Title
Dirichlet Branes and Nonperturbative aspects of supersymmetric string and gauge theories

Author
Yin, Zheng

Publication Date
1999-05-01

LBNL-41827

ERNEST ORLANDO LAWRENCE BERKELEY NATIONAL LABORATORY

Dirichlet Branes and Nonperturbative Aspects of Supersymmetric String and Gauge Theories

Zheng Yin

Physics Division

May 1998

Ph.D. Thesis

DISCLAIMER

LBNL-41827

Dirichlet Branes and Nonperturbative Aspects of Supersymmetric String and Gauge Theories

Zheng Yin
Ph.D. Thesis

Department of Physics
University of California, Berkeley

and

Physics Division
Ernest Orlando Lawrence Berkeley National Laboratory
University of California
Berkeley, CA 94720

May 1998

This work was supported in part by the Director, Office of Science, Office of High Energy and Nuclear Physics, Division of High Energy Physics, of the U.S. Department of Energy under Contract No. DE-AC03-76SF00098.

Dirichlet Branes and Nonperturbative Aspects of Supersymmetric String and Gauge Theories

by

Zheng Yin

B.A. (University of California, Berkeley) 1993

M.A. (University of California, Berkeley) 1995

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Physics

in the

GRADUATE DIVISION

of the

UNIVERSITY of CALIFORNIA at BERKELEY

Committee in charge:

Professor Hirosi Ooguri, Chair

Professor Korkut Bardakci

Professor Nicolai Reshetikhin

Spring 1998

This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098.

Contents

List of Figures

List of Tables

Preface

Throughout the ages, the compulsively curious and inquisitive amongst our ancestors have quested for ever deeper insights into the natural world. Today we have a labyrinth of scientific theories capable of explaining and predicting a vast array of natural phenomena covering every aspect of the material existence, from the mundane to the extraordinary, from the palpable to the imperceptible. This is an impressive feat, but even more significant is that they are not just a hotchpotch of arbitrary and independent rules. Rather, they fit within a very tightly constrained framework. Using the language of mathematics they can be reconstructed from an amazingly small set of principles in physics. Yet this process of distilling is still incomplete. Beyond our sights lies temptingly the pinnacle atop this pyramid of ideas, a mythical "final" theory of all matters and energy in their most elemental forms that connects the divergent regimes of subnucleonic particles and of the whole cosmos itself.

No rationale justifies the optimism that intelligent creatures such as ourselves can one day comprehend this all encompassing scheme in its entirety. Yet it is the hope for its existence that perpetuates us in the cycle of formulating, refining, verifying, and modifying proposals after proposals for its interpretation. For the last decade and half, the most promising amongst them is **superstring theory**, or string theory in short. It may still be very far from being tested in any sort of experiment or observation, but it has already claimed one glory denied to all other valiant attempts: a consistent quantum mechanical framework for the theory of gravity, which describes the very fabric of our space-time. As a result, superstring theory has grown into an impressive intellectual enterprise. We devote this report to but a few aspects of it, those of a certain type of dynamical objects known as Dirichlet branes, or D-branes in short. They weld crucial and previously missing links connecting drastically different physical conditions yet at the same yield to classical string theory methods. As a result they have played a pervasive role in recent developments.

In §1 we review some elements of string theory relevant to the rest of this report. We shall touch on both the "classical," i.e. perturbative, string physics before D-branes rise to prominence, and some of the progresses they brought forth. In §2 we proceed to give an exact algebraic formulation of D-branes in curved spaces. This allows us to classify them in backgrounds of interest and study their geometric properties. We apply this formalism to string theory on Calabi-Yau and other supersymmetry preserving manifolds. Then we study the behaviour of the D-branes under mirror symmetry in §3. Mirror symmetry is known to be a symmetry of string theory perturbatively. We find evidence for its nonperturbative validity when D-branes are also considered and compute some dynamical consequences. In §4 we turn to examine the consistency of curved and/or intersecting D-brane configurations. They have been used recently to extract information about the field theories that arise in certain limits. It turns out that there are potential quantum mechanical inconsistencies associated with them. What save the day are certain subtle topological properties of D-branes. This resolution has implications for the conserved charges carried by the D-branes, which we compute for the cases studied in §2. In §5 we use intersecting brane configurations to study three dimensional supersymmetric gauge theories. There is also a mirror symmetry there that, among other things, exchanges classical and quantum mechanical quantities of a (mirror) pair of theories. It has an elegant realization in term of a symmetry of string theory involving D-branes. We employ it to study a wide class of 3d models. We also predict new mirror pairs and unconventional 3d field theories without Lagrangian descriptions.

Acknowledgments

Now is a moment to pause and reflect, as I close one chapter of my story and prepare writing the next. Looking back at the road I trekked along conjures up images of those who helped shape my graduate life.

It was Professor Hirosi Ooguri's arrival at Berkeley that made my interest in modern string theory into something more than a hobby. As my advisor, he introduced me to the realm of research and showed me the path forward with his guidance and example. Here I would like to register my profound gratitude toward him.

I also want to recount what a pleasure it was to have worked with Ofer Aharony, Katrin Becker, Melanie Becker, Yeuk-Kwan Cheung, Jan de Boer, Kentaro Hori, Shamit Kachru, David Morrison, Hirosi Ooguri, and especially Yaron Oz, with whom I had many dinners together although we never seem to agree on what counts as good food. I learned a lot from my collaborators as well as Professor Korkut Bardakci, Martin Halpern, Stanley Mandetstam, Hitoshi Muryama, Nicolai Reshetikhin, Majorie Shapiro, and Bruno Zumino. Just as important was help from fellow graduate students Luis Bernardo, Chong-Sun Chu, Pei-Ming Ho, and Bogdan Morariu.

I would have long lost my sanity to this world had there not been the understanding, support, and encouragement from Edna and Sang-Yup. I have the best wishes for their future as well as that they would continue listening to my soliloquies.

Chapter 1

Strings and p-Branes

1.1 Perturbative string theories

1.1.1 Motivations and intuitions

It is customary to formulate physical models in terms of particles. For macroscopic objects such as billiard balls or microscopic objects such as nuclei, this is just a simplifying approximation that captures their most salient physical properties. Occasionally and gradually, when it becomes necessary to take into account the nuances resulting from their finite spatial extension, one use model of strings, membranes, or clumps. As candidates for the most basic building block of matter and energy, however, one often takes for granted that the objects of interest are **exactly** point particles. More than being a mere simplifying assumption, it also fits the notion of being fundamental, bypassing the perpetual question "what is it that makes up this object?" The advent of quantum mechanics brought the realization that perhaps this is a meaningless question to ask. Quantum theories lead naturally to the quantization of angular momentum, energy, as well as charges. A string that carries a minimal unit of charge can be just as **fundamental** as a point particle.

With this insight, models using strings as fundamental objects have been formulated over the past decades. A complete introduction to this field requires an extensive treatise, such as [1][1]. Here we shall just outline some fundamental features relevant to understanding this report.

1.1.2 Bosonic string theories

Worldsheet action

We now formulate string theory as a quantum field theory on a two-dimensional **worldsheet**, parameterized by σ^0 and σ^1. The cylindrical coordinate σ^0 is the "time" on the string worldsheet, and σ^1 is the "space" coordinate along the closed string, with the identification $\sigma^1 + 2\pi \sim \sigma^1$. In this section we shall consider *orientable closed* strings, which means that the worldsheet has no boundary and can be assigned a definite orientation. The (1+1)d field theory on it should contain information about the embedding (position, size, and shape) of the worldsheet in space-time. Therefore one introduces the worldsheet fields $X^\mu(\sigma^0, \sigma^1)$, $\mu =$

[1]For a quick introduction to perturbative string theory, see [2]

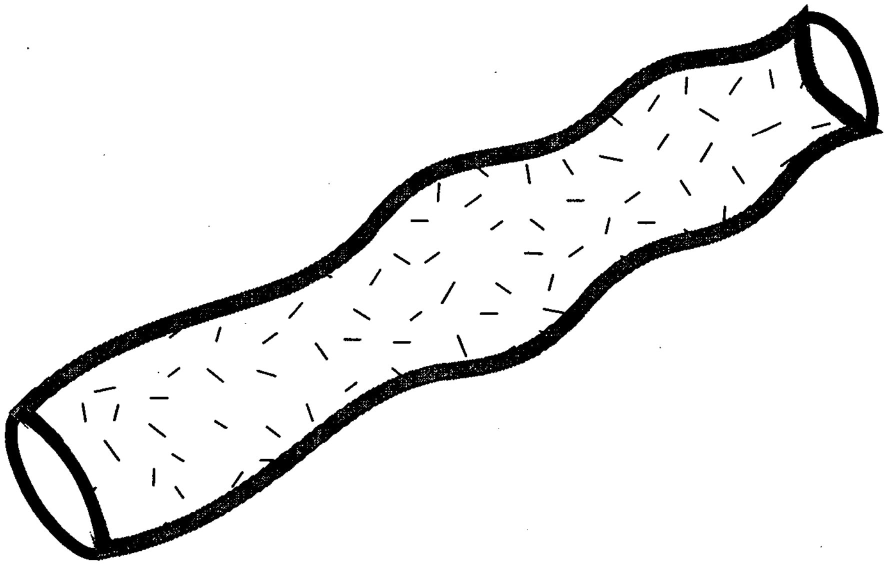


Figure 1.1: The worldsheet of a string propagating in space-time.

$0, ..., D-1$. We also want the theory to be invariant under reparameterization of the worldsheet. The most familiar way to achieve this is to introduce a metric g_{ab} for the worldsheet as a dynamical variable. However, the Einstein-Hilbert action in 2d is topological, so for the purpose of canonical quantization, one only need to consider the action

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X_\mu, \tag{1.1.1}$$

where both, the metric g_{ab} as well as X^μ, are treated as dynamical variables.

The worldsheet metric g_{ab} has no local propagating degrees of freedom and acts mainly as Lagrangian multipliers. Classically, the equation for g requires it to be proportional to the induced metric $\hat{g}_{ab}$

$$\hat{g}_{ab} = \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b}; \quad a, b = 0, 1; \quad \hat{g} \equiv \det \hat{g}_{ab}. \tag{1.1.2}$$

Substitute this back to (eq. 1.1.1) and we obtain the Nambu-Goto action

$$S = \frac{-1}{2\pi\alpha'} \int d^2\sigma \sqrt{-\hat{g}}. \tag{1.1.3}$$

This is the 2d generalization of the action for a relativistic point particle:

$$S = -m \int d\tau \sqrt{-\dot{X}^2}, \tag{1.1.4}$$

It can be shown from (eq. 1.1.3) that the dimensionful constant $T \equiv 1/(2\pi\alpha')$ gives the tension of the string.

Consistent with the topological nature of the Einstein-Hilbert action for g, the worldsheet metric consists almost purely of gauge degrees of freedom. First the worldsheet metric has three independent degrees of freedom, two of which can be gauged away using worldsheet diffeomorphism, bringing the metric into the standard form

$$(g_{ab}) = \begin{pmatrix} -\lambda & 0 \\ 0 & \lambda \end{pmatrix}, \tag{1.1.5}$$

in what is known as conformal coordinates. Furthermore, the Polyakov action (eq. 1.1.1) has the Weyl rescaling symmetry which allows us to scale λ to, say, 1. This is known as the conformal gauge. Note that although this choice of gauge breaks diffeomorphism invariance, it still preserves the global Poincare invariance on the worldsheet. The equation of motion for g is as usual the vanishing of the energy momentum tensor. After fixing g, this must imposed as a constraint, known as the Virasoro constraint.

In conformal gauge, there is still a residual gauge symmetry. It is called **conformal symmetry** because it only rescales the induced metric. To exhibit it, define the light-cone coordinates $\sigma^{\pm} \equiv \sigma^0 \pm \sigma^1$. It is not difficult to show that a coordinate transformation preserving the conformal gauge condition (eq. 1.1.5) must be of the form

$$\sigma^+ \to \sigma^{+\prime} = f(\sigma^+), \qquad \sigma^- \to \sigma^{-\prime} = h(\sigma^-). \tag{1.1.6}$$

In the light-cone coordinates,

$$-(d\sigma^0)^2 + (d\sigma^1)^2 = -d\sigma^+ d\sigma^-.$$

Since

$$d\sigma'^+ = f' d\sigma^+, \qquad d\sigma'^- = h' d\sigma^-,$$

(eq. 1.1.5) is indeed preserved as

$$d\sigma'^+ d\sigma'^- = f'h' d\sigma^+ d\sigma^-.$$

The worldsheet of a freely propagating string clearly looks like a tube. Choosing L_n and $\tilde{L}_n$, the Fourier components of f and h respectively, as the generators of conformal transformation on a cylinder, it is not difficult to find their commutators:

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m}, \\ \left[\tilde{L}_n, \tilde{L}_m\right] &= (n-m)\tilde{L}_{n+m}, \\ \left[L_n, \tilde{L}_m\right] &= 0. \end{aligned} \tag{1.1.7}$$

In the conformal gauge, the action (eq. 1.1.1) becomes

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu \\ &= \frac{1}{\pi\alpha'} \int d^2\sigma\, \partial_+ X^\mu \partial_- X_\mu. \end{aligned} \tag{1.1.8}$$

There are two complications to this story. First, in general (eq. 1.1.5) can only be enforced in each coordinate patch. Between patches there can be global degrees of freedom left. Roughly speaking they describe the shape of the world-sheet and are known as **complex moduli** since they parameterize the choice of a complex structure. Second, quantum mechanically the Weyl rescaling symmetry may became anomalous, and the algebra of conformal transformation (eq. 1.1.7) is not realized on the Hilbert space. It is deformed to be the Virasoro algebra, which is the conformal algebra (eq. 1.1.7) with a nontrivial **central extension**:

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0}, \\ \left[\tilde{L}_n, \tilde{L}_m\right] &= (n-m)\tilde{L}_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0}, \\ \left[L_n, \tilde{L}_m\right] &= 0. \end{aligned} \tag{1.1.9}$$

The central charge c measures the violation of conformal invariance. The central charge for (eq. 1.1.8) is D, equal to the dimension of the space-time. However, as in gauge fixing for Yang-Mill theories, choosing the conformal gauge introduces Faddeev-Popov ghosts. Their action, which provide for the correct normalization for the path integral respecting the reparameterization invariance, carry an additional central charge -26. Since the conformal anomaly is additive, only when $D = 26$ does the anomaly from the X's cancel against that from the ghosts and give us a consistent theory.

First quantization of string

For point particles, there are two roads from classical physics to quantum physics. The first quantization quantizes the worldline action and yields quantum mechanics (i.e. one-dimensional QFT) of the particles. The second quantization quantizes their space-time action and yields a $(1, D-1)$-dimensional QFT. In string theory, the worldsheet is already two-dimensional, so we have a $(1, 1)$-dimensional QFT theory already in the first quantization.

Let us use the conformal gauge (eq. 1.1.5) and quantize the action (eq. 1.1.8), which is just an action of D free scalars. The equation of motion for $X^\mu(\sigma)$ is

$$(\partial_0^2 - \partial_1^2)X^\mu \equiv \partial\bar{\partial}X^\mu = 0, \tag{1.1.10}$$

and its general solution is

$$X^\mu = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-).$$

The X's must be periodic in σ^1 with period 2π. After Fourier decomposition, we separate and recover the center of mass and the oscillating modes:

$$X^\mu = x^\mu + \alpha' p^\mu \sigma^0 + \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{i}{n}\{\alpha^\mu_n e^{-in\sigma^+} + \tilde{\alpha}^\mu_n e^{-in\sigma^-}\} \tag{1.1.11}$$

Canonical quantization gives[2]

$$\left[\dot{X}^\mu(\sigma^0,\sigma^1), X^\nu(\sigma^0,\sigma^{1'})\right] = 2\pi\alpha' i\eta^{\mu\nu}\delta(\sigma^{1'}-\sigma^1) \tag{1.1.12}$$

In term of the Fourier modes, one has

$$\begin{gathered}[x^\mu, p^\nu] = i\eta^{\mu\nu} \\ [\alpha_n^\mu, \alpha_m^\nu] = n\eta^{\mu\nu}\delta_{n+m,0}, \quad [\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu] = n\eta^{\mu\nu}\delta_{n+m,0} \\ (\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu, \qquad (\tilde{\alpha}_n^\mu)^\dagger = \tilde{\alpha}_{-n}^\mu. \end{gathered} \tag{1.1.13}$$

Note the left and right moving fields are completely symmetric. The α's and $\tilde{\alpha}$'s are raising and lowering operators for the harmonic oscillators associated with the oscillation modes on the string. The above commutation relations are also captured in the **operator product expansion** (OPE) of the relevant fields. For example, the above commutation relations are equivalent to

$$\partial X^\mu(z)\partial X^\nu(w) \sim \frac{G^{\mu\nu}}{(z-w)^2}. \tag{1.1.14}$$

So the Hilbert space is the tensor product of $2\times D$ infinite towers of harmonic oscillators, each labeled by positive integers (coming from α_n and $\tilde{\alpha}_n$) and that of the D-dimensional quantum mechanics (coming from the zero modes X^μ and P^μ):

$$\bigotimes_{0\le\mu<D}^{n>0} \left\{(\alpha_{-n}^\mu)^i\,|0\rangle\,|i=0\ldots\infty\right\} \bigotimes_{0\le\mu<D}^{n>0} \left\{(\tilde{\alpha}_{-n}^\mu)^i\,|0\rangle\,|i=0\ldots\infty\right\} \bigotimes \Phi(X^\mu).$$

The operator α_{-n}^μ (α_n^μ), with $n>0$, creates (destroys) a quantum of left moving oscillation with angular frequency n along the X^μ direction in space-time. $\tilde{\alpha}_{-n}^\mu$ ($\tilde{\alpha}_n^\mu$) does the same for the right movers. This decomposition of degrees of freedom into essentially decoupled left and right movers is what makes many two-dimensional field theories so much more manageable compared to theories in higher dimensions.

It remains to impose the Virasoro constraints. By varying the worldsheet metric away from γ_{ab}, we can find its (worldsheet) energy-momentum tensor $T^{ab} \sim \frac{\delta S}{\delta\gamma_{ab}}$. Since the action is conformally invariant, the trace of the classical energy-momentum tensor T vanishes. The remaining two components are

$$T_{++} = \frac{1}{\alpha'}(\partial_+X)^2, \qquad T_{--} = \frac{1}{\alpha'}(\partial_-X)^2. \tag{1.1.15}$$

Classically the Virasoro constraints is that they must also vanish. As mentioned earlier, there are quantum mechanical anomalies. To exhibit the nature of this anomaly, it is convenient to Fourier transform the T's:

$$T \equiv T_{++} = \frac{1}{\alpha'}(\partial_+X)^2 \equiv \sum_n L_n e^{-in\sigma^+};$$

[2]In this report, i denotes $\sqrt{-1}$.

$$\tilde{T} \equiv T_{--} = \frac{1}{\alpha'}(\partial_- X)^2 \equiv \sum_n \tilde{L}_n e^{in\sigma^+};$$

$$L_n = \sum_m \frac{1}{2}\alpha_{n-m}\alpha_m, \quad \tilde{L}_n = \sum_m \frac{1}{2}\tilde{\alpha}_{n-m}\tilde{\alpha}_m,$$

$$\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu.$$

These L_n and $\tilde{L}_n$ are well defined except for $n = 0$, for which there is a normal ordering ambiguity. If we define

$$\begin{aligned} L_0 &= \sum_{n>0} \alpha_{-n}\alpha_n + \frac{1}{2}(\alpha_0)^2, \\ \tilde{L}_0 &= \sum_{n>0} \tilde{\alpha}_{-n}\tilde{\alpha}_n + \frac{1}{2}(\tilde{\alpha}_0)^2, \end{aligned} \tag{1.1.16}$$

the constraint for the $n = 0$ part would be $L_0 - a = 0$, $\tilde{L}_0 - \tilde{a} = 0$ where a and $\tilde{a}$ are constants reflecting the normal ordering ambiguity. The combination $(L_0 + \tilde{L}_0)$ is the Hamiltonian of the system generating a translation in σ^0 direction and $(L_0 - \tilde{L}_0)$ is the worldsheet momentum. Since

$$[L_0, \alpha_{-n}] = n\alpha_{-n},$$

the n-th oscillator has energy n, equal to its angular frequency. The same holds for the right movers.

It can be checked that the L's form a representation of the **Virasoro algebra** (eq. 1.1.9). In our case, the central charge c is equal to D, the space-time dimension.

We shall not review the detail for imposing the Virasoro constraints, which gives another derivation that D=26 is a critical value that gives a consistent spectrum. Suffices it to say that the Virasoro operator L's and $\tilde{L}$'s are very important for building a consistent string theory in this formulation. They will useful again when we define D-branes in the next chapter. Some of their properties will be reviewed briefly later in this introduction.

String propagation and interactions

Point particles propagate in a straight line with amplitude given by their Feynman propagators. They interact at a well-defined point in space-time, where straight lines intersect at vertices. Each vertex also has some coupling constant associated with it. We calculate a scattering amplitude of them by drawing the corresponding Feynman diagrams, and multiplying together all the propagators and the coupling constants at each vertex figure 1.2. In string theory, the picture is similar figure 1.3. Propagation of string is represented by a tube. A slice of the worldsheet at any time determines a string state at that instant. However, because

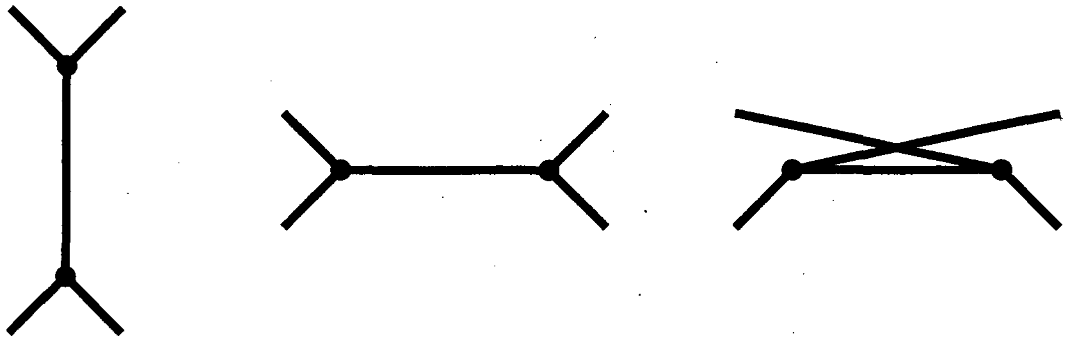


Figure 1.2: Some Feynman diagrams for point particles

of worldsheet reparameterization invariance, no scheme of time slicing is preferred over others. This and the smooth joining and splitting of string tubes mean that there is no freedom in assigning coupling constants to any particular point. Indeed it will soon become clear that there is only one measure of string coupling, which is however a field carried by and distributed over the strings themselves.

To study string worldsheets of various topologies, it is convenient to choose the worldsheet metric to be Euclidean rather than Lorentzian. This can be done by performing a Wick rotation on the worldsheet:

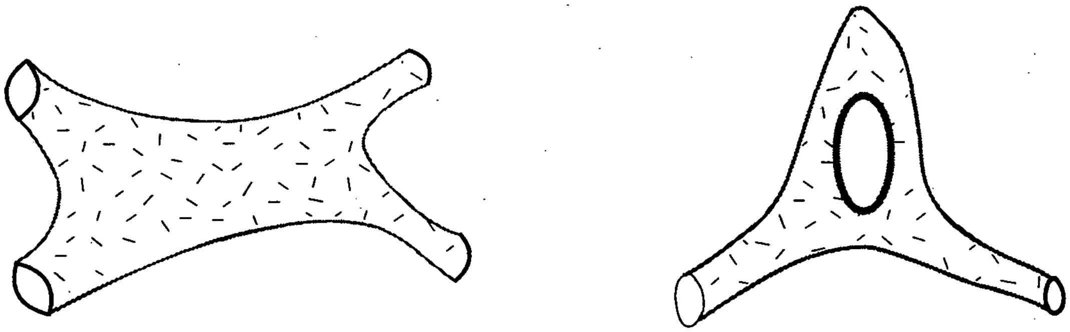
$$\sigma^0 = -i\sigma^2$$

$$z \equiv i\sigma^+ = \sigma^2 + i\sigma^1, \quad \bar{z} = i\sigma^- = \sigma^2 - i\sigma^1,$$

$$X^\mu = x^\mu - i\alpha' p^\mu \mathrm{Re} z + \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{i}{n} \{\alpha_n^\mu e^{-nz} + \tilde{\alpha}_n^\mu e^{-n\bar{z}}\}$$

We will use this Euclidean notation from now on.

figure 1.3a shows the worldsheet for a tree level string-string scattering. Its amplitude is calculated by evaluating the Polyakov path integral over it. After gauging away arbitrary reparameterizations, the integration over the worldsheet metric g of Polyakov action is reduced to a sum of over all possible shapes and sizes of worldsheets of a given topology. Since the size of the worldsheet can be gauged away for critical string theory, this reduces to a finite dimensional integral over its moduli space, the space that parameterizes the shape of worldsheet with this topology. Worldsheet actions themselves do not tell us which topology of worldsheet we should choose, but analogy with Feynman diagrams suggests that handles in the worldsheet represent internal loops and we should sum over all



a. Tree-level 4-string scattering

b. One-loop 2-string scattering

Figure 1.3: Some worldsheet for string interactions

number of handles. In fact the unitarity of the S-matrix dictates how to sum over topologies of the worldsheet. As another simple illustration, consider the one-loop vacuum to vacuum string amplitude figure 1.3b. This has the physical interpretation of calculating the vacuum energy.

Conformal field theory and vertex operator

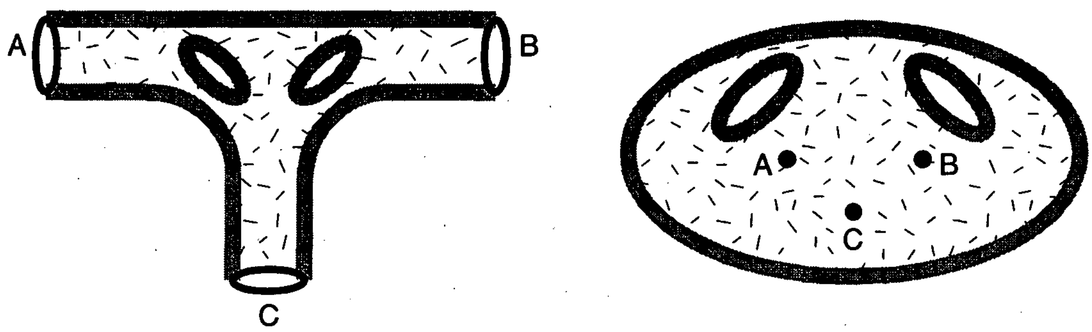
The conformal gauge action (eq. 1.1.8) is an example of a 2d **conformal field theory** (CFT)[3]. Each CFT has two copies of the Virasoro algebra. One each for left and right movers[4]. The decoupling between the left and right movers is an essential feature of all 2d conformal field theories. They possess conformal invariance quantum mechanically with a mild, controllable anomaly. [5]. Many conformal field theories of interest here or in the literature have additional symmetries. Their algebra are also infinite dimensional and decoupled into independent left and right movers. They are known as chiral algebras. We will now introduce some facts and concepts that will be useful. Consider a path integral calculation of a CFT over some Riemann surface, with some tubes extending to infinity. The field configurations at the ends of the tube correspond to states in the CFT Hilbert space. In string theory they represent external, asymptotic string states in a scattering

[3]For an extensive discussion on the subject, see for example [3] and [4]

[4]They are decoupled as far as the action of the two copies of Virasoro algebra are concerned. The actual states in the spectrum generically transform under both. In this sense the left and right movers are intertwined.

[5]To be more precise, conformal invariance is realized projectively.

process. We can perform arbitrary conformal transformations when evaluating the path integral of a CFT. Let us choose one that brings the tube C in (fig. 7) from infinity to within a finite distance from the scattering region. Because this would



a. Before conformal transformation: asympototic states coming from infinity

b. After conformal transformation: vertex operators inserted

Figure 1.4: Two worldsheets for the same 2-loop 3-string amplitude, related by a conformal transformation.

involve an infinite rescaling in the neighborhood of the end circle of tube C, the end circle, which has finite radius, will shrink to a point. Its effect should therefore be represented by the insertion of a local field operator at that point. It is called a **vertex operator**. Therefore there is a one-to-one correspondence between states and operators in CFT. In string theory, for example, a vertex operator taking momentum k has the form, : (oscillator part) $\times e^{ik\cdot X}$:, where :: denotes the normal ordering. The oscillator part of the operator is determined by its counterpart for the corresponding state. For example, the operator that creates an insertion of a massless operator of momentum k is

$$\xi_{\mu\nu} : \partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X} : .$$

For the tachyon, the oscillator part is just the identity, so the vertex operator is simply : $e^{ik\cdot X}$:. Of course, not all vertex operators correspond to insertion of physical states. They have to obey the operatorial version of the Virasoro constraints. For an operator Φ, the constraints can be summarized in the **singular parts** of its operator product expansion (OPE) with the energy-momentum tensor:

$$T(z)\phi(w,\bar{w}) \sim \frac{a\phi(w,\bar{w})}{(z-w)^2} + \frac{\partial\phi(w,\bar{w})}{(z-w)},$$

$$\tilde{T}(\bar{z})\phi(w,\bar{w}) \sim \frac{\tilde{a}\phi(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\phi(w,\bar{w})}{(\bar{z}-\bar{w})}. \tag{1.1.17}$$

with

$$a = 1 = \tilde{a}. \tag{1.1.18}$$

The last one is known as the level matching condition. If it is not satisfied but (eq. 1.1.17) holds, ϕ is known as a Virasoro primary field with conformal weight $(a, \tilde{a})$.

The Virasoro algebra (eq. 1.1.9) itself can be written as

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}, \tag{1.1.19}$$

and similarly for $\tilde{T}$ with no singularity between T and $\tilde{T}$. Thus T is almost a Virasoro primary field of weight $(2,0)$ except for its conformal anomaly. It is a fundamental property of a conformal field theory that its Hilbert space and operator content is a direct sum of often infinitely many irreducible representations of the left $\times$ right Virasoro algebra, each of which is generated by the action of the algebra on a highest weight state. The Virasoro primary fields of a CFT and their operator product expansion (OPE) completely characterize it.

1.1.3 Superstrings

The bosonic string theory reviewed in the last section displays some very interesting structures, yet it conspicuously lacks one important ingredient: fermion. In the real world, we of course know that fermions, such as quarks and electrons, are the basic constituents of matter. So we should find some way to incorporate them into string theory if the latter is to become a theory of reality. By the theorem on the connection of spin and statistics [5], we want space-time spinors. Over the years physicist found a way to build a string theory containing space-time spinors by introducing worldsheet spinors. Besides gaining fermionic degree of freedom, the annoying tachyon has disappeared. Moreover a symmetry between space-time bosons and fermions emerges naturally. This symmetry is known as supersymmetry and the string with it is called superstring. Some versions of supersymmetry have been studied by phenomenologists as a promising extension to their "standard model". Interestingly, the worldsheet action for superstring also has worldsheet supersymmetry. As it turns out, this is not a coincidence.

Superstring action and its quantization

The symmetric version of the action (eq. 1.1.1) is:

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{-g}\,\left\{g^{ab}(\partial_a X^\mu \partial_b X_\mu + i\bar{\Psi}_\mu \lambda^a \partial_a \Psi^\mu) \right.$$
$$\left. + \bar{\chi}_a \lambda^b \lambda^a (\partial_b X^\mu + \frac{1}{2}\bar{\Psi}_\mu \Psi^\mu \chi_b)\right\}.$$

Here λ^a are the worldsheet Dirac matrices. New to the field content are D worldsheet spinors Ψ^μ that transform in space-time as a tangent vector, and a worldsheet Rarita-Schwinger field χ_a with no space-time index. The action has four local symmetries: the worldsheet diffeomorphism and Weyl rescaling symmetries already present for the bosonic string, and their superpartners: local supersymmetry and super-Weyl transformation. Classically they together allow one to gauge away the metric g and the Rarita-Schwinger field χ_a, and impose constraints on the physical phase space. In the **superconformal** gauge, g_{ab} can be set to $\lambda\gamma_{ab}$ and χ_a to 0. Again, there are potential anomalies. The new Faddeev-Popov ghosts introduced by gauge fixing the local fermionic symmetries raise the central charge for the ghost sector to -15. On the other hand, the contribution from the Ψ's increases the matter sector central charge to $\frac{3}{2}D$. Therefore the critical dimension for them to cancel is now $D = 10$.

Like the conformal gauge, the superconformal gauge is preserved by some residual gauge symmetries, which are called **superconformal transformations**. The superconformal gauge action,

$$S = \frac{1}{\pi\alpha'}\int d^2\sigma\,\{\partial_+X\cdot\partial_-X + i\psi_L\cdot\partial_-\psi_L + i\psi_R\cdot\partial_+\psi_R\}, \tag{1.1.20}$$

is the supersymmetric extension of (eq. 1.1.8). Here the worldsheet fermions Ψ^μ split into the left moving ψ^μ's and the right moving $\tilde{\psi}^\mu$'s. (eq. 1.1.20) is a **superconformal field theory** (SCFT), a conformal field theory with additional structures and algebra reflecting its superconformal symmetry. Quantizing this theory one finds

$$\{\psi^\mu_r, \psi^\nu_s\} = 2G^{\mu\nu}\delta_{r+s,0}, \tag{1.1.21}$$

with the Fourier modes of ψ defined as

$$\psi^\mu(z) = \sum_r \Psi^\mu_r e^{-rz}. \tag{1.1.22}$$

This is also captured in their OPE:

$$\psi^\mu(z)\psi^\mu(w) \sim \frac{G^{\mu\nu}}{(z-w)} \tag{1.1.23}$$

As a worldsheet fermion, there are two possibilities for the (anti-)periodicity of the ψ's. Using the cylindrical coordinate $\sigma_0(\sigma_2)$ and σ_1 for the string worldsheet, in the Ramond (R) sector the Ψ's are periodic along σ_1 while in the Neveu-Schwarz (NS) sector they are antiperiodic. Space-time Lorentz covariance requires all the left (right) moving fermions to be in the same sector, but the choice for the left and right movers be independent. Hence the superstring has 4 sectors: NS-NS, NS-R, R-NS and R-R. As usual, left and right moving operators decouple, and we will concentrate on the left movers:

$$T = \sum_n L_n e^{-nz} = -\frac{1}{2}\partial X\cdot\partial X - \frac{1}{2}\psi\cdot\partial\psi,$$

$$G = \sum_n G_n e^{-nz} = i\psi \cdot \partial X.$$

Because ∂X's have integer moding, the moding of G is the same as that of ψ's: $r \in \mathbb{Z}$ in R sector; $r \in \mathbb{Z} + \frac{1}{2}$ in NS sector. The superconformal algebra is

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{D}{8}(n^3 - n)\delta_{n+m,0}.$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{D}{2}(r^2 - \frac{1}{4})\delta_{r+s,0}$$

$$[L_n, G_r] = (\frac{1}{2}n - r)G_{n+r}. \tag{1.1.24}$$

The corresponding OPE's can be found in §12 of [4]. $T \sim 0 \sim \tilde{T}$ and $G \sim 0\tilde{T}$ are the equations of motion for g and χ in the super-conformal gauge. Therefore they make up the super-Virasoro constraints one must imposes in that gauge. In particular they contain the Dirac and the Klein-Gordon equations, as well as equations of motions for all other fields.

In R sector, the ψ's have even moding. From (eq. 1.1.21) ψ_0's form a Clifford algebra. The R sector Hilbert space, in particular the ground state, form a representation under it, therefore they transform as space-time spinors. One can define **spin fields** S^α's that map the unique NS ground state to those R grounds states. The construction of S^α is involved and can be found in [6], but we shall not need it for this report. It is worth noting, however, that the spin fields are very special vertex operators and are not completely local with respect to the ψ's, as seen in the fractional power in their OPE:

$$\psi(z)S^\alpha(w) \sim \frac{\Gamma^\alpha_\beta S^\beta(w)}{(z-w)^{1/2}}. \tag{1.1.25}$$

To maintain overall consistency of the worldsheet theory, one has to include both NS and R sectors yet at the same time take what is known as the **GSO projection**. The latter project out spinors of, say, negative 10d chirality and odd powers of ψ. Now consider both the left and the right movers. The spin fields are essentially the generators for space-time supersymmetry. Thus type II superstrings have 10d N=2 supersymmetry, or 32 real supercharges.

The five perturbative superstring theories

In taking the GSO projection, if the same space-time chirality is used for both the left and the right movers, one obtains the nonchiral **type IIA** string theory; otherwise, it is **type IIB** string theory. Both have 32 global supersymmetry charges and no non-Abelian gauge symmetries.

Now let us examine the massless particles in superstring theory for their space-time meaning. We will use the language of the covariant superconformal gauge, therefore our counting will be off-shell. For NS-NS sector, we clearly get the same fields as for bosonic string: the dilaton Φ, the metric $G_{\mu\nu}$ and the antisymmetric tensor field $B_{\mu\nu}$. For the NS-R and R-NS sectors, the Ramond parts transform as space-time spinors λ_L or λ_R. In fact they are Majorana-Weyl spinors. The NS parts are of course vectors, so we have two 10-dimensional Rarita-Schwinger fields. The only known way to incorporate such fields consistently is to couple them to the supergravity current. They are therefore the gravitinos. So a GSO projected superstring theory contains $N = 2$ supergravity. Depending on the choice of the relative sign in defining $(-1)^{F_L}$ and $(-1)^{F_R}$, we have two inequivalent possibilities, corresponding to the relative chirality of the surviving λ_L and λ_R. If we choose opposite chiralities, we obtain the type IIA superstring theory whose low energy effective theory is the type IIA supergravity. The type IIA theory is non-chiral and can be obtained by dimensional reduction from 11-dimensional supergravity. This is the first and simplest evidence for the relation between type IIA string theory and a theory in eleven dimensions, "**M theory**." If we choose the same chirality for both left and right movers, we obtain the type IIB superstring theory. The corresponding type IIB supergravity is chiral and potentially anomalous. Cancellation of gravitational anomaly in type IIB supergravity was shown by Alvarez-Gaumé and Witten (ref. 20 in [1], Vol 1).

More novelties come from the R-R sectors. Here the massless states transform as the products of two spinors. Contracting them with antisymmetrized products of gamma matrices, we see that they are related to antisymmetric tensors of rank 0 to 10. However, because the spinors making the products are chiral, not all the possibilities can appear. For the type IIA theory, λ_L and λ_R are of the opposite chiralities, and we obtain even rank tensors

$$F^{\{0\}} \equiv \bar{\lambda}_L \lambda_R, \quad {F^{\{2\}}}_{\mu\nu} \equiv \bar{\lambda}_L \Gamma_{\mu\nu} \lambda_R, \cdots.$$

On the other hand, the type IIB theory contains odd rank tensors

$$F^{\{1\}}{}_{\mu} \equiv \bar{\lambda}_L \Gamma_\mu \lambda_R, \quad F^{\{3\}}{}_{\mu\nu\rho} \equiv \bar{\lambda}_L \Gamma_{\mu\nu\rho} \lambda_R, \cdots.$$

Here $\Gamma_{\mu_1 \ldots \mu_n}$ is the antisymmetrized product of n Gamma matrices. Moreover they are not all independent. There is an important Γ-matrix relation:

$$\epsilon_{\mu_1 \ldots \mu_n}{}^{\rho_{n+1} \ldots \rho_{10}} \Gamma_{\rho_{n+1} \ldots \rho_{10}} \sim \Gamma^{10} \Gamma_{\mu_1 \ldots \mu_n},$$

where Γ^{10} is the 10d chirality operator. Because of the GSO projection, Ψ_L and Ψ_R both have definite eigenvalue of Γ^{10}. Therefore

$$F^{\{n\}} \sim *F^{\{10-n\}}. \tag{1.1.26}$$

In particular, $F^{\{5\}}$ is self-dual. The readers can verify that the number of independent components of the antisymmetric tensor fields, taking into account these relations, is equal to that of the tensor product of two Majorana-Weyl spinors.

What kind of fields are they? It is not difficult to show that the massless Dirac equations for λ_L and λ_R are equivalent to

$$d^* F^{\{n\}} = 0, \quad dF^{\{n\}} = 0.$$

They are the equations of motion and Bianchi identities for antisymmetric tensors fields $A^{\{n-1\}}$ such that $F^{\{n\}} = dA^{\{n-1\}}$. Note that $A^{\{n-1\}}$ and $A^{\{9-n\}}$ are related by electric-magnetic duality, which exchanges equations of motion and Bianchi identities. The way they arise out of string theory places them on equal footing.

There is also an antisymmetric tensor field B in NS-NS sector, but the way it is coupled to the string is very different from the R-R fields. The VEV of its potential $B_{\mu\nu}$ couples directly to the vertex operator for it. Its contribution to the string action is just the integral of the pullback of B over the worldsheet. By analogy with the minimal coupling of the usual 1-form potential A_μ to the worldline of a charged point particle, we see that this means a string carries unit "electric" charge with respect to B. However, the coupling of R-R fields with string involves only the field strength. This means elementary string states cannot carry any charge with respect to the R-R fields. However, it was discovered by Polchinski that there are (p+1)-dimensional solitonic objects called **Dp-branes**, which do carry such charges [7]. They have played a very significant role in string theory recently. For example, D0-brane is at the core of the proposed equivalence between type IIA and M theory. They will be the protagonist in this report and starting at the next section we shall turn our attention to them.

Type II string theories are not the whole story. Exploiting the decoupling of left and right movers of the worldsheet theory, one can use bosonic string theory for the left moves and superstring for the right movers and obtains what is called **heterotic superstring theory**. The boundary condition kills half of the space-time supersymmetry so its fields fall into 10 N=1 supermultiplet. At the massless level it contains a N=1 gravity multiplet. To cancel the ensuing gravitational anomaly it is necessary to introduce gauge multiplets (besides returning to the type II complement). Such theory possess non-Abelian gauge symmetries. It turns out that there are two discrete choices: Spin$(32)/\mathbb{Z}_2$ and $E_8 \times E_8$.

To understand the remaining possibility[6] one has to allow the worldsheet to have boundaries. This procedure is be reviewed later in this chapter. Because it intertwine the left movers and right movers, it can only be performed within type IIB theory, where there is a symmetry between the two. Even there consistency turns out to require a certain projection on close string spectrum. The resulting theory is known as type I and has gauge symmetry $Spin(32)/\mathbb{Z}_2$[7] Its spectrum

[6]There are other theories of supersymmetric strings not mentioned here. They all have some idiosyncrasies such as possessing more than one time directions and/or have no propagating gravitons. They are of interest in their own sake and may even be indirectly relevant to the usual superstring theory. However, they do not yet lend themselves to standard physical interpretations.

[7]Perturbative formulation itself only establish that the gauge group be one of the cover groups of SO$(32)/Z_2$, but consideration of D1-brane vacuum structure fixes it to be Spin$(32)/\mathbb{Z}_2$.

include part of that of type IIB string. Among the massless bosons, the NS-NS B field, the R-R 0 and 4-form potentials are projected out, and what remains belong again a 10 N=1 gravity supermultiplet. The remainder comes from the gauge multiplet. Therefore the massless fields are the same as that of Heterotic $\mathrm{Spin}(32)/\mathbb{Z}_2$.

1.2 Solitonic p-branes

1.2.1 p-Forms and p-branes

There are supergravity solutions carrying charges with respect to the antisymmetric tensor field A's [8]. Associated to $A^{\{p+1\}}$ are solitonic objects known as p-branes with p+1 dimensional worldvolume M. They couple to $A^{\{p+1\}}$ in the same minimal fashion as electrons couple to an $U(1)$ gauge potential:

$$\int_M A.$$

The antisymmetric tensors are distinguished also by whether they come from NS-NS sector or R-R sector of the strings. The brane correspondingly there are called NS and D-brane respectively. Table 1.1 gives all the possibilities for all five perturbative formulation of string theories. An entry with "NS" denote the existence of a (p+1) form from the NS-NS sector and the corresponding NS p-branes. An entry with "D" similarly denote R-R forms and Dp-branes.

Table 1.1: p-Branes and (p+1)-forms

p	0	1	2	3	4	5	6	7	8	9
IIA	D	NS	D		D	NS	D		D	
IIB		D,NS		D		D,NS		D		
I		D				D				D
Het(both)		Het				Het				

A more substantial difference between the NS-NS and the R-R fields lies in their the low energy effective actions[1]. That of the NS-NS fields is the same as that of the bosonic string:

$$S = \frac{1}{2\kappa^2}\int d^{10}X\,\sqrt{-G}e^{-2\Phi}\{R - \frac{1}{12}H^2 + 4(\nabla\Phi)^2 + O(\alpha')\},$$

where $H = dB$. The variation of S with respect to B gives

$$e^{2\Phi}\nabla^\mu\left(e^{-2\Phi}H_{\mu\nu\rho}\right) = (\nabla^\mu - 2\partial^\mu\Phi)H_{\mu\nu\rho} = 0.$$

[1]The following argument was given by Polchinski and possibly others.

The origin of the coupling between H and Φ can be traced to the way the dilaton couples to the string worldsheet, $\sqrt{g}R\Phi$. Since $T \sim \frac{1}{\sqrt{g}}\frac{\delta S}{\delta g^{zz}}$, if the dilaton is not constant, the energy-momentum tensor T is modified as

$$T \sim -\frac{1}{2}(\partial X)^2 + \partial_\mu \Phi \, \partial_z^2 X^\mu.$$

The equation of motion for H can then be obtained from the Virasoro constraint on physical states, which receives the additional contribution from Φ.

Now let us find out what happens to the antisymmetric tensor fields in the R-R sector. The dilaton field also modifies the supercurrent as

$$G \sim \imath \psi_\mu \partial X^\mu + \psi^\mu \partial_\mu \Phi.$$

As we recall, the zero mode of the super-Virasoro constraint yields the massless Dirac equation in the constant dilaton background. If the dilaton is not constant, the Dirac operator is modified as

$$G_0 \sim \not{\partial} - \not{\partial}\Phi = e^{\Phi} \not{\partial} e^{-\Phi}.$$

Correspondingly, the equations of motion for the R-R fields are

$$d^*(e^{-\Phi}F^{\{n\}}) = 0, \quad d(e^{-\Phi}F^{\{n\}}) = 0.$$

Therefore it is the rescaled fields

$$\hat{F}^{\{n\}} \equiv e^{-\Phi}F^{\{n\}}$$

which obey the usual Bianchi identity and equations of motion for an antisymmetric tensor. We can then write $\hat{F}^{\{n\}} = d\hat{A}^{\{n-1\}}$ and their space-time action is

$$\int d^{10}X \, \hat{F}^{\{n\}} \wedge *\hat{F}^{\{n\}},$$

without the usual $e^{-2\Phi}$ factor. Thus, we find that the R-R fields do not couple to the dilaton if they are suitably defined. This is contrary to the case of the NS-NS B field, for which such rescaling is not possible. It protects certain quantities associated with D-branes from string loop and nonperturbative corrections.

1.2.2 Description of NS and Dp-branes

NS 1-brane is simply the fundamental string itself. NS 5-branes are described as nontrivial conformal field theory background [9, 10], as are the 5-branes in heterotic theories [11, 12], although such descriptions are often not adequate or convenient. Dp-branes, on the other hand, has a elegant description in terms of worldsheets with boundaries within the framework of perturbative string theory and amenable to many calculations. A precise formulation for general situation will be given in §2. Here we review the simple case of flat D-branes in flat space.

Recall that on a worldsheet there are ten fields parameterize its embedding in the 10d space-time, $X^\mu, \mu = 0, \ldots, 9$. If in addition to closed worldsheets, one allows the possibility of worldsheets with boundaries, i.e. open strings, then one also have to specify boundary condition for the fields X. There are two distinguished choices: Neumann boundary condition

$$\partial_\perp X^\mu = 0 = \partial_L X^\mu - \partial_R X^\mu \tag{1.2.1}$$

and Dirichlet boundary condition

$$\partial_\| X^\mu = 0 = \partial_L X^\mu - \partial_R X^\mu. \tag{1.2.2}$$

One can see that the latter means the tangential derivative of X^μ along the boundary vanishes. In other words, the string is stuck at the boundary to some fixed value for X^μ. Thus $p+1$ Neumann boundary condition and $9-p$ Dirichlet boundary condition describes a fundamental string ending on (p+1) dimensional object which has been named a **Dp-brane** as depicted in figure 1.5[2]. In any case, supersymmetry between X's and the worldsheet fermions ψ's lead to the boundary conditions

$$\psi_L^\mu = \pm\psi_R^\mu, \tag{1.2.3}$$

with the **same** choice of sign for all μ.

Impose the boundary conditions as a constraint one may proceed to quantize the theory of **open strings**. The collective excitation of a Dp-brane is described by the fluctuation of such open strings. Here we shall sketch the results when the two boundary of the string share the identical conditions. More general scenarios will be discussed in §4.3 Using (half) cylindrical coordinate to parameterize the open string so that the two boundaries are at $\sigma^1 = 0$ and π respectively, then (eq. 1.1.11) and (eq. 1.1.22) still hold but with left and right oscillators no longer independent — only one set is. Roughly one gets the Hilbert space of only, say, the left movers of close superstring. The zero modes, x and p, , are also affected. Quantizing a coordinate with Dirichlet boundary condition yields a massless space-time scalar along the associated direction in its spectrum. This is a Goldstone boson signalling the spontaneous breaking of translation invariance along that direction — it parameterizes fluctuation. As will be discussed in §2.1.3, Neumann boundary condition preserves translation invariance. Quantizing coordinates under its influence yields a gauge potential propagating in the D-brane worldvolume.

The choice of sign in (eq. 1.2.3) divides the Hilbert space into two sectors analogous to the situation for close superstring. The overall choice of sign is immaterial as it can be changed by a field redefinition of ψ, but the relative sign does. If one chooses opposite signs, the fermion oscillators' mode number are half integral. The Hilbert space is that of NS sector. If one chooses the same sign, those mode numbers are even and the Hilbert space is that of the Ramond

[2]In particular, if $p = 9$, the D9-brane fills the whole space. For this case there are additional consistency conditions, which, when satisfied, yields the type I string theory.

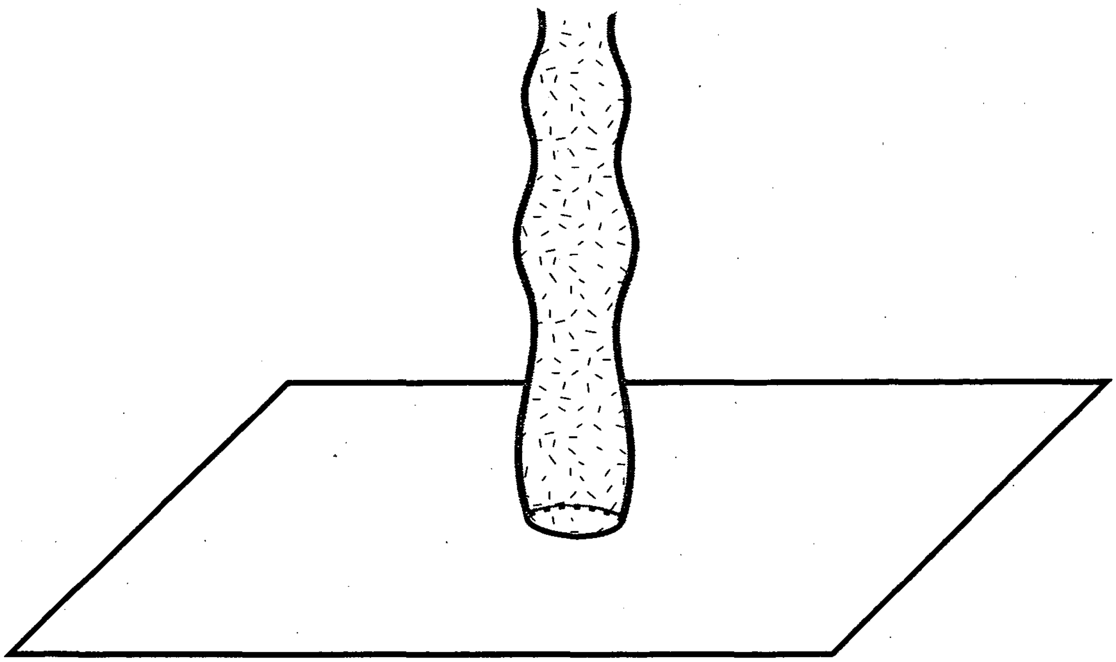


Figure 1.5: A D-brane is where strings can end

sector. As expected the ground state of the Ramond sector transform as spinors of Spin$(1,9)$, which is broken by the boundary conditions into the products of the Lorentz groups for the worldvolume and the space transverse to it, $\mathrm{Spin}_{||} \times \mathrm{Spin}_{\perp}$. One must also take the GSO projection to keep only spinor of a definite 10d chirality. The fields surviving the projection complement the fields from the NS sector to make complete supermultiplets with 16 supercharges. This suggests, and will be shown in §2.1.3, that exactly half of the 32 space-time supercharges in flat 10d space are broken by the presence of a D-brane. They generate Goldstone fermions.

Partial breaking of supersymmetry is one of the salient attributes of a p-brane. As supersymmetry transformations close into translation, this follows from their tautological property of breaking translation invariance. On the other hand, for p-branes that are flat and have not boundary, some translation invariance remains. It turns out that for those p-branes, precisely one half of the original (vacuum) supersymmetry are broken. This remaining supersymmetry imposes severe kinematic constraint on the property of the p-branes that are often expressed as nonrenormalization theorem.

1.3 Unifying string theories and p-branes

One of the unexpected consequences of studying the rich variety of p-branes in string theory has been a set of duality transformations that relate them among each other and unify the different perturbative forms of string theories.

1.3.1 T-duality

The simplest of them is the T-duality. Consider a type II string theory (either type A or B) in space-time $R^9 \times S^1$ with the radius of circle being R. An immediately consequence of this compactification is that the string can wind around the circle for an arbitrary number of times

$$X^9(\sigma + 2\pi R) = X^9(\sigma) + 2m\pi R.$$

This means the general solution to the equation of motion for X^9 is no longer (eq. 1.1.11) but

$$X^9 = X^9_L + X^9_R;$$

with independent left and right moving part, even for the zero modes:

$$X^9_L = x^9_L - ip^9_L z + \sum_{n\neq 0} \frac{i}{n}\alpha^i_n e^{-nz}, \qquad X^9_R = x^9_R - ip^9_R \bar{z} + \sum_{n\neq 0} \frac{i}{n}\tilde{\alpha}^i_n e^{-n\bar{z}};$$
$$p^9_L = \frac{n}{R} + \frac{mR}{2}; \qquad p^9_R = \frac{n}{R} - \frac{mR}{2}. \tag{1.3.1}$$

Here n is the quantum of center of mass momentum along the circle. Then allowed values for the momenta are simply

$$p_L = \frac{n}{R} + \frac{mR}{2}, \qquad p_R = \frac{n}{R} - \frac{mR}{2}, \tag{1.3.2}$$

Consider another theory compactified on radius $R' = \frac{2}{R}$. If we interchange n and m in (1.3.2), then we can identify the momentum operator for $R' = \frac{2}{R}$ with that of R with the isomorphism

$$p_L \leftrightarrow p'_L; \qquad p_R \leftrightarrow -p'_R \tag{1.3.3}$$

by interchanging the labels n and m. Now extending this to an isomorphism of the fields in the two theories, the commutation relation between $x_{L,R}$ and $p_{L,R}$ forces us to require also

$$x_L \leftrightarrow x'_L; \qquad x_R \leftrightarrow -x'_R. \tag{1.3.4}$$

In order to have the space-time interpretation of this duality as inverting the radius of (or equivalently the metric G_{ij} on) the circle, we need to transform the oscillators as well:

$$\alpha_n \leftrightarrow \alpha'_n; \qquad \tilde{\alpha}_n \leftrightarrow -\tilde{\alpha}'_n. \tag{1.3.5}$$

(eq. 1.3.3), (eq. 1.3.4), and (eq. 1.3.5) can be combined into a more compact form[1]:

$$X_L \leftrightarrow X'_L; \qquad X_R \leftrightarrow -X'_R. \tag{1.3.6}$$

This isomorphism of operators clearly translates into an isomorphism between Hilbert spaces. As a check, one can evaluate the path integral of the worldsheet theory on closed Riemann surfaces of arbitrary genus. There $R \to \frac{2}{R}$ is an invariance provided one shifts the constant dilaton field appropriately. See [13] for more details. To show that the two theories are actually equivalent, we have also to show that this map is an operator algebra isomorphism. This is easy, since both theories are free and their operator product expansions can be computed exactly. Thus $R \to \frac{2}{R}$ is an exact symmetry of the string action, on arbitrary Riemann surfaces. But is it really a symmetry of the space-time theory that the worldsheet action describes? From earlier discussion of string perturbation, we see that it is a symmetry of string theory order by order in string perturbation expansion. In fact, it is a **gauge symmetry** of the bosonic string theory.

Now let us briefly review how the T-duality $R \to \frac{2}{R}$ acts on superstring compactified on $M^9 \times S^1$. Recall that this duality involves the isomorphism $\partial X^9_L \leftrightarrow \partial X^{9\prime}_L$ and $\partial X^9_R \leftrightarrow -\partial X^{9\prime}_R$. The same clearly carries over to superstrings, but we also have to respect the worldsheet supersymmetry. It is clear that the isomorphism for the worldsheet fermions should be $\psi^9_L \leftrightarrow \psi^{9\prime}_L$ and $\psi^9_R \leftrightarrow -\psi^{9\prime}_R$. In particular, the zero mode of ψ^9 in R sector, which acts as Γ^9 on the right movers, changes its sign. This means that the relative chirality between the left and right movers is flipped. Therefore $R \to \frac{2}{R}$ maps type II A superstring compactified on a circle of radius R to type IIB superstring on a circle of radius $\frac{2}{R}$. This is an identification of two different types of theories, rather than a gauge symmetry as in the case of bosonic string.

T-duality also connects the two type of Heterotic strings. When they are compactified on a circle, one can break the gauge symmetries of both theories down to SO(16) $\times$ $SO(16)$ by turning on their respective Wilson lines appropriately. The resulting configurations are related by a T-duality transformation.

1.3.2 Dp-branes from type I string theory

So far we have only considered T-duality for closed strings. When boundary conditions are involved, T-duality introduces more surprises — it exchange Neumann (eq. 1.2.1) and Dirichlet (eq. 1.2.2) boundary conditions. Therefore Dp-branes and D(p+1)-branes are mapped into each other. The massless scalar on a Dp-brane that corresponds to the Goldstone mode for translation along the T-dualized direction is exchanged with the Wilson line along the same direction for the gauge potential on the the dual D(p+1)-brane. In particular, one can obtain Dp-branes for all possible values of p by repeatedly compactifying type I string

[1] As a side remark, we note that this is a two-dimensional version of the "electro-magnetic" duality.

theory on circles and perform T-dualities. These mappings of branes constitute an important part of the nonperturbative definition of T-duality. D-branes are solitonic objects whose tension are inversely proportional to the coupling constant of string perturbation expansion. Excitations on them correspond to nonperturbative states of string theory. At the same time they are also charged under the appropriate R-R fields. If T-duality holds nonperturbatively, it maps the states and the fields of the dual pairs in a consistent way.

Because NS5-branes are represented in string theory by a nontrivial conformal field theory in the space transverse to the branes, T-duality along worldvolume direction of a NS5-brane just returns another NS5-brane wrapping around the dual circle. T-duality along a circle transverse to a NS5-brane maps it to some other space-time (metric) background [14].

1.3.3 S duality of type IIB superstring

While T-duality can be formulated perturbatively, there are also duality transformations that are inherently nonperturbative. A famous example is the S-duality of type IIB string theory [15]. All other nonperturbative string dualities can be constructed by conjugating with T-dualities. In type IIB string theory's massless spectrum, in addition to the graviton, there are two scalar fields, the dilaton ϕ and axion χ, two rank two antisymmetric tensor fields, $B^i_{\mu\nu}, i = 1, 2$, and a rank four antisymmetric tensor field. Organizing the two real scalars into one complex scalar:

$$\rho = \chi + ie^{i2\phi}, \tag{1.3.7}$$

then the S-duality acts on it by

$$\rho \to -1/\rho, \tag{1.3.8}$$

which is why it can not be seen perturbatively. It transforms the two 2-forms by

$$B^2 \to B^1, B^1 \to -B^2 \tag{1.3.9}$$

while leaving the 4-form invariant. Therefore it exchanges fundamental strings with D 1-branes, NS5-brane with D5-brane, but leaves D3 branes unchanged.

Type I string theory can be obtained by introducing 32 D9 branes in type IIB string theory and impose a projection. This projection eliminates the axion, one of the two 2-forms, and the 4-form. Therefore this S duality cannot be a symmetry of type I string theory; rather it map it to another string theory, Heterotic $Spin(32)/\mathbb{Z}_2$. Again it maps D1-branes to fundamental strings and D5-branes to heterotic 5-branes.

Therefore, we see that all five types of perturbative string theory and the p-branes they admit are unified by T and S dualities.

Chapter 2

General Formulation of Dirichlet p-Branes

2.1 D-branes and boundary conformal field theory

As mentioned in chapter 1, D-branes are Ramond-Ramond charged BPS solitons in type II string theories [7]. In the presence of a D-brane, the boundary conditions for open strings are modified in such a way that Dirichlet boundary conditions are allowed in addition to the Neumann boundary conditions. Earlier study of D-branes has been mainly restricted to the cases where the D-brane worldvolume is flat. In [16], a study of D-branes wrapped on curved spaces has been carried out in the long wavelength limit.

In this chapter we will present a framework at the SCFT level for the study of D-branes on Calabi-Yau spaces. Perturbative string computations in the presence of a D-brane can be formulated by using a boundary state which describes how closed strings are emitted or absorbed on the D-brane worldvolume. In the case of the fully Neumann boundary condition near the flat background, the boundary state was constructed in [17]. One of our objects of study is the boundary state for a D-brane wrapping on a non-trivial supersymmetric cycle in a Calabi-Yau space. In particular, we examine how the geometric data on the cycle are encoded in the boundary state.

Although Calabi-Yau compactifications are the most thoroughly studied, there are other, "exceptional" compactifications. They are of particular interest in application to M and F theories and result in an amount of supersymmetry that is phenomenologically interesting. Again we use the boundary SCFT approach to study supersymmetry preserving D-branes in these spaces. Along the way, we shall also find an exceptional type of supersymmetric cycles in Calabi-Yau 4-fold with a surprising property

This chapter is organized as follows. In section 1 we review the notion of string compactification and give the general formulation of D-brane in term of boundary conformal field theory. In section 2 we classify boundary conditions for $N=2$ SCFT which preserves half of the space-time supersymmetry and the $N=1$ worldsheet supersymmetry. We then examine how these boundary conditions are realized by D-branes wrapping on cycles in a Calabi-Yau manifold. In section 3 we will study the algebraic and geometrical structures of the boundary states of D-branes wrapped on supersymmetric cycles in Calabi-Yau spaces. We will distinguish between the middle-dimensional and even-dimensional cycles, and find the dependence of the boundary states on the choice of the cycles as well as the complex and Kähler moduli of the Calabi-Yau space. For illustration we discuss the construction of boundary states for Gepner models, exhibiting the relation between the boundary conditions for the model and the supersymmetric cycles. In section 4 we will analyze some exceptional types of supersymmetric cycles: the Cayley cycles in 8d $Spin(7)$ holonomy and Calabi-Yau manifolds, the associative and coassociative cycles in seven-dimensional G_2 holonomy manifolds. We will find that the Cayley 4-cycle in $SU(4)$ holonomy Calabi-Yau 4-fold is novel in that it preserves only one quarter of space-time supersymmetry, while the others preserve as usual half of the supersymmetry. We also present some simple examples of supersymmetric cycles in Calabi-Yau 4-folds.

2.1.1 String compactification in curved spaces

The restriction on space-time dimension by requiring quantum mechanical consistency that we reviewed in the last chapter is a striking result. Some analog of it may one day tell us why we live in three spatial and one temporal dimensions. However, as a candidate theory of everything, string theory faces the immediate criticism that it gives us *too many* dimensions. Naturally one entertains the possibility that the true space-time takes the form of a direct product $F \times M$, where F is the "observable" space-time, in real life the 4-dimensional Minkowski space, and M an extremely tiny compact manifold that our crude probes of nature have so far failed to reveal. This idea was formulated in the form of Kaluza-Klein program long before string theory was invented. However, string compactifications introduce interesting "stringy" effects not seen in the usual Kaluza-Klein schemes.

For a string propagating in a $F \times M$ background space-time with constant VEV Φ for the dilaton, we may absorb Φ into the string coupling constant. The conformal gauge action is then

$$S = \frac{1}{4\pi} \int d^2z \, (G_{\mu\nu} + B_{\mu\nu}) \partial X^\mu \bar{\partial} X^\nu, \tag{2.1.1}$$

where we have set α' to 2 by choosing a unit of length. Because of the direct product structure of $F \times M$, S can be split into an external part S_F involving coordinates on F and an internal part S_M on M, which can be studied separately. The analysis of S_F is trivial and all the interesting consequences of compactification come from S_M. In the last chapter we reviewed briefly string theory in flat space-time, including toroidal compactifications. However, many more interesting and potentially phenomenologically relevant physics appear for more complicated choices of M. In this chapter we define D-branes for general compactifications. Although the ultimate goal of string theory is to describe the $D = 4$ world we live in, it turns out to be very instructive and enlightening to consider choices of M other than six dimensional manifolds.

From the last chapter we see that conformal invariance and hence cancellation of conformal anomaly is crucial for a consistent string theory. Generic conformal field theories do not have a space-time interpretation. Since only the space-time in the uncompactified Minkowski space is observable, one may consider using arbitrary CFT to represent the effects of "compactification" even if they do not have any space-time interpretation like that of (2.1.1). This is consistent as long as they have the right amount of central charge so that the total conformal anomaly still cancels. The formulation of D-branes presented in this chapter is based on general properties of conformal field theories and therefore applicable to these types of compactifications as well. However, when a space-time interpretation is available for the CFT, we will still be able to recover the geometric features of the D-branes. Almost all works on string compactification being studied preserve some amount of supersymmetry, for both phenomenological and technical reasons. We shall also specialize to D-branes in supersymmetric compactifications that preserves some supersymmetry, known as **supersymmetric cycles**. Such D-branes share many of the properties of BPS states in supersymmetric field theories. For example, their energy density in the uncompactified space and the multiplet structure of their long wavelength collective excitation are both protected form radiative corrections. (In certain compactifications and limits they truly become BPS states in the approximating field theories.) This allows one to study them within perturbative string theory yet obtain results that are nonperturbatively correct [18].

2.1.2 Boundary conformal field theory

A D-brane is where strings can end. This most succinctly characterizes a D-brane from the worldsheet viewpoint. We are thus led to examine the ramification of boundaries for worldsheet theories. For the worldsheet Lagrangian (eq. 1.1.8), conformal invariance is of uttermost importance. It is a residual gauge symmetry and generated by two copies of the same algebra (eq. 1.1.7) or its quantum deformation (eq. 1.1.9). In the presence of a boundary, half of this gauge symmetry is fixed by a choice in representing it, e.g. the real axis of the upper complex plane or the perimeter of a disc. The rest remains and generates a single copy of the virasoro algebra. This half breaking of conformal invariance is expressed by the boundary condition

$$T = \tilde{T} \tag{2.1.2}$$

along the boundary of the worldsheet.

There are two ways to implement this type of boundary conditions. If the boundaries are timelike on the worldsheet, as in figure 2.1a, one imposes (eq. 2.1.2) or rather its solution in terms of boundary conditions for fundamental fields, as constraints in quantizing this open string theory. If the boundary is spacelike, as in figure 2.1b, it can be realized as a boundary "state" $|B\rangle\rangle$ that satisfies

$$(T - \tilde{T})\, |B\rangle\rangle \tag{2.1.3}$$

Written in terms of the Fourier modes of T and $\tilde{T}$, this can be written as

$$(L_n - \tilde{L}_{-n})\, |B\rangle\rangle \tag{2.1.4}$$

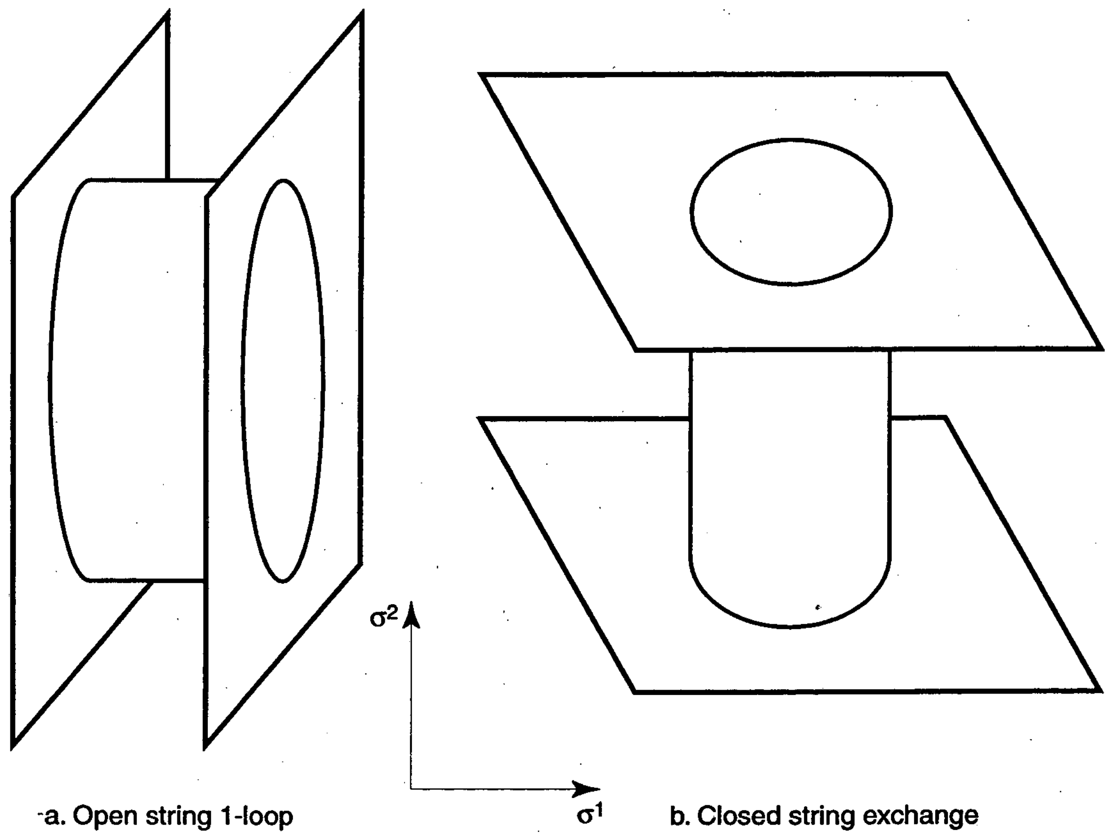


Figure 2.1: Open string one-loop vs. closed string exchange.

General solutions of this equation have been found [19]. Associated with every representation j of the Virasoro algebra is an independent solution of (eq. 2.1.4). Denote by $|j,n\rangle$, $\overline{|j,n\rangle}$ an orthonormal basis of j. The corresponding solution is

$$|j\rangle\rangle = \sum_n |j,n\rangle \otimes U\overline{|j,n\rangle}, \tag{2.1.5}$$

where U be an antiunitary matrix that preserves the highest weight state $|j\rangle$ and commutes with the $\tilde{L}$'s. The general solution to (eq. 2.1.4) is an arbitrary linear combination of such $|j\rangle\rangle$'s.

The above two ways of formulating boundary conditions in conformal field theory are equivalent. Some worldsheet diffeomorphism not connected to identity can exchange them so that the same amplitude can have two different interpretations. For instance, in figure 2.1, an one-loop diagram from the open string fields between two D-branes is related to an exchange of close string fields between them. They fit the above mentioned two situations respectively. In practice, the boundary state formalism provides a concrete object to associate with a D-brane and yields more easily to computation. We shall use it in §2.3 and the next chapter. The relative phase between the left and right movers in the boundary conditions can change, depending on the spin of the worldsheet fields involved, after a worldsheet diffeomorphism swapping time and space. As a result, open string constraint is slightly more intuitively at the level of current algebra and will be implicitly used in other parts of this chapter unless stated otherwise.

For all the cases studied here, there are also additional generators in the chiral algebra. For instance, $N = 1$ superconformal invariance (eq. 1.1.24) adds the generate G and $\tilde{G}$. In general, a conformal field theory we study will have some chiral algebra of interest $\mathcal{A} \times \tilde{\mathcal{A}}$. $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are the isomorphic symmetry algebra for the left and right movers respectively. The boundary condition breaks the left $\times$ right algebra down to a diagonal part in the following fashion. Let $\mathcal{A}$ be generated by $\mathcal{J}^i$ and $\tilde{\mathcal{A}}$ by $\tilde{\mathcal{J}}^i$. The general boundary conditions are

$$\mathcal{J}^i = U^i_j \tilde{\mathcal{J}}^j. \tag{2.1.6}$$

U^i_j is not an arbitrary matrix. (eq. 2.1.6) must respect both the left and the right moving current algebra $\mathcal{A}$ and $\tilde{\mathcal{A}}$. Therefore U^i_j must be an automorphism of $\mathcal{A}$. The allowed choices for U classify the possible boundary conditions.

Again take the $N = 1$ super-Virasoro algebra (eq. 1.1.24) as example. Because it generates the residual gauge symmetry of the superstring worldsheet, it enters in every case we shall study here and must always be half preserved in the above fashion independent of any additional chiral field of interest. This algebra contain a $\mathbb{Z}_2$ automorphism, associated with the choice of sign for G and $\tilde{G}$. We already encountered it when discussing the (anti-)periodicity of the fermionic elements. Thus the additional boundary condition is

$$G = \pm \tilde{G}. \tag{2.1.7}$$

In the situations of figure 2.1, only the relative choice of signs on the two boundary matters, as the $\mathbb{Z}_2$ automorphism of the algebra changes the overall sign.

2.1.3 Preserving supersymmetry

We now apply the above formalism to reexamine the flat D-branes introduced in the last chapter. For those simple case the language of boundary conformal theory is not mandatory since the worldsheet theory is free. Nonetheless it is an instructive preparation for more complicated compactifications. The close string worldsheet theory has (8,8) superconformal symmetry. In addition 10d super-Poincare symmetry is also reflected on the worldsheet as a chiral algebra. Its generators are ∂X^μ and ψ^μ, $\mu = 0, \ldots, 9$, and, in the Ramond sectors, the Spin$(1,9)$ spin fields S^α, along with their right moving counterparts. Their current algebras are (eq. 1.1.14), (eq. 1.1.23), and (eq. 1.1.25). Space-time translation are generated by

$$\partial X^\mu + \tilde{\partial} X^\mu \tag{2.1.8}$$

and Lorentz rotation by

$$\sim X^{[\mu,}\partial X^{\nu]} + \psi^{[\mu}\psi^{\nu]} + X^{[\mu,}\tilde{\partial} X^{\nu]} + \tilde{\psi}^{[\mu}\tilde{\psi}^{\nu]}. \tag{2.1.9}$$

Since the Poincare symmetries are generated by a particular linear combination of left and right moving chiral fields, the only allowed boundary conditions for ∂X's and ψ's are

$$\begin{aligned} \partial X^\mu &= \tilde{\partial} X^\mu, \\ \psi^\mu &= \pm\tilde{\psi}^\mu. \end{aligned} \tag{2.1.10}$$

So far the choices of signs for different μ are independent. However, the N=1 super-Virasoro algebra must be separately preserved in half as mentioned earlier. As $G \sim \psi\partial X$, this forces the the same sign to be chosen for all μ in (eq. 2.1.10).

The space-time supersymmetries are generated simply by S^α and $\tilde{S}^\alpha$, **independently**. (eq. 1.1.25) thus allows

$$S = \pm\tilde{S}. \tag{2.1.11}$$

for the choice of "+" sign in (eq. 2.1.10) and

$$S = \pm\chi^{10}\tilde{S}. \tag{2.1.12}$$

for "-" sign. Here χ^{10} is the 10d chirality operator. We have to take into account the GSO projection. The above boundary conditions on spin fields are therefore only sensible in type IIB theory, in which the surviving S and $\tilde{S}$ have the same chirality, say, positive under χ^{10}. So (eq. 2.1.11) is sufficient. Note that exactly half of the supersymmetry of type II B theory survives it.

Now we would like to relax the condition and require that only (1+p)-dimensional Poincare invariance and an Spin$(9-p)$ global symmetry of the original 10d Poincare invariance of the space-time. Some analysis shows that this allows some new possibility:

$$\partial X^\mu = R^\mu{}_\nu \tilde{\partial} X^\nu; \tag{2.1.13}$$

$$\psi_L^\mu = \pm R^\mu{}_\nu \psi_R^\nu. \tag{2.1.14}$$

Here R is a real 10 by 10 matrix that can be diagonalized into two blocks of size $p+1$ by $p+1$ and $9-p$ by $9-p$ respectively:

$$R = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} \tag{2.1.15}$$

The eigen-subspace of R with $+1$ eigenvalue retains the (p+1)d Poincare invariance. Let it be parameterized by X^{ρ_A}, $A = 0, \ldots, p$. The transverse space has only rotational invariance. Note that (eq. 2.1.10) is the special case of (eq. 2.1.13) with $p = 9$.

The boundary condition for the spinor part is slightly more involved. For that we define the worldvolume chirality operator

$$\chi^{p+1} \equiv (-1)^{(p+2)(p-1)/2}\Gamma_{\rho_0\ldots\rho_p}. \tag{2.1.16}$$

and a modified version of it,

$$\hat{\chi}^{p+1} \equiv \begin{cases} \chi^{p+1} & (p = \text{even}); \\ \chi^{10}\chi^{p+1} & (p = \text{odd}). \end{cases} \tag{2.1.17}$$

$\hat{\chi}^{p+1}$ has the useful property

$$\hat{\chi}^{p+1}\Gamma^{\mu}\hat{\chi}^{p+1} = R^{\mu}_{\nu}\Gamma^{\nu}. \tag{2.1.18}$$

Now we can write down the boundary condition for the spin fields consistent with (eq. 1.1.25):

$$S = \pm\hat{\chi}^{p+1}\tilde{S}. \tag{2.1.19}$$

Note that once again exactly half of the supersymmetry are broken. This constraint makes sense for type IIA if p is even and type IIB if p is odd. The special case $p = 9$ reduces to earlier results (eq. 2.1.11).

2.1.4 Supersymmetric cycles

In general, (eq. 2.1.19) tells us how to realize the breaking of space-time supersymmetry with worldsheet boundaries. In curved space-time, a smaller amount of supersymmetry exists then in flat space. The spin fields representing them take the form

$$\zeta^{\alpha}S^{\alpha} \tag{2.1.20}$$

and

$$\tilde{\zeta}^{\alpha}\tilde{S}^{\alpha} \tag{2.1.21}$$

where both ζ and $\tilde{\zeta}$ are covariantly constant spinors. The boundary condition (eq. 2.1.19) can be translated to a statement meaningful within a low energy effective theory approximation:

$$\zeta = \hat{\chi}^{p+1}\tilde{\zeta} \tag{2.1.22}$$

Therefore at least half of the space-time supersymmetry is broken by the presence of a D-brane. We shall explore its ramification in the remainder of this subsection. Along the way we shall gain useful insights into the physical properties and geometric attributes of supersymmetric cycles.

For D-brane of general shape in general space-time, the covariantization of (eq. 2.1.22) is tantamount to replacing χ^{p+1} with its covariantized form

$$\chi^{p+1} \equiv \frac{1}{(p+1)!}\frac{\epsilon^{A_0\ldots A_p}}{\sqrt{g}}\partial_{A_0}X^{\mu_0}\cdots\partial_{A_p}X^{\mu_p}\Gamma_{\mu_0\ldots\mu_p} \tag{2.1.23}$$

A space-time supersymmetry transformation is preserved by the presence of the D-brane only in if (eq. 2.1.22) is satisfied and both ζ and $\tilde{\zeta} = \hat{\chi}^{p+1}\zeta$ are preserved by the compactification.

Unless explicitly stated to the contrary, we shall assume in this chapter that the D-brane worldvolume lie entirely within the internal space M. It is therefore at a point in F[1]. One can always write a 10d space-time spinors can as a linear combination of

$$\zeta = \zeta_F \otimes \zeta_M. \tag{2.1.24}$$

Here ζ_F and ζ_M are spinors under Spin_F and Spin_M, the spin groups for the external and internal spaces respectively. The criterion for space-time supersymmetry surviving compactification is that ζ_M be a covariantly constant spinor of M[2]. (eq. 2.1.22) means that given a covariantly constant spinor ζ_1 of M, to get a space-time supersymmetry preserved by the D-brane, it must pair up with another covariantly constant spinor ζ_2 so that

$$\zeta_2 = \hat{\chi}^{p+1}\zeta_1. \tag{2.1.25}$$

If this is satisfied, because $\left(\hat{\chi}^{p+1}\right)^2 = 1$,

$$\hat{\chi}^{p+1}\zeta_\pm = \pm\zeta_\pm \tag{2.1.26}$$

with

$$\zeta_\pm \equiv \zeta^1 \pm \zeta^2 \tag{2.1.27}$$

Define the chiral projection operators for the Dp-brane

$$P_\pm^{p+1} \equiv \frac{1}{2}\left(1 \pm \hat{\chi}^{p+1}\right). \tag{2.1.28}$$

(eq. 2.1.26) can be rewritten as

$$P_\pm^{p+1}\zeta_\mp = 0. \tag{2.1.29}$$

The sign choices in the same equation are correlated here and below.

(eq. 2.1.29) has an interesting interpretation. Consider the inequality

$$\zeta_\mp^\dagger P_\pm^{p+1} P_\pm^{p+1}\zeta_\mp = \zeta_\mp^\dagger P_\pm^{p+1}\zeta_\mp \geq 0. \tag{2.1.30}$$

Scale ζ to unit norm, and the last inequality turns into

$$\left|(f^*\rho)_{A_0\ldots A_P}\right| \leq \sqrt{g}\epsilon_{A_0\ldots A_P}. \tag{2.1.31}$$

Here

$$\rho_{\mu_0\ldots\mu_p} = \zeta_M^\dagger \Gamma_{\mu_0\ldots\mu_p}\zeta_M. \tag{2.1.32}$$

[1]Supersymmetry condition for the D-brane with worldvolume in F can be readily studied separately or obtained from T-duality.

[2]Here we are assuming vanishing background for antisymmetry tensor field strengths

and f^* is the pullback associated with the embedding map $X^\mu(A)$ of the D-brane:

$$(f^*(\omega))_{A_0 \dots A_P} \equiv \partial_{A_0} X^{\mu_0} \cdots \partial_{A_p} X^{\mu_p} \omega_{\mu_0 \dots \mu_p}. \tag{2.1.33}$$

The spectrum of antisymmetric tensors from decomposing the products of covariantly constant spinors are an important characteristic of the compactification manifold M [20, 21]. In later sections of this chapter we will see examples of them and their physical applications.

Because ζ and $\tilde{\zeta}$ are covariantly constant, so are the forms ρ's. In particular they are closed. Mathematicians refer closed differential forms that satisfy (eq. 2.1.31) as calibrated [22, 34]. Since $\sqrt{g}\epsilon_{A_0 \dots A_P}$ is the volume form of Dp-branes, integrating (eq. 2.1.31) over any submanifold γ of M yields

$$\left| \int_\gamma \rho \right| \le \mathrm{vol}(\gamma). \tag{2.1.34}$$

If (eq. 2.1.31) saturates for some γ, γ is said to be calibrated by ρ. It has the minimal volume in its homology class: if γ' differs from γ by a boundary $\partial\beta$, one has

$$\mathrm{vol}(\gamma') \ge \int_{\gamma'} \rho = \int_\gamma \rho = \mathrm{vol}(\gamma). \tag{2.1.35}$$

(eq. 2.1.34) and (eq. 2.1.35) have physical interpretations. (eq. 2.1.34) resembles a Bogomolnyi bound — the mass or tension is greater than or equal to the charge of a Dp-brane, in appropriate units. (eq. 2.1.35) says no smooth deformation can make a supersymmetry preserving Dp-brane smaller in volume. This fits with the notion that supersymmetry implies stability — the configuration of a D-brane can hardly be stable if it has finite tension yet can shrink in size smoothly. To conclude, **supersymmetric cycles are submanifolds calibrated by forms coming from products of covariantly constant spinors.**

2.2 Supersymmetric cycles in Calabi-Yau manifolds

The most thoroughly studied class of string compactification is that over Calabi-Yau manifolds, and among them Calabi-Yau 3-folds (6 real dimensional) in particular. All Calabi-Yau compactifications have the common feature that the conformal field theory for the internal space possesses at least N=2 superconformal symmetry.

In this section we will classify the boundary conditions for $N = 2(4)$ SCFT which preserves half of the space-time supersymmetry as well as the worldsheet superconformal symmetry. We will then examine how these boundary conditions are realized by D-branes wrapping on cycles in a Calabi-Yau manifold. Here we will consider the case when the sigma-model for the Calabi-Yau manifold has one set of $N = 2(4)$ superconformal algebra for the left-movers and one set for the right movers. It is straightforward to extend this analysis to the case where we have more than one set of $N = 2(4)$ algebras, such as T^{2d} with $d \ge 2$.

The supersymmetric cycles we find below are universal to all Calabi-Yau compactifications. They share the property of preserving exactly one half of the supersymmetry of type II string theory with those compactifications. Generically they are the only possibility for preserving space-time supersymmetry. For Calabi-Yau 4-fold, however, there are an exceptional type that breaks 3/4 of the space-time supersymmetry, to which we shall to in §2.4.2.

2.2.1 Boundary conditions for $N = 2$ SCFT

The supersymmetric sigma-model for a Calabi-Yau manifolds has $N = 2$ superconformal algebra (SCA). Throughout this chapter, we set the signs of the left and the right $U(1)$ currents to be

$$J_L = g_{i\bar{j}}\psi_L^i\psi_L^{\bar{j}}, \qquad J_R = g_{i\bar{j}}\psi_R^i\psi_R^{\bar{j}}, \tag{2.2.1}$$

which determines the convention for $G^{\pm}$ as

$$\begin{aligned} G_L^+ &= g_{i\bar{j}}\psi_L^i\partial X^{\bar{j}}, \quad G_L^- = g_{i\bar{j}}\psi_L^{\bar{j}}\partial X^i, \\ G_R^+ &= g_{i\bar{j}}\psi_R^i\bar{\partial} X^{\bar{j}}, \quad G_R^- = g_{i\bar{j}}\psi_R^{\bar{j}}\bar{\partial} X^i. \end{aligned} \tag{2.2.2}$$

In addition, in order to preserve half of the space-time supersymmetry, we should take into account the spectral flow operator $e^{i\phi_L}$ defined by

$$e^{i\phi_L} = \Omega_{i_1\ldots i_d}\psi^{i_1}\ldots\psi^{i_d}, \tag{2.2.3}$$

This and its complex conjugate are the squares of the spin fields associated with the covariantly constant spinor on the Calabi-Yau. They are more convenient to use than the spin fields. Here Ω is the holomorphic d-form on the Calabi-Yau d-fold and $J_L = i\partial\phi_L$. Note that, in this convention, the $N = 1$ supercurrent is generated by

$$G = G_L^+ + G_L^-. \tag{2.2.4}$$

In order to represent a BPS saturated state in space-time, the boundary must preserve half of the space-time supersymmetry. Thus we require the boundary state to be invariant under a linear combination of the left and right $N = 2$ algebra extended by the spectral flow operators. As discussed earlier, consistency restricts the linear combination to correspond to the automorphism group of the algebra. The automorphism is $O(2)$ for $N = 2$ SCA and Z_2 for $N = 1$. Since the supercurrent G is gauged, its form should be preserved. Thus we are left with a $Z_2 \times Z_2$-wise choice:

A-type boundary condition:[1]

$$J_L = -J_R, \quad G_L^+ = \pm G_R^-, \quad e^{i\phi_L} = e^{-i\phi_R}. \tag{2.2.5}$$

[1]In this section we write boundary conditions in the notation appropriate for the open string channel.

B-type boundary condition:

$$J_L = +J_R, \quad G_L^+ = \pm G_R^+, \quad e^{i\phi_L} = (\pm 1)^d e^{i\theta} e^{i\phi_R}. \tag{2.2.6}$$

The phase factor $e^{i\theta}$ will be determined later. In the A-type boundary condition, it can be absorbed in the definition of Ω. This is why we did not put the phase factor in (eq. 2.2.5). Clearly both A-type and B-type boundary conditions preserve the $N=1$ SCA

$$T_L = T_R, \qquad G_L = \pm G_R, \tag{2.2.7}$$

where T denotes the stress tensor. It should be noted that the mirror symmetry exchanges the A-type and the B-type boundary conditions.

2.2.2 $N=4$ SCFT

In the case of string compactification on K3, the spectral flow operators have the conformal weight 1. Combined with the $U(1)$ current J, they form the affine $SU(2)$ algebra and $N=2$ SCA is extended to $N=4$. For later convenience, let us write the holomorphic 2-form and the Kähler form as

$$\Omega = k^1 + ik^2, \qquad k = k^3. \tag{2.2.8}$$

The $SU(2)$ currents are then

$$J^I = k^I_{\mu\nu} \psi^\mu_L \psi^\nu_L \quad (I = 1, 2, 3), \tag{2.2.9}$$

where the indices μ, ν refer to real coordinates on K3.

In addition to $G^\pm$, we have two more supercurrents, which together with the original two form a $\underline{4}$ of $SO(4)$, the automorphism group of $N=4$ SCA. The automorphism consists of the internal and the external parts, $SU(2)_c \times SU(2)_f$, where $SU(2)_c$ is generated by the $SU(2)$ currents J^a and $SU(2)_f$ is the external automorphism of the N=4 SCA [23]. We can then organize the four supercurrents as $(\underline{2}, \underline{2})$ of $SU(2)_c \times SU(2)_f$ as

$$\begin{aligned} G^{+-} &= g_{i\bar{j}} \psi^i_L \partial X^{\bar{j}}, \quad G^{++} = \Omega_{ij} \psi^i_L \partial X^j, \\ G^{-+} &= g_{i\bar{j}} \psi^{\bar{j}}_L \partial X^i, \quad G^{--} = \bar{\Omega}_{\bar{i}\bar{j}} \psi^{\bar{i}}_L \partial X^{\bar{j}}. \end{aligned} \tag{2.2.10}$$

In this notation, the $N=1$ supercurrent G is

$$G = G^{+-} + G^{-+}, \tag{2.2.11}$$

which is a singlet under the diagonal action of $SU(2)_c \times SU(2)_f$. Since G is fixed, a general boundary condition which preserves both the N=4 and N=1 should only involve the diagonal subgroup of $SU(2)_c \times SU(2)_f$, i.e. $SO(3)$ in the full automorphism $SO(4)$. By decomposing the four supercurrents into $\underline{3}$ and $\underline{1}$ of $SO(3)$, the most general boundary condition is written as

$$J^I_L = U^I{}_J J^J_R, \quad G^I_L = \pm U^I{}_J G^J_R, \quad G_L = \pm G_R, \quad (I, J = 1, 2, 3), \tag{2.2.12}$$

where $U \in SO(3)$.

2.2.3 Geometric realization – general case

We would like to find out how the above classification of supersymmetric boundary conditions corresponds to that of D-branes in a Calabi-Yau manifold M. In this section, we seek this identification in the large volume limit of M, where we can treat the sigma-model semi-classically.

We begin by noting that (eq. 2.2.7) is solved by

$$\partial X^\mu = R^\mu{}_\nu \bar{\partial} X^\nu, \qquad \psi_L^\mu = \pm R^\mu{}_\nu \psi_R^\nu. \tag{2.2.13}$$

for some matrix R provided it satisfies

$$g_{\mu\nu} R^\mu{}_\rho R^\nu{}_\sigma = g_{\rho\sigma}. \tag{2.2.14}$$

The eigenvector of R with eigenvalue (-1) gives the Dirichlet boundary condition for X, and thus should correspond to directions normal to the D-brane. If the matrix R is symmetric, the orthogonal directions are also eigenvectors of R with eigenvalues $(+1)$, and thus they obey the Neumann boundary condition corresponding to the tangential directions to the D-brane. In general, however, R does not have to be symmetric, and this gives rise to a mixed Neumann-Dirichlet condition. As we will see, this corresponds to the case when the $U(1)$ gauge field on the D-brane worldvolume has non-zero field strength.

In the neighborhood of a $(p+1)$-cycle γ on the Calabi-Yau d-fold, we can choose local coordinates such that x^A $(A = 1, ..., (p+1))$ are coordinates on the cycle and y^a $(a = 1, ..., 2d-(p+1))$ are for the directions normal to γ. Clearly $(2d-(p+1))$ is equal to the number of (-1) eigenvalues of R.

Suppose the D-brane wrapping on γ gives the B-type boundary condition. It follows from (eq. 2.2.6) that R should satisfy

$$\begin{aligned} k_{\mu\nu} R^\mu{}_\rho R^\nu{}_\sigma &= k_{\rho\sigma}, \\ \Omega_{\mu_1 \ldots \mu_d} R^{\mu_1}_{\nu_1} \ldots R^{\mu_d}_{\nu_d} &= e^{i\theta} \Omega_{\nu_1 \ldots \nu_d}. \end{aligned} \tag{2.2.15}$$

The first of these equations implies

$$k_{Ab} = 0, \tag{2.2.16}$$

namely the Kähler form k must be block diagonal on γ in the tangential and the normal directions to γ. Since k is nondegenerate, k_{AB} and k_{ab} must also be nondegenerate. This means the dimensions $(p+1)$ of the cycle must be even. Because k is block diagonal, we can use it to define almost complex structure on the cycle. In fact it is integrable and defines a complex structure on the cycle. Thus γ is a holomorphic submanifold of M. In the complex coordinates, the nonvanishing components of the top form Ω has $(p+1)/2$ holomorphic indices tangential to γ and $d-(p+1)/2$ holomorphic indices normal to it. This determines the phase $e^{i\theta}$ in (2.2.15) in terms of the background gauge field on γ. In particular when the gauge field is flat, we find $e^{i\theta} = (-1)^{d-(p+1)/2}$.

On the other hand, if the cycle corresponds to the A-type boundary condition, (eq. 2.2.5) implies

$$\begin{aligned} k_{\mu\nu} R^{\mu}{}_{\rho} R^{\nu}{}_{\sigma} &= -k_{\rho\sigma}, \\ \Omega_{\mu_1 \ldots \mu_d} R^{\mu_1}_{\nu_1} \ldots R^{\mu_d}_{\nu_d} &= \bar{\Omega}_{\nu_1 \ldots \nu_d}. \end{aligned} \tag{2.2.17}$$

If the background gauge field on γ is flat, R squares to the identity matrix. In this case, the first of the above equations implies

$$k_{ab} = 0, \qquad k_{AB} = 0. \tag{2.2.18}$$

Since k is nondegenerate, this is possible only if $(p+1) = d$. Thus a cycle without a gauge field must be middle-dimensional. In this case, all the components of the holomorphic d-form Ω are related to $\Omega_{A_1 \cdots A_d}$ as

$$\Omega_{a_1 \cdots a_m A_{m+1} \cdots A_d} \sim k_{a_1}{}^{A_1} \cdots k_{a_m}{}^{A_m} \Omega_{A_1 \cdots A_d}, \tag{2.2.19}$$

for $m = 1, ..., d$. Since $\Omega \wedge \bar{\Omega}$ is proportional to the volume form of the d-fold, it follows that the pullback of Ω onto the cycle is proportional to its volume form. It is easy to generalize this to the case with background gauge field. One can see that (eq. 2.2.17) implies $(p+1) = d, d+2, ..., 2d$. The reason for this will become clear in the next chapter.

We can make contact with earlier discussion of minimal cycles. In Calabi-Yau manifold, there are two universal types of calibrated forms. One consists of powers of the Kähler form,

$$\hat{k}^p \equiv \frac{1}{((p+1)/2)!} k^{(p+1)/2}, \tag{2.2.20}$$

and the other is constructed from the top holomorphic form,

$$\hat{\Omega}_\theta \equiv \mathrm{Re}(e^{i\theta}\Omega). \tag{2.2.21}$$

They appear in the decomposition of products of the two covariantly constant spinors on a Calabi-Yau manifold. In the above we imposed the boundary conditions on Ω, the square of the corresponding spin fields. By looking at the condition for these spin fields directly, one finds that supersymmetric cycles of A and B types are calibrated by ω^θ and K^{p+1} respectively. They are known to the mathematicians as special Lagrangian and Kähler cycles. The geometric condition for supersymmetric cycles in the case of $p + 1 = 3$ also arises from the low-energy effective worldvolume action of the supermembrane [24].

2.2.4 Geometric realization - K3

In the case of K3, (eq. 2.2.12) states that k^I $(I = 1, 2, 3)$ behave as

$$k^I_{\mu\nu} R^{\mu}{}_{\rho} R^{\nu}{}_{\sigma} = U^I{}_J k^J_{\rho\sigma}. \tag{2.2.22}$$

on the cycle γ. By going through some linear algebra, we find that the conjugacy class of the rotation U is completely determined by the gauge field. For example, in the absence of the gauge field, the matrix U is equal to 1 for 0-cycle and 4-cycle while it is in the conjugacy class of π-rotation for 2-cycle. To understand this more geometrically, we diagonalize U as

$$U = M^t \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} M. \tag{2.2.23}$$

By introducing a new basis by $M \in SO(3)$ rotation

$$\tilde{k}^I = M^I{}_J k^J, \tag{2.2.24}$$

(eq. 2.2.22) is expressed as

$$\begin{aligned} \tilde{k}^3_{\mu\nu} R^\mu{}_\rho R^\nu{}_\sigma &= \tilde{k}_{\rho\sigma}, \\ \tilde{k}^{\pm}_{\mu\nu} R^\mu{}_\rho R^\mu{}_\sigma &= e^{\pm i\theta} \tilde{k}^{\pm}_{\rho\sigma}. \end{aligned} \tag{2.2.25}$$

Comparing this with the analysis of the B-type boundary condition in the previous subsection, we see that the cycle γ is a holomorphic submanifold of K3 with respect to the complex structure such that $\tilde{k}^3$ is a Kähler form and $\tilde{k}^+$ is a holomorphic 2-form. Namely the $SO(3)$ rotation by U reflects the $SO(3)$-wise choice of complex structure for a given metric on K3. This result also agrees with the analysis in [24], [16].

2.2.5 Summary

We now summarize our classification of supersymmetric cycles that are universal to all Calabi-Yau compactifications. For Calabi-Yau 4-fold, additional possibility appear and will be discussed in §2.4.2. For each complex dimension d of the Calabi-Yau manifold, we designate allowed values of $(p+1)$ (real dimensions of the cycle) and their possible boundary conditions by type A, B or the one parameterized by $SO(3)$. This table is for the case with flat gauge field on γ. It is straightforward to generalize this to the case with non-zero gauge field strength.

Table 2.1: Supersymmetric cycles in Calabi-Yau Manifolds

d	1			2			3					4				
$p+1$	0	1	2	0	2	4	0	2	3	4	6	0	2	4	6	8
Type	B	A	B	B	$SO(3)$	B	B	B	A	B	B	B	B	A/B	B	B

2.3 Boundary states for D-branes

In this section, we examine the properties of the boundary states for D-branes wrapping on the supersymmetric cycles discussed in the previous section. We will show how the geometric data of the cycles are encoded in the boundary states.

2.3.1 Supersymmetric boundary states

Type II strings compactified on Calabi-Yau spaces possesses the worldsheet $N=2$ SCA in both the left and right sectors. As we saw in the previous section, a D-brane wrapping on a supersymmetric cycle preserves a linear combination of the left and right $N=2$ algebras. We would like to study the correspondence, D-branes $\leftrightarrow$ boundary states, for D-branes wrapped on supersymmetric cycles in Calabi-Yau spaces. In particular, given a D-brane, we would like to find the highest weight states that appear in its boundary state and their multiplicity, and conversely for a given boundary state we would like to find the D-brane configuration.

Recall from the analysis of section 2 that, for the closed strings, there are two types of supersymmetric boundary conditions: For middle-dimensional cycles, we have

$$G_L^+ = \pm i G_R^-, \qquad G_L^- = \pm i G_R^+, \qquad J_L = J_R, \tag{2.3.1}$$

and for even-dimensional cycles

$$G_L^+ = \pm i G_R^+, \qquad G_L^- = \pm i G_R^-, \qquad J_L = -J_R. \tag{2.3.2}$$

Here we are using the notation appropriate for the closed string channel[1]. They are called the A-type and the B-type boundary conditions. For the K3 case, the boundary conditions are parameterized by $SO(3)$ corresponding to the $SO(3)$-wise choice of complex structures for a given metric on K3. The boundary states realizing the A and B-type conditions should then satisfy

$$(G_L^+ \mp i G_R^-)|B\rangle = 0, \quad (G_L^- \mp i G_R^+)|B\rangle = 0, \quad (J_L - J_R)|B\rangle = 0, \tag{2.3.3}$$

or

$$(G_L^+ \mp i G_R^+)|B\rangle = 0, \quad (G_L^- \mp i G_R^-)|B\rangle = 0, \quad (J_L + J_R)|B\rangle = 0, \tag{2.3.4}$$

depending on whether the boundary conditions are A-type or B-type. Let us examine the properties of these boundary states.

[1] $J_R \to -J_R$ and $G_R^\pm \to i G_R^\pm$ compared to the notation in section 2.

2.3.2 A-type boundary condition

Let us consider first the A-type boundary condition corresponding to middle-dimensional cycles. The boundary state can be expanded in terms of the Ishibashi states as

$$|B\rangle = \sum_a c^a |a\rangle\rangle, \tag{2.3.5}$$

where the sum is over the highest weight states of the $N = 2$ algebra which appear in the Hilbert space of the sigma-model for the Calabi-Yau space M. They may be chiral primary states or non-chirals. According to our convention (2.2.2), complex moduli of M are associated to (c, c) and (a, a) primary states and Kähler moduli are included in and (c, a) and (a, c).

The requirement that $(J_L - J_R) = 0$ at the boundary implies $q_L = q_R$ for the $U(1)$ charges and thus a selection rule for the conformal fields that can contribute to the boundary state. In particular, this means that the coefficients in front of the (c, a) and (a, c) primaries are zero. In the following we will find an explicit form for the coefficients c^a for the (c, c) and (a, a) chiral primary states.

For the sigma-model, the (c, c) primaries with charge (q, q) correspond to elements of the middle cohomology $H^{q,d-q}(M)$ where $d = \dim_C M$. It is straightforward to show that the coefficient c^a corresponding to the (c, c) primary state is given by

$$c^a = \eta^{ab} \langle 0_{top} | \phi_b(z, \bar{z}) | B \rangle_{Ramond-Ramond}, \tag{2.3.6}$$

where $\langle 0_{top}|$ is the topological vacuum of the A-model, η^{ab} is the topological metric, and ϕ_b is the (c, a) primary field associated to $\omega_b \in H^{q,d-q}(M)$. By the A-model, we mean the one with the topological twist such that G_L^+ and G_R^- become one-forms on the worldsheet[2]. Since ϕ_b is physical in the A-model, and one may regard $c_a = \eta_{ab} c^b$ as a topological string amplitude on a disk with a puncture at z.

The coefficient c_a may in principle depends on the Kähler moduli $(t^i, \bar{t}^i)$ $(i = 1, ..., h^{1,1})$ as well as the complex moduli of M. To compute $\partial_{\bar{t}}$ of c_a, we insert $G_L^+ G_R^- \bar{\varphi}_i$ onto the disk, where $\bar{\varphi}_i$ is an (a, c) primary field with $(q_L, q_R) = (-1, 1)$. Since both G_L^+ and G_R^- are one-forms in the B-model, we can employ the standard contour deformation argument in the topological field theory. Taking into account the boundary condition $G_L^+ = \pm i G_R^-$, one finds that the result of this insertion is zero. Thus c_a is holomorphic in t^i and therefore the instanton approximation to c_a is exact.

Furthermore one can also show that c_a is independent of t^i. One way to show this is to do the instanton expansion explicitly and verify that the instanton correction vanishes due to the fermion zero modes.

Another way to show this is to insert $G_L^- G_R^+ \varphi_i$ where φ_i is a (c, a) primary field with $(q_L, q_R) = (-1, 1)$. In this case, both G_L^- and G_R^+ are two-forms on

[2] Thus the topological vacuum $\langle 0_{top}|$ has charges $(-d/2, +d/2)$. Since the (c, a) primary field ϕ_b carries charges $(q, q - d)$, the total charges of $\langle 0_{top}|\phi_b$ is $(q - d/2, q - d/2)$ satisfying the selection rule.

the disk and we cannot immediately deform their contours. On the disk with one puncture at z, there is a global holomorphic (-1) form $\xi(w) = (w-z)(w-\bar{z})$. By multiplying ξ, we can convert G_L^- into one-form and we can use the contour deformation argument. Since $\xi(w)$ vanishes at $w = z$, where ϕ_a is located, we can move the contour to the Dirichlet boundary where we can convert ξG_L^- into $\bar{\xi} G_R^+$ since ξ is real-valued on the boundary (We chose the boundary to be $Im\ w = 0$.). We can them move $\bar{\xi} G_R^+$ back and the contour slips out of the disk. Thus we have shown that ∂_{t^i} of c_a also vanishes. This reasoning is similar to the one which shows that the topological metric of the A-model does not receive the instanton correction.

Since c_a is independent of the Kähler moduli, we can take the large volume limit in (2.3.6) to show

$$c_a(\gamma) = \int_\gamma \omega_a, \tag{2.3.7}$$

where γ is the supersymmetric cycle in question. Thus the chiral primary part of the boundary state is determined entirely by the homology class of the cycle γ.

This in particular means that the chiral primary part

$$|\gamma\rangle = \sum_{\phi_a:(c,c)} c^a |a\rangle_{Ramond-Ramond}, \tag{2.3.8}$$

of the boundary state is a flat section of the so-called improved connection [25, 26, 27] for the bundle of Ramond vacua over the moduli space of $N = 2$ superconformal field theories (for a review, see also section 2 of [28]). Since it plays an important role in the case of the B-type boundary condition in the following, let us demonstrate this fact explicitly here. Let us organize the basis of $H^d(M)$ as $\omega_0 \in H^{d,0}$, $\omega_\alpha \in H^{d-1,1}$ $(\alpha = 1, ..., h^{d-1,1})$, etc. Then we find

$$\frac{\partial c_0}{\partial \bar{y}^\alpha} = 0, \quad \frac{D c_0}{D y^\alpha} = c_\alpha, \quad \text{etc}, \tag{2.3.9}$$

where y^α are the complex moduli of M and D is the covariant derivative on the vacuum line bundle $\mathcal{L}$ over the moduli space of the $N = 2$ theories. These equations can be summarized as

$$\nabla_\alpha |\gamma\rangle = 0, \quad \bar{\nabla}_{\bar{\alpha}} |\gamma\rangle = 0, \tag{2.3.10}$$

where

$$\nabla_\alpha = D_\alpha - C_\alpha, \quad \bar{\nabla}_{\bar{\alpha}} = \bar{D}_{\bar{\alpha}} - \bar{C}_{\bar{\alpha}}, \tag{2.3.11}$$

and C_α is the multiplication by the Yukawa coupling.

This in particular means that c_a for $\omega_a \in H^{d-1,1}$ etc, is obtained by acting with D_α on c_0. Thus the chiral primary part of the coefficients in (2.3.5) is completely determined by computing the period

$$c_0(\gamma) = \int_\gamma \Omega, \tag{2.3.12}$$

of the holomorphic $(d,0)$-form. To be precise, this is the case when the complex dimension of the Calabi-Yau manifold is less than 4. When $d \geq 4$, there is some subtlety since there may be an element ω_a of $H^{d-q,q}$ with $q \geq 2$ which is not generated by differentiating $H^{d,0}$ with respect to the complex moduli. If that is a case, we have to evaluate (2.3.7) for such ω_a separately. Understanding how this procedure works for $d \geq 4$ would help clarify issues on the mirror symmetry in higher dimensions [29].

2.3.3 B-type boundary condition

For an even-dimensional cycle $\tilde{\gamma}$, the boundary states satisfy the B-type condition $(J_L + J_R)|\tilde{B}\rangle = 0$. Thus the coefficients $\tilde{c}^a$ for the expansion

$$|\tilde{B}\rangle = \sum_a \tilde{c}^a |a\rangle\rangle, \tag{2.3.13}$$

vanish for the (c,c) and (a,a) primary states. On the other hand, the coefficients for the (c,a) primaries are obtained by

$$\tilde{c}^a = \tilde{\eta}^{ab} \langle \tilde{0}_{top} | \tilde{\phi}_b(z,\bar{z}) | \tilde{B} \rangle_{Ramond-Ramond}, \tag{2.3.14}$$

where $\langle \tilde{0}_{top}|$ is the topological vacuum of the B-model, $\tilde{\eta}^{ab}$ is the topological metric and $\tilde{\phi}_a(z,\bar{z})$ is the (c,c) primary field associated to $\tilde{\omega}_a$ in the vertical series of the cohomologies $H^{vertical}(M) = \oplus_{q=0}^{d} H^{q,q}(M)$. The B-model is defined in such a way that G_L^+ and G_R^+ behave as one-forms[3].

By repeating the contour deformation argument as in the case of the A-type boundary condition, one finds that c^a is independent of the complex moduli y, but may depend on the Kähler moduli $(t,\bar{t})$. We now present two arguments to show that the (c,a) primary part of the boundary state

$$|\tilde{\gamma}\rangle = \sum_{\tilde{\phi}_a:(c,a)} \tilde{c}^a |a\rangle_{Ramond-Ramond}, \tag{2.3.15}$$

is "flat" with respect to the improved connection over the Kähler moduli space. This determines the $(t,\bar{t})$ dependence of $\tilde{c}^a$.

A simple way to show this is to use the mirror symmetry. Since the mirror symmetry transforms the A-type boundary condition into the B-type, the flatness property of the state $|\gamma\rangle$ over the complex moduli space for the middle-dimensional cycle γ should imply the flatness of $|\tilde{\gamma}\rangle$ over the Kähler moduli space for the even-dimensional cycle $\tilde{\gamma}$ provided γ and $\tilde{\gamma}$ are related to each other by the mirror transform.

In the next section, we will use the flatness of $|\tilde{\gamma}\rangle$ to study the mirror symmetry between the D-branes. For the sake of completeness, we therefore give another

[3]Thus the topological vacuum $\langle \tilde{0}_{top}|$ has charges $(-d/2,-d/2)$ while $\tilde{\omega}_a$ carries $(d-q,q)$. Combined, they satisfy $q_L = -q_R$ as required.

argument for the flatness which stands independently of the mirror symmetry. To take a derivative of $\tilde{c}_a$ with respect to the Kähler moduli t^i, we insert $G_L^- G_R^+ \varphi_i$ on the disk, where ϕ_i is a (c,a) primary field corresponding to an element of $H^{1,1}$. Unlike the case of the complex moduli derivative, however, this does not yet give us $D_i \tilde{c}_a$ since $G_L^- G_R^+ \varphi_i$ is divergent at the Dirichlet boundary. The covariant derivative D_i must be defined in such a way that the contribution from the boundary is removed. Since G_R^+ is a one-form in the B-model, we can deform its contour on the disk. By taking into account the boundary condition (2.3.4), one finds that $G_L^- G_R^+ \varphi_i$ becomes $\partial \varphi_i$. The integral of $\partial \varphi_i$ over the disk with the puncture reduces to two surface integrals, one around the puncture at z and another around the Dirichlet boundary. The former can be evaluated using the Yukawa coupling since it is related to the OPE of $H^{1,1}$ and $H^{q,q}$. The latter is canceled by the covariantization. This shows

$$(D_i - C_i)|\tilde{\gamma}\rangle = 0, \tag{2.3.16}$$

and similarly

$$(\bar{D}_{\bar{i}} - \bar{C}_{\bar{i}})|\tilde{\gamma}\rangle = 0. \tag{2.3.17}$$

The flatness of $|\tilde{\gamma}\rangle$ implies that the coefficient $\tilde{c}_0$ corresponding to the top cohomology $H^{d,d}$ is holomorphic with respect to the Kähler moduli. It also implies that the rest of $\tilde{c}_a$ is obtained by taking derivatives of $\tilde{c}_0$ with respect to t. Since $\tilde{c}_0$ is holomorphic in t, the instanton approximation is exact, i.e. $\tilde{c}_0$ can be expressed as a sum over holomorphic maps from the disk to M such that the boundary of the disc is mapped to the cycle $\tilde{\gamma}$. When $\tilde{\gamma}$ is $2q$-dimensional, the contribution from the constant map can be evaluated by taking the large volume limit as

$$\tilde{c}_0(\tilde{\gamma}) \sim \int_{\tilde{\gamma}} k^q + O(e^{2\pi i t}), \tag{2.3.18}$$

where $k = \sum_i t^i k_i$ and we choose k_i to be the basis of $H^{1,1}(M;Z)$.

The instanton corrections to $\tilde{c}_0$ are obtained by replacing the classical intersections in (2.3.18) by quantum ones in an appropriate sense. This in particular implies that $\tilde{c}_0$ for 0 or 2-cycle does not receive an instanton correction since the image of the holomorphic map of the disc does not intersect with the homology dual to k_i in these cycles. In the next section, we will find that this in fact is consistent with the mirror symmetry.

The expressions (2.3.18) in particular means that the large volume limit of $\tilde{c}_0$ is a homogeneous polynomial of t and the dimensions of the cycle $\tilde{\gamma}$ is characterized by the degree of the polynomial. One may be worried that this statement is not invariant under the integral shift of the theta parameters of the sigma-model, $t^i \to t^i + m^i$ $(m^i \in Z)$. In fact this shift should mix cycles of different dimensions. Consider a cycle $\tilde{\gamma} \in H_{vertical}(M;Z)$ and decompose it as

$$\tilde{\gamma} = \sum_{q=0}^{d} \tilde{\gamma}_q, \tag{2.3.19}$$

where $\tilde{\gamma}_q \in H_{q,q}(M;Z)$. The equation (2.3.18) can then be rewritten as

$$\begin{aligned} \tilde{c}_0(\tilde{\gamma}) &= \sum_q \int_M k^q \wedge \tilde{\gamma}_q^* \\ &= \int_M e^k \wedge \left(\sum_q q! \tilde{\gamma}_q^* \right), \end{aligned} \tag{2.3.20}$$

where $\tilde{\gamma}_q^* \in H^{d-q,d-q}(M;Z)$ is the Poincare dual of $\tilde{\gamma}_q$. One then finds that the shift $k \to k + \omega$ with $\omega \in H^2(M;Z)$ mixes $\tilde{\gamma}_q$'s as

$$\tilde{\gamma}_q^* \to \sum_n {}_{q+n}C_n \; \omega^n \wedge \tilde{\gamma}_{q+n}^*. \tag{2.3.21}$$

As we will see in the next chapter, this mixing is in accord with the mirror symmetry.

2.3.4 Example: boundary states for Gepner models

A Gepner model [30] can be viewed as an orbifold construction in which we project out states that do no satisfy the required conditions and add twisted sectors to the Hilbert space. This suggests that the way to construct the boundary state for a Gepner model is to take the product of the boundary states for the minimal model parts with the appropriate projection and addition of twisted sectors.

In the following we consider the simplest example: The $(k = 1)^3$ Gepner model. This corresponds to a sigma-model on T^2 with Z_3 symmetry. In this case, each minimal model can be constructed by a free boson. Thus we have ϕ_i, $i = 1, 2, 3$. Let us construct the boundary state for a D-brane wrapped on a supersymmetric 1-cycle in T^2. Imposing the A-type boundary conditions implies

$$\phi_L^i = \phi_R^i + c_i, \tag{2.3.22}$$

with constants c_i

$$c_i = \frac{2\pi}{\sqrt{3}} n_i + (0 \text{ or } \frac{2\pi}{2\sqrt{3}}), \tag{2.3.23}$$

where n_i are integers and the choice of 0 or $\frac{2\pi}{2\sqrt{3}}$ corresponds to the sign of the Ramond-Ramond charge (i.e. BPS or anti-BPS). For each choice of c_i, the boundary state is uniquely constructed by the standard oscillator procedure.

It is instructive to interpret this from the sigma-model viewpoint. The sigma-model for T^2 consists of complex free boson X and a complex free fermion ψ which are related to ϕ_i by

$$\begin{aligned} \psi &= \exp\left[\frac{i}{\sqrt{3}} (\phi_1 + \phi_2 + \phi_3) \right], \\ \partial X &= \exp\left[\frac{i}{\sqrt{3}} (-2\phi_1 + \phi_2 + \phi_3) \right] + (\text{permutations in } 1,2,3). \end{aligned} \tag{2.3.24}$$

The boundary conditions (2.3.22),(2.3.23) correspond in the sigma model to

$$\begin{aligned}
\psi_L &= \pm e^{\frac{2\pi i}{3}(n_1+n_2+n_3)}\psi_R, \\
\partial X &= e^{\frac{2\pi i}{3}(n_1+n_2+n_3)}\bar{\partial} X.
\end{aligned} \tag{2.3.25}$$

The case $n_1 + n_2 + n_3 = 0 \bmod 3$ corresponds to the Neumann boundary condition on the $\{X = \text{real}\}$ cycle of T^2, while $n_1 + n_2 + n_3 = 1$ or $2 \bmod 3$ correspond to Neumann boundary conditions on the Z_3 related 1-cycles. We see that the different choices of boundary conditions for the Gepner model correspond to the different choices of supersymmetric 1-cycles. We expect that such relations between the algebraic and the geometric structures should exist in general.

The boundary state takes the form $|B\rangle = |B\rangle_X |B\rangle_\psi$ where

$$\begin{aligned}
|B\rangle_X &= \exp\left[-e^{\frac{2\pi i}{3}(n_1+n_2+n_3)}(\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{L,-n}\alpha_{R,-n} + \text{c.c})\right]|0\rangle, \\
|B\rangle_\psi &= \exp\left[\pm i e^{\frac{2\pi i}{3}(n_1+n_2+n_3)}(\sum_{n}\psi_{L,-n}\psi_{R,-n} + \text{c.c})\right]|0\rangle.
\end{aligned} \tag{2.3.26}$$

Note that from the chiral primary states only the (c, c) ring $\{1,\ \psi_L\psi_R\}$ and its complex conjugate (a, a) ring contribute to the boundary state as expected.

2.4 Exceptional supersymmetric cycles

In previous sections we studied two types of supersymmetric cycles. Their existence and characteristics are intimately tied with the properties of Calabi-Yau manifold. Our aim in this section is to study supersymmetric cycles of exceptional type that are not complex or special Lagrangian submanifolds. For this we turn our attention to compactification manifolds with the exceptional Ricci-flat manifold with *Spin*(7) and G_2 holonomy [31]. *Spin*(7) manifold have real dimensional 8 and $G(2)$ manifold 7. They both have only one covariantly spinor on them. The analysis in §2.1.4 still applies and we look for closed forms for these manifolds that arise from the products of covariantly constant spinors on them. This is a straightforward group theoretic exercise. On *Spin*(7) one finds manifolds a self-dual 4-form known as the Cayley calibration on eight-dimensional manifolds On $G(2)$ manifolds there are the associative calibration, a 3-form, and its dual the coassociative calibration, a 4-form.

2.4.1 *Spin*(7) holonomy

Let M be an eight-manifold. A *Spin*(7) structure on M is given by a closed self-dual *Spin*(7) invariant 4-form Φ. This defines a metric g with holonomy group $Hol(g) \subset Spin(7)$. Such a metric is Ricci-flat. Compact *Spin*(7) holonomy manifolds have been constructed in [32] by resolving the singularities of T^8/Γ

orbifolds. Here T^8 is equipped with a flat $Spin(7)$ structure and Γ is a finite group of isometries of T^8 preserving that structure. On a $Spin(7)$ holonomy manifold there exists one covariantly constant spinor, which will provide us, upon compactification, with one space-time supersymmetry.

The extended symmetry algebra of sigma models on $Spin(7)$ manifolds has been found in [31]. In addition to the stress momentum tensor T and its superpartner G, it contains two operators $\tilde{X}$ and $\tilde{M}$ with spins 2 and $\frac{3}{2}$ respectively. The presence of the spin 2 operator $\tilde{X}$ may be understood along the following lines: Recall that corresponding to the covariantly constant spinor there exists a dimension $\frac{1}{2}$ Majorana-Weyl spin field operator Ψ_L mapping the Neveu-Schwarz (NS) sector to the Ramond sector. It implies the existence of a dimension 2 operator $\tilde{X}$, which is the energy-momentum tensor for the $c = \frac{1}{2}$ Majorana-Weyl fermion (Ising model), mapping the NS to NS sectors. In the large volume limit of the manifold M, $\tilde{X}$ takes the form [31]

$$\tilde{X}_L = \frac{1}{2} g_{\mu\nu} \psi_L^\mu \partial_z \psi_L^\nu + \Phi_{\mu\nu\rho\sigma} \psi_L^\mu \psi_L^\nu \psi_L^\rho \psi_L^\sigma, \tag{2.4.1}$$

with a similar formula for $\tilde{X}_R$. The ψ's in (2.4.1) are the left handed fermions in the sigma-model. This $\tilde{X}$ and its superpartner $\tilde{M}$ together with T and G make a closed algebra, and we will refer to it as the Ising superconformal algebra (ISCA).

Let us impose now the boundary conditions. In order to preserve the $N = 1$ SCA we require

$$T_L = T_R, \qquad G_L = \pm G_R. \tag{2.4.2}$$

Also, we have to preserve a linear combination of the left and right spectral flow operators. The ISCA algebra implies that

$$\tilde{X}_L = \tilde{X}_R, \qquad \tilde{M}_L = \pm \tilde{M}_R. \tag{2.4.3}$$

Thus, there is only one type of boundary condition in this case.

The conditions (2.4.2) are solved in the large volume limit by

$$\partial X^\mu = R^\mu_\nu \bar{\partial} X^\nu, \qquad \psi_L^\mu = \pm R^\mu_\nu \psi_R^\nu, \tag{2.4.4}$$

where

$$g_{\mu\nu} R^\mu_\rho R^\nu_\sigma = g_{\rho\sigma}. \tag{2.4.5}$$

Here X^μ and ψ^μ denote coordinates and vielbein one-forms on the manifold. The eigenvectors of R with eigenvalues (-1) give the Dirichlet boundary condition and thus correspond to the directions normal to the D-brane. As noted above, in the large volume limit $\tilde{X}$ takes the form (2.4.1). Using (2.4.4),(2.4.5) and (2.4.1) we see that the condition (2.4.3) reads

$$\Phi_{\mu\nu\alpha\beta} R^\mu_\rho R^\nu_\sigma R^\alpha_\gamma R^\beta_\delta = \Phi_{\rho\sigma\gamma\delta}. \tag{2.4.6}$$

One can derive this by applying the boundary condition (eq. 2.1.26) on the unique covariant spinor with the help of (eq. 2.1.18). From the same equations one readily finds that the submanifold is calibrated by Φ: it is a Cayley cycle. The physical content of (eq. 2.1.22) in this case is again that half of the space supersymmetry is preserved by the D-brane. Let type IIB string theory be compactified on $Spin(7)$ holonomy manifold. It possesses $(2,0)$ space-time supersymmetry in the flat 2d space-time. Now consider a D-brane wrapping a Cayley cycle and spanning the 2d external space-time.[1] The final supersymmetry is $(1,0)$.

2.4.2 $SU(4)$ holonomy

A Calabi-Yau 4-fold with $SU(4)$ holonomy possesses two covariantly constant spinors of the same chirality. Thus, there exist two corresponding spin field operators Ψ_L and Ψ_L^* of dimension $\frac{1}{2}$. Combined with Ψ_R and Ψ_R^* we have four spin field operators which means that type IIB string compactified on a Calabi-Yau 4-fold to $1+1$ dimensions has $(4,0)$ space-time supersymmetry.

As shown in §2.2, supersymmetric cycles of special Lagrangian and holomorphic types are associated with A and B types of boundary conditions respectively [33]. These boundary conditions preserve two linear combinations of the spin field operators $\{\Psi_L, \Psi_L^*, \Psi_R, \Psi_R^*\}$ which implies that wrapping D-branes on these cycles breaks half of the space-time supersymmetry. For Calabi-Yau 4-fold,the $(4,0)$ space-time supersymmetry for the (1+1)-dimensional external space is broken down to $(2,0)$. Unlike the others cases, there is room to break supersymmetry further down to (1,0). Can a single D-brane do this?

To answer this question, we apply the methods developed in §2.1.4. The number of supersymmetry preserved by the D-brane boundary conditions is that of independent solutions for (eq. 2.1.29). On Calabi-Yau 2 and 3-fold, the two covariantly constant spinors are correlated through an overall reality condition when tensored with spinors of the external space F to make a 10d Majorana spinor. ON Calabi-Yau 8-fold, however, they are completely independent. Therefore one find three scenarios that preserves some supersymmetry.

Let ϵ_1 and ϵ_2 be two orthonormal covariantly spinors on the 4-fold. The first possibility is

$$P_+^{p+1}\epsilon_i = 0, i = 1, 2. \tag{2.4.7}$$

Here the choice of P_+^{p+1} instead of P_-^{p+1} merely reflects a preference for the orientation of the D-brane. The calibrated forms that can come out of the left hand side of (eq. 2.1.31) are the powers of Kähler form $\hat{k}^p$ (eq. 2.2.20). Therefore the supersymmetric cycle is Kähler .

Another possibility is for

$$P_+^{p+1}\epsilon_1 = 0 = P_-^{p+1}\epsilon_2. \tag{2.4.8}$$

[1] We shall ignore the problem with uncancelled charge here.

There is a angular parameter θ in the choice of ϵ_1 and ϵ_2 that was implicit in this equation. One readily infers from this by means of (eq. 2.1.31) that the cycle is calibrated by $\hat{\Omega}_\theta$ defined in (eq. 2.2.21).

So far we reproduced the standard supersymmetric cycles for Calabi-Yau manifold. For 4-fold, however, it is possible to impose only

$$P_+^{p+1}\epsilon_1 = 0 \tag{2.4.9}$$

without any condition on ϵ_2. Clearly this means only one supersymmetry is preserved by the D-brane. Again an angular parameter θ is implicit in the above equation. The corresponding Bogomolnyi-type equation says the cycle is calibrated with respect

$$\Phi_\theta = \frac{1}{2}k^2 + Re(e^{i\theta}\Omega), \tag{2.4.10}$$

known as the Cayley calibration.

Now we turn to a boundary SCFT formulation of this supersymmetric cycle. In view of the previous results, we know a Cayley cycle in a $Spin(7)$ manifold preserve the same absolute amount of supersymmetry as this Cayley cycle in Calabi-Yau 4-fold that we are studying. Since Calabi-Yau 4-fold are special cases of Spin(7) manifold, we should the ISCA algebra in the $N = 2$ SCA and then preserve only the linear combination of spin fields in the $N = 2$ SCA that correspond the Ψ of ISCA. Thus in particular we need to preserve the spin 2 operator $\tilde{X}$ corresponding to the energy momentum tensor of the preserved spin field operator. The embedding of the algebra goes as follows:

$$T = T_{N=2}, \qquad G = G^+_{N=2} + G^-_{N=2}, \qquad \tilde{X} = \frac{1}{2}J^2 + Re(e^{i\theta}\Omega), \tag{2.4.11}$$

with $\tilde{M}$ as the superpartner of $\tilde{X}$. In the large volume limit $\tilde{X}$ takes the form

$$\tilde{X}_L = \frac{1}{2}g_{\mu\nu}\psi_L^\mu\partial_z\psi_L^\nu + (\frac{1}{2}k^2 + Re(e^{i\theta}\Omega))_{\mu\nu\rho\sigma}\psi_L^\mu\psi_L^\nu\psi_L^\rho\psi_L^\sigma, \tag{2.4.12}$$

where we used the large volume limit expressions $J_L = g_{\mu\nu}\psi_L^\mu\psi_L^\nu$ and $\Omega = \Omega_{\mu\nu\rho\sigma}\psi_L^\mu\psi_L^\nu\psi_L^\rho\psi_L^\sigma$. Equation (2.4.12) is expected since as noted in (2.4.1), $\tilde{X}$ consists of two parts: The energy momentum tensor for the fermions and the Cayley calibration form, and the latter is given in (2.4.10). Note that in fact (2.4.11) defines an S^1 family of embeddings as suggested by (2.4.10).

Let us also verify that $\tilde{X}$ is indeed the energy-momentum tensor for the Ising model. One way to see that is to bosonize the $U(1)$ current $J = i\partial_z\phi$ and use $\Omega = e^{i\phi}$. Thus,

$$\tilde{X}_L = \frac{1}{2}(\partial_z\phi)^2 + \cos(\phi+\theta). \tag{2.4.13}$$

Combining the two spin field operators as $e^{i(\phi+\theta)} = \Psi_1 + i\Psi_2$ we see that $\tilde{X}_L = \Psi_1\partial_z\Psi_1$, namely $\tilde{X}_L$ is the energy-momentum tensor for the Majorana-Weyl spinor Ψ_1 with $c = \frac{1}{2}$[2].

[2]That the energy-momentum tensor of the Ising model is given by (2.4.13) was shown in [35].

The boundary condition that corresponds to a Cayley submanifold which is neither special Lagrangian nor complex is that of (2.4.2) and (2.4.3). Thus, as we discussed, we are only preserving the energy-momentum tensor for one linear combination of spin field operators and break the rest of the $N = 2$ SCA. This leaves us with one quarter of the supersymmetry. The S^1 family of Cayley calibrations corresponds to the choice of the preserved linear combination of the spin field operators.

Until now the only known way for D-branes to break more than half of the space-time supersymmetry was to use a configuration of intersecting branes [36]. The Cayley submanifold provides the first and the only example of a supersymmetric cycle on which a single wrapped D-brane breaks three quarters of the space-time supersymmetry.

Examples

The simplest examples of supersymmetric 4-cycles can be found in flat space by explicitly solving

$$\int \frac{1}{2} k \wedge k + \mathrm{Re}(e^{i\theta} \int \Omega) = \mathrm{vol}_4. \tag{2.4.14}$$

Here:

$$\begin{aligned} \Omega &= dX^1 \wedge dX^2 \wedge dX^3 \wedge dX^4, \\ k &= dX^1 \wedge dX^{\bar{1}} + \ldots + dX^4 \wedge dX^{\bar{4}}. \end{aligned} \tag{2.4.15}$$

An example of a Lagrangian submanifold is the surface described by $X^i = X^{\bar{i}}$ for $i = 1, \ldots, 4$. In that case (eq. 2.4.14) is satisfied because the pullback of Ω satisfies

$$\partial_\alpha X^m \partial_\beta X^n \partial_\gamma X^p \partial_\delta X^q \Omega_{mnpq} = \epsilon_{\alpha\beta\gamma\delta}, \tag{2.4.16}$$

while the pullback of k vanishes.

A more complicated example, that is not in flat space, can be found as a 4-cycle in the sextic hypersurface

$$\sum_{i=1}^{6} (X^i)^6 = 0. \tag{2.4.17}$$

in CP^5. This 4-cycle is the four-dimensional submanifold on which all the X^i's are real [37, 24].

An example of a complex submanifold is given by the surface described by $X_3 = X_4 = 0$. Here the pullback of Ω vanishes and the pullback of $k \wedge k$ is

$$\partial_\alpha X^m \partial_\beta X^n \partial_\gamma X^{\bar{p}} \partial_\delta X^{\bar{q}} k_{m\bar{p}} k_{n\bar{q}} = 2\epsilon_{\alpha\beta\gamma\delta}, \tag{2.4.18}$$

so that (eq. 2.4.14) holds.

An example of a Cayley geometry, for which both the pullback of the holomorphic 4-form and the pullback of $k \wedge k$ are non-vanishing is described by

$X^2 = \sqrt{2}e^{i\varphi}(X^1 + X^{\bar{1}})$ and $X^4 = \sqrt{2}e^{i\varphi}(X^3 + X^{\bar{3}})$, for every value of the angle φ. More generally, every Cayley plane that is neither special Lagrangian nor holomorphic will give an example of this type.

2.4.3 G_2 holonomy

Let M be an seven-manifold. A G_2 structure on M is given by a closed G_2 invariant 3-form Φ. This defines a metric g with holonomy group $Hol(g) \subset G_2$. Such a metric is Ricci-flat. Compact G_2 holonomy manifolds have been constructed in [38, 39] in analogy with the $Spin(7)$ holonomy case by resolving the singularities of T^7/Γ orbifolds. Here T^7 is equipped with a flat G_2 structure and Γ is a finite group of isometries of T^7 preserving that structure. On a G_2 holonomy manifold there exists one covariantly constant spinor. The 3-form Φ and its Hodge dual 4-form ${}^*\Phi$ define the associative and coassociative calibrations respectively.

The extended symmetry algebra of sigma models on G_2 manifolds has been constructed in [31]. In addition to the stress tensor T and its superpartner G, it contains the superpartners (K, Φ) with spins $(2, \frac{3}{2})$ and (X, M) with spins $(2, \frac{5}{2})$. In the large volume limit, Φ corresponds to the associative calibration 3-form and X is the sum of the coassociative calibration 4-form ${}^*\Phi$ and the stress tensors for seven Majorana-Weyl fermions. In analogy with the $Spin(7)$ holonomy case where we viewed $\tilde{X}$ as the stress tensor corresponding to the dimension $\frac{1}{2}$ spin field operator, here we can view X as the stress tensor corresponding to the dimension $\frac{7}{16}$ spin field operator which is the that of the $c = \frac{7}{10}$ tri-critical Ising model.

In addition to the $N = 1$ boundary condition (2.4.2), the G_2 algebra implies the boundary conditions

$$\begin{aligned} \Phi_L &= \Phi_R, \qquad K_L = \pm K_R, \\ X_L &= X_R, \qquad M_L = \pm M_R. \end{aligned} \tag{2.4.19}$$

In the large volume limit we have

$$\Phi_L = \Phi_{ijk}\psi_L^i\psi_L^j\psi_L^k, \qquad X_L = \frac{1}{2}g_{ij}\psi_L^i\partial_z\psi_L^j +^* \Phi_{ijkl}\psi_L^i\psi_L^j\psi_L^k\psi_L^l. \tag{2.4.20}$$

Thus the boundary conditions (2.4.19) take the form

$$\Phi_{ijk}R_l^iR_m^jR_n^k = \Phi_{lmn}, \qquad {}^*\Phi_{ijkl}R_m^iR_n^jR_o^kR_p^l =^* \Phi_{mnop}, \tag{2.4.21}$$

These conditions can also be found in the familiar vein by requiring

$$P_+^{p+1}\epsilon = 0 \tag{2.4.22}$$

from the unique covariantly constant spinor *epsilon*. Therefore these cycles inherit one half of the space-time supersymmetry endowed by a $G(2)$ manifold. Product of ϵ with itself generate Φ and $*\Phi$. Thus it also follows that the supersymmetric cycles can be either 3 or 4 dimensional. On a 3-cycle Φ is the volume form while on a 4-cycle ${}^*\Phi$ is the volume form. They are the associative and coassociative cycles respectively.

Chapter 3

Applications to Mirror Symmetry

In §1.3.1, we reviewed T-duality, an equivalence between two string theories on different geometric background and D-branes. In general, it requires the existence of a compact Abelian isometry. For most curved backgrounds, such as generic Calabi-Yau manifolds, this is not available. However, a refined version of it does exist for Calabi-Yau. It is known as the **mirror symmetry** for superstring theory. It is to be distinguished from the mirror symmetry for 3d supersymmetric gauge theory which we shall turn to in §5, but we shall refer to it simply as mirror symmetry in this chapter.

To see how it works, recall that string theory on a Calabi-Yau manifold possesses $N = 2$ SCA. To be precise it contains two copies of it, one each for the left and right movers respectively. Each has a Z_2 automorphism of particular interest. It simply amounts to changing the sign of the $U(1)$ R-current and hence exchanging G^+ with G^-. It is the same automorphism that we used to distinguish between the A and B type of boundary conditions in §2.2. At the level of conformal field theory, this automorphism is trivial. However, it does have an impact on the space-time interpretation, much like the more mundane T-duality. Given a conformal field theory describing string propagating on Calabi-Yau manifold M. If we act on the right mover the nontrivial element of the Z_2 automorphism, the resulting theory describe string propagating on a Calabi-Yau manifold $\widetilde{M}$. In general M and $\widetilde{M}$ are different [40].[1]

Like T-duality, mirror symmetry is established perturbatively at each level of string loop. If it also holds nonperturbatively, it must map nonperturbative objects such as D-branes of the dual theories in a well defined and consistent manner. Interestingly, its perturbative representation as an isomorphism between the Hilbert spaces of the sigma-models on M and $\widetilde{M}$ provides a mean to study this nonperturbative effect. If the cycles γ and $\tilde{\gamma}$ are related to each other by mirror symmetry, the corresponding boundary states $|B\rangle$ and $|\tilde{B}\rangle$ should be identified by the isomorphism[2].

Mirror symmetry transforms type IIA string on a Calabi-Yau 3-fold M into type IIB string on the mirror $\widetilde{M}$. Since type IIA string has even-dimensional D-branes while type IIB has odd-dimensional ones, we expect that mirror symmetry to transform middle ($= 3$) dimensional cycles on M into even-dimensional cycle

[1]In some cases $\tilde{M}$ might not exist as a manifold although M does. However, the dual conformal field theory still makes sense.

[2]To be precise, the boundary state $|B\rangle$ does not belong to the Hilbert space since it is not normalizable. This problem can be easily avoided by considering $q^{L_0}\bar{q}^{\bar{L}_0}|B\rangle$ for $|q| < 1$, for example.

on $\widetilde{M}$. From the point of view of SCFT, mirror symmetry transforms the A-type boundary condition (2.3.3) for the 3-cycle to the B-type boundary condition (2.3.4) for the even-dimensional cycle. In this section, we will examine how this transformation between the supersymmetric cycles takes place. The analysis of the boundary state in §2.3 will be our main tool.

It has been observed that, for a Calabi-Yau 3-fold M, mirror symmetry not only maps the even cohomology of M to the odd cohomology of its mirror $\widetilde{M}$ (with complex coefficient), as an Hilbert space isomorphism would require, but it does so while respecting the integral structure of the cohomologies [41]. Based on this, it was conjectured by Aspinwall and Morrison [42] that the Ramond-Ramond field on a Calabi-Yau space must have a certain periodicity reflecting this integral structure. This way, the mirror map can be extended to the Ramond-Ramond field configurations. We will verify that this conjecture is consistent with the mirror map between D-brane configurations. This by itself is an evidence for the nonperturbative validity of mirror symmetry for it means mirror symmetry commutes with charge quantization conditions for the Ramond-Ramond fields.

The precise understanding of mirror symmetry between D-branes enables us to study open string worldsheet instanton effects. We will find that the chiral primary part of the boundary states for 0, 2 and 3-cycles in a Calabi-Yau 3-fold does not receive instanton corrections while the instanton corrections for 4 and 6-cycles can be expressed in term of the closed string worldsheet instantons on the same manifold.

Section 1 will be devoted to the analysis of the mirror transformation of D-brane configurations. In section 2 we will present examples where mirror symmetry is realized as T-duality on tori and Calabi-Yau orbifolds. In section 3 we discuss the implications of the exceptional supersymmetric cycles found in §2.4 for mirror symmetry in higher dimensions.

3.1 Mirror map between cycles

Suppose the boundary state $|B\rangle$ for a 3-dimensional cycle γ in M is mapped to the boundary state $|\tilde{B}\rangle$ for an even-dimensional cycle $\tilde{\gamma}$ in $\widetilde{M}$ under the mirror transformation. Since the chiral primary part of the boundary states are characterized by c_0 and $\tilde{c}_0$ given in the previous section, they should be related to each other under the mirror map. For the 3-cycle γ, c_0 is given by

$$c_0(\gamma) = \int_\gamma \Omega. \tag{3.1.1}$$

Since we know the large volume limit of $\tilde{c}_0$ as in (2.3.18), we should compare it with c_0 in the corresponding limit, which is called the large complex structure limit [43] of M.

In this limit, $H^{0,3}(M)$ aligns with the lattice of $H^3(M;Z)$ [26], [41]. Thus we have a filtration of $H^3(M;Z)$ in a form of

$$H^{0,3} \subset H^{0,3} \oplus H^{1,2} \subset H^{0,3} \oplus H^{1,2} \oplus H^{2,1} \subset H^3(M;Z), \qquad (3.1.2)$$

called the monodromy weight filtration [44]. Accordingly we can choose a symplectic basis $\{\alpha_I, \beta^I\}_{I=0,\dots,h^{2,1}}$ for $H_3(M;Z)$,

$$\alpha_I \cap \alpha_J = 0, \quad \beta^I \cap \beta^J = 0, \quad \alpha_I \cap \beta^J = \delta_I^J, \qquad (3.1.3)$$

such that α_0 is the unique cycle dual to $H^{0,3}$ and $\{\alpha_0, \dots, \alpha_{h^{2,1}}\}$ spans the dual of $H^{0,3} \oplus H^{1,2}$. The cycle α_0 may also be characterized by the fact that it is invariant under the monodromy of $H_3(M;Z)$ at the large complex structure limit [45], [46]. Note, on the other hand, α_i with $i = 1, \dots, h^{2,1}$ may be shifted by α_0 under the monodromy transformation.

With this choice of the basis for H_3, the flat coordinates of the complex moduli space are given by

$$s^i = \frac{X^i}{X^0} \quad (i = 1, \dots, h^{2,1}(M) = h^{1,1}(\widetilde{M})), \qquad (3.1.4)$$

where

$$X^0 = \int_{\alpha_0} \Omega, \quad X^i = \int_{\alpha_i} \Omega. \qquad (3.1.5)$$

In the large complex structure limit $s \to \infty$ the Schmid orbit theorem [47] yields

$$\begin{aligned} c_0(\beta^0) = \int_{\beta^0} \Omega &= \frac{1}{3!} X^0 d_{ijk} s^i s^j s^k + \cdots, \\ c_0(\beta^i) = \int_{\beta^i} \Omega &= -\frac{1}{2!} X^0 d_{ijk} s^j s^k + \cdots, \end{aligned} \qquad (3.1.6)$$

where d_{ijk} is the large complex structure limit of the Yukawa coupling.

In order to construct the mirror map, we choose the standard gauge of the special geometry,

$$c_0(\alpha_0) = \int_{\alpha_0} \Omega = 1. \qquad (3.1.7)$$

In this gauge, the flat coordinates are

$$c_0(\alpha_i) = \int_{\alpha_i} \Omega = s^i. \qquad (3.1.8)$$

By the mirror map, we may also use it as the flat coordinates for the Kähler moduli space of $\widetilde{M}$. In the large complex structure limit, this mirror symmetry maps the Yukawa coupling d_{ijk} in (3.1.6) to

$$d_{ijk} = \int_{\widetilde{M}} k_i \wedge k_j \wedge k_k. \qquad (3.1.9)$$

By comparing large volume limit (2.3.18) of $\tilde{c}_0$ for even-dimensional cycles in $\widetilde{M}$ with the large complex structure limit (3.1.6) – (3.1.9) of c_0 for $\{\alpha_I, \beta^I\}$, we can immediately see how mirror symmetry transforms a D-brane wrapping on a 3-cycle in M to a D-brane wrapping on an even-dimensional cycle in $\widetilde{M}$. In particular, the 3-cycle α_0 dual to $H^{0,3}$ in M is a mirror image of a 0-cycle in $\widetilde{M}$, and the 3-cycles α_i ($i = 1, ..., h^{1,2}$) correspond to 2-cycles in $\widetilde{M}$. Thus the mysterious correspondence between the integral structures of $H^3(M)$ and $H^{vertical}(\widetilde{M})$ pointed out in [41] is now understood as the mirror map between the D-brane configurations.

After the research reported in this chapter is finished, we received a preprint [48] by Strominger, Yau and Zaslow where it is argued that the mirror of a 0-cycle in $\widetilde{M}$ should be a *toroidal* 3-cycle in M. Our analysis here shows a mirror of the 0-cycle should be the 3-cycle α_0 dual to $H^{0,3}$ in the large complex structure limit of M. In the case of the quintic defined by,

$$p(x) = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - 5\psi x_1 x_2 x_3 x_4 x_5 = 0, \tag{3.1.10}$$

such a 3-cycle is in fact known to be T^3 [49]. In the large complex structure limit $\psi \to \infty$, the holomorphic 3-form becomes

$$\Omega = 5\psi \frac{x_5 dx_1 \wedge dx_2 \wedge dx_3}{\partial p / \partial x_4} \to -\frac{dx_1 dx_2 dx_3}{x_1 x_2 x_3}, \tag{3.1.11}$$

and the 3-cycle dual to $\bar{\Omega}$ is T^3 surrounding $x_1 = x_2 = x_3 = 0$. It would be very interesting to see whether this feature of $H^{0,3}$ is true for a general M with a mirror partner.

So far we have only looked at the large volume limit of $\widetilde{M}$ and the corresponding large complex structure limit of M. Fortunately, since the state $|\tilde{\gamma}\rangle$ and $|\tilde{\gamma}\rangle$ are flat sections over the moduli spaces, their correspondence can be traced to interiors on the moduli spaces following the mirror map. We will demonstrate this through examples in section 5. If we go around a non-trivial cycle over the moduli space, we have to deal with the monodromy problem, which we will discuss below.

3.1.1 Open string instantons

For the A-type boundary condition, the classical formula

$$c_0(\gamma) = \int_\gamma \Omega \tag{3.1.12}$$

is exact. On the other hand, the formula (2.3.18) for $\tilde{c}_0(\tilde{\gamma})$ for the B-type boundary condition is corrected by open string worldsheet instantons, i.e. holomorphic maps from a disk to $\widetilde{M}$ such that the boundary of the disk is mapped to the cycle $\tilde{\gamma}$. Mirror symmetry suggests that such open string instanton effects are expressed in terms of the closed string instantons on $\widetilde{M}$.

Mirror symmetry gives another proof for the fact that the formula (2.3.18) for $\tilde{c}_0$ does not receive the instanton correction when the cycle $\tilde{\gamma}$ is 0 or 2-dimensional. This is because the corresponding formulae (3.1.7) and (3.1.8) for α_I ($I = 0, ..., h^{2,1}$) are, by definition, exact.

On the other hand, $\tilde{c}_0$ for 4 or 6-cycle does receive instanton corrections. In the mirror picture, the exact formulae for $c_0(\gamma)$ for β^I $(I = 0, ..., h^{2,1})$ can be written in terms of the prepotential $\mathcal{F}$ for M as

$$c_0(\beta^0) = 2\mathcal{F} - s^i \frac{\partial}{\partial s^i}\mathcal{F},$$
$$c_0(\beta^i) = \frac{\partial}{\partial s^i}\mathcal{F}, \tag{3.1.13}$$

where we are working in the $X^0 = 1$ gauge appropriate for mirror symmetry. In $\widetilde{M}$, the prepotential is related to the sum over closed string instantons as[1]

$$-\frac{\partial^3}{\partial s^i \partial s^j \partial s^k}\mathcal{F} = d_{ijk} + \sum_n N(n) n_i n_j n_k \frac{e^{2\pi i n_i s^i}}{1 - e^{2\pi i n_i s^i}}, \tag{3.1.14}$$

where $N(n)$ is the number of rational curves on $\widetilde{M}$ of the type $n = \{n_1, ..., n_{h^{1,1}(\widetilde{M})}\}$. By integrating this, we find

$$\mathcal{F} = -\frac{1}{3!} d_{ijk} s^i s^j s^k + a - \sum_n \sum_{m=1}^{\infty} \frac{N(n)}{(2\pi i m)^3} e^{2\pi i m n_i s^i}, \tag{3.1.15}$$

where a is a constant, presumably related to the four-loop term in the β-function of the sigma-model [49]. Substituting this into (3.1.13), we can extract the open string instanton corrections to $\tilde{c}_0$ and express them in terms of of the number of the closed string instantons $N(n)$.

This suggests a relation between the moduli spaces of open and closed string instantons and the corresponding intersection theories. One way to find such a relation may be to regard a closed string instanton intersecting a supersymmetric cycle as a pair of open string instantons glued on the cycle.

3.1.2 Integral structure and monodromy

It has been observed that in the large radius limit, mirror symmetry maps the integer valued homology $H_3(M; Z)$ to $\oplus_q H_{2q}(\widetilde{M}; Z)$ in such a way that the monodromy is preserved [41], [45]. Based on this, it was conjectured by Aspinwall and Morrison [42, 46] that the Ramond-Ramond fields on the Calabi-Yau 3-fold should have periodicity under the discrete shift reflecting these integral structures. This would guarantee that mirror symmetry can be extended to the Ramond-Ramond fields configurations. This periodicity should be a consequence of the coupling of the Ramond-Ramond field to the worldvolume of the D-brane. In fact the mirror map between the D-branes we found in the above is consistent with this picture.

[1]We are using the same coordinates s^i for both the complex moduli of M and the Kähler moduli of $\widetilde{M}$ related to each other by mirror symmetry.

By requiring that the monodromy be preserved, Morrison also pointed out [46] that the shift of the NS-NS B-field by $H^2(M; Z)$ should cause a certain rearrangement of the integral structure of the Ramond-Ramond fields of even ranks. This is also consistent with the mixing of the even dimensional cycles we found in (2.3.21).

Although the mixing of the cycles is required by mirror symmetry, one can also explain it without invoking the mirror. For the sigma-model without a boundary, the shift of B-field by $H^2(M; Z)$ is a discrete symmetry. However, in the presence of a boundary, the coupling of the B-field to the string world-sheet is accompanied by the coupling of a $U(1)$ gauge field A to the boundary [50]. Since the gauge invariant field strength is $\mathcal{F} = F - B$ where $F = dA$, the shift $B \to B + \omega$ with $\omega \in H^2(M; Z)$ is compensated by $F \to F + \omega$. This effectively mixes cycles of different dimensions as in (2.3.21). Below we will demonstrate this explicitly through examples.

3.2 Two case studies

In this section we will present several examples to illustrate the general results of the previous sections. We will show explicitly how starting with a D-brane wrapped on a middle-dimensional supersymmetric cycle, depending on the D-brane configuration and T-duality or mirror transformation, we can obtain different dimensionalities for the dual configuration with gauge fields background.

3.2.1 T-duality on tori

Let us start with a general discussion of the duality map for tori and orbifolds. As we discussed in section 2, the condition for $N = 1$ SCA yields

$$\partial X^\mu = R^\mu{}_\nu \bar{\partial} X^\nu, \tag{3.2.1}$$

where R is an orthogonal matrix. The requirement for having a geometrical interpretation of a D-brane without gauge fields background is more restrictive and implies that R has to be a symmetric matrix and squares to the identity matrix. In this case, its eigen-values are $(+1)$ or (-1) corresponding to the tangential and normal vectors to the D-brane respectively. To preserve the $N = 2$ SCA, R should further obey

$$k_{\mu\nu} R^\mu{}_\rho R^\nu{}_\sigma = \pm k_{\rho\sigma}, \tag{3.2.2}$$

where $\pm$ refers to the A and B-type boundary conditions and thus to middle and even-dimensional cycles.

T-duality transformation is realized by

$$\partial X^\mu \to \partial X^\mu, \qquad \bar{\partial} X^\mu \to T^\mu_\nu \bar{\partial} X^\nu, \tag{3.2.3}$$

where T is the symmetric matrix implementing the duality transformation and $T^2 = 1$. In order for this to induce the mirror transformation, the sign of J_R

should be reversed while J_L remain invariant. This means

$$k_{\mu\nu}T^{\mu}_{\ \rho}T^{\nu}_{\ \sigma} = -k_{\rho\sigma}. \tag{3.2.4}$$

Thus, starting with a D-brane configuration and performing T-duality transformation we will end up with a configuration satisfying the boundary condition

$$\partial X^{\mu} = \tilde{R}^{\mu}_{\ \nu}\bar{\partial}X^{\nu}, \tag{3.2.5}$$

where $\tilde{R} = RT$ is an orthogonal matrix. If the matrix $\tilde{R}$ is symmetric and thus squares to the identity matrix, the boundary condition has geometrical realization as a D-brane without the $U(1)$ gauge field. This occurs if and only if

$$[R, T] = 0, \tag{3.2.6}$$

namely T-duality transformation commutes with the original D-brane configuration.

When (3.2.6) is not satisfied, we get a mixing between the Neumann and Dirichlet boundary conditions of the type induced by a background gauge field. Since $\tilde{R} = RT$ is orthogonal, by a coordinate transformation, we can alway bring it into the standard form,

$$\tilde{R} = \begin{pmatrix} -1_{(2d-p)\times(2d-p)} & 0 \\ 0 & \left(\frac{1-F}{1+F}\right)_{p\times p} \end{pmatrix}, \tag{3.2.7}$$

where for some p and an anti-symmetric matrix F. This implies the Dirichlet boundary condition for the first $(2d-p)$ directions, while the boundary condition for the second p directions is

$$\partial X^{\mu} = \left(\frac{1-F}{1+F}\right)^{\mu}_{\ \nu}\bar{\partial}X^{\nu}. \tag{3.2.8}$$

Therefore the matrix $\tilde{R}$ describes a p-cycle with a background gauge field F.

Whether F is zero or not, mirror symmetry exchanges odd and even-dimensional cycles when $d = \dim_C M$ is odd. In this case, the condition (3.2.4) for T-duality to be the mirror symmetry implies $\det T = -1$. On the other hand, $\det R = -1$ for an odd dimensional cycle since the rotation matrix $(\frac{1-F}{1+F})$ has determinant $(+1)$. Thus $\tilde{R} = RT$ for its mirror obeys $\det\tilde{R} = \det R \cdot \det T = +1$, i.e. the mirror of the odd dimensional cycle is even-dimensional. If R and T commute, $F = 0$ in the original cycle implies $F = 0$ for its mirror.

Let us construct now a simple example to illustrate the above. Consider the torus T^2 with real coordinates (x, y), and a D-brane configuration defined by the Pauli matrix $R = \sigma_1$. The Neumann boundary condition is imposed on the 1-cycle defined by the vector $(1, 1)$, while the Dirichlet boundary condition is imposed on the vector orthogonal to it. Then the mirror transformation is generated by T-duality transformation along the x coordinate, i.e. $T = -\sigma_3$. Clearly this T does not commute with R. In fact $\tilde{R} = -i\sigma_2 = F$, and this has no (-1) eigen-value, namely there are no Dirichlet boundary conditions. The configuration we got is that of a 2-cycle with background gauge field F.

It is instructive to consider this example from a different viewpoint. In the limit of the large complex structure, $\tau \to i\infty$, the cohomology $H^{0,1}$ generated by $d\bar{z} = dx + \bar{\tau}dy$ gets aligned with the lattice $H^1(T^2; Z)$ generated by dx and dy. In this limit, the cycle $(1,0)$ becomes dual to $H^{0,1}$ and the mirror map transforms it to a 0-cycle, as expected. On the other hand, either $(0,1)$ or $(1,1)$ can be combined with $(1,0)$ to make the symplectic basis of $H^1(T^2; Z)$. Since $(0,1)$ is mirror to a 2-cycle without a gauge field, one may regard $(1,1) = (0,1) + (1,0)$ as mirror to the 2-cycle with a 0-cycle on it. Though the filtration $H^{0,1} \subset H^1(T^2; Z)$ makes sense only in the large complex structure limit, the mirror map between the cycles holds even for finite value of τ. The reason for this can be traced back to the fact that the chiral primary part of the boundary state $|\gamma\rangle$ is a flat section over the moduli space of complex structure, as we explained in section 4.

This picture is correct as far as the homology goes, but a sum of the straight lines, $(0,0) \to (1,0)$ and $(1,0) \to (1,1)$, is not actually supersymmetric since the combined cycle is not minimal. The diagonal line $(0,0) \to (1,1)$ is shorter and thus costs less energy. In the mirror picture, this means that the 2-cycle with the $U(1)$ gauge field should be regarded as a ground state of the 0-cycle on the 2-cycles.

This simple example illustrates the mixing of cycles (2.3.21). The D-brane worldvolume action has terms of the form [51]

$$S = \int_{2-cycle} C_0 \mathcal{F} + C_2, \tag{3.2.9}$$

where C_0 and C_2 are the Ramond-Ramond fields and $\mathcal{F} = F - B$. A shift of B by $H^2(T^2; Z)$ then mixes C_0 and C_2 corresponding to the mixing of cycles. In the mirror picture, the shift $B \to B + 1$ becomes the modular transformation $\tau \to \tau + 1$. This sends the cycle $(0,1)$ (the 2-cycle in the mirror) to $(1,1)$ (the 0-cycle on the 2-cycle in the mirror). Thus the mixing of the cycle (2.3.21) is natural from the point of view of the coupling of the D-brane to the B field [50] as well as the mirror symmetry.

3.2.2 Calabi-Yau orbifold

In this section we discuss an example of a mirror pair of Calabi-Yau orbifolds. In fact the phenomena is basically similar to the tori cases, with some technicality related to the correct choice of a ground state. As an explicit example we will consider the mirror of the Calabi-Yau orbifold $(T^2)^3/(Z_2 \times Z_2)$ which is constructed by the inclusion of a discrete torsion [52]. Let us first discuss the orbifold without a discrete torsion. The Calabi-Yau orbifold $(T^2)^3/\Gamma$ where $\Gamma = Z_2 \times Z_2$ is defined by $z_i \to (-1)^{\varepsilon_i} z_i,\;\; i = 1,2,3$ such that $\prod_i (-1)^{\varepsilon_i} = 1$. Supersymmetric 2 cycles can constructed by projecting a T^2 in $(T^2)^3$ with respect to Γ. Similarly, supersymmetric 4-cycles can be obtained by projecting a product of two T^2's with respect to Γ. The even-dimensional supersymmetric cycles are interesting in this example since the twisted Ramond ground states contribute to $H^{1,1}$ and $H^{2,2}$. Thus the latter can show up in their boundary states.

Consider, for instance, a 2-cycle boundary state where Neumann boundary conditions are imposed on the z_3 coordinate and Dirichlet boundary conditions on z_1, z_2. Orbifold boundary states are simply constructed as a sum of contributions from the untwisted and twisted sectors

$$|B\rangle_{orbifold} = |B\rangle_{untwisted} + \sum_{twisted\ sectors} |B\rangle_{twisted}, \tag{3.2.10}$$

with an appropriate projection on invariant states.

untwisted sector:

The boundary state takes the form

$$|B\rangle_{untwist} = \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{i=1}^{2}\alpha^i_{L,-n}\alpha^i_{R,-n} + \alpha^{*3}_{L,-n}\alpha^3_{R,-n}\right) + \text{c.c}\right)|0\rangle, \tag{3.2.11}$$

and projection is not required since the boundary state is Γ-invariant. The fermionic part works similarly.

twisted sectors:

There exist three twisted sectors corresponding to the three Γ group elements. Consider, for instance, the twisted sector corresponding to the generator α, $\alpha(z_1, z_2, z_3) = (-z_1, -z_2, z_3)$, where the β and γ are defined by a permutation of the signs. This implies half integer moding for the first two coordinates and integer moding for the third. The other twisted sectors are simply permutations of that.

Let us consider now the inclusion of a discrete torsion. This simply amounts to a change in the projection operators in the twisted sectors. Thus in the sector twisted by α it amounts to an inclusion of another minus sign in the transformation of states under $z_3 \to -z_3$. This has the effect that only twisted Ramond ground states that contribute to $H^{1,2}$ and $H^{2,1}$ survive the projection. Thus we end up with a Hodge diamond mirror to that of the orbifold without discrete torsion. It was argued in [52] that these indeed constitute a mirror pair, where the mirror map is T-duality.

Upon inclusion of a discrete torsion, the interesting supersymmetric cycles are the middle dimensional ones. The construction of a boundary state is standard and we can follow the duality map. There is, however, a delicate point. The discrete torsion changes the projection operator, and for instance in the α twisted sector it takes the form

$$P = \frac{1}{4}(1 + \alpha - \beta - \gamma), \tag{3.2.12}$$

which naively annihilates the twisted sector boundary state. This is resolved by picking the correct ground state. Consider the Ramond sector: Related to z_3 we have the fermionic zero modes $\psi^3_{L,0}, \psi^3_{R,0}$ with the boundary condition

$$(\psi^3_{L,0} + i\eta\psi^{3*}_{R,0})|0\rangle = 0, \qquad (\psi^{3*}_{L,0} + i\eta\psi^3_{R,0})|0\rangle = 0, \tag{3.2.13}$$

with $\eta = \pm 1$.

Of the possible Ramond ground states only $(i\eta\psi^3_{L,0} + \psi^3_{R,0} + \text{c.c})|0\rangle$ survives the projection and should be picked. This is to be contrasted with the case without discrete torsion where the correct twisted sector Ramond ground state is $(i\eta + \psi^3_{L,0}\psi^3_{R,0} + \text{c.c})|0\rangle$.

Consider now the D-brane matrix $R = diag[\sigma_1, \sigma_1, \sigma_1]$. A mirror symmetry transformation is defined by:

$$\partial z_i \to \partial z_i, \qquad \bar{\partial} z_i \to \bar{\partial} \bar{z}_i. \tag{3.2.14}$$

Thus the matrix T takes the form $T = diag[\sigma_3, \sigma_3, \sigma_3]$ and does not commute with R. Since both R and the mirror symmetry T are equivariant with respect to the $Z_2 \times Z_2$ discrete group, the same applies for the Calabi-Yau orbifold $(T^2)^3/(Z_2 \times Z_2)$, and we get the mixing phenomena as we discussed before.

In the orbifold models, we may consider gauge field strength which belongs to the twisted sectors, namely localized on a particular fixed point. In this case we should expect that the particular twisted sector corresponding to this fixed point will be influenced. Thus, we are led to consider different boundary conditions R in (3.2.1) for the untwisted and twisted sectors. It would be interesting to further explore this structure.

3.3 Mirror symmetry in higher dimensions

Mirror symmetry for 4-folds has several new features which distinguish it from the three-dimensional case [29]. Mirror symmetry is expected to map $H^4(X) = \bigoplus_p H^{p,4-p}(X)$ to $\bigoplus_p H^{p,p}(Y)$ and $\bigoplus_p H^{p,p}(X)$ to $H^4(Y) = \bigoplus_p H^{p,4-p}(Y)$; one of the new features is that $H^{2,2}(X)$ appears in both of these spaces. (These spaces were referred to as the "horizontal" and "vertical" cohomology in [29].)

The special Lagrangian submanifolds of X define classes in $H^4(X)$ which lie in the so-called *primitive cohomology*, that is, they are classes which are orthogonal to the Kähler class. Since the classes of special Lagrangian submanifolds are also classes in integer cohomology, the natural space to consider for these manifolds is $H^4(X)_{prim} \cap H^4(X, \mathbb{Z})$. It is not clear how much of this space will actually be represented by special Lagrangian submanifolds.

On the other hand, the complex submanifolds of X define classes which have Hodge type (p,p) and are also integer cohomology classes; the natural space to consider for them is $\bigoplus_p H^{p,p}(X) \cap H^{even}(X, \mathbb{Z})$. The celebrated "Hodge conjecture" in mathematics asserts that if we pass to $\mathbb{Q}$-coefficients instead of $\mathbb{Z}$-coefficients, then all classes in this space are represented by complex submanifolds; it is not known if this conjecture holds for Calabi–Yau 4-folds.

We are thus faced with the situation of having an unknown subspace of $H^4(X)_{prim} \cap H^4(X, \mathbb{Z})$ represented by special Lagrangian submanifolds, and an unknown subspace of $\bigoplus_p H^{p,p}(X) \cap H^{even}(X, \mathbb{Z})$ represented by complex submanifolds. In fact, it is quite possible that the appropriate pieces of these subspaces fall short of filling out all of $H^{2,2}(X)$ (even though both will contribute subspaces of $H^{2,2}(X)$). Cayley submanifolds provide another potential source of cohomology classes which could help to fill out $H^{2,2}(X)$: it may be that some of the classes which cannot be represented by either special Lagrangian or complex submanifolds will instead be represented by Cayley submanifolds.

Such a possibility meshes well with mirror symmetry: we observe that the mirror of a Cayley submanifold will be another Cayley submanifold. (This is because any D-brane on X—which defines some type of boundary condition for open strings—should map to a D-brane on Y.) If the first Cayley submanifold is neither special Lagrangian nor a complex submanifold, then since it preserves only 1/4 of the supersymmetry, its mirror will have the same property. It would be interesting to find explicit examples of this phenomenon.

Finally, we would like to mention an implication for mirror symmetry in higher dimensions that becomes evident by considering the spectrum of BPS soliton states. Strominger, Yau and Zaslow [48] showed that every Calabi-Yau 3-fold that has a mirror admits a supersymmetric T^3-fibration. The basic assumption of this argument is quantum mirror symmetry [53, 42, 24, 46], where the isomorphism between the type IIA theory compactified on a 3-fold X and the IIB theory compactified on the mirror Y of X is extended to the non-perturbative BPS states in $D = 4$. Since these BPS states are constructed as D-branes, the quantum mirror symmetry is actually a consequence of the classical mirror symmetry of the bulk CFT [33]. It is then natural to wonder if the previous argument can be extended to higher dimensional Calabi-Yau manifolds. Some precise mathematical aspects of this generalization have been recently considered in [54]. We consider the type IIA theory compactified on a large Calabi-Yau n-fold X and its mirror Y. Quantum mirror symmetry implies that both theories are isomorphic. On the X side there are BPS objects in $D = (10 - 2n)$ which arise from the ten-dimensional 0-brane. These states arise from a supersymmetric n-brane wrapping a n-cycle in Y. This n-cycle corresponds to a special Lagrangian submanifold. This is because the 0-brane corresponds to B-type boundary conditions and by mirror symmetry these are transformed to the A-type boundary conditions that correspond to the special Lagrangian submanifold [33]. Extending the arguments of [48] to n-folds, we arrive at the conclusion that the n-cycles corresponding to special Lagrangian submanifolds are toroidal. This leads us to the conclusion that every Calabi-Yau n-fold that has a mirror admits a supersymmetric T^n-fibration. This suggests that mirror symmetry for the n-fold is equivalent to a T-duality on the T^n-fibers.

Chapter 4

Consistency of D-Brane Configurations

4.1 Anomaly, the dark secret of brane engineering

In recent studies of string theory, configurations of curved D-branes and/or intersecting D-branes play a very important role. Examples of the latter type of **brane engineering** at work are as presented in the next chapter. The low energy physics of those configurations are that of field theories, which often possess both gauge and global symmetries. In such constructions, some global symmetries, usually the R symmetries that act on the supercharges, originate from the rotation symmetry of the bulk string theory restricted to the normal bundles of the branes. They are gauged in the bulk space-time and therefore must be free of anomalies, just as the symmetries gauged on the branes. However, there is generically chiral asymmetry with respect to these global symmetries on a D-brane or the intersection of a pair of D-branes, known as an **I-brane**. It brings about pure and mixed anomalies involving these global symmetries in the effective brane worldvolume theory. If this were the only story, such brane configurations would be inconsistent.

The mechanism to cancel the anomaly in an otherwise anomalous theory is to compensate it with an "anomalous" variation of the classical action. An example is the Green-Schwarz mechanism for type I and heterotic string theories [55]. More generally, the anomalous theory can be embedded in a higher dimensional theory. The anomalous variation of the classical action of the bigger theory is localized at ("flows" to) the worldvolume for the anomalous theory and cancels its anomaly, hence the name **anomaly inflow** [56, 57]. More recently it has been applied to derive the Chern-Simons type of actions on D-branes, whose classical variations cancel the Yang-Mills and gravitational anomalies that appear on a certain class of I-branes [58]. However, there are additional anomalies associated with the global R symmetries as mentioned earlier. They exist for generic D-branes and their intersections. If D-branes are wrapped around nontrivial cycles of a curved compactification manifold [16, 33, 59], the anomalies can manifest themselves as nonvanishing variation of the effective action under a local gauge transformation. Such scenarios have appeared in studies of string dualities [60, 14] as well as field theory dualities [61, 62, 63, 64]. They have also found use in studying topological field theories [16, 65, 66]. However, anomaly cancellation for them has not been investigated until now.

In generalizing the inflow method to such cases, one inevitably runs up against a serious obstruction. Factorizability of an anomaly, as defined precisely later, is crucial for it to be cancelled via the inflow mechanism. However, for the additional anomalies we study, factorizability is apparently lost. To recover it we shall encounter a classic result from differential topology[1]. It allows us to cancel the new anomalies in all cases as long as the D-brane Chern-Simons actions are well-defined.

The D-brane Chern-Simons actions derived in [58] imply that topological defects on D-branes carry their own Ramond-Ramond charges determined by their topological ("instanton") numbers. This observation has far reaching consequences [69, 51, 70]. To cancel the new anomalies that we study, the Chern-Simons actions are modified. This can change the induced Ramond-Ramond charges on a D-brane if it is wrapped around some cycle of a nontrivial compactification manifold.

The plan of this chapter is as follows. In section 2 we discuss how the inflow mechanism works. In addition to a review of some known results, we shall uncover subtleties in the choice of the kinetic action for the Ramond-Ramond field that have not been addressed in the literature. We also define carefully the notion of brane current. For describing flat D-branes, it is just a very convenient notation, but in the anomaly cancellation considered later in this chapter, it plays an essential role. In section 3 we consider the chiral asymmetry induced by twisting the normal bundle and compute the resulting anomaly. We then point out the apparent obstruction to cancelling such anomaly. In section 4 this difficulty is overcome with the help of some interesting topological information encoded in the brane current. Then in section 5 we give examples where the normal bundles of D-branes are nontrivial and calculate the induced Ramond-Ramond charge. In the appendix we comment on the relevance of brane stability and supersymmetry to our anomaly analysis.

4.2 Help from beyond: the inflow mechanism

The inflow mechanism was originally discovered in the context of gauge theory [56], where the action in space-time has a gauge noninvariant term. Its variation is concentrated on topological defects and cancels the anomalies produced by their chiral fermion zero modes. It was recognized in [58] that this mechanism also applies to the Yang-Mills and gravitational anomalies that arise for a certain class of intersecting D-branes in string theory. In this section, we present systematically the details of the inflow mechanism. Although much of it is a review of the earlier results cited above, there are some salient departures. The most important one being our use of a kinetic action manifestly symmetric with respect to all Ramond-Ramond potentials. Its use is really required by the way the inflow mechanism works for D-branes and turns out to be important for reproducing the correct Ramond-Ramond charge.

[1]This result, the relation between Thom class and Euler class, has also been used in a different context: anomaly analysis for the NS5-branes in type IIA string theory and the 5-branes in M theory [67, 68].

As it shall become clear, an anomaly must be factorizable in an appropriate sense in order to be cancelled by inflow. One of the difficulty associated with the anomalies we consider here is their apparent lack of factorizability, and the key to cancelling them involves rewriting them in a factorized form.

4.2.1 Branes and currents

Before discussing the detail of the inflow mechanism, we first introduce a notion that is very convenient here and will prove essential later. Usually a brane is introduced into the bulk theory by adding to the bulk action

$$\int_M \mathcal{L}_M$$

where M is the m-dim worldvolume of the brane and $\mathcal{L}_M$ the Lagrangian density governing the dynamics on the brane. One may rewrite this into an integration over total (bulk) space-time X, with the help of a "differential form" τ_M, defined by

$$\int_M \zeta \equiv \int_X \tau_M \wedge \zeta \tag{4.2.1}$$

for all rank m form ζ defined on M[1]. Thus the rank of τ_M is equal to the codimension of M in X. To be precise, (eq. 4.2.1) defines τ_M as an element in the dual of the space of forms, known to mathematicians as the space of **currents** [71]. Currents are differential-form analogue of distributions; likewise, τ_M is the generalization of Dirac's delta function[2]. Obviously, τ_M must have singular support on M and integrate to 1 in the transverse space of M.

In (eq. 4.2.1), the form ζ is allowed to be any form on M. If instead it is restricted to be closed, the same equation only defines a cohomology class $[\tau_M]$, known as the **Poincare dual** of M. It contains topological information about M. τ_M can be defined as a particular representative of $[\tau_M]$ that is supported only on M.

Here we shall call τ_M the brane current associated with a brane wrapped around M, for a very physical reason. For illustration, consider a d-dim gauge theory with a conventional 2-form field strength F. Let M be the worldline trajectory of an electrically charged particle embedded in the total space-time X. The kinetic term for the gauge field F is

$$S_{\text{gauge}} = -\frac{1}{2}\int_X F \wedge *F. \tag{4.2.2}$$

[1]This definition makes sense because any form ζ on M can be extended to be a form on X by a suitable bump function with support on a tubular neighborhood of M. Conversely, if ζ is a form defined on X to start with, pullback to M is implicit on the LHS of (eq. 4.2.1), as in similar expressions throughout this chapter.

[2]In this language, a delta function in R^d is really a rank d current that maps a function (0-form) into a number.

The coupling of the potential to the electron is

$$\begin{aligned} S_{\text{matter}} &= -\int_M A \\ &= -\int_X \tau_M \wedge A. \end{aligned} \tag{4.2.3}$$

Then the equation of motion for A yields

$$\begin{aligned} *j_{ele} &\equiv d * F \\ &= (-1)^d \tau_M. \end{aligned} \tag{4.2.4}$$

So the usual physical current (source) is related to τ_M by a Hodge $*$ operation. Similarly, if $\hat{M}$ is the $(d-3)$-dim worldvolume of a magnetically charged object, the Bianchi identity would read something like

$$*j_{mag} \equiv dF = \pm \tau_{\hat{M}}.$$

Now return to string theory. Let M be the worldvolume of a D-brane. It couples to the Ramond-Ramond potential C of the appropriate rank just as in (eq. 4.2.3) but with A replaced by C. Then (eq. 4.2.4) gives the definition of the brane current τ_M with F replaced by the appropriate Ramond-Ramond field strength H.

On M, the tangent bundle $T(X)$ of the total space-time X, decomposes into the Whitney sum of $T(M)$ and $N(M)$, the tangent and normal bundles to M respectively. Note that within each fiber of $N(M)$ (eq. 4.2.4) is just the usual Poisson equation. Its RHS has Dirac's δ-type singular support on the zero section. Thus τ_M can be constructed **locally** as

$$\tau_M \overset{naive}{=} \delta(x^1)dx^1 \wedge \cdots \wedge \delta(x^{\dim N(M)})dx^{\dim N(M)}, \tag{4.2.5}$$

where x^μ are Gaussian normal coordinates in the transverse space of M, or equivalently Cartesian coordinates in the fiber of $N(M)$. We emphasize that this expression is naive and in general ill-defined globally.

Now consider the intersection $M_{12} \equiv M_1 \cap M_2$ of two brane-worldvolumes M_1 and M_2. In the literature M_{12} has been called I-brane. For simplicity we shall concentrate on I-branes from intersections at right angle, but the results apply to other cases as well[3]. The right angle condition implies that on the I-brane $M_1 \cap M_2$, the tangent bundle of the total space-time $T(X)$ decomposes as follows:

$$\begin{aligned} T(X) &= T(M_1) \cap T(M_2) \oplus T(M_1) \cap N(M_2) \\ &\oplus N(M_1) \cap T(M_2) \oplus N(M_1) \cap N(M_2), \end{aligned} \tag{4.2.6}$$

[3]The basic reason is that the relevant quantum numbers of the massless fermions are determined by $T(M_1) \cap T(M_2)$ and $N(M_1) \cap N(M_2)$, which are well defined even for oblique intersections.

where $\cap$ denotes fiberwise set theoretic intersection. It is clear that

$$T(M_{12}) = T(M_1) \cap T(M_2) \tag{4.2.7}$$

and

$$N(M_{12}) = T(M_1) \cap N(M_2) \oplus N(M_1) \cap T(M_2) \oplus N(M_1) \cap N(M_2). \tag{4.2.8}$$

Then (eq. 4.2.5) implies that

$$\begin{aligned} \tau_{M_1} \wedge \tau_{M_2} &= \tau_{M_{12}} \qquad \text{if } N(M_1) \cap N(M_2) = \emptyset, \\ &\overset{naive}{=} 0 \qquad \text{otherwise,} \end{aligned} \tag{4.2.9}$$

where in the second line we have used the anticommutivity of exterior multiplication. Here again we emphasize that the second equation is naive, because it uses the naive expression (eq. 4.2.5). The correct statement and its important implication will be given in section 4. Intersections on which $N(M_1) \cap N(M_2) = \emptyset$ are known as **transversal**.

4.2.2 The inflow

Suppose the anomaly on an I-brane M_{12} can be written in the following form:

$$I_{12} = \pi \int \tau_{M_{12}} \wedge \left(Y_1 \wedge \tilde{Y}_2 + Y_2 \wedge \tilde{Y}_1\right)^{(1)}, \tag{4.2.10}$$

where Y_i and $\tilde{Y}_i$, $i = 1, 2$, are some invariant polynomials of the Yang-Mills field strengths and gravitational curvatures defined on M_i. The expression $Z^{(1)}$ denote the Wess-Zumino descent [72, 73] of an invariant curvature polynomial Z: if N is the constant part of Z,

$$Z \equiv N + Z_0,$$

and $Z^{(0)}$ is its secondary characteristic,

$$Z_0 \equiv dZ^{(0)},$$

then the gauge variation of $Z^{(0)}$ is

$$\delta_g Z^{(0)} \equiv dZ^{(1)}.$$

Y_i and $\tilde{Y}_i$ must be defined entirely by the D-branes wrapping M_i. For example, Y_1's dependence on the gravitational curvature from $T(M_1)$ and $N(M_1)$ may be different, but it must not distinguish, say, between the contributions from $T(M_1) \cap T(M_2)$ and $T(M_1) \cap N(M_2)$. In this report such an anomaly is called factorizable.

To cancel the anomaly (eq. 4.2.10), one introduces the following ansatz for a Chern-Simons type action on D-branes [58][4]:

$$-\frac{\mu}{2}\sum_i \int_{M_i} N_i C-(-1)^q H\wedge Y_i^{(0)} = -\frac{\mu}{2}\sum_i \int_X \tau_{M_i}\wedge(N_i C-(-1)^q H\wedge Y_i^{(0)}). \quad (4.2.11)$$

Here q is 1 for IIA and 0 for IIB string theory. i labels the D-brane wrapping worldvolume M_i, whose brane current is τ_{M_i}. N_i is the constant part of Y_i. Anomaly computation in section 3 will show that it is the multiplicity of the D-branes wrapping M_i. μ, rather than $\frac{\mu}{2}$ as one would naively expect, is the brane charge, for reason to be explained shortly. C and H are the formal sums of all the Ramond-Ramond antisymmetric tensor potentials and field strengths respectively. Integration automatically picks out products of forms with the appropriate total rank. In the following we shall often denote by $Z_{(n)}$ the rank n part of any formal sum Z. For example,

$$C = C_{(1)} + C_{(3)} + C_{(5)} + C_{(7)} + C_{(9)}$$

for type IIA string theory and

$$C = C_{(0)} + C_{(2)} + C_{(4)} + C_{(6)} + C_{(8)}$$

for type IIB string theory. It is important to remark that unlike the usual Chern-Simons action, in (eq. 4.2.11) one cannot use integration by parts to reduce the RHS to the more uniform expression of $-\frac{\mu}{2}\int_{M_i} C\wedge Y_i$. The reason is, as we shall see, $dH_{(n)} \neq 0$, even away from any magnetic D(8-n) brane. So H has corrections to its usual expression of dC:

$$H = dC + \cdots. \quad (4.2.12)$$

Therefore a brane Lagrangian in the form of $-\frac{\mu}{2}C\wedge Y$ is different from (eq. 4.2.11) by some additional terms. In fact, only (eq. 4.2.11) can cancel the factorized anomaly (eq. 4.2.10).

In a theory that treats electric and magnetic potentials on equal footing, there could be ambiguity in deriving the equations of motion using the conventional kinetic action. Since (eq. 4.2.11) explicitly involves both electric and magnetic sources, it must be understood to be part of an action that is a manifestly **electromagnetically symmetric**. The detail of the action and its ramifications are interesting in their own rights and presented in the next subsection. The relevant results can be summarized as follows: given the coupling in (eq. 4.2.11), with the factor of $\frac{1}{2}$, the equations of motion are

$$d * H = \mu\sum_i \tau_i \wedge Y_i, \quad (4.2.13)$$

[4]T-duality relates the charge μ for D-branes of different dimensions. With a suitable choice for the unit of length, they are all equal [75].

and the Bianchi identities are

$$dH = -\mu \sum_i \tau_i \wedge \tilde{Y}_i, \tag{4.2.14}$$

with

$$\tilde{Y}_{j(l)} = -(-1)^{\frac{dim(M_j)-q}{2}}(-1)^{l/2} Y_{j(l)}, \tag{4.2.15}$$

without any factor of $\frac{1}{2}$! Note Y and $\tilde{Y}$ are in general different. It will become apparent later that the factor $(-1)^{l/2}$ relates $\tilde{Y}$ to Y by complex conjugation of the group representation of the associated Yang-Mills gauge group, while the factor $(-1)^{\frac{dim(M_j)-q}{2}}$ chooses an orientation for the I-brane.

The Bianchi identities (eq. 4.2.14) impose very strong conditions on the terms represented by $\cdots$ in (eq. 4.2.12). The minimal expression for H is

$$H = dC - \mu(-1)^q \tau_{M_j} \wedge \tilde{Y}_j^{(0)}, \tag{4.2.16}$$

where $\tilde{N}_j$ is the constant part of $\tilde{Y}_j$, and $\tilde{Y}_j^{(0)}$ its secondary characteristic (similar notations apply to the untilded Y's). Since the field strengths H are physical observables, they must be invariant under gauge transformations. Thus C must have compensating gauge variations:

$$\delta_g C = \mu \sum_j \tau_{M_j} \wedge \tilde{Y}_j^{(1)}, \tag{4.2.17}$$

where $\tilde{Y}_j^{(1)}$ is the Wess-Zumino descent of $\tilde{Y}_j$.

Now we can compute the variation of (eq. 4.2.11) under gauge transformations to be

$$\begin{aligned}
\delta_g S &= -\frac{\mu^2}{2} \sum_{ij} \int_X \tau_{M_i} \wedge \tau_{M_j} \wedge \left(\tilde{Y}_j^{(1)} N_i + \tilde{Y}_j (Y_i)^{(1)}\right) \\
&= -\frac{\mu^2}{2} \sum_{ij} \int_X \tau_{M_i} \wedge \tau_{M_j} \wedge \left(Y_i \wedge \tilde{Y}_j\right)^{(1)}.
\end{aligned} \tag{4.2.18}$$

For a particular pair of **distinct** D-brane worldvolume M_1 and M_2, this gives an anomalous variation

$$\begin{aligned}
\delta_g S_{12} &= -\frac{\mu^2}{2} \int_X \tau_{M_1} \wedge \tau_{M_2} \left(\tilde{Y}_2^{(1)} N_1 + \tilde{Y}_2 (Y_1)^{(1)} + \tilde{Y}_1^{(1)} N_2 + \tilde{Y}_1 (Y_2)^{(1)}\right) \\
&= -\frac{\mu^2}{2} \int_X \tau_{M_1} \wedge \tau_{M_2} \left(Y_1 \wedge \tilde{Y}_2 + Y_2 \wedge \tilde{Y}_1\right)^{(1)}.
\end{aligned} \tag{4.2.19}$$

According to the first equation in (eq. 4.2.9), when $N(M_1) \cap N(M_2) = \emptyset$, this **inflow** precisely cancels the anomaly (eq. 4.2.10) if

$$\frac{\mu^2}{2} = \pi. \tag{4.2.20}$$

So the anomaly and inflow analysis also constitutes an independent verification of brane charge computed in [7]. The factor of $\frac{1}{2}$ in (eq. 4.2.11) relative to (eq. 4.2.13) and (eq. 4.2.14) is crucial for agreement[5].

The cases with $N(M_1) \cap N(M_2) = \emptyset$ were considered in [58]. When this does not hold, the second equation in (eq. 4.2.9) suggests that the inflow (eq. 4.2.19) vanishes. However, we shall show in section 3 that on the corresponding I-branes there still exist anomalies. Fortunately, in section 4, we shall find the correction to (eq. 4.2.9) that keeps the inflow finite and cancels the anomaly.

4.2.3 Electro-magnetically symmetric action

In this subsection we derive the equations of motion (eq. 4.2.13) and justify the relative factor of $\frac{1}{2}$ in (eq. 4.2.11)[6]. As mentioned earlier, this factor is essential for obtaining the correct brane charges required by string duality [7, 75]. The kinetic action for antisymmetric tensors we shall use is the one proposed in [76] for source free situations. After we couple it to sources, it is well suited for (eq. 4.2.11) because it treats both electric and magnetic potentials on the same footing. The price to pay is the loss of manifest Lorentz invariance — the action has only manifest rotation invariance in the spatial dimensions, although it possesses additional symmetries that reduce on shell to the usual Lorentz transformations [76]. More recently, there has been progress in covariantizing it[7]. However, for the present discussion the simpler noncovariant version suffices.

First consider just one electro-magnetic dual pair of RR fields $H_{(n)}$ and $\check{H}_{(d-n)}$, where the subscripts, often omitted, denote the ranks of forms. Their respective potentials are $C_{(n-1)}$ and $\check{C}_{(d-n-1)}$. Now let

$$C = \Phi + A \tag{4.2.21}$$

and

$$H = E + B \tag{4.2.22}$$

so that the components of Φ and E consist of those of C and H respectively with a temporal index, while A and B have only spatial indices. Similarly we can also decompose the space-time exterior derivative d into the spatial exterior derivative ∇ and the temporal part d_t:

$$d = d_t + \nabla, \tag{4.2.23}$$

with

$$\{d_t, \nabla\} = 0 = d_t^2 = \nabla^2. \tag{4.2.24}$$

[5]In [58], there was no factor of $\frac{1}{2}$ in the Chern-Simons action, but the total anomaly was computed to be twice as large, so the same value for μ was obtained. We would like to thank the authors of [58] for useful communications regarding this issue.

[6]A similar factor of $\frac{1}{2}$ in the coupling to sources has also been suggested recently in [74]. However, the detailed form of the action used there seems to be different.

[7]See, for example, [77].

Then

$$
\begin{aligned}
E &= d_t A + \nabla\Phi. && (4.2.25)\\
B &= \nabla A. && (4.2.26)
\end{aligned}
$$

The analogy with the usual non-manifestly Lorentz covariant formulation of electrodynamics should be clear. The same can be carried out for the dual fields:

$$
\begin{aligned}
\breve{H} &= \breve{E} + \breve{B},\\
\breve{C} &= \breve{\Phi} + \breve{A}. && (4.2.27)
\end{aligned}
$$

Consider now the action [76]:

$$
S_{BE} = -\frac{1}{2}\int (B \wedge \breve{E} - E \wedge \breve{B} + B \wedge *B + \breve{B} \wedge *\breve{B}). \qquad (4.2.28)
$$

In the absence of sources, the fields satisfy the following Bianchi identities in light of (eq. 4.2.24):

$$
\begin{aligned}
\nabla B &= 0 = \nabla\breve{B}, && (4.2.29)\\
d_t B + \nabla E &= 0 = d_t\breve{B} + \nabla\breve{E}. && (4.2.30)
\end{aligned}
$$

By using the first of them one finds that the equations of motion for Φ and $\breve{\Phi}$ are trivially satisfied — they only enter the action as parts of total exterior derivatives. This implies a larger set of gauge transformations than in the usual formulation:

$$
\begin{aligned}
\delta_g A = \nabla\Gamma, &\qquad \delta_g\breve{A} = \nabla\breve{\Gamma}; && (4.2.31)\\
\delta_g\Phi = \Psi, &\qquad \delta_g\breve{\Phi} = \breve{\Psi} && (4.2.32)
\end{aligned}
$$

with independent Γ, $\breve{\Gamma}$, Ψ, and $\breve{\Psi}$. The gauge transformations (eq. 4.2.32) allow Φ and $\breve{\Phi}$ to be set to 0, corresponding to the usual temporal gauge. Applying (eq. 4.2.30), the equations of motion for A and $\breve{A}$ are found to be

$$
\begin{aligned}
\nabla(\breve{E} + *B) &= 0; && (4.2.33)\\
\nabla(E - (-1)^{n(d-n)} * \breve{B}) &= 0 && (4.2.34)
\end{aligned}
$$

respectively: the expressions inside the parenthesis are closed. By using the gauge transformations (eq. 4.2.31), one can choose a gauge so that they vanish:

$$
\begin{aligned}
\breve{E} &= - * B;\\
E &= (-1)^{n(d-n)} * \breve{B}. && (4.2.35)
\end{aligned}
$$

They then give the duality relation between H and $\breve{H}$. Substituting them for the Bianchi identities (eq. 4.2.29) and (eq. 4.2.30) one finally recovers the conventional equations of motion for antisymmetric tensors:

$$
\begin{aligned}
\nabla * E &= 0 = d_t * E + \nabla * B;\\
\nabla * \breve{E} &= 0 = d_t * \breve{E} + \nabla * \breve{B}. && (4.2.36)
\end{aligned}
$$

Note that although the action (eq. 4.2.28) is not Lorentz invariant, the equations of motion obtained from it are. Furthermore, one can recover from (eq. 4.2.28) the conventional action for one of the gauge potential, say A, in temporal gauge by solving the duality equation (eq. 4.2.35) for its dual $\breve{A}$ and make the gauge choice

$$\Phi = 0 = \breve{\Phi}. \tag{4.2.37}$$

Now let us put in the sources. In the conventional action formalism, where only one potential is used, the potential remains single valued if just electric sources are present. When there is also magnetic source, the potential can only be defined over patches — it is a connection of a nontrivial bundle [78]. The Bianchi identities must be modified. When one switches to the dual description, the meaning of electric and magnetic sources are interchanged, as are the equations of motion and the Bianchi identities. In the symmetric formalism we use here, because both of the dual pair of potentials are used, some Bianchi identities must be modified whichever type of sources is introduced — there is no longer a meaningful distinction between "electric" and "magnetic" sources. However they are called, the same set of equations for the field strengths must obtain in all three approaches if they are equivalent.

Let the brane current for the the sources be proportional to

$$\lambda = \omega + \sigma, \tag{4.2.38}$$
$$\breve{\lambda} = \breve{\omega} + \breve{\sigma}, \tag{4.2.39}$$

with the decomposition into the temporal parts (ω and $\breve{\omega}$) and the spatial parts (σ and $\breve{\sigma}$) understood. They are normalized so that the Bianchi identities are now

$$\begin{aligned} \nabla B &= \breve{\sigma}, \\ d_t B + \nabla E &= \breve{\omega}; \\ \nabla \breve{B} &= \sigma, \\ d_t \breve{B} + \nabla \breve{E} &= \omega. \end{aligned} \tag{4.2.40}$$

These brane currents also make a contribution, denoted by S_j, to the total action. One can derive the form of S_j by using the modified Bianchi identities (eq. 4.2.40). The equations of motion for Φ and $\breve{\Phi}$ require the dependence of S_j on them to be

$$S_j = \frac{1}{2}\int \left((-1)^{n+1}\Phi \wedge \sigma + \frac{1}{2}(-1)^{(n+1)(d-n)} \int \breve{\Phi} \wedge \breve{\sigma} + \cdots\right). \tag{4.2.41}$$

This is necessary for the consistency of the theory and ensures that the gauge transformations (eq. 4.2.32) continue to hold. Note the factor of $\frac{1}{2}$. It comes from the same factor in (eq. 4.2.28).

Turning now to the equations of motion for A and $\breve{A}$, we demand that the duality relation (eq. 4.2.35) holds again. This completely fixes the dependence of

S_j on them:

$$S_j = \frac{1}{2}\int\Big((-1)^{n+1}A\wedge\omega + \frac{1}{2}(-1)^{(n+1)(d-n)}\int\breve{A}\wedge\breve{\omega}+\cdots\Big). \tag{4.2.42}$$

Now S_j is completely determined and has a Lorentz invariant expression:

$$S_j = \frac{1}{2}\int\Big((-1)^{n+1}C\wedge\lambda + (-1)^{(n+1)(d-n)}\breve{C}\wedge\breve{\lambda}\Big). \tag{4.2.43}$$

The conventional equations of motion are again determined from the Bianchi identities (eq. 4.2.40) and the duality relation (eq. 4.2.35):

$$\begin{aligned}\nabla*\breve{E} &= (-1)^{n(d-n)}\breve{\sigma},\\ d_t*\breve{E}+\nabla*\breve{B} &= (-1)^{n(d-n)}\breve{\omega};\\ \nabla*E &= -\sigma,\\ d_t*E+\nabla*B &= -\omega.\end{aligned} \tag{4.2.44}$$

When, say, $\breve{\lambda}=0$, one can recover the conventional action in temporal gauge for C just as for the source free case. The resulting source term is found to be conventionally normalized, i.e. **without** the factor $\frac{1}{2}$. When both an electric brane of charge q_e and a magnetic brane of charge q_m are present, deforming the worldvolume of, say, the electric brane around the magnetic brane by a complete revolution shifts the action (eq. 4.2.42) by a constant. The electric and magnetic parts of (eq. 4.2.42) each makes an equal contribution of $\frac{1}{2}q_eq_m$. Requiring $\exp(iS_j)$ to be single-valued reproduces the standard Dirac quantization: $q_eq_m=2m\pi$.

Finally, we shall write down the electro-magnetically symmetric action for the Ramond-Ramond fields, which is directly relevant for the inflow mechanism. In string theory, a Ramond-Ramond field strength $H_{(n)}$ and its dual $*H_{(n)}$ appear on equal footing. The formal sum H actually includes all electro-magnetic dual pairs of Ramond-Ramond field strengths, and so does $*H$. To find their relation, recall that these field strengths can be defined as follows in terms of the decomposition of bispinors:

$$H_{\mu_1\ldots\mu_n} = S_L^T\Gamma_{\mu_1}\ldots\Gamma_{\mu_n}S_R. \tag{4.2.45}$$

Here S_L has positive $Spin(1,9)$ chirality, while S_R has positive or negative chirality for IIB and IIA string respectively. It is straightforward to infer from this

$$H_{(n)} = (-1)^{(n+q-1)/2}*(H_{(10-n)}). \tag{4.2.46}$$

Recall that q is 0 for IIB and 1 for IIA theory. These duality relations can be obtained from the action

$$S_{BE} = -\frac{1}{2}\int d^{10}x\sum_n\Big((-1)^{(n-q+1)/2}B_{(n)}\wedge E_{(10-n)} + B_{(n)}\wedge *B_{(n)}\Big). \tag{4.2.47}$$

Then if S_j is the Chern-Simons coupling in (eq. 4.2.11), it can be shown that the Bianchi identities must be (eq. 4.2.14) and the equations of motion must be (eq. 4.2.13).

4.3 Brane anomalies

As usual, the anomalies on D-branes and I-branes result from the chiral asymmetry of massless fermions on them. These fermions are in one-to-one correspondence with the ground states of the relevant open string Ramond sectors. In the case of N D-branes wrapping M, the relevant open strings start and end on identical but possibly distinct D-brane. Open string quantization[1] as in §1.2.2 requires that the Ramond ground states be the sections of the spinor bundle lifted from $T(X) = T(M) + N(M)$, tensored with a vector bundle in the $(N, \bar{N})$ representation (adjoint) of the gauge group $U(N)$ on the brane. The latter is dictated by the usual Chan-Paton factors. Because the adjoint representation is real, these fermions are CPT self-conjugate. We shall be interested in perturbative gauge anomalies, so consider $\dim(M)$ to be even. The GSO projection restricts the fermions to have a definite $SO(1,9)$ chirality. If $N(M) = \emptyset$, one is dealing with D9-branes. The worldvolume theory is the super-Yang-Mills part of the type I string theory [7]. It is chiral and anomalous but its anomaly is cancelled by that of the gravitinos and the inflow from the close string sector via the Green-Schwarz mechanism [55].

When $N(M) \neq \emptyset$, the fermions have the quantum number $(+,+) \oplus (-,-)$ under the worldvolume Lorentz group $\mathrm{Spin}(1,p)$ and the space-time Lorentz group restricted to $N(M)$: $\mathrm{Spin}(9-p)$. The latter is now the global R symmetry of the worldvolume theory. If $N(M)$ is flat, left and right moving fermions as sensed by the worldvolume are treated equally and the theory is nonchiral. However, when $N(M)$ has curvature, chiral asymmetry on the worldvolume is induced. The point is that the worldvolume chiralities of the fermions are correlated with their representations under the global R symmetry. Therefore a distinction arises between $(+,+)$ and $(-,-)$. The resulting perturbative anomaly can be calculated by the family index theorem [79, 80, 81, 82, 83]. For $\dim(M) = 4k+2$, the $(+,+)$ and $(-,-)$ fermions are independent and separately Majorana. The total anomaly associated with them is

$$\begin{aligned} I_{D\text{-}brane} &= \frac{2\pi}{2} \int_M \Big(\mathrm{ch}[U(N)_{(N,\bar{N})}] \wedge \hat{A}[M] \\ &\quad \wedge \Big(\mathrm{ch}[S^+_{N(M)}] - \mathrm{ch}[S^-_{N(M)}] \Big) \Big)^{(1)}. \end{aligned} \tag{4.3.1}$$

Here $\mathrm{ch}[E]$ denotes the Chern character of a vector bundle E. $U(N)_{(N,\bar{N})}$ denotes the vector bundle in the $(N, \bar{N})$ representation of the structure group $U(N)$ associated with the N D-branes . $S^{\pm}_{N(M)}$ is the spin bundle lifted from $N(M)$ with $\pm$ chirality. $\hat{A}$ is the Dirac genus. The factor of $\frac{1}{2}$ in front reflects the reality of the fermions. Since $U(N)$ is unitary,

$$\mathrm{ch}[U(N)_{(N,\bar{N})}] = \mathrm{ch}(F) \wedge \mathrm{ch}(-F^*)$$

[1] See the appendix for a discussion of the issue of stability and supersymmetry of brane configurations.

$$= \mathrm{ch}(F) \wedge \mathrm{ch}(-F), \tag{4.3.2}$$

where

$$\mathrm{ch}(F) \equiv \exp(\frac{F}{2\pi}). \tag{4.3.3}$$

F is the properly normalized Hermitian field strength for the $U(N)$ connection on the D-brane in the **fundamental** representation. Using

$$\mathrm{ch}[S_E^+] - \mathrm{ch}[S_E^-] = \frac{e(E)}{\hat{A}(E)}, \tag{4.3.4}$$

which holds for any spin and orientable real vector bundle E, one can rewrite the anomaly as

$$I_{D-brane} = \frac{2\pi}{2} \int_M \left(\mathrm{ch}(F) \wedge \mathrm{ch}(-F) \wedge \frac{\hat{A}[T(M)]}{\hat{A}[N(M)]} \wedge e[N(M)] \right)^{(1)}. \tag{4.3.5}$$

In the special case when $N(M)$ is null, $e[N(M)]$ as well as $A[N(M)]$ is 1.

For $\dim(M) = 4k$, $(+,+)$ and $(-,-)$ are both complex and related by conjugation. Anomaly can be calculated by the contribution from either (but should not be doubly counted) as

$$\begin{aligned} I_{D-brane} &= 2\pi \int_M \left(\mathrm{ch}(F) \wedge \mathrm{ch}(-F) \wedge \hat{A}(M) \wedge \mathrm{ch}[S^+_{N(M)}] \right)^{(1)} \\ &= 2\pi \int_M \Big(\mathrm{ch}(F) \wedge \mathrm{ch}(-F) \wedge \hat{A}(M) \\ &\quad \wedge \frac{1}{2} \Big(\mathrm{ch}[S^+_{N(M)}] + \mathrm{ch}[S^-_{N(M)}] \\ &\quad + \mathrm{ch}[S^+_{N(M)}] - \mathrm{ch}[S^-_{N(M)}] \Big) \Big)^{(1)}. \end{aligned} \tag{4.3.6}$$

Because $\mathrm{ch}[S^+_{N(M)}] + \mathrm{ch}[S^-_{N(M)}]$ is a sum of Pontrjagin classes, it is made up of forms of ranks in multiples of 4. The same is true $\mathrm{ch}(F) \wedge \mathrm{ch}(-F)$. So only $\mathrm{ch}[S^+_{N(M)}] - \mathrm{ch}[S^-_{N(M)}]$ can contribute in (eq. 4.3.6) and we obtain (eq. 4.3.5) again as the expression for the anomaly.

When two D-branes intersect, additional massless fermions arise from the open string sectors with two ends on the two D-branes respectively. Consider a configuration with N_1 D-branes wrapping around M_1 and N_2 around M_2. In the sector with the string starting on M_1 and ending on M_2, the difference in the boundary conditions on the two ends of the string modifies its zero point energy and shifts the moding of some of its worldsheet operators [84, 36]. In the Ramond section, the worldsheet fermions have integral, in particular zero, mode numbers only along the directions either tangential to both D-branes or transverse to both. The result is that the massless fermions are a section of the chiral spinor bundle lifted from

$$T(M_1) \cap T(M_2) \oplus N(M_1) \cap N(M_2),$$

tensored with the $(N_1, \bar{N}_2)$ vector bundle due to their Chan-Paton quantum numbers. The anomaly can be calculated in the same fashion as before:

$$\begin{aligned} I_{I-brane} &= 2\pi \int_{M_{12}} \left(\mathrm{ch}(F_1) \wedge \mathrm{ch}(-F_2) \wedge \frac{\hat{A}[T(M_1) \cap T(M_2)]}{\hat{A}[N(M_1) \cap N(M_2)]} \right. \\ & \left. \wedge e[N(M_1) \cap N(M_2)] \right)^{(1)}. \end{aligned} \tag{4.3.7}$$

Since $(N_1, \bar{N}_2)$ is complex, the fermions are not self-conjugate, and there is no factor of $\frac{1}{2}$ in front. Note that (eq. 4.3.5)is precisely one half of the special case of (eq. 4.3.7) with $M_1 = M = M_2$.

Using brane currents and (eq. 4.2.1), we can rewrite the anomalies (eq. 4.3.5) and (eq. 4.3.7) in forms that will prove useful:

$$\begin{aligned} I_{D-brane} &= \pm\frac{2\pi}{2} \int \tau_M \wedge \left(e[N(M)] \right. \\ & \left. \wedge \mathrm{ch}(F) \wedge \mathrm{ch}(-F) \wedge \frac{\hat{A}[T(M)]}{\hat{A}[N(M)]} \right)^{(1)}, \end{aligned} \tag{4.3.8}$$

$$\begin{aligned} I_{I-brane} &= \pm 2\pi \int \tau_{M_{12}} \wedge \left(e[N(M_1) \cap N(M_2)] \right. \\ & \left. \wedge \mathrm{ch}(F_1) \wedge \mathrm{ch}(-F_2) \wedge \frac{\hat{A}[T(M_1) \cap T(M_2)]}{\hat{A}[N(M_1) \cap N(M_2)]} \right)^{(1)}. \end{aligned} \tag{4.3.9}$$

Here we have left their signs undetermined because, being integrals of differential forms, they really depend on choices of orientation that are not yet fixed by any consideration so far. This ambiguity will soon be resolved by the requirement of factorizability.

In [58], the cases in which M_{12} is the transversal intersection of M_1 and M_2, i.e. $N(M_1) \cap N(M_2) = \emptyset$, were considered. Then the expression for I-brane anomaly (eq. 4.3.7) can be further simplified as

$$I_{I-brane} = \pm 2\pi \int \tau_{M_1} \wedge \tau_{M_2} \wedge \left(\mathrm{ch}(F_1) \wedge \mathrm{ch}(-F_2) \frac{\hat{A}[T(M_1) \cap T(M_2)]}{\hat{A}[N(M_1) \cap N(M_2)]} \right)^{(1)}, \tag{4.3.10}$$

where we have evaluated $e(\emptyset)$ to be 1 but kept $\hat{A}([N(M_1) \cap N(M_2)]$ for future comparison.

It is easy to check that (eq. 4.3.10) is factorizable in the sense of (eq. 4.2.10), with

$$Y_i = \mathrm{ch}(F_i) \wedge \sqrt{\frac{\hat{A}[T(M_i)]}{\hat{A}[N(M_i)]}} \tag{4.3.11}$$

and

$$\tilde{Y}_j = -(-1)^{\frac{dim(M_j)-q}{2}} \mathrm{ch}(-F_j) \sqrt{\frac{\hat{A}[T(M_j)]}{\hat{A}[N(M_j)]}}. \tag{4.3.12}$$

Hence this anomaly can be cancelled by the inflow (eq. 4.2.19). The sign factor in (eq. 4.3.12) is determined by (eq. 4.2.15). As promised before, this fixes the choice of orientation for the anomaly, and (eq. 4.3.10) becomes

$$\begin{aligned} I_{I-brane} &= -\pi \int \tau_{M_1} \wedge \tau_{M_2} \wedge \Big(\big((-1)^{\frac{dim(M_2)-q}{2}} \mathrm{ch}(F_1) \wedge \mathrm{ch}(-F_2) \\ &\quad + \{1 \leftrightarrow 2\} \big) \wedge \frac{\hat{A}[T(M_1) \cap T(M_2)]}{\hat{A}[N(M_1) \cap N(M_2)]} \Big)^{(1)} . \end{aligned} \tag{4.3.13}$$

After some manipulation one can show that the two terms in the integrand of (eq. 4.3.13) contribute equally, rather than cancelling each other, to the anomaly:

$$\begin{aligned} I_{I-brane} &= -(-1)^{\frac{dim(M_2)-q}{2}} 2\pi \int \tau_{M_1} \wedge \tau_{M_2} \wedge \Big(\mathrm{ch}(F_1) \wedge \mathrm{ch}(-F_2) \\ &\quad \wedge \frac{\hat{A}[T(M_1) \cap T(M_2)]}{\hat{A}[N(M_1) \cap N(M_2)]} \Big)^{(1)} . \end{aligned} \tag{4.3.14}$$

(eq. 4.3.10) is also trivially correct when $N(M_1) \cap N(M_2)$ is nonempty but trivial, because the RHS' of both (eq. 4.3.7) and (eq. 4.3.10) vanish. However, (eq. 4.2.9) would want one to believe that (eq. 4.3.10) fails for a nontrivial $N(M_1) \cap N(M_2)$ because its RHS would seem to vanish, although the anomaly does not in general. There are similar difficulties for the D-brane anomaly (eq. 4.3.5). Consider D-branes with worldvolume M. For $N(M) = \emptyset$, the anomaly is that of Type I string theory and cancelled via the Green-Schwarz mechanism [55]. For $N(M) \neq \emptyset$, the closest thing would be (eq. 4.2.18) with $M_1 = M = M_2$. However, $\tau_M \wedge \tau_M$ naively vanishes.

4.4 Topology to the rescue

It is clear from the earlier discussions that factorizability in the sense of (eq. 4.2.10) is crucial for an anomaly to be cancelled via this inflow method. However, when the relevant normal bundle is nontrivial, it can be shown that the **integrand** of (eq. 4.3.7) is no longer factorizable because of the Euler class. In other words, it is not factorizable unless $N(M_1) \cap N(M_2)$ is empty. The same can be said about the D-brane anomaly (eq. 4.3.5). A related puzzle on the other side of the inflow mechanism has also been shown. The second equation in (eq. 4.2.9) would imply vanishing inflow for M_{12} as long as $N(M_1) \cap N(M_2) \neq \emptyset$, regardless of the twisting of the normal bundle. It could cancel no anomaly, factorized or not.

The origin of all these difficulties can be traced back to the properties of brane currents. Being a physical observable, τ_M must be globally defined over M. However, (eq. 4.2.5) only makes sense within each coordinate patch, because between patches the transversal coordinates are defined only up to the transition functions for the normal bundle. To it one must add additional terms, which vanish when $N(M)$ is trivial but turn τ_M into a globally defined form when $N(M)$ is not. Therefore if such correction can be found, it must carry topological information about $N(M)$, and from (eq. 4.2.5) it must have components with indices tangential to M. Mathematicians have found an elaborate construction for this correction [85]. By pulling τ_M back to M, only parts from the correction can survive. It is remarkable that the result is cohomologically the Euler class $e[N(M)]$ of $N(M)$.

Before proceeding further it is convenient to introduce some notations. First observe that τ_M is determined by $N(M)$, because it should be defined as the limit of nonsingular differential forms with shrinking compact supports in the neighborhood of M, which is approximated by the neighborhood of the zero section of $N(M)$. As such τ_M can be defined for any oriented real orientable vector bundle E by taking M to be the zero section E. To emphasize this we define[1]

$$\Phi[E] \equiv \tau_M \tag{4.4.1}$$

for any vector bundle $\pi, E \to M$. The important property just mentioned can be written as

$$\tau_M \wedge \tau_M = \tau_M \wedge \Phi[N(M)] = \tau_M \wedge \Big[e[N(M)]\Big] \tag{4.4.2}$$

where $[e]$ denotes some representative of the cohomology class of e. Another useful property is [85]:

$$\Phi(A \oplus B) = \Phi(A) \wedge \Phi(B). \tag{4.4.3}$$

This can be seen as Euler class also factorizes under Whitney sum. Now by (eq. 4.2.6), for the I-brane worldvolume $M_{12} = M_1 \cap M_2$ we have

$$\begin{aligned}
\tau_{M_1} \wedge \tau_{M_2} &= \Phi[T(M_1) \cap N(M_2) \oplus N(M_1) \cap N(M_2)] \\
&\quad \wedge \Phi[N(M_1) \cap T(M_2) \oplus N(M_1) \cap N(M_2)] \\
&= \Phi[T(M_1) \cap N(M_2) \oplus N(M_1) \cap T(M_2) \oplus N(M_1) \cap N(M_2)] \\
&\quad \wedge \Phi[N(M_1) \cap N(M_2)] \\
&= \tau_{M_{12}} \wedge e[[N(M_1) \cap N(M_2)]],
\end{aligned} \tag{4.4.4}$$

where in the last step we have used (eq. 4.4.3) again along with (eq. 4.4.2). This is the correct replacement for the naive equation in (eq. 4.2.9). Now returning to the I-brane anomaly (eq. 4.3.9), one notes that as long as $\dim[T(M_1) \cap T(M_2)] + 2 > \dim[N(M_1) \cap N(M_2)]$, one can use the freedom to add local counterterms to choose to make the Wess-Zumino descent on terms **other than** the Euler form. The I-brane anomaly then becomes

$$\begin{aligned}
I_{I\text{-}brane} &= \pm 2\pi \int \tau_{M_{12}} \wedge e[N(M_1) \cap N(M_2)] \\
&\quad \wedge \left(\mathrm{ch}(F_1) \wedge \mathrm{ch}(-F_2) \frac{\hat{A}[T(M_1) \cap T(M_2)]}{\hat{A}[N(M_1) \cap N(M_2)]} \right)^{(1)}.
\end{aligned} \tag{4.4.5}$$

[1] Actually for our purpose, knowledge of the cohomology class of $\Phi(E)$ is sufficient. It is called the Thom class of E.

By the same token, only the cohomology class of e is important here. Substituting for (eq. 4.4.4), one obtains again (eq. 4.3.10) as the expression for anomaly. But now it is clearly valid even when the normal bundle is nontrivial. Furthermore, the D-brane anomaly can also be written in this form with $M_1 = M_2 = M$, as long as $\dim[T(M)] + 2 > \dim[N(M)]$. When $\dim[T(M_1) \cap T(M_2)] + 2 < \dim[N(M_1) \cap N(M_2)]$, both the anomaly and the inflow vanish. The case of $\dim[T(M_1) \cap T(M_2)] + 2 = \dim[N(M_1) \cap N(M_2)]$ is an intriguing one and we will comment on it shortly. We have shown that except for that case, the inflow (eq. 4.2.18) not only does not vanish identically but cancels precisely the anomalies (eq. 4.3.9) and (eq. 4.3.8).

There is a nice topological characterization of our results. It has emerged that the anomaly, written as an integral over the total space-time, is always proportional to

$$\tau_{M_1} \wedge \tau_{M_2}. \tag{4.4.6}$$

Its cohomology class is the Poincare dual of the transversal intersection of M_1 and M_2. Transversal intersection, unlike geometric or set-theoretic intersection, has the property of stability: because there is no common transverse direction, small perturbation can only move the intersection around but never make it disappear. Consider now a **nontransversal** intersection $M_{12} = M_1 \cap M_2$. Because $N(M_1) \cap N(M_2) \neq \emptyset$, a small perturbation in those directions would naively separate them and lift the intersection altogether. This is the meaning of the second line in (eq. 4.2.9). Such perturbation is given by a global section of $N(M_1) \cap N(M_2)$. However, a global section of a sufficiently twisted vector bundle will necessarily have nonempty zero locus. For $N(M_1) \cap N(M_2)$, this means that M_1 and M_2 cannot be completely separated. Any small perturbation will leave intact some submanifold of M_{12}, the zero locus of the corresponding section of $N(M_1) \cap N(M_2)$, which is now stable. That is precisely the **traversal** intersection of M_1 and M_2. It can be shown that the Poincare dual of the zero locus of an orientable real vector bundle E is none other than $e(E)$. This gives another derivation of (eq. 4.4.4). For $M_1 = M = M_2$, the story is similar. $e[N(M)]$ is the Poincare dual of the zero locus of $N(M)$. So $\tau_M \wedge \tau_M$ measures the self-intersection of M. To recapitulate, D-brane and I-brane anomalies are associated with transversal intersections, even when the pertinent geometric intersections are not transversal. In light of this, it seems worthwhile to introduce the notion of **transversal I-brane**, whose brane current is simply $\tau_{M_1} \wedge \tau_{M_2}$.

Now turning to the special case of

$$\dim[T(M_1) \cap T(M_2)] + 2 = \dim[N(M_1) \cap N(M_2)].$$

This implies that $\dim[T(M_1)] + \dim[T(M_2)] = 8$, or that the two D-branes make up an electro-magnetic dual pair. An example would be a D-string intersecting with a D5-brane at 0 angle. For $M_1 = M = M_2$, the condition $\dim[T(M)] + 2 = \dim[N(M)]$ means one is dealing with the self-dual D3-brane in IIB theory. For these examples the anomaly (eq. 4.3.9) is finite but the inflow, even after taking into account the nontriviality of the normal bundles, still seems to vanish. But one should not rush to conclude that anomaly does not cancel for them, because the intersection of electric and magnetic sources introduces an additional subtlety: the Chern-Simons action (eq. 4.2.11) is no longer well defined. A more powerful approach is needed but will not be pursued in the present work.

4.5 Induced brane charges, the silverlining

An important consequence of the inflow mechanism, besides lending support to the consistency of various brane configurations, is that charges for the bulk Ramond-Ramond fields are induced by the gauge fields and gravitational curvatures as in (eq. 4.2.13). Let M be the worldvolume of some Dp-branes with gauge field strength F. Consider a m-cycle γ of M[1]. Then

$$Q_{\text{ind}} = \int_\gamma \text{ch}(F) \wedge \sqrt{\frac{\hat{A}[T(M)]}{\hat{A}[N(M)]}} \tag{4.5.1}$$

gives the induced charge, in integral unit, for the Ramond-Ramond $(p+1-m)$-form potential. From the viewpoint of the field theory on the Dp-brane, the characteristic class on the RHS measures the topological charge of a gravitational/Yang-Mills "instanton". Let us call it Y as before. Then (eq. 4.2.13) shows that $\tau_M \wedge Y$ can be thought of as the brane current for a "fat" D$(p-m)$-brane bound to and spread out on the Dp-brane. When the instanton shrink to zero size, Y also acquires Dirac's δ singularity. $\tau_M \wedge Y$ is just like a brane current. One might well wonder if the instanton can be lifted off the brane and become a physical D-brane in its own right. At least for Yang-Mills instantons there has been much evidence in support of this idea: field theory instantons and branes are continuously connected by transitions between different branches of the moduli space of the I-brane field theory [69, 51, 70, 86]. Recently, more complicated configurations involving gravitational curvatures on the D-brane were used to study geometric engineering and realizations of field theory dualities employing brane configurations [61, 64]. In this section we consider specific examples in which the twisting of the normal bundle modifies the induced charge.

As discussed in the appendix, our analysis seems to apply, a posteriori, to nonsupersymmetric brane configurations as well. However, in most applications considered in the literature there are some supersymmetries left so as to have control over radiative corrections. Therefore here we shall only consider Type II compactifications over d-dimensional manifolds S that preserve some supersymmetries. A D-brane wraps around a m-dimensional submanifold M of S can preserve some of the supersymmetries of the compactification provided M satisfies some conditions. Such a M is called a supersymmetric cycle [24]. All supersymmetric

[1]In this section we count in complex unit the dimensions of compactification manifolds S if it is Calabi-Yau and in real units those of other types as well as all submanifolds of S.

cycles have been analyzed and classified in [33, 59]. We shall consider them one by one. We shall also only consider S' with irreducible holonomy because the analysis for the other cases can be reduced to them. The forms in $\hat{A}(N)$ all have ranks in multiples of 4. On the other hand, to have nontrivial normal bundle, the D-brane must wrap a **proper** submanifold of M. By counting dimensions and ranks, the contribution of $N(M)$ to Q_{ind} comes from the rank 4 component of

$$\frac{1}{\sqrt{\hat{A}(N(M))}}.$$

For convenience we shall group it together with the contribution from $T(M)$ at the same rank, so the characteristic class we shall be computing is

$$\lambda \equiv \frac{p_1[N(M)] - p_1[T(M)]}{48}. \tag{4.5.2}$$

Let the Chern roots of $T(M)$ be

$$\pm x_i, \quad i = 1 \ldots \frac{m}{2}. \tag{4.5.3}$$

For the cases considered here m is always even. Let the Chern roots of $N(M)$ be

$$\pm y_j, \quad j = 1 \ldots \lfloor \frac{d-m}{2} \rfloor, \tag{4.5.4}$$

with an additional 0 if $d - m$ is odd. Then (eq. 4.5.2) can be written via the splitting principle as

$$-\frac{1}{48}\left(\sum_i x_i^2 - \sum_j y_j^2\right). \tag{4.5.5}$$

Of particular interest is whether λ and hence Q_{ind} can be expressed purely in terms of x's, information which is encoded in $T(M)$.

The first nontrivial compactification is K3. However, for this case there cannot be any additional contribution to the induced brane charge from a twisted normal bundle, for dimensional reasons mentioned above.

The next case is for S to be a generic Calabi-Yau 3-fold. According to [33] a supersymmetric cycle is either a Lagrangian submanifold (3-cycle) or a Kähler submanifold (2n-cycle) of the Calabi-Yau 3-fold. For the reasons discussed above, only for Kähler 4-cycles does $N(M)$ make a contribution to Q_{ind}. The holonomy of $T(M)$is $U(2)_T$ and that of $N(M)$ is $U(1)_N$. The Calabi-Yau condition requires

$$x_1 + x_2 + y = 0. \tag{4.5.6}$$

The relevant charge is proportional to

$$\begin{aligned} \lambda &= \frac{p_1[N(M)] - p_1[T(M)]}{48} \\ &= \frac{2x_1x_2}{48} = \frac{2e(T(M))}{48}. \end{aligned} \tag{4.5.7}$$

The remaining type of Calabi-Yau compactification is over a generic Calabi-Yau 4-fold. It can have three types of supersymmetric cycles: Lagrangian (4-cycle), Kähler (2n-cycle), and Cayley (4-cycle). A special Lagrangian submanifold has the property that the holonomy of its normal bundle is the same as that of its tangent bundle. Therefore the effect of $N(M)$ on the induced charge completely cancels whatever contribution from $T(M)$: $\lambda = 0$.

Among the Kähler (2n)-cycles, 4-cycles and 6 cycles will see contribution from $N(M)$. The holonomy group of $T(M)$ is $U(n)$. The holonomy group of $N(M)$ is $U(4-n)$. The Calabi-Yau condition says that

$$\sum_i x_i + \sum_j y_j = 0. \tag{4.5.8}$$

Using this we can calculate

$$\begin{aligned}\lambda &= \frac{1}{48}\left(2\left(\sum_{i_1<i_2} x_{i_1}x_{i_2} - \sum_{j_1<j_2} y_{j_1}y_{j_2}\right)\right) \\ &= \frac{2c_2[T_+(M)] - 2c_2[N_+(M)]}{48}.\end{aligned} \tag{4.5.9}$$

where $T_+(M)$ and $N_+(M)$ are the holomorphic tangent and normal bundles of M respectively, and c_2 denotes the second Chern class. For a Kähler 6-cycle, $c_2[N(M)]$ is 0, so (eq. 4.5.9) is entirely determined by information encoded in $T(M)$. This is not so for a Kähler 4-cycle, for which (eq. 4.5.9) reduces to

$$\lambda_{4\text{-cycle}} = \frac{2\big(e[T(M)] - e[N(M)]\big)}{48} \tag{4.5.10}$$

but cannot be expressed in terms of x alone.

Calabi-Yau 4-folds admit one more type of supersymmetric cycles [59]. It is to date the only known case where a single D-brane breaks the supersymmetries of a type II compactification by $\frac{3}{4}$ instead of $\frac{1}{2}$. They are known as Cayley submanifolds [87]. They are 4-dimensional and satisfy the conditions [88, 59]

$$x_1 + x_2 + y_1 + y_2 = 0, \tag{4.5.11}$$

and

$$x_1 - x_2 = y_2 - y_1. \tag{4.5.12}$$

These conditions are sufficiently restrictive to imply the vanishing of λ.

There are two other cases of string compactifications: S may be a seven dimensional manifold with $G(2)$ holonomy or an eight dimensional manifold with Spin(7) holonomy [31, 89]. A generic Spin(7) manifold supports only Cayley submanifolds as supersymmetric cycles [59]. It is again 4-dimensional. With a suitable choice of orientations, the curvature is subject to (eq. 4.5.11) but not (eq. 4.5.12). Then (eq. 4.5.10) follows again [59].

Finally we come to the case of $G(2)$ manifold. It admits two types of supersymmetric cycles [59]. They are known as coassociative (4-cycle) and associative (3-cycle) submanifolds respectively. Only for the coassociative submanifold will Q_{ind} be affected by the gravitational curvature. With a suitable choice of orientations, they satisfy the condition [88] that

$$x_1 + x_2 + y = 0. \tag{4.5.13}$$

Hence

$$\lambda = \frac{2x_1x_2}{48} = \frac{2e(T(M))}{48}. \tag{4.5.14}$$

The results in this section are summarized in the following table.

Table 4.1: Induced Ramond-Ramond charges

Holonomy of S	Type of M	λ
SU(3)	Kähler 4	$2e(M)/48$
SU(4)	Special Lagrangian	0
SU(4)	Cayley	0
SU(4)	Kähler 4	$2[e(M) - e(N)]/48$
SU(4)	Kähler 6	$2c_2[T_+(M)]/48$
G(2)	Coassociative	$2e(M)/48$
Spin (7)	Cayley	$2[e(M) - e(N)]/48$

Appendix 4.A Comments on Brane Stability and Supersymmetry

It is appropriate to address the issue of stability of brane configurations and its relevance to the anomaly analysis[2]. For a generic brane configuration, there are forces between nonparallel branes. If they do not cancel, this configuration is not stable. One can no more trust string perturbation theory in an unstable brane configuration than one can trust perturbative expansion around a false vacuum in field theory. Anomaly calculations is in some sense more robust than many other perturbative calculations, but one must still know the correct spectrum of massless fermions in **some** true vacuum to correctly compute the anomaly. Of course this was the original motivation for t'Hooft's anomaly matching conditions. In the above, we have relied on string perturbation when we obtained the massless fermion contents and their quantum numbers. When the brane configuration is unstable, there is no known reason to expect a priori that such analysis captures correctly the spectrum.

[2]We would like to thank K. Bardakci for useful conversations regarding this issue.

On the other hand, supersymmetry is the only general condition under which the forces between branes cancel. If supersymmetry is completely broken in a brane configuration, the latter is **generically** unstable. For N identical D-branes to preserve some supersymmetry in a string compactification, they must wrap around the supersymmetric cycles classified in [33, 59]. Between a pair of D-branes, the pattern of supersymmetry breaking depends on their relative arrangement. For the case of intersection at right angle, some supersymmetries survive provided that [58]

$$\dim[T(M_1) \cap N(M_2)] + \dim[N(M_1) \cap T(M_2)] = 0 \ (\text{mod} 4). \tag{4.5.15}$$

The expression on the LHS of this equation is sometime denoted $nd + dn$ in the literature because it is the number of space-time coordinates for which the boundary condition of the relevant open string is Neumann on one end and Dirichlet on the other. When (eq. 4.5.15) is not satisfied, anomaly calculation based on perturbative string theory does not have to be reliable. For example, if $nd + dn = 2$, it may be shown that the force between the two D-branes is attractive. It is believed that in this case there exists a stable nonmarginal bound state [75]. There seems a priori to be no reason to expect that the correct degrees of freedom of the bound state to be obtained from a perturbative string analysis carried out at the unstable configuration.

On the other hand, (eq. 4.5.15) was not needed in the analysis carried out in this chapter. In fact it follows through as long as

$$\dim[T(M_1) \cap T(M_2)] + \dim[N(M_1) \cap N(M_2)] = 0 \ (\text{mod} 2), \tag{4.5.16}$$

a condition satisfied by any pair of D-branes that can coexist in the same string theory. This seems to suggest that even for nonsupersymmetric brane configurations, at least the massless fermion contents might be captured correctly.

Chapter 5

p-Branes and 3d Gauge Field Theories

5.1 Introduction

3d mirror symmetry for was first proposed in [90] as a duality between certain pairs of generally different 3d N=4 theories at the infrared limit. Infinite sequences of new mirror pairs and strong field theoretic evidence for them were found in [91]. It is a nonperturbative duality that in particular equates certain quantities receiving large quantum corrections with some that are determined entirely classically. Naturally one asks whether this can be the consequence of some string dualities. There are several different approaches [90, 91, 92, 93, 94, 95, 96], each of which has it own advantage and is related to the others by some sequences of dualities. In this chapter we shall present one that is particularly intuitive, first proposed in [93, 94]. As mentioned in the last chapter, the global R-symmetry of N=4 theories appears as geometric rotations. R-symmetry breaking part of the moduli space of vacua and the parameters space of the field theory are realized as the moduli space of D-branes configurations[1]. Mirror symmetry itself is implemented by the S duality of type IIB string theory [93]. This construction allows us to engineer a large class of theories and find their mirror duals [94]. Since then it has been generalized to 3d N=2 theories [98, 99]. Similar ideas of constructing field theories in 4d have also been very fruitful[2]. In this chapter, we will focus on 3d N=4 theories. After reviewing the rules for "model building" via "brane engineering," we will show how the mirror pairs emerge from this prescription. As a bonus we can predict an infinite number of 3d field theories without conventional Lagrangian descriptions. Some of them are dual to ordinary Lagrangian theories via mirror symmetry, but the rest are not; yet they can be smoothly connected in the moduli space of brane configurations. We will also discuss how open string instanton configurations appear as field theory instantons giving corrections to the 3d gauge theories.

5.2 What is 3d mirror symmetry?

The structure of N=4 supersymmetric gauge theories in three dimensions can be easily obtained by dimensionally reducing the minimally supersymmet-

[1] The anomalies associated with such R-symmetry in brane configurations have been analyzed in [97].

[2] See, for example, [100, 101, 102, 103, 104].

ric 6d Yang-Mills Lagrangian [105]. The global R-symmetry of the 3d theory is $SU(2)_{345} \times SU(2)_{789}$. The reason for choosing these subscripts will become clear in the brane realization described later. The field content consists of vector multiplets and hypermultiplets. For each vector multiplet associated with a $U(1)$ factor of the gauge group there are 3 real Fayet-Iliopoulos parameters , $\vec{\zeta}$, which can be thought as coming from the VEV's of a background hypermultiplet. For each hypermultiplet, there are 3 real mass parameters, $\vec{m}$. They are the VEV's of a background vector multiplet. There are also gauge coupling constants, which also come from background vector multiplets. The transformation properties of the parameters and VEV's under R-symmetry are summarized in table 5.1.

Note that usually the scalars in a hypermultiplet are written as a doublet under $SU(2)_{789}$. However, it is convenient, by a change of variables, to rearrange them into a singlet b and a triplet $\vec{r}$. On the other hand, an interesting feature peculiar to three dimensions is that a vector potential is dual to a scalar by the usual electro-magnetic duality . Of course this duality transformation can be precisely formulated only for a free $U(1)$ gauge fields, but this is what is available for the low energy effective theory at generic points of the vector multiplet branch of moduli space. Therefore it is meaningful to include the dualized scalar in considering the moduli space of vacua. By supersymmetry, the moduli space must be hyper-Kähler for both the hypermultiplets and the (dualized) vector multiplets . Because of their different patterns of global R-symmetry breaking, however, the VEV's of the vector and hyper multiplets respectively are distinct order parameters of the theory, even after taking into account of quantum fluctuations. Vacua of N=4, D=3 gauge theories always contain a vector multiplet branch in which the gauge group is generically broken down to $U(1)^N$ where N is the total rank of the gauge group. This branch is parameterized by the $4N$ real scalars from the N corresponding vector multiplets. If it has a sufficient number of hypermultiplets, there can also be a hypermultiplet branch and/or mixed branches.

Table 5.1: R charges of the VEV's and parameters

Multiplets/Parameters	Notation	$SU(2)_{345} \times SU(2)_{789}$
Vector	$\vec{a} : a^3, a^4, a^5$	(3,1)
	$A_\mu \to \sigma$	(1,1)
Hyper	$\vec{r} : r^7, r^8, r^9$	(1,3)
	b	(1,1)
Fayet-Iliopoulos	$\vec{\zeta} : \zeta^7, \zeta^8, \zeta^9$	(1,3)
mass	$\vec{m} : m^3, m^4, m^5$	(3,1)
(electric) coupling	e	(1,1)

The low energy effective action up to two derivatives and four fermions is controlled by the metric of the moduli space, which depends on the parameters of the theory as well as the position on the moduli space. The dependence is constrained by extended supersymmetry: the Kähler potential is the sum of a term that depends only on the hypermultiplet scalar and one only on vector multiplet scalars [106, 107]. So a mixed branch is the direct product of a vector branch $\mathcal{M}_V$ and a "hyper" branch $\mathcal{M}_H$. By reinterpreting the parameters of the theory as the VEV's of background superfields, one can further deduce the effects of tuning them on the metric. The gauge coupling $\frac{1}{g^2}$ lives in a vector multiplet, so it can continuously deform the metric of $\mathcal{M}_V$ but not that of $\mathcal{M}_H$. Since g^2 also plays the role of $\hbar$, one concludes that the the metric of $\mathcal{M}_H$ is determined purely classically whereas that of $\mathcal{M}_V$ in general receives quantum corrections. Mass parameters also live in a vector multiplet, so they also can continuously deform $\mathcal{M}_V$ but not $\mathcal{M}_H$ - they can only affect $\mathcal{M}_H$'s dimensionality by changing the number of massless hypermultiplets. Fayet-Iliopoulos parameters, on the other hand, live in hypermultiplet, and therefore can only deform $\mathcal{M}_H$ and change the dimensions of $\mathcal{M}_V$. This is summarized in table 5.2.

Table 5.2: Parameters' influence on moduli space

Branch	$\vec{m}$	$\vec{\zeta}$	e	m (?)
$\mathcal{M}_V$	deform	reduce	deform	no effect (?)
$\mathcal{M}_H$	reduce	deform	no effect	deform (?)

Looking at the above two tables, the pattern for a possible duality emerges. Starting with some theory that we call A model, if we switch what what we mean by $SU(2)_{345}$ and $SU(2)_{789}$, and at the same time exchange masses with Fayet-Iliopoulos parameters, will we end up with an apparently different theory, model B, that is nonetheless equivalent to A? $\mathcal{M}_V$ of one theory would have to be mapped to $\mathcal{M}_H$ of the other. Classically this is definitely not true: $\mathcal{M}_H$ is a complicated space obtained via the hyper-Kähler quotient construction [108] while $\mathcal{M}_V$ is just a flat space quotiented by the Weyl group for the gauge symmetry. Furthermore, when some of the mass terms $\vec{m}_A$ of, say, A, vanish, enhanced global (flavor) symmetries emerge and act on $\mathcal{M}_H^A$. For B there would have to be corresponding global symmetries emerging at vanishing Fayet-Iliopoulos parameters $\vec{\zeta}_B$ and acting on $\mathcal{M}_V^B$. Classically there is no such symmetry. Therefore this hypothetical duality, named **mirror symmetry in three dimensions** in [90], must be a quantum equivalence. However, a glance at tables 5.1 and 5.2 reveals a missing dual of the gauge coupling constant e, whose property as predicted by mirror symmetry is listed in table 5.2 with question marks. Brane interpretations of this parameter, known as the **magnetic coupling** m (without an arrow), have been proposed [93, 95], but it has yet to be found in any Lagrangian formulation. Given such, mirror symmetry can only manifest for ordinary field theory when e approach a particular value[1]. Being dimensionful (e^2 has the dimension of mass in 3d), the only natural candidates are 0 and ∞. $e = 0$, the classical limit, is already ruled out. So we are left with the strong coupling limit, which, since it is in 3d, is also the infrared limit. This also leads us to one of the most striking aspects of this proposed mirror symmetry: it maps from one theory the metric for $\mathcal{M}_V$, a quantity that receives very large quantum corrections, to, in the dual theory, the metric for $\mathcal{M}_H$, which is given by purely classical expressions. As many physicists have suggested, this may imply the line between quantum and classical physics is more blurred than previously thought.

Because of its strong coupling nature, proving mirror symmetry within the context of field theory will be difficult. Embedding the field theory in the dynamics of branes [70, 109, 110, 111, 112] renders many aspects of mirror symmetry manifest [93, 94], if one assumes the S duality of type IIB string. This will be reviewed extensively in the rest of this talk. Before that, we want to give some example of mirror pairs and one of the many pieces of field theoretic evidence supporting it [91]. They are logically independent of any conjectures about string theory.

A model has gauge group $U(K)$ with N fundamentals and 1 adjoint hypermultiplets. B models has gauge group $U(K)^N$, which we label as $U(K)_\alpha$, $\alpha = 0, \ldots, N-1$. Its hypermultiplets consist of one fundamental charged under $U(K)_1$ and N bi-fundamentals. The latter type of fields are each charged respectively under $U(K)_\alpha \times U(K)_{\alpha+1}$ in representation $(\bar{K}, K)$, with cyclic identification $\alpha \sim \alpha + N$. These field contents are nicely encoded in the "quiver" diagrams [113] of figure 1. Each inner node with a number K represents a $U(K)$ gauge group. Each link connecting a pair of them represents a bi-fundamental charged under the pair as (fundamental, fundamental*). An outer node with number N represents fundamentals with multiplicity N charged under the gauge group associated with the inner node to which it is attached.

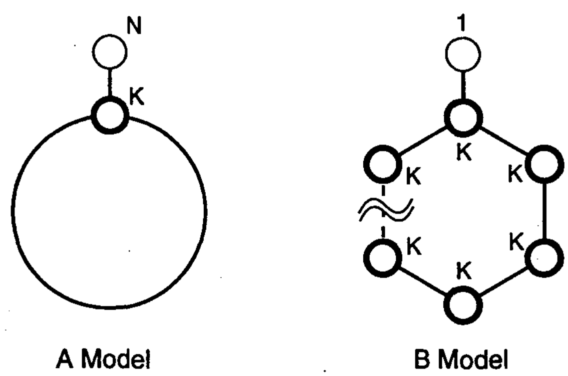


Figure 5.1: Quiver for A and B models.

[1]However, it is possible, and the brane realization discussed later strongly suggests, that m can be a field theory parameter that has no Lagrangian representation.

Using the notation given in table 1, the metric for $\mathcal{M}_V^A$ takes the form

$$ds^2 = g_{ij}d\vec{a}_i d\vec{a}_j + (g^{-1})_{ij}(d\sigma_i + \vec{\omega}_{il} \cdot d\vec{a}_l)(d\sigma_j + \vec{\omega}_{jl} \cdot d\vec{a}_l), \tag{5.2.1}$$

under the constraint

$$\begin{aligned} \vec{\nabla}_i g_{jl} &= \vec{\nabla}_j g_{il} && (5.2.2)\\ \frac{\partial}{\partial a_i^x}\omega_{jl}^y - \frac{\partial}{\partial a_j^y}\omega_{il}^x &= \epsilon_{xyz}\frac{\partial}{\partial a_i^z}g_{jl}. && (5.2.3) \end{aligned}$$

Here $i,\ldots = 0,\ldots,k-1$ index the Cartan of $U(k)$; $x,\ldots = 3,4,5$. In [91], g_{ij} is computed. Perturbatively it is one-loop exact:

$$\begin{aligned} g_{i\neq j} &= \frac{2}{|\vec{a}_i - \vec{a}_j|} - \frac{1}{|\vec{a}_i - \vec{a}_j + \vec{m}_{\text{adj}}|} - \frac{1}{|\vec{a}_i - \vec{a}_j - \vec{m}_{\text{adj}}|}, \\ g_{ii} &= \frac{1}{e^2} + \sum_{\alpha=0}^{N-1}\frac{1}{|\vec{a}_i - \vec{m}_\alpha|} && (5.2.4)\\ &+ \sum_{j=1\ldots K}^{j\neq i}\left(\frac{-2}{|\vec{a}_i - \vec{a}_j|} + \frac{1}{|\vec{a}_i - \vec{a}_j + \vec{m}_{\text{adj}}|} + \frac{1}{|\vec{a}_i - \vec{a}_j - \vec{m}_{\text{adj}}|}\right). \end{aligned}$$

(5.2.5)

Here $\vec{m}_{\text{adj}}$ is the triplet mass for the adjoint hypermultiplet; $\vec{m}_\alpha$ those of the fundamentals, indexed by $\alpha = 1,\ldots,N$. When $\vec{m}_{\text{adj}} = 0$, there is no instanton correction to the metric and (eq. 5.2.4) is also nonperturbatively exact. For illustration, consider this simpler case, so that $g_{i\neq j}$ vanishes. (eq. 5.2.1) and (eq. 5.2.4) state that $\mathcal{M}_V^A$ is the direct product of K multi-Taub-NUT space with charge N. After quotiented by the Weyl group of $U(K)$, the direct product becomes a symmetric product.

It turns out that setting $\vec{m}_{\text{adj}} = 0$ for A model is mapped by mirror symmetry to a condition on the Fayet-Iliopoulos parameters $\vec{\zeta}_\alpha$ of B model:

$$\sum_\alpha \vec{\zeta}_\alpha = 0.$$

In this case, $\mathcal{M}_H^B$ is given by the symmetric product of K ALE spaces with A_{N-1} singularity [114]. The metric of each ALE space is given by

$$ds^2 = g d\vec{r}^2 + \frac{1}{g}(db + \vec{\omega}(\vec{r}) \cdot d\vec{r})^2$$

with

$$g(\vec{r}) = \sum_{\alpha=0}^{N-1}\frac{1}{\left|\vec{r} - \sum_{\beta=0}^{\alpha}\vec{\zeta}_\beta\right|}.$$

Now the metric for each Taub-NUT factor of $\mathcal{M}_V^A$ can be read from (eq. 5.2.4) after setting $\vec{m}_{\text{adj}} = 0$.

$$g(\vec{a}) = \frac{1}{e^2} + \sum_{\alpha=0}^{N-1} \frac{1}{|\vec{a} - \vec{m}_\alpha|}.$$

It is clear that $\mathcal{M}_V^A = \mathcal{M}_H^B$ if and only if $e = \infty$ and one makes the following identification[2]

$$\vec{\zeta}_\alpha = \vec{m}_\alpha - \vec{m}_{\alpha-1}.$$

5.3 Setting the branes to work

Consider in type IIB string theory, a configuration that includes 3 types of branes, whose worldvolume configurations are given in table 5.3 [93]. Such a con-

Table 5.3: Configurations of the branes

	0	1	2	3	4	5	6	7	8	9
D3-brane	×	×	×				×			
D5-brane	×	×	×					×	×	×
NS5-brane	×	×	×	×	×	×				

figuration can preserve up to 8 supercharges, corresponding to N=4 in 3d. Of the original $Spin(1,9)$ Lorentz symmetry, only the (1+2)d Lorentz group $SL(2,R)_{012}$ and the R-symmetry group $SU(2)_{345} \times SU(2)_{789}$ remain manifest. we will now review some basic rules for "model building" from branes and their justifications. Many of them first appeared in [93].

- By sending M_{planck} to ∞, the worldvolume theory on the branes decouple from the bulk fields such as graviton.

- D3-branes can break and end on D5 or NS5-branes without violating RR charge conservation [115, 116] by becoming a magnetic "monopole" on the 5-branes, as depicted on the left in figure 5.2. There the horizontal lines represent D3-branes; solid and dashed vertical lines represent NS and D5-branes respectively. Conversely, two D3-branes ending on the same 5-brane from opposite sides can rejoin.

[2]To see the counting of parameters matches, note that the "center" of the mass parameters can be absorbed by shifting the origin of $\mathcal{M}_V$ on both sides. Here this is used to set $\vec{m}_{N-1} = 0$ for A model.

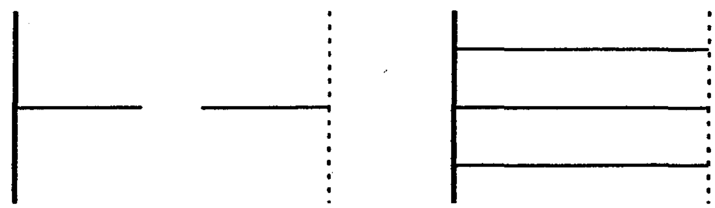


Figure 5.2: D3-brane ending on 5-branes and forbidden configurations.

- To the worldvolume theory on the D3-branes, breaking and ending are tantamount to imposing boundary conditions, reducing N=4, D=4 supermultiplets to N=4, D=3 supermultiplets.

- By taking the appropriate scaling limit, the Kaluza-Klein modes along the 6th direction can be kept massive and integrated out. The effective infrared theory on the D3-brane is therefore a (1+2)d QFT.

- The worldvolume theories on the 5-branes are weakly coupled in the infrared. The fields on them have an infinite volume coefficient in their kinetic terms as compared to D3-brane fields due to their relative sizes. As a result they are frozen as background fields and their VEV's are parameters of the effective 3d theory. Gauge symmetries on the 5-branes become global symmetries.

- When a NS5-brane crosses a D5-brane, a D3-brane is created and stretched in between. This is a consequence of charge conservation [93][1].

- Certain configurations are believed to be forbidden. They involve more than one D3 brane stretched between the same NS-D5 pair, such as the one in the right of figure 5.2.

Following the above rules one can build brane configurations of arbitrary complexity. To read off the contents of the resulting field theory, one also needs to know the following.

- Dynamical fields of the decoupled (1+2)d theory arise out of the lightest excitations of open strings starting and ending on the D-branes. There are three types: open strings connecting between D3-branes in the same "cubicle" (figure 5.3a) give vector multiplets; those between D3-branes in adjacent "cubicle" give bi-fundamental hypermultiplets (figure 5.3b);

[1]For some other treatments of this phenomenon, see [117] and the references therein.

while open strings connecting between D3 and D5 give fundamental hypermultiplets (figure 5.3c). These can be read off from perturbative open string quantization and the boundary conditions. Their transformation properties under R-symmetry agree with the assignment given in the last section. Their end points are electric sources on the D3-branes.

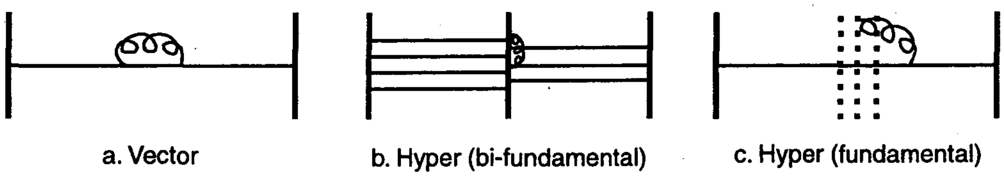


Figure 5.3: Field content from open fundamental strings.

- Configuration of the D3-branes selects the gauge symmetries and the vacuum, as illustrated in figure 5.4.

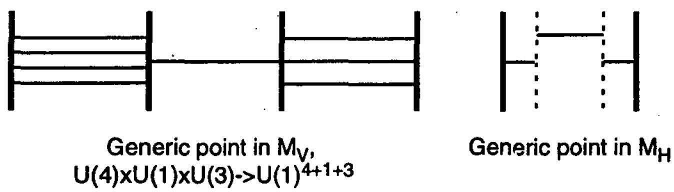


Figure 5.4: D3-brane configuration selects gauge group and vacuum

- Configuration of the 5-branes determines the parameters (figure 5.5) and global symmetries of the 3d QFT. Note the identification of the magnetic coupling constant as the inverse square root of the distance between adjacent D5-branes along the 6th direction. The frozen gauge dynamics of the D5-brane gives rise to the flavor global symmetry acting on the fundamental hypermultiplets. Since in conventional Lagrangian field theories, this global symmetry is restored by setting the masses to zero, the magnetic coupling is fixed to be infinite. At the same time, the global symmetry resulting from the NS5-branes is broken unless they coincide, which amounts to setting Fayet-Iliopoulos to zero and e to infinity — it can only appear at a nontrivial infrared fixed point.

- Type IIB string theory has a nonperturbative **S** duality [15] that inverts the string coupling constant and exchange NS5-branes with D5-branes as well as fundamental strings with D-strings. It leaves invariant D3-branes

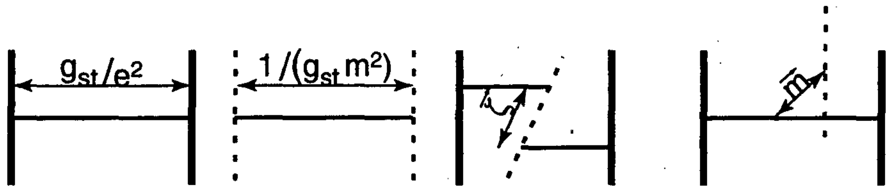


Figure 5.5: 5-Brane configuration fixes parameters and global symmetries

but acts on their worldvolume theories as the S duality for N=4, D=4 SYM [118].

- We therefore should also consider excitations of open D-strings starting and ending on D3-branes and/or NS5-branes. They are simply obtained from the S dual of figure 5.3. However, on D3-branes, rather than generating additional degree of freedom, they are related to the open string fields nonlocally — end points of these two types of string on D3-branes are respectively magnetic and electric sources. Each appear as solitonic excitations of the other and are exchanged by the field theoretic S duality.

- Therefore there are two descriptions of the same 3d theory: one uses open fundamental string fields as the canonical variables while the other uses open D-string fields. Their equivalence is the 3d mirror symmetry, and it follows from the S duality type IIB string theory.

- It is convenient to rephrase this duality as an operation combing an S duality transformation with the interchange of 345 and 789 directions [93]. This makes explicit the interchange of the two R-symmetry factors, $\mathcal{M}_V$ with $\mathcal{M}_H$, masses with Fayet-Iliopoulos parameters, and e with m. For reasons just stated, this is reflected on ordinary Lagrangian field theories only in their infrared limit.

5.4 Mirror pairs

Now we will present a few examples of mirror pairs constructed in [94]. Their field contents are best described by the type of quiver diagrams introduced earlier. They are obtained using the brane-engineering rules outlined above, but with a compactified 6th direction, i.e. with periodic identification along X^6. As a warm-up, let's look at the A and B models given in figure 1. The corresponding brane configurations are sketched in figure 5.6

For A model, the N fundamentals of $U(K)$ originate from open strings stretched between the N D5 and the K D3-branes. A special feature of this

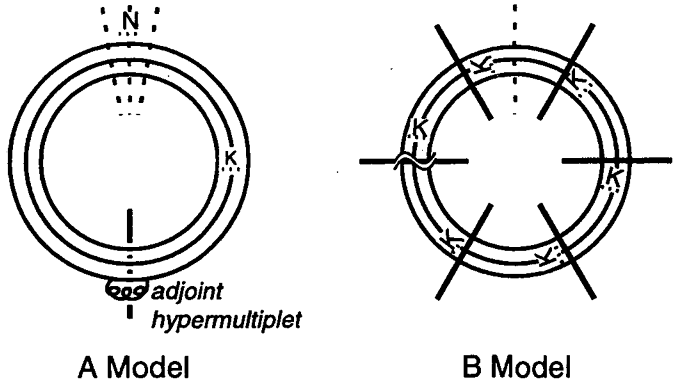


Figure 5.6: Brane configurations for the quivers in figure 5.1.

configuration is that the bi-fundamental, coming from open string stretching between "nearest-neighbor" D3-branes, become the adjoint of $U(N)$ because there is only one gauge groups. For B model, more generic situation prevails and there are N bi-fundamentals. The correspondence between the moduli spaces of vacua of A and B as well as the identification (eq. 5.2) is evident. As it is, there is one constraint on the field theory parameters, namely

$$\vec{m}_{\text{adj}} = 0 = \sum_{\alpha} \vec{\zeta}_{\alpha}.$$

A way to relax this condition has been given in [101].

Now let's look at mirror pairs of more complex theories. As depicted by the quiver diagrams in figure 5.7, model A again has gauge group $U(K)^N$ and N bi-fundamentals, but each $U(K)$ now has fundamentals with an arbitrary number of flavors w_i. Its mirror, model B, has gauge group $U(K)^M$ with

$$M = \sum_i w_i.$$

Besides the M bi-fundamentals, it has N fundamentals arranged as shown in figure 5.7. If all $w_i > 0$, each $U(K)$ factor has at most one fundamental. If some $w_i = 0$, the corresponding nodes in B model's quiver coalesce and give rise to fundamentals of higher flavor. Such mirror pairs are again constructed via the S duality of type IIB string theory [94]. The mirror map relates the moduli spaces of the two theories, as well as their parameters, in the same manner as the simpler case discussed above.

It is natural to generalize to the cases with A model having gauge group $\prod_{i=0}^{N-1} U(K_i)$, N bi-fundamentals, and arbitrary fundamentals. Its quiver is shown in figure 5.8, along with its brane realization. However, here one encounters an important subtlety. Although one can always perform a S duality transformation

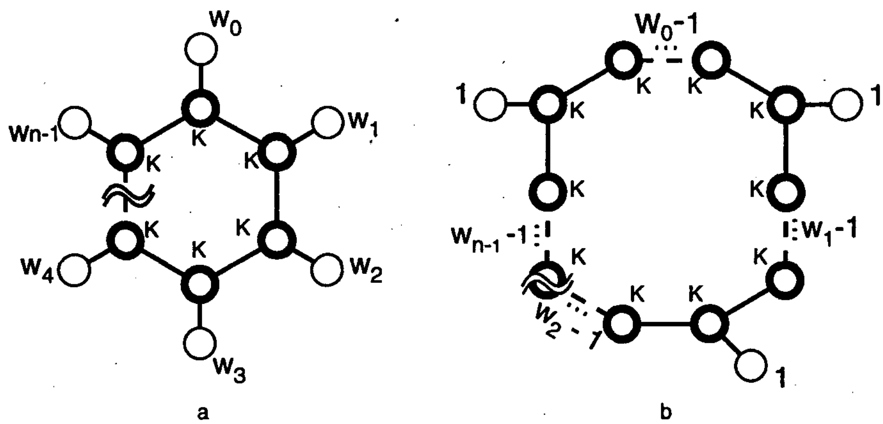


Figure 5.7: Arbitrary flavor of fundamentals and the mirror

and obtain the mirror configuration, the result does not always correspond to a gauge theory. To see this, recall that mirror symmetry exchanges $\mathcal{M}_V$ with $\mathcal{M}_H$. A universal property of N=4, D=3 super-Yang-Mills theories is the existence of the Coulomb phase, a branch of $\mathcal{M}_V^B$ with $4r$ dimensions, where r is the total rank of the gauge group. This is mapped under mirror symmetry to a branch of $\mathcal{M}_H^A$: the completely Higgsed phase. Therefore a necessary condition for model B to have an ordinary gauge theoretic Lagrangian description is for model A to admit complete Higgsing. This amounts to requiring [119, 120, 91, 94]:

$$2k_i - k_{i-1} - k_{i+1} \leq w_i. \tag{5.4.1}$$

When this is satisfied, the mirror gauge theory can be constructed along the same vein as before. The details become complicated and can be found in [94].

5.5 Phases and transitions

What happens if (eq. 5.4.1) is not satisfied? S duality still gives a mirror configuration, but one without a gauge theoretic description. An example of this is shown in figure 5.9a.

To understand this phenomenon, note that from the field theory perspective, mirror symmetry corresponds to the Z_2-wise freedom in labeling the two $SU(2)$ R-symmetries. While the vector and hypermultiplets transform distinctly though somewhat symmetrically under them, their interactions enter in the Lagrangian in rather different form. The Lagrangian descriptions of a theory and its dual would in general be quite different, as the examples above show. Actually, there is no reason a priori to expect that both sides of a mirror pair have Lagrangian

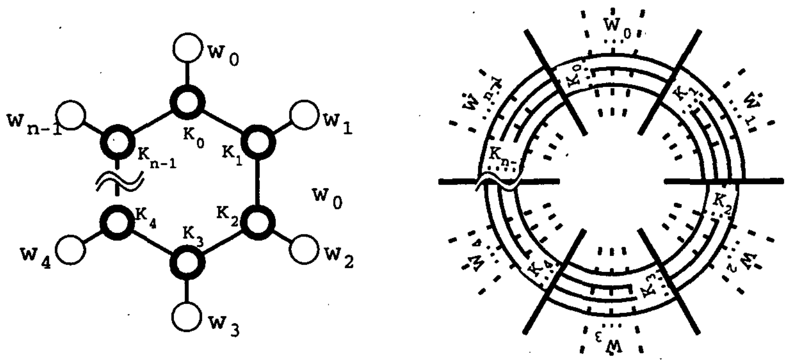
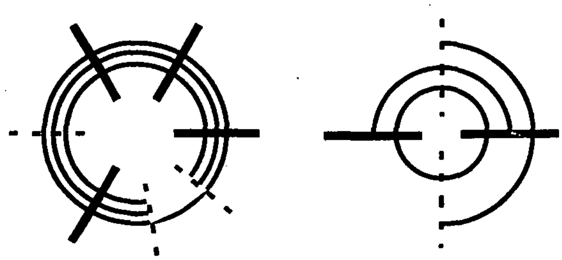


Figure 5.8: Quiver and brane realization of $\prod_i U(K_i)$



a b

Figure 5.9: Brane configurations giving rise to novel field theories

descriptions at all. It is natural to conjecture this is the what is happening here[1]. 3d N=4 theories with Lagrangian description can therefore be classified into those that have two, related by mirror symmetry in the infrared, and those that have only one[2].

An example of the latter type, in which the non-Lagrangian description is that with E_n tensionless string [121, 122], was conjectured already in [90]. It

[1]In fact, barring an unexpected way to incorporate the mysterious magnetic coupling into a Lagrangian formulation, even for the cases in which both of the dual pair have a Lagrangian description, mirror symmetry is manifest only at the infinite coupling, i.e. infrared limit, as mentioned earlier. Here we shall be referring to that limit implicitly unless otherwise stated.

[2]One should note that the lack of a completely higgsed phase is necessary but not sufficient for a non-Lagrangian dual. For example, a free $U(1)$ vector multiplet, which obviously does not have any Higgs phase, is dual to a free hypermultiplet. However, such free theories cannot help explain interacting fixed points, e.g. when the D3-branes in figure 5.9a or b coincide.

has recently been explicitly engineered using one of the alternative formulations of mirror symmetry from string theory [95, 96]. Here we have a very simple prescription for engineering an infinite number of such Lagrangian-non-Lagrangian mirror pairs. As noted in [90], these are local quantum field theories, simply because on one side of the mirror there is a Lagrangian description that flows to it. However, experience in 2, 4, 5, and 6 dimensions has shown a Lagrangian description, though convenient in many ways, may not be a necessary condition for a local quantum field theory (see, for example, [123, 124, 125, 126]. Indeed, one can easily engineer using branes a third class of theories that have **no** Lagrangian description on either side of the mirror. An example is shown in figure 5.9b. Since the decoupling of bulk as well as Kaluza-Klein modes works just as in the more mundane cases, they should still be interacting local quantum field theories, but with no known Lagrangian description flowing to it.

Such interesting phenomena deserve an explanation from string theory. In that context mirror symmetry is the equivalence of two descriptions of the same physics related by S duality. The degrees of freedom on the D3-brane theory can be captured both by open fundamental string and by open D-string fields. Starting with, say, a description using only open fundamental string fields, one can employ standard string perturbation theory to obtain their interactions and write a Lagrangian for the field theory modes in the decoupling limit. The same prescription goes through for a description based solely on open D-string fields. Suppose, however, that there is no way to capture the full degrees of freedom by using only, say, open D-string fields. The description on the B model side may not be in the form of a local action, as open D-strings and open fundamental strings ending on D3-branes are mutually nonlocal. If neither open fundamental string fields nor open D-string fields can account for the full degrees of freedom by themselves respectively, we end up with the third class of theories.

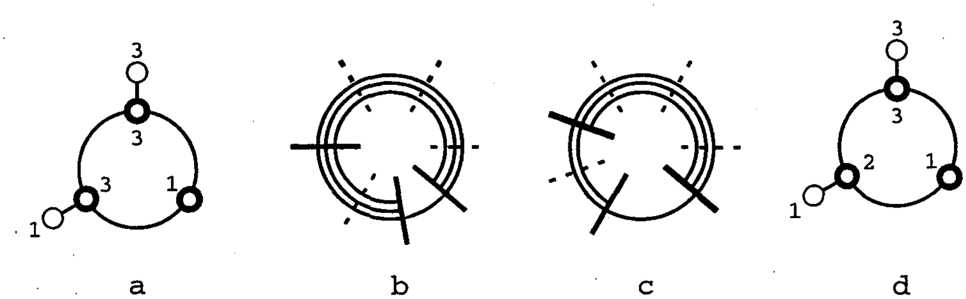


Figure 5.10: Transition between different classes of field theories via brane motion.

Remarkably, with branes one can not only engineer examples of all three types of theories, but also interpolate between them continuously by moving the 5-branes around. Shown in figure 5.10a, b is the mirror of figure 5.9a. It is a theory of the second type (Lagrangian-non-Lagrangian mirror pair). By moving one NS5-branes past another, we arrive at the theory depicted in figure 5.10c, d, which is of the first type (Lagrangian-Lagrangian mirror pair). Similar transition can be engineered between theories of the third class and the first two as well. To appreciate the meaning of such a process, recall that the electric and magnetic coupling constants are inversely proportional to the square root of the distance between adjacent NS5 and adjacent D5-branes respectively. Moving 5-branes of the same type past each other effectively makes coupling constants of the corresponding type imaginary. This is indicative of a change of the effective degrees of freedom describing the system, namely a phase transition. That smooth movements in the moduli space of brane configurations can connect distinct classes of field theories via some type of phase transitions is one of the most important lessons to be learned from this work.

5.6 Superpotentials and open D-string instantons

It has been argued that nonperturbative dynamics of supersymmetric gauge theories in three dimensions is controlled by instantons [127]. In this section we will study instantons from string theory viewpoint with the aid of open D-strings. In three dimensions the instanton carries a magnetic charge. The magnetic charge is mediated by the scalar dual to the photon σ [128]. The instantons from the string theory viewpoint are the D-strings that end on the D3 branes[93, 94]. The boundary of a D-string is the worldline of a magnetic monopole in the D3 brane [129]. To break half of the supersymmetry, it must be holomorphically embedded [33], which in this case means being flat and orthogonally intersecting other branes. To qualify as an instanton configuration for the effective three dimensional theory, the D-string worldvolume must be Euclidean and compact. Therefore it must be bounded on all sides.

One such instanton is illustrated in figure 5.11, a D-string stretched (shaded region) between parallel pairs of D3 and NS 5-branes. They are the $SL(2, Z)$ dual of the open fundamental string instantons of [48, 33]. Here we consider a generic point in the Coulomb branch of the moduli space, so the $U(N_c)$ gauge group on the D3 worldvolume is broken to its maximal Abelian subgroup. By the convention of §5.2, the $\vec{a}$'s are the VEV's of the expectation values of the scalars in the vector multiplets. They parameterize the positions of the D3 branes in the x^3 direction.

The instanton contributions to the path integral take the form of $K'e^{-S_0-i\sigma}$ [128], where K' is a factor that includes the one loop determinant and S_0 is the classical action for the instanton background $S_0 \sim -\frac{1}{\vec{a}}|e^2$. σ is the dual to the photon of the unbroken $U(1)$. It emerges from field theory after summing the instantons in the dilute instanton gas approximation [128, 130]. It is also expected by holomorphy arguments. All these have counterparts in string theory language.

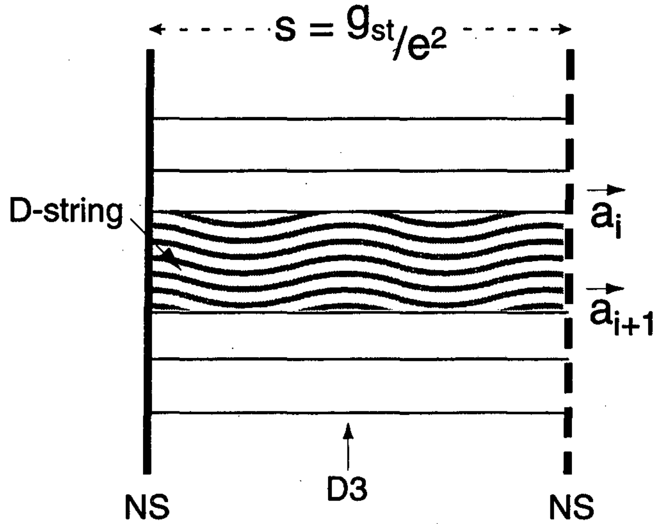


Figure 5.11: Open D-string instanton generation of a superpotential.

Naturally, instanton corrections in string theory come in the form of

$$Ke^{-S_{\text{D-string}}} \tag{5.6.1}$$

where K is a factor that includes the one-loop determinant of the massive fields on the D-string worldsheet, and $S_{\text{D-string}}$ is the D-string worldsheet action. This action contains two pieces:

$$S_0 = S_{\text{Nambu-Goto}} + i\int_{\text{boundary}} \tilde{A}\cdot dX, \tag{5.6.2}$$

The Nambu-Goto action simply yields the area of the Euclidean D-string divided by the tension of the D-string [94]. Thus

$$S_{Nambu-Goto} = \text{area} \times \text{tension} = \frac{[|\vec{a}_i - \vec{a}_{i+1}|\alpha'] \times [g_{\text{st}}/e^2]}{g_{st}\alpha'} = \frac{|\varphi_i - \varphi_{i+1}|}{e^2}, \tag{5.6.3}$$

where we used the relation between the three dimensional gauge coupling e, the string coupling g_{st} and the distance s between the NS 5-branes in the x^6 direction: $\frac{1}{e^2} = \frac{s}{g_{\text{st}}}$.

In addition, there is the contribution from the boundary of the D-string. It couples to the electric and magnetic gauge potential on the D5 and D3 branes with coupling constants g of the respective theories. The former is not dynamical but the latter is important. Denote the magnetic and electric gauge potentials and field strengths by tilded and untilded symbols respectively, then

$$\epsilon_{ijk}F_{ij} = g_{\text{st}}\tilde{F}_{k6} = g_{\text{st}}(\partial_k\tilde{A}_6 - \partial_6\tilde{A}_k), \tag{5.6.4}$$

where i, j, k take value among $0, 1, 2$. Applying $SL(2, Z)$ to the discussion in [93], one deduces that when a D3 brane ends on two NS 5-branes, the magnetic gauge field vanishes in the effective three dimensional theory but $\tilde{A}_6$ survives. Equation (5.6.4) now reads

$$\epsilon_{ijk} F_{ij} = g_{\mathrm{st}} \partial_k \tilde{A}_6. \tag{5.6.5}$$

Thus $g_{\mathrm{st}} \tilde{A}_6 = e^2 \sigma$ is the dual of the photon. The contribution of the second term in (5.6.4) is now

$$\int_{boundary} \tilde{A} \cdot dX = \sigma_i - \sigma_{i+1} \tag{5.6.6}$$

Therefore the correction from such an instanton is proportional to

$$e^{-((\varphi_i - \varphi_{i+1})/e^2 + i(\sigma_i - \sigma_{i+1}))} = e^{(Z_{i+1} - Z_i)}, \qquad Z_i \equiv \frac{\varphi_i}{e^2} + i\sigma_i, \tag{5.6.7}$$

in agreement with field theoretic expectation. Note that $S_{Nambu-Goto}$ is insensitive to the orientation of the D-string but the $i\sigma$ term is. For anti-(D-string)instanton it changes sign so an anti-instanton correction is anti-holomorphic. Note also that the factor K cannot have any dependence on the fields Z.

The instantons, being BPS objects, break one half of the supersymmetry of the gauge theory. This is consistent with the stringy interpretation as the D-string configuration in figure 5.11 breaks by a further half the supersymmetry preserved by the the NS5-D3 configuration. This yields four zero modes. Hence such instanton configurations correct the superpotential. Indeed when $m_{adj} \neq 0$, the perturbative expression (eq. 5.2.4) for the vector multiplet metric ceases to be positive definite for sufficiently small $|\vec{a}_i - \vec{a}_j|$. It is believed that instanton corrections of the form (eq. 5.6.7) keeps the metric meaningful.

It is not obvious how to compute the contribution of D-string instanton more explicitly then we have done above, least of all the quantum fluctuation K. However mirror symmetry may give us an indirect approach. The mirror dual of figure 5.11 is an open fundamental string bounded on D5 and D3-branes. One can interpret this as an **open string** exchange between two "monopoles" on D5-branes By analogy with the formalism discussed in §2.3, one can formally define boundary states using open string Hilbert space and compute the amplitude in figure 5.11 as a matrix element of the form $\langle B' | q^{L_0} \bar{q}^{\bar{L}_0} | B \rangle$.

Bibliography

[1] M. Green, J. Schwarz and E. Witten, *Superstring Theory*, Vol 1 & 2, Cambridge University Press, (1987).

[2] H. Ooguri, Z. Yin, in *Fields, Strings, and Duality*, Ed. Costas Efthimiou, Brian Greene, World Scientific, Singapore, hep-th/9612254

[3] P. Ginsparg, in Proceedings of Les Houches Summer School, *Fields, Strings, and Critical Phenomena*, Les Houches, France, 1988, eds. E. Brezin and J. Zinn-Justin, North-Holland (1990).

[4] D. Lüst, S. Theisen, *Lectures on String Theory*, Springer-Verlag (1989).

[5] R. F. Streater, A.S. Wightman, *PCT, Spin and Statistics and All That*, Addison-Wesley (1989).

[6] Daniel Friedan, Emil Martinec, Stephen Shenker, *Nucl. Phys.* **B271** (1986) 93.

[7] J. Polchinski, *Phys. Rev. Lett.* **75** (1995) 4724.

[8] Gary T. Horowitz, Andrew Strominger, *Nucl. Phys.* **B360** (1991) 197.

[9] Jr. Curtis G. Callan, Jeffrey A. Harvey, Andrew Strominger. *Nucl. Phys.* **B367** (1991) 60.

[10] Jr. Curtis G. Callan, Jeffrey A. Harvey, Andrew Strominger. in Proceedings, String theory and quantum gravity '91, Trieste 1991, hep-th/9112030.

[11] Andrew Strominger. *Nucl. Phys.* **B343** (1990) 167.

[12] Jr. Curtis G. Callan, Jeffrey A. Harvey, Andrew Strominger. *Nucl. Phys.* **B359** (1991) 611.

[13] A. Giveon, M. Porrati, E. Rabinovici, *Phys. Rept.* **244C** (1994) 77.

[14] Hirosi Ooguri, Cumrun Vafa. *Nucl. Phys.* **B463** (1996) 55, hep-th/9511164.

[15] J. H. Schwarz, *Phys. Lett.* **360B** (1995) 13, hep-th/9508143.

[16] M. Bershadsky, V. Sadov, C. Vafa, *Nucl. Phys.* **B463** (1996) 420, hep-th/9511222.

[17] J. Polchinski, Y. Cai, *Nucl. Phys.* **B296** (1988) 91; C. G. Callan, C. Lovelace, C. R. Nappi, S. A. Yost, *Nucl. Phys.* **B308** (1988) 221.

[18] Andrew Strominger, Cumrun Vafa. *Phys. Lett.* **379B** (1996) 99.

[19] N. Ishibashi, *Mod. Phys. Lett.* **A4** (1989) 251.

[20] McKenzie Y. Wang, *Ann. Global Anal. Geom.* **7** (1989) 59.

[21] R. Harvey, *Spinors and Calibrations*, Academic Press (1990).

[22] H. Federer, *Geometric Measure Theory*, Springer Verlag, 1969.

[23] N. Berkovits C. Vafa, *Nucl. Phys.* **B433** (1995) 123, hep-th/9407190.

[24] K. Becker, M. Becker, A. Strominger, *Nucl. Phys.* **B456** (1995) 130, hep-th/9507158.

[25] A. Strominger, *Comm. Math. Phys.* **133** (1990) 163.

[26] P. Candelas, X. C. de la Ossa, *Nucl. Phys.* **B355** (1991) 455.

[27] S. Cecotti, C. Vafa, *Nucl. Phys.* **B367** (1991) 359.

[28] M. Bershadsky, S. Cecotti, H. Ooguri, C. Vafa, *Comm. Math. Phys.* **165** (1994) 311, hep-th/9309140.

[29] B. R. Greene, D. R. Morrison, M. R. Plesser, *Comm. Math. Phys.* **173** (1995) 559, hep-th/9402119.

[30] D. Gepner, *Phys. Lett.* **199B** (1987) 380.

[31] S. Shatashvili and C. Vafa, *Selec. Math.* **1** (1995) 347, hep-th/9407025.

[32] D. Joyce, *Invent. Math.* **123** (1996) 507.

[33] H. Ooguri, Y. Oz, Z. Yin, *Nucl. Phys.* **B477** (1996) 407, hep-th/9606112.

[34] R. Harvey and H. B. Lawson, *Acta. Math.* **148** (1982) 47.

[35] T. Eguchi and K. Higashijima, in Proceedings of the Niels Bohr Centennial Symposium, Copenhagen, Denmark, May 6-10, 1985.

[36] M. Berkooz, M. R. Douglas and R. G. Leigh, *Nucl. Phys.* **B480** (1996) 265, hep-th/9606139.

[37] R. L. Bryant, ' in Differential Geometry and Differential Equations, Shanghai 1985, Lecture Notes in Math., vol. 1255, Springer-Verlag, 1985, p. 1.

[38] D. Joyce, *J. Amer. Math. Soc.* **43** (1996) 291.

[39] D. Joyce, *J. Amer. Math. Soc.* **43** (1996) 329.

[40] B. R. Greene and M. R. Plesser, *Nucl. Phys.* **B338** (1990) 15.

[41] P. S. Aspinwall and C. A. Lütken, *Nucl. Phys.* **B355** (1991) 482.

[42] P. S. Aspinwall and D. R. Morrison, *Phys. Lett.* **355B** (1995) 141, hep-th/9505025.

[43] D. R. Morrison, in Proceedings of Strings 93, hep-th/9311049.

[44] D. Morrison, *J. Diff. Geom.* **6** (1993) 223.

[45] D. R. Morrison, in *Essays on Mirror Manifolds II*, alg-geom 9504013.

[46] D. R. Morrison, Trieste Conference on S-Duality and Mirror Symmetry, *Nucl. Phys. B Proc. Suppl.* **46** (1996), 146, hep-th/9512016.

[47] W. Schmid, *Invent. Math.* **22** (1973) 211.

[48] A. Strominger, S.-T. Yau and E. Zaslow, hep-th/9606040.

[49] P. Candelas, X. C. De La Ossa, P. S. Green and P. Parkes, *Nucl. Phys.* **B359** (1991) 21.

[50] E. Witten, *Nucl. Phys.* **B460** (1996) 335, hep-th/9510135.

[51] M. R. Douglas, hep-th/9512077.

[52] C. Vafa and E. Witten, *J. Geom. Phys.* **15** (1995) 189, hep-th/9409188.

[53] A. Strominger, *Nucl. Phys.* **B451** (1995) 96.

[54] D. R. Morrison, alg-geom/9608006.

[55] M. B. Green, J. H. Schwarz, *Phys. Lett.* **149B** (1984) 117.

[56] C. G. Callan, J. A. Harvey, *Nucl. Phys.* **B250** (1985) 427.

[57] J. Blum, J. A. Harvey, *Nucl. Phys.* **B416** (1994) 119,hep-th/9310035.

[58] M. B. Green, J. A. Harvey, G. Moore, *Class. Quantum Grav.* **14** (1997) 47, hep-th/9605033.

[59] K. Becker, M. Becker, D. R. Morrison, H. Ooguri, Y. Oz, Z. Yin, *Nucl. Phys.* **B480** (1996) 225, hep-th/9608116.

[60] M. Bershadsky, V. Sadov, C. Vafa, *Nucl. Phys.* **B463** (1996) 398, hep-th/9510225.

[61] M. Bershadsky, A. Johansen, T. Pantev, V. Sadov, C. Vafa, *Nucl. Phys.* **B448** (1995) 166, hep-th/9612052.

[62] H. Ooguri, C. Vafa, hep-th/9702180.

[63] C. Vafa, B. Zwiebach, hep-th/9701015.

[64] K. Hori, Y. Oz, hep-th/9702173.

[65] M. Blau, G. Thompson, *Nucl. Phys.* **B492** (1997) 545, hep-th/9612143.

[66] L. Baulieu, A. Losev, N. Nekrasov, hep-th/9707174.

[67] E. Witten, hep-th/9610234.

[68] L. Bonora, C. S. Chu, M. Rinaldi, hep-th/9710063.

[69] E. Witten, *Nucl. Phys.* **B460** (1996) 541, hep-th/9511030.

[70] M. R. Douglas, hep-th/9604198.

[71] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Chap. 3, Wiley-Interscience, New York 1978.

[72] J. Wess, B. Zuimino, *Phys. Lett.* **37B** (1971) 95.

[73] B. Zuimino, Lectures given at Les Houches Summer School on Theoretical Physics, 1983.

[74] S. Deser, A. Gomberoff, M. Henneaux, C. Teitelboim, *Phys. Lett.* **400B** (1997) 80, hep-th/9702184.

[75] J. Polchinski, Lectures given at TASI Summer School on Fields, Strings and Duality, 1996, hep-th/9611040.

[76] J. H. Schwarz, A. Sen, *Nucl. Phys.* **B411** (1994) 35, hep-th/9304154.

[77] G. Dall'Agata, K. Lechner, D. Sorokin, hep-th/9707044.

[78] T. Y. Wu, C. N. Yang, *Phys. Rev.* **D12** (1975) 3845.

[79] M. F. Atiyah, I. M. Singer, *Proc.Nat.Acad.Sci.* **81** (1984) 2597.

[80] O. Alvarez, I. M. Singer, B. Zuimino, *Comm. Math. Phys.* **96** (1984) 409.

[81] T. Sumitani, *J. Phys. A.* **17** (1984) L811.

[82] L. Alvarez-Gaume, P. Ginsparg, *Ann. Physics* **161** (1985) 423, Erratum *ibid.* **171** (1986) 233.

[83] J. Manes, R. Stora, B. Zuimino, *Comm. Math. Phys.* **102** (1985) 157.

[84] J. Polchinski, E. Witten, *Nucl. Phys.* **B460** (1996) 525, hep-th/9510169.

[85] R. Bott, L. W. Tu, *Differential Forms in Algebraic Topology*, Chap. 2, Springer-Verlag, New York, 1982.

[86] C. Vafa, *Nucl. Phys.* **B463** (1996) 35, hep-th/9512078.

[87] R. Harvey, B. Lawson, *Acta. Math.* **148** (1982) 47.

[88] R. C. McLean, http://www.math.duke.edu/preprints/96-01.ps.

[89] G. Papadopoulos, P. K. Townsend, *Phys. Lett.* **357B** (1995) 300, hep-th/9506150

[90] K. Intriligator, N. Seiberg, *Phys. Lett.* **387B** (1996) 513, hep-th/9607207.

[91] J. de Boer, K. Hori, H. Ooguri, Y. Oz, *Nucl. Phys.* **B493** (1997) 101, hep-th/9611063.

[92] M. Porrati, A. Zaffaroni, *Nucl. Phys.* **B490** (1997) 107, hep-th/9611201.

[93] A. Hanany, E. Witten, *Nucl. Phys.* **B492** (1997) 152, hep-th/9611230.

[94] J. de Boer, K. Hori, H. Ooguri, Y. Oz, Z. Yin, *Nucl. Phys.* **B493** (1997) 148, hep-th/9612131.

[95] K. Hori, H. Ooguri, C. Vafa, *Nucl. Phys.* **B504** (1997) 147, hep-th/9705220.

[96] S. Katz, P. Mayr, C. Vafa, hep-th/9706110.

[97] Y.-K. E. Cheung, Z. Yin, *Nucl. Phys.* **B517** (1998) 69, hep-th/9710206.

[98] J. de Boer, K. Hori, Y. Oz, Z. Yin, *Nucl. Phys.* **B502** (1997) 107, hep-th/9702154.

[99] O. Aharony, A. Hanany, K. Intriligator, N. Seiberg, M. J. Strassler, *Nucl. Phys.* **B499** (1997) 67, hep-th/9703110.

[100] S. Elitzur, A. Giveon, D. Kutasov, *Phys. Lett.* **400B** (1997) 269, hep-th/9702014.

[101] E. Witten, *Nucl. Phys.* **B500** (1997) 3, hep-th/9703166.

[102] K. Hori, H. Ooguri, Y. Oz, hep-th/9706082.

[103] E. Witten, hep-th/9706109.

[104] J. de Boer, K. Hori, H. Ooguri, Y. Oz, hep-th/9711143.

[105] N. Seiberg, E. Witten, hep-th/9607163.

[106] B. de Wit, P. G. Lauwers, A. van Proeyen, *Nucl. Phys.* **B255** (1985) 569.

[107] P. Argyres, M. R. Plesser, N. Seiberg, *Nucl. Phys.* **B471** (1996) 159, hep-th/9603042.

[108] N. Hitchin, A. Karlhede, U. Lindström, M. Roček, *Comm. Math. Phys.* **108** (1987) 535.

[109] A. Sen, *Nucl. Phys.* **B475** (1996) 562, hep-th/9605150.

[110] T. Banks, M. R. Douglas, N. Seiberg, *Phys. Lett.* **387B** (1996) 278, hep-th/9605199.

[111] M. R. Douglas, D. Kabat, P. Pouliot, S. H. Shenker, *Nucl. Phys.* **B485** (1997) 85, hep-th/9608024.

[112] N. Seiberg, *Phys. Lett.* **384B** (1996) 81, hep-th/9606017.

[113] M. R. Douglas, G. Moore, hep-th/9603167.

[114] P. B. Kronheimer, H. Nakajima, *Math. Ann.* **288** (1990) 263.

[115] A. Strominger, *Phys. Lett.* **383B** (1996) 44, hep-th/9512059.

[116] P. K. Townsend, *Phys. Lett.* **373B** (1996) 68, hep-th/9512062.

[117] I. Klebanov, hep-th/9709160.

[118] M. B. Green, M. Gupertle, *Phys. Lett.* **377B** (1996) 28.

[119] H. Nakajima, *Duke Math. J.* **76** (1994) 365.

[120] H. Nakajima, Quiver Varieties and Kac-Moody algebras, preprint.

[121] O. J. Ganor, A. Hanany, *Nucl. Phys.* **B474** (1996) 122, hep-th/9602120.

[122] N. Seiberg, E. Witten, *Nucl. Phys.* **B471** (1996) 121.

[123] P. C. Argyres, M. R. Douglas, *Nucl. Phys.* **B448** (1995) 93, hep-th/9505062.

[124] N. Seiberg, *Phys. Lett.* **388B** (1996) 753, hep-th/9608111.

[125] N. Seiberg, *Phys. Lett.* **390B** (1997) 169, hep-th/9609161.

[126] N. Seiberg, hep-th/9705117.

[127] E. Witten, hep-th/9604030.

[128] A.M. Polyakov, *Nucl. Phys.* **B** (1977) 429.

[129] M. R. Douglas and M. Li, hep-th/9604041.

[130] I. Affleck, J. Harvey, E. Witten, *Nucl. Phys.* **B206** (1982) 413.

Ernest Orlando Lawrence Berkeley National Laboratory
One Cyclotron Road | Berkeley, California 94720

Prepared for the U.S. Department of Energy under Contract No. DE-AC03-76SF00098