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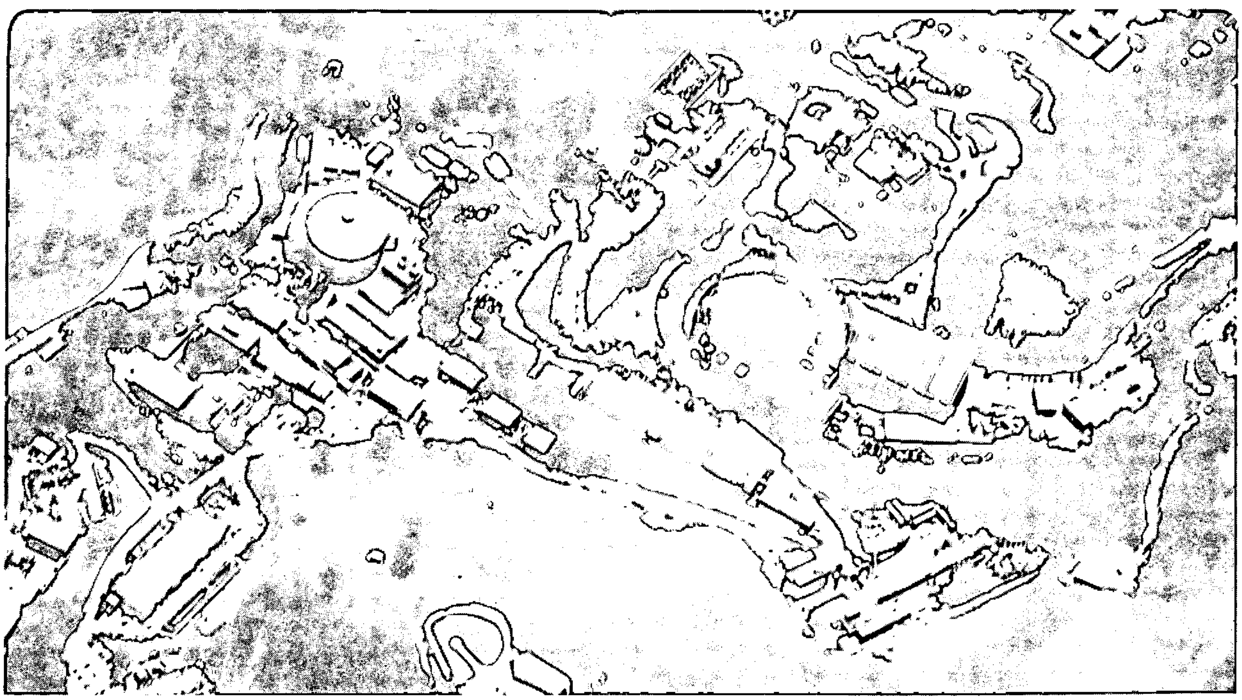

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CAPILLARY WEDGES REVISITED*

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Capillary wedges revisited

Paul Concus and Robert Finn

This paper is devoted to the results announced in our earlier note [1], concerning existence and nonexistence of capillary surfaces over domains with corners, when the data on the two sides of the corner may differ. The behavior of the solutions can differ in significant qualitative ways from that which occurs in the previously considered case of constant data; we are able to a large extent to characterize the conditions under which such qualitative changes must occur.

1. For background considerations, we refer the reader to our earlier papers [2, 3] and to Chapters 1, 5, and 6 in [4]. In general terms, we consider a cylindrical capillary tube Z with section Ω, closed at one end and partly filled with fluid in the absence of gravity, forming a free surface $\mathcal{S}$. We suppose the boundary Σ of Ω to be piesewise smooth and to have an isolated corner P of opening 2α, $0 < 2\alpha < \pi$, forming a local "wedge domain" at P, see Figure 1. We seek conditions under which, for prescribed constant (contact) angles γ_1 and γ_2 in the interval $[0,\pi]$, there will exist an $\mathcal{S}$ that can be (locally) represented by a function $z = u(x, y)$ over a neighborhood Ω^* of P in Ω, and which meets the sides Z_1 and Z_2, over adjacent segments Σ_1 and Σ_2 of $\partial\Omega$ that abut at P, in the angles γ_1 and γ_2.

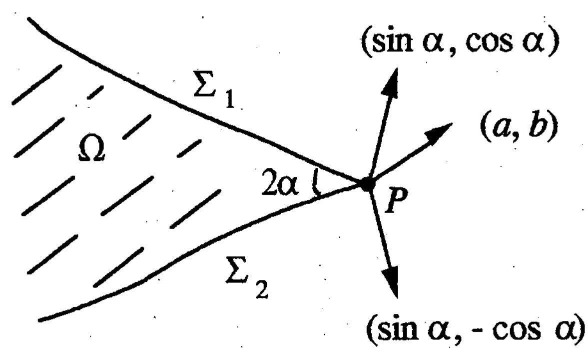

Figure 1: The wedge configuration

Specifically, we seek a solution of

$$\operatorname{div} Tu = 2H \tag{1}$$

in some Ω^*, with

$$Tu = \frac{Du}{\sqrt{1+|Du|^2}} \tag{2}$$

and H an arbitrary prescribed constant, such that

$$\begin{aligned} \nu \cdot Tu &= \cos\gamma_1 \qquad \text{on } \Sigma_1 \\ \nu \cdot Tu &= \cos\gamma_2 \qquad \text{on } \Sigma_2 \;, \end{aligned} \tag{3}$$

ν being unit exterior normal vector. Geometrically, H is the *mean curvature* of $\mathcal{S}$; when $H = 0$, $\mathcal{S}$ becomes a minimal surface. In a physical situation, H is determined by the global configuration of Ω and by the boundary conditions over the entire boundary. One sees easily that meaningful physical conditions can give rise to any desired value of H. It is worth noting that if $\gamma_1 = \gamma_2 \neq \pi/2$ over the entire boundary, then $H \neq 0$; if $\gamma_1 = \gamma_2 = \pi/2$ over the entire boundary, then the global problem admits the solution $u \equiv 0$ in Ω, which is unique up to an additive constant.

It is important to observe that in the statement of the problem, $\mathcal{S}$ is not assumed to be defined over P, and no growth conditions are imposed as P is approached from within Ω.

2. In earlier work [2,3], we have shown that if $\gamma_1 = \gamma_2 = \gamma$, then a solution of the local problem (1), (2), (3) can exist only if $\alpha \geq \left|\frac{\pi}{2} - \gamma\right|$; if $H \neq 0$ and if Σ_1, Σ_2 are linear segments then this condition also suffices, while if $H = 0$ then $\alpha > \left|\frac{\pi}{2} - \gamma\right|$ suffices for existence. Again we emphasize that no growth restriction is required at P. Tam [6] showed that whenever a solution exists, then the surface $\mathcal{S}$ is continuous and has a continuous unit normal $\vec{N}$ up to P, see also Miersemann [7] and Lieberman [8] for further developments. This remarkable behavior is the underlying reason that Vreeburg obtained in [9] the identical expression $\alpha \geq \left|\frac{\pi}{2} - \gamma\right|$ as condition for existence of a surface with normal vector continuous to the vertex, without any use of the differential equation.

In the interim Keller, King, and Merchant [10] studied again the question of a capillary surface $u(x,y)$ defined in a wedge, with (possibly) differing angles γ_1, γ_2 on the two sides. They assume that the sides are linear and extend to infinity and that the surface extends to the entire infinite wedge with the same boundary condition; this is possible only in the particular case $H = 0$, thus limiting the physical interest of their discussion. For this among other reasons that we indicate below, the assertion of the authors on p.161 of [10] that they provide a simple derivation of our result described above is in our view misleading. (We find additionally the authors' justification that $H = 0$ to be unconvincing; however, the result can be proved, using methods developed in [11], [12], and [13].)

The authors of [10] then assume further that the solution is uniquely determined by the data. From a mathematical point of view this question is unsettled, see, e.g., Theorem 4 in [14]. They conclude from the assumed uniqueness that the surface must be ruled, and give a reasoning to the effect that every ruled minimal surface must necessarily be a plane. The right helicoid $z = \arctan y/x$ provides a counterexample to their assertion.

Nevertheless, it is known [15] that the right helicoid is the only ruled minimal surface other than the plane; since the helicoid does not meet the wedge walls in constant angles, the boundary condition excludes it from consideration. Hence, subject to the missing proof of uniqueness, we may conclude with the authors of [10] that any surface $u(x,y)$ satisfying their conditions interior to the (infinite) wedge and on its edges is a plane. We emphasize here that uniqueness cannot routinely be expected, see, e.g., [16], where configurations with a continuum of solutions are discussed, and especially the remarks in the Postscript of that reference.

The authors proceed in [10] to derive the formal condition that a plane covering the infinite wedge can meet both wedge walls in the prescribed angles, and then to use that criterion as the physical one for the (local) existence of a capillary surface near the vertex. But the condition they thus obtain excludes, for example, the case $\alpha = \left|\frac{\pi}{2} - \gamma\right|$ in the equal angle configuration, for which a lower hemisphere provides an explicit solution; the existence and particular properties of this solution are crucial in our own discussion of the behavior of capillary surfaces at corner points, and we believe them to be crucial for a correct understanding of what can occur.

The criterion of [10] excludes many other significant solutions. In our Example 2 (and in Theorem 3) below, we point out the existence of surfaces $u(x,y)$ of constant H, defined in a neighborhood of P in wedge domains and meeting the walls in (different) constant angles, but for which the unit surface normals are discontinuous at P (thus they behave locally very differently than do planes). These surfaces satisfy the correct physical conditions and it must be expected that they will be observed in practice; they are however not encompassed in the discussion of [10]. For this reason, we cannot concur with the proposal at the end of §2 in [10], to use that paper's criterion as a basis for a procedure to measure contact angle; we believe that such a procedure would lead to erroneous results in a large range of cases of particular interest.

It seems appropriate here to reiterate for the present context, the point made in our earlier note [1] with regard to Vreeburg's paper [9]. Although the circumstances differ in detail, the two discussions [9] and [10, §2] have in common that the authors have unduly restricted the class of surfaces admitted into consideration, thus overlooking solutions of the equation and boundary conditions that are mathematically (and also physically) significant. It is a curious accident that when the contact angles on the two wedge sides are equal, all procedures lead to criteria that look formally similar. Despite the apparent similarities, the criteria do differ in an important way, even in that particular case; it is presumably a failure to recognize this distinction that led the authors in [10] to the erroneous statement in their Abstract that "*the height of the free surface at the corner tends to infinity as the wedge angle decreases to a critical value dependent on the contact angle.*" This assertion is in conflict with the discontinuous disappearance of the surface that is a feature of the discussion in our work to which they refer; it seems clear on the basis of experiments already conducted (cf., [4] p. 137, also [5]) that the discontinuous behavior will be observed in practice.

When differing contact angles on the two sides are contemplated, the distinctions become still more marked, and a whole range of solutions appears that is envisaged neither in [9] nor in [10, §2]. The criteria for existence of these solutions are very different from the ones established in those references, and lead in particular to different predictions as to results of experiments. It should be of considerable interest to design such experiments, which could be carried out in a suitable microgravity environment.

3. In what follows, we discuss solutions under the generality introduced at the beginning of this paper.

In the (B_1, B_2) plane we introduce the closed elliptical domain

$$\mathcal{E}: \qquad B_1^2 + B_2^2 + 2B_1B_2\cos 2\alpha \leq \sin^2 2\alpha \tag{4}$$

inscribed in a square $\mathcal{Q}$ as indicated in Figure 2 . $\mathcal{E}$ cuts off domains $\mathcal{D}_1^+$, $\mathcal{D}_1^-$ of $\mathcal{Q}$ that are interior to the strip $\mathcal{A}$: $|B_1 - B_2| < 2\cos^2\alpha$, and domains $\mathcal{D}_2^+$, $\mathcal{D}_2^-$ of $\mathcal{Q}$ that are exterior to $\mathcal{A}$. Note that the lines $B_1 - B_2 = \pm 2\cos^2\alpha$ pass through the intersection points of $\partial\mathcal{E}$ with $\partial\mathcal{Q}$. We then have

<u>**Theorem**</u> **1:** *Set* $B_1 = \cos\gamma_1$, $B_2 = \cos\gamma_2$. *A necessary condition for existence of a solution surface* $\mathcal{S}$: $u(x,y)$ *of* (1),(2),(3) *with unit normal* $\vec{N}$ *continuous up to P is that the point* (B_1, B_2) *lie in* $\mathcal{E}$; *the boundary of* $\mathcal{E}$ *corresponds to those configurations for which* $\mathcal{S}$ *is vertical (*$\vec{N}$ *horizontal) at P. On* $\partial\mathcal{E} \cap \partial\mathcal{D}_1^+$ *there holds* $\gamma_1+\gamma_2 = \pi - 2\alpha$; *on* $\partial\mathcal{E} \cap \partial\mathcal{D}_1^-$ *there holds* $\gamma_1+\gamma_2 = \pi + 2\alpha$. *On* $\partial\mathcal{E} \cap \partial\mathcal{D}_2^+$, $\partial\mathcal{E} \cap \partial\mathcal{D}_2^-$ *there hold, respectively,* $\gamma_1 - \gamma_2 = \pi - 2\alpha$ *and* $\gamma_1 - \gamma_2 = -\pi + 2\alpha$. *For existence of such a solution in a domain* Ω^* *of the type considered and for arbitrary H, it suffices that* Σ_1, Σ_2 *be linear segments, and that* (B_1, B_2) *lie interior to* $\mathcal{E}$. *If* $(B_1, B_2) \in \partial\mathcal{E} \cap \partial\mathcal{D}_1^+$ *then there is a solution (in some* Ω^**) for any* $H > 0$; *if* $(B_1, B_2) \in \partial\mathcal{E} \cap \partial\mathcal{D}_1^-$ *there is a solution for any* $H < 0$.

Any solution u(x,y), corresponding to interior points of $\mathcal{E}$ *or to points of* $\partial\mathcal{E}$ *interior to* $\mathcal{A}$, *is continuous and admits a continuous unit normal vector up to P.*

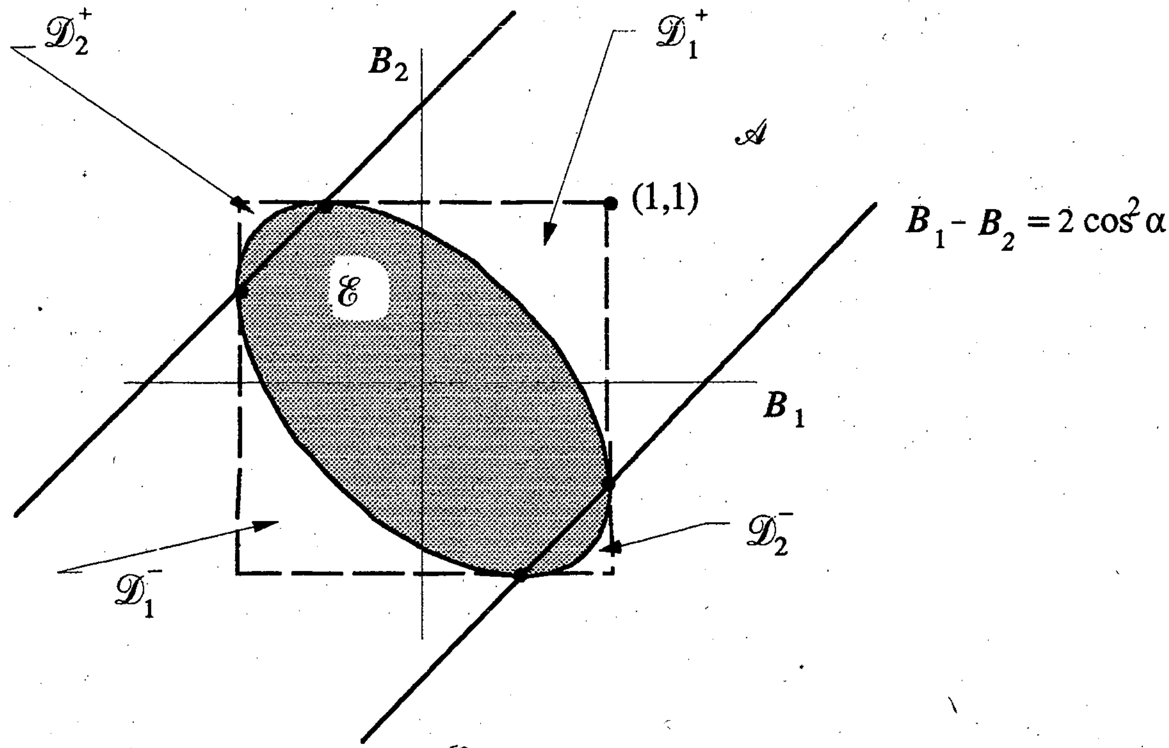

Figure 2: Elliptical domain $\mathscr{E}$, reference strip $\mathscr{A}$, square $\mathscr{Q}$, and domains $\mathscr{D}_i^{\pm}$; case $2\alpha < \pi/2$. If $2\alpha > \pi/2$, the directions of major and minor axes interchange.

<u>**Proof of Theorem 1**</u>: Write $\vec{N} = \langle a, b, c \rangle$ with $c \le 0$, $a^2 + b^2 + c^2 = 1$. Referring to Figure 1, we find

$$\begin{aligned} \cos\gamma_1 &= a\sin\alpha + b\cos\alpha \\ \cos\gamma_2 &= a\sin\alpha - b\cos\alpha \end{aligned},$$

and the first sentence of the necessary condition follows immediately from the observation that $a^2 + b^2 \le 1$, equality holding if and only if $c = 0$. For any $(B_1, B_2) \in \partial\mathscr{E} \cap \partial\mathscr{Q}_1^+$, there corresponds a unique (γ_1, γ_2) with γ_1, γ_2 in $[0, \pi]$. We rewrite

$$B_1^2 + B_2^2 + 2B_1B_2\cos 2\alpha = \sin^2 2\alpha \tag{5}$$

in the form

$$(1-B_1^2)(1-B_2^2) = (B_1B_2 + \cos 2\alpha)^2. \tag{6}$$

Since on the indicated arc we have

$$(B_1-B_2)^2 < 4\cos^4\alpha\,, \tag{7}$$

there follows from (5)

$$B_1B_2 + \cos 2\alpha > 0\,. \tag{8}$$

Thus, using positive square roots, we obtain from (6)

$$\sqrt{1-B_1^2}\sqrt{1-B_2^2} = B_1B_2 + \cos 2\alpha\,, \tag{9}$$

which is equivalent to

$$\cos(\gamma_1+\gamma_2) = \cos(\pi-2\alpha)\,, \tag{10}$$

so that either $\gamma_1+\gamma_2 = \pi - 2\alpha$ or $\gamma_1+\gamma_2 = \pi + 2\alpha$. But from (5), we find at the symmetry point $B_1 = B_2 = B > 0$ on $\partial\mathscr{E} \cap \partial\mathscr{D}_1^+$ that $B = \sin\alpha = \cos(\pi/2 - \alpha)$. Thus, the former relation must hold at the symmetry point, and hence it holds throughout the arc. Similarly, on $\partial\mathscr{E} \cap \partial\mathscr{D}_1^-$, there holds $\gamma_1+\gamma_2 = \pi + 2\alpha$.

On the remaining two arcs $\partial\mathscr{E} \cap \partial\mathscr{D}_2^+$ and $\partial\mathscr{E} \cap \partial\mathscr{D}_2^-$, we obtain by analogous reasoning that $\gamma_1 - \gamma_2 = \pi - 2\alpha$ and $-\pi + 2\alpha$, respectively.

To prove the sufficiency, observe that if (B_1, B_2) is interior to $\mathscr{E}$, then a, b, c are uniquely determined by the conditions just given, and that $c < 0$. The plane Π through P with normal $\vec{N}$ then solves the problem when $H = 0$. If $H > 0$ then a lower hemisphere of radius $1/H$ and tangent to Π at P provides an explicit local solution, while if $H < 0$ then an upper hemisphere yields a solution (see Figure 3).

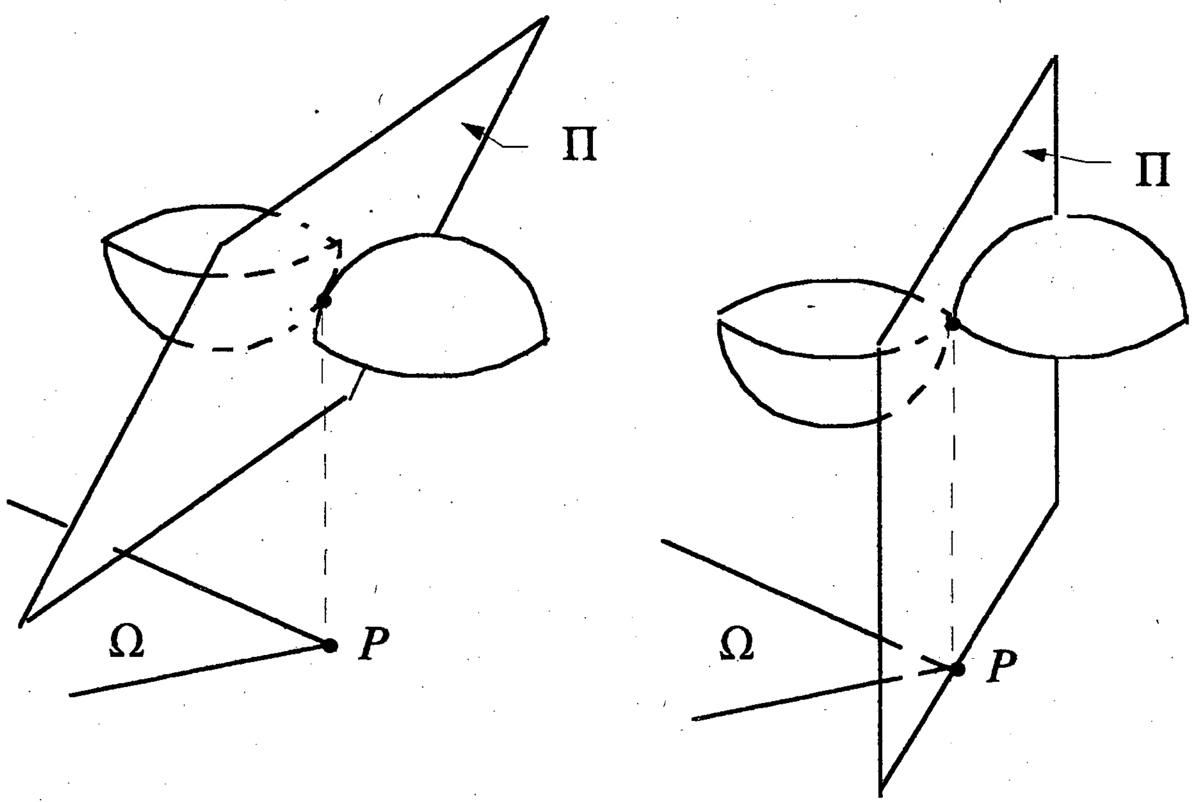

Figure 3: Covering of neighborhoods of P by hemispheres; when Π is vertical (corresponding to points on $\partial\mathscr{E}$) only one of the two hemispheres achieving the boundary data at P covers a neighborhood of P in Ω.

If (B_1, B_2) is a boundary point of $\mathscr{E}$, then this procedure always fails when $H = 0$; if $H \neq 0$ then the procedure can under some circumstances yield a solution, provided the trace of Π on the plane of Ω does not enter the (closed) wedge domain. That is, the vector $<a,b>$ of Figure 1 must be a linear combination, with positive coefficients, of the two other vectors in the figure. Since $c = 0$ in this case, the condition becomes $|b| < \cos\alpha$, or equivalently $|B_1 - B_2| < 2\cos^2\alpha$; that is, $(B_1, B_2) \in \mathscr{A}$. But even with that restriction, not all possibilities can be achieved, as the signs of u_x, u_y now reverse for the two hemispheres tangent to Π at P that cover a neighborhood of P in Ω (see Figure 3), and thus changing the sign of H also reverses the signs of u_x, u_y. Nevertheless, on $\partial\mathscr{E} \cap \partial\mathscr{D}_1^+$ the condition $\gamma_1+\gamma_2 = \pi - 2\alpha$ can be realized by an explicit construction with a lower hemisphere of arbitrary radius; the construction is indicated in Figure 4.

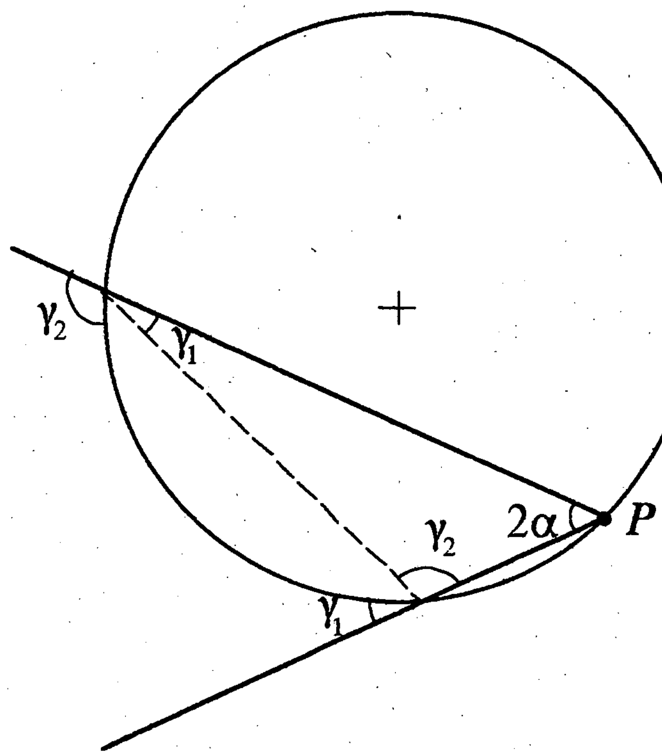

Figure 4: Construction of solution as lower hemisphere; $\gamma_1 + \gamma_2 = \pi - 2\alpha$, $H > 0$.

Similarly, on $\partial\mathscr{E} \cap \partial\mathscr{D}_1^-$, there must hold $\gamma_1+\gamma_2 = \pi + 2\alpha$, and an explicit construction can be achieved with an upper hemisphere of arbitrary radius.

With regard to the remaining two arcs $\partial\mathscr{E} \cap \partial\mathscr{D}_2^+$ and $\partial\mathscr{E} \cap \partial\mathscr{D}_2^-$, it will be shown below (in §5) that solutions exist, at least at the symmetry points of these arcs; these solutions are however not known explicitly.

The final statement of the theorem follows from the method of Tam [6], which applies without essential change to the extended situation considered here.□

It should be emphasized that we have not excluded the possibility of solutions with negative H achieving the data on the segment $\partial \mathscr{E} \cap \partial \mathscr{D}_1^+$, or with positive H achieving the data on the segment $\partial \mathscr{E} \cap \partial \mathscr{D}_1^-$.

In view of the four relations just obtained for γ_1 and γ_2 on $\partial \mathscr{E}$, we see that $\mathscr{E}$ appears as a rectangle in the γ_1, γ_2 coordinates, with sides inclined at 45° to the axes (Figure 5).

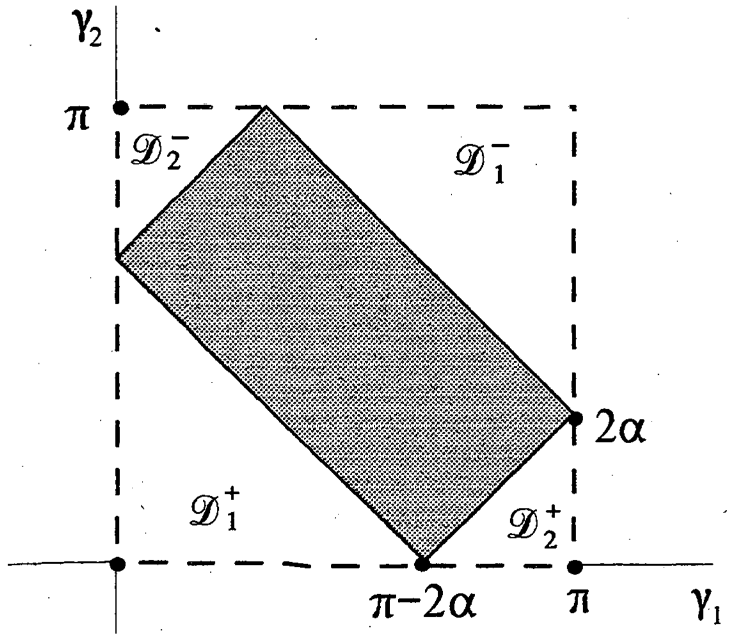

Figure 5: Image of $\mathscr{E}$ and of $\mathscr{D}$ in (γ_1, γ_2) coordinates

4. In the above discussion, the requirement that Σ_1, Σ_2 be linear was introduced solely to facilitate a simple explicit sufficiency proof; it is not essential to the substance of the problem. It is less clear under what conditions solutions with discontinuities at P are excluded, as happens in the equal angle case. To study that point, we attempt to extend the method we introduced for that case to this more general situation. Following in general outline our earlier procedure, we apply Green's Identity to (1) in the subdomain $\Omega^{(\Lambda)}$ indicated in Figure 6, cut off by Γ and Λ (the segment Λ is introduced to exclude possible singularities at the vertex P). We obtain

$$2H\left|\Omega^{(\Lambda)}\right| = \left|\Sigma_1^{(\Lambda)}\right|\cos\gamma_1 + \left|\Sigma_2^{(\Lambda)}\right|\cos\gamma_2 + \int_\Gamma \nu\cdot Tu\,ds + \int_\Lambda \nu\cdot Tu\,ds\ . \qquad (11)$$

The crucial observation in what follows is that $|\nu\cdot Tu| < 1$ for any function $u(x, y)$. This inequality permits us initially to move Λ to the vertex P, with the integral over that segment disappearing in the limit. Our next step is to replace $\nu\cdot Tu$ in the other integral

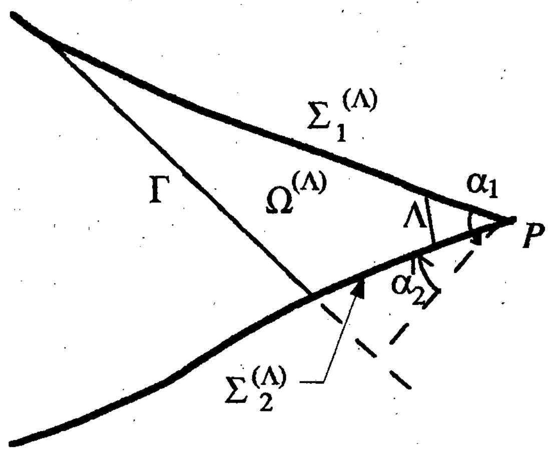

Figure 6: Configuration for Theorem 2. Note that $\alpha_1 > 0$, $\alpha_2 < 0$; both are in the range $(2\alpha - \pi/2\,,\, \pi/2)$.

by its positive and negative bounds, and then to let Γ move to P by parallel translation. Referring to Figure 6, we choose α_1, α_2 in $(2\alpha - \frac{\pi}{2}, \frac{\pi}{2})$ such that $\alpha_1 + \alpha_2 = 2\alpha$. Since the area term in (11) tends to zero faster than any of the lengths and since Σ_1, Σ_2 are asymptotically linear, we are led to the inequality

$$\left|\frac{\cos\gamma_1}{\cos\alpha_1} + \frac{\cos\gamma_2}{\cos\alpha_2}\right| \le \tan\alpha_1 + \tan\alpha_2$$

as a necessary condition for existence of a solution. Setting $A_1 = \cos\alpha_1$, $A_2 = \cos\alpha_2$ and introducing B_1, B_2 as above, we are led to

Lemma 1: *If* α_1, α_2 *are as above and* $|B_2A_1 + B_1A_2| > \sin 2\alpha$ *then there is no solution to the problem, regardless of growth conditions at P.*

Clearly the conditions of Lemma 1 cannot be satisfied when (B_1, B_2) is interior to $\mathcal{E}$, as Theorem 1 would then imply existence of a solution. We ask whether the conditions are necessarily satisfied for points exterior to $\mathcal{E}$. In formal terms, we have the

Question: *Given* (B_1, B_2) *disjoint from* $\mathcal{E}$, *do there exist* α_1, α_2 *in* $(2\alpha - \frac{\pi}{2}, \frac{\pi}{2})$ *such that* $\alpha_1 + \alpha_2 = 2\alpha$ *and* $|B_2A_1 + B_1A_2| > \sin 2\alpha$?

To answer the question, we prove first:

Lemma 2: *For* α_1, α_2 in $(-\pi/2, \pi/2)$, *the constraint* $\alpha_1 + \alpha_2 = 2\alpha$ *implies the relation*

$$A_1^2 + A_2^2 - 2A_1A_2\cos 2\alpha = \sin^2 2\alpha\ ;\ A_1\,, A_2 > 0 \qquad (12)$$

describing that portion of an elliptical arc $\mathcal{E}$ *partly inscribed in a unit square in the* (A_1, A_2) *plane, that lies in the first quadrant* (see Figure 7). *Conversely, whenever* (12) *holds there is a unique pair* α_1, α_2 *(up to permutation) such that* $\alpha_1 + \alpha_2 = 2\alpha$ *and* α_1, α_2 *are in* $(2\alpha - \frac{\pi}{2}, \frac{\pi}{2})$.

Proof: From $\alpha_1 + \alpha_2 = 2\alpha$ we find $A_1A_2 - \cos(2\alpha) = \pm\sqrt{1-A_1^2}\sqrt{1-A_2^2}$, from which (12) follows on squaring both sides. Conversely, if (12) holds it can be rewritten in the form just indicated. Choosing $\alpha_1 = \cos^{-1}(A_1)$, $\alpha_2 = \cos^{-1}(A_2)$ in $[0, \pi/2)$ we find $\cos(\alpha_1 \pm \alpha_2) = \cos 2\alpha$, from which $\alpha_1 \pm \alpha_2 = \pm 2\alpha$. By changing the signs of α_1 or α_2 or both we can arrange to have $\alpha_1 + \alpha_2 = 2\alpha$, with α_1, α_2 in $(-\pi/2, \pi/2)$. If $\alpha_1 < 2\alpha - \pi/2$ or $\alpha_2 < 2\alpha - \pi/2$ then $\alpha_1 + \alpha_2 < 2\alpha$. This contradiction completes the proof. □

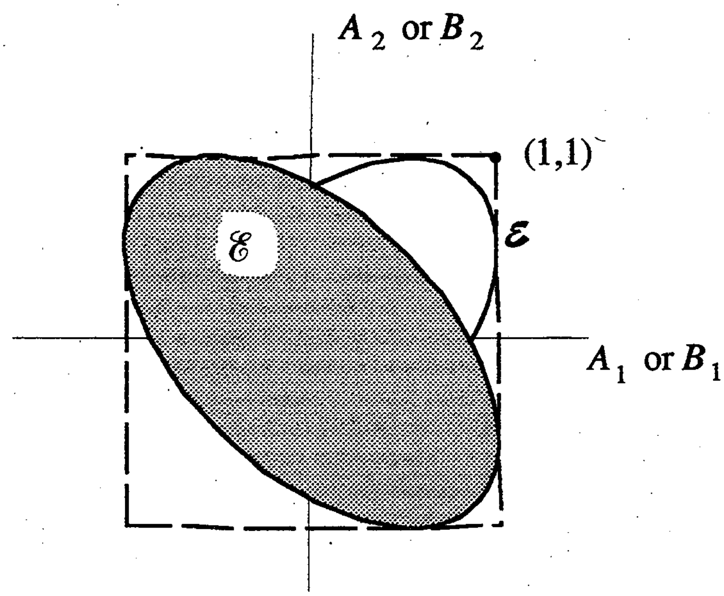

Figure 7: Elliptical domain $\mathscr{E}$, elliptical constraint arc $\mathcal{E}$; case $2\alpha < \pi/2$.

We set

$$A_1 = \frac{x}{\cos\alpha} + \frac{y}{\sin\alpha}$$
$$A_2 = \frac{-x}{\cos\alpha} + \frac{y}{\sin\alpha} \qquad (13)$$

transforming the elliptical arc (12) into a circular arc C centered at the origin, of radius $R = \sin\alpha\cos\alpha$, and restricted to the upper sector between the lines

$$y = \pm x \tan\alpha \qquad (14)$$

(see Figure 8). The two lines L_1, L_2 determined by $B_2A_1 + B_1A_2 = \pm\sin 2\alpha$

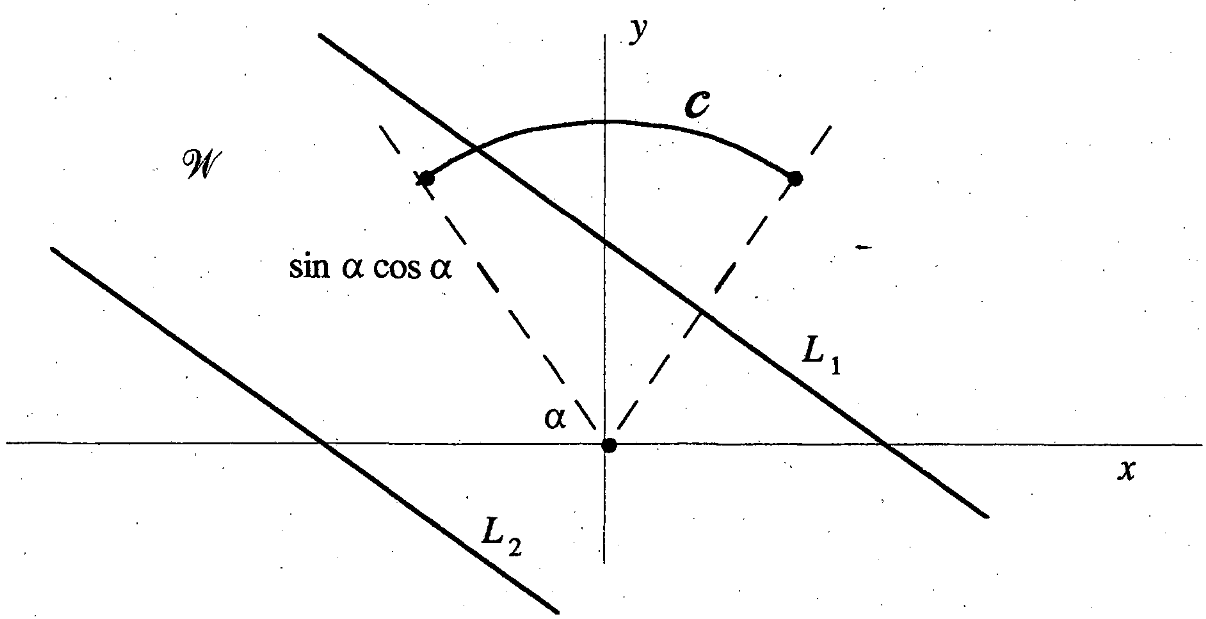

Figure 8: Normalized configuration (tentative); reference lines.

become now

$$\frac{B_1 - B_2}{\cos \alpha} x + \frac{B_1 + B_2}{\sin \alpha} y = \pm \sin 2\alpha \ . \tag{15}$$

The inequality $|B_2 A_1 + B_1 A_2| > \sin 2\alpha$ holds if and only if (x, y) lies outside the strip $\mathcal{W}$ bounded by the lines, and each line has distance

$$d = \frac{\sin^2 2\alpha}{2\sqrt{B_1^2 + B_2^2 + 2B_1 B_2 \cos 2\alpha}} \tag{16}$$

from the origin. When (B_1, B_2) is exterior to $\mathcal{E}$, we find

$$d < \sin \alpha \cos \alpha = R \ . \tag{17}$$

Despite the inequality in this last result, it can happen that $\boldsymbol{C}$ lies strictly interior to $\mathcal{W}$, as $\boldsymbol{C}$ contains only a portion of the full circle. In such a case, the method yields no information. But it can also occur that interior points of $\boldsymbol{C}$ lie exterior to $\mathcal{W}$; whenever that happens, any such point of $\boldsymbol{C}$ yields by Lemma 2 a suitable pair α_1, α_2 and excludes the possibility of any solution to the original problem. We summarize what we have found:

__Theorem__ 2: *If $\boldsymbol{C}$ contains points exterior to the closed strip $\mathcal{W}$ determined by* (15), *then there is no solution of* (1), (2), (3) *in any neighborhood Ω^* of P in Ω, for any constant H; this result holds without growth condition at P. If $\boldsymbol{C}$ lies interior to $\mathcal{W}$, then the method provides no information.*

The final statement of the theorem does not reflect a technical failure of the method, but arises rather from actual properties of the solutions. This will be apparent from the second of the following examples.

5. **Example 1**: Σ_1 and Σ_2 are linear, $\gamma_1 = \gamma_2 = \gamma$ (equivalently, $B_1 = B_2 = B$). Then (15) becomes the pair of horizontal lines $y = \pm \frac{\sin^2\alpha \cos\alpha}{B}$. The arc $\boldsymbol{C}$ is independent of B and is indicated in Figure 9. If (B,B) is exterior to $\mathcal{E}$ then one of the lines crosses $\boldsymbol{C}$ as indicated and points of $\boldsymbol{C}$ will lie exterior to $\mathcal{W}$; hence by Theorem 2 no

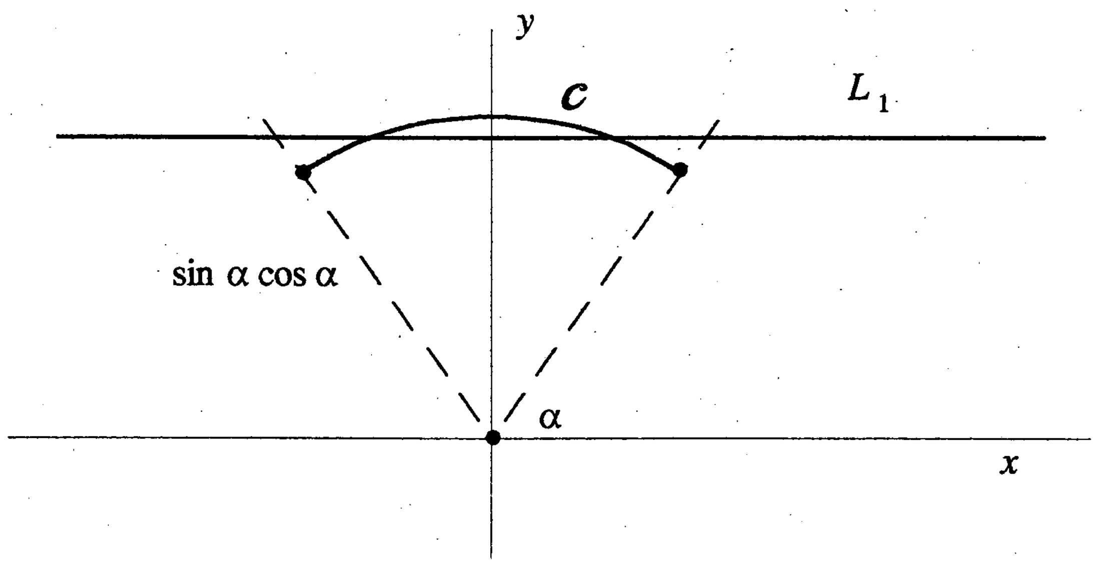

Figure 9: Configuration for Example 1.

solution can exist. If (B,B) is in the closure of $\mathcal{E}$ then $\boldsymbol{C}$ lies in the closed strip and Theorem 2 yields no information. However in this case we clearly have $|B_1 - B_2| < 2\cos^2\alpha$ and hence, by Theorem 1, a solution with continuous normal exists. Since (B,B) is exterior to $\mathcal{E}$ if and only if $\alpha < \left|\frac{\pi}{2} - \gamma\right|$ we retrieve exactly our earlier result for the constant angle case, from a more general point of view.

Example 2: Σ_1 and Σ_2 are linear, $\gamma_1 = \pi - \gamma_2 \neq \frac{\pi}{2}$ $(B_1 = -B_2 = B \neq 0)$. Now $\boldsymbol{C}$ is as before, but (15) now yields the two vertical lines $Bx = \pm \sin\alpha \cos^2\alpha$. Since $|B| \leq 1$, $\boldsymbol{C}$ always lies interior to $\mathcal{W}$, and thus Theorem 2 yields no information.

If in addition $B > \cos\alpha$, then $(B, -B)$ is exterior to $\mathcal{E}$, and according to Theorem 1 no solution with continuous normal can exist. Nevertheless, a solution to the original problem without growth hypotheses can exist, at least in a significant family of cases. Examples with $B = 1$ and any α are provided by the "moonies" whose existence is proved in [17]. These surfaces have $\gamma_1 = 0$, $\gamma_2 = \pi$ on adjacent circular arcs of differing radius, see Figure 10. Theorem 1 provides a new proof independent of the one given in [17], that these surfaces have discontinuous normals at P.

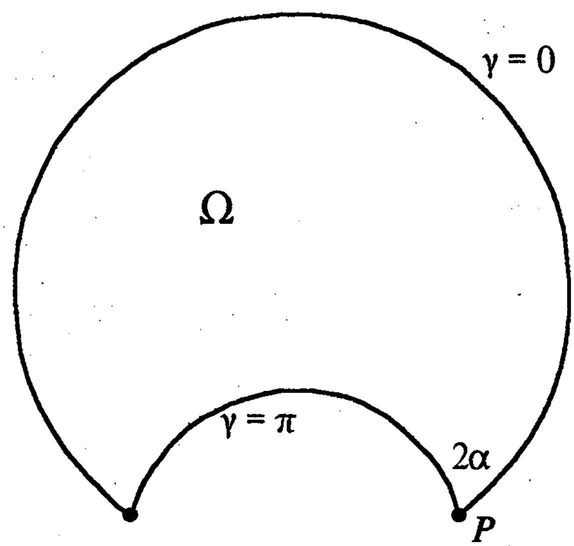

Figure 10: Domain for Moonie

The existence proof in [17] can be modified without essential change to show that if the data $\gamma = 0$ and $\gamma = \pi$ are modified to γ and $\pi - \gamma$, with $0 \le \gamma \le \pi/2$, then a solution exists in the identical domain. Thus we obtain a solution of the problem just formulated for any B with $0 \le B \le 1$; these solutions have normal vectors discontinuous at P if $B > \cos\alpha$. It can be shown that if $B < 1$ then the surface is bounded above and below in Ω; if $B = 1$ then $u(x,y) \to -\infty$ for any approach to the smaller circle, but remains bounded below on the larger one.

Thus, in a configuration with differing contact angles, solutions may appear whose behavior at P is very different from that which can occur in the equal angle case. These solutions could not have been obtained by the procedures used for Theorem 1. □

We observe that if $B_1 = -B_2$, then according to Theorem 1 solutions that are smooth up to P can be obtained with successively larger values of $|B|$ tending to unity, as the opening angle 2α closes down to zero. This is despite the discontinuity that occurs when $|B| > \cos\alpha$, and in contrast with the equal angle case, where the admissible B necessarily become small in magnitude with α.

<u>Example</u> 3: Σ_1 and Σ_2 are linear, $B_1 = B$, $B_2 = 0$, $\alpha < \pi/4$. We obtain once more the same $\boldsymbol{C}$, but (15) yields the sloping lines

$$\frac{B}{\cos\alpha}x + \frac{B}{\sin\alpha}y = \pm \sin 2\alpha\ .$$

These two lines will enclose $\boldsymbol{C}$ if and only if $|B| \le \sin 2\alpha$ (see Figure 11). This is exactly *if* $B_1 = -1$. *The left hand endpoint of* $\boldsymbol{C}$ *lies on* L_1 *if and only if* $B_2 = 1$; *it lies on* L_2 *if and only if* $B_2 = -1$. *Both endpoints lie always interior to the closed strip* $\mathcal{W}$.

Proof: Setting

$$F(x,y) = \frac{B_1 - B_2}{\cos\alpha} x - \frac{B_1 + B_2}{\sin\alpha} y - \sin 2\alpha$$
$$G(x,y) = \frac{B_1 - B_2}{\cos\alpha} x - \frac{B_1 + B_2}{\sin\alpha} y + \sin 2\alpha$$

the lines L_1, L_2 are characterized, respectively, by $F(x,y) = 0$ and by $G(x,y) = 0$. Choosing for x,y the coordinates of the right hand end point of $\boldsymbol{C}$ we find $F(x,y) = (B_1 - 1)\sin 2\alpha$, $G(x,y) = (B_1 + 1)\sin 2\alpha$. This proves the assertions relating to B_1; those relating to B_2 are proved analogously. The same relations show that if $|B_1| < 1$ then the right hand endpoint of $\boldsymbol{C}$ lies strictly between the two lines; similarly, the left hand endpoint lies between the lines when $|B_2| < 1$. □

As a consequence of Lemma 3, we see that *the configuration indicated in* Figure 8, *which was drawn to be indicative of a general situation, cannot occur as shown, as one of the endpoints of* $\boldsymbol{C}$ *lies exterior to the strip in that configuration.*

Referring to Figure 2, we introduce $\mathcal{D}_1^+$, $\mathcal{D}_1^-$, $\mathcal{D}_2^+$, $\mathcal{D}_2^-$ as in that figure. We adjoin to these domains all the boundary points that lie on the boundary of the square. On the line segment $B_1 = B_2 = B > 0$, L_1 takes the form $y = \sin^2\alpha \cos\alpha / B$, a horizontal line that is tangent to $\boldsymbol{C}$ at its midpoint when (B,B) is on the boundary of $\mathcal{E}$, and cuts through $\boldsymbol{C}$ when (B,B) lies exterior to $\mathcal{E}$, as in Figure 9. Thus, according to Theorem 2, the wedge problem admits no solution corresponding to the segment $1 \geq B > \sin\alpha$ (cf., Example 1). We now allow (B_1, B_2) to move along the arc of $\partial\mathcal{E}$ between the two nearest contact points with the square. According to (16), the distance of L_1 to the origin remains unchanged, and thus we obtain a family of lines tangent to the circle on which $\boldsymbol{C}$ lies. Since $|B_1| < 1$, $|B_2| < 1$ interior to the arc of $\partial\mathcal{E}$ considered, we find by Lemma 3 that all corresponding contact points with the circle actually lie interior to $\boldsymbol{C}$. Again by Lemma 3, as the points on $\partial\mathcal{E}$ move to the contact points with the square, L_1 becomes tangent to $\boldsymbol{C}$ at the respective endpoints; thus, all of $\boldsymbol{C}$ is covered. It is easy to see that the covering is 1-1.

Through each of the considered points of $\partial\mathcal{E}$, we construct the extended line segment from the origin. Repeating the reasoning given above for the $B_1 = B_2 > 0$ configuration, we find that for all points of that line segment exterior to $\mathcal{E}$ there exists no solution to the wedge problem. Since these lines sweep out $\mathcal{D}_1^+$, there can be no solution for any point of $\mathcal{D}_1^+$. An identical reasoning excludes solutions for any point of $\mathcal{D}_1^-$.

We now consider the two remaining complementary domains $\mathcal{D}_2^+$, $\mathcal{D}_2^-$, which have (1,–1) and (–1,1) as boundary points. In Example 2 above, we have shown the existence of solutions for every point on the line segment $-1 \le B_1 = - B_2 \le 1$; thus $\boldsymbol{C}$ lies between L_1 and L_2 for all points on that segment (see Figure 12). These solution surfaces have discontinuous normals at P for all points exterior to $\mathscr{E}$. Essentially a repetition of the reasoning directly above shows that $\boldsymbol{C}$ lies between L_1 and L_2 for all points of $\mathscr{D}_2^+$ and of $\mathscr{D}_2^-$. We have proved:

Theorem 3: *For any points* (B_1, B_2) *in the domains* $\mathscr{D}_1^+$, $\mathscr{D}_1^-$ *defined above, there are points of* $\boldsymbol{C}$ *exterior to the strip* $\mathscr{W}$, *and hence there exists no solution to the wedge problem* (1), (2), (3) *in any neighborhood of the vertex P, for any constant H. In* $\mathscr{D}_2^+$, $\mathscr{D}_2^-$, $\boldsymbol{C}$ *lies interior to* $\mathscr{W}$; *solutions do exist, at least on the symmetry line* $B_1 = - B_2$ *of those domains. For all points of that line exterior to* $\mathscr{E}$, *the unit normals to the solution surfaces are discontinuous at P.*

For all interior points of $\mathscr{E}$ and any H, solutions exist and all such solutions are smooth up to P. We conjecture that for any α in the range $0 < \alpha < \pi/2$ considered, solutions (with discontinuous normal) exist for all (B_1, B_2) lying in $\mathscr{D}_2^+$ or in $\mathscr{D}_2^-$.

We close with:

Theorem 4: *Whenever a bounded solution exists in a wedge domain, then every solution in that domain is bounded.*

This is an immediate formal consequence of the general comparison principle for capillary surfaces, see [20] Section 2 or [4] Chapter 5. Unbounded solutions can occur, as in the "moonie" example above. In such a case, every solution is unbounded.

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