P}^n$, we obtain an exact sequence

$$0 \longrightarrow \mathcal{F}_i(\mathbb{P}^n) \longrightarrow \mathcal{F}_i(\mathbb{A}^n) \xrightarrow{\partial_\infty \circ i} H_\infty^{i+1}(\mathbb{P}^n, \mathbb{Q}/\mathbb{Z}(i-1))$$

where i is the injection $\mathcal{F}_i(\mathbb{A}^n) \hookrightarrow \mathcal{F}_i(F(\mathbb{A}^n)) = \mathcal{F}_i(F(\mathbb{P}^n))$. Now, we would like to apply Lemmas A.6 and A.7 of [**BM**] to the dominant morphism $\pi_1 : \mathbb{P}^n \times \mathbb{A}^n \longrightarrow \mathbb{P}^n$, given by the projection onto the first coordinate. Letting x denote the generic point of $\mathbb{A}^n$, we have $\pi_1(\infty \times x) = \infty$, a point of codimension 1 in $\mathbb{P}^n$. Thus we can apply Lemmas A.6 and A.7 to get the commutative diagram below where the top row is exact and where j denotes the injection $\mathcal{F}_i(\mathbb{A}^n \times \mathbb{A}^n) \hookrightarrow \mathcal{F}_i(F(\mathbb{A}^n \times \mathbb{A}^n)) = \mathcal{F}_i(F(\mathbb{P}^n \times \mathbb{A}^n))$.

$$\begin{array}{ccccccc} 0 \longrightarrow \mathrm{Br}(\mathbb{P}^n) & \xrightarrow{h} & \mathrm{Br}(\mathbb{A}^n) & \xrightarrow{\partial_\infty \circ i} & H^3_\infty(\mathbb{P}^n, \mathbb{Q}/\mathbb{Z}(1)) \\ & & \downarrow q_1^* & & \downarrow {\pi_1}^*_{\infty\times x} \\ & & \mathrm{Br}(\mathbb{A}^n \times \mathbb{A}^n) & \xrightarrow{\partial_{\infty\times x}\circ j} & H^3_{\infty\times\mathbb{A}^n}(\mathbb{P}^n \times \mathbb{A}^n, \mathbb{Q}/\mathbb{Z}(1)) \end{array}$$

The projection onto the second coordinate $\pi_2 : \mathbb{P}^n \times \mathbb{A}^n \longrightarrow \mathbb{P}^n$, induces maps $q_2^* : \mathcal{F}_i(\mathbb{A}^n) \longrightarrow \mathcal{F}_i(\mathbb{A}^n \times \mathbb{A}^n)$ and $\pi_2^* : \mathcal{F}_i(F(\mathbb{P}^n)) \longrightarrow \mathcal{F}_i(F(\mathbb{P}^n \times \mathbb{A}^n))$ in a similar way. So, we have the commutative diagram:

$$\begin{array}{ccccc} \mathcal{F}_i(\mathbb{A}^n) & = & \mathcal{F}_i(\mathbb{A}^n) & & \\ \downarrow \alpha_2 & & \downarrow q_2^* & & \\ \mathcal{F}_i(\mathbb{P}^n \times \mathbb{A}^n) & \longrightarrow & \mathcal{F}_i(\mathbb{A}^n \times \mathbb{A}^n) & \xrightarrow{\partial_{\infty\times x}\circ j} & H^{i+1}_{\infty\times\mathbb{A}^n}(\mathbb{P}^n \times \mathbb{A}^n, \mathbb{Q}/\mathbb{Z}(i-1)) \end{array}$$

where α_2 is map induced by the second projection and the bottom row is exact from (3.1.3) above. Thus, $\partial_{\infty\times x} \circ j \circ q_2^* = 0$, and so we have the following commutative diagram whose top row is exact.

$$\begin{array}{ccccccc} 0 \longrightarrow \mathcal{F}_i(\mathbb{P}^n) & \longrightarrow & \mathcal{F}_i(\mathbb{A}^n) & \xrightarrow{\partial_\infty \circ i} & H^{i+1}_\infty(\mathbb{P}^n, \mathbb{Q}/\mathbb{Z}(i-1)) \\ & & \downarrow q_1^* - q_2^* & & \downarrow {\pi_1}^*_{\infty\times x} \\ & & \mathcal{F}_i(\mathbb{A}^n \times \mathbb{A}^n) & \xrightarrow{\partial_{\infty\times x}\circ j} & H^{i+1}_{\infty\times x}(\mathbb{P}^n \times \mathbb{A}^n, \mathbb{Q}/\mathbb{Z}(i-1)) \end{array}$$

So $\mathcal{F}_i(\mathbb{A}^n)_{\mathrm{bal}} = \ker(q_1^* - q_2^*) = \mathcal{F}_i(\mathbb{P}^n) = \mathcal{F}_i(F)$. Thus, we obtain the the the exact sequences:

$$0 \longrightarrow H_i \longrightarrow \mathcal{F}_i(U/G)_{\mathrm{bal}} \xrightarrow{\phi_0} \mathcal{F}_i(F) \tag{3.15}$$

It remains to show that ϕ_0 in the exact sequence (6.4) is a split epimorphism, and so we have a description $\mathrm{Inv}(G, \mathcal{F}_i) = H_i \oplus \mathrm{Br}(F)$.

Consider X, a scheme over F. Then letting $r_1^*, r_2^* : \mathcal{F}_i(X) \longrightarrow \mathcal{F}_i(X \times X)$ denote the morphisms induced by the projections $r_1, r_2 : X \times X \longrightarrow X$, we have for i=1,2, a commutative diagram

$$\begin{array}{ccc} \mathcal{F}_i(F) & \xrightarrow{1} & \mathcal{F}_i(F) \\ \downarrow & & \downarrow \\ \mathcal{F}_i(X) & \xrightarrow{r_i^*} & \mathcal{F}_i(X \times X) \end{array}$$

where the vertical maps are induced by the structure morphisms. Thus, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{F}_i(F) & \xrightarrow{0} & \mathcal{F}_i(F) \\ \downarrow & & \downarrow \\ \mathcal{F}_i(X) & \xrightarrow{r_1^* - r_2^*} & \mathcal{F}_i(X \times X) \end{array}$$

and so $\mathcal{F}_i(F) \hookrightarrow \ker(r_1^* - r_2^*) = \mathrm{Br}(X)_{\mathrm{bal}}$. Taking $X = U$, $X = U/G$ $g : U \longrightarrow U/G$ induces the commutative diagram

$$\begin{array}{ccc} \mathcal{F}_i(F) & = & \mathcal{F}_i(F) \\ \downarrow & & \downarrow \simeq \\ \mathcal{F}_i(U/G)_{\mathrm{bal}} & \xrightarrow{\phi_0} & \mathcal{F}_i(U)_{\mathrm{bal}} \end{array}$$

And so we have ϕ_0 is a split surjection, as desired. □

3.2 Evaluation Map for Degree-Two Invariants

We now have a description of degree-two cohomological invariants of a finite group G over a field F in terms of $H^2(G, F^\times)$. We would now like to understand explicitly the map

$$H^2(G, F^\times) \to \mathrm{Br}(K)$$

corresponding to the evaluation of an invariant in $\mathrm{Inv}^2(G, \mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}}$ on a Galois G-algebra $L/K \in H^1(K, G)$.

$$\begin{array}{ccc} H^2(G, F^\times) & \xrightarrow{\simeq} & \mathrm{Inv}^2(G, \mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}} \\ \downarrow & & \downarrow \mathrm{eval}_{L/K} \\ H^2(G, L^\times) & \longrightarrow & \mathrm{Br}(K) \end{array}$$

We will see that this map is given by the crossed product construction.

3.2.1 Crossed Product for Rings

To an element $u \in H^2(G, L^\times)$ we can associate an Azumaya algebra $\Delta(G, L/K, u)$ over K, where $\Delta(L/K, G, u) = \bigoplus_{\sigma \in G} Le_\sigma$, with $e_\sigma e_\tau = u(\sigma, \tau)e_{\sigma\tau}$, and for $\alpha \in L$, $e_\sigma \alpha = \sigma(\alpha)e_\sigma$. Since K is a field, $\Delta(G, L/K, u)$ is actually a central simple algebra. This gives rise to a map $H^2(G, L^\times) \longrightarrow \mathrm{Br}(K)$ defined by sending u to $[\Delta(G, L/K, u)]$, the class of $\Delta(G, L/K, u)$ in $\mathrm{Br}(K)$.

3.2.2 Crossed Product for Schemes

The map

$$H^2(G, F[U]^\times) \longrightarrow \mathrm{Br}(U/G)$$

from the Chase-Harrison-Rosenberg exact sequence is given by a closely related construction to the above, also called the crossed product construction. For an element $u \in H^2(G, F[U]^\times)$, associate to u the $\mathcal{O}_U$-algebra $\Delta'(G, g, u) = \amalg_{\sigma \in G} g_* \mathcal{O} e_\sigma$ with multiplication defined as follows. For each $W \subset U$, we have the restriction map $F[U] \longrightarrow \mathcal{O}_U(W)$, which provides an action of $F[U]^\times$ on $g_*\mathcal{O}$. Thus, we can set $e_\sigma e_\tau = u(\sigma, \tau)e_{\sigma\tau}$, as above. Also, if $s \in \mathcal{O}_U(W)$, we can define $e_\sigma s = \sigma(s)e_\sigma$, with the trivial G-action. The claim is that the $\mathcal{O}_{U/G}$-algebra $\Delta'(G, g, u)$ is an Azumaya algebra.

Since Δ' is a finite-type as a $\mathcal{O}_{U/G}$-module is an Azumaya algebra, by [21, Proposition 2.1] we just need to show

(1) that Δ' is locally free as a $\mathcal{O}_{U/G}$-module and

(2) that $\Delta'_x \otimes F(x)$ is a central algebra over $F(x)$ for each $x \in U/G$.

Since $g : U \longrightarrow U/G$ is a torsor, g is faithfully flat and finite, which implies that $g_*\mathcal{O}$ is a locally free $\mathcal{O}_{U/G}$-module. It remains to show (2). The point $x \in U/G$ corresponds to a morphism $x : \mathrm{Spec}(F(x)) \longrightarrow U/G$. Since $g : U \longrightarrow U/G$ is a classifying torsor, the pullback of g along x is some G-torsor $\mathrm{Spec}(l) \longrightarrow \mathrm{Spec}(F(x))$ where l is a Galois extension of $F(x)$.

$$\begin{array}{ccc} \mathrm{Spec}(l) & \xrightarrow{\ y\ } & U \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ \mathrm{Spec}(F(x)) & \xrightarrow{\ x\ } & U/G \end{array}$$

So we have $(g_*\mathcal{O}_U)_x \otimes F(x) \simeq x^* g_* \mathcal{O}_U \simeq f_* y^* \mathcal{O}_U \simeq \mathcal{O}_{\mathrm{Spec}(l)} = l$, where l is an $F(x)$-module via f. So $\Delta'_x \otimes F(x) \simeq \amalg_{g \in G} l e_\sigma$, with $e_\sigma e \tau = y^*(u(\sigma, \tau)) e \sigma \tau$. This is a central simple algebra over $F(x)$.

3.2.3 Describing the Image of ψ

For an invariant $I \in \mathrm{Inv}(G, \mathrm{Br})$, with $G = \mathrm{Gal}(L/K)$, we would like to associate an element of $\mathrm{Br}(K)$ via

$$\psi : \mathrm{Inv}(G, \mathrm{Br}) \longrightarrow \mathrm{Br}(K)$$

defined by $I \mapsto I(L/K)$. Notice, however, that there is another natural map from $\mathrm{Inv}(G, \mathrm{Br})$ to $\mathrm{Br}(K)$: a normalized invariant $I \in \mathrm{Inv}(G, \mathrm{Br})$ corresponds to an element $u \in H^2(G, F^\times)$ via the crossed product construction, which we can map to its image in $H^2(G, L^\times)$ under the map induced by $F \hookrightarrow L$, and then this element corresponds to some equivalence class in $\mathrm{Br}(K)$ by the crossed product construction for rings. We would like to show that these two associations are the same for normalized invariants; i.e. that the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{Inv}(G, \mathrm{Br})_{\mathrm{norm}} & \xrightarrow{\ \simeq\ } & H^2(G, F^\times) & \overset{s}{\hookrightarrow} & \mathrm{Br}(U/G)_{\mathrm{bal}} \\ & & \big\downarrow{\scriptstyle j} & & \big\downarrow{\scriptstyle x^*} \\ & & H^2(G, L^\times) & \overset{t}{\hookrightarrow} & \mathrm{Br}(K) \end{array}$$

In the diagram above,

- $x^* \circ s$ sends an element $u \in H^2(G, F^\times)$ to $I_u(L/K)$ where I_u is the invariant associated to the element u via the crossed product construction.

- $t \circ j$ is the map induced by the inclusion $F \hookrightarrow L$ followed by the crossed product construction.

- An element $u \in H^2(G, F^\times)$ corresponds to an invariant I_u, which corresponds to a pullback of some $x : \mathrm{Spec}(K) \longrightarrow U/G$ along g, since $G : U \longrightarrow U/G$ is a classifying torsor. This x induces x^* which is given by $x^*([A]) = [A_x \otimes k(x)]$ for an Azumaya algebra A over U/G.

The same argument used above to show that $\Delta'_x \otimes F(x)$ is a central algebra over $F(x)$ for each $x \in U/G$ also shows that this diagram commutes. So for a normalized invariant $I_u \in \mathrm{Inv}(G, \mathrm{Br})_{\mathrm{norm}}$ corresponding to $u \in H^2(G, F^\times)$, $I(L/K)$ is given by the equivalence class of the associated crossed product corresponding to the image of $u \in H^2(G, L^\times)$ under the map induced by the inclusion $F^\times \hookrightarrow L^\times$.

CHAPTER 4

Unramified Degree One Invariants of a Finite Group and Decomposable Invariants

As mentioned in the introduction, we are interested in determining whether there exist nontrivial normalized unramified invariants for a finite group G over a field F. We will now use Theorem (**??**) to show that if G is a finite group over a field F, there are no nontrivial normalized unramified degree 1 invariants with coefficients in $\mathbb{Q}/\mathbb{Z}$.

Lemma 3. *Let G be a finite group over a field F, and let H a subgroup of G. Then the restriction of a character $\chi : G \to \mathbb{Q}/\mathbb{Z}$ to a character $\sigma = \chi|_H$ coincides with the restriction of the invariant $I_\chi \in Inv(G, Ch)_{norm}$ to $I_\sigma \in Inv^1(H, \mathbb{Q}/\mathbb{Z})_{norm}$. In other words, the following diagram commutes.*

$$\begin{array}{ccc} Inv^1(G, \mathbb{Q}/\mathbb{Z})_{norm} & \xrightarrow{res} & Inv^{(}H, \mathbb{Q}/\mathbb{Z})_{norm} \\ \simeq\downarrow & & \downarrow\simeq \\ H^1(G, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{res} & H^1(H, \mathbb{Q}/\mathbb{Z}) \end{array}$$

Proof. Recall from Theorem (**??**), we have

$$\mathrm{Inv}^1(G, \mathbb{Q}/\mathbb{Z})_{\mathrm{norm}} \simeq H^2(G, \mathbb{Z}).$$

From the short exact sequence

$$1 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 1$$

we get the exact sequence

$$H^1(G,\mathbb{Q}) \to H^1(G,\mathbb{Q}/\mathbb{Z}) \to H^2(G,\mathbb{Z}) \to H^2(G,\mathbb{Q}).$$

However, since $\mathbb{Q}$ is a divisible group, $H^i(G,\mathbb{Q}) = 0$ for $i \geq 1$. Thus, we have isomorphisms

$$H^2(G,\mathbb{Z}) = H^1(G,\mathbb{Q}/\mathbb{Z})$$

as in the figure above. For any H-torsor x, $\text{res}(I_\chi)(x) = I_\chi(\text{Ind}_H^G(x))$. Here $\text{Ind}_H^G(x)$ is the image of x under the map $H^1(K,G) \to H^1(K,H)$ induced by the inclusion $H \hookrightarrow G$. Since the action of Γ_K on H is trivial, x corresponds to a homomorphism $\phi_x : \Gamma_K \to H$, and so $\text{Ind}_H^G(x)$ corresponds to the homomorphism $i \circ \phi_x$ where $i : H \hookrightarrow G$ is the inclusion. So we have $\text{res}(I_\chi)(x)$ is the map $\chi \circ (i \circ \phi_x) = (\chi \circ i) \circ \phi_x = \sigma \circ \phi_x = I_\sigma(x)$. So, $\text{res}(I_\chi) = I_\sigma$.

Theorem. *Let G be a finite group over a field F. There are no nontrivial normalized unramified degree one $\mathbb{Q}/\mathbb{Z}$-invariants for G; i.e. $Inv^1_{nr}(G,\mathbb{Q}/\mathbb{Z})_{norm} = 0$.*

Proof. Let $I_\chi \in \text{Inv}^1(G,\mathbb{Q}/\mathbb{Z})_{\text{norm}}$ be nontrivial, where χ is the corresponding element of $H^1(G,\mathbb{Q}/\mathbb{Z})$. We need to show that there exists an extension K/F and a Galois G-algebra L/K such that the fixed field E/K of the character $I_\chi(L/K)$ is ramified. We first consider the case when G is a cyclic group of order n. In this case, it is enough to construct a degree n cyclic extension L/K over F, which is totally ramified with respect to some discrete valuation v on K/F. Indeed, the character $I(L/K)$ is given by the composition of the map $\rho_{L/K} : \Gamma_K \to G$ with $\chi : G \to \mathbb{Q}/\mathbb{Z}$. Since L/K is an extension of fields, $\rho_{L/K}$ is a surjection and so $I(L/K)$ is nonzero for all nontrivial χ. Thus, the fixed field of the kernel of $I(L/K)$ is a nontrivial, and thus ramified, subextension of L/K.

Suppose first that n is prime to the characteristic of F. Let K/F be an extension containing a primitive n^{th} root of unity, ζ_n, and let $\pi \in K$ be the uniformizer with respect to a valuation v on K. Then $K(\alpha)$, with α a root of the polynomial $X^n - \pi$, is a totally ramified degree n extension of K. $K(\alpha)/K$ is cyclic since K contains ζ_n.

Now assume that the characteristic of F is p, and that v is a valuation on an extension K/F. When $n = p$, by Artin-Shreier, a root of the polynomial $X^p - X - a$ generates a degree

p cyclic extension of K whenever a is not of the form $b^p - b$. Let π be a uniformizer of K/F. Claim that π^{-1} is not of this form. If $b^p - b = \pi^{-1}$, then $v(b^p - b) = v(\pi^{-1}) = -1$ and so the minimum of $v(b)$ and $v(b^p)$ is equal to -1. But in this case, both $v(b)$ and $v(b^p) = pv(b) < v(b)$ are negative, and so $v(b^p) = -1$. This is a contradiction, since $v(b^p) \in p\mathbb{Z}$ but $v(b) = -1$. Thus, adjoining a root of $X^p - X - \pi^{-1}$ generates a degree p cyclic extension of K. This extension is totally ramified

Now suppose $n = p^k$. The exact sequence

$$0 \to \mathbb{Z}/p^{k-1}\mathbb{Z} \to \mathbb{Z}/p^k\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$$

yields the exact sequence

$$H^1(K, \mathbb{Z}/p^n\mathbb{Z}) \to H^1(K, \mathbb{Z}/p\mathbb{Z}) \to H^2(K, \mathbb{Z}/p^{k-1}\mathbb{Z})$$

but since the p-cohomological dimension of K is at most one, $H^2(K, \mathbb{Z}/p^{k-1}\mathbb{Z}) = 0$. Thus every degree p field extension of K extends to a cyclic extension of degree p^k for $k > 1$. In particular, the ramified degree p extension constructed above extends to a totally ramified degree p^k cyclic extension of K, as desired.

Now using the preceeding paragraphs, for any n divisible by $p = \mathrm{char}(F)$, we can use the Chinese Remainder Theorem to construct a totally ramified degree n cyclic extension.

Now let G be any finite group, and let $I_\chi \in \mathrm{Inv}(G, \mathrm{Ch})_{\mathrm{norm}}$ be nontrivial. Since $\chi : G \to \mathbb{Q}/\mathbb{Z}$ is nonzero, there is some $x \in G$ such that $\chi(x)$ is not zero. Denote by H the subgroup of G generated by x, and set $\sigma = \chi|_H$. Since H is cyclic, there is a Galois extension L/K with Galois group H such that $I_\sigma(L/K)$ is ramified. By the lemma, we have that $I_\sigma(L/K) = I_\chi(\mathrm{Ind}_H^G(L/K))$, and thus we have that the invariant I_χ is ramified. $\square$

4.1 An Example

While there are no nontrivial unramified degree-one invariants for a finite group G with coefficients in $\mathbb{Q}/\mathbb{Z}(0)$, there do exist nontrivial unramified decomposable degree-two invariants for certain choices of F and G. Serre gives the following example in [9].

Example. Let $G = \mathbb{Z}/8\mathbb{Z}$ and let $F = \mathbb{Q}$. Consider the invariant given by the image of $\chi \otimes 16$ under the cup product $G^* \otimes \mathbb{Q} \to H^2(G, \mathbb{Q}^\times) \simeq \mathrm{Inv}(G, \mathrm{Br})_{\mathrm{norm}}$, where $\chi : G \to \mathbb{Q}/\mathbb{Z}$ is the character with $\chi(\bar{1}) = \frac{\bar{1}}{8}$. Notice that $G^* \otimes \mathbb{Q} \simeq \mathbb{Z}/8\mathbb{Z} \otimes \mathbb{Q} \simeq \mathbb{Q}^\times/\mathbb{Q}^{\times 8}$. Since $\chi \otimes 16 = \chi \otimes 2^4$ is not an eighth power in $G^* \otimes \mathbb{Q}$, it is nontrivial. By the injectivity of the cup product, we have that the invariant $\chi \cup 16$ is nontrivial.

First note that if K is an extension of $\mathbb{Q}$ and $L/K \in H^1(K, \mathbb{Z}/8\mathbb{Z})$ is not a field, then $(\chi \cup 16)(L/K) = 0$. To see this, let $\rho : \Gamma_K \to \mathbb{Z}/8\mathbb{Z}$ be the homomorphism corresponding to L/K. Notice that $(\chi \cup 16)(L/K) = (4\chi \cup 2)(L/K) = (4\rho \circ \chi) \cup 2$. But since L/K is not a field, ρ is not surjective, and so $\rho \circ \chi$ has order dividing 4, and so $(\chi \cup 16)(L/K) = 0$.

Now we will see that $\chi \cup 16$ is unramified. To get a contradiction, suppose that $\chi \cup 16$ is ramified. By the previous paragraph, there exists some extension of fields L/K with $\mathrm{Gal}(L/K) = \mathbb{Z}/8\mathbb{Z}$ such that $(\chi \cup 16)(L/K) = [L/K, \sigma, 16]$ is ramified with respect to a discrete valuation v on K/F. This implies that the field extension L/K is ramified with respect to v. Then since $\chi \cup 16 = 4\chi \cup 2$, the class of the cyclic algebra $[L/K, \sigma, 16] = (\chi \cup 16)(L/K)$ is the same as that of $[E/K, \tau, 2]$, where E/K is the quadratic extension fixed by 4χ and τ generates $\mathrm{Gal}(E/K)$. Since the class of the cyclic algebra $[E/K, \tau, 16]$ is a ramified element of $\mathrm{Br}(K)$, the extension E/K is ramified. This implies that the extension L/K is totally ramified, since the maximal unramified subextension of L/K must be a proper subfield of the quadratic extension E/K, as $\mathrm{Gal}(L/K) = \mathbb{Z}/8\mathbb{Z}$.

Let $\hat{K}$ and $\hat{L}$ denote the completions of K and L with respect to v and v_L respectively. Then because L/K is totally ramified, we have that $\hat{L}/\hat{K}$ must also be totally ramified. If π denotes the uniformizer of $\hat{K}$, and Π denotes the uniformizer of $\hat{L}$, we have $\pi = u\Pi^8$ for some $u \in K^\times$. However, applying the generator σ of $\mathrm{Gal}(L/K)$, we have $\sigma\pi = (\sigma u)(\sigma\Pi)^8$, and so $\pi = u(\sigma\Pi)^8$. Thus, $\Pi^8 = (\sigma\Pi)^8$, and so we must have that $\sigma\Pi = \zeta\Pi$ for some eighth root of unity $\zeta \in \hat{L}$.

Now we will see that ζ must be a primitive eighth root of unity. If ζ is not a primitive eighth root, then $(\sigma\Pi)^4 = \Pi^4$. We will show that this implies that $\sigma^4 = e$, which is a contradiction since $\langle\sigma\rangle = \mathbb{Z}/8\mathbb{Z}$. Let $\alpha \in \hat{L}$ be nonzero. Then $\alpha = u\Pi^k$, for some $k \geq 0$ with $u \in \hat{L}$ a unit in $\mathcal{O}_{\hat{L}}$. Thus, $\sigma^4(\alpha,\sigma) = \sigma^4(u)(\sigma(\Pi)^{4k}) = \sigma^4(u)\Pi^k$. We need to show that $\sigma^4(\alpha,\sigma) = \alpha$, so it is enough to show that $\sigma(u) = u$. Since $\hat{L}/\hat{K}$ is totally ramified, the residue fields $\bar{L}$ and $\bar{K}$ are equal, and so $\bar{u} \in \bar{K}$. So $\bar{u}$ lifts to an element $w \in K \subset \hat{L}$. Because $\bar{u} = \bar{w}$ in $\bar{K}$, $v_{\hat{L}}(u-w)\rangle 0$, but $v_{\hat{L}}(w) = v_{\hat{L}}(u) = 0$, and so $u = w \in \hat{K}$. Thus $\sigma(u) = u$, and so we have $\sigma^4(\alpha,\sigma) = \alpha$ for all $\alpha \in \hat{L}$, a contradiction.

As before, the fact that $\chi \cup 16 = 4\chi \cup 2$ implies that $(\chi \cup 16)(\hat{L}/\hat{K})$ evaluates to the class of the cyclic algebra $[\hat{E}/\hat{K},\tau,2]$. Notice that since the primitive eighth root of unity $\zeta_8 = \frac{1+\zeta_8^2}{\sqrt{2}}$ is in $\hat{K}$, $\sqrt{2}$ is also in $\hat{K}$. However, $[\hat{E}/\hat{K},\tau,2]$ is the twisted algebra $\hat{E}\langle x\rangle/\langle x^2\rangle$ defined by $ax = x\eta(a,\sigma)$ for all $a \in \hat{E}$, which is just the field extension $\hat{E}/\hat{K}$, since $\sqrt{2} \in \hat{K}$. Thus, $(\chi \cup 16)(\hat{L}/\hat{K})$ is trivial and so $(\chi \cup 16)(L/K)$ splits over $\hat{K}$; but this means that $(\chi \cup 16)(L/K)$ is unramified with respect to v. □

4.2 Decomposable Invariants

The invariant produced above is an example of a decomposable invariant. We have seen that $\mathrm{Inv}^1(G,\mathbb{Q}/\mathbb{Z}(0))_{\mathrm{norm}} \simeq H^2(G,\mathbb{Z})$ and that $\mathrm{Inv}^2(G,\mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}} \simeq H^2(G,F^\times)$. Thus, we have a map

$$\mathrm{Inv}^1(G,\mathbb{Q}/\mathbb{Z}(0))_{\mathrm{norm}} \otimes F^\times \to \mathrm{Inv}^2(G,\mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}} \tag{4.1}$$

induced by the cup product

$$\cup : H^2(G,\mathbb{Z}) \otimes H^0(G,F^\times) \to H^2(G,F^\times) \tag{4.2}$$

Proposition. *The cup product* $\cup : H^2(G,\mathbb{Z}) \times H^0(G,F^\times) \to H^2(G,F^\times)$ *is injective.*

Proof. Since $F^\times$ is abelian, it has has a free resolution

$$0 \to A_2 \to A_1 \to F^\times \to 0$$

for some abelian groups A_1, A_2. So for each $n \geq 0$ we have the induced exact sequence

$$H^n(G, A_2) \to H^n(G, A_1) \to H^n(G, F^\times) \to H^{n+1}(G, A_2) \to H^{n+1}(G, A_1)$$

In the case where $A_i = \mathbb{Z}$, the cup product is given by sending $[\phi] \otimes a \mapsto [a\phi]$ in this case, the same as the usual isomorphism $H^n(G, \mathbb{Z}) \otimes \mathbb{Z} \simeq H^n(G, \mathbb{Z})$. The case for A_i a general free abelian group follows from induction. Replacing the above appropriately, we get an exact sequence

$$H^n(G, \mathbb{Z}) \otimes A_2 \xrightarrow{f_1} H^n(G, \mathbb{Z}) \otimes A_1 \to H^n(G, F^\times) \to H^{n+1}(G, \mathbb{Z}) \otimes A_2 \xrightarrow{f_2} H^{n+1}(G, \mathbb{Z}) \otimes A_1$$

From this, we have the exact sequence

$$0 \to \operatorname{coker}(f_1) \to H^n(G, F^\times) \to \ker(f_2) \to 0$$

which becomes the exact sequence

$$0 \to H^n(G, \mathbb{Z}) \otimes F^\times \xrightarrow{\cup} H^n(G, F^\times) \to \operatorname{Tor}(H^{n+1}(G, \mathbb{Z}), F^\times) \to 0$$

showing that the cup product (4.2) is injective. Thus, we also have that the map (4.1) is injective. □

In particular, the proof of the proposition (4.2) above gives the following exact sequence.

$$0 \to \operatorname{Inv}^1(G, \mathbb{Q}/\mathbb{Z}(0))_{\text{norm}} \otimes F^\times \xrightarrow{\cup} \operatorname{Inv}^2(G, \mathbb{Q}/\mathbb{Z}(1))_{\text{norm}} \to \operatorname{Tor}(H^3(G, \mathbb{Z}), F^\times) \to 0 \quad (4.3)$$

Elements of $\operatorname{Inv}^2(G, \mathbb{Q}/\mathbb{Z}(1))_{\text{norm}}$ in the image of the map

$$\operatorname{Inv}^1(G, \mathbb{Q}/\mathbb{Z}(0))_{\text{norm}} \otimes F^\times \xrightarrow{\cup} \operatorname{Inv}^2(G, \mathbb{Q}/\mathbb{Z}(1))_{\text{norm}}$$

in (4.3) are called degree-two *decomposable* invariants. The group of degree-two decomposable invariants for a group G with coefficients in $\mathbb{Q}/\mathbb{Z}(1)$ is denoted $\operatorname{Inv}^2_{\text{dec}}(G, \mathbb{Q}/\mathbb{Z}(1))_{\text{norm}}$. The quotient

$$\mathrm{Inv}^2(G, \mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}}/\mathrm{Inv}^2_{\mathrm{dec}}(G, \mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}} \simeq \mathrm{Tor}(H^3(G, \mathbb{Z}), F^\times) \tag{4.4}$$

is called the group of *indecomposable* invariants. For the rest of this dissertation, we will be concerned with computing the group of degree-two unramified normalized decomposable invariants of a finite group G with coefficients in $\mathbb{Q}/\mathbb{Z}(1)$.

In prime characteristic, we have the following consequence of (4.4).

Corollary. *Let G be a finite group over a field F of characteristic $p > 0$. If p divides the order of G, then all cohomological invariants of G of degree 2 with coefficients in $\mathbb{Q}/\mathbb{Z}(1)$ with p-primary exponent are decomposable.*

Proof. If p divides the order of G, each element of $H^1(K, G)$ is p-torsion for any K/F. Since the homomorphism

$$\mathrm{Inv}^2(G, \mathbb{Q}/\mathbb{Z}(1)) \to \mathrm{Tor}(H^3(G, \mathbb{Z}), F^\times)$$

is surjective, and the image of p-torsion elements are also p-torsion,

$$\mathrm{Tor}(H^3(G, \mathbb{Z}), F^\times) = \mathrm{Tor}(H^3(G, \mathbb{Z}), F^\times)_p = \mathrm{Tor}(H^3(G, \mathbb{Z})_p, F_p^\times).$$

But $F_p^\times$ is the group of the p^{th} roots of unity of F, which is trivial since F has characteristic p. So

$$\mathrm{Tor}(H^3(G, \mathbb{Z}), F^\times) = \mathrm{Tor}(H^3(G, \mathbb{Z}), 1) = 0,$$

and so the map

$${}_p\mathrm{Inv}^2_{\mathrm{dec}}(G, \mathbb{Q}/\mathbb{Z}(1))^{\mathrm{norm}} \to_p \mathrm{Inv}^2(G, \mathbb{Q}/\mathbb{Z}(1))^{\mathrm{norm}}$$

in (4.4) is an isomorphism. □

Proposition. *For a finite group G over a field F,*

$$Inv^2_{dec}(G, \mathbb{Q}/\mathbb{Z}(1))_{norm} \simeq G^* \otimes F^\times$$

where G^* denotes the group of characters $Hom_{cts}(G, \mathbb{Q}/\mathbb{Z})$.

Proof. From the discussion above, we have

$$\mathrm{Inv}^2_{\mathrm{dec}}(G, \mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}} \simeq H^2(G, \mathbb{Z}) \otimes F^\times$$

and so it remains to show that $H^2(G, \mathbb{Z}) \simeq H^1(G, \mathbb{Q}/\mathbb{Z})$. From the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

we get the exact sequence

$$H^1(G, \mathbb{Q}) \to H^1(G, \mathbb{Q}/\mathbb{Z}) \to H^2(G, \mathbb{Z}) \to H^2(G, \mathbb{Q})$$

and since $\mathbb{Q}$ is divisible, $H^1(G, \mathbb{Q}) = H^2(G, \mathbb{Q}) = 0$. So $H^2(G, \mathbb{Z}) = H^1(G, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{\mathrm{cts}}(G, \mathbb{Q}/\mathbb{Z})$. □

The lemma above suggests that for a finite group G over a field F, we can think about decomposable invariants in $\mathrm{Inv}^2_{\mathrm{dec}}(G, \mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}}$ as finite sums of elements of the form $\chi \cup a$ for characters $\chi : G \to \mathbb{Q}/\mathbb{Z}$ and elements $a \in F^\times$. In fact, for a G-torsor $L/K \in H^1(K, G)$, a decomposable invariant $\chi \cup a$ evaluates on L/K to the class of the cyclic algebra given by $E/K, a, \sigma)$, where $E/K = (L/K)^{\mathrm{Ker}(\chi)}$ and σ is a generator of $\mathrm{Gal}(E/K)$.

4.3 Cyclic Algebras

We will show that on extensions of fields, decomposable invariants evaluate to the classes of cyclic algebras. More precisely,

Proposition. *For a finite group G over a field F, a decomposable invariant $I \in Inv^2(G, \mathbb{Q}/\mathbb{Z}(1))_{norm}$ and an extension of fields $L/K \in H^1(K, G)$, $I(L/K) \in Br(K)$ corresponds to the equivalence class of a cyclic algebra.*

Proof. We need to show that elements in the image of the composition

$$H^2(G, \mathbb{Z}) \otimes F^\times \xrightarrow{\cup} H^2(G, F^\times) \to H^2(G, L^\times) \hookrightarrow \mathrm{Br}(K)$$

correspond to the equivalence classes of cyclic algebras in $\mathrm{Br}(K)$. Here, the first map is given by the cup product, the second is induced by the inclusion $F \hookrightarrow L$, and the third is given by the crossed product construction. We have seen that the composition of the latter two is evaluation of an invariant at the Galois G-algebra L/K, which is given by the crossed product construction.

Let $w \in H^2(G, \mathbb{Z})$. We can assume w is nonzero since in that case the image of w in $\mathrm{Br}(K)$ is $[K]$, which is trivially cyclic. Since G is finite and $\mathbb{Q}$ uniquely divisible,

$$H^1(G, \mathbb{Q}) = H^2(G, \mathbb{Q}) = 0$$

and so the connecting homomorphism

$$\delta : H^1(G, \mathbb{Q}/\mathbb{Z}) \to H^2(G, \mathbb{Z})$$

is an isomorphism. Thus we can write $w = \delta\chi$ for some $\chi \in H^1(G, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{cts}(G, \mathbb{Q}/\mathbb{Z})$. Let H denote $G/\mathrm{ker}\chi$. By the inflation-restriction sequence, we have

$$H^1(H, \mathbb{Q}/\mathbb{Z}) \to H^1(G, \mathbb{Q}/\mathbb{Z}),$$

and so we can consider χ as an element of $H^1(H, \mathbb{Q}/\mathbb{Z})$, and w as an element of $H^2(G, \mathbb{Z})$. Since G is finite, the image of χ is of the form $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$ for some n, and so H is cyclic of order n with generator σ. Note that since H is a quotient of $G = \mathrm{Gal}(L/K)$, $H = \mathrm{Gal}(E/K)$ for some extension $K \subset E \subset L$, and we have the following commutative diagram.

$$\begin{array}{ccccccc}
H^2(H, \mathbb{Z}) \otimes F^\times & \xrightarrow{\cup} & H^2(H, F^\times) & \longrightarrow & H^2(H, E^\times) & \longrightarrow & \mathrm{Br}(K) \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
H^2(G, \mathbb{Z}) \otimes F^\times & \xrightarrow{\cup} & H^2(G, F^\times) & \longrightarrow & H^2(G, L^\times) & \longrightarrow & \mathrm{Br}(K)
\end{array}$$

We need to show that for $a \in F^\times$, the image of $w \otimes a$ under the composition of maps in the top row corresponds to the class of a cyclic algebra in $\mathrm{Br}(K)$. To get a more explicit description of w, we choose a lift $\hat{\chi} : G \to \mathbb{Q}$ of χ, given by $\hat{\chi}(\sigma^i) = \frac{i}{n}$ if $0 \le i < n$, and $\hat{\chi}(\sigma^i) = 0$ otherwise. Then $w = \partial\hat{\chi} : G \times G \to \mathbb{Z}$ is given by $w(\sigma^i, \sigma^j) = \hat{\chi}(\sigma^i) - \hat{\chi}(\sigma^{i+j}) + \hat{\chi}(\sigma^j)$. Thus for $0 \le i, j < n$, we have

$$w(\sigma^i, \sigma^j) = \begin{cases} 0 & \text{if } i + j < n \\ 1 & \text{if } i + j \ge n \end{cases}$$

Now the cup product $w \cup a$ is given by $w \cup a(\sigma^i, \sigma^j) = w(\sigma^i, \sigma^j) \otimes_{\mathbb{Z}} a = a^{w(\sigma^i, \sigma^j)}$. So by the crossed product construction, $w \cup a$ defines the algebra $\amalg_{0 \le i < n} E e_i$ with

$$e_i e_j = \begin{cases} e_{i+j} & \text{if } i + j < n \\ a e_{i+j} & \text{if } i + j \ge n \end{cases}$$

and for all $\alpha \in E$, $e_i \alpha = \sigma^i(\alpha, \sigma) e_i$. Thus, $w \otimes a$ gets sent to the cyclic algebra defined by the pair $(E/K, a, \sigma)$.

Let L/K be a G-Galois extension over F and let $I \in \mathrm{Inv}^1(G, \mathbb{Q}/\mathbb{Z})_{\mathrm{norm}}$. Then we have the following commutative diagram, where $\mathrm{Ch}(K)$ denotes $H^1(K, \mathbb{Q}/\mathbb{Z})$.

$$\begin{array}{ccc} \mathrm{Inv}^1(G, \mathbb{Q}/\mathbb{Z})_{\mathrm{norm}} \otimes F^\times & \longrightarrow & \mathrm{Ch}(K) \otimes F^\times \\ \downarrow & & \downarrow \\ \mathrm{Inv}^2(G, \mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}} & \longrightarrow & \mathrm{Br}(K) \end{array} \tag{4.5}$$

Here, the two horizontal maps are given by evaluation at the G-Galois algebra L/K and the left map is induced by the cup product. The right map is given by the crossed product construction as follows. Let $\chi \otimes a \in \mathrm{Ch}(K) \otimes F^\times$ be nontrivial. Then $\chi : \Gamma_K \to \mathbb{Q}/\mathbb{Z}$ has image $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$ for some n with canonical generator $\frac{1}{n}$, and so kerχ fixes some degree n cyclic extension E/K with generator σ of Gal(E/K). Then, as in the proof above, we associate to $\chi \otimes a$ the cyclic algebra $(E/K, a, \sigma) = \amalg_{0 \leq i < n} E e_i$ where

$$e_i e_j = \begin{cases} e_{i+j} & \text{if } i + j < n \\ a e_{i+j} & \text{if } i + j \geq n \end{cases}$$

and and for all $\alpha \in E$, $e_i \alpha = \sigma^i(\alpha, \sigma) e_i$. □

CHAPTER 5

Degree-Two Decomposable Invariants in Characteristic Prime to $|G|$

5.1 Unramified Degree-Two Decomposable Invariants of a Finite Cyclic Group

We now compute the group $\mathrm{Inv}^2_{\mathrm{dec}}(G, \mathbb{Q}/\mathbb{Z}(1))^{\mathrm{nr}}_{\mathrm{norm}}$ of unramified normalized decomposable invariants of a finite cyclic group G of degree two, with coefficients in $\mathbb{Q}/\mathbb{Z}(1)$ in the case when the characteristic of the base field is prime to the order of G. We use the notation F_n to denote the extension of a field F by adjoining a primitive n^{th} root of unity.

Theorem. *Let G be a finite cyclic group with 2-primary component of degree n over a field F of characteristic prime to the order of G. Then*

$$Inv^2_{dec}(G, \mathbb{Q}/\mathbb{Z}(1))^{norm}_{nr} = \begin{cases} 0 & \text{if } Gal(F_n/F) \text{ is cyclic} \\ \mathbb{Z}/2\mathbb{Z} & \text{otherwise} \end{cases}$$

Before proving the theorem, we note that since G is a finite cyclic group, we will reduce to the case when G is a cyclic p-group for a prime p.

Lemma 4. *Let G and H be finite groups over a field F and let $f : H \to G$ be a group homomorphism. Then the following diagram commutes,*

$$\begin{array}{ccc} Inv^2_{dec}(G,\mathbb{Q}/\mathbb{Z}(1))_{norm} & \longrightarrow & Inv^2_{dec}(H,\mathbb{Q}/\mathbb{Z}(1))_{norm} \\ \uparrow & & \uparrow \\ Inv^2_{dec}(G,\mathbb{Q}/\mathbb{Z}(1))^{nr}_{norm} & \longrightarrow & Inv^2_{dec}(H,\mathbb{Q}/\mathbb{Z}(1))^{nr}_{norm} \end{array}$$

where the map $Inv^2_{dec}(G) \to Inv^2_{dec}(H)$ above sends a decomposable invariant $\chi \cup a$ to $(\chi \circ f) \cup a$.

Proof. We need to show that if $(\chi \circ f) \cup a$ is ramified, then the decomposable invariant $\chi \cup a$ is also ramified. Thus, it is enough to show that for any Galois H-algebra L/K, there is a Galois G-algebra E/K such that $(\chi \circ f) \cup a(L/K) = \chi \cup a(E/K)$; i.e. that

$$(\chi \circ f)(L/K) = \chi(E/K).$$

Since the set of Galois H-algebras over K can be identified with the set of continuous homomorphisms $\Gamma_K \to H$, L/K corresponds to some map $\phi_{L/K}$. Composing with $f : H \to G$, we get $f \circ \phi_{L/K} \in H^1(K, G)$ which must correspond to some Galois G-algebra E/K. Then since $\chi(L/K)$ defined as $\chi \circ \phi_{L/K}$, by construction we have $(\chi \circ f)(L/K) = \chi(E/K)$.

Lemma 5. *Let G and H be groups. Then*

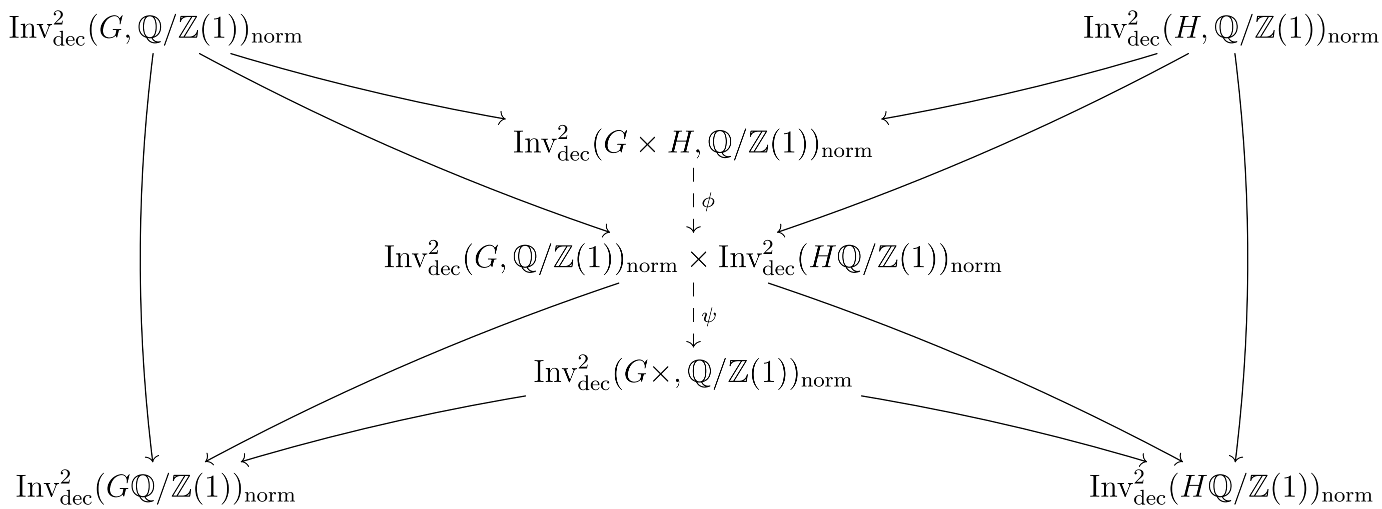
$$Inv^2_{dec}(G \times H,\mathbb{Q}/\mathbb{Z}(1))^{nr}_{norm} \simeq Inv^2_{dec}(G,\mathbb{Q}/\mathbb{Z}(1))^{nr}_{norm} \times Inv^2_{dec}(H,\mathbb{Q}/\mathbb{Z}(1))^{nr}_{norm}$$

Proof. The functor $\mathrm{Inv}^2_{\mathrm{dec}}(-,\mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}} : \mathrm{Groups} \to \mathrm{Ab}$ induces an isomorphism which is given explicitly by the restriction.

$$\phi : \mathrm{Inv}^2_{\mathrm{dec}}(G \times H,\mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}} \to \mathrm{Inv}^2_{\mathrm{dec}}(G,\mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}} \times \mathrm{Inv}^2_{\mathrm{dec}}(H,\mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}}$$

$$\phi(\chi \cup a) = (\chi|_G \cup a, \chi|_H \cup a)$$

Because finite product and coproduct of two abelian groups coincide, we have the following



where $\psi \circ \phi = \mathrm{id}$. From the lemma above, we then have induced maps on the unramified subgroups:

$$\mathrm{Inv}^2_{\mathrm{dec}}(G\times H, \mathbb{Q}/\mathbb{Z}(1))^{\mathrm{nr}}_{\mathrm{norm}} \xrightarrow{\phi^{nr}} \mathrm{Inv}^2_{\mathrm{dec}}(G, \mathbb{Q}/\mathbb{Z}(1))^{\mathrm{nr}}_{\mathrm{norm}} \times \mathrm{Inv}^2_{\mathrm{dec}}(H, \mathbb{Q}/\mathbb{Z}(1))^{\mathrm{nr}}_{\mathrm{norm}} \xrightarrow{\psi^{nr}} \mathrm{Inv}^2_{\mathrm{dec}}(G\times H, \mathbb{Q}/\mathbb{Z}(1))^{\mathrm{nr}}_{\mathrm{norm}} \quad (5.1)$$

Since ψ is an isomorphism, ψ^{nr} is injective, and since $\phi_{nr} \circ \phi^{nr} = \mathrm{id}$, we must have that ψ_{nr} is surjective. Thus ψ^{nr} is an isomorphism and so $\mathrm{Inv}^2_{\mathrm{dec}}(G \times H, \mathbb{Q}/\mathbb{Z}(1))^{\mathrm{nr}}_{\mathrm{norm}} \simeq \mathrm{Inv}^2_{\mathrm{dec}}(G, \mathbb{Q}/\mathbb{Z}(1))^{\mathrm{nr}}_{\mathrm{norm}} \times \mathrm{Inv}^2_{\mathrm{dec}}(H, \mathbb{Q}/\mathbb{Z}(1))^{\mathrm{nr}}_{\mathrm{norm}}$.

Now, we reduce to the case when G is a finite cyclic group of order $n = p^k$ for a prime p. If G be a finite cyclic group over F of order n, and G_p is the p-primary component of G corresponding to a prime p, group unramified normalized decomposable invariants decomposes as

$$\mathrm{Inv}^2_{\mathrm{dec}}(G, \mathbb{Q}/\mathbb{Z}(1))^{\mathrm{nr}}_{\mathrm{norm}} = \amalg_{p|n} \mathrm{Inv}^2_{\mathrm{dec}}(G_p, \mathbb{Q}/\mathbb{Z}(1))^{\mathrm{nr}}_{\mathrm{norm}} \quad (5.2)$$

since $G = \Pi_{p|n} G_p$.

So we have reduced to computing the unramified decomposable invariants in the case when G is a cyclic p-group. The proof of Theorem (5.1) will now follow from the following lemmas.

Lemma 6. *Let $n = p^k$ for some prime p and $k > 0$, and let F be a field of characteristic different from p. If $Gal(F_n/F)$ is cyclic and is generated by the automorphism σ where $\sigma(\zeta_n) = \zeta_n^5$, then $H^1(Gal(F_n/F), \mu_n) = 0$.*

Proof. Assume that $\mathrm{Gal}(F_n/F)$ is cyclic. Since $\mu_n(F_{sep})$ is finite, the Herbrand quotient

$$|\hat{H}^0(\mathrm{Gal}(F_n/F), \mu_n)|/|H^1(\mathrm{Gal}(F_n/F), \mu_n)| = 1$$

and so it is enough to show that $|\hat{H}^0(\mathrm{Gal}(F_n/F), \mu_n)| = 1$. Thus we need to show that every $\zeta \in \mu_n(F)$ is the norm of some $\zeta' \in \mu_n(F_{sep})$. Since the norm is multiplicative, it suffices to show that if the primitive root $\zeta_{p^d} \in F$, then ζ_{p^d} is a norm.

First consider the case when $p = 2$. We have $[F_n : F] = 2^l$ for some $0 \leq l < k$. If the primitive root $\zeta_{2^d} = \zeta_n^{2^{k-d}}$ is in F, then we must have $k - d \geq l$, so it is enough to show that ζ_{2^d} is a norm whenever $d \leq k - l$. Explicitly, we need to show that there exists $\zeta = \zeta_n^s \in F_n$ such that $\Pi\sigma^i(\zeta) = \zeta_{2^d}$, where σ generates $\mathrm{Gal}(F_n/F)$ and $\sigma(\zeta_n) = \zeta_n^t$. This is equivalent to showing that whenever $d \leq k - l$, there exists some $s \in \mathbb{N}$ such that

$$2^k | s(1 + t + t^2 + \cdots + t^{2^l - 1}) - 2^{k-d}$$

It is possible to choose such an s whenever 2^{k-d+1} does not divide $\alpha = 1+t+t^2+\cdots+t^{2^l-1}$. Since $k - d + 1 \geq l + 1$, it is enough to show that 2^{l+1} does not divide α.

Note that since σ generates $\mathrm{Gal}(F_n/F)$, t is odd and so $t = 1 + 2i$ for some $i \geq 1$. Thus we have

$$\begin{aligned}\alpha = 1 + (1+2i) + (1+2i)^2 + \cdots + (1+2i)^{2^l-1} &= \frac{(1+2i)^{2^l}-1}{2i} \\ &= \binom{2^l}{1} + \binom{2^l}{2}(2i) + \cdots + \binom{2^l}{2^l}(2i)^{2^l-1} \\ &= 2^l + (\binom{2^l}{2}(2i) + \cdots + \binom{2^l}{2^l}(2i)^{2^l-1})\end{aligned}$$

If we can show that 2^{l+1} divides every summand in the parentheses in the last line above, we will be done. Notice that since we can take $t = 5$ and so $i = 2$ by assumption, it suffices to check that the m^{th} binomial coefficient divisible by 2^{l-m+1} for $2 \leq m \leq l$.

$$\binom{2^l}{m} = \frac{2^l!}{(2^l-m)!m!} \tag{5.3}$$

The equation for the number of factors of 2 in $N!$ ($N \geq 1$), is given by

$$\left\lfloor \frac{N}{2} \right\rfloor + \left\lfloor \frac{N}{2^2} \right\rfloor + \left\lfloor \frac{N}{2^3} \right\rfloor + \cdots = N - \Sigma_j a_j$$

where the a_j are coefficients of N when expressed in base 2. Applying this, we see that the number of factors of two in the numerator of (6.3) is $2^l - 1$. The number of factors of 2 in the denominator is given by the expression

$$((2^l - m) - \Sigma_j a_j) + (m - \Sigma_i c_i) = 2^l - \Sigma_j a_j - \Sigma_i c_i$$

where the a_j and c_i are coefficients of $2^l - m$ and m, respectively, when expressed in base 2. Thus we need to show that $\Sigma_j a_j + \Sigma_i c_i \geq l - m + 2$. We do this in the lemma below.

Now suppose that p is odd and that $\zeta_{p^d} \in F$. If F does not contain a primitive p^{th} root, then $\mu_n(F) = \{1\}$, and 1 is a norm. If $\zeta_p \in F$, then $|\text{Gal}(F_n/F)| = p^l$ for some $l < k$. In this situation the argument (and computation) is nearly identical to the $p = 2$ case. □

Lemma 7. *Let l, m be integers such that $2 \leq m \leq l$. Suppose that $2^l - m = \Sigma_j a_j 2^j$, $m = \Sigma_i c_i 2^i$ are base two expansions. Then $\Sigma_j a_j + \Sigma_i c_i \geq l - m + 2$.*

Proof. Notice that since $(2^l - m) + m = 2^l$ has $l+1$ digits when expressed in base two, at least one of the expansions of $2^l - m$ and m must have l digits. Since $2^l \geq l \geq m$, the expansion of $2^l - m$ must have l digits and the base two expansion of m has $r \leq l$ digits. This means that the first $l - r$ digits of $2^l - m$ must be 1's.

$$\begin{array}{r} 1\,1 \cdots 1\ \ast\ \ast \ldots \ast \\ +\qquad\quad 1\ \ast\ \ast \ldots \ast \\ \hline 1\,0\,0\ 0\ 0 \cdots 0\ 0\ 0 \end{array}$$

In particular, there are always at least $l - r + 1$ digits equal to 1 in the base two expansions of m and $2^l - m$, so it is enough to show that $l - r + 1 \geq l - m + 2$, i.e. that $m \leq r + 1$. But $r \leq log_2(m)$, and $m \leq log_2(m) + 1$, and we are done. □

Lemma 8. *Let F be a field of characteristic different from p, and let $n = p^k$ where $k > 0$. If $Gal(F_n/F)$ is not cyclic, then $|H^1(Gal(F_n/F), \mu_n)| = 2$.*

Proof. If $\text{Gal}(F_n/F)$ is not cyclic, then $p = 2$, and $F_n/F(i)/F$ with $F_n/F(i)$ cyclic. By the inflation-restriction sequence, we then have

$$0 \to H^1(\text{Gal}(F(i)/F), \mu_4) \to H^1(\text{Gal}(F_n/F), \mu_n) \to H^1(\text{Gal}(F_n/F(i)), \mu_n) \to 0$$

Since $F_n/F(i)$ cyclic, we have $H^1(\text{Gal}(F(i)/F), \mu_4) = H^1(\text{Gal}(F_n/F), \mu_n)$. Since $H^1(\text{Gal}(F(i)/F), \mu_4)$ is $\mu_n(F_{sep})$ modulo squares,

$$H^1(\text{Gal}(F(i)/F), \mu_4) = H^1(\text{Gal}(F_n/F), \mu_n) = \mathbb{Z}/2\mathbb{Z}$$

. □

The following lemma completes the proof of the theorem.

Lemma 9. *Let G be a cyclic group of order $n = p^k$, where $k > 0$ and p is prime, over a field F of characteristic different from p. Then $Inv^2_{dec}(G, \mathbb{Q}/\mathbb{Z}(1))^{norm}_{nr} = H^1(Gal(F_n/F), \mu_n)$.*

Proof. By the inflation restriction sequence, $H^1(\mathrm{Gal}(F_n/F), \mu_n) = \ker(f_n)$, where

$$f_n : F^\times/F^{\times^n} \to F_n^\times/F_n^{\times^n}$$

is the natural map. So it is enough to show that $\mathrm{Inv}^2_{dec}(G, \mathbb{Q}/\mathbb{Z}(1))^{norm}_{nr} = \ker(f_n)$. Note that any decomposable invariant can be written uniquely as $\chi \cup a$ with $\chi : G = \mathbb{Z}/n\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ defined by $\chi(1) = [\frac{1}{n}]$ and $a \in F^\times$. We will show that $\chi \cup a \in \mathrm{Inv}^2_{dec}(G, \mathbb{Q}/\mathbb{Z}(1))^{\mathrm{norm}}_{\mathrm{nr}}$ if and only if $a \in \ker(f_n)$.

Suppose first that $\chi \cup a \in \mathrm{Inv}^2_{dec}(G, \mathbb{Q}/\mathbb{Z}(1))^{\mathrm{norm}}_{\mathrm{nr}}$. Then $(\chi \cup a)(L/K)$ is split for any Galois G-algebra L over a complete field K. In particular, taking $K = F_n((t))$ and $L = K(t^{\frac{1}{n}})$, $(\chi \cup a)(L/K)$ split means that $a \in F^\times$ is an n^{th} power in K, and thus in $F_n^\times$. So $a \in \ker(f_n)$.

Now suppose that $a \in \ker(f_n)$. It is clear that $\chi \cup a$ is normalized. To get a contradiction, suppose that $\chi \cup a$ is a nontrivial ramified invariant. Since $a \in \ker(f_n) = H^1(\mathrm{Gal}(F_n/F), \mu_n)$, either $a = 1$ or $|a| = 2$ by the above lemmas. Since $\chi \cup 1$ is trivial, we must have that a has order 2. Thus, $a = b^{n/2}$ for some $b \in F^\times$, $\sqrt{b} \notin F^\times$.

Since $\chi \cup a$ is ramified, there exists a field extension L/K such that

$$(\chi \cup a)(L/K) = (\frac{n}{2} b\chi \cup b)(L/K) = [E/K, \tau, b]$$

is ramified, where E/K is the quadratic subextension of L/K fixed by $\frac{n}{2}\chi$. So the extension E/K is ramified. Since L/K is cyclic of degree $n = p^k$ with a ramified quadratic subextension, L/K must be totally ramified. So the completion $\hat{L}/\hat{K}$ is also a totally ramified degree n cyclic extension and so we must have $\zeta_n \in \hat{K}$. Thus $F_n \subset \hat{K}$. However, since $a \in \ker(f_n)$ this implies that $\chi \cup a$ splits over F_n, a contradiction since we assumed $(\chi \cup a)(L/K)$ is ramified. $\square$

We have now computed the degree two unramified normalized decomposable invariants of a finite cyclic group with coefficients in $\mathbb{Q}/\mathbb{Z}(1)$ when the order of G is prime to the characteristic of the base field F. We can use this to compute $\mathrm{Inv}^2_{dec}(G, \mathbb{Q}/\mathbb{Z}(1))^{norm}_{nr}$ in the case when G is a finite abelian group over F.

5.2 $\mathbf{Inv}^2(G, \mathbb{Q}/\mathbb{Z}(1))$ for Finite Abelian G in Characteristic Prime to $|G|$

We now will apply the results of the previous section to obtain a formula for $\mathrm{Inv}^2_{\mathrm{dec}}(G, \mathbb{Q}/\mathbb{Z}(1))^{\mathrm{nr}}_{\mathrm{norm}}$ for an arbitrary finite abelian group G over a field of characteristic prime to the order of G.

Theorem. *Let G be a finite abelian group over a field F of characteristic prime to $|G|$. Let*

$$G_2 = \Pi_i \mathbb{Z}/2^{k_i}\mathbb{Z}$$

be the 2-primary component of G. Then

$$Inv^2_{dec}(G, \mathbb{Q}/\mathbb{Z}(1))^{nr}_{norm} = (\mathbb{Z}/2\mathbb{Z})^r$$

where r is the number of k_i with $Gal(F_{k_i}/F)$ cyclic.

Proof. This follows immediately from (5) and (5.1) of the previous section.

We now compute the group of unramified normalized decomposable invariants of certain nonabelian finite groups G over a field of characteristic prime to the order of G. We have the following lemma.

Lemma 10. *Let G be a finite group over a field F, and let $G^{ab} = G/[G, G]$ denote the abelianization of G. Then the map*

$$p^* : Inv^2(G^{ab}, \mathbb{Q}/\mathbb{Z}(1))_{norm} \to Inv^2(G, \mathbb{Q}/\mathbb{Z}(1))_{norm} \tag{5.4}$$

induced by the projection

$$p : G \to G^{ab} \tag{5.5}$$

sends decomposable invariants to decomposable invariants and

$$p^* : Inv^2_{dec}(G^{ab}, \mathbb{Q}/\mathbb{Z}(1))_{norm} \to Inv^2_{dec}(G, \mathbb{Q}/\mathbb{Z}(1))_{norm} \tag{5.6}$$

is surjective.

Proof. Let $I \in \mathrm{Inv}^2_{\mathrm{dec}}(G^{\mathrm{ab}}, \mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}}$. Then by definition, $p^*(I)$ is the normalized invariant for G given by

$$p^*(I)(L/K) = I(L/K^{\mathrm{Ker}(p)})$$

where $\mathrm{Ker}(p)$ is the kernel of the projection p. Since the invariant I is decomposable, we have seen that $I = \chi \cup a$ for some character $\chi \in H^1(G, \mathbb{Q}/\mathbb{Z})$ and some $a \in F^\times$. Recall that $(\chi \cup a)(L/K)$ for a G^{ab}-Galois algebra L/K is equal to the cyclic algebra $(\chi, E/K, a)$ where E/K is the subalgebra of L/K fixed by the kernel of

$$\chi : G \to \mathbb{Q}/\mathbb{Z}.$$

So $p^*(I)(M/K)$ for a G-Galois algebra M/K, is given by the cyclic algebra $(\chi, E'/K, a)$ where E' is the subalgebra of $M^{\mathrm{Ker}(p)}$ fixed by the kernel of χ. Thus, p^* sends decomposable invariants to decomposable invariants.

Now we have the following diagram

$$\begin{array}{ccc}
\mathrm{Inv}^2_{\mathrm{dec}}(G^{\mathrm{ab}}, \mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}} & \xrightarrow{p^*} & \mathrm{Inv}^2_{\mathrm{dec}}(G, \mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}} \\
\simeq \downarrow & & \downarrow \simeq \\
{G^{\mathrm{ab}}}^* \otimes F^\times & \xrightarrow[-\circ p]{} & G^* \otimes F^\times
\end{array}$$

in which the two vertical maps are the isomorphisms given in the previous chapter, and the bottom horizontal map is given by

$$\chi \cup a \mapsto (\chi \circ p) \cup a.$$

The diagram commutes since if $I \simeq \chi \cup a$,

$$p^*(I)(L/K) = I((L/K)^{\mathrm{Ker(p)}}) = (\chi, L'/K, a) = (\chi \circ p) \cup a(L/K),$$

where L'/K is the subalgebra of $(L/K)^{\mathrm{Ker(p)}}$ fixed by the kernel of p. Additionally, the bottom map

$$G^{\mathrm{ab}^*} \otimes F^\times \to G^* \otimes F^\times$$

is surjective since any character $\chi : G \to \mathbb{Q}/\mathbb{Z}$ factors uniquely through the projection $p : G \to G^{\mathrm{ab}}$. Thus, equation (5.6) is surjective. □

5.3 Quaternion Group Q_8

We now consider the quaternion group $Q_8 = \langle u, v : u^2 = v^2, u^4 = 1, v^{-1}uv = u^{-1}v^{-1}\rangle$ and will show the following.

Theorem. *Let F be a field of characteristic different from two. Then there are no unramified normalized decomposable degree two invariants for Q_8 over F with coefficients in $\mathbb{Q}/\mathbb{Z}(1)$; i.e.*

$$Inv^2_{dec}(G, \mathbb{Q}/\mathbb{Z}(1))^{nr}_{norm} = 0.$$

Proof. Since the lemma above gives us that the map

$$p^* : \mathrm{Inv}^2_{\mathrm{dec}}(Q_8^{\mathrm{ab}}, \mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}} \to \mathrm{Inv}^2_{\mathrm{dec}}(Q_8, \mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}}$$

is surjective, it is enough to show that for every nontrivial normalized decomposable Q_8^{ab}-invariant I, there exists a Q_8^{ab}-Galois extension E/K such that $I(E/K)$ is ramified and such that E/K embeds in some Q_8-Galois extension L/K. We first determine a condition for a Q_8^{ab}-Galois extension to embed in a Q_8-Galois.

The commutator subgroup $[Q_8, Q_8] = \langle u^2 \rangle$, and so the abelianization of Q_8 is given by

$$Q_8^{\text{ab}} = \langle \bar{u}, \bar{v} \rangle \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Thus, the exact sequence

$$0 \to [Q_8, Q_8] \to Q_8 \to Q_8^{\text{ab}} \to 0 \tag{5.7}$$

reads

$$0 \to \mathbb{Z}/2\mathbb{Z} \to Q_8 \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to 0.$$

Because $[Q_8, Q_8] = \langle u^2 \rangle$ is central in Q_8, for any field K/F, we get the exact sequence in Galois cohomology

$$H^1(K, Q_8) \to H^1(K, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \to H^2(K, \mathbb{Z}/2\mathbb{Z}) =_2 \text{Br}(K)$$

where the first map is given by the restriction and the second map is given by the crossed product construction. Thus, a biquadratic extension in $H^1(K, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ embeds in a Q_8 extension when its image under the crossed product vanishes in ${}_2\text{Br}(K)$.

To compute the image $[A]$ of a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-Galois algebra E/K in ${}_2\text{Br}(K)$, we first determine the cocycle $f \in H^2(Q_8^{\text{ab}}, [Q_8, Q_8])$ corresponding to the exact sequence (5.7). For any $h, k \in Q_8^{\text{ab}}$, $f(a, b) = \hat{h} \cdot \hat{k} \cdot \widehat{hk}^{-1} \in [Q_8, Q_8]$, where $\hat{h}$ denotes a chosen lift of h in Q_8. Write $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \langle e_1, e_2 \} \rangle$, where u, v and uv in Q_8 are lifts of e_1, e_2 and e_1e_2 respectively. Let $g \in H^2(Q_8^{\text{ab}}, E^\times)$ be the composition of f with the inclusion $\mathbb{Z}/2\mathbb{Z} \hookrightarrow E^\times$. Then $[A] \in_2 \text{Br}(K)$ is given by

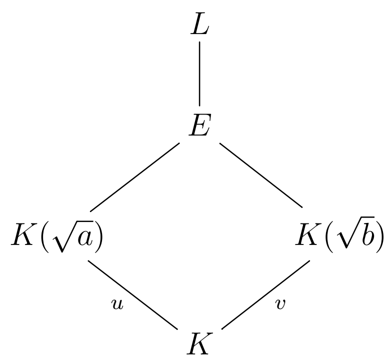
$$[A] = \amalg_{h \in Q_8^{\text{ab}}} E x_h$$

subject to the relations

1. $x_h \cdot x_k = g(a, b) x_{hk}$ for all $h, k \in Q_8^{\text{ab}}$ and

2. $\alpha \cdot x_h = x_h \cdot h(\alpha)$ for all $h \in Q_8^{\text{ab}}$ and $\alpha \in E$.

The biquadratic extension E can be written $E = K(\sqrt{a}, \sqrt{b})$ for some $a, b \in K$ where $\text{Gal}(K(\sqrt{a})/K) = \langle u \rangle$ and $\text{Gal}(K(\sqrt{b})/K) = \langle v \rangle$.



Notice that since

$$\begin{aligned} g(e_1, e_1) &= u^2 \cdot (u^2)^{-1} = 1 \\ g(e_2, e_2) &= v^2 \cdot (v^2)^{-1} = 1 \\ g(e_1, e_2) &= u \cdot v \cdot (uv)^{-1} = 1 \\ g(e_2, e_1) &= v \cdot u \cdot (uv)^{-1} = vuv^{-1}u^{-1} = -1 \end{aligned} \tag{5.8}$$

we have

$$\begin{aligned} x_u \cdot x_u &= 1 \\ x_v \cdot x_v &= 1 \\ x_u \cdot x_v &= -x_v \cdot x_u \end{aligned} \tag{5.9}$$

and so the subalgebra $\langle x_u, x_v \rangle$ of A is isomorphic to the split quaternion algebra $(1, 1)$. By the centralizer theorem, we have

$$A = (1, 1) \otimes C_A(\langle x_u, x_v \rangle)$$

and since the centralizer of $\langle x_u, x_v \rangle$ in A is given by the subalgebra $\langle \sqrt{a}x_v \sqrt{b}x_u \rangle$, we have

$$A = (1, 1) \otimes \langle \sqrt{a}x_v \sqrt{b}x_u \rangle.$$

Now, we see that since

$$\begin{aligned} \sqrt{a}x_v \cdot \sqrt{a}x_v &= a \\ \sqrt{b}x_u \cdot \sqrt{b}x_u &= b \\ \sqrt{a}x_v \cdot \sqrt{b}x_u &= -\sqrt{b}x_u \cdot \sqrt{a}x_v \end{aligned} \tag{5.10}$$

we have that $\langle \sqrt{a}x_v \sqrt{b}x_u \rangle$ is isomorphic to the the quaternion algebra (a, b). Thus, $[A] = [(a, b)]$, and so $E = K(\sqrt{a}, \sqrt{b})$ embeds into a Q_8 extension L/K when the quaternion algebra (a, b) splits over K.

Now, in order to show that there are no unramified normalized decomposable Q_8-invariants, we must show that for every nontrivial normalized decomposable Q_8^{ab}-invariant (c, d) there exists some field K/F and a $Q_8^{\mathrm{ab}} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ extension $K(\sqrt{a}, \sqrt{b})$ of K which embeds in a Q_8 extension of K and such that $(c, d)(K(\sqrt{a}, \sqrt{b}))$ is ramified.

Let K be a field with valuation v trivial on F, with uniformizer π, and let (c, d) be a nontrivial element of

$$\mathrm{Inv}^2_{\mathrm{dec}}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Q}/\mathbb{Z}(1))^{\mathrm{nr}}_{\mathrm{norm}} \simeq Q_8^{\mathrm{ab}} \otimes F^\times = F^\times/F^{\times 2} \times F^\times/F^{\times 2}.$$

Since (c, d) is nontrivial, either c or d is not a square. If c is not a square, choose $(a, b) = (\pi, 1)$. Then the image of $(c, d)(K(\sqrt{a}, \sqrt{b}))$ under the residue map ∂_v is given by

$$\partial_v([(c, \pi)] + [(d, 1)]) = c$$

which is nontrivial. On the other hand, if c is a square in F, then d is not, and so we can choose $(a, b) = (1, \pi)$ and we have

$$\partial_v([(c, 1)] + [(d, \pi)]) = d$$

which is nontrivial. In either case, $(c, d)(K(\sqrt{a}, \sqrt{b}))$ is ramified and the quaternion algebra (a, b) splits. Thus, $\mathrm{Inv}^2_{\mathrm{dec}}(G, \mathbb{Q}/\mathbb{Z}(1))^{\mathrm{nr}}_{\mathrm{norm}} = 0$. □

Note. It is actually known that a Q_8 is stably rational over $\mathbb{Q}$ [14], [11]. It was proven by Grobner in 1934, but the result was lost after World War II, unfortunately.

5.4 Dihedral Group D_8

We now consider the case of the dihedral group D_8. We will show that $\mathrm{Inv}^2_{\mathrm{dec}}(G, \mathbb{Q}/\mathbb{Z}(1))^{\mathrm{norm}}_{\mathrm{nr}} = 0$ when F has characteristic different from 2. In fact, the stronger statement that D_8 is stably rational over $\mathbb{Q}$ is known [14].

Theorem. *Let F be a field of characteristic not equal to two. Then there are no nontrivial unramified normalized decomposable degree two cohomological invariants of D_8 over F with coefficients in $\mathbb{Q}/\mathbb{Z}(1)$.*

Proof. The map

$$p^* : \mathrm{Inv}^2_{\mathrm{dec}}(D_8^{\mathrm{ab}}, \mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}} \to \mathrm{Inv}^2_{\mathrm{dec}}(D_8, \mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}} \tag{5.11}$$

induced by the projection of D_8 onto its abelianization $D_8^{\mathrm{ab}} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ sends a decomposable D_8^{ab}-invariant I to the D_8-invariant defined by $p^*(I)(L/K) = I(L/K^{[D_8,D_8]})$ for any D_8-Galois extension L/K. Since p^* is surjective, in order to show that there are no unramified normalized decomposable D_8-invariants, it is enough to show that for every every normalized decomposable D_8^{ab}-invariant I there is a D_8^{ab}-Galois extension E/K such that $I(E/K)$ is ramified and E/K embeds in a D_8-Galois extension L/K.

First, we determine an equivalent condition for a $D_8^{\mathrm{ab}} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ extension to extend to a D_8-Galois extension. Writing $D_8 = \langle \sigma, \tau | \sigma^4 = \tau^2 = 1, \tau\sigma\tau = \sigma^{-1} \rangle$, we can write the commutator subgroup $[D_8, D_8] = \langle \sigma^2 \rangle \simeq \mathbb{Z}/2\mathbb{Z}$. Since $\langle \sigma^2 \rangle$ is central in D_8, the exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to D_8 \to D_8^{\mathrm{ab}} \to 1 \tag{5.12}$$

yields the exact sequence for each field extension K/F

$$H^1(K, D_8) \to H^1(K, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \to H^2(K, \mathbb{Z}/2\mathbb{Z}) \simeq_2 \mathrm{Br}(K) \tag{5.13}$$

where the first map is given by the restriction. So a D_8^{ab}-Galois algebra E/K extends to D_8-Galois algebra L/K if and only if its image in ${}_2\mathrm{Br}(K)$ is trivial. The map above from

$$\delta : H^1(K, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \to_2 \mathrm{Br}(K) \tag{5.14}$$

is given by the crossed product. So we need to characterize the kernel of this crossed product.

Let $E = K(\sqrt{a}, \sqrt{b})$ be a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-Galois extension of K. If $g \in H^2(K, \mathbb{Z}/2\mathbb{Z})$ is the 2-cocycle corresponding to the extension D_8 of D_8^{ab} in (5.12), and $f \in H^2(D_8^{\text{ab}}, E^\times)$ is the composition of g with the inclusion $\mathbb{Z}/2\mathbb{Z} \hookrightarrow E^\times$, then δ sends E/K to the Brauer class of the Azumaya algebra $A = (E, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, f)$.

We now compute explicitly the Brauer class $[A]$. To simplify notation, let $\alpha = \sqrt{a}$, $\beta = \sqrt{b}$. As an algebra over K, A is generated by the elements

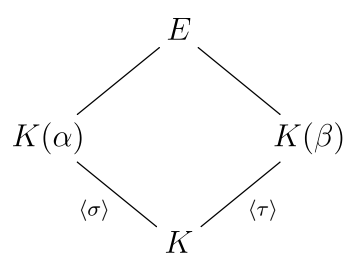
$$A = \langle 1, x_\alpha, x_\beta, x_\sigma, x_\tau, x_{\sigma\tau} \rangle \tag{5.15}$$

subject to the relations

i. $x_s \cdot x_t = f(s,t)x_{st}$ for any $s, t \in D_8$ and

ii. $a \cdot x_s = s(a)a$ for any $a \in E$, and any $s \in D_8$.

with σ and τ are generators of $D_8^{\text{ab}} = \mathbb{Z}/2\mathbb{Z}$ as below.



From (5.12), we have a lift of each $h \in D_8^{\text{ab}}$ to D_8. Letting $\hat{h}$ denote the lift of h, we have

$$\begin{aligned} \hat{\sigma} &= \sigma \\ \hat{\tau} &= \tau \\ \hat{\sigma} &= \sigma \\ \widehat{\sigma\tau} &= \sigma\tau \end{aligned} \tag{5.16}$$

where the elements on the right sides of the equal signs are the elements of D_8. Since for any $s, t \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $g(s,t)$ is explicitly given by

$$g(s,t) = \hat{s} \cdot \hat{t} \cdot \widehat{st}^{-1} \in \langle \sigma^2 \rangle = \mathbb{Z}/2\mathbb{Z}$$

we see that

$$\begin{aligned}
g(\sigma,\sigma) &= \hat{\sigma}\cdot\hat{\sigma}\cdot\hat{1}^{-1} = \sigma^2 \\
g(\tau,\tau) &= \hat{\tau}\cdot\hat{\tau}\cdot\hat{1}^{-1} = 1 \\
g(\sigma,\tau) &= \hat{\sigma}\cdot\hat{\tau}\cdot\widehat{\sigma\tau}^{-1} = \sigma\tau(\sigma\tau)^{-1} = 1 \\
g(\tau,\sigma) &= \hat{\tau}\cdot\hat{\sigma}\cdot\widehat{\tau\sigma}^{-1} = \tau\sigma(\sigma\tau)^{-1} = \sigma^2
\end{aligned} \tag{5.17}$$

Thus, as $f(s,t)$ is the image of $g(s,t)$ after composition with the inclusion $\mathbb{Z}/2\mathbb{Z} \hookrightarrow E^\times$, we have

$$x_\sigma\cdot x_\sigma = -1 \qquad x_\tau\cdot x_\tau = 1 \qquad x_\sigma\cdot x_\tau = -x_\tau\cdot x_\sigma \tag{5.18}$$

From (5.18), we see that the subalgebra B of A generated over K by the elements $\{1, x_\sigma, x_\tau, x_{\sigma\tau}\}$ is the split quaternion algebra $(-1,1)$. So by the Centralizer Theorem,

$$[A] = [(-1,1)\otimes C_A(B)] = [C_A(B)]$$

where $C_A(B)$ denotes the centralizer of B in A. Computing this centralizer, we find $C_A(B)$ is generated over K by the elements $\{1, \alpha x_\tau, \beta x_\sigma, \alpha\beta x_{\sigma\tau}\}$. Indeed, each of these elements commute with x_σ and x_τ and for dimension considerations we know $C_A(B)$ has dimension 4 over K. Notice that

$$\alpha x_\tau\cdot\alpha x_\tau = \alpha^2 = a \qquad \beta x_\sigma\cdot\beta x_\sigma = -\beta^2 = -b \qquad \alpha x_\tau\cdot\beta x_\sigma = -\beta x_\sigma\cdot\alpha x_\tau \tag{5.19}$$

Thus, $C_A(B)$ is isomorphic to the quaternion algebra $(a,-b)$ over K, and we have

$$[A] = [(a,-b)] \tag{5.20}$$

and so $E = K(\sqrt{a},\sqrt{b})$ extends to a D_8 extension of K if and only if the quaternion algebra $(a,-b)$ splits over K.

Now, to show that there are no nontrivial unramified normalized decomposable degree two cohomological invariants of D_8 over F with coefficients in $\mathbb{Q}/\mathbb{Z}(1)$, it suffices to find for each nontrivial normalized decomposable $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$-invariant (c,d) a field K/F and a pair a, b such that $(c,d)(K(\sqrt{a},\sqrt{b}))$ is ramified. Take K to be an extension of F with valuation v trivial on F and uniformizer π. Since the evaluation map

$$\mathrm{Inv}^2_{\mathrm{dec}}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Q}/\mathbb{Z}(1))^{\mathrm{norm}} \otimes H^1(K, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Br}(K) \tag{5.21}$$

is given by

$$((c,d), K(\sqrt{a},\sqrt{b})) \mapsto [c \cup a] + [d \cup b] = [(c,a)] + [(d,b)] \tag{5.22}$$

we must find for any $(c,d) \in F^\times/F^{\times 2} \times F^\times/F^\times$ nontrivial, a pair (a,b) so $\partial([(c,a)]+[(d,b)])$ is not a square. If c is not a square, we can choose $(a,b) = (\pi, 1)$, and we have the residue at v, $\partial((a,\pi)+(b,1)) = a$ is nontrivial. Similarly, if c is a square, we take $(a,b) = (1,\pi)$, and so (c,d) is ramified. □

We now use the proof above to compute the nontrivial unramified normalized decomposable degree two cohomological invariants of D_{16} over F with coefficients in $\mathbb{Q}/\mathbb{Z}(1)$.

5.5 Dihedral Group D_{16}

We now show that there are no unramified normalized decomposable degree two cohomological invariants of D_{16} over F with coefficients in $\mathbb{Q}/\mathbb{Z}(1)$ when the characteristic of F is different from two. We mention, however, that a stronger result is know: that D_{16} is stably rational even over an arbitrary field F [15]. Our strategy is similar to the D_8 case: we will show that the map

$$p^* : \mathrm{Inv}^2_{\mathrm{dec}}(D_{16}, \mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}} \to \mathrm{Inv}^2_{\mathrm{dec}}(D_{16}^{\mathrm{ab}}, \mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}}$$

is surjective by finding for each element $(c,d) \in \mathrm{Inv}^2_{\mathrm{dec}}(D_{16}^{\mathrm{ab}}, \mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}}$ a biquadratic extension $E = K(\sqrt{a},\sqrt{b})$ of K such that $(c,d)(E/K)$ is ramified and E/K can be embedded in a D_{16} extension of K. The following lemma gives conditions for such an E/K to be embedded into a D_{16} extension. The proof relies on the fact given by section 3.4 and Theorem 2.2.2 of [28] that in this particular case, such an L/K can be embedded into D_{16} when the compatibility condition that the crossed product algebra $A = D_{16} \times K\sqrt{a}, \sqrt{b}$ described in the proof is isomorphic to a matrix algebra of order 4 over a subalgebra.

Lemma 11. *Let $E = K(\sqrt{a}, \sqrt{b})$ be a biquadratic extension of a field K. Then if the quaternion algebra $(a, -b)$ splits over K and the quaternion algebra $(a, 2)$ splits over $K[\sqrt{-b}]$, then E can be embedded in a D_{16}-Galois extension of K.*

Proof. Consider the projection

$$\pi : D_{16} \to D_{16}^{\mathrm{ab}} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \tag{5.23}$$

which sends σ and τ to generators of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We let $A = D_{16} \times K\sqrt{a}, \sqrt{b}$ denote the algebra consisting of sums $\Sigma_{g \in D_{16}} u_g x_g$ where $x_g \in K\sqrt{a}, \sqrt{b}$ and

1. $x_g \cdot x_h = x_{gh}$ for $g, h \in D_{16}$, and
2. $c \cdot x_g = x_g \cdot \pi(g)(c)$ for $c \in K(\sqrt{a}, \sqrt{b})$.

Since the kernel of π is cyclic of degree 4, section 3.4 and Theorem 2.2.2 of [28] together say that in order to show that such an embedding exists it is enough to show that

$$D_{16} \times K(\sqrt{a}, \sqrt{b}) \simeq M_4(Z)$$

where Z is the center of A. Let

$$y = \Sigma_{g \in D_{16}} u_g x_g \in D_{16} \times K(\sqrt{a}, \sqrt{b})$$

be an element of Z. Then

$$\sqrt{a} \cdot \Sigma_{g \in D_{16}} u_g x_g = \Sigma_{g \in D_{16}} \pm u_g x_g \cdot \sqrt{a}$$

where the sign depends on the action of g on $\sqrt{a}$. Since if y commutes with $\sqrt{a}$, each u_g must commute with $\sqrt{a}$. Similarly, since y commutes with $\sqrt{b}$, then each u_g must commute with $\sqrt{b}$. So, in order for y to be in the center of $D_{16} \times K(\sqrt{a}, \sqrt{b})$, each u_g with x_g nonzero is an element of $\{1, u_{\sigma^2}, u_{\sigma^4}, u_{\sigma^6}\}$. Now, write

$$y = x_1 + u_{\sigma^2} x_{\sigma^2} + u_{\sigma^4} x_{\sigma^4} + u_{\sigma^4} x_{\sigma^4}$$

and notice that

$$x_\tau y = y x_\tau$$

implies that x_1 and x_{σ^4} are fixed by τ, and $\tau(x_{\sigma^2}) = x_{\sigma^6}$. Similarly, from

$$x_\sigma y = y x_\sigma$$

we have that σ fixes each of $x_1, x_{\sigma^2}, x_{\sigma^4}, x_{\sigma^6}$. Together, we have that $x_1, x_{\sigma^4} \in K$ and $x_{\sigma^2}, x_{\sigma^6} \in K(\sqrt{a}, \sqrt{b})$. In particular, $\dim_K(Z) < 8$.

On the other hand,

$$\{1, x_{\sigma^4}, x_{\sigma^2} + x_{\sigma^6}, \sqrt{b}x_{\sigma^2} - \sqrt{b}x_{\sigma^6}\} \subset Z.$$

To verify this, if is sufficient to check that each of the above elements commutes with x_σ, x_τ, $\sqrt{a}$ and $\sqrt{b}$. Since $\dim_K(Z)$ divides $\dim_K(A) = 64$ and $4 \leq \dim_K(Z) < 8$, we must have $\dim_K(Z) = 4$, and so Z is generated over K by $\{1, x_{\sigma^4}, x_{\sigma^2} + x_{\sigma^6}, \sqrt{b}x_{\sigma^2} - \sqrt{b}x_{\sigma^6}\}$.

Notice that the elements

$$e_1 = \frac{1}{2}(1 + x_{\sigma^4})$$
$$e_2 = \frac{1}{2}(1 - x_{\sigma^4})$$

of Z are primitive idempotents. So identifying x_{σ^2} with and indeterminate X over K, we have

$$Z = Z/e_1 Z \times Z/e_2 Z = K[X]/\langle X^2 - 1\rangle \times K[\sqrt{-b}] = K \times K \times K[\sqrt{-b}] \tag{5.24}$$

So, from the above discussion, $K(\sqrt{a}, \sqrt{b})$ embeds into a D_{16}-Galois extension of K when

$$A = D_{16} \times K(\sqrt{a}, \sqrt{b}) = M_4(Z) = M_4(K) \times M_4(K) \times M_4(K[\sqrt{-b}]).$$

Now, we have

$$A = Ae_1 \times Ae_2$$

but notice that the elements

$$f_1 = \frac{1}{2}(1 + x_{\sigma^2})e_1$$
$$f_2 = \frac{1}{2}(1 - x_{\sigma^2})e_1$$

are primitive idempotents in Ae_1. Thus,

$$A = Ae_1 f_1 \times Ae_1 f_2 \times Ae_2.$$

Claim. *Each of $Ae_1 f_1$, $Ae_1 f_2$, and Ae_2 is a central simple algebra over K.*

Proof. Since A has no central nilpotent elements, by [20] we have that A is semisimple.

Consider first $Ae_2 \simeq A/\langle 1 + x_{\sigma^4} \rangle$. Notice that since $x_{\sigma^4} = -1$ in Ae_2, we have

$$(x_\sigma - x_{\sigma^3})^2 = x_{\sigma^2} - 2x_{\sigma^4} + x_{\sigma^6} = x_{\sigma^2} + 2 - x_{\sigma^2} = 2$$

and as $(x_\sigma - x_{\sigma^3})$ anticommutes with $\sqrt{a}$, the subalgebra $Q = \langle \sqrt{a}, x_\sigma - x_{\sigma^3} \rangle$ is isomorphic to the quaternion algebra $(a, 2)$. Since Ae_2 is a central simple algebra over K, we have

$$Ae_2 = Q \otimes C_{Ae_2}(Q)$$

Since

$$\begin{aligned} C_{Ae_2}(Q) &= \langle 1, \sqrt{b}, x_\tau, x_{\sigma^2}, \sqrt{b}x_\tau, \sqrt{b}x_{\sigma^2}, x_{\sigma^2\tau}, \sqrt{b}x_{\sigma^2\tau}\rangle \\ &= \langle \sqrt{b}, x\tau\rangle \otimes K[\sqrt{b}x_{\sigma^2}]/\langle x_{\sigma^4}+1\rangle \\ &= (b,1)\otimes K[\sqrt{-b}] \end{aligned} \tag{5.25}$$

we have

$$Ae_2 = Q\otimes C_{Ae_2}(Q) = (a,2)\otimes(b,1)\otimes K[\sqrt{-b}] = (a,2)\otimes M_2(K)\otimes F[\sqrt{-b}] = M_2((a,2)\otimes K[\sqrt{-b}]) \tag{5.26}$$

Now consider $Ae_1f_1 = A/\langle x_{1-\sigma^2}\rangle$. Since Ae_1f_1 is central simple over K and the subalgebra $\langle x_\sigma, \sqrt{a}\rangle$ of Ae_1f_1 is isomorphic to the quaternion algebra $(1,a)$, we have

$$Ae_1f_1 = (1,a)\otimes C_{Ae_1f_1}(\langle x_\sigma, \sqrt{a}\rangle).$$

But $C_{Ae_1f_1}(\langle x_\sigma, \sqrt{a}\rangle) = \langle x_\tau, \sqrt{b}\rangle$ which is isomorphic to the split quaternion algebra $(1,b)$, and so

$$Ae_1f_1 = (1,a)\otimes(1,b) = M_4(K). \tag{5.27}$$

Finally, consider the algebra $Ae_1f_2 = A/\langle x_{1+\sigma^2}\rangle$. Since the subalgebra $\langle\sqrt{a}, \sqrt{b}x_\sigma\rangle$ is isomorphic to $(a,-b)$ and the centralizer of $\langle\sqrt{a}, \sqrt{b}x_\sigma\rangle$ in Ae_1f_2 is $\langle\sqrt{b}, x_\tau\rangle \simeq (b,1)$, we have

$$Ae_1f_2 = (a,-b)\otimes(b,1) = (a,-b)\otimes M_2(K) = M_2((a,-b)) \tag{5.28}$$

Thus,

$$A = M_4(K)\otimes M_2((a,-b))\otimes M_2((a,2)\otimes K[\sqrt{-b}]) \tag{5.29}$$

and since $K(\sqrt{a},\sqrt{b})$ embeds into a D_{16}-Galois extension of K when

$$A = M_4(K) \times M_4(K) \times M_4(K[\sqrt{-b}]),$$

Thus, $K(\sqrt{a}, \sqrt{b})$ extends to a D_{16}-extension of K when $(a, -b)$ splits over K and $(a, 2)$ splits over $K[\sqrt{-b}]$. $\square$

Theorem. *Let F be a field of characteristic not equal to two. Then there are no nontrivial unramified normalized decomposable degree two cohomological invariants of D_{16} over F with coefficients in $\mathbb{Q}/\mathbb{Z}(1)$.*

Proof. To show that $\mathrm{Inv}^2_{\mathrm{dec}}(D_{16}, \mathbb{Q}/\mathbb{Z}(1))^{\mathrm{nr}}_{\mathrm{norm}} = 0$, it is enough to show that for every nontrivial normalized decomposable D_{16}^{ab}-invariant (c, d) over F, there is an field K/F and an extension $K(\sqrt{a}, \sqrt{b})/K$ such that

$$(c, d)(K(\sqrt{a}, \sqrt{b})/K) = (c, a) + (d, b)$$

is ramified and which extends to a D_{16}-extension of K. From the previous lemma, this means we must find for each nontrivial $(c, d) \in F^\times/F^{\times 2} \times F^\times/F^{\times 2}$ some pair $(a, b) \in K^\times \times K^\times$ satisfying:

1. the quaternion algebra $(a, 2)$ splits over $K(\sqrt{-b})$,
2. the quaternion algebra $(a, -b)$ splits over K, and
3. $(c, d)(a, b) = (c, a) + (d, b) \in H^2(K, \mathbb{Q}/\mathbb{Z}(1))$ is ramified.

Let K be a field with valuation v trivial on F and let π denote the uniformizer of K. Since (c, d) is nontrivial, not both c and d are squares. Suppose first that d is not a square. Then we can take $a = 1$,$b = \pi$, since $(a, 2) = (1, 2)$ and $(a, -b) = (1, -\pi)$ are both split over K and since

$$\partial_v((c,a)+(d,b)) = \partial_v((c,1)+(d,\pi)) = d \tag{5.30}$$

which is nontrivial, making $(c,d)(a,b)$ ramified. On the other hand, if d is a square, we can choose $a=b=\pi$. In order to show that the quaternion algebra (a,b) splits over a field k, by [10, Proposition 1.1.7], it is equivalent to show that $b = N_{k(\sqrt{a})/k}(u)$ for some $u \in k$. So quaternion algebra $(a,2) = (\pi,2)$ splits over $K(\sqrt{-b}) = K(\sqrt{-\pi})$ if $2 = x^2 - \pi y^2$ for some $x, y \in K(\sqrt{-\pi})$. Taking $x=1$ and $y = \frac{1}{\sqrt{-\pi}}$,

$$x^2 - \pi y^2 = 1 + 1 = 2 \tag{5.31}$$

as desired. So $(a,2)$ splits over $K(\sqrt{-b})$ in this case. Similarly, to see that $(a,-b) = (\pi,-\pi)$ is split over K, it is enough to see that $-\pi$ is a norm of the extension $K(\sqrt{\pi})/K$; i.e. that $-\pi = r^2 - \pi s^2$ for some choice of $r, s \in K$. But this is the case for $r=0$, $s=1$. So $(a,-b)$ splits over K. Finally,

$$\partial_v((c,a)+(d,b)) = \partial_v((c,-\pi)+(d,\pi)) = c \tag{5.32}$$

as d is trivial, and since c is not a square, $(c,d)(a,b)$ is ramified. $\square$

5.6 Dihedral Group D_{2n} for n Odd

We will show that when $G = D_{2n}$ for n odd, $\mathrm{Inv}_{\mathrm{dec}}(G, \mathbb{Q}/\mathbb{Z}(1))^{\mathrm{nr}}_{\mathrm{norm}} = 0$. Note that in this case, the commutator subgroup $[D_{2n}, D_{2n}]$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$, and the abelianization D_{2n}^{ab} of D_{2n} is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Theorem. *Let F be a field of characteristic not equal to two. Then there are no nontrivial unramified normalized decomposable degree-two cohomological D_{2n}-invariants over F with coefficients in $\mathbb{Q}/\mathbb{Z}(1)$ if n is odd.*

Our method will be similar to the two dihedral examples above. First, we will use the following lemma which will give us a condition for a quadratic extension to extend to a D_{2n}

extension. Then since the map

$$p^* : \mathrm{Inv}^2_{\mathrm{dec}}(D_{2n}^{\mathrm{ab}}, \mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}} \to \mathrm{Inv}^2_{\mathrm{dec}}(D_{2n}, \mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}} \tag{5.33}$$

is surjective, it is enough to find for each decomposable D_{2n}^{ab} invariant I a quadratic extension E/K such that $I(E/K)$ is ramified and E/K extends to a D_{2n} extension L/K. The following gives us a condition for a quadratic extension to embed in D_{2n} extension when n is odd.

Lemma 12. *Let $n \geq 3$ be an odd integer, and let K be a field containing a primitive n^{th} root of unity. Then a $\mathbb{Z}/2\mathbb{Z}$-Galois extension E/K with automorphism group $\langle \tau \rangle$ exetnds to a D_{2n} extension of K if and only if there is some $x \in E^\times$ satisfying $x\tau(x) \in {K^\times}^n$.*

Proof. Suppose first that E/K embeds in the D_{2n} extension L/K. Then $L = E(\alpha)$ for some $\alpha \in L^\times$.

$$\begin{array}{c} L = E(\alpha) \\ | \\ E \\ | \\ K \end{array}$$

Set $\beta = \tau(\alpha)$ and let $x = \alpha^n$. We will show that $x\tau(x)$ is an n^{th} power in $K^\times$.

Notice that

$$x\tau(x) = \alpha^n \beta^n = (\alpha\beta)^n,$$

and so it suffices to show that $\alpha\beta \in K^\times$. Since K contains a primitive n^{th} root of unity ζ, $\mathrm{Gal}(L/E)$ is cyclic of order n. Let σ be a generator of $\mathrm{Gal}(L/E)$, sending α to $\zeta\alpha$. The element $\alpha\beta \in L$ is fixed by σ since

$$\sigma(\alpha\beta) = \zeta\alpha \cdot \sigma(\beta) = \zeta\alpha \cdot \sigma(\tau(\alpha)) = \zeta\alpha \cdot \tau(\sigma^{-1}(\alpha)) = \zeta\alpha \cdot \tau(\zeta^{-1}\alpha) = \alpha \cdot \tau(\alpha) = \alpha\beta$$

Also,

$$\tau(\alpha\beta) = \tau(\alpha\tau(\alpha)) = \tau(\alpha)\alpha = \alpha\beta$$

and so $\alpha\beta$ is fixed by τ. Since $\alpha\beta \in K$ and is nonzero, $\alpha\beta \in K^\times$.

Conversely, assume that there is some $x \in E^\times$ and some $y \in K^\times$ such that

$$x\tau(x) = y^n.$$

We will show that E/K embeds in a D_{2n} extension L/K. Set $\alpha = x^{1/n}$, $L = E(\alpha)$, and let $\sigma : L \to L$ be the automorphism of L/E defined by $\sigma(\alpha) = \zeta\alpha$ for some primitive root of unity $\zeta \in K$. Note that the automorphism τ of E/K can be extended to an automorphism of L/K via

$$\tau(\alpha) = y\alpha^{-1}$$

Let $G = \langle\sigma, \tau\rangle$. To show that L/K is $G-Galois$, we need to show that $L^G = K$, $|G| = [L : K]$, and that $\mathrm{Aut}(L/K) = G$. It is clear that $L^G = K$. Also, $G \subset \mathrm{Aut}(L/K)$, and since

$$2n = |G| \leq |\mathrm{Aut}(L/K)|[L : K] = 2n$$

we have $\mathrm{Aut}(L/K) = G$. Finally, to see that $G = D_{2n}$, we need to check that σ and τ satisfy the dihedral relations. Note that since $y \in K$, $\tau(y) = y$. Then

$$\tau^2(\alpha) = \tau(y\alpha^{-1}) = y\tau(\alpha^{-1}) = y(y\alpha^{-1})^{-1} = \alpha$$

and so the automorphism τ has order two. Since $\sigma^n = 1$, $\tau^2 = 1$, it remains to check that $\tau \cdot \sigma = \sigma^{-1} \cdot \tau$. Let $\gamma \in E^\times$ be an element such that $\tau(\gamma) = -\gamma$, so that $L = K(\alpha, \gamma)$. Then

$$\tau \cdot \sigma(\alpha) = \tau(\zeta\alpha) = \zeta \cdot y\alpha^{-1} = y(\zeta^{-1}\alpha)^{-1} = y(\sigma^{-1}(\alpha))^{-1} = y\sigma^{-1}(\alpha^{-1}) = \sigma^{-1}\tau(\alpha)$$

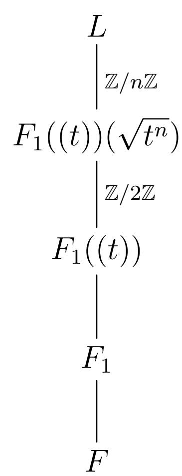
and

$$\tau \cdot \sigma(\gamma) = \tau(\gamma) = -\gamma = \sigma^{-1}(-\gamma) = \sigma^{-1}(\tau(\gamma))$$

and so $G = D_{2n}$. □

Now for the proof of (5.6).

Proof. Let (a), where $a \in F \times / F^{\times 2}$, be a nontrivial normalized decomposable invariant for D_{2n} over F with coefficients in $\mathbb{Q}/\mathbb{Z}(1)$, where n is odd and $|G|$ is prime to the characteristic of F. Then we will find a field K/F and a quadratic extension E/K such that E/K embeds in a D_{2n} extension L/K and $(c,d)(E/K)$ is ramified. Let F_1 be a field extension of F containing a primitive n^{th} root of unity. Choose $K = F_1((t))$ with the usual valuation, and $E = K(\sqrt{t^n})$. Let τ be the generator of $\mathrm{Gal}(E/K)$.



Then by the above lemma, E/K embeds in a D_{2n} extension since $\sqrt{t^n}\tau\sqrt{t^n} = -t^n = (-1)^n t^n \in K^{\times n}$. Also, $(a)(E/K)$ evaluates to the quaternion algebra (a, t^n) which is ramified. Indeed, $\partial((a, t^n)) = a^n$ which is nontrivial in $F^\times / F^{\times 2}$ since n is odd. Thus, the decomposable invariant (a) is ramified. □

CHAPTER 6

Degree-Two Invariants in Characteristic Dividing $|G|$

In the previous chapter, we obtained formulas for the normalized unramified decomposable invariants for various finite G in the case when the characteristic of the base field F does not divide the order of G. We now consider the case when the characteristic of F does divide $|G|$.

6.1 Case for Cyclic G of Prime Order

Let $G = \mathbb{Z}/p\mathbb{Z}$ over a field F of characteristic p. In this section, we will show that there are no nontrivial normalized unramified decomposable invariants for G over F with coefficients in $\mathbb{Q}/\mathbb{Z}(1)$, i.e. $\mathrm{Inv}^2_{\mathrm{dec}}(G, \mathbb{Q}/\mathbb{Z}(1))^{\mathrm{nr}}_{\mathrm{norm}} = 0$.

Recall that when the characteristic of F does not divide the order of G, for any field extension K/F, an element of $H^i(K, \mathbb{Q}/\mathbb{Z}(i-1))$ is called unramified if it is in the kernel of the residue maps

$$\partial_v : H^i(K, \mathbb{Q}/\mathbb{Z}(i-1)) \to H^{i-1}(\kappa_v, \mathbb{Q}/\mathbb{Z}(i-2))$$

for each discrete valuation v on K trivial on F. Here κ_v is the residue field. These residue maps fit into an exact sequence [6]

$$\cdots \to H^i(\mathcal{O}_v, \mathbb{Q}/\mathbb{Z}(i-1)) \xrightarrow{\alpha_v} H^i(K, \mathbb{Q}/\mathbb{Z}(i-1)) \xrightarrow{\partial_v} H^{i-1}(\kappa_v, \mathbb{Q}/\mathbb{Z}(i-2)) \to \cdots \quad (6.1)$$

where $\mathcal{O}_v$ is the valuation ring of v. Here $H^i(\mathcal{O}_v, \mathbb{Q}/\mathbb{Z}(i-1))$ denotes the tale cohomology group $H^i_{\text{t}}(\text{Spec}(O_v), \mathbb{Q}/\mathbb{Z}(i-1))$. In the case when $i = 2$, $H^2(\mathcal{O}_v, \mathbb{Q}/\mathbb{Z}(1))$ can be interpreted as $\text{Br}(\mathcal{O}_v)$ which is the group of equivalence classes of Azumaya algebras over O_v just as $H^2(K, \mathbb{Q}/\mathbb{Z}(1))$ is $\text{Br}(K)$. The map

$$\alpha_v : H^2(\mathcal{O}_v, \mathbb{Q}/\mathbb{Z}(1)) \to H^2(K, \mathbb{Q}/\mathbb{Z}(1)) \tag{6.2}$$

is given by sending an Azumaya algebra $[A] \in \text{Br}(\mathcal{O}_v)$ to $[A \otimes_{\mathcal{O}_v} K] \in \text{Br}(K)$.

In the case when the characteristic of F divides the order of G, residue maps

$$\partial_v : H^2(K, \mathbb{Q}/\mathbb{Z}(1)) \to H^1(K, \mathbb{Q}/\mathbb{Z}(0))$$

are not defined. However, there is still a notion of an uramified element of $H^2(G, \mathbb{Q}/\mathbb{Z}(1))$; namely, $x \in H^2(K, \mathbb{Q}/\mathbb{Z}(1))$ is said to be unramified if x is in the image of the map α_v described in (6.1) for each discrete valuation v of K trivial on F.

Theorem. *Let F be a field of characteristic p and let $G = \mathbb{Z}/p\mathbb{Z}$. Then there are no normalized, unramified degree-two invariants for G; i.e. $Inv^2(G, \mathbb{Q}/\mathbb{Z}(1))^{norm}_{nr} = 0$.*

Recall from Chapter 5 that we have the following exact sequence

$$0 \to \text{Inv}^2_{\text{dec}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Q}/\mathbb{Z}(1))_{\text{norm}} \to \text{Inv}^2(\mathbb{Z}/p\mathbb{Z}, \mathbb{Q}/\mathbb{Z}(1))_{\text{norm}} \to \text{Tor}(H^3(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}), F^\times).$$

When F has characteristic p, $\text{Tor}(H^3(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}), F^\times) = 0$, and so

$$\text{Inv}^2_{\text{dec}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Q}/\mathbb{Z}(1))_{\text{norm}} = \text{Inv}^2(\mathbb{Z}/p\mathbb{Z}, \mathbb{Q}/\mathbb{Z}(1))_{\text{norm}};$$

i.e. every invariant is decomposable.

To prove the above lemma, it is enough to show that there exists a G-Galois algebra L/K over F such that for for any normalized decomposable invariant $\chi \cup a \in G^* \otimes F^\times (\simeq$

$\mathrm{Inv}_{dec}(G, \mathrm{Br})^{\mathrm{norm}})$, $(\chi \cup a)(L/K)$ is nontrivial and ramified whenever $\chi \cup a$ is nontrivial. We will show that the extension $L = K[x]/\langle x^p - x - t^{-1}\rangle$ of $K = F((t))$ satisfies this property. In this case the crossed product construction gives us that $(\chi \cup a)(L/K)$ is the p-algebra $[t^{-1}, a)$.

Lemma 13. *Let F be a field of characteristic p, and let $a \in F^{\times}$. Then $[t^{-1}, a)$ is split if and only if $\chi \cup a$ is trivial.*

Proof. It is clear that if $\chi \cup a$ is trivial, then $[t^{-1}, a)$ split. Suppose that $[t^{-1}, a)$ is split. We will show that a is a p^{th} power in $F^{\times}$, which, since χ is p-torsion, implies that $\chi \cup a$ is trivial. If $[t^{-1}, a)$ is split, then a theorem of Teichmüller says

$$t^{-1} = \gamma_0^p - \gamma_0 + \gamma_1^p a + \gamma_2^p a^2 + \cdots + \gamma_{p-1}^p a^{p-1} \tag{6.3}$$

for some $\gamma_0, \cdots, \gamma_{p-1} \in F((t))$. To get a contradiction, suppose that a is not a p^{th} power in F. Then $[K(\sqrt[p]{a}) : K] = p$ and so $\{1, \sqrt[p]{a}, (\sqrt[p]{a})^2, \cdots, (\sqrt[p]{a})^{p-1}\}$ are linearly independent over K. Thus, equation (6.3) implies that $\gamma_0^p - \gamma_0 = t^{-1}$. Letting v be the valuation on K/F defined by the degree of its lowest degree term in t, we have that

$$-1 = v(t^{-1}) = v(\gamma_0^p - \gamma_0) \tag{6.4}$$

Since $\min\{v(\gamma_0), v(\gamma_0^p)\} \leq -1$, we cannot have $v(\gamma_0) = 0$, and so we must have $v(\gamma_0) \neq v(\gamma_0^p)$ and so

$$-1 = \min\{v(\gamma_0), v(\gamma_0^p)\} = pv(\gamma_0).$$

But $v(\gamma_0) \in \mathbb{Z}$, a contradiction. Thus, we must have that a is a p^{th} power and so $\chi \cup a$ is trivial.

□

The following two lemmas complete the proof of the proposition by showing that when the p-algebra $B = [t^{-1}, a)$ is nontrivial, it is not in the image of the map $- \otimes_{\mathcal{O}_v} K : \mathrm{Br}(\mathcal{O}_v) \to \mathrm{Br}(K)$; i.e. that it is ramified.

Lemma 14. *Let F be a field of characteristic p, v a valuation on the extension K/F with Noetherian valuation ring $\mathcal{O}_v$, and B a p-algebra over K. If $B = A \otimes_{\mathcal{O}_v} K$ for some Azumaya algebra A over $\mathcal{O}_v$, then $A = \mathcal{O}_B$.*

Proof. The valuation v extends to B via

$$w(b) = \frac{1}{\mathrm{Ind}(B)} v(\mathrm{Nrd}(b)) = \frac{1}{p} v(\mathrm{Nrd}(b)). \tag{6.5}$$

First we show that $\mathcal{O}_B$ is a finitely generated $\mathcal{O}_v$-module. Let $\{b_1, b_2, \ldots, b_{p^2}\}$ be a K-basis for B. Perhaps after multiplication by an element of $\mathcal{O}_v$, we can assume that each $b_i \in \mathcal{O}_B$. Consider the reduced trace map $\mathrm{Trd} : B \to K$.Since B is a division algebra, the reduced trace form is nondegenerate, and so there exists a dual basis $\{c_1, c_2, \ldots, c_{p^2}\}$ of B such that $\mathrm{Trd}(b_i c_j) = \delta_{ij}$.

We will show that $\mathcal{O}_B \subseteq \Sigma_{i=1}^{p^2} \mathcal{O}_v c_i \simeq \mathcal{O}_v^{p^2}$. Let $z \in \mathcal{O}_B$. We can write

$$z = a_1 c_1 + a_2 c_2 + \cdots + a_{p^2} c_{p^2}$$

for some coefficients $a_1, \cdots a_{p^2} \in K$. Since $z \in \mathcal{O}_B$ and each $b_i \in \mathcal{O}_B$ by assumption, each product $b_i z \in \mathcal{O}_B$. From Proposition 1.5 in [Tignol-Wadsworth], $b_i z \in \mathcal{O}_B$ implies that the coefficients of the minimal polynomial of $b_i z$, in particular $\mathrm{Trd}(b_i z)$, belong to $\mathcal{O}_v$. But

$$\mathrm{Trd}(b_i z) = \mathrm{Trd}(b_i a_1 c_1 + b_i a_2 c_2 + \cdots + b_i a_{p^2} c_{p^2}) = a_i \mathrm{Trd}(b_i c_i) = a_i$$

since Trd is K-linear. Thus, each coefficient $a_i \in \mathcal{O}_v$, and so $\mathcal{O}_B \subseteq \Sigma_{i=1}^{p^2} \mathcal{O}_v c_i$. Since $\mathcal{O}_v$ is a Noetherian ring, $\mathcal{O}_v^{p^2}$ is a Noetherian module over $\mathcal{O}_v$, and so $\mathcal{O}_B \subseteq \mathcal{O}_v^{p^2}$ is a finitely generated $\mathcal{O}_v$-module, as desired.

Suppose that $B = A \otimes_{\mathcal{O}_v} K$ for some Azumaya algebra A over $\mathcal{O}_v$. We will now show that we must have $A = \mathcal{O}_B$. Since the reduced norm of any element of A lies in $\mathcal{O}_v$, from (6.5), we have that $A \subset \mathcal{O}_B$. Also, since $\mathcal{O}_B \subset A \otimes_{\mathcal{O}_v} K$ is a finitely generated $\mathcal{O}_v$-module, there exists some $m \in \mathbb{N}$ such that $B \subset \pi^m A$, where π is the uniformizer of $\mathcal{O}_v$.

$$A \subseteq \mathcal{O}_B \subseteq \pi^m A \tag{6.6}$$

Consider the exact sequence

$$0 \to A \xrightarrow{i} \mathcal{O}_B \to \operatorname{coker}(i) \to 0 \tag{6.7}$$

Tensoring over $\mathcal{O}_v$ with $\mathcal{O}_v/\mathfrak{m}$, we have the exact sequence

$$A/\mathfrak{m} \xrightarrow{\bar{i}} \mathcal{O}_B/\mathfrak{m} \to \operatorname{coker}(i)/\mathfrak{m} \to 0 \tag{6.8}$$

Because A is *Azumaya*, $A/\mathfrak{m}$ is a central simple algebra over $\mathcal{O}_v/\mathfrak{m}$. In particular, $\bar{i}$ is injective. But (6.6) implies that $A/\mathfrak{m}$ and $\mathcal{O}_B/\mathfrak{m}$ are $\mathcal{O}_v/\mathfrak{m}$-vector spaces of the same dimension, and so we must have $\operatorname{coker}(i)/\mathfrak{m} = 0$. Applying Nakayama's Lemma, $\operatorname{coker}(i) = 0$, and so $A = \mathcal{O}_B$.

□

Lemma 15. *Let F be a field of characteristic p, set $K = F((t))$, and let B be the p-algebra $[t^{-1}, a)$ where $a \in F^\times$ is not a p^{th} power. Then $\mathcal{O}_B$ is not an Azumaya algebra over $\mathcal{O}_v$.*

Proof. We can write B in term of generators and relations as

$$B = K[x, y : x^p - x - t^{-1} = 0, y^p = a, yx = (x+1)y].$$

In the previous lemma we saw that $\mathcal{O}_B$ is a free $\mathcal{O}_v$-module. Now, we show that $\{x^{-i}y^j\}$, $0 \le i, j \le p-1$, forms an $\mathcal{O}_v$-basis for $\mathcal{O}_B$. Since $w(x^{-i}) = \frac{i}{p}$ and $w(y^j) = 0$ (where w is the extension of v to B), each $x^{-i}y^j \in \mathcal{O}_B$. Using the relation

$$x^i = x^{i-p} + t^{-1}x^{i-p+1} \text{ for } 1 \le i \le p-1$$

and the fact that $\{x^i y^j\}$ forms a K-basis for B, we can express each element $z \in \mathcal{O}_B$ uniquely as $z = \Sigma_i \Sigma_j \alpha_{ij} x^{-i} y^j$ where each α_{ij} is in K. We would like to show that each coefficient α_{ij} is actually in $\mathcal{O}_v$.

We can write $z = \Sigma_i x^{-i}\gamma_i$ where $\gamma_i = \Sigma_j \alpha_{ij} y^j$. Notice that since $w(\gamma_i x^{-i}) \equiv \frac{i}{p} \mod \mathbb{Z}$,

$$w(z) = \min\{w(\gamma_i x^{-i})\}.$$

In particular, since $z \in \mathcal{O}_B$, $w(\gamma_i) \geq 0$ for each i. To get a contradiction, suppose that some $\alpha_{ij} \notin \mathcal{O}_v$; say $w(\alpha_{ij}) = -m$. Consider the element $(\pi^m\gamma_i)^p$ of $\mathcal{O}_v$, where π is the uniformizer of $\mathcal{O}_v$:

$$(\pi^m\gamma_i)^p = (\pi^m\alpha_{i0})^p + (\pi^m\alpha_{i1}y)^p + \cdots + (\pi^m\alpha_{i(p-1)}y^{p-1})^p = (\pi^m\alpha_{i0})^p + (\pi^m\alpha_{i1})^p a + \cdots + (\pi^m\alpha_{i(p-1)})^p a^{p-1}$$

Since $w((\pi^m\gamma_i)^p) > 0$, the image of $(\pi^m\gamma_i)^p$ in $F = \mathcal{O}_v/\mathfrak{m}$ is zero. So we have

$$0 = (\overline{\pi^m\alpha_{i0}})^p + (\overline{\pi^m\alpha_{i1}})^p a + \cdots (\overline{\pi^m\alpha_{i(p-1)}})^p a^{p-1}. \tag{6.9}$$

In the extension $F(\sqrt[p]{a})/F$, equation (6.9) yields

$$0 = \overline{\pi^m\alpha_{i0}} + \overline{\pi^m\alpha_{i1}}\sqrt[p]{a} + \cdots \overline{\pi^m\alpha_{i(p-1)}}\sqrt[p]{a}^{p-1}.$$

But because a is not a p^{th} power in F, $\{1, \sqrt[p]{a}, \cdots, \sqrt[p]{a}^{p-1}\}$ are linearly independent over F, and so we must have that each coefficient $\overline{\pi^m\alpha_{i0}}, \overline{\pi^m\alpha_{i1}}, \cdots, \overline{\pi^m\alpha_{i(p-1)}}$ is zero. However, this contradicts our assumption that $w(\alpha_{ij}) = -m$ for some j. So each coefficient α_{ij} is in $\mathcal{O}_v$, as desired.

Now, we have seen that every element of $\mathcal{O}_B$ can be uniquely expressed as an $\mathcal{O}_v$-linear combination of the elements $\{x^{-i}y^j\}$, and so these elements form a basis for $\mathcal{O}_B$ as an $\mathcal{O}_v$-module. We want to show that $\mathcal{O}_B$ is not an Azumaya algebra over $\mathcal{O}_v$. It is enough to show that $\mathcal{O}_B \otimes_{\mathcal{O}_v} \mathcal{O}_v/\mathfrak{m}$ is not a central simple algebra over $F = \mathcal{O}_v/\mathfrak{m}$. Notice that the relation $x^p - x - t^{-1} = 0$ in B yields the relation

$$t(1 - x^{1-p}) = x^{-p}, \tag{6.10}$$

and the relation $yx = (x+1)y$ gives the relation

$$x^{-1}yx^{-1} = x^{-1}y - yx^{-1}. \tag{6.11}$$

We have

$$\mathcal{O}_B = \mathcal{O}_v[x^{-1}, y : y^p = a, x^{-p} = t(1 - x^{1-p}), x^{-1}yx^{-1} = x^{-1}y - yx^{-1}]$$

and so

$$\mathcal{O}_B \otimes_{\mathcal{O}_v} \mathcal{O}_v/\mathfrak{m} = F[x^{-1}, y : y^p = a, x^{-p} = 0, x^{-1}yx^{-1} = x^{-1}y - yx^{-1}]$$

To get a contradiction, suppose that $\mathcal{O}_B \otimes_{\mathcal{O}_v} \mathcal{O}_v/\mathfrak{m}$ were a central simple algebra over F. Then since $x^{1-p} \neq 0$, the ideal $\langle x^{1-p} \rangle$ generates the whole ring. In particular, there exist $f, g \in \mathcal{O}_B \otimes_{\mathcal{O}_v} \mathcal{O}_v/\mathfrak{m}$ such that

$$1 = fx^{1-p} + x^{1-p}g.$$

But this means that $x^{-2} = 0$. So,

$$0 = x^{-2} = x^{-1}(x^{-1}yx^{-1}) = x^{-1}(x^{-1}y - yx^{-1}) = -x^{-1}yx^{-1} = -(x^{-1}y - yx^{-1})$$

where we repeatedly use the relation (6.11). But this means that $x^{-1}y = yx^{-1}$ and so x and y commute. So $\mathcal{O}_B \otimes_{\mathcal{O}_v} \mathcal{O}_v/\mathfrak{m}$ cannot be a central simple algebra over F.

□

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