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Fourier-Bessel Analysis for Ordinary and Graded 2×2 Hermitean Matrices*

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Abstract

Chain-like integrals in matrix spaces play an important part in high energy and solid state physics and in general random matrix theory. In the special case of ordinary and graded 2×2 Hermitean matrices, a method is proposed to integrate out all angular variables. The essence of this method is a Fourier-Bessel analysis in these matrix spaces which is formulated in this paper. Close formal similarities are found between the ordinary and the graded case. The main differences arise from the fact that the ordinary case can be reduced to the study of a vector space whereas no analogous feature is present in the graded case.

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1 Introduction

In various fields of mathematical physics where integrations over matrix spaces are required, some relevant integrals have a chain-like structure. More precisely, if one has to integrate over M matrices H_m, $m = 1, \ldots, M$ of dimension $N \times N$ with the Cartesian volume elements $d[H_m]$ that is the product of all independent variables of H_m we call chain-like a multiple integral of the form

$$\int \prod_{m=1}^{M} d[H_m] \, f_m(H_m) \prod_{m=1}^{M-1} \exp\left(v_{m(m+1)} \mathrm{tr}\, H_m H_{m+1}\right) \quad . \tag{1}$$

In the exponential term, the matrices with neighbouring indices are coupled with, all together $M - 1$, strength parameters $v_{m(m+1)}$. The matrices H_m usually have internal symmetries, i.e. they are for example real symmetric, Hermitean or quaternion self adjoint. In that case, the functions $f_m(H_m)$ can often be thought of as invariant under the corresponding group transformations, implying $f_m(H_m) = f_m(x_m)$ where x_m is the diagonal matrix of the eigenvalues of H_m. An important physical example for such a chain-like model appears in the theory of the so called planar approximation in high energy physics as discussed by Itzykson and Zuber [1]. In the case of $M = 2$ Hermitean matrices these authors derived a powerful formula, the Itzykson-Zuber integral, to integrate over the angular degrees of freedom, i.e. over the two diagonalizing unitary matrices. Using the Itzykson-Zuber integral, Mehta and Pandey [2] performed a complete and analytical discussion of the time reversal invariance breaking in random matrix theory. However, except for trivial functions $f_m(x_m)$, the evaluation of the integral (1) for arbitrary M is still an unsolved problem, even within a saddle-point approximation. The main difficulty is apparently connected to the fact that the dimension N of the matrices is a large number.

Nevertheless, there are also chain-like models involving integrations over graded matrices with relatively small dimensions $2k$. Graded or supermatrices contain commuting and anticommuting variables, they were introduced by Berezin [3] and first used in solid state physics by Efetov [4]. The structure of the relevant integrals remains formally unchanged if the ordinary $N \times N$ matrices H_m in the expression (1) are replaced by graded $2k \times 2k$ matrices σ_m and the traces tr by graded traces trg. Similar to the ordinary case, the most interesting internal symmetries are graded real symmetric, Hermitean and quaternion self adjoint. In the case of $M = 2$ graded Hermitean matrices, the analogy of the Itzykson-Zuber integral was evaluated and used, in the framework of random matrix theory, to construct a kind of irreducible representation for the correlation functions of the Gaussian Unitary Ensemble (GUE) [5]. Moreover, employing the graded Itzykson-Zuber integral, the breaking of a quantum number, for example isospin, was discussed in a complete analytical calculation for the GUE [6]. For larger and arbitrary values of M, graded chain-like models are used in precompound nuclear scattering [7] and in the theory of mesoscopic fluctuations [8] and localization. In this context, a saddle-point approximation that removes the degrees of freedom related to the level densities can be performed and is often highly advantageous. Recently it was shown that the chain-like model in the remaining coset degrees of freedom has a direct and very useful connection to the theory of Fourier transforms and convolutions in curved spaces [9]. There are, however, situations where this kind of saddle-point approximation is not the best choice, an example is the already mentioned symmetry breaking [6]. Furthermore, in the framework of this saddle-point method, the extremely helpful determinantal structure [10] of all fluctuation functions of random matrix theory does not become obvious for higher correlation. Hence we feel that there is still a need to discuss chain-like models starting from the Cartesian formulation (1) and aiming at a method that allows the integration of all angular degrees of freedom leaving an integral that involves only the eigenvalues. Although this is highly ambitious in the general case for arbitrary M and arbitrary dimension N or $2k$, respectively, it is, to begin with, certainly worth to study the much simpler problem for arbitrary M and 2×2 matrices. The main idea is to consider the $M-1$ expressions $\exp\left(v_{m(m+1)} \mathrm{tr}\, H_m H_{m+1}\right)$ or $\exp\left(v_{m(m+1)} \mathrm{trg}\, \sigma_m \sigma_{m+1}\right)$, respectively, as something like a plane wave in which the trace of the product of the two matrices plays formally the same part as the scalarproduct in a vector model. These plane waves are expanded in the spherical functions of the angular variables, whereas the coefficients are functions of the eigenvalues alone. Inserting everything the angular degrees of freedom can be integrated by using properties of the spherical functions. In this paper, we evaluate these expansions of the plane waves and show that this is just a Fourier-Bessel analysis in matrix spaces. In section 2 we discuss the space of ordinary Hermitean 2×2 matrices. Although the results are somewhat straightforward because of special $SU(2)$ properties we do that in some detail, firstly, to make the reader acquaint with the ideas in the framework of the familiar $SU(2)$ and, secondly, to allow a direct comparison with the case of graded Hermitean 2×2 matrices that is studied in section 3. The ordinary and the graded case show both, striking similarities and considerable differences. In order to make the comparison easy, the subsections in sections 2 and 3 are organized fully analogously. Our findings are discussed in section 4. The results of the graded case allow a study of the GUE level density of the localization problem. This will be treated in a separate paper [11]. We hope that our ideas might be of some relavance for chain-like models with arbitrary dimensional Hermitean matrices, both ordinary and graded. Further investigations are under way.

2 Ordinary 2×2 Hermitean Matrices

After introducing the Cartesian formulation in subsection 2.1, we go to eigenvalue-angle coordinates in subsection 2.2. The eigenfunctions of the Laplacian in these coordinates are calculated in subsection 2.3 and used to construct a Fourier-Bessel transformation in subsection 2.4. The gradient formula is discussed in subsection 2.5

2.1 Cartesian Coordinates

We consider ordinary 2×2 Hermitean matrices parametrized as

$$H = \begin{bmatrix} H_{11} & H_{21}^* \\ H_{21} & H_{22} \end{bmatrix} \quad . \tag{2}$$

The trace of two such Hermitean matrices H and K,

$$\mathrm{tr}\, HK = H_{11}K_{11} + H_{22}K_{22} + 2\mathrm{Re}H_{21}\mathrm{Re}K_{21} + 2\mathrm{Im}H_{21}\mathrm{Im}K_{21} \tag{3}$$

will in the following play the part of the scalarproduct. Note that it is real since real and imaginary parts do not appear mixed. The differential matrix dH is found from the matrix H by replacing each matrix element with its differential. Hence the square of the invariant length element is given by

$$\mathrm{tr}\, dH^2 = dH_{11}^2 + dH_{22}^2 + 2dH_{21}^*dH_{21} \quad . \tag{4}$$

It is possible and very helpful to define a gradient operator in this matrix space, it is obtained by replacing the matrix elements of H with the corresponding derivatives and transposing the result,

$$\frac{\partial}{\partial H} = \begin{bmatrix} \partial/\partial H_{11} & \partial/\partial H_{21} \\ \partial/\partial H_{21}^{*} & \partial/\partial H_{22} \end{bmatrix} \quad . \tag{5}$$

The Laplacian is defined as the squared invariant length of the gradient

$$\Delta = \mathrm{tr}\frac{\partial^2}{\partial H^2} = \frac{\partial^2}{\partial H_{11}^2} + \frac{\partial^2}{\partial H_{22}^2} + 2\frac{\partial^2}{\partial H_{21}^{*}\partial H_{21}} \tag{6}$$

In the context of the Fourier transform, the most important function is the plane wave $\exp(i\mathrm{tr}\, HK)$. The gradient of the plane wave is given by

$$\frac{\partial}{\partial H}\exp(i\mathrm{tr}\, HK) = iK\exp(i\mathrm{tr}\, HK) \quad , \tag{7}$$

and applying the Laplacian yields

$$\Delta\exp(i\mathrm{tr}\, HK) = -\mathrm{tr}\, K^2\exp(i\mathrm{tr}\, HK) \quad . \tag{8}$$

Thus, the plane wave $\exp(i\mathrm{tr}\, HK)$ is eigenfunction of the gradient operator and the Laplacian to the eigenvalues iK and $-\mathrm{tr}\, K^2$, respectively. All this corresponds directly to the usual case of vectors, here, however, the trace plays the part of the scalarproduct.

2.2 Eigenvalues and Angles

In order to go to a Fourier-Bessel type of analysis we now have to express our formulas in Cartesian coordinates in terms of eigenvalues and diagonalizing angles,

$$H = U^{\dagger}\, x\, U \qquad \text{where} \qquad x = \mathrm{diag}(x_1, x_2) \quad . \tag{9}$$

It is of course advantageous to choose the non-canonical Euler angle parametrization for the $SU(2)$ matrix U,

$$U = \begin{bmatrix} \exp(i\psi/2) & 0 \\ 0 & \exp(-i\psi/2) \end{bmatrix} \begin{bmatrix} \cos(\vartheta/2) & \sin(\vartheta/2) \\ -\sin(\vartheta/2) & \cos(\vartheta/2) \end{bmatrix} \begin{bmatrix} \exp(i\varphi/2) & 0 \\ 0 & \exp(-i\varphi/2) \end{bmatrix} \tag{10}$$

since the angle ψ drops out in equation (9). We can write the matrix H in the form

$$H = \frac{x_1 + x_2}{2} 1 + \frac{x_1 - x_2}{2} \vec{e}_H \cdot \vec{\tau} \tag{11}$$

where 1 is the 2×2 unit matrix and

$$\vec{e}_H \cdot \vec{\tau} = \begin{bmatrix} \cos\vartheta & \exp(-i\varphi)\sin\vartheta \\ \exp(i\varphi)\sin\vartheta & -\cos\vartheta \end{bmatrix} \quad . \tag{12}$$

is nothing else but the quaternion representation of the unit vector $\vec{e}_H = (\cos\varphi\sin\vartheta, \sin\varphi\sin\vartheta, \cos\vartheta)$, usually called $\vec{e}_r$ in spherical coordinates. We have introduced the three component vector $\vec{\tau}$ of the Pauli matrices

$$\tau_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \quad \tau_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} , \quad \tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad . \tag{13}$$

This must imply a close relation to the Fourier-Bessel analysis in the usual three dimensional vector space. We now introduce a similar parametrization for the matrix K,

$$K = V^\dagger k V \qquad \text{where} \qquad k = \operatorname{diag}(k_1, k_2) \quad . \tag{14}$$

The $SU(2)$ matrix V is obtained by replacing $(\psi, \vartheta, \varphi)$ in equation (10) with (ζ, η, ξ) and the unit vector is given by $\vec{e}_K = (\cos\xi\sin\eta, \sin\xi\sin\eta, \cos\eta)$. Using the relation $\operatorname{tr}(\vec{e}_H\cdot\vec{\tau})(\vec{e}_K\cdot\vec{\tau}) = 2\vec{e}_H\cdot\vec{e}_K$ we can reduce the scalarproduct of the matrices H and K to the scalarproduct of the unit vectors,

$$\begin{aligned} \operatorname{tr} HK &= \frac{(x_1 + x_2)(k_1 + k_2)}{2} + \frac{(x_1 - x_2)(k_1 - k_2)}{2} \vec{e}_H \cdot \vec{e}_K \\ \vec{e}_H \cdot \vec{e}_K &= \cos\vartheta\cos\eta + \sin\vartheta\sin\eta\cos(\varphi - \xi) \quad . \end{aligned} \tag{15}$$

This will be very important in the next subsection.

We now have to derive the gradient operator and the Laplacian in the new coordinates. There are, of course, various ways of doing it. We choose the following procedure that proves highly efficient for the present and also later for the graded case: The matrix differential can be written in the form

$$dH = E_{x_1} dx_1 + E_{x_2} dx_2 + \frac{x_1 - x_2}{\sqrt{2}} \left(E_\vartheta d\vartheta + \sin\vartheta E_\varphi d\varphi \right) \tag{16}$$

where the four matrices E_a, $a = x_1, x_2, \vartheta, \varphi$ are explicitly given by

$$\begin{aligned} E_{x_1} &= U^\dagger \tau_{11} U = \frac{1}{2}\,(1 + \vec{e}_H \cdot \vec{\tau}) \\ E_{x_2} &= U^\dagger \tau_{22} U = \frac{1}{2}\,(1 - \vec{e}_H \cdot \vec{\tau}) \\ E_\vartheta &= \frac{1}{\sqrt{2}} U^\dagger \tau_1 U = \frac{1}{\sqrt{2}} \vec{e}_\vartheta \cdot \vec{\tau} \\ E_\varphi &= \frac{1}{\sqrt{2}} U^\dagger \tau_2 U = \frac{1}{\sqrt{2}} \vec{e}_\varphi \cdot \vec{\tau} \quad . \end{aligned} \tag{17}$$

Here, we use the 2×2 matrices τ_{ij} with unity in the position (i, j) and zeros elsewhere as a second basis besides the Pauli matrices. Moreover, the unit vectors $\vec{e}_\vartheta = (\cos\vartheta\cos\varphi, \cos\vartheta\sin\varphi, -\sin\vartheta)$ and $\vec{e}_\varphi = (-\sin\varphi, \cos\varphi, 0)$ in spherical coordinates have been introduced. As the vectors $\vec{e}_H, \vec{e}_\vartheta, \vec{e}_\varphi$ form an orthonormal set, the same is true for the four matrices E_a, $a = x_1, x_2, \vartheta, \varphi$, we find

$$\mathrm{tr}\, E_a E_b = \delta_{ab} \quad , \qquad a, b = x_1, x_2, \vartheta, \varphi \quad . \tag{18}$$

We will call these matrices basis vectors in the following, too. In the vector space we have one spherical coordinate, here we have in some sense two, the two eigenvalues x_1 and x_2. At this point, one might consider to introduce the new coordinates $x_\pm = (x_1 \pm x_2)/\sqrt{2}$ since from the results derived so far it is clear that these reflect the symmetries. We decided not to do so because, firstly, the relation is obvious and, secondly, we want to compare our results with the graded case where no similar feature exists.

Equation (16) allows together with the orthonormality (18) an easy calculation of the squared invariant length element,

$$\mathrm{tr}\, dH^2 = dx_1^2 + dx_2^2 + \frac{(x_1 - x_2)^2}{2} \left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right) \quad . \tag{19}$$

Much more important, however, equation (16) gives directly the gradient just by inverting the prefactors since the basis vectors are orthonormal,

$$\frac{\partial}{\partial H} = E_{x_1}\frac{\partial}{\partial x_1} + E_{x_2}\frac{\partial}{\partial x_2} + \frac{\sqrt{2}}{x_1 - x_2}\left(E_\vartheta \frac{\partial}{\partial \vartheta} + \frac{1}{\sin\vartheta} E_\varphi \frac{\partial}{\partial \varphi}\right) \quad , \tag{20}$$

again, this result corresponds to the vector case. Hence, we find immediately for the Laplacian

$$\Delta = \operatorname{tr} \frac{\partial^2}{\partial H^2} = \Delta_x^{(0)} + \frac{2}{(x_1 - x_2)^2} \vec{L}^2 \tag{21}$$

where we have defined

$$\Delta_x^{(0)} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{2}{x_1 - x_2} \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right) \tag{22}$$

as the pure eigenvalue part and the usual $SU(2)$ Casimir operator

$$\vec{L}^2 = \frac{\partial^2}{\partial \vartheta^2} + \operatorname{cotan} \vartheta \frac{\partial}{\partial \vartheta} + \frac{1}{\sin^2 \vartheta} \frac{\partial^2}{\partial \varphi^2} \tag{23}$$

as the pure angular part. Observe that we were able to derive the Laplacian in eigenvalue-angle coordinates without explicit inversion of some kind of a Jacobi matrix, a procedure that is always required in the standard formulas for the transformation of the Laplacian to new coordinates [12]. Here, the use of the orthonormal basis (17) made it possible to bypass such a matrix inversion.

2.3 Eigenfunctions of the Laplacian

The spherical harmonics Y_{LM} are the eigenfunctions of the pure angular part. The orthonormality and completeness, i.e.

$$\langle LM | L'M' \rangle = \delta_{LL'} \delta_{MM'} \quad \text{and} \quad \sum_{LM} |LM\rangle\langle LM| = 1 \tag{24}$$

in Dirac notation, allow the construction of the hybrid operator $\Delta_x^{(L)}$,

$$\langle LM | \Delta | L'M' \rangle = \Delta_x^{(L)} \delta_{LL'} \delta_{MM'} \quad \text{with} \quad \Delta_x^{(L)} = \Delta_x^{(0)} + \frac{2L(L+1)}{(x_1 - x_2)^2} \tag{25}$$

whose eigenfunctions to the eigenvalue $-(k_1^2 + k_2^2)$ can be easily calculated by introducing the coordinates $x_\pm$, we find

$$T_L(x,k) = \exp\left(i \frac{(x_1 + x_2)(k_1 + k_2)}{2} \right) j_L \left(\frac{(x_1 - x_2)(k_1 - k_2)}{2} \right) \tag{26}$$

where j_L is the spherical Bessel function of order L. Hence the functions $T_L(x,k) Y_{LM}(\Omega_H)$ with Ω_H shorthand for the angles (ϑ, φ) are eigenfunctions of the Laplacian in eigenvalue angle coordinates.

2.4 Fourier-Bessel Transformation

Since the scalarproduct (3) does not mix real and imaginary parts of the matrix elements, it makes sense to define the Fourier transform of a function $f(H)$ in Cartesian coordinates by

$$F(K) = \frac{1}{2(2\pi)^2} \int \exp(i \mathrm{tr}\, HK)\, f(H)\, d[H] \tag{27}$$

where the volume element

$$d[H] = dH_{11}\, dH_{22}\, d\mathrm{Re}H_{21}\, d\mathrm{Im}H_{21} \tag{28}$$

should not be confused with the differential matrix dH. The inverse transform is given by

$$f(H) = \frac{1}{2(2\pi)^2} \int \exp(-i \mathrm{tr}\, HK)\, F(K)\, d[K] \quad . \tag{29}$$

Using the results of the last two subsections, it is now very easy to go to a Fourier-Bessel type of analysis. We only need the familiar expansion of the plane wave

$$\exp(iz\vec{e}_H \cdot \vec{e}_K) = 4\pi \sum_{L=0}^{\infty} \sum_{M=-L}^{L} i^L j_L(z)\, Y^*_{LM}(\Omega_H)\, Y_{LM}(\Omega_K) \tag{30}$$

where Ω_K is shorthand for the angles (η, ξ). This yields together with equation (15) an expansion for the plane wave in the matrix space,

$$\exp(i \mathrm{tr}\, HK) = 4\pi \sum_{LM} i^L T_L(x,k)\, Y^*_{LM}(\Omega_H)\, Y_{LM}(\Omega_K) \quad . \tag{31}$$

Using the orthonormality we find the integral representation for our Bessel functions

$$T_L(x,k) = \frac{1}{i^L \sqrt{4\pi(2L+1)}} \int \exp(i \mathrm{tr}\, U^\dagger x U k)\, Y_{L0}(\Omega_H)\, d\Omega_H \tag{32}$$

where $d\Omega_H = \sin\vartheta d\vartheta d\varphi$. This is in perfect formal agreement with the vector case just by replacing the scalarproducts in the exponent.

Every well behaved function $f(H)$ has an harmonic expansion of the form

$$\begin{aligned} f(H) &= \sum_{LM} f_{LM}(x)\, Y_{LM}(\Omega_H) \\ f_{LM}(x) &= \int f(H)\, Y^*_{LM}(\Omega_H)\, d\Omega_H \end{aligned} \tag{33}$$

and analogously for $F(K)$ with coefficients $F_{LM}(k)$. We also need the transformed volume element

$$d[H] = \frac{1}{8}\,\Delta_2^2(x) d[x]\, d\Omega_H \qquad \text{where} \qquad \Delta_2^2(x) d[x] = (x_1 - x_2)^2 dx_1 dx_2 \quad , \tag{34}$$

we used the common notation $\Delta_2(x)$ for the, in this case trivial, Vandermonde determinant. Inserting now equations (31) and (33) into equation (27) we obtain the relation between these coefficients, i.e. the Bessel transforms

$$\begin{aligned} F_{LM}(k) &= \frac{i^L}{16\pi} \int T_L(k,x)\, f_{LM}(x)\, \Delta_2^2(x) d[x] \\ f_{LM}(x) &= \frac{(-i)^L}{16\pi} \int T_L(x,k)\, F_{LM}(k)\, \Delta_2^2(k) d[k] \quad . \end{aligned} \tag{35}$$

From this, we also find a version of Hankel's integral

$$\frac{1}{(16\pi)^2} \int T_L(x,k)\, T_L(k,x')\, \Delta_2^2(k) d[k] = \frac{\delta(x-x')}{\Delta_2(x)\Delta_2(x')} \tag{36}$$

which can be considered as a special case of the Bessel transform.

2.5 Gradient Formula

The raising and lowering operators for the Bessel functions $T_L(x,k)$ follow directly from equation (26). There is, however, a very instructive way of deriving them without using the explicit form of the functions $T_L(x,k)$. The previous considerations imply the decomposition

$$\Delta_x^{(L)} = \sum_{L'M'} \mathrm{tr}\, \langle LM| \frac{\partial}{\partial H} |L'M'\rangle \langle L'M'| \frac{\partial}{\partial H} |LM\rangle \tag{37}$$

of the hybrid operator. The matrix elements of the gradient are first order differential operators in the eigenvalues and they thus have to be related directly to the raising and lowering operators. In the vector space, such considerations are known as gradient formulas [13]. Hence, in our matrix space, the only tool we need is the eigenvalue-angle expression (20) for the gradient. Due to the close relation between the vector space and our matrix space we can use many of the results derived in reference [13]. We therefore skip the derivation here, the result is

$$\begin{aligned} \langle LM|\frac{\partial}{\partial H}|L'M'\rangle &= \mathcal{T}_0^{(L)}\,\delta_{LL'}\delta_{MM'}\,\frac{1}{\sqrt{2}}1 \\ &+ \sqrt{\frac{L}{2L+1}}\,\mathcal{T}_-^{(L)}\,\delta_{L(L'-1)}\,A_{LMM',-} - \sqrt{\frac{L-1}{2L+1}}\,\mathcal{T}_+^{(L)}\,\delta_{L(L'+1)}\,A_{LMM',+} \end{aligned} \quad (38)$$

where we have defined the matrices

$$A_{LMM',\pm} = \frac{1}{\sqrt{2}}\sum_{q=-1}^{+1} \vec{e}_q \cdot \vec{\tau}\,\delta_{M(M'-q)}(LM1q|(L\mp 1)(M+q)) \quad . \quad (39)$$

Here, we introduced the tensor basis vectors $\vec{e}_q$, $q = 0, \pm 1$ where $\vec{e}_0 = \vec{e}_z$ and $\vec{e}_{\pm 1} = \mp(\vec{e}_x \pm i\vec{e}_y)/\sqrt{2}$, moreover, $(LM1q|(L\mp 1)(M+q))$ are Clebsch-Gordan coefficients. The operators $\mathcal{T}_q^{(L)}$, $q = 0, \pm$ are explicitly given by

$$\begin{aligned} \mathcal{T}_0^{(L)} &= \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right) \\ \mathcal{T}_-^{(L)} &= \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right) + \frac{\sqrt{2}(L+1)}{x_1 - x_2} \\ \mathcal{T}_+^{(L)} &= \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right) - \frac{\sqrt{2}L}{x_1 - x_2} \quad , \end{aligned} \quad (40)$$

in the coordinates $x_\pm$, the operators $\mathcal{T}_\pm^{(L)}$ are of course the usual raising and lowering operators for the spherical Bessel functions. But, remarkably, even without knowing the form of the functions $T_L(x, k)$, the raising and lowering character of the operators $\mathcal{T}_q^{(L)}$ is directly obvious through the Kronecker symbols next to the operators in equation (38). The properties

$$\begin{aligned}
\mathcal{T}_0^{(L)}\, T_L(x,k) &= i\frac{k_1+k_2}{\sqrt{2}}\, T_L(x,k) \\
\mathcal{T}_-^{(L)}\, T_L(x,k) &= \frac{k_1-k_2}{\sqrt{2}}\, T_{L-1}(x,k) \\
\mathcal{T}_+^{(L)}\, T_L(x,k) &= -\frac{k_1-k_2}{\sqrt{2}}\, T_{L+1}(x,k) \quad . \qquad (41)
\end{aligned}$$

are easily checked. The hybrid operator is then given by

$$\Delta_x^{(L)} = \left(\mathcal{T}_0^{(L)}\right)^2 + \mathcal{T}_+^{(L-1)}\mathcal{T}_-^{(L)} = \left(\mathcal{T}_0^{(L)}\right)^2 + \mathcal{T}_-^{(L+1)}\mathcal{T}_+^{(L)} \quad . \qquad (42)$$

We emphasize that the operators $\mathcal{T}_q^{(L)}$ are scalar, they do not show a matrix structure. Note that the matrix elements (38) in the space of spherical harmonics are not diagonal in the projection M, i.e. the matrices $A_{LMM',\pm}$ are not proportional to $\delta_{MM'}$.

3 Graded 2 × 2 Hermitean Matrices

There are different ways of defining complex conjugation, integration and other properties of anticommuting variables. Throughout this section we use the notations and conventions of references [4, 5, 14]. Concerning the derivative with respect to an anticommuting variable γ we use for convenience a notation slightly different from the literature. We always write $\partial f(\gamma)/\partial\gamma$ but the derivative is meant to act from the right. Hence, for example, for the function $f(\gamma) = \exp(a\gamma) = 1 + a\gamma$ we find $\partial f(\gamma)/\partial\gamma = a$ no matter whether a is commuting or anticommuting. This makes the notation easier and at the same time it often prevents trouble with signs. This section is organized fully analogously to section 2.

3.1 Cartesian Coordinates

Now, we consider graded 2 × 2 Hermitean matrices

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{21}^* \\ \sigma_{21} & i\sigma_{22} \end{bmatrix} \qquad (43)$$

where σ_{11}, σ_{22} are real commuting and σ_{21} is complex anticommuting. The imaginary unit in front of σ_{22} is introduced for convergence reasons. Similar to the ordinary case, the graded trace [3] of two such matrices σ and ρ,

$$\mathrm{trg}\,\sigma\rho \;=\; \sigma_{11}\rho_{11} + \sigma_{22}\rho_{22} + \sigma_{21}^*\rho_{21} - \sigma_{21}\rho_{21}^* \quad , \tag{44}$$

which again is real, will have the meaning of the scalarproduct. The differential matrix $d\sigma$ is defined as in the ordinary case and thus the square of the invariant length element is

$$\mathrm{trg}\, d\sigma^2 \;=\; d\sigma_{11}^2 + d\sigma_{22}^2 + 2d\sigma_{21}^* d\sigma_{21} \quad . \tag{45}$$

The gradient operator is given by

$$\frac{\partial}{\partial\sigma} \;=\; \begin{bmatrix} \partial/\partial\sigma_{11} & \partial/\partial\sigma_{21} \\ \partial/\partial\sigma_{21}^* & \partial/\partial\sigma_{22} \end{bmatrix} \quad , \tag{46}$$

note the transposition compared to σ. Again, the Laplacian is defined as the squared invariant length of the gradient

$$\Delta \;=\; \mathrm{trg}\frac{\partial^2}{\partial\sigma^2} \;=\; \frac{\partial^2}{\partial\sigma_{11}^2} + \frac{\partial^2}{\partial\sigma_{22}^2} + 2\frac{\partial^2}{\partial\sigma_{21}^*\partial\sigma_{21}} \tag{47}$$

For the properties of the plane wave $\exp(i\mathrm{tr}\,\sigma\rho)$ we find

$$\frac{\partial}{\partial\sigma}\exp(i\mathrm{trg}\,\sigma\rho) \;=\; i\rho\exp(i\mathrm{trg}\,\sigma\rho) \quad , \tag{48}$$

and the Laplacian gives

$$\Delta\exp(i\mathrm{trg}\,\sigma\rho) \;=\; -\mathrm{trg}\,\rho^2\exp(i\mathrm{trg}\,\sigma\rho) \quad . \tag{49}$$

These are eigenequations with eigenvalues $i\rho$ and $-\mathrm{trg}\,\rho^2$, respectively. Up to now, everything is fully analogous to the ordinary case.

3.2 Eigenvalues and Angles

The diagonalization of the graded matrix σ reads

$$\sigma \;=\; u^\dagger s u \qquad \text{where} \qquad s \;=\; \mathrm{diag}(s_1, is_2) \quad , \tag{50}$$

where u is an $U(1/1)$ matrix. The most convenient parametrization is

$$u = \exp(i\chi/2) \begin{bmatrix} \exp(i\omega/2) & 0 \\ 0 & \exp(-i\omega/2) \end{bmatrix} \begin{bmatrix} 1+\alpha\alpha^*/2 & \alpha \\ \alpha^* & 1+\alpha^*\alpha/2 \end{bmatrix} \quad (51)$$

with two real commuting angles χ and ω and a complex anticommuting angle α. This parametrization is somewhat in the same spirit as the non-canonical Euler angles parametrization (10), some interesting properties from a grouptheoretical viewpoint are discussed in a separate paper [15]. The commuting angles drop out and we are left with

$$\begin{aligned} \sigma &= s + (s_1 - is_2) \begin{bmatrix} \alpha\alpha^* & \alpha \\ -\alpha^* & \alpha\alpha^* \end{bmatrix} \\ &= s_1\varepsilon_{s_1} + s_2\varepsilon_{s_2} \end{aligned} \quad (52)$$

where we have introduced the graded matrices

$$\begin{aligned} \varepsilon_{s_1} &= u^\dagger \tau_{11} u = \begin{bmatrix} 1+\alpha\alpha^* & \alpha \\ -\alpha^* & \alpha\alpha^* \end{bmatrix} \\ \varepsilon_{s_2} &= u^\dagger i\tau_{22} u = i \begin{bmatrix} \alpha\alpha^* & -\alpha \\ \alpha^* & 1-\alpha\alpha^* \end{bmatrix} \end{aligned} \quad (53)$$

whose properties will be discussed below. There is no obvious relation to a vector as in equation (11). Similarly, for the matrix ρ we have

$$\rho = v^\dagger r v \qquad \text{where} \qquad r = \mathrm{diag}(r_1, ir_2) \quad , \quad (54)$$

and the $U(1/1)$ matrix v is found by replacing (χ, ω, α) in equation (51) with (κ, λ, β). In eigenvalue and angel coordinates, the scalarproduct is given by

$$\mathrm{trg}\,\sigma\rho = s_1 r_1 + s_2 r_2 + (s_1 - is_2)(r_1 - ir_2)(\alpha - \beta)(\alpha^* - \beta^*) \quad . \quad (55)$$

This tells us that the action of v in this scalarproduct is just a simple translation of the anticommuting variables of u and vice versa. Another remarkable point is that the first two terms cannot be rewritten in sum and difference coordinates $s_1 \pm is_2$ and $r_1 \pm ir_2$ in such a way that these coordinates become decoupled as in equation (15). This will have important consequences. For the plane wave we find

$$\exp(i\mathrm{trg}\,\sigma\rho) = \exp(i\mathrm{trg}\,sr)\,(1 + i(s_1 - is_2)(r_1 - ir_2)(\alpha - \beta)(\alpha^* - \beta^*)) \quad (56)$$

by expanding in the anticommuting variables.

In order to derive now the gradient and the Laplacian we use the method of subsection 2.2. The differential matrix can be written in the form

$$d\sigma \;=\; \varepsilon_{s_1} ds_1 \,+\, \varepsilon_{s_2} ds_2 \,+\, (s_1 - is_2)\,(\varepsilon_\alpha d\alpha \,+\, \varepsilon_{\alpha^*} d\alpha^*) \quad . \tag{57}$$

The matrices ε_α and ε_{α^*} are defined by

$$\begin{aligned} \varepsilon_\alpha d\alpha \;&=\; u^\dagger \tau_{12} d\alpha\, u \;=\; u^\dagger \tau_{12}\, u\, p\, d\alpha \\ \varepsilon_{\alpha^*} d\alpha^* \;&=\; u^\dagger \tau_{21} d\alpha^*\, u \;=\; -u^\dagger \tau_{21}\, u\, p\, d\alpha^* \end{aligned} \tag{58}$$

where the matrix $p = \mathrm{diag}(-1, +1)$ takes care of the signs when commuting the differentials. Hence we have explicitly

$$\begin{aligned} \varepsilon_\alpha \;&=\; u^\dagger \tau_{12}\, u\, p \;=\; \begin{bmatrix} -\alpha^* & 1 \\ 0 & -\alpha^* \end{bmatrix} \\ \varepsilon_{\alpha^*} \;&=\; -u^\dagger \tau_{21}\, u\, p \;=\; \begin{bmatrix} \alpha & 0 \\ -1 & \alpha \end{bmatrix} \end{aligned} \tag{59}$$

implying the relation $\varepsilon_\alpha^\dagger = \varepsilon_{\alpha^*}$. Observe that these matrices are not graded in the usual sense, they have anticommuting entries on the diagonal and commuting ones on the off-diagonal. Including the differentials, however, the matrices (58) are graded in the usual sense. Similar to the ordinary case, the four matrices ε_a, $a = s_1, s_2, \alpha, \alpha^*$ form an orthonormal basis. Since ε_α and ε_{α^*} are by construction related to shift-operators we find slightly different from equation (18)

$$\mathrm{trg}\, \varepsilon_{a^*} \varepsilon_b \;=\; \delta_{ab} \quad , \qquad a, b \;=\; s_1, s_2, \alpha \tag{60}$$

and we emphazise $\mathrm{trg}\, \varepsilon_{\alpha^*} \varepsilon_\alpha = -\mathrm{trg}\, \varepsilon_\alpha \varepsilon_{\alpha^*}$. The crossing feature (60) for the basis vectors (59) can be overcome by introducing coordinates $\alpha_\pm = (\alpha \pm \alpha^*)/\sqrt{2}$ with $\alpha_-^* = \alpha_+$. However, we do not use them since they destroy the translation property mentioned concerning equation (55).

We easily find for the squared invariant length element

$$\mathrm{trg}\, d\sigma^2 \;=\; ds_1^2 \,+\, ds_2^2 \,+\, 2(s_1 - is_2)^2 d\alpha d\alpha^* \quad . \tag{61}$$

As in the ordinary case, the gradient follows directly from the differential matrix (57),

$$\frac{\partial}{\partial \sigma} \;=\; \varepsilon_{s_1} \frac{\partial}{\partial s_1} + \varepsilon_{s_2} \frac{\partial}{\partial s_2} + \frac{1}{s_1 - is_2} \left(\varepsilon_\alpha \frac{\partial}{\partial \alpha^*} - \varepsilon_{\alpha^*} \frac{\partial}{\partial \alpha} \right) \quad , \tag{62}$$

the only difference is the interchange of α and α^* in the derivatives which is of course due to the crossing feature (60). Using results like

$$\frac{\partial \varepsilon_{s_1}}{\partial a} = \varepsilon_a \quad , \qquad \frac{\partial \varepsilon_{s_2}}{\partial a} = -i\varepsilon_a \quad , \qquad a = \alpha, \alpha^* \tag{63}$$

we find for the Laplacian

$$\Delta = \operatorname{trg} \frac{\partial^2}{\partial \sigma^2} = \Delta_s^{(0)} + \frac{2}{(s_1 - is_2)^2} \Lambda^2 \tag{64}$$

where we have defined

$$\Delta_s^{(0)} = \frac{\partial^2}{\partial s_1^2} + \frac{\partial^2}{\partial s_2^2} - \frac{2}{s_1 - is_2} \left(\frac{\partial}{\partial s_1} - i\frac{\partial}{\partial s_2} \right) \tag{65}$$

as the pure eigenvalue part and

$$\Lambda^2 = \frac{\partial^2}{\partial \alpha \partial \alpha^*} \tag{66}$$

as the pure angular part. It is shown in reference [15] that this is indeed the Casimir operator on the coset parameter space. Again, these results correspond directly to those in subsection 2.2.

3.3 Eigenfunctions of the Laplacian

In order to go now to a Fourier-Bessel type of analysis we cannot resort to the results of a corresponding vector case as in the ordinary case. Hence we have to solve the eigenequation for the Laplacian (64) in eigenvalue-angle coordinates. We do this by the separation ansatz $t(s)y(\omega_\sigma)$ with ω_σ shorthand for (α, α^*). The eigenequation for the angular function is

$$\Lambda^2 y(\omega_\sigma) = -\mu\mu^* y(\omega_\sigma) \tag{67}$$

with an eigenvalue $-\mu\mu^*$ that is real since Λ^2 is real and that is the product of an anticommuting variable μ and its complex conjugate since it has to reflect the structure of the operator Λ^2. Obviously, there is a freedom of choice for the sign of this eigenvalue, we could write $+\mu\mu^*$ as well. The implications of this observation are discussed in reference [15]. Here, we always use the form (67) of the eigenequation. Formally, the eigenvalue is something like the squared length of the anticommuting variable μ and we also write $|\mu|^2 = \mu\mu^*$. The most general form of the function is $y(\omega_\sigma) = y_0 + y_{11}\alpha + y_{12}\alpha^* + y_2\alpha\alpha^*$, this ansatz gives equations for the coefficients y_i, $i = 0, 11, 12, 2$. The first solutions are

$$y(\omega_\sigma) = y_{\mu\mu^*}(\omega_\sigma) = 2\pi \exp(\mu\alpha - \mu^*\alpha^*) \quad , \tag{68}$$

the second solutions are the complex conjugates $y^*_{\mu\mu^*}(\omega_\sigma) = y_{\mu\mu^*}(-\omega_\sigma)$. The factor 2π will turn out the proper normalization. These angular functions have some simple properties like

$$\begin{aligned} y_{\mu\mu^*}(\omega_\sigma)\, y_{\mu\mu^*}(\omega_\rho) &= 2\pi\, y_{\mu\mu^*}(\omega_\sigma + \omega_\rho) \quad , \\ y_{\mu\mu^*}(\omega_\sigma)\, y^*_{\mu\mu^*}(\omega_\rho) &= 2\pi\, y_{\mu\mu^*}(\omega_\sigma - \omega_\rho) \quad , \\ y_{\mu\mu^*}(\omega_\sigma)\, y_{\mu'\mu'^*}(\omega_\sigma) &= 2\pi\, y_{(\mu+\mu')(\mu+\mu')^*}(\omega_\sigma) \quad , \\ y_{\mu\mu^*}(\omega_\sigma)\, y^*_{\mu'\mu'^*}(\omega_\sigma) &= 2\pi\, y_{(\mu-\mu')(\mu-\mu')^*}(\omega_\sigma) \end{aligned} \tag{69}$$

which imply the integral relations

$$\begin{aligned} \int y^*_{\mu\mu^*}(\omega_\sigma)\, y_{\mu'\mu'^*}(\omega_\sigma)\, d\omega_\sigma &= \delta(\mu^* - \mu'^*)\, \delta(\mu - \mu') \\ \int y_{\mu\mu^*}(\omega_\sigma)\, y^*_{\mu\mu^*}(\omega_\rho)\, d[\mu] &= \delta(\omega_\sigma - \omega_\rho) \end{aligned} \tag{70}$$

where $d\omega_\sigma = d\alpha d\alpha^*$ and $d[\mu] = d\mu d\mu^*$. The δ function of an anticommuting variable γ is defined [4] by $\delta(\gamma) = \sqrt{2\pi}\gamma$. The equations (70) can be considered as orthonormality and completeness relations of the functions (68). Hence, these functions form something like a Hilbert space. However, they are not a countable infinite set labeled by indices like LM as in the ordinary case. The indices here are the Grassmann variables μ and μ^* and the double sum is replaced by a Grassmann double integral. Introducing a Dirac type of notation we have in full formal analogy to equations (24)

$$\langle\mu\mu^*|\mu'\mu'^*\rangle = \delta([\mu] - [\mu']) \quad \text{and} \quad \int d[\mu]\, |\mu\mu^*\rangle\langle\mu\mu^*| = 1 \tag{71}$$

where we used $\delta([\mu] - [\mu']) = \delta(\mu^* - \mu'^*)\delta(\mu - \mu')$. Thus, the functions (68) might be called graded spherical harmonics.

We now can construct an hybrid operator $\Delta_s^{(|\mu|)}$ by calculating the matrix elements of the full Laplacian,

$$\begin{aligned} \langle \mu\mu^* | \Delta | \mu'\mu'^* \rangle &= \Delta_s^{(|\mu|)} \, \delta([\mu] - [\mu']) \\ \Delta_s^{(|\mu|)} &= \Delta_s^{(0)} - \frac{2|\mu|^2}{(s_1 - is_2)^2} \quad . \end{aligned} \tag{72}$$

Observe that the hybrid operator depends only on the length of the anticommuting variable μ. The corresponding eigenequation is

$$\Delta_s^{(|\mu|)} \, t(s) = -(r_1^2 + r_2^2) \, t(s) \tag{73}$$

to the eigenvalue $-(r_1^2 + r_2^2)$. This partial differential equation is entirely different from the ordinary case. To show this we introduce the complex coordinate $z = (s_1 + is_2)/\sqrt{2}$ and similarly $w = (r_1 + ir_2)/\sqrt{2}$, this gives

$$\left(2\frac{\partial^2}{\partial z^* \partial z} - \frac{2}{z^*}\frac{\partial}{\partial z} - \frac{|\mu|^2}{z^{*2}} \right) t(z, z^*) = -2|w|^2 \, t(z, z^*) \tag{74}$$

Note that the first order derivative occurs with a minus sign. More important, however, due to the imaginary unit in front of s_2, mixed expressions in z and z^* appear whereas the coordinates $x_\pm$ and $k_\pm$ decouple in the ordinary case in such a way that the solution $T_L(x, k)$ is a product of two functions depending on the pairs (x_+, k_+) and (x_-, k_-), respectively. Here, the solution cannot be a product of functions that depend only on (z, w) and (z^*, w^*), respectively. The ansatz $t(z, z*) = \exp(iwz^* + iw^*z)h(z^*)$ couples (z, w^*) and (z^*, w), the remaining function $h(z^*)$ which is solution of

$$i2w^* \left(\frac{\partial}{\partial z^*} - \frac{1}{z^*} \right) h(z^*) = \frac{|\mu|^2}{z^{*2}} \, h(z^*) \tag{75}$$

has to depend on (z^*, w^*). We find $h(z^*) = 2w^*z^* + i|\mu|^2$ which is symmetric in z^* and w^*. Hence, collecting everything and going back to the eigenvalues we find including a normalization factor

$$t(s) = t_{|\mu|}(s, r) = \frac{1}{2\pi} \exp(i\,\mathrm{trg}\, sr) \left((r_1 - ir_2)(s_1 - is_2) + i|\mu|^2 \right) \quad . \tag{76}$$

These functions will play a part fully analogous to the functions $T_L(x, k)$ in the ordinary case and will thus be called graded Bessel functions. Observe the symmetry $t_{|\mu|}(s,r) = t_{|\mu|}(r,s)$ corresponding to $T_L(x,k) = T_L(k,x)$. Hence the functions $t_{|\mu|}(s,r)y_{\mu\mu^*}(\omega_\sigma)$ are eigenfunctions to the Laplacian in eingenvalue angle coordinates.

At this point, an important remark is in order. Due to the Efetov-Wegner theorem [4, 16], the formula (76) is not yet complete if an integration over the whole matrix space is required. In case we want to integrate over the whole ρ space, we find using results of reference [5]

$$t_{|\mu|}(s,r) = (1-\eta(s))\,(r_1 - ir_2)^2 \exp\left(\frac{1}{2}\mathrm{trg}\,(r^2 - s^2)\right)\,\delta(r) + \frac{1}{2\pi}\exp(i\mathrm{trg}\,sr)\,\left((r_1 - ir_2)(s_1 - is_2) + i|\mu|^2\right) \quad (77)$$

where we have defined the function

$$\eta(s) = \begin{cases} 0 & \text{if } s = 0 \\ 1 & \text{else} \end{cases} \quad . \quad (78)$$

The first term in equation (77) is not symmetric in s and r. For integrations over the whole σ space, s and r have to be interchanged. The practial relevance of the Efetov-Wegner term is shown in the appendix.

3.4 Fourier-Bessel Transformation

The graded Fourier transform of a function $f(\sigma)$ in Cartesian coordinates is well defined through the equation

$$F(\rho) = \int \exp(i\mathrm{trg}\,\sigma\rho)\,f(\sigma)\,d[\sigma] \quad (79)$$

where the volume element is given by

$$d[\sigma] = d\sigma_{11}\,d\sigma_{22}\,d\sigma_{21}^*\,d\sigma_{21} \quad . \quad (80)$$

The inverse transform is

$$f(\sigma) = \int \exp(-i\mathrm{trg}\,\sigma\rho)\,F(\rho)\,d[\rho] \quad , \quad (81)$$

the necessarily occuring δ functions in anticommuting variables are also well defined [4]. In order to go to a Fourier-Bessel type of analysis, we need the expansion of the plane wave which is given by

$$\exp(i\mathrm{trg}\,\sigma\rho) = \int t_{|\mu|}(s,r)\,y_{\mu\mu^*}^*(\omega_\sigma)\,y_{\mu\mu^*}(\omega_\rho)\,d[\mu] \quad (82)$$

as can be proven in a direct calculation. The graded Bessel functions in this expansion are those from equation (76). Observe the complete formal similarity of this expansion to the expansion (31) in the ordinary case. Using the orthonormality of the graded spherical harmonics we find the integral representation for the graded Bessel functions,

$$t_{|\mu|}(s,r) = \frac{1}{2\pi}\int \exp\left(i\,\mathrm{trg}\,u^{\dagger}sur\right)\, y_{\mu\mu^*}(\omega_\sigma)\, d\omega_\sigma \tag{83}$$

similar to equation (32).

Due to the relations (71) we obtain in full analogy to the formulas (33) the harmonic expansion of any well behaved function $f(\sigma)$ in the graded spherical harmonics

$$\begin{aligned} f(\sigma) &= \int f_{\mu\mu^*}(s)\, y_{\mu\mu^*}(\omega_\sigma)\, d[\mu] \\ f_{\mu\mu^*}(s) &= \int f(\sigma)\, y^*_{\mu\mu^*}(\omega_\sigma)\, d\omega_\sigma \end{aligned} \tag{84}$$

and similarly for $F(\rho)$ with coefficients $F_{\mu\mu^*}(r)$. The transformed volume element reads [5]

$$d[\sigma] = B_1^2(s)d[s]\, d\omega_\sigma \qquad \text{where} \qquad B_1^2(s)d[s] = \frac{ds_1 ds_2}{(s_1 - is_2)^2} \quad . \tag{85}$$

Inserting now equations (82) and (84) into equation (79) we obtain the relation between the coefficients of the harmonic expansion, i.e. the graded Bessel transforms

$$\begin{aligned} F_{\mu\mu^*}(r) &= \int t_{|\mu|}(r,s)\, f_{\mu\mu^*}(s)\, B_1^2(s)d[s] \\ f_{\mu\mu^*}(s) &= \int t_{|\mu|}(s,r)\, F_{\mu\mu^*}(r)\, B_1^2(r)d[r] \quad . \end{aligned} \tag{86}$$

The corresponding graded version of Hankel's integral

$$\int t_{|\mu|}(s,r)\, t_{|\mu|}(r,s')\, B_1^2(r)d[r] = \frac{\delta(s-s')}{B_1(s)B_1(s')} \tag{87}$$

can again be considered as a special case of the graded Bessel transform.

3.5 Gradient Formula

Fully analogously to the ordinary case, the hybrid operator can be decomposed in the matrix elements of the gradient,

$$\Delta_s^{(|\mu|)} = \int d[\mu'] \,\mathrm{trg}\, \langle \mu\mu^* | \frac{\partial}{\partial\sigma} | \mu'\mu'^* \rangle \langle \mu'\mu'^* | \frac{\partial}{\partial\sigma} | \mu\mu^* \rangle \quad . \tag{88}$$

Again, the matrix elements of the gradient are first order differential operators in the eigenvalues. Hence, from our experience in the ordinary case, we conclude that, provided they exist, the raising and lowering operators of the functions $t_{|\mu|}(s,r)$ have to show up in these matrix elements. It is of course clear that raising and lowering operators fully analogous to the ordinary case cannot exist since the index $|\mu|$ of the graded Bessel functions does not span a countable set whatsoever. But it is worthwhile to construct the graded gradient formula in order to see how the first order operators in the eigenvalues show up. To do so, we first evaluate the auxiliary relations

$$\begin{aligned}
\langle \mu\mu^* | \alpha | \mu'\mu'^* \rangle &= \sqrt{2\pi}\, \delta(\mu^* - \mu'^*) \\
\langle \mu\mu^* | \alpha\alpha^* | \mu'\mu'^* \rangle &= -2\pi \\
\langle \mu\mu^* | \frac{\partial}{\partial\alpha} | \mu'\mu'^* \rangle &= \sqrt{2\pi}\, \mu\mu'\, \delta(\mu^* - \mu'^*) \\
\langle \mu\mu^* | \alpha\frac{\partial}{\partial\alpha} | \mu'\mu'^* \rangle &= \sqrt{2\pi}\, \mu'\, \delta(\mu^* - \mu'^*) \quad .
\end{aligned} \tag{89}$$

Now collecting everything we find for the graded gradient formula

$$\begin{aligned}
\langle \mu\mu^* | \frac{\partial}{\partial\sigma} | \mu'\mu'^* \rangle &= \begin{bmatrix} \partial/\partial s_1 & 0 \\ 0 & i\partial/\partial s_2 \end{bmatrix} \delta(\mu - \mu')\delta(\mu^* - \mu'^*) \\
&- \sqrt{2\pi} \left(\frac{\partial}{\partial s_1} - i\frac{\partial}{\partial s_2} \right) \begin{bmatrix} \sqrt{2\pi} & -\delta(\mu^* - \mu'^*) \\ -\delta(\mu - \mu') & \sqrt{2\pi} \end{bmatrix} \\
&\frac{1}{s_1 - is_2} \left(\begin{bmatrix} 1 & -\mu^* \\ 0 & 1 \end{bmatrix} \delta(\mu'^*)\delta(\mu - \mu') - \begin{bmatrix} 1 & 0 \\ -\mu & 1 \end{bmatrix} \delta(\mu')\delta(\mu^* - \mu'^*) \right)
\end{aligned} \tag{90}$$

which inserted in equation (88) yields of course the correct hybrid operator. Comparing now equations (38) and (90), we find considerable differences in their structure. In the ordinary case, the operators that involve the eigenvalues are scalar, i.e. they do not show a matrix structure. Here, however, this is obviously different. Hence, we draw the conclusion that there are now scalar operators somehow similar to the ordinary case. It is of course possible to combine the eigenvalue operators appearing in the gradient formula (90) in such a way that they yield the hybrid operator $\Delta_s^{(|\mu|)}$, but apparently they do not allow an interpretation in any sense as raising and lowering operators.

4 Summary and Discussion

We formulated the Fourier-Bessel analysis in the space of ordinary and graded 2×2 Hermitean matrices. In the space of ordinary matrices, the close relation to the three dimensional vector space simplifies the analysis considerably. Remarkably, the Laplacian could be constructed without explicitly inverting any kind of Jacobi matrix. Instead, an orthonormal basis was introduced that allowed an easy calculation of the Laplacian in eigenvalue angle coordinates and thus the bypassing of such an inversion. This method is independent of any possible relation between the matrix and a vector space. However in this case, the relevant angular operator in the Laplacian is precisely the $SU(2)$ Casimir operator. Consequently, the angular part of the eigenfunctions is given by the usual spherical harmonics. The eigenvalue part could be reduced to a simple exponential and the spherical Bessel functions by introducing proper coordinates which decouple the trace, i.e. the scalarproduct in the matrix space. Hence, the formulas for the Fourier-Bessel analysis reflect essentially those in the three dimensional vector space. A gradient formula in the matrix space was evaluated giving the raising and lowering operators for the eigenvalue functions. Generally, this is possible without knowing these functions explicitly.

In the space of graded matrices, however, there is no direct correspondence to a vector space. Although it is possible to construct vector spaces with related properties, the correpondences found so far are never as close as in the ordinary case. The reason might be the very special properties of the algebra of $SU(2)$, or $SO(3)$: it is the only one whose generators can be efficiently ordered into a vector, the angular momentum. Apparently, there are no similar features for the algebra of the graded group $U(1/1)$. The method of evaluating the gradient and thus the Laplacian in eigenvalue angle coordinates that was derived in the ordinary case works very well in the graded case, too. In particular, an orthonormal basis was found in graded matrices.

The eigenfunctions of the angular part of the Laplacian span a space that has orthonormality and completeness relations and hence formally something in common with an Hilbert space. The indices of those graded spherical harmonics are an anticommuting number and its complex conjugate but not a countable infinite set of integers. The eigenvalue part of the eigenfunctions satisfies a partial differential equation entirely different from the ordinary case, especially, there is no decoupling in suitable coordinates similar to the latter. The index of the resulting graded Bessel functions is the length of the anticommuting indices of the graded spherical harmonics. All formulas concerning the Fourier-Bessel analysis in the ordinary case have a direct analogy in the graded case. However, due to the lack of a corresponding vector space and thus a decoupling feature, there are no scalar operators that could be interpreted in any sense as analogous to raising and lowering operators. This became obvious in the calculation of the graded analogy of the gradient formula.

This investigations provide a tool to study chain-like models in the special case of 2×2 matrices. The application to the study of the GUE level density of the localization problem will be discussed in a separate paper [11]. Furthermore, these considerations and some of our methods might be relevant for the study chain-like models with arbitrarily dimensional matrices, in particular we have in mind the procedure of constructing the Laplacian without explicit inversions and the gradient formulas which might , in the sector of commuting variables, allow the calculation of the higher eigenvalue function from the lowest one, i.e. without evaluating integrals.

Appendix: Relevance of the Efetov-Wegner Term

In Cartesian coordinates the Fourier transform of the Gaussian function is again Gaussian,

$$\int \exp\left(-\frac{1}{2}\mathrm{trg}\,\rho^2\right)\,\exp\left(i\mathrm{trg}\,\rho\sigma\right)\,d[\rho] \;=\; \exp\left(-\frac{1}{2}\mathrm{trg}\,\sigma^2\right) \quad . \qquad (91)$$

We will now demonstrate that this result is achieved in eigenvalue-angle coordinates only if the Efetov-Wegner term is included. Inserting the expansion (82) and using the integral representation [5]

$$\eta(s) \;=\; \frac{1}{2\pi B_1(s)} \int \exp\left(-\frac{1}{2}\mathrm{trg}\,(r-s)^2\right) B_1(r) d[r] \tag{92}$$

and the orthonormality of the graded spherical harmonics we find

$$\begin{aligned}
&\int \exp\left(-\frac{1}{2}\mathrm{trg}\,\rho^2\right) \exp\left(i\mathrm{trg}\,\rho\sigma\right)\, d[\rho] \\
&= \int d[r] B_1^2(r) \exp\left(-\frac{1}{2}\mathrm{trg}\,r^2\right) \int d\omega_\rho \\
&\qquad \int d[\mu]\, t_{|\mu|}(s,r)\, y_{\mu\mu^*}(\omega_\sigma)\, y_{\mu\mu^*}(\omega_\rho) \\
&= \int d[r] B_1^2(r) \exp\left(-\frac{1}{2}\mathrm{trg}\,r^2\right) \int d[\mu]\, t_{|\mu|}(s,r)\, y_{\mu\mu^*}(\omega_\sigma)\, \frac{\delta(\mu^*)\delta(\mu)}{2\pi} \\
&= \int d[r] B_1^2(r) \exp\left(-\frac{1}{2}\mathrm{trg}\,r^2\right) t_0(s,r) \\
&= (1-\eta(s)) \exp\left(-\frac{1}{2}\mathrm{trg}\,s^2\right) \\
&\qquad + \frac{1}{2\pi B_1(s)} \int d[r] B_1(r) \exp\left(-\frac{1}{2}\mathrm{trg}\,r^2\right) \exp\left(i\mathrm{trg}\,sr\right) \\
&= (1-\eta(s)) \exp\left(-\frac{1}{2}\mathrm{trg}\,s^2\right) + \eta(s) \exp\left(-\frac{1}{2}\mathrm{trg}\,s^2\right) \\
&= \exp\left(-\frac{1}{2}\mathrm{trg}\,\sigma^2\right) \quad ,
\end{aligned} \tag{93}$$

i.e. the required result.

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