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SEMIQUANTUM GEOMETRY

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Dedicated to Yu. I. Manin on the occasion of his 60th birthday

In this paper we study associative algebras with a Poisson algebra structure on the center acting by derivations on the rest of the algebra. These structures appeared in the study of quantum groups at roots of 1 and related algebras. They also appeared in the study of the representation theory of affine Lie algebras at the critical level. We also believe that these *Poisson fibred algebras* play an essential role in Fedosov's quantization of matrix-valued functions on symplectic manifolds. They are also implicit in the work of Emmrich and one of the authors [10] on multicomponent WKB approximations. In all these cases, this special Poisson algebra structure is induced by a one parameter family of deformations.

We also take up the general study of noncommutative spaces which are close to enough commutative ones so that they contain enough points to have interesting commutative geometry. One of the most striking uses of our noncommutative spaces is the quantum Borel-Weil-Bott Theorem 6.1 for quantum $\mathfrak{sl}_q(2)$ at a root of unity, which comes as a calculation of the cohomology of actual sheaves on actual topological spaces. The idea of such noncommutative spaces is not new and has been approached by many mathematicians, starting with A. Grothendieck [14], who treated the most general situation of a scheme ringed with a sheaf of noncommutative algebras. Later work [5, 6, 16, 19] treated a particular case, supermanifolds or $\mathbb{Z}_2$-graded manifolds, which are in some sense quantum manifolds at $q = -1$, a square root of unity. The idea of unifying supermathematics with noncommutative geometry was made explicit by Manin, see [22].

In a series of papers [2, 3, 4], M. Artin, J. Tate and M. Van den Bergh worked out the case of quantum projective planes, being mostly interested in the ones corresponding to an elliptic curve in $\mathbb{P}^2$ and an

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automorphism of the curve. When the automorphism was of finite order, they were in the situation of the present paper, but only in dimension two. Yu. I. Manin [21] noticed the same pattern, working with quantum abelian varieties at roots of unity. B. Parshall and Jian Pan Wang [23] made a very comprehensive study of quantum groups and homogeneous spaces at roots of unity at the level of Hopf algebras and their comodules. And finally, C. De Concini, V. G. Kac and C. Procesi [9] studied orbits of the coadjoint representation of quantum groups at roots of unity and came across similar spaces.

For a quantum geometric object X_q (a deformation of a classical object), it would be interesting to consider as a whole the hierarchy of quantum objects X_ϵ over roots ϵ of unity of all possible orders. The analogy with Diophantine geometry beyond the Frobenius morphism (which in this work relates X_ϵ with X_1), that is, for instance, studying the corresponding Galois groups and relating them to $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, defining the corresponding ζ-functions, etc., seems to be very promising, both for noncommutative geometry and number theory.

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1. Poisson fibred algebras

1.1. Here and below we assume that all objects are defined over a field κ of characteristic zero.

As usual, by a Poisson algebra we mean a commutative algebra with a Lie algebra structure on it (given by a Poisson bracket), such that the Poisson bracket acts on the commutative algebra by derivations. A Poisson manifold is a smooth manifold X with a Poisson algebra structure on the smooth functions on X under pointwise multiplication.

Definition 1.1. A *Poisson fibred algebra* is a triple $(A, Z, \{\cdot,\cdot\})$, where A is an associative algebra with a unit 1, Z is its center, and $\{\cdot,\cdot\}$ is a bracket

$$\begin{aligned} Z \times A &\to A, \\ (f, a) &\mapsto \{f, a\} \end{aligned}$$

which has the following properties.

1. Its restriction to $Z \times Z$ provides Z with the structure of a Poisson algebra.
2. It gives an action by derivations of this Poisson algebra on the whole algebra A, such that

$$\{f, ab\} = \{f, a\}b + a\{f, b\},$$
$$\{fg, a\} = f\{g, a\} + g\{f, a\},$$

for $f, g \in Z$, $a, b \in A$.

Notice that the last two equations imply

$$\{f, 1\} = \{1, a\} = 0 \qquad \text{for all } f \in Z,\ a \in A.$$

Definition 1.2. The obstruction $\Phi(f, g) \in Der(A)$ to A being a Lie algebra module over Z

$$\Phi(f, g)(a) = \{\{f, g\}, a\} - \{f, \{g, a\}\} + \{g, \{f, a\}\}, \tag{1}$$

where $f, g \in Z$, $a \in A$, is called the *curvature of the Poisson fibred algebra*.

It is obviously Z-linear in f, g and a and skew in f and g.

Notice that any linear map $\theta : A \to A$ determines a new bracket

$$\{f, g\}_\theta = \{f, g\} + [\theta(f), g], \tag{2}$$

which also provides a structure of Poisson fibred algebra on A.

We will call two Poisson fibred algebras *equivalent* if their brackets are related by a transformation (2). The curvature $\Phi(f, g)_\theta$ of the Poisson fibred algebra with the bracket (2) is related to (1) as follows

$$\Phi(f, g)_\theta(a) = \Phi(f, g)(a) + [\theta(\{f, g\}), a] - [\{f, \theta(g)\}, a] + [\{g, \theta(f)\}, a] - [[\theta(f), \theta(g)], a]. \tag{3}$$

Thus, strictly speaking, by the curvature of a Poisson fibred algebra we should understand the equivalence class of Φ under the transformations (3). In particular, one can say that a Poisson fibred algebra is flat if its curvature can be removed by a "gauge" transformation (2).

1.2. **The geometric interpretation of Poisson fibred algebras.** Poisson fibred algebras arise from the following geometric situation.

Recall that a *Lie algebroid* is a vector bundle L over a smooth manifold X together with the structure $[,]$ of a Lie algebra on its smooth sections and a vector-bundle morphism $\rho : L \to TX$ (called the *anchor* of the Lie algebroid) satisfying the following conditions:

$$[\rho(a), \rho(b)] = \rho([a, b]),$$
$$[a, fb] = f[a, b] + (\rho(a)f)b, \tag{4}$$

where a and b are sections of L and f is a function on X. For more details about Lie algebroids, see [18].

Suppose that we have a Poisson manifold X. The Poisson structure on X defines the structure of a Lie algebroid on its cotangent bundle. The anchor is defined as contraction ι with the bivector field $P \in \Gamma(X, \Lambda^2 TX)$ defining the Poisson structure:

$$\rho : T^*X \to TX,$$
$$\rho(\omega) = \iota(\omega)P = \omega(P).$$

The bracket on 1-forms is defined as the one extending the bracket

$$[df, dg] = d\{f, g\}$$

by the Leibniz rule (4). This yields the following formula for arbitrary 1-forms:

$$[\omega, \varphi] = L_{\rho(\omega)}\varphi - L_{\rho(\varphi)}\omega + d\iota(\omega \wedge \varphi)P.$$

Thinking of a Lie algebroid as a generalization of the tangent bundle, we would like to define the notion of a Lie algebroid connection in a vector bundle V over X. Consider the *Atiyah algebra* $\mathcal{A}(V)$ of V, which is the bundle of first-order differential operators on V with symbol $\sigma = \mathrm{Id}_V \otimes \xi$, where ξ is a vector field on X. There is a natural short exact sequence

$$0 \to \mathrm{End}(V) \to \mathcal{A}(V) \xrightarrow{\sigma} TX \to 0$$

of vector bundles. The Atiyah algebra is a Lie algebroid with respect to the commutator of differential operators and the anchor map σ.

If L is a Lie algebroid over X, an anchor preserving vector bundle mapping

$$\nabla : L \to \mathcal{A}(V) \tag{5}$$

will be called an *L-connection on* V; we define its *curvature* as the section Φ_∇ of the bundle $\Lambda^2 L^* \otimes \mathcal{A}(V)$ by the formula

$$\Phi_\nabla(\omega \wedge \varphi) = \nabla[\omega, \varphi] - [\nabla\omega, \nabla\varphi].$$

An L-connection ∇ is a morphism of Lie algebroids if and only if its curvature vanishes. In this case, the "flat connection" ∇ is also called a *representation of the Lie algebroid* L *in* V. (When X is a point, L is just a Lie algebra, and this is the usual definition of a representation.)

Now assume that V is a bundle of associative algebras over a Poisson manifold X and that ∇ is a T^*X-connection on V such that

$$\nabla_\omega(ab) = (\nabla_\omega a)b + a(\nabla_\omega b) \tag{6}$$

for all 1-forms ω and $a, b \in C^\infty(X, V)$.

Theorem 1.1. *The bracket*

$$\{f, a\} = \nabla_{df} a$$

where f is a function on X and a is a section of V, defines the structure of a Poisson fibred algebra on sections of V. Conversely, let $Z = C^\infty(X)$ be the algebra of smooth functions on a manifold X and $A = C^\infty(X, V)$ the Z-module of smooth sections of a vector bundle V over X. Then any local structure of a Poisson fibred algebra on the pair (A, Z) gives rise to a T^*X-connection ∇ on V satisfying (6). *The curvature of this Poisson fibred algebra is related to the curvature of ∇ by the formula*

$$\Phi(f, g) = \Phi_\nabla(df, dg).$$

Proof. The fact that a T^*X-connection satisfying (6) gives a Poisson fibred algebra is a direct verification of axioms. The converse statement is proved by defining a T^*X-connection by the formula

$$\nabla_\omega a(x) = \{f, a\}(x),$$

where ω is a 1-form, a is a section of V and f is a function on X, such that $df(x) = \omega(x)$ at a point $x \in X$. The fact that for a fixed section a of V, the bracket $\{f, a\}$ is a first-order differential operator on X ensures that $\{f, a\}(x)$ depends only on the first jet of f at $x \in X$, which guarantees correctness of our definition of ∇. □

1.3. **Formal deformations.** Poisson fibred algebras appear naturally in the context of formal deformations.

Consider an associative algebra A with the center Z. Let A_h be a formal deformation of A such that A_h is a $\kappa[[h]]$-algebra isomorphic to $A[[h]]$ as a module over $\kappa[[h]]$. Choose such an isomorphism $\phi : A[[h]] \to A_h$. Let $\phi(1) = 1$. In the sequel we will identify $A[[h]]$ with A_h via this isomorphism.

After this identification the multiplication in A_h, sometimes called $*$-multiplication, is given by a formal power series

$$f * g = fg + hB_1(f, g) + h^2 B_2(f, g) + \dots . \tag{7}$$

Here $B_i(f, g)$ are A-valued bilinear forms on A. The following proposition follows from identities for commutators in the associative algebra A_h.

Proposition 1.2. 1. *The bracket*

$$\{a, b\} = B_1(a, b) - B_1(b, a) \tag{8}$$

determines a Poisson fibred algebra structure on (A, Z).

2. *The curvature of this Poisson fibred algebra is given by*

$$\Phi(f,g)(a) = [a, B_2(f,g) - B_2(g,f)].$$

If we compose the identification map $\phi : A[[h]] \to A_h$ with any $\kappa[[h]]$-linear map $\psi : A[[h]] \to A[[h]]$ and assume that $\psi = \mathrm{Id} + h\psi_1 + \dots$, the bracket (8) will change to

$$\{a,b\}' = \{a,b\} + [\psi_1(a), b].$$

The curvature will change according to (3) with $\theta = \psi_1$.

2. Some examples

Example 2.1. Any Poisson algebra is a flat Poisson fibred algebra with $A = Z$.

Example 2.2. Consider a two dimensional quantum torus. For any $q \in \mathbb{C}$, the two dimensional quantum torus is the associative algebra A_q generated by two invertible elements u and v subject to the relations

$$uv = qvu \tag{9}$$

When $q = \epsilon$, where $\epsilon^l = 1$, the elements u^l and v^l and their inverses generate the center of the algebra A_ϵ, and the algebra A_ϵ becomes a Poisson fibred algebra.

The Poisson bracket between elements u^l and v^l can be easily computed from (8). If we choose the isomorphism between vector spaces A_ϵ and A_q identifying monomials $u^i v^j$ in both algebras, we obtain the following brackets:

$$\{u^l, v^l\} = l^2 u^l v^l \tag{10}$$

It is easy to check that in this case $B_2(f,g)$ is central if f and g are central. Therefore the Poisson fibred algebra A_ϵ has zero curvature.

Example 2.3 ([8])**.** Let $\kappa = \mathbb{C}$ and q be a formal parameter. The quantum universal enveloping algebra of $\mathfrak{s}l_2$ is the associative algebra $U_q(\mathfrak{s}l_2)$ over $\mathbb{C}$ with generators E, F, K and K^{-1} subject to the following relations:

$$\begin{aligned} KEK^{-1} &= q^2 E, \\ KFK^{-1} &= q^{-2} F, \\ [E,F] &= \frac{K - K^{-1}}{q - q^{-1}}. \end{aligned}$$

It has a Hopf algebra structure reflecting its group properties. Here we will focus on its associative algebra properties when q is a root of unity.

It is well known that the center of $U_q(\mathfrak{s}l_2)$ for a generic q is generated by the element

$$c = EF + \frac{Kq^{-1} + K^{-1}q}{(q - q^{-1})^2}.$$

Let $l > 2$ be an odd integer. Specify $q = \epsilon$, a primitive lth root of unity. According to Section 1.3, the algebra $A = U_\epsilon = U_\epsilon(\mathfrak{s}l_2)$ is an example of a Poisson fibred algebra. Here we will describe a slight modifictaion (with Z being a certain central subalgebra of A, as opposed to the whole center) of this algebra explicitly. The case of an even root of 1 is technically a little more complicated, but absolutely similar.

The center of $U_\epsilon(\mathfrak{s}l_2)$ is generated by the elements E^l, F^l, K^l, K^{-l}, c. Consider the subalgebra $\mathbb{C}[E^l, F^l, K^l, K^{-l}] \subset U_q$. At $q = \epsilon$ it degenerates to a subalgebra Z, which lies in the center of $A = U_\epsilon$, and U_ϵ becomes a free Z-module of rank l^3. The algebra Z has a canonical Poisson bracket

$$\{a, b\} = \lim_{q \to \epsilon} \frac{\tilde{a}\tilde{b} - \tilde{b}\tilde{a}}{q - \epsilon},$$

where $a, b \in Z$ and $\tilde{a}, \tilde{b} \in U_q$ are liftings of these elements: $a = \tilde{a} \pmod{q - \epsilon}$, $b = \tilde{b} \pmod{q - \epsilon}$. It also acts by derivations on U_ϵ:

$$\{a, u\} = \lim_{q \to \epsilon} \frac{\tilde{a}\tilde{u} - \tilde{u}\tilde{a}}{q - \epsilon},$$

but the bracket $\{a, u\}$ is defined up to the addition of $vu - uv$ for some $v \in U_\epsilon$, because of the ambiguity of lifting of a modulo $q - \epsilon$.

The monomials $F^i K^j E^k$, $i, j, k \in \mathbb{Z}$, $i, k \geq 0$, form a basis of U_q, and this choice of basis identifies U_ϵ with a subspace of U_q, thereby defining the liftings $\tilde{a}$, etc. This allows to compute explicitly the Poisson structure on $X = \mathbb{C}^2 \times \mathbb{C}^*$ in coordinates $x = E^l$, $y = F^l$ and $z = K^l$ and the action of Z on the vector bundle V over X associated with the free Z-module A in coordinates E, F and K on A. Here we use explicit formulas from De Concini-Kac [8] for commutators of elements in A.

$$\begin{aligned}
\{x,y\} &= dc^{-l}(z-z^{-1}),\\
\{x,z\} &= -dc^{l}xz,\\
\{y,z\} &= dc^{l}yz,\\
\{x,E\} &= 0,\\
\{x,F\} &= (dc^{l-2}/l)(K\epsilon - K^{-1}\epsilon^{-1})E^{l-1},\\
\{x,K\} &= -(dc^{l}/l)xK,\\
\{z,E\} &= (dc^{l}/l)zE,\\
\{z,K\} &= 0,
\end{aligned}$$

etc., where

$$d = \lim_{q\to\epsilon}\frac{[l]!}{q-\epsilon}$$

$$[l]! = [l][l-1]\dots[1] = \frac{q^l-q^{-l}}{q-q^{-1}}\frac{q^{l-1}-q^{1-l}}{q-q^{-1}}\dots\frac{q-q^{-1}}{q-q^{-1}},$$

$$c = \epsilon - \epsilon^{-1}.$$

Thus the Poisson tensor on X is equal to

$$P = dc^{-l}(z-z^{-1})\,\partial/\partial x\wedge\partial/\partial y - dc^{l}xz\,\partial/\partial x\wedge\partial/\partial z + dc^{l}yz\,\partial/\partial y\wedge\partial/\partial z.$$

The curvature is also computable:

$$\Phi(x,y)\,a = [\,a,\ \sum_{j=1}^{l-1}\frac{d^2}{([l-j]!)^2[j]!}F^{l-j}\prod_{r=j-2l+1}^{2j-2l}[K;r]E^{l-j}\,],$$

where q is specialized to ϵ in $[l-j]!$ and $[j]!$ and

$$[K;r] = \frac{K\epsilon^r - K^{-1}\epsilon^{-r}}{\epsilon-\epsilon^{-1}}.$$

In particular, it is clear that the curvature is nonzero. However

$$\Phi(x,z) = \Phi(y,z) = 0.$$

Example 2.4 (Quantum tori and abelian varieties, Manin [21])**.** A *quantum torus* $T(H,\alpha)$ is a pair consisting of a free finitely generated abelian group H and a multiplicatively skew-symmetric bilinear form $\alpha : H\times H \to \mathbb{C}^*$, i.e., $\alpha(\chi,\eta) = \alpha(\eta,\chi)^{-1}$, $\alpha(\chi+\chi',\eta) = \alpha(\chi,\eta)\cdot\alpha(\chi',\eta)$. The *quantum function ring of* *the torus* is the vector space $A(H, \alpha)$ freely generated over $\mathbb{C}$ by symbols e_χ, $\chi \in H$, with the associative multiplication law

$$e_\chi e_\eta = \alpha(\chi, \eta) e_{\chi+\eta}.$$

which equips $A(H, \alpha)$ with the structure of an associative algebra.

When $\alpha \equiv 1$, we get the commutative multiplication law for functions on the usual complex torus $T(H, 1)$, i.e., $\mathrm{Hom}(H, \mathbb{C}^*)$. Otherwise it is deformed, with α being thereby a quantization (multi)parameter.

Let us now introduce the structure of a Poisson fibred algebra on a quantum torus.

Suppose we came to the root of unity case by deforming parameters of the quantum torus, that is, deforming the form α along a tangent vector γ to the parameter space, at a point where α takes values in roots of unity. The tangent space to the parameter space at α can be identified with the space of skew-symmetric bilinear forms (in the usual additive sense) γ on H: a tangent vector γ represents the 1-jet $\alpha(1 + t\gamma + \dots)$ of a curve.

Then we can define a Poisson fibred algebra structure at a root of unity as follows:

$$\begin{aligned}
&\{e_\chi, e_\eta\} \\
&\qquad = D_\gamma [e_\chi, e_\eta]^\sim \\
&\qquad = \lim_{t \to 0} (\alpha(\chi, \eta)(1 + t\gamma(\chi, \eta)) - \alpha^{-1}(\chi, \eta)(1 + t\gamma(\chi, \eta))^{-1}) e_{\chi+\eta}/t \\
&\qquad = 2\gamma(\chi, \eta) e_{\chi+\eta},
\end{aligned}$$

where $\chi, \eta \in H$, at least one of them being in $H' = \mathrm{Ker}\, \alpha$, D_γ is the directional derivative, and $[,]^\sim$ is the commutator evaluated at a point $\alpha(1 + t\gamma + \dots)$. When χ and η are in H', we get the structure of a Poisson algebra on $A(H', 1)$, implying a Poisson structure on the torus $T(H', 1)$, and when $\chi \in H'$ and $\eta \in H$, we get an action of $A(H', 1)$ by derivations on $A(H, \alpha)$, which defines the structure of a Poisson fibred algebra on $(Z, A) = (A(H', 1), A(H, \alpha))$, according to the generalities of Section 1.3. The curvature of this Poisson fibred algebra vanishes, because away from the root of unity point, the pair $(A(H', 1), A(H, \alpha))$ deforms as a pair, i.e., $A(H', 1)$ lifts to a *subalgebra*, $A(H', \alpha|_{H'})$ of $A(H, \alpha)$ for generic α. This implies $B_2(f, g) \in A(H', 1)$ and therefore $\Phi(f, g) = 0$, see Section 1.3.

Remark 1. It would be interesting to study the similar structure on a quantum abelian variety, where there are not enough global functions and their replacement, the quantum theta functions do not have a natural associative algebra structure. To make the problem more concrete, we recall Manin's construction [21] of quantum abelian varieties.

The complex torus $T(H,1)$ acts on the quantum torus $T(H,\alpha)$. More precisely, $T(H,1)$ acts by ring homomorphisms of the function ring: $f^*: A(H,\alpha) \to A(H,\alpha)$, $f \in A(H,1) = \mathrm{Hom}(H,\mathbb{C}^*)$, by

$$(f^* e_\chi) = f(\chi) e_\chi.$$

If $B \subset T(H,1)$ is a subgroup (a period subgroup), then a *quantized theta function* is a formal series $\theta = \sum_\chi a_\chi e_\chi$ which is automorphic with respect to B with "linear" multiplicators λe_χ:

$$b^*\theta = \lambda_b e_{\chi_b} \theta,$$

$\lambda_b \neq 0$. In the classical $\alpha \equiv 1$ theory, when a type L of multiplicators is fixed, theta functions are formal sections $\Gamma(L)$ of the corresponding line bundle L on the abelian variety $T(H,1)/B$. When L is a polarization, i.e., a positive line bundle, the graded ring $\bigoplus_{n\geq 0} \Gamma(L^n)$ is the ring of homogeneous functions on the abelian variety. Therefore, we can think of the collection of quantum theta functions as defining a *quantum abelian variety*.

Now suppose the form α takes values at roots of unity. Then we get a large central subalgebra $Z = A(H',1)$, where $H' = \mathrm{Ker}\,\alpha$ (the center is $A(\mathrm{Ker}\,\alpha^2)$), of the algebra $A(H,\alpha)$. Thus, the latter forms a bundle V of noncommutative algebras over the usual torus $T(H',1) = \mathrm{Spec}\, Z$. Any type of multiplicators such that $\chi_b \in H'$ for $b \in B$ determines a line bundle L_0 on the abelian variety $\mathcal{A} = T(H',1)/B$. If in addition L_0 is a polarization and L an arbitrary type of quantum multiplicators, then the quantum theta functions of the types $L \otimes L_0^n$, $n \geq 0$, form a graded module over the homogeneous ring $\bigoplus_{n\geq 0} \Gamma(\mathcal{A}, L_0^n)$ and therefore define a coherent sheaf of quantum theta functions on the classical abelian variety $\mathcal{A}$.

It would be interesting to study related Poisson structures on the abelian variety $\mathcal{A}$ and the sheaf of quantum theta functions on it.

3. Poisson modules

The goal of this section is to introduce notions of modules over Poisson algebras. The idea is to take the "infinitesimal part" of the corresponding notions for formal deformations.

A Poisson fibred algebra with zero curvature carries the important algebraic structure of a Poisson module. While Poisson fibred algebras arise under deformation of an algebra A and a central subalgebra Z of it, Poisson modules show up by deformation of a pair (A,Z), where Z

is a commutative algebra and A a module over it. Geometrically, this corresponds to the "semiclassical limit" of a quantization of a manifold along with a vector bundle over it.

3.1. **Left and right modules over formal deformations.** Let A_h be a formal deformation of a Poisson algebra A. We assume that this is a torsion free deformation and will identify A_h with $A[[h]]$ as vector spaces. Let M_h be a module over A_h which is a formal deformation of the module M over the commutative algebra A ($M = M_h/hM_h$). Again we assume that M_h is isomorphic to $M[[h]]$ as a vector space and fix this isomorphism. Then the multiplication in A_h and the module structure are given by formal power series:

$$f * g = fg + (1/2)\{f, g\}h + B(f, g)h^2 + \dots,$$
$$f * m = fm + (1/2)\{f, m\}h + \tilde{B}(f, m)h^2 + \dots$$

Remark 2. In physical applications, we would work over $\mathbb{C}$ and let h be i times Planck's constant.

Assume that the correspondence $\sigma : h \mapsto -h$ defines an anti-involution

$$\sigma(f * g) = \sigma(g) * \sigma(f) \tag{11}$$

of the associative algebra $A[[h]]$. This means in particular that the bracket $\{f, g\}$ is skew and $B(f, g)$ is symmetric.

It is obvious that for $f \in A[[h]]$ and $m \in M[[h]]$ the formula

$$m * f := \sigma(\sigma(f) * \sigma(m)) \tag{12}$$

defines on $M[[h]]]$ the structure of a right $A[[h]]$-module. For $f \in A$ and $m \in M$, we have

$$m * f = fm - (1/2)\{f, m\}h + \tilde{B}(f, m)h^2 - \dots \tag{13}$$

This property of formal deformations generalizes the fact that any left module over a commutative algebra is also a right module over this algebra, and that any left module over Lie algebra carries a right module structure as well.

If we follow the "deformation philosophy", it is natural to ask the question: which modules over a commutative algebra A can be deformed to modules over formal deformations of A? This question suggests the following definition.

Definition 3.1. A vector space M is called a *left Poisson module* over a Poisson algebra A if a scalar multiplication fm and a bracket $\{f, m\}$ between any elements $f \in A$ and $m \in M$ are defined, such that

1. the scalar multipication defines on M the structure of a module over the commutative associative algebra A;
2. $\{f,g\}m = f\{g,m\} - \{fg,m\} + \{f,gm\}$;
3. there exists $\tilde{B} \in \mathrm{Hom}(A \otimes M, M)$, such that

$$2\{\{f,g\},m\} - \{f,\{g,m\}\} + \{g,\{f,m\}\} \\ = 4(f\tilde{B}(g,m) - \tilde{B}(g,fm) + \tilde{B}(f,gm) - g\tilde{B}(f,m)).$$

Remark 3. If we regard $\mathrm{Hom}(M,M)$ naturally as a module over the associative algebra A and consider the corresponding Hochschild complex $\mathrm{Hom}(A^{\otimes\bullet}, \mathrm{Hom}(M,M))$ with a differential d, then Property 2 of the above definition is equivalent to saying that the two-cocycle $\{f,g\}m$ is exact:

$$\{f,g\}m = (d\alpha)(f \otimes g, m), \quad \text{with } \alpha(f,m) = \{f,m\}. \tag{14}$$

Likewise, Property 3 is equivalent to the existence of a one-cocycle $\tilde{B}$, such that

$$2\{\{f,g\},m\} - \{f,\{g,m\}\} + \{g,\{f,m\}\} = 4(d\tilde{B})(f \otimes g - g \otimes f, m). \tag{15}$$

Unfortunately, we do not have any good geometric interpretation of Properties 2 and 3 of left Poisson modules.

From the deformational point of view, left Poisson modules admit first order deformations extendable to the second order as left associative modules over formal deformations of A.

Similarly *right Poisson modules* are those which admit second order deformations as right modules over associative formal deformation of A.

3.2. **Poisson modules.** A vector space M_h over $\mathbb{C}[[h]]$ is a bimodule over A_h if it has a structure of left and right modules over A_h and these two actions commute.

Since any right module over a formal deformation A_h of a Poisson algebra (provided that A_h has the property (11)) is also a left module, the bimodule structure on M_h is equivalent to two commuting left module structures.

The special class of bimodules emerges when we assume that the left and right actions are equal modulo h:

$$f * m - m * f = h\{f,m\}' + O(h^2) \ .$$

Let $(1/2)\{f,m\}$ and $(1/2)\{m,f\}$ be the first order terms of the left and right actions respectively. Then

$$\{f,m\}' = (1/2)(\{f,m\} - \{m,f\}) \ .$$

The brackets $\{f,m\}'$ satisfy the following identities:

$$
\begin{aligned}
\{\{f,g\},m\}' &= \{f,\{g,m\}'\}' - \{g,\{f,m\}'\}', \\
\{f,gm\}' &= g\{f,m\}' + \{f,g\}m, \\
\{fg,m\}' &= f\{g,m\}' + g\{f,m\}'.
\end{aligned}
\tag{16}
$$

This is why the following definition seems natural.

Definition 3.2. A vector space M is called a *Poisson module* over Poisson algebra A if M is a module over a commutative algebra A and if a bilinear bracket $\{\,,\,\}' : A \otimes M \to M$ which satisfies (16) is given.

Proposition 3.1. *Any formal deformation of a pair* (A, M) *to a left* $A[[h]]$*-module* $M[[h]]$ *with* $\sigma : h \mapsto -h$ *defining an anti-involution on* $A[[h]]$ *and such that the right* $A[[h]]$*-module structure on* $M[[h]]$ *defined by* (12) *makes it a bimodule defines the structure of a Poisson algebra on* A *and a Poisson module on* M *over* A *with respect to the bracket* $\{\,,\,\}$ *of* (13).

Under the assumptions of this proposition, $\{\,,\,\} = \{\,,\,\}'$, and therefore, it follows from the properties of $\{\,,\,\}'$.

Here is one more argument why this definition is "natural". Poisson modules describe Poisson extensions of Poisson algebras (as bimodules over associative algebras can be used to construct associative extensions). The proof of the next proposition is left to the reader.

Proposition 3.2. *Given a Poisson algebra* A *and a vector space* M*, the structure of a Poisson module on* M *is equivalent to the structure of a Poisson algebra on* $A \oplus M$ *extending that on* A *and such that* $M \cdot M = \{M, M\} = 0$.

Poisson modules also have a nice operadic meaning, cf. [13, 17]. Consider the Poisson operad, that is the operad whose nth component $\mathcal{P}(n)$ is the vector space generated by elements of degree one in each variable $X_1, \ldots, X_n$ of the free Poisson algebra generated by $X_1, \ldots, X_n$. $\{\mathcal{P}(n) \mid n \geq 1\}$ is a quadratic operad with two generators $X_1 \cdot X_2$ and $\{X_1, X_2\}$, such that $X_1 \cdot X_2 = X_2 \cdot X_1$ and $\{X_1, X_2\} = -\{X_2, X_1\}$ and relations

$$\begin{aligned} X_1(X_2X_3) &= (X_1X_2)X_3, \\ (X_1X_3)X_2 &= X_1(X_3X_2), \\ \{\{X_1,X_2\},X_3\} + \{\{X_2,X_3\},X_1\} &+ \{\{X_3,X_1\},X_2\} = 0, \\ \{X_1,X_2X_3\} &= \{X_1,X_2\}X_3 + X_2\{X_1,X_3\}, \\ \{X_2,X_1X_3\} &= \{X_2,X_1\}X_3 + X_1\{X_2,X_3\}, \\ \{X_3,X_1X_2\} &= \{X_3,X_1\}X_2 + X_1\{X_3,X_2\}. \end{aligned}$$

A Poisson algebra is nothing but an algebra over the operad $\mathcal{P}(n)$ and a Poisson module over a Poisson algebra is exactly a module over an algebra over the Poisson operad.

4. Noncommutative spaces of finite type

Considerations of Section 1 arise when a usual quantum space, e.g., a quantum group $U_q(\mathfrak{g})$, gets a large center at a certain value of the parameter q. This is how the Poisson structures enter the picture. On the other hand, even the undeformed part of that picture, an associative algebra of finite type over a central subalgebra, is interesting. By analogy with supermanifolds, we would like to consider a more general *noncommutative space of finite type*, which is either a smooth manifold with a finite rank bundle of associative algebras or a pair $(X, \mathcal{Q}_X)$, where X is a scheme with a structure sheaf $\mathcal{O}_X$ and $\mathcal{Q}_X$ is a sheaf of $\mathcal{O}_X$-algebras coherent as a sheaf of $\mathcal{O}_X$-modules. Depending on the context (smooth or algebraic), we will use either kind of noncommutative spaces. From this section on, we will leave the "semiclassical" Poisson world for the "semiquantum" world of noncommutative spaces of finite type.

In the algebraic context, every algebra which is finite-dimensional over a central subalgebra is an example:

Lemma 4.1. *Let A be an algebra, Z a subalgebra of its center and A be finitely generated as a Z-module. Then there exists a coherent sheaf of noncommutative algebras over* $\operatorname{Spec} Z$*, such that the stalk of the sheaf over a point $\mathfrak{p} \in \operatorname{Spec} Z$ is exactly the localization $A_{\mathfrak{p}}$ at the corresponding prime ideal $\mathfrak{p}$ of Z.*

Proof. The proof just repeats the commutative construction of the structure sheaf on $\operatorname{Spec} Z$. For every element f in the center Z, take the localization $A[f^{-1}]$ of the algebra A and glue them together over the principal open subsets $D(f) = \operatorname{Spec} Z[f^{-1}] \subset \operatorname{Spec} Z$. □

Example 4.1 (The quantum space $\mathbb{C}^n_\epsilon$ at a root ϵ of unity)**.** Consider the associative algebra $A = \mathbb{C}\langle x_1, \dots, x_n\rangle / (x_i x_j - \epsilon x_j x_i) \mid i < j)$, where $\epsilon \in \mathbb{C}$, $\epsilon^l = 1$. The center of A is the subalgebra Z generated by $x_1^l, \dots, x_n^l$, and the whole algebra A is a free Z-module of rank l^n.

Example 4.2 (The quantum group $\mathrm{SL}_\epsilon(n)$ at a root of unity)**.** This is a group of linear transformations of $\mathbb{C}^n_\epsilon$, see Manin [20] or Faddeev-Reshetikhin-Takhtajan [11]. To avoid cumbersome notation, consider the case $n = 2$. The quantum group $\mathrm{SL}_q(2)$ is the associative algebra $A = \mathbb{C}[q, q^{-1}]\langle a, b, c, d\rangle / (ab = q^{-1}ba,\ ac = q^{-1}ca,\ cd = q^{-1}dc,\ bd = q^{-1}db,\ bc = cb,\ ad - q^{-1}bc = da - qcb = 1)$. Of course, it has a quantum group, i.e., a Hopf algebra structure coming from the matrix multiplication of the variables arranged in a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, which is of no interest so far, as long as we are concerned only with the noncommutative space properties of $\mathrm{SL}_q(2)$. When $q = \epsilon$ is an lth root of 1, the subalgebra generated by the entries of M^l is central and the whole algebra is finite-dimensional over it. The noncommutative space is then the sheaf corresponding to the algebra A over the usual algebraic group $\mathrm{SL}(2)$. This example is in a certain sense dual to the quantum algebra $U_\epsilon(\mathfrak{sl}_2)$ of Example 2.3.

5. Quantum projective spaces

Let $S = \bigoplus_{n \ge 0} S_n$ be a $\mathbb{Z}$-graded associative algebra, Z a graded central subalgebra of it, such that S is finitely generated as a Z-module. Then one can construct the *noncommutative projective spectrum* $\operatorname{Proj}_Z S$ as the ringed space $(X, \mathcal{Q})$, where $X = \operatorname{Proj} Z$ and the sheaf $\mathcal{Q}$ is constructed as in Lemma 4.1: for every homogeneous element $f \in Z$, consider the localization $S_{(f)} = S[f^{-1}]_0$ and glue them over open subsets $D((f)) = \operatorname{Spec} Z[f^{-1}]_0$. For a finite-type graded module M over S, we can similarly construct a "coherent" sheaf $\mathcal{M}$ of $\mathcal{Q}$-modules over X and define the *functor of global sections*

$$\Gamma(X, \mathcal{M}) = H^0(X, \mathcal{M}) = \operatorname{Hom}_{\mathcal{Q}}(\mathcal{Q}, \mathcal{M}) = \operatorname{Hom}_{\mathcal{O}}(\mathcal{O}, \mathcal{M})$$

just as usual and the *cohomology* as its derived functors or as

$$H^q(X, \mathcal{M}) = \operatorname{Ext}^q_{\mathcal{Q}}(\mathcal{Q}, \mathcal{M}) = \operatorname{Ext}^q_{\mathcal{O}}(\mathcal{O}, \mathcal{M}),$$

where $\mathcal{O}$ is the structure sheaf of $X = \operatorname{Proj} Z$. We can define an *invertible sheaf* $\mathcal{Q}(n)$, $n \in \mathbb{Z}$, as the sheaf of $\mathcal{Q}$-modules associated with the graded free S-module $S(n)$ of rank one defined by $S(n)_m = S_{m+n}$.

5.1. **The quantum projective space $\mathbb{P}^n_\epsilon$.** Consider the graded algebra

$$S = \mathbb{C}_q\langle X_0, \dots, X_n\rangle/(X_iX_j = qX_jX_i \text{ for } i < j), \tag{17}$$

over $\mathbb{C}_q = \mathbb{C}[q, q^{-1}]$. We can think of it as the homogeneous ring of a quantum projective space $\mathbb{P}^n_q$. When q is specified to a root ϵ of unity, S gets a huge center, the subalgebra Z generated by $X_0^l, \dots, X_n^l$. This subalgebra lifts to a subspace closed under the deformed multiplication at an arbitrary value q. It is still the subalgebra generated by the l-th powers of the generators, with the same kind of multiplication law as for the original projective space, but with q replaced by q^{l^2}, which is not 1 when q is not equal to an l^2'th root of unity. Thus, we are in the situation of a Poisson fibred algebra with zero curvature, which we studied before. We will save studying those structures for a future work; for now, we would like to focus on the scalar multiplication structure, which is already very rich.

The *quantum projective space* $X = \mathbb{P}^n_\epsilon$ is the pair $(X, \mathcal{Q}_X)$, where $X = \mathbb{P}^n$ and the sheaf $\mathcal{Q}_X$ of noncommutative algebras obtained from the noncommutative graded $\mathbb{C}[X_0^l, \dots X_n^l]$-algebra (17). The sheaf $\mathcal{Q}_X$ can be described as follows. Consider the degree l^n ramified covering

$$\pi : \mathbb{P}^n \to \mathbb{P}^n,$$
$$(X_0 : \dots : X_n) \mapsto (X_0^l : \dots : X_n^l).$$

The sheaf $\bar{\mathcal{Q}}_X = \pi_* \mathcal{O}$ on $\mathbb{P}^n$ is

$$\bar{\mathcal{Q}}_X = \mathcal{O} \oplus m_1\mathcal{O}(-1) \oplus m_2\mathcal{O}(-2) \oplus \dots \oplus m_n\mathcal{O}(-n), \tag{18}$$

where $m_1 = p(l, n+1; l) = \binom{n+l}{n} - n - 1$ is the number of partitions of l into $n+1$ nonnegative parts, each strictly smaller than l, $m_2 = p(2l, n+1; l) = \binom{n+2l}{2l} - (n+1)m_1 - \binom{n+2}{2}$, $m_3 = p(3l, n+1; l) = \binom{n+3l}{3l} - (n+1)m_2 - \binom{n+2}{2} m_1 - \binom{n+3}{3}, \dots, m_n = p(nl, n+1; l)$. $\bar{\mathcal{Q}}_X$ is a sheaf of algebras with the product coming from that on $\mathcal{O}$ upstairs. It can be described in terms of the decomposition (18). For instance, when $n = 1$, we obtain $\bar{\mathcal{Q}}_X = \mathcal{O} \oplus (l-1)\mathcal{O}(-1)$, the $l-1$ components $\mathcal{O}(-1)$ being generated by $X_0X_1^{l-1}, X_0^2X_1^{l-2}, \dots, X_0^{l-1}X_1$

in degree 1, with the following multiplication table on the generators

$$X_0^p X_1^{l-p} \cdot X_0^q X_1^{l-q}$$
$$= X_0^{p+q} X_1^{2l-p-q} = \begin{cases} X_1^l (X_0^{p+q} X_1^{l-p-q}) & \text{if } p+q<l, \\ X_0^l X_1^l \cdot 1 & \text{if } p+q=l, \\ X_0^l (X_0^{p+q-l} X_1^{2l-p-q}) & \text{if } p+q>l, \end{cases}$$

which induces the following mappings of sheaves:

$$\mathcal{O}(-1) \otimes \mathcal{O}(-1) \quad \begin{cases} \xrightarrow{X_1^l \cdot} \mathcal{O}(-1), \\ \xrightarrow{X_0^l X_1^l \cdot} \quad \mathcal{O}, \\ \xrightarrow{X_0^l \cdot} \mathcal{O}(-1), \end{cases}$$

respectively.

Thus when $n = 1$, we get

$$X_0^p X_1^{l-p} \cdot X_0^q X_1^{l-q} = \epsilon^{(p-l)q} X_0^{p+q} X_1^{2l-p-q}.$$

Note that this multiplication law is still commutative — what else can we expect from a one-dimensional space? The sheaves Q_X of quantum functions on $\mathbb{P}_\epsilon^n$ will be, of course, noncommutative, starting from $n = 2$.

Remark 4. One can think of the morphism of ringed spaces

$$F : \mathbb{P}_\epsilon^n = (X, \mathcal{Q}) \to \mathbb{P}^n = (X, \mathcal{O})$$

defined by the natural embedding

$$\mathbb{C}[Z_0, \dots, Z_n] \to S,$$
$$Z_i \mapsto X_i^l,$$

as a *quantum Frobenius morphism.*

5.2. **Cohomology of invertible sheaves on $\mathbb{P}_\epsilon^n$.** The quantum projective spaces $\mathbb{P}_\epsilon^n$ are the simplest examples of compact quantum homogeneous spaces G/P at roots of unity. Contrary to the case of generic q, where such spaces were studied at the level of their noncommutative algebras of functions, see Lakshmibai-Reshetikhin [15], Soibelman [24] and Chari-Pressley [7], our root-of-unity spaces are honest topological spaces, ringed with rather small sheaves of noncommutative algebras. This has the advantage that one can use the usual cohomology theory of sheaves of abelian groups on them.

As usual, the cohomology of invertible sheaves on $\mathbb{P}_\epsilon^n$ comprise a class of highest-weight representations of the quantum group $\mathrm{SL}_\epsilon(n)$. In case of $\mathrm{SL}_\epsilon(2)$, when $\mathbb{P}_\epsilon^1$ is nothing but the complete flag space G/B, we will obtain the Borel-Weil-Bott theorem, which realizes all irreducible representations in the cohomology.

First, we should describe the "Picard group", which is here nothing more than a pointed set a priori, of $\mathbb{P}^n_\epsilon$.

Proposition 5.1. *The Picard set* $\operatorname{Pic}\mathbb{P}^n_\epsilon$ *of isomorphism classes of invertible left* $\mathcal{Q}$*-modules on* $\mathbb{P}^n_\epsilon$ *is isomorphic to* $\mathbb{Z}$*, the sheaves* $\mathcal{Q}(j)$*,* $j \in \mathbb{Z}$*, corresponding to the graded* S*-modules* $S(j) = \bigoplus_{m \ge -j} S(j)_m = \bigoplus_{m \ge -j} S_{j+m}$ *making up a complete list of representatives.*

Proof. As usual, the Picard set can be computed as the nonabelian cohomology $H^1(\mathbb{P}^n, \mathcal{Q}^*)$, which we will compute here for the standard covering of $\mathbb{P}^n$ with open sets $U_i = \{X_i^l \ne 0\}$, $i = 0, 1, \dots, n$, in terms of homogeneous coordinates X_i on $\mathbb{P}^n_\epsilon$. A cocycle defining a class in $H^1(\mathbb{P}^n, \mathcal{Q}^*)$ can be represented by functions $f_{ij} \in \Gamma(U_{ij}, \mathcal{Q}^*) \mid 0 \le i, j \le n$. Each function f_{ij} is an invertible degree 0 element of $\mathbb{C}[X_0, \dots, X_n; X_i^{-l}, X_j^{-l}]$. Therefore, $f_{ij} = c_{ij} X_i^d X_j^{-d}$. Note that d is independent of the choice of a pair ij. This defines a natural degree mapping

$$\operatorname{Pic}\mathbb{P}^n_\epsilon \to \mathbb{Z}, \tag{19}$$

which assigns the degree d in X_i of the function $f_{ij} = c_{ij} X_i^d X_j^{-d}$ representing a cocycle in $H^1(\mathbb{P}^n, \mathcal{Q}^*)$.

We claim that (19) is an isomorphism of pointed sets. Indeed, the inverse of (19) is given by assigning the class of the class of $\mathcal{Q}(d)$ to each $d \in \mathbb{Z}$. It is a true inverse, because any class $\{f_{ij}\} \in H^1(X, \mathcal{Q}^*)$ is of the form $c_{ij} X_i^d X_j^{-d}$, with c_{ij} forming a 1-cocycle with coefficients in the constant sheaf $\mathbb{C}^*$. Since the cohomology of a noetherian topological space with constant coefficients is always zero in all degrees higher than zero, the 1-cocycle $\{c_{ij}\}$ is a coboundary and thus, $\{f_{ij}\}$ is equivalent to the class of $\{X_i^d X_j^{-d}\}$, which is the class of $\mathcal{Q}(d)$ in $\operatorname{Pic}\mathbb{P}^n_\epsilon$. □

The quantum group $\mathrm{SL}_\epsilon(n+1)$ acts naturally on $\mathbb{P}^n_\epsilon$ and the sheaves $\mathcal{Q}(j)$. This means the sheaf $\mathcal{Q}$ of functions and the sheaves of sections of $\mathcal{Q}(j)$ have a (right) $\mathrm{SL}_\epsilon(n+1)$-comodule structure, coming from the natural comodule structure on the graded algebra S and grading shifts of it.

Theorem 5.2. *The cohomology groups* $H^i(X, \mathcal{Q}(j))$ *of the invertible sheaves on the quantum projective space* $X = \mathbb{P}^n_\epsilon$ *as representations of* $\mathrm{SL}_\epsilon(n+1)$ *are given by the following formula:*

$$H^i(X, \mathcal{Q}(j)) = \begin{cases} S^j(V^*), & \text{for } j \ge 0,\ i = 0, \\ S^{-n-1-j}(V) & \text{for } j \le -n-1,\ i = n, \\ 0 & \text{otherwise,} \end{cases}$$

where V^* is the standard $n+1$-dimensional representation of $\mathrm{SL}_\epsilon(n+1)$ *in the space $V = S_1$, a graded component of the algebra $S = \mathbb{C}\langle X_0, \dots, X_n\rangle/(X_iX_j = \epsilon X_jX_i$ for $i < j)$. The jth symmetric power is understood as the representation of* $\mathrm{SL}_\epsilon(n+1)$ *in S_j.*

Proof. The space $H^0(X, \mathcal{Q}(j))$ can be naturally identified with the space of globally defined quantum rational functions of X_i of degree j. These are just quantum polynomials in X_i of degree j, that is the space S_j. The computation of H^n is given by the natural SL_ϵ-equivariant pairing $\mathcal{Q}(j) \otimes \mathcal{Q}(-n-1-j) \to \mathcal{Q}(-n-1)$ and the trace mapping $H^n(X, \mathcal{Q}(-n-1)) \to \mathbb{C}$, which follows from the decomposition

$$\mathcal{Q}(j) = \bigoplus_{k\in\mathbb{Z}} p(kl+j, n+1; l)\mathcal{O}(-k)$$

(only a finite number of terms do not vanish), generalizing (18). The same decomposition and the computation of cohomology of $\mathcal{O}(j)$ implies vanishing of the other cohomology groups. □

6. The Borel-Weil-Bott theorem

Here we are going to apply Theorem 5.2 to deduce Corollary 6.1, the quantum version of the Borel-Weil-Bott (BWB) theorem in the simplest case of the quantum group $\mathrm{SL}_\epsilon(2)$ when ϵ is an lth root of unity for l odd. Thus, it will complete the program indicated in Manin [20] and started in Parshall-Wang [23], Andersen [1], and Lakshmibai-Reshetikhin [15] entirely in terms of the functor of induction from the quantum Borel subgroup, without any use of the geometry of homogeneous spaces.

We need to recall some basic facts of representation theory of $G = \mathrm{SL}_\epsilon(2)$, see [23]. There is a natural sequence of quantum group embeddings, i.e., surjective inverse morphisms of Hopf algebras,

$$T \subset B \subset G,$$

where T is the quantum subgroup of diagonal matrices, which is just the usual torus $\mathbb{C}^*$, and B is the subgroup of upper triangular matrices. Given a character $\lambda \in \mathbb{Z}$ of T, we can extend it trivially to a character of B and then form an induced representation $\mathrm{ind}_B^G \lambda$, a highest weight module. It is not equal to zero if and only if $\lambda \ge 0$. If $\lambda < l$, it is irreducible. Another, purely root-of-unity construction of irreducibles starts from an irreducible module $L(\lambda)$, $\lambda \ge 0$, of the ordinary group $\mathrm{SL}(2)$ and pulls it back to a representation of $\mathrm{SL}_\epsilon(2)$ via the *quantum Frobenius morphism*

$$F : \mathrm{SL}_\epsilon(2) \to \mathrm{SL}(2)$$

sending the standard generators A, B, C, D of the algebra of regular functions on $\mathrm{SL}(2)$ to a^l, b^l, c^l, d^l in the Hopf algebra $\mathrm{SL}_\epsilon(2)$, respectively. The pullback $F^*L(\lambda)$ is an irreducible representation $L(l\lambda)$ of $G = \mathrm{SL}_\epsilon(2)$, sitting inside $\operatorname{ind}_B^G l\lambda$. Finally, the complete list of irreducibles is given by tensor products

$$L(\lambda_0 + l\lambda_1) = L(\lambda_0) \otimes F^*L(\lambda_1) \subset \operatorname{ind}_B^G(\lambda_0 + l\lambda_1),$$

for $0 \le \lambda_0 < l, \quad 0 \le \lambda_1$.

In view of these facts, our computation of cohomology in Theorem 5.2 yields the following result.

Corollary 6.1 (Quantum BWB Theorem for $G = \mathrm{SL}_\epsilon(2)$)**.** 1. *As a representation of* $G = \mathrm{SL}_\epsilon(2)$,

$$H^i(X, \mathcal{Q}(\lambda)) = \begin{cases} \operatorname{ind}_B^G \lambda & \text{if } \lambda \ge 0 \text{ and } i = 0, \\ \operatorname{ind}_B^G(-2-\lambda) & \text{if } \lambda \le -2 \text{ and } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

2. *Any irreducible representation of* G *is*

$$L(\lambda) = H^0(X, \mathcal{Q}(\lambda)) \qquad \text{for } 0 \le \lambda < l,$$
$$L(l\lambda) = H^0(X, \mathcal{O}(\lambda)) \qquad \text{for } 0 \le \lambda$$

or the image $L(\lambda_0 + l\lambda_1)$ *of the natural product*

$$H^0(X, \mathcal{Q}(\lambda_0)) \otimes H^0(X, \mathcal{O}(\lambda_1)) \to H^0(X, \mathcal{Q}(\lambda_0 + l\lambda_1))$$

coming from $\mathcal{Q}(\lambda_0) \otimes \mathcal{O}(\lambda_1) \to \mathcal{Q}(\lambda_0 + l\lambda_1)$.

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