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Solution of the Contractive Projection Problem

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In this paper we show that the class of J^*-algebras (a class of concrete Jordan triple systems) is stable under the action of norm one projections. This result constitutes a general solution to a problem considered by the authors and others. Specifically our main result is:

THEOREM 2. *Let P be an arbitrary contractive projection defined on a J^*-algebra M. Then $P(M)$ is a Jordan triple system in the triple product $(a, b, c) \to \{a, b, c\} \equiv P(\frac{1}{2}(ab^*c + cb^*a))$, for $a, b, c \in P(M)$; and $(P(M), \{\ \})$ has a faithful representation as a J^*-algebra.*

A *J^*-algebra* is a norm closed complex linear subspace of $\mathscr{L}(H, K)$, the bounded linear operators from a Hilbert space H to a Hilbert space K, which is closed under the operation $a \to aa^*a$. A J^*-algebra is a concrete example of a *Jordan triple system*, i.e., a complex vector space V together with a map $\{\cdot, \cdot, \cdot\}$: $V \times V \times V \to V$ which is linear in each outer variable, symmetric in the outer variables, conjugate linear in the middle variable and satisfies the identity:

$$\{xy\{uvz\}\} - \{uv\{xyz\}\} = \{\{xyu\}\,vz\} - \{u\{yxv\}\,z\}. \tag{0.0}$$

In a J^*-algebra, $\{xyz\} = \frac{1}{2}(xy^*z + zy^*x)$.

By a *contractive projection* we mean an idempotent linear map of norm one, i.e., $P^2 = P$, $\|P\| = 1$.

The class of J^*-algebras includes all Jordan operator algebras and has connections with the theory of bounded symmetric domains and with mathematical physics.

Although the natural setting for Theorem 2 is the class of J^*-algebras, the result is of interest and new even in the particular case when M is a C^*-algebra. Particular cases of Theorem 2 are known if additional assumptions are made on the space M and/or on the projection P.

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Choi and Effros prove in [5] that if M is a C^*-algebra and if P is completely positive and unital then $P(M)$ is a C^*-algebra in a product given by

$$a * b = P(ab), \qquad a, b \in P(M).$$

Arazy and Friedman in [4] completely classified the range $P(M)$ of an arbitrary contractive projection in case M is the C^*-algebra C_∞ of compact operators on a separable complex Hilbert space. Using their classification, Theorem 2 (with $M = C_\infty$) can be verified on a case by case basis. Effros and Størmer in [6] prove that if M is a JC-algebra and P is positive and unital, then $P(M)$ is a JC-algebra in the product

$$a \circ b = P(\tfrac{1}{2}(ab + ba)), \qquad a, b \in P(M).$$

In [7], the authors proved Theorem 2 in case M is a commutative C^*-algebra and they gave a complete description of all contractive projections in this case.

Influenced by these results, the authors suggested in [8] that the range of an arbitrary contractive projection should have a faithful representation as a J^*-algebra. In [9], the authors developed tools needed for this problem, and solved it in case $P(M)$ is finite dimensional. Inspired by ideas in the paper [1] we are now able to extend the techniques from [9] to the general case. Although our result contains the main result of [6], our proof uses that result.

Throughout this paper, with the exception of Theorem 2 itself, our results deal with a contractive projection Q on the dual M' of a J^*-algebra M. All of these results can be generalized with no change in proofs to contractive projections on the pre-dual of a von Neumann J^*-algebra, i.e., a weakly closed J^*-algebra.

As a by-product of our investigation we obtain geometric properties of the unit ball of the range of a contractive projection Q on the dual of a J^*-algebra. These properties are entirely analogous to the properties developed in [1] for the state space of a Jordan operator algebra (which corresponds to the case $M =$ Jordan algebra, $Q = \mathrm{Id}$).

The state space of any algebraic system is important in mathematical physics. Contractive projections are related to state spaces by virtue of the induced action on the unit ball of the space and of its dual, and their study has given rise to fundamental geometric properties of the dual ball of a Jordan triple system. Further work in this direction will lead to a geometric characterization of the dual ball of a J^*-algebra. Such characterizations have recently been obtained for the state spaces of JB-algebras and C^*-algebras [1, 2].

Recall the main theorem of [1]: a compact convex set K is the state space of a JB-algebra iff

(i) K has the Hilbert ball property;

(ii) K splits into an atomic and a non-atomic part;

(iii) every norm exposed face of K is projective;

(iv) every continuous real affine function on K is the difference of orthogonal positive continuous affine functions on K.

It is also shown in [1] that (i) and (ii) can be replaced by the following "pure state properties," which are less geometric but more physical:

(v) every extreme point of K is norm exposed (each pure state is prepared by a filter);

(vi) every P-projection preserves extreme rays of K (filters change pure states to pure states of smaller intensity);

(vii) for every pair f, g of extreme points of K with support projective units v, u, respectively, we have $f(u) = g(v)$ (symmetry of transition probabilities).

The contents of this paper are the following. In Section 1 we generalize the key lemma of [9] to arbitrary dimensions (Proposition 1) and improve on two other lemmas from [9, Sect. 4]. We also use [6, Corollary 1.5] to prove a local decomposition of a functional (Lemma 1.4), which is an important tool in our investigation.

In Section 2 we begin by proving an analog (Lemma 2.1) of the property called "symmetry of transition probabilities" in [1], and use it to show that a certain J^*-algebra is of rank $\leqslant 2$ (Lemma 2.4). This and Proposition 1 then show that the Peirce projections associated with an atom of Q leave the canonical and atomic parts of Q' invariant (Proposition 2).

In Section 3 we prove a fundamental result (Proposition 3) analogous to the property: "filters change pure states to pure states of smaller intensity" of [1]. This result is needed for proving that the local decomposition of a functional is in fact a global decomposition. To prove it we develop a generalization of the Hilbert ball property (Lemma 3.2) and make use of the classification of finite dimensional Jordan triple systems.

Our main results appear in Section 4. In Theorem 1 we decompose the range of Q into an atomic and a non-atomic part and collect various consequences of this decomposition which are used in the proof of Theorem 2. In Theorem 3 we summarize the above mentioned geometric properties which were obtained on the way to Theorems 1 and 2.

We now recall some notation and results from [9] which will be used repeatedly in this paper.

Let M be a J^*-algebra. For each f in M' let $v = v(f)$ be the unique partial isometry in M'' occurring in the enveloping polar decomposition of f [9, Theorem 1]. Then $l(f) = vv^*$ and $r(f) = v^*v$ are projections in the von Neumann algebra A'', where A is any C^*-algebra containing M as a J^*-subalgebra. More generally, for any partial isometry v in M'', the Peirce projections are defined by $E(v)x = lxr$, $F(v)x = (1-l)\,x(1-r)$, $G(v)x = lx(1-r) + (1-l)\,xr$, where $l = vv^*$ and $r = v^*v$. We shall write $E(f)$ for $E(v(f))$ and similarly for $G(f)$ and $E(f)$.

The Peirce projections also act on linear functionals by duality: $E(v)g = g \circ E(v)$, etc. We use the same notation for $E(v)$ and its dual, etc. More generally, in a binary algebra we use the notation $x \cdot g$ for the functional $z \to g(xz)$. Functionals f and g are *orthogonal*, in symbols $f \perp g$, if $F(f)g = g$ or equivalently $F(g)f = f$. We shall write $E(f) \geqslant E(g)$ if $E(f)\,E(g) = E(g)\,E(f) = E(g)$.

The following two results from [9], concerning functionals on a J^*-algebra M, will be used frequently ([9, Lemma 2.7, Lemma 2.9]).

If $f \in M'$ and $x \in M''$ satisfy $f(x) = \|f\|$ and $\|x\| = 1$, then $x = v(f) + F(f)x$. (0.1)

If f, $g \in M'$ satisfy one of the three mutually exclusive relations $f = E(g)f$, $f = F(g)f$, $f = G(g)f$ then $l(f)\,l(g) = l(g)\,l(f)$ and $r(f)\,r(g) = r(g)\,r(f)$. Therefore $\{E(f), F(f), G(f), E(g), F(g), G(g)\}$ is a commutative family of operators. (0.2)

The following commutativity formulas from [9] are fundamental: let Q be a contractive projection on the dual M' of a J^*-algebra M and let $f \in Q(M')$. Then

$$QE(f) = E(f)\,QE(f) \qquad ([9, \text{Proposition } 3.3]); \tag{0.3}$$

$$F(f)\,Q = F(f)\,QF(f) = QF(f)\,Q \qquad ([9, \text{Proposition } 3.5]); \tag{0.4}$$

$$G(f)\,Q = QG(f)\,Q \qquad ([9, \text{Proposition } 4.3]); \tag{0.5}$$

$$E(f)\,Q = QE(f)\,Q \qquad ([9, \text{Proposition } 4.3]). \tag{0.6}$$

Let Q be a contractive projection on the dual M' of M. By an *atom* of Q is meant any extreme point of the unit ball $Q(M')_1$ of $Q(M')$. Define

$$L = \sup\{l(f): f \in Q(M')\}, \qquad R = \sup\{r(f): f \in Q(M')\},$$

$$L_0 = \sup\{l(f): f \text{ atom of } Q\}, \qquad R_0 = \sup\{r(f): f \text{ atom of } Q\}.$$

Then L, R, L_0, R_0 are projections in A'' (where A is any C^*-algebra containing M as a J^*-subalgebra) and they define contractive projections $\mathscr{E}$

and $\mathscr{E}_0$ on A'' by $\mathscr{E}z = LxR$, $\mathscr{E}_0 z = L_0 z R_0$ for $z \in A''$. We shall also make use of the contractive projections $\mathscr{F}$, $\mathscr{F}_0$ on A'' defined by $\mathscr{F}z = (1-L)\,z(1-R)$, $\mathscr{F}_0 z = (1-L_0)\,z(1-R_0)$, for $z \in A''$.

We shall call $\mathscr{E}Q'$ the *canonical projection* associated with Q. (If $M = A$ is a C^*-algebra, $\mathscr{E}Q'$ is obviously a projection. In general $\mathscr{E}Q'$ maps M'' into A''.) Also $\mathscr{E}_0 Q'(M'')$ will be called the *atomic part* of Q'.

Two fundamental properties of atoms proved in [9] are the following: Let f be an atom of Q; then, with $v = v(f)$,

$$QE(f)\,g = \langle g, v\rangle f \qquad \text{for} \quad g \in M'; \tag{0.7}$$

$$E(f)\,Q'x = \langle f, x\rangle\, v \qquad \text{for} \quad x \in M'' \qquad ([9, \text{Proposition } 3.7]). \tag{0.7$'$}$$

For any element $g \in G(f)\,Q(M')$, either

$$f = E(g)f \qquad \text{or} \qquad f = G(g)f \qquad ([9, \text{Lemma } 4.7]). \tag{0.8}$$

We shall assume the reader has a basic familiarity with the essentials of JBW-algebra theory. In particular, we will need to use [1, Lemmas 5.2, 5.6] and the "halving lemma" [3, Theorem 6.10]. A *JBW^*-algebra* is the complexification of a JBW-algebra.

Finally, by a Hahn–Banach extension we mean a norm preserving extension of a functional, and if ξ, η are vectors in a Hilbert space H, $\omega(\xi, \eta)$ denotes the restriction of the linear functional $x \to (x\xi, \eta)$ to an appropriate subspace of $\mathscr{L}(H)$ which will be clear from the context.

1. Minimal Tripotents in the Range of a Canonical Projection

In this section we generalize [9, Lemma 3.8] to arbitrary dimensions and add more information to [9, Lemma 4.10].

Proposition 1. *Let Q be a contractive projection on the dual of a J^*-algebra M and let f be an atom of Q. Then $Q'v = v + \mathscr{F}Q'v$, where $v = v(f)$, and consequently $\mathscr{E}Q'v = v$ and $\mathscr{E}_0 Q'v = v$.*

The proof of this statement for finite dimensional range in [9, Lemma 3.8] used the finite dimensionality of $Q(M')$ only to be able to write each element of $G(f)\,Q(M')$ as a finite sum of atoms of $G(f)Q$. This latter fact will be proved in Lemma 1.5 below for arbitrary Q. Thus Lemma 1.5 together with the work in [9, Sect. 4] constitute the proof of Proposition 1.

The following remark, which follows simply from the definition of the Peirce projections, will be needed later.

REMARK 1.1. *Let g and h be orthogonal functionals on a J^*-algebra M. Then*

(a) $E(g+h)=E(g)+E(h)+G(g)\,G(h)$;

(b) *if* $f\in M'$ *satisfies* $f=G(g)\,G(h)f$; *then* $E(f)\,E(g+h)=E(f)$.

Our first lemma puts an upper bound (namely two) on the number of orthogonal non-zero elements in the range of $G(f)Q$.

LEMMA 1.2. *Let Q be a contractive projection on the dual of a J^*-algebra M and let f be an atom of Q.*

(a) *If* $g=G(f)\,g$ *and* $f=E(g)f$, *then* $F(g)\,G(f)=0$;

(b) *let g and h be orthogonal non-zero elements of $G(f)Q$. Then* $f=E(g+h)f$, *and therefore* $F(g+h)\,G(f)=0$.

Proof. (a) By [9, Lemma 2.9], $f=E(g)f$ implies $l(f)\leqslant l(g)$ and $r(f)\leqslant r(g)$. Thus for arbitrary x in M'',

$$\begin{aligned}F(g)\,G(f)x &= (1-l(g))\,G(f)\,x(1-r(g))\\ &= (1-l(g))[(1-l(f))\,xr(f)\\ &\quad + l(f)\,x(1-r(f))](1-r(g))=0.\end{aligned}$$

(b) If either $f=E(g)f$ or $f=E(h)f$, then by using the fact that $E(g+h)\geqslant E(g)$ and $E(g+h)\geqslant E(h)$ we have $E(g+h)f=f$. Thus by (0.8), we may assume that $f=G(g)f$ and $f=G(h)f$. By Remark 1.1(b), $f=E(f)f=E(g+h)\,E(f)f=E(g+h)f$. By using (a), with g replaced by $g+h$, we have $F(g+h)\,G(f)=0$. ∎

The next remark follows easily from [9, Lemma 3.1] and (0.3).

Remark 1.3. *Let Q be a contractive projection on the dual of a J^*-algebra and let $g=Qg$. Then any atom of $QE(g)$ or of $E(g)\,Q$ is also an atom of Q.*

We now introduce a local decomposition of a functional in the range of a contractive projection Q, into atomic and non-atomic parts. This decomposition will be an important tool for our investigation. The proof is based the powerful result of Effros and Størmer [6, Corollary 1.5] and on [1, Lemma 5.6].

LEMMA 1.4. *Let Q be a contractive projection on the dual M' of a J^*-algebra M and let $g=Qg$. Then*

(1) $E(g)\,Q'(M'')$ *is a J^*-subalgebra of* M'';

(2) *$E(g)\,Q'(M'')$ has a structure of JBW^*-algebra and the restriction of g to $E(g)\,Q'(M'')$ is a faithful positive normal functional;*

(3) *there is a sequence $\{g_i\}$ of pairwise orthogonal atoms of Q and an element $h \in Q(M')$ orthogonal to each g_i such that*

$$g = \sum_i \lambda_i g_i + h \tag{1.1}$$

where $\lambda_i \geqslant 0$, $\|g\| = \sum \lambda_i + \|h\|$, and $E(h)\,Q'(M'')$ is the purely non-atomic part of the JBW^*-algebra $E(g)\,Q'(M'')$;

(4) *if g is not an atom of Q, then there are at least two orthogonal non-zero elements of $E(g)\,Q(M')$.*

Proof. (1) $E(g)(M'')$ is a J^*-subalgebra of M'' and $E(g)\,Q'$ is a contractive projection on $E(g)(M'')$ satisfying the hypothesis of [9, Theorem 2]. Thus $E(g)\,Q'(M'')$ is a J^*-subalgebra of $E(g)(M'')$.

(2) By [9, Remark 3.2] and the proof of [9, Theorem 2] (which used [6, Corollary 1.5]) the map $x \to v^*x$ (where $v = v(g)$) is an isometry of $E(g)\,Q'(M'')$ onto a weakly closed Jordan $*$-algebra of operators. It is easy to check that the restriction $\tilde{g}$ of g to $E(g)\,Q'(M'')$ is a faithful positive normal functional in the JBW^*-algebra structure on $E(g)\,Q'(M'')$.

(3) Decompose $\tilde{g}$ in the JBW^*-algebra $B \equiv E(g)\,Q'(M'')$ according to [1, Lemma 5.6]. Thus $\tilde{g} = \sum_i \lambda_i \tilde{g}_i + \tilde{h}$, where $\lambda_i \geqslant 0$, the $\tilde{g}_i$ are pairwise orthogonal normal pure states of B, $\tilde{h}$ is purely nonatomic and $\|\tilde{g}\| = \sum \lambda_i + \|\tilde{h}\|$. Define $g_i = \tilde{g}_i \circ E(g) \circ Q'$, $h = \tilde{h} \circ E(g) \circ Q'$. Then g_i, $h \in Q(M')$ and extend $\tilde{g}_i$ and $\tilde{h}$, respectively. Clearly $g = \sum \lambda_i g_i + h$, $\|g\| = \sum \lambda_i + \|h\|$, and $E(h)\,Q'(M'')$ is the purely nonatomic part of B. By (1) and the pairwise orthogonality of $\tilde{g}_i$, $\tilde{h}$ in B we have the pairwise orthogonality of g_i, h in M'. Since $\tilde{g}_i$ is a normal pure state of B, g_i is an atom of $QE(g)$, so by Remark 1.3 g_i is an atom of Q.

(4) If this statement were false, there could be at most one non-zero term in (1.1) and since g is not an atom we must have $g = h$ in (1.1). Therefore B is a purely non-atomic JBW^*-algebra. The identity element of B can then be written as a sum of two orthogonal non-zero projections in B. For this use any non-trivial central projection, or if $B_{\text{s.a.}}$ is a JB-factor use the "halving lemma" [3, Theorem 6.10]. Since $\tilde{g} = \tilde{h}$ is faithful, there exist two orthogonal non-zero normal states of B. By transferring back to $E(g)\,Q'(M'')$ we obtain two orthogonal non-zero elements of $QE(g)(M')$.

As remarked above, the following lemma, together with [9, Sect. 4] proves the first assertion in Proposition 1. The second assertion follows from the obvious relations $\mathscr{E}\mathscr{E}^{\sim} = 0$ and $\mathscr{E}_0\mathscr{E} = \mathscr{E}_0$.

LEMMA 1.5. *Let Q be a contractive projection on the dual of a J^*-algebra M, let f be an atom of Q and let g be a non-zero element of $G(f)Q$. Then either*

(1) $g = \alpha_1 g_1$ *with* g_1 *an atom of* $G(f)Q$, $\alpha_1 > 0$; *or*

(2) $g = \alpha_1 g_1 + \alpha_2 g_2$ *with* g_1, g_2 *orthogonal atoms of* $G(f)Q$ *and* $\alpha_1 > 0$, $\alpha_2 > 0$.

Proof. Apply Lemma 1.4 to the projection $G(f)Q$ to obtain a decomposition $g = \sum \lambda_i g_i + h$ with $\{g_i\}$ pairwise orthogonal atoms of $G(f)Q$ and h orthogonal to each g_i. By Lemma 1.2(b) there are at most two non-zero terms in the decomposition. It suffices now to prove that $h = 0$. In the notation of Lemma 1.4, $\tilde{h}$ is a faithful normal positive functional on the purely non-atomic JBW^*-algebra $B \equiv E(h)\,Q'G(f)(M'')$. Let e denote the unit of B. If $e \neq 0$, we can choose, using [3, Theorem 6.10] if necessary, three non-zero pairwise orthogonal projections e_1, e_2, e_3 in B such that $e = e_1 + e_2 + e_3$. The elements e_1, e_2, e_3 are partial isometries in $E(g)\,Q'(M'')$ and if we define $h_i = E(e_i)h$ then we have three pairwise orthogonal non-zero functionals in $G(f)Q$. This contradicts Lemma 1.2(b).

With Proposition 1 now proved we turn to some other consequences of the results in this section, which will be needed later.

The first part of the next remark generalizes [9, Lemma 4.8] which required that g be an atom of $G(f)Q$.

REMARK 1.6. *Let Q be a contractive projection on the dual of a J^*-algebra, let f be an atom of Q, and let g be a non-zero element of $G(f)Q$.*

(a) *If $f = G(g)f$, then $\|g\|^{-1}g$ is an atom of Q.*

(b) *If there exists a non-zero element h of $G(f)Q$ orthogonal to g, then $\|g\|^{-1}g$ and $\|h\|^{-1}h$ are atoms of Q.*

Proof. (a) If $\|g\|^{-1}g$ is an atom of $G(f)Q$, then $\|g\|^{-1}g$ is an atom of Q by [9, Lemma 4.8]. If $\|g\|^{-1}g$ is not an atom of $G(f)Q$, then by Lemma 1.5(2) and Lemma 1.2(b) $f = E(g)f$, contradiction.

(b) By (0.8) either $f = E(h)f$ or $f = G(h)f$. If $f = E(h)f$ then by Lemma 1.2(a) $F(h)\,G(f) = 0$ and since g is orthogonal to h, $g = F(h)\,g = F(h)\,G(f)\,g = 0$, a contradiction. Therefore $f = G(h)f$. Similarly $f = G(g)f$ and by (a) $\|g\|^{-1}g$ and $\|h\|^{-1}h$ are atoms of Q.

The next remark conveniently summarizes the form of an arbitrary non-zero element g of $G(f)Q$, where f is an atom of Q.

REMARK 1.7. *For a non-zero element g in $G(f)Q$ with f an atom of Q, there are two possibilities*:

(1) $g = \alpha_1 g_1 + \alpha_2 g_2$ *where* g_1, g_2 *are atoms of* Q, $\alpha_1 > 0$ *and* $\alpha_2 \geqslant 0$;

or

(2) $\|g\|^{-1} g$ *is an atom of* $G(f)Q$ *and* $E(g)f = f$.

The following lemma completes [9, Lemma 4.10].

LEMMA 1.8. *Let* Q *be a contractive projection on the dual of a* J^*-*algebra* M, *let* f *be an atom of* Q, *let* g *be an atom of* $G(f)Q$, *and suppose* $f = E(g)f$. *Then*:

(a) *There exists an atom* h *of* Q *such that* $E(h) = E(g)F(f)$ *and* $E(g) = E(f+h)$;

(b) $g = \alpha_1 g_1 + \alpha_2 g_2$, *where* g_1, g_2 *are orthogonal atoms of* Q *and* $\alpha_1 > 0$, $\alpha_2 > 0$.

Proof. In the proof of [9, Lemma 4.10] a functional h is constructed satisfying $h \in F(f)Q$, $l(f+h) = l(g)$, $r(f+h) = r(g)$. To prove (a) it remains to show that any such h of norm 1 is an atom of Q. By Lemma 1.4(4) if h is not an atom of Q, then there are two orthogonal non-zero elements, say h_1, h_2 of $E(h)\,Q(M')$. Now $h_1 = E(g)\,h_1 = F(f)\,h_1$ so by (0.2), (0.3), and (0.5), $F(h_1)\,g = G(f)\,QE(g)\,F(h_1)\,g = \lambda g$, the latter by (0.7), since g is an atom of $G(f)Q$. If $\lambda \neq 0$, g is orthogonal to h_1 and $h_1 = F(g)\,h_1 = F(g)\,E(g)\,h_1 = 0$, contradiction. If $\lambda = 0$, then since $F(h_1)\,g = 0$ and $E(h_1)\,g = E(h_1)\,E(h)\,G(h)\,g = 0$ we have $g = G(h_1)\,g$ and by Remark 1.1(b) $E(g) = E(g)\,E(f+h_1)$. Since $E(f+h_1) \leqslant E(f+h) = E(g) \leqslant E(f+h_1)$ we have $E(h_1) = E(h)$, which contradicts the existence of h_2.

(b) By Lemma 1.4, $B \equiv E(f+h)\,Q'(M'')$ has a structure of JBW^*-algebra with pre-dual $B_* \cong QE(f+h)(M')$. Because f and h are orthogonal atoms of Q, the identity element of B is a sum of two orthogonal minimal projections, Since $g \in QE(f+h)(M') \cong B_*$ we can therefore write $g = \alpha_1 g_1 + \alpha_2 g_2$ with g_1, g_2 orthogonal atoms of $QE(f+h)$ and $\alpha_1 > 0$, $\alpha_2 > 0$. By Remark 1.3 g_1, g_2 are atoms of Q. ∎

For a contractive projection Q on the dual of a J^*-algebra M recall that $\mathscr{E}Q'$ is called the *canonical projection* associated with Q. An arbitrary partial isometry u in M'' is said to be a *minimal* tripotent of Q', if $\mathscr{E}Q'u = u$ and $E(u)\,Q'(M'') = \mathbb{C}u$.

The following lemma implies the existence of sufficiently many minimal tripotents of Q'. We shall see in §4 that the norm closure of the *linear* span of the minimal tripotents of Q' forms a J^*-algebra which is weakly dense in the range of $\mathscr{E}_0 Q'$.

LEMMA 1.9. *Let Q be a contractive projection on the dual M' of a J-algebra M and let u be a partial isometry in M''. The following are equivalent*:

(a) $u = v(g)$ *for some atom* g *of* Q;

(b) u *is a minimal tripotent of* Q'.

Proof. (a) $\Rightarrow$ (b). By Proposition 1 and (0.7).

(b) $\Rightarrow$ (a). For each x in M'', determine a number $\lambda(x)$ by the rule $E(u)\,Q'x = \lambda(x)u$. Then λ is an element of the unit ball of M', and g, defined by $g = \lambda \circ Q'$, belongs to $Q(M')$. Now $g(u)\,u = \lambda(Q'u)\,u = E(u)\,Q'u = u$ so that $g(u) = 1$ and $\|g\| = 1$. By (0.1), $u = v(g) + F(g)u$ so that $E(g) = E(g)\,E(u)$. Therefore, for $x \in M''$, $E(g)\,Q'x = E(g)\,E(u)\,Q'x = E(g)\,\lambda(x)u = \lambda(x)\,v(g)$. Thus $QE(g)(M')$ is one dimensional and g, being an element of norm one of $QE(g)(M')$, is an atom of $QE(g)$. By Remark 1.3, g is an atom of Q. By Proposition 1, $v(g) = \mathscr{E}Q'v(g)$ and so $v(g) = E(u)\,v(g) = E(u)\,\mathscr{E}Q'v(g) = \lambda(v(g))u$, i.e., $v(g) = u$. ∎

2. INVARIANCE OF THE ATOMIC PART UNDER THE PEIRCE PROJECTIONS OF AN ATOM

In Section 1 we showed that the Peirce projection $E(f)$, with f an atom of a contractive projection Q on the dual of a J^*-algebra, leaves invariant the canonical and atomic parts of Q', i.e., $RE(f)R = E(f)R$, where R is $\mathscr{E}Q'$ or $\mathscr{E}_0 Q'$. In this section we prove

$$RG(f)R = G(f)R. \tag{2.0}$$

It then follows that all three of the Peirce projections associated with an atom f of Q leave the canonical and atomic parts of Q invariant. The fact that the Peirce projections of an arbitrary φ in $Q(M')$ leave $Q(M')$ invariant is less deep and is proved by using (0.4), (0.5), and (0.6).

PROPOSITION 2. *Let Q be a contractive projection on the dual M' of a J^*-algebra M and let f be an atom of Q. Then $\mathscr{E}_0 Q'a = a$ and $\mathscr{E}Q'a = a$ for all $a \in G(f)\,Q'(M'')$.*

We shall prove Proposition 2 by showing that every element of $G(f)\,Q'(M'')$ is a linear combination of two orthogonal minimal tripotents of Q', i.e., $G(f)\,Q'(M'')$ is a J^*-algebra of rank $\leqslant 2$. Proposition 2 and (2.0) then follow from Proposition 1.

In each of the following lemmas, Q will denote a contractive projection on the dual of a J^*-algebra M.

The first part of the following lemma is the analog of the "pure state property" called "symmetry of transition probabilities" in [1].

LEMMA 2.1. (1) *Let f and g be atoms of Q. Then $f(v(g))=\overline{g(v(f))}$.*

(2) *Let $\mathcal{A}_0$ denote the linear span of the atoms of Q. There is a linear map $\tau: \mathcal{A}_0 \to \mathcal{A}_0'$ such that for any atoms f_i of Q and complex α_i we have*

$$\left\langle \tau\left(\sum_{i=1}^{n} \alpha_i f_i\right), g\right\rangle = \left\langle g, \sum_{i=1}^{n} \bar{\alpha}_i v(f_i)\right\rangle, \quad \textit{for} \quad g \in \mathcal{A}_0. \tag{2.1}$$

Proof. (1) Let $v=v(f)$. Then for any $x \in M''$, $\{v, \mathscr{E}Q'x, v\} = v(L(Q'x)R)^*v = vR(Q'x)^*Lv = v(Q'x)^*v = \overline{f(x)}v$ by (0.7′). Let $w=v(g)$. By Proposition 1, $w=\mathscr{E}Q'w$ and so $vw^*v = \overline{f(w)}\,v$. Similarly $wv^*w = \overline{g(v)}\,w$. In general $\{xy\{xzx\}\} = \{x\{yxz\}x\}$ holds. Since $\{wv\{wvw\}\} = \{w, v, \overline{g(v)}\,w\} = \overline{g(v)}^2 w$ and $\{w\{vwv\}w\} = \overline{f(w)}\{wvw\} = \overline{f(w)}\,\overline{g(v)}\,w$ we have $\overline{g(v)}^2 = \overline{g(v)}\,\overline{f(w)}$. Similarly $f(w)^2 = \overline{f(w)}\,g(v)$. Thus $g(v)=0$ implies that $f(w)=0$; and $g(v)\neq 0$ implies $f(w) = \overline{g(v)}$.

(2) For $f \in \mathcal{A}_0$ of the form $f=\sum_{i=1}^{n} \alpha_i f_i$, define τ formally by (2.1). We need only to prove that $\sum_{i=1}^{n} \alpha_i f_i = 0$ with f_i atoms of Q and α_i complex implies $\langle g, \sum \bar{\alpha}_i v(f_i)\rangle = 0$ for all $g \in \mathcal{A}_0$. This follows easily from (1). Let $g=\sum_{j=1}^{m} \beta_j g_j$, let $v_i = v(f_i)$ and $w_j = v(g_j)$. Then $\overline{\langle g, \sum \bar{\alpha}_i v_i\rangle} = \overline{\langle \sum_j \beta_j g_j, \sum_i \bar{\alpha}_i v_i\rangle} = \sum_j \sum_i \bar{\beta}_j \alpha_i \overline{g_j(v_i)} = \sum_j \sum_i \bar{\beta}_j \alpha_i f_i(w_j) = \sum_j \bar{\beta}_j \langle \sum_i \alpha_i f_i, w_j\rangle = 0$.

The following commutativity formula is now needed. Unlike (0.3) to (0.6), it requires that f be an atom.

LEMMA 2.2. *Let f be an atom of Q. Then, with $G=G(f)$ we have $QG = GQG$ and QG is a contractive projection. In particular, QG and GQ have the same range.*

Proof. If $QG=GQG$ then $(QG)^2 = QGQG = Q(QG) = QG$ so that QG is a projection: To prove $QG=GQG$, write $QG = EQG + GQG + FQG$, where $F=F(f)$, $E=E(f)$. By [9, Lemma 3.4] $FQG=0$ so it remains to prove $EQG=0$. Let $h \in M'$ and let $g=Gh$. We shall prove that $EQg=0$. Let $\tilde{h} \in A'$ be a Hahn–Banach extension of h and write $g = g_1 + g_2$, where $g_1 = (1-l)\cdot \tilde{h} \cdot r \,|\, M$, $g_2 = l \cdot \tilde{h} \cdot (1-r) \,|\, M$ and $l=l(f)$, $r=r(f)$. We shall show $EQg_1 = 0$ and a similar proof will yield $EQg_2=0$. Write $g_1 = \omega(\xi, \eta)$ with $\xi = r\xi$, $\eta = (1-l)\eta$, $\|\xi\|=1$. With $\omega = \omega(\xi, v\xi)$, $v=v(f)$, we have $\omega = E(f)\omega$ and $Q\omega = QE(f)\omega = \langle \omega, v\rangle f = f$ by (0.7). By (0.6), we have $EQg_1 = QEQg_1 = \lambda f$ for some scalar λ. Thus $EQ(\omega(\xi, v\xi) + t\omega(\xi, \eta)) = (1+t\lambda) f$ for arbitrary scalar t and therefore $|1+t\lambda| \leqslant \|\omega(\xi, v\xi + \bar{t}\eta)\| \leqslant \|v\xi + \bar{t}\eta\| = (1+|t|^2\|\eta\|^2)^{1/2}$, since $v\xi$ and η are orthogonal. Since t is arbitrary, $\lambda = 0$.

The formula $QG = GQG$ just proved and (0.5) now show that GQ and QG have the same range.

In Lemma 2.4 we shall obtain properties of a concrete realization of the map τ defined in Lemma 2.1, with Q replaced by $G(f)Q$. In this case, by Lemma 1.5, $\mathcal{A}_0 = G(f)\,Q(M')$ and therefore by Lemma 2.2, $\mathcal{A}_0' = [G(f)\,Q(M')]' = [QG(f)(M')]'$, which is canonically isomorphic to $G(f)\,Q'(M'')$. Let σ be the mapping from $\mathcal{A}_0'$ to $G(f)\,Q'(M'')$ and let $\pi = \sigma \circ \tau\colon G(f)\,Q(M') \to G(f)\,Q'(M'')$, which is a concrete realization of τ. To obtain these properties we need the following lemma.

LEMMA 2.3. *Let f be an atom of Q, and let g be an atom of $G(f)Q$. Then $G(f)\,Q'v(g) = v(g)$.*

Proof. By Lemma 2.2, $QG(f)$ is a projection with the same range as $G(f)Q$. Therefore g is also an atom of $G(f)Q$. By (0.8), either $f = E(g)f$ or $f = G(g)f$.

If $f = E(g)f$ then $G(f)\,F(g) = 0$ by Lemma 1.2(a). But by Proposition 1, $G(f)\,Q'v(g) = v(g) + F(g)\,G(f)\,Q'v(g) = v(g)$.

If $f = G(g)f$ then by Remark 1.6(a), g is an atom of Q and therefore by Proposition 1, $Q'v(g) = v(g) + \mathscr{E}Q'v(g) = v(g) + F(f)\,\mathscr{E}Q'v(g)$ so $G(f)\,Q'v(g) = G(f)\,v(g) + G(f)\,F(f)\mathscr{E}Q'v(g) = v(g)$. ∎

LEMMA 2.4. *Let f be an atom of Q. The linear mapping π: $QG(f)(M') \to G(f)\,Q'(M'')$ defined above satisfies:*

(1) $\pi(\sum_{i=1}^{n} \alpha_i g_i) = \sum_{i=1}^{n} \bar{\alpha}_i v(g_i)$, *for g_i atoms of $G(f)Q$ and α_i complex;*

(2) *the image of π is a J^*-algebra of* rank $\leqslant 2$;

(3) *the mapping π is onto $G(f)\,Q'(M'')$.*

Proof. (1) For $\Phi \in [G(f)\,Q(M')]'$, $\sigma(\Phi)$ is the unique element x of $G(f)\,Q'(M'')$ satisfying $\Phi(h) = h(x)$ for all $h \in G(f)\,Q(M')$. Let $\Phi = \tau(\sum_{i=1}^{n} \alpha_i g_i)$, defined by (2.1). By Lemma 2.3, $\sum_{i=1}^{n} \bar{\alpha}_i v(g_i) \in G(f)\,Q'(M'')$, so $\sigma(\Phi) = \sum \bar{\alpha}_i v(g_i)$, proving (1).

(2) By Lemma 1.5 each element in the image of π is a linear combination of at most two orthogonal partial isometries in $G(f)\,Q'(M'')$ and therefore the image of π is closed under the operation $x \to xx^*x$. Lemma 1.5 also implies that π satisfies $\frac{1}{2}\|g\| \leqslant \|\pi(g)\| \leqslant \|g\|$ for any $g \in G(f)\,Q(M')$. Thus the image of π is norm closed.

(3) Since a J^*-algebra of finite rank is weakly closed [11, Theorem 6.2], it will suffice to prove that the image of π is σ-weakly dense in $G(f)\,Q'(M'')$. Let $g \in G(f)\,Q(M')$ vanish on the image of π. Since by Lemma 1.5 $g = \alpha_1 g_1 + \alpha_2 g_2$, with g_1, g_2 orthogonal atoms of $G(f)Q$, the

conditions $g(v(g_1)) = g(v(g_2)) = 0$ force $\alpha_1 = \alpha_2 = 0$. Therefore the image of π is weak-*dense in $G(f)\,Q'(M'')$ so the result follows from [13, Theorem II 2.6]. ∎

Proof of Proposition 2. Let $a = \pi(g)$ for some $g \in G(f)\,Q(M')$, $g \neq 0$. By Remark 1.7 either

$$g = \alpha_1 g_1 + \alpha_2 g_2 \tag{2.2}$$

with g_1, g_2 orthogonal atoms of Q, or $\|g\|^{-1} g$ is an atom of $G(f)Q$ satisfying $f = E(g)f$. In the latter case Lemma 1.8 says that g still has the form (2.2). Thus in either case, $a = \pi(g) = \bar{\alpha}_1 v(g_1) + \bar{\alpha}_2 v(g_2)$ with g_1, g_2 atoms of Q. By Proposition 1, $\mathscr{E}Q'a = a$ and $\mathscr{E}_0 Q'a = a$. ∎

3. The Extreme Ray Property

In this section we shall prove the following proposition, which is an important tool for proving global properties from local ones. We shall use Proposition 3 in Section 4 to show that the decomposition (1.1) in Lemma 1.4 is global in the sense that the functional h occurring in (1.1) is orthogonal to *all* atoms of Q.

Proposition 3 (Extreme Ray Property). *Let Q be a contractive projection on the dual M' of a J^*-algebra M, let f be an atom of Q and let $\varphi \in Q(M')$. Then $E(\varphi)f$ is a scalar multiple of some atom of Q.*

A property similar to Proposition 3 has occurred in the context of Jordan algebras [1, Sect. 4]. Since Proposition 3 is of fundamental importance and its proof is rather lengthy we shall begin by sketching its proof.

Let S be the symmetry determined by φ, i.e., $S = E(\varphi) + F(\varphi) - G(\varphi)$. We show first (Lemma 3.1) that S is a linear isometry and therefore a J^*-automorphism of M [10]. Since S' leaves $Q(M')$ invariant, $g \equiv S'f$ is also an atom of Q.

A simple algebraic argument (Remark 3.3) results in the remarkable fact that any Jordan triple system generated by a pair of minimal tripotents e_1, e_2 is *linearly* spanned by the four elements e_1, e_2, $G(e_1)\,e_2$, $G(e_2)\,e_1$. (A tripotent e is *minimal* if $\{exe\} = \lambda e$ for arbitrary x.)

Using Propositions 1 and 2 and a similar algebraic argument applied to $u = v(f)$, $v = v(g)$, it is shown (Lemma 3.2) that the J^*-algebra J generated by u, v is of dimension at most 4, and lies in the canonical and atomic parts of Q'. This results can be considered as a generalization of the celebrated Hilbert ball property of [1].

The classification of finite dimensional Jordan triple systems [12] then implies (Remark 3.4) that J is isomorphic to one of the following: $\mathbb{C}$,

$M_{1,2}(\mathbb{C})$, $\mathbb{C}\oplus\mathbb{C}$, $S_{2,2}(\mathbb{C})$, $M_{2,2}(\mathbb{C})$. Lemma 3.5 and Remark 3.6 prove Proposition 3 in the first two cases, and Lemma 3.7 and Lemma 3.8 prove Proposition 3 in the remaining three cases.

LEMMA 3.1. (1) *For each partial isometry v in a J^*-algebra M, the map $S_v = E(v) + F(v) - G(v)$ is asymmetry (i.e., S_v is norm preserving and $S_v^2 = \mathrm{Id}$).*

(2) *If Q is a contractive projection on M' and $v = v(\varphi) \in M''$ for some $\varphi \in Q(M')$ then $S_v' Q(M') = Q(M')$. Therefore if f is an atom of Q, so is $g \equiv S_v' f$ and we have $S_v(v(f)) = v(g)$.*

Proof. (1) Writing $r = r(v)$ and $x = xr + x(1-r)$ for $x \in M''$, we have $\|x\|^2 = \|xx^*\| = \|xr - x(1-r)\|^2$. Similarly with $l = l(v)$, $\|x\| = \|lx - (1-l)x\|$. Therefore

$$\begin{aligned}\|x\| &= \|xr - x(1-r)\| \\ &= \|lxr - lx(1-r) - (1-l)\,xr + (1-l)\,x(1-r)\| = \|S_v x\|.\end{aligned}$$

By [10, Theorem 4] S_v is a J^*-isomorphism and clearly $S_v^2 = I$.

(2) By (0.4)–(0.6), we have

$$\begin{aligned}S_v' Q &= (E + F - G)Q = EQ + FQ - GQ \\ &= QEQ + QFQ - QGQ = QS_v' Q.\end{aligned}$$

Therefore $S_v' Q(M') \subset Q(M')$ and since S_v' has order 2, equality holds. Since S_v' is an isometry, atoms of Q are preserved and since S_v is a J^*-isomorphism all polar decompositions are preserved. In particular $S_v(v(f)) = v(g)$. ∎

In the next lemma we shall need the following purely algebraic identities which are valid in an arbitrary Jordan triple system (see [12, JP7, 9, 10, 16]). Recall that in a J^*-algebra, $\{xyz\} = \frac{1}{2}(xy^*z + zy^*x)$.

$$\{\{xyz\}\,ya\} = \{z\{yxy\}a\} + \{x\{yzy\}a\}, \tag{3.1}$$

$$\{xy\{zya\}\} = \{x\{yay\}z\} + \{x\{yzy\}a\}, \tag{3.2}$$

$$\{x(yxa\}z\} = \{\{xyx\}\,az\} + \{zy\{xax\}\}, \tag{3.3}$$

$$\{\{xyu\}\,va\} - \{u\{yxv\}a\} = \{x\{vuy\}a\} - \{\{uvx\}\,ya\}. \tag{3.4}$$

LEMMA 3.2. *Let Q be a contractive projection on the dual of a J^*-algebra M, let f and g be atoms of Q, let $u = v(f)$, $v = v(g)$ and let $J(u, v)$ be the J^*-algebra generated by u and v in M''. Then*

(1) *$J(u, v)$ is linearly spanned by $u, v, \{uuv\}, \{vvu\}$;*

(2) *$J(u, v)$ is linearly spanned by $u, v, G(u)v, G(v)u$;*

(3) *$\mathscr{E}_0 Q'a = a$ for all $a \in J(u, v)$.*

Proof. By Proposition 1 $\mathscr{E}_0 Q'u = u$, $\mathscr{E}_0 Q'v = v$ and by Lemma 1.9 u and v are minimal tripotents of Q'. In general $\{uuv\} = E(u)v + \frac{1}{2}G(u)v$ and $\{vvu\} = E(v)u + \frac{1}{2}G(v)u$. Now $G(u)Q'v = G(u)(v + \mathscr{E}Q'v) = G(u)v$ by Prop. 1. It follows from Propositions 1 and 2 that $\mathscr{E}_0 Q'(\{uuv\}) = \{uuv\}$. Similarly $\mathscr{E}_0 Q'(\{vvu\}) = \{vvu\}$. Therefore the two sets in (1) and (2) have the same linear span, which is fixed elementwise by $\mathscr{E}_0 Q'$.

To complete the proof it suffices to show that any triple product $\{abc\}$ with a, b, c in $\{u, v, \{uuv\}, \{vvu\}\}$ is a linear combination of $\{u, v, \{uuv\}, \{vvu\}\}$. By symmetry of u and v it suffices to consider only the 15 elements $a_1, a_2, a_3, a_4, a_5, b_1, \ldots, b_7, c_1, c_2, c_3$ listed below. Using (3.1) to (3.4), respectively, we have

$$a_1 = \{\{vuu\}\,uv\} = \{u\{uvu\}v\} + \{v\{uuu\}v\} = \lambda_1\{uuv\} + \lambda_2 v;$$

$$a_2 = \{\{vuu\}\,vu\} = \{u\{uuu\}v\} + \{u\{vvv\}u\} = \{uuv\} + \lambda_3 u;$$

$$a_3 = \{u\{vuu\}v\} = \{\{uvu\}\,uv\} + \{vv\{uuu\}\} = \lambda_4\{uuv\} + \{vvu\};$$

$$a_4 = \{\{vvu\}\,uu\} = \{u\{vvu\}u\} + \{v\{uuv\}u\} - \{\{uuv\}\,vu\}$$
$$= \lambda_5 u + a_3 - a_2.$$

Using (3.4) again, we have

$$a_5 = \{\{uuv\}\,uu\} = \{v\{uuu\}u\} + \{u\{uvu\}u\} - \{\{vuu\}\,uu\}$$

so that $a_5 = \frac{1}{2}\{vuu\} + \frac{1}{2}\lambda_6 u$.

Using (0.0), the following elements can be shown to lie in the linear span of u, v, $\{uuv\}$, $\{vvu\}$

$$b_1 = \{u\{uuv\}\{uuv\}\}, \qquad b_2 = \{u\{uuv\}\{vvu\}\},$$

$$b_3 = \{u\{vvu\}\{uuv\}\}, \qquad b_4 = \{u\{vvu\}\{vvu\}\},$$

$$b_5 = \{\{uuv\}\,u\{uuv\}\}, \qquad b_6 = \{\{uuv\}\,u\{vvu\}\},$$

$$b_7 = \{\{vvu\}\,u\{vvu\}\}, \qquad c_1 = \{\{uuv\}\{uuv\}\{uuv\}\},$$

$$c_2 = \{\{uuv\}\{uuv\}\{vvu\}\}, \qquad c_3 = \{\{uuv\}\{vvu\}\{uuv\}\}.$$

Remark 3.3. *Lemma* 3.2 *has the following purely algebraic counterpart which is essentially the case $Q = I$: any two minimal tripotents in a Jordan triple system generate a Jordan triple subsystem of linear dimension at most* 4.

Remark 3.4. *The Jordan triple system J in Lemma* 3.2 *or in Remark* 3.3 *is simple unless u is orthogonal to v.* For if $J = J_1 \oplus J_2$, then u, v must each lie in a component J_1 or J_2.

Therefore according to the classification of simple finite dimensional complex Jordan triple systems [12], J is isomorphic to one of the following:

(1)	$\mathbb{C}$	dimension 1, rank 1
(2)	$M_{1,2}(\mathbb{C})$	dimension 2, rank 1 (1 by 2 complex matrices)
(3)	$\mathbb{C} \oplus \mathbb{C}$	dimension 2, rank 2
(4)	$A_3(\mathbb{C})$	dimension 3, rank 1 (3 by 3 antisymmetric matrices)
(5)	$M_{1,3}(\mathbb{C})$	dimension 4, rank 1
(6)	$S_2(\mathbb{C})$	dimension 3, rank 2 (2 by 2 symmetric matrices)
(7)	$M_{2,2}(\mathbb{C})$	dimension 4, rank 2

Recall that the rank of a J^*-algebra is the largest number of mutually orthogonal tripotents. We note next that cases (4) and (5) cannot occur. To see this note that J of rank 1 implies $F(u)v = 0$ and $F(v)u = 0$ and therefore $v = E(u)v + G(u)v + F(u)v = \lambda u + G(u)v$ so that $G(u)v$ (and by symmetry $G(v)u$) lies in the span of u and v, so $\dim J \leqslant 2$.

The following notation will be used in the rest of this section: Q is a contractive projection on the dual of a J^*-algebra M, f is an atom of Q, φ is an element of $Q(M')$, g is the atom of Q defined by $g = Sf$, where S is the symmetry determined by φ, $u = v(f)$, $v = v(g)$, and J is the J^*-subalgebra of M'' generated by u and v. Since $f = E(\varphi)f + F(\varphi)f + G(\varphi)f$ and $g = E(\varphi)f + F(\varphi)f - G(\varphi)f$, we have

$$\tfrac{1}{2}(f + g) = E(\varphi)f + F(\varphi)f. \tag{3.5}$$

LEMMA 3.5. *If $J \simeq M_{1,2}(\mathbb{C})$, then every linear combination of f and g is a multiple of an atom of Q.*

Proof. Let w be a partial isometry in J which is orthogonal to u in the Hilbert space structure of J induced by $M_{1,2}(\mathbb{C})$. The orthogonality of u and w is equivalent to $E(u)w = 0 = E(w)u$. Since J has rank 1, $F(w)u = 0 = F(u)w$. Therefore

$$G(u)w = w \qquad \text{and} \qquad G(w)u = u. \tag{3.6}$$

By Lemma 3.2(3) and Proposition 2 we have $w = \mathscr{E}Q'w = G(f)\mathscr{E}Q'w = \mathscr{E}G(f)Q'w = G(f)Q'w$. Therefore, by Lemma 2.4, $w = \pi(h)$ for some element $h \in QG(f)(M')$ and $E(h)f = E(w)f = 0$. Thus by (0.8), $G(h)f = f$ and by Remark 1.6(a) h is an atom of Q with $v(h) = w$.

	r_1	r_2	r_3	
ℓ_1		f	h	
ℓ_2	f			
ℓ_3	h			

FIGURE 1

We show next that for every $\alpha, \beta \in \mathbb{C}$,

$$\|\alpha f + \beta h\| = (|\alpha|^2 + |\beta|^2)^{1/2}. \tag{3.7}$$

Since $f = G(h)f$ and $h = G(f)h$, (0.2) implies that the support projections of f and h commute and that we have the block matrix representation shown in Fig. 1, where $r_1 = r(f)\,r(h)$, $r_2 = r(f)(1 - r(h))$, etc.

If $r_2 \neq 0$ let $\xi_2 = r_2\xi_2$, $\|\xi_2\| = 1$, and let $\eta = u\xi_2$ so that $l_1\eta = \eta$. Then $Q(\omega(\xi_2, \eta)) = QE(f)(\omega(\xi_2, \eta)) = (u\xi_2, \eta)f = f$. Now define $\xi_3 = w^*(\eta)$. Then it follows similarly that $Q(\omega(\xi_3, \eta)) = h$. Therefore $\|\alpha f + \beta h\| = \|Q(\alpha\omega(\xi_2, \eta) + \beta\omega(\xi_3, \eta))\| \leqslant \|\omega(\alpha\xi_2 + \beta\xi_3, \eta)\| \leqslant \|\alpha\xi_2 + \beta\xi_3\| = (|\alpha|^2 + |\beta|^2)^{1/2}$. Since $f|J, h|J$ is the orthonormal basis for J' dual to u, w, $\|(\alpha f + \beta h) \,|\, J\| = (|\alpha|^2 + |\beta|^2)^{1/2}$ so (3.7) is proved in case $r_2 \neq 0$. In case $r_2 = 0$, a similar proof of (3.7) can be given that begins by choosing a nonzero vector in the range of l_3.

We next show that g belongs to the linear span of f and h and therefore $\operatorname{span}\{f, h\} = \operatorname{span}\{f, g\}$.

There are scalars α, β such that $g \,|\, J = (\alpha f + \beta h) \,|\, J$, and therefore $\|g\| = g(v) = \|g \,|\, J\| = (|\alpha|^2 + |\beta|^2)^{1/2}$. Let $\tilde{g}$ denote $\alpha f + \beta h$. Since $\tilde{g}(v) = \|\tilde{g}\|$, $\|E(v)\,\tilde{g}\| = \|\tilde{g}\|$ so [9, Lemma 3.1] implies $\tilde{g} = E(v)\,\tilde{g}$. Because g is an atom of Q, $\tilde{g} = E(v)\,\tilde{g} = QE(v)\,\tilde{g} = \langle \tilde{g}, v\rangle g$ and therefore $g = \tilde{g} = \alpha f + \beta h$.

To prove our lemma it is enough to show that every linear combination of f and h of unit norm is an atom of Q. Let $\varphi = \gamma f + \delta h$ and $|\gamma|^2 + |\delta|^2 = \|\varphi\| = 1$. Suppose $\varphi = \frac{1}{2}(\varphi_1 + \varphi_2)$ with $\varphi_1, \varphi_2 \in Q(M')_1$. We shall show that $\varphi = \varphi_1$. Since J is a Hilbert space and $\|\varphi \,|\, J\| = 1$, φ and φ_1 agree on J. Therefore $\varphi_1(u) = \varphi(u) = \gamma$ and $\varphi_1(w) = \varphi(w) = \delta$ so that by (0.7), $E(f)\,\varphi_1 = E(f)\,Q\varphi_1 = \langle \varphi_1, u\rangle f = \gamma f$ and similarly $E(h)\,\varphi_1 = \delta h$. Moreover φ assumes its norm on $s \equiv \bar{\gamma}u + \bar{\delta}w$ and therefore $E(s)\,\varphi_1 = \varphi_1$. Since $\varphi_1 = (E(h) + F(h) + G(h))(E(f) + F(f) + G(f))\,\varphi_1 = \gamma f + \delta h + G(h)\,G(f)\,\varphi_1$, φ_1 has the block matrix representation shown in Fig. 2, where $\psi = G(h)\,G(f)\,\varphi_1$. Thus $\varphi_1 = \varphi + \psi$. Let E_1, F_1, G_1 be the Peirce projections on A'' defined by the projections l_1, r_1. Then $\varphi_1(M \cap E_1(A'')) = 0$ and therefore there exists a Hahn–Banach extension $\tilde{\varphi}_1$

	r_1	r_2	r_3	
ℓ_1		γf	δh	
ℓ_2	γf		ψ	
ℓ_3	δh	ψ		

FIGURE 2

of φ_1 to A which vanishes on $E_1(A'')$. We have $\|G_1\tilde{\varphi}_1\| \geqslant \|\varphi\| = \|\varphi_1\| = \|\tilde{\varphi}_1\| = \|(G_1+F_1)\tilde{\varphi}_1\|$ and by [9, Corollary 4.2], $F_1\tilde{\varphi}_1 = 0$ and therefore $\psi = \tilde{\varphi}_1 \mid (M \cap F_1(A'')) = 0$. ▮

REMARK 3.6. *If J is isomorphic to $\mathbb{C}$ or to $M_{1,2}(\mathbb{C})$, then $E(\varphi)f$ is a multiple of some atom of Q, i.e., Proposition* 3 *is true in cases* (1) *and* (2) *of Remark* 3.4.

Proof. If J is isomorphic to $\mathbb{C}$, then u is a multiple of v, f is a multiple of g, so $\frac{1}{2}(f+g)$ is a multiple of the atom g of Q. If J is isomorphic of $M_{1,2}(\mathbb{C})$, then $\frac{1}{2}(f+g)$ is a multiple of an atom of Q by Lemma 3.5. Since $E(\varphi)f$ and $F(\varphi)f$ are orthogonal, (3.5) implies that $E(\varphi)f \in E(f+g)(M')$. Since $E(\varphi)f \in Q(M')$ by (0.6), and $\|f+g\|^{-1}(f+g)$ is atom of Q, (0.7) implies $E(\varphi)f$ is a multiple of $f+g$. ▮

LEMMA 3.7. *If J is isomorphic to $\mathbb{C} \oplus \mathbb{C}$, $S_2(\mathbb{C})$, or $M_{2,2}(\mathbb{C})$, then there exist two orthogonal atoms f_1, f_2 of Q such that $E(f_1+f_2)a = a$ for all $a \in J$.*

Proof. If J is isomorphic to $\mathbb{C} \oplus \mathbb{C}$ we may choose $f_1 = f$ and $f_2 = g$. Now suppose that J is isomorphic to either $S_2(\mathbb{C})$ or $M_{2,2}(\mathbb{C})$. Let u_1, u_2, u_3 in J correspond, respectively, to the matrices $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$. We may assume $u_1 = u = v(f)$.

Obviously $u_2 = G(u_1)u_2 = G(f)u_2 = G(f)\mathscr{E}Q'u_2 = G(f)Q'u_2$ so by Lemma 2.4 there exists $h \in QG(f)(M')$ such that $\pi(h) = u_2$. Since $E(h)u_1 = E(u_2)u_1 = u_1 \neq u_2$, h is not an atom of Q and therefore by Remark 1.7 and Lemma 1.8, $h = \alpha_1 f_1 + \alpha_2 f_2$ for two orthogonal atoms f_1, f_2 of Q. Finally, for any $a \in J$, $E(f_1+f_2)a = E(h)a = E(u_2)a = a$. ▮

LEMMA 3.8. *If J is ismorphic to $\mathbb{C} \oplus \mathbb{C}$, $S_2(\mathbb{C})$, or $M_{2,2}(\mathbb{C})$, then $E(\varphi)f$ is a multiple of some atom of Q, i.e., Proposition* 3 *is true in cases* (3), (6) *and* (7) *of Remark* 3.4.

Proof. Let f_1 and f_2 be two orthogonal atoms of Q satisfying $E(f_1+f_2)a = a$ for all $a \in J$. The existence of f_1, f_2 is given in Lemma 3.7.

For notation's sake let $w=v(f_1+f_2)$, $\varphi_1=E(\varphi)f, \varphi_2=F(\varphi)f$. Then $E(w)u=u$, $E(w)v=v$ implies $E(w)f=f$, $E(w)g=g$ and therefore $E(w)(\frac{1}{2}(f+g))=\frac{1}{2}(f+g)$. By (3.5) and the orthogonality of φ_1, φ_2 we have $E(w)\varphi_1=\varphi_1$, $E(w)\varphi_2=\varphi_2$. Thus φ_1 and φ_2 belong to $QE(w)(M')$, which is the predual of the JBW^*-algebra $E(w)Q'(M'')$ (cf. Lemma 1.4). Since $w=v(f_1)+v(f_2)$ with f_1, f_2 orthogonal atoms of Q, the JBW^*-algebra $E(w)Q'(M'')$ has rank 2, i.e., its identity is the sum of two orthogonal minimal (self-adjoint) idempotents.

We now use the following fact about such Jordan algebras. Every element ψ of norm 1 in the predual is either extremal, or $E(\psi)$ is the identity, i.e., $l(\psi)=r(\psi)=1$. Indeed if $r(\psi)\neq 1$, then $r(\psi)$ is a minimal projection so that $|\psi|$ is a pure state in this case and $\psi=v(\psi)\cdot|\psi|$ is extremal. Similarly if $l(\psi)\neq 1$, $|\psi^*|$ is a pure state and $\psi=|\psi^*|\cdot v(\psi)$ is extremal.

If $\|\varphi_1\|^{-1}\varphi_1$ is not an atom of Q, then $E(\varphi_1)a=a$ for all $a\in J$. Since $\varphi_1=E(\varphi)f$ we have $E(\varphi_1)\leqslant E(\varphi)$ and therefore $E(\varphi)a=a$ for all $a\in J$. In particular $E(\varphi)u=u$ and therefore $E(\varphi)f=f$. Thus $g=E(\varphi)f+F(\varphi)f-G(\varphi)f=f$, and $J\simeq\mathbb{C}$, contradiction.

Proposition 3 is now proved. Using the same proof we have the following.

REMARK 3.9. *Let f be an atom of Q, and let $\varphi=Q\varphi$. Then $F(\varphi)f$ is a scalar multiple of an atom of Q.*

4. THE MAIN RESULTS

In this section we shall prove our main results. The first lemma will be used to show uniqueness of a decomposition in Theorem 1.

LEMMA 4.1. *Let M be a J^*-algebra and let $g_1, g_2\in M'$ satisfy $E(g_2)g_1=g_2$ and $E(g_1)g_2=g_1$. Then $g_1=g_2$.*

Proof. For $x\in M''$, $g_1(x)=g_2(l_1xr_1)=g_1(l_2l_1xr_1r_2)$, where $l_i=l(g_i)$, $r_i=r(g_i)$. Thus $\|g_1\|=g_1(v_1)=g_1(l_2l_1v_1r_1r_2)=g_1(l_2v_1r_2)$. Since $\|l_2v_1r_2\|\leqslant 1$, (0.1) implies that $l_2v_1r_2=v_1+(1-l_1)l_2v_1r_2(1-r_1)$. Therefore $v_1=l_1l_2v_1r_2$ and $v_1=l_2v_1r_2r_1$ so that $l_1=v_1v_1^*=l_2v_1r_2r_1r_2v_1^*l_2$, i.e., $l_1\leqslant l_2$; and $r_1=v_1^*v_1=r_2v_1^*l_2l_1l_2v_1r_2$, i.e., $r_1\leqslant r_2$. By symmetry, $l_2\leqslant l_1$, $r_2\leqslant r_1$ so $g_2=E(g_2)g_1=E(g_1)g_1=g_1$. ∎

THEOREM 1. *Let Q be a contractive projection on the dual M' of a J^*-algebra M, let $\mathcal{O}\!\mathcal{l}$ be the norm closure of the linear span of the atoms of Q, and let $\mathcal{N}=\{\Psi\in Q(M')\colon \Psi$ is orthogonal to all atoms of $Q\}$. Then $Q(M')=\mathcal{O}\!\mathcal{l}\oplus_{l_1}\mathcal{N}$, and the unit ball of $\mathcal{N}$ contains no extreme points.*

Proof. Let $\varphi \in Q(M')$. By Lemma 1.4 $\varphi = h + \psi$ with $h \in \mathcal{A}$ and $\psi \in Q(M')$ such that $\|\varphi\| = \|h\| + \|\psi\|$, $h \perp \psi$, and $E(\psi)\, Q'(M'')$ is a purely non-atomic JBW^*-algebra. Let f be an arbitrary atom of Q. We show next that ψ is orthogonal to f, and the decomposition $\varphi = h + \psi$ is unique. To this end we first show that $E(\psi)f = 0$. Indeed by Proposition 3 $E(\psi)f = \lambda g$ for some $\lambda \in \mathbb{C}$ and some atom g of Q. If $\lambda \neq 0$ then $g \in QE(\psi)(M')$, which is the predual of the purely non-atomic JBW^*-algebra $E(\psi)\, Q'(M'')$. Since g is an atom of Q it corresponds to an extremal elements of $QE(\psi)(M')$, contradiction. Therefore $E(\psi)f = 0$. We show next a strong uniqueness property: Suppose $\varphi = h_1 + \psi_1 = h_2 + \psi_2$ with $h_1, h_2 \in \mathcal{A}$ and $E(\psi_1)f = 0$ and $E(\psi_2)f = 0$ for all atoms f of Q. Then $E(\psi_i)\, h_j = 0$ for $i, j = 1, 2$ and therefore $\psi_1 = E(\psi_1)\varphi = E(\psi_1)\, \psi_2$ and $\psi_2 = E(\psi_2)\varphi = E(\psi_2)\, \psi_1$. By Lemma 4.1, $\psi_1 = \psi_2$ and therefore $h_1 = h_2$. To prove that ψ is orthogonal to an arbitrary atom f of Q, consider $f + \psi$. By Lemma 1.4 we can write $f + \psi = h_1 + \psi_1$ with $h_1 \in \mathcal{A}$, $\psi_1 \in Q(M')$ and $\psi_1 \perp h_1$. By the uniqueness property just proved, $f = h_1$ and $\psi = \psi_1$. In particular ψ is orthogonal to f.

Since the norms in the direct sum add, each extreme point of the unit ball of $\mathcal{N}$ would also be extremal in $Q(M')$. ∎

REMARK 4.2. *Let $\varphi = h + \psi$ be the decomposition of an element $\varphi \in Q(M')$ given by Theorem* 1. *Then $h = \mathscr{E}_0 \tilde{\varphi} \,|\, M$ and $\psi = \mathscr{E}_0 \tilde{\varphi} \,|\, M$, where $\tilde{\varphi}$ is any Hahn–Banach extension of φ to A.*

COROLLARY 4.3. $\mathcal{N} = \{\varphi \in Q(M') \colon \varphi(v(f)) = 0$ *for all atoms f of Q*$\}$.

Proof. Suppose $\varphi \in Q(M')$ satisfies $\varphi(v(f)) = 0$ for all atoms f of Q. We shall show that $\varphi \in \mathcal{N}$. Let g be an arbitrary atom of Q, and write $g + \varphi = h + \psi$, where $h \in \mathscr{A}$, $\psi \in \mathcal{N}$. Since φ and ψ both vanish on $v(f)$, where f is an arbitrary atom of Q, we have

$$g(v(f)) = h(v(f)) \qquad \text{for all atoms } f \text{ of } Q.$$

We shall prove that $g = h$, hence $\varphi = \psi \in \mathcal{N}$. In the first place $\|h\| = 1$ since on the one hand, $h(v(g)) = g(v(g)) = 1$ and on the other hand, writing $h = \sum \lambda_i h_i$ with h_i orthogonal atoms of Q, $\|h\| = h(\sum_i v(h_i)) = \sum_i h(v(h_i)) = \sum_i f(v(h_1)) = f(\sum_i v(h_i)) \leqslant \|f\| = 1$. In the second place, $\langle E(g)h, v(f)\rangle = 1$ implies $\|E(g)h\| \geqslant 1$, and $1 \leqslant \|E(g)h\| \leqslant \|h\| = 1$ and by [9, Lemma 3.1] $E(g)h = h$. Now by (0.6) and (0.7), $h = E(g)h = E(g)\, Qh = QE(g)\, Qh = \langle h, v(g)\rangle g = g$. ∎

The following notation will be used in order to obtain other important consequences of Theorem 1. First, write $[Q(M')]' = \mathcal{N}^{\perp} \oplus_{l\infty} \mathcal{A}^{\perp}$, where $\mathcal{N}^{\perp} = \{\Phi \in [Q(M')]' \colon \Phi(\mathcal{N}) = 0\}$, and $\mathcal{A}^{\perp} = \{\Phi \in [Q(M')]' \colon \Phi(\mathcal{A}) = 0\}$.

Then let $T: Q'(M'') \to [Q(M')]'$ be the linear isometry onto given by $Tx(\varphi) = \varphi(x)$ for $x \in Q'(M'')$ and $\varphi \in Q(M')$. We then have

$$Q'(M'') = T^{-1}(\mathcal{N}^{\perp}) \oplus_{l^\infty} T^{-1}(\mathcal{O}^{\perp}). \tag{4.1}$$

Note that T can be considered as a restriction operator and T^{-1} as an extension operator.

COROLLARY 4.4. *Let M_{fin} be the set of all finite linear combinations of minimal tripotents of Q': $M_{\text{fin}} = \{\sum_{i=1}^{n} \alpha_i v_i$: α_i complex, $v_i = v(f_i)$, f_i atom of Q: $n = 1, 2,...\}$. Then*

(a) *$Q'(M_{\text{fin}})$ is σ-weakly dense in $T^{-1}(\mathcal{N}^{\perp})$;*

(b) *M_{fin} is σ-weakly dense in $\mathscr{E}_0 T^{-1}(\mathcal{N}^{\perp})$.*

Proof. (a) Since $T^{-1}(\mathcal{N}^{\perp})$ is σ-weakly closed in M'', it suffices by [13, Theorem II 2.6], to show that $Q'(M_{\text{fin}})$ is weak*-dense in $T^{-1}(\mathcal{N}^{\perp})$, equivalently that $TQ'(M_{\text{fin}})$ is weak *-dense in $\mathcal{N}^{\perp}$. Let $\varphi \in Q(M')$ vanish on $TQ'(M_{\text{fin}})$. Then $\varphi(v(f)) = 0$ for every atom of Q so by Corollary 4.3, $\varphi \in \mathcal{N}$. Therefore φ vanishes on $\mathcal{N}^{\perp}$.

(b) By Proposition 1, $M_{\text{fin}} = \mathscr{E}_0 Q'(M_{\text{fin}})$, and by (a) it is σ-weakly dense in $\mathscr{E}_0 T^{-1}(\mathcal{N}^{\perp})$. ∎

We shall now obtain a splitting of $Q'(M'')$ into orthogonal parts, analogous to the splitting of $Q(M')$ in Theorem 1.

LEMMA 4.5. *For each $x \in T^{-1}(\mathcal{O}^{\perp})$, we have $x = \bar{\mathscr{E}}_0 x$. Therefore by (4.1), $\mathscr{E}_0 Q'(M'') = \mathscr{E}_0 T^{-1}(\mathcal{N}^{\perp})$.*

Proof. By definition of $\bar{\mathscr{E}}_0$ it suffices to prove that $F(f)x = x$ for each atom f of Q. We note first that by definition of $T^{-1}(\mathcal{O}^{\perp})$, $\langle g, x \rangle = 0$ for every atom g of Q. Therefore, by (0.7), $E(f)x = E(f)\, Q'x = \langle f, x \rangle\, v(f) = 0$. It remains to show that $G(f)x = 0$. For arbitrary $\varphi \in M'$, we have by Remark 1.7, Lemma 1.8, and Lemma 2.2,

$$\langle G(f)x, \varphi \rangle = \langle x, QG(f)\varphi \rangle = \langle x, \alpha_1 g_1 + \alpha_2 g_2 \rangle = 0$$

where g_1, g_2 are atoms of Q. ∎

PROPOSITION 4. *Let Q be a contractive projection on the dual of a J^*-algebra M. Then for each x in $Q'(M'')$, $x = \mathscr{E}_0 x + \bar{\mathscr{E}}_0 x$.*

Proof. For x in $Q'(M'')$, we have by (4.1), $x = x_1 + x_2$ with $x_1 \in T^{-1}(\mathcal{N}^{\perp})$ and $x_2 \in T^{-1}(\mathcal{O}^{\perp})$. By Corollary 4.4 the element x_1 can be approximated σ-weakly by elements of the form $\sum_{i=1}^{n} \alpha_i Q' v_i =$

$\mathscr{E}_0 \sum_{i=1}^n \alpha_i v_i + \mathscr{E} \sum \alpha_i Q' v_i$ (by Proposition 1). Therefore $x_1 = \mathscr{E}_0 x_1 + \mathscr{E} x_1$. By Lemma 4.5 $x_2 = \bar{\mathscr{E}}_0 x_2$ so $x = \mathscr{E}_0 x + \bar{\mathscr{E}}_0 x$. ∎

Proposition 4 says that $Q'(M'') \subseteq \mathscr{E}_0 Q'(M'') + \bar{\mathscr{E}}_0 Q'(M'')$. We prove next that $\mathscr{E}_0 Q'(M'')$ is a J^*-subalgebra of A''. For this we shall use the following lemma.

Recall that $\mathscr{A}_0$ is the linear span of the atoms of Q and M_{fin} is the linear span of the minimal tripotents of Q'.

LEMMA 4.6. *There is a linear bijection $\pi\colon \mathscr{A}_0 \to M_{\text{fin}}$ given by*

$$\pi\left(\sum_{i=1}^n \alpha_i f_i\right) = \sum_{i=1}^n \bar{\alpha}_i v(f_i) \tag{4.2}$$

where f_i are atoms of Q and $\alpha_i \in \mathbb{C}$.

Proof. We show first that $\sum \alpha_i f_i = 0$, with f_i atoms of Q and $\alpha_i \in \mathbb{C}$, implies $\sum \bar{\alpha}_i v_i = 0$ where $v_i = v(f_i)$. Since $b \equiv \sum \bar{\alpha}_i v_i \in \mathscr{E}_0 Q'(M'')$, it suffices by Lemma 4.5 to show that $b \in T^{-1}(\mathscr{A}^\perp)$, i.e., that $g(b) = 0$ for all atoms g of Q. This follows from Lemma 2.1. It now follows that the map $\pi\colon \mathscr{A}_0 \to M_{\text{fin}}$ defined by (4.2) is well defined, linear, and onto. Finally we shall show that it is one-to-one. Suppose $\sum_{i=1}^n \alpha_i v_i = 0$. We are to prove that $\sum_{i=1}^n \bar{\alpha}_i f_i = 0$. By Lemma 2.1, if g is an atom of Q, we have $\langle v(g), \sum \bar{\alpha}_i f_i \rangle = \langle g, \sum_i \alpha_i v_i \rangle = 0$. By Corollary 4.3, $\sum \bar{\alpha}_i f_i \in \mathscr{N}$. Since $\sum \bar{\alpha}_i f_i \in \mathscr{A}_0 \subseteq \mathscr{A}$, the proof is complete. ∎

PROPOSITION 5. *Let Q be a contractive projection on the dual of a J^*-algebra M. Then $\mathscr{E}_0 Q'(M'')$ is a J^*-subalgebra of A''.*

Proof. We show first that if $a \in M_{\text{fin}}$ then $aa^*a \in \mathscr{E}_0 Q'(M'')$. Indeed if $a = \sum_{i=1}^n \alpha_i v_i$ then $\pi^{-1}(a) \in \mathscr{A}_0$ can be written as a (possibly infinite) linear combination of pairwise orthogonal atoms of Q, say $\pi^{-1}(a) = \sum_{i=1}^n \bar{\alpha}_i f_i = \sum \lambda_i g_i$, where g_i are orthogonal atoms of Q, $\lambda_i \geqslant 0$. Let $b = \sum \lambda_i v(g_i)$, which exists as a strong limit in $\mathscr{E}_0 T^{-1}(\mathscr{N}^\perp) = \mathscr{E}_0 Q'(M'')$. We shall show that $b = a$. Since $a, b \in \mathscr{E}_0 Q'(M'')$ it suffices to show that $\langle g, a \rangle = \langle g, b \rangle$ for every atom g of Q. This follows from Lemma 2.1: $\overline{\langle g, a \rangle} = \sum \bar{\alpha}_i \overline{\langle g, v_i \rangle} = \sum \bar{\alpha}_i \langle f_i, v(g) \rangle = \langle \sum \lambda_i g_i, v(g) \rangle = \sum \lambda_i \overline{\langle g, v(g_i) \rangle} = \overline{\langle g, \sum \lambda_i v(g_i) \rangle} = \overline{\langle g, b \rangle}$. Thus $a = b$, and by the orthogonality of g_i we have $aa^*a = bb^*b = \sum \lambda_i^3 v(g_i) \in \mathscr{E}_0 Q'(M'')$.

It now follows from the polarization identities [10, p. 17] that $a, b, c \in M_{\text{fin}}$ implies $ab^*c + cb^*a \in \mathscr{E}_0 Q'(M'')$. By the separate σ-weak continuity of multiplication and the σ-weak continuity of involution, the Proposition follows from Corollary 4.4 and Lemma 4.5. ∎

REMARK 4.7. *Proposition* 3 *and* [1, *Lemma* 5.1] *show that the orthogonal decomposition* $b = \sum \lambda_i v(g_i)$ *in the proof of Proposition* 5 *is actually finite, i.e.,* M_{fin} *is closed under the operation* $x \to xx^*x$.

We are now able to prove Theorem 2, which is a complete solution to the contractive projection problem.

THEOREM 2. *Let* P *be an arbitrary contractive projection defined on a* J^**-algebra* M. *Then* $P(M)$ *is a Jordan triple system in the triple product* $(a, b, c) \to \{a, b, c\} \equiv P(\frac{1}{2}(ab^*c + cb^*a))$, *for* $a, b, c \in P(M)$; *and* $P(M), \{\ \})$ *has a faithful representation as a* J^**-algebra.*

Proof. From Krein and Milman, the mapping $\mathscr{E}_0$ is norm preserving on $P(M)$. We prove that $\mathscr{E}_0 P(M)$ is a J^*-subalgebra of $\mathscr{E}_0 P''(M'')$, which, by Proposition 5, is a J^*-subalgebra of A''. Let $y \in \mathscr{E}_0 P(M)$, say $y = \mathscr{E}_0 x$ with $x \in P(M)$. By Proposition 4 $x = y + z$, where $z = \bar{\mathscr{E}}_0 z$. Therefore

$$xx^*x = yy^*y + zz^*z \tag{4.3}$$

and $\bar{\mathscr{E}}_0(zz^*z) = zz^*z$. Now $yy^*y \in \mathscr{E}_0 P''(M'')$ so by Proposition 1, Corollary 4.4 and Lemma 4.5, $\mathscr{E}_0 P''(yy^*y) = yy^*y$. On the other hand, with $\tilde{z} \equiv zz^*z$, $\bar{\mathscr{E}}_0 \tilde{z} = \tilde{z}$ implies that $F(f)\tilde{z} = \tilde{z}$ for all atoms f of P'. Thus by (0.4) $P''\tilde{z} = P''F(f)\tilde{z} = F(f)P''F(f)\tilde{z}$, i.e., $P''\tilde{z} = \bar{\mathscr{E}}_0 P''\tilde{z}$ so $\mathscr{E}_0 P''\tilde{z} = 0$. Finally, applying $\mathscr{E}_0 P$ to (4.3) results in

$$\mathscr{E}_0 P(xx^*x) = yy^*y. \tag{4.4}$$

This proves that $\mathscr{E}_0 P(M)$ is a J^*-algebra and therefore a Jordan triple system. We transfer this Jordan triple system structure from $\mathscr{E}_0 P(M)$ to $P(M)$ by defining $[a, b, c] = \frac{1}{2}\mathscr{E}_0^{-1}(\mathscr{E}_0 a(\mathscr{E}_0 b)^* \mathscr{E}_0 c + \mathscr{E}_0 c(\mathscr{E}_0 b)^* \mathscr{E}_0 a)$ for a, b, c in $P(M)$. Note that by (4.4),

$$[a, a, a] = \mathscr{E}_0^{-1}(\mathscr{E}_0 a(\mathscr{E}_0 a)^* \mathscr{E}_0 a) = \mathscr{E}_0^{-1}(\mathscr{E}_0 P(aa^*a)) = P(aa^*a).$$

By polarization, $[a, b, c] = P(\frac{1}{2}(ab^*c + cb^*a))$ for a, b, c in $P(M)$. ∎

As a by-product of this investigation we obtain the following properties of the unit ball of the range of a contractive projection Q on the dual of a J^*-algebra. These properties are entirely analogous to the properties developed in [1] for the state space of a Jordan operator algebra (which corresponds to the case $M =$ Jordan algebra, $Q =$ Id.).

THEOREM 3. *Let* Q *be a contractive projection on the dual* M' *of a* J^**-algebra* M *and let* K *be the solid unit ball of* $Q(M')$. *Then*

(1) *the* σ*-convex hull of the extreme points of* K *forms a split face in* K;

(2) *every extreme point of K is norm exposed*;

(3) *for each $\varphi \in Q(M')$, the operator $E(\varphi)$ preserves extreme rays of K, i.e., $E(\varphi)$ maps an atom into a multiple of an atom*;

(4) *for each pair f, g of extreme points of K, $f(v(g)) = \overline{g(v(f))}$.*

Proof. (1) is a restatement of Theorem 1; (2) follows from (0.7); (3) is Proposition 3; (4) is Lemma 2.1. ∎

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