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Table of Contents

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PION CONDENSATION IN A FIELD THEORY CONSISTENT WITH BULK PROPERTIES OF NUCLEAR MATTER*

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ABSTRACT

Pion condensation has not previously been investigated in a theory that accounts for the known bulk properties of nuclear matter, its saturation energy and density and compressibility. We have formulated and solved self-consistently, in the mean field approximation, a relativistic field theory that possesses a condensate solution and reproduces the correct bulk properties of nuclear matter. The theory is solved in its relativistically covariant form for a general class of space-time dependent pion condensates. Self-consistency and compatibility with bulk properties of nuclear matter turn out to be very stringent conditions on the existence and energy of the condensate, but they do allow a weak condensate energy to develop. The spin-isospin density oscillations, on the other hand, can be large. It is encouraging, as concerns the possible existence of new phases of nuclear matter, that this is so, unlike the Lee-Wick density isomer, that appears to be incompatible with nuclear matter properties.

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I. INTRODUCTION

Interest in the theory of matter at densities above that of nuclei has been stimulated in the last few years by developments in both astrophysics and nuclear physics. In astrophysics, the equation of state of matter over a very wide range is needed to calculate supernova explosions and neutron star properties. In nuclear physics, collisions between nuclei at relativistic energies may create dense nuclear matter for the first time in the laboratory. At high density, new states of matter become possible, involving additional particles or field configurations than are present in the ground state. Several such states have been discussed in the literature, the pion condensate, and the density isomer.[1] At sufficiently high energy density, excitations of the internal structure of the nucleons become possible,[2] leading perhaps ultimately to a quark matter phase.[3]

The pion condensate has been studied in two approaches. Each has some advantages and disadvantages compared to the other. In one of these, the pion propagator in the nuclear medium is studied.[1] The singularities of the propagator occur at the energies of excitation of the medium with the quantum numbers of the pion. At the density for which the lowest excitation becomes degenerate with the ground state, a symmetry of the original ground state is lost. In this case it is parity. This density is the critical density, above which the ground state has a finite amplitude for the condensate. In this approach, which is equivalent to the small-amplitude random phase approximation, one can incorporate a number of physical effects on the particle-hole amplitudes describing the condensed state at threshold. Short-range correlations between nucleons, the excitation of the Δ-resonance, and finite-size form factors are the most important effects.

The other approach constructs a relativistic field theory of the constituents of matter, and solves for the self-consistent fields in the mean field approximation. The advantage here is that one can calculate the equation of state of matter through the critical density into the fully developed condensate region. The disadvantage is that the effect of short-range correlations and finite-size form factors for the particles of the medium would require a major additional calculation using the spectrum provided by the solution of the mean field equations as a starting point. Fortunately, as we shall discuss fully at a later point, at the critical density the two approaches can be related. This permits the advantages of the propagator approach to be carried into the field theoretical approach through a renormalization of the pion-nucleon coupling constant. It turns out that the renormalization is almost independent of density.

It is the second of these approaches, a field theory of matter, that we explore here. Our study differs from earlier research along this line in the choice of the interacting fields. The choice in earlier work was motivated by a desire to preserve a certain elementary particle property, chiral symmetry, which is explictly realized in the chiral σ-model.[4,5] One drawback in these calculations, which however is not inherent in the model, is that they were carried out in a non-relativistic approximation. One expects that at higher densities and for finite pion momentum, this will lead to significant error. We find in fact that already at normal density, the non-relativistic approximation is serious. However, the most disturbing aspect of the earlier work is that while purporting to discuss new states of nuclear matter, the chiral lagrangians employed are not able to describe the normal nuclear state, as Kerman and Miller showed.[6]

The motivation behind our choice of interacting fields is first to account for the known bulk properties of nuclei, and thus constrained, to extrapolate to moderately higher density, to learn what the theory implies about the existence of a pion condensate. We make four contributions to the theory of abnormal states in hadronic matter.[7]

1) We have formulated and solved self-consistently in the mean field approximation a field theory of nuclear matter that possesses a pion condensate;
2) The theory is constrained to reproduce the known bulk properties of nuclear matter, namely, its satruation energy, density and compressibility;
3) It is solved in its relativistically covariant form;
4) A continuous class of space-time dependent pion condensate solutions is exhibited.

In the next sections we formulate the theory, derive the connection with the propagator approach that allows us to determine renormalized coupling constants, calculate the source currents, and finally present and discuss the numerical results. We will emphasize the implications of self-consistency for the existence of condensate solutions, the dependence of the condensate energy on the nuclear equations of state within acceptable uncertainties and we will test the non-relativistic approximation which has been used in previous work.

II. RELATIVISTIC FIELD THEORY OF NUCLEAR MATTER

A. Discussion of the Relevant Fields

Field theory is the appropriate mathematical scheme in which to discuss matter under conditions where new particles can spontaneously appear. Since we are interested here in matter near the density of normal nuclei, then the nucleons enjoy a special role. The state of such matter is determined by expectation values of the various nucleon current operators.

$$J_\Gamma(x) = \langle\bar{\psi}(x)\Gamma\psi(x)\rangle \quad ,$$

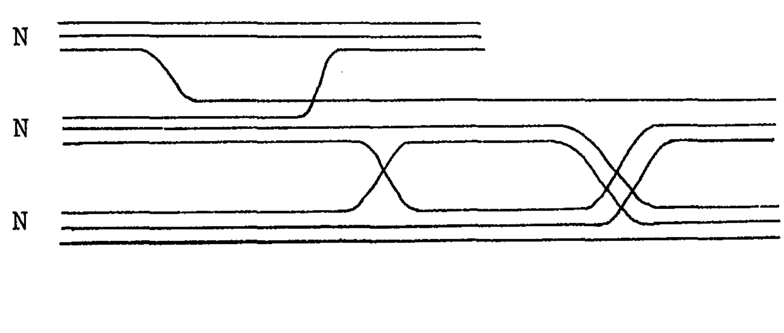
$$\Gamma = \{1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \sigma_{\mu\nu}\} \times \{1, \underset{\sim}{\tau}\} \quad . \tag{1}$$

The normal state of symmetric matter has a very uncomplicated structure in which only $\Gamma = 1$ and γ_0 are non-vanishing currents. The second is the nucleon density, and the first reduces to the same in the non-relativistic limit. Other states of matter are characterized by non-vanishing expectation values of additional currents. The pion condensate has non-vanishing current

$$\underset{\sim}{J}_{\mu 5} = \langle\bar{\psi}\gamma_\mu\gamma_5\underset{\sim}{\tau}\psi\rangle \qquad \text{or} \qquad \underset{\sim}{J}_5 = \langle\bar{\psi}\gamma_5\underset{\sim}{\tau}\psi\rangle \quad .$$

Our problem is to determine whether and under what conditions the field equations will give rise to such non-vanishing currents.

If quarks and gluons are the fundamental fields, then the interaction between the nucleons would be described by such diagrams as



This is intended to indicate not only the exchange of quarks but a space-time structure to the exchange and some involvement of more than two nucleons at a time. This theory of matter (QCD) is incomplete, especially as concerns the large distance behavior, and so does not provide a means of calculating the nucleon currents, Eq. (1), at the present. However, nature provides a partial (asymptotic) representation of the exchange quanta through the physical mesons and resonances. This is the historic approach to nuclear forces. By introducing a set of meson fields of various spins and isospins with Yukawa coupling to the nucleons, one should be able to represent the interaction between nucleons as long as the intrinsic quark structure can be ignored. The internal structure can be ignored presumably at densities such that the nucleon bags do not overlap, i.e., for densities $\rho < \rho_c \equiv (4/3\,\pi\, R_B^3)^{-1}$. Estimates[8] for the bag radius vary between 1/3 to 1 fm, corresponding to ρ_c/ρ_o between 30 and 1 respectively. Here $\rho_o = 0.145\ \mathrm{fm}^{-3}$ is the normal nuclear matter density,[9] corresponding to a radius parameter $r_o = 1.18$ fm. If the bag radius were really as large as 1 fm, then a description of ordinary nuclei in terms of nucleons would be a poor one. Since, however, this would contradict our experience with the shell model and direct nuclear reactions, we assume here that up to moderately high densities, say $4\rho_o$, the nuclear forces can be adequately represented by the exchange of mesons. For higher densities (or temperature), the internal structure of the nucleon would need to be considered, at least in the approximation of introducing the resonances as new fields.[2]

In accord with the above discussion, a set of meson fields in the various spin-isospin channels is introduced to represent the interaction of nucleons in a medium up to intermediate density. These meson fields are

$$\sigma(J=0^+,\ I=0)\ ,\qquad \omega_\mu(1^-,0)\ ,\qquad \pi(0^-,1)\ ,\qquad \rho_\mu(1^-,1)\ \ldots$$

and they are Yukawa coupled to the nucleon field ψ:

$$\mathcal{L}^{int} = g_\sigma \sigma\bar{\psi}\psi - g_\omega \omega^\mu \bar{\psi}\gamma_\mu\psi - g_\pi(\partial^\mu \underset{\sim}{\pi})\cdot(\bar{\psi}\gamma_5\gamma_\mu \underset{\sim}{\tau}\psi)$$

$$- g_\rho \underset{\sim}{\rho}^\mu \cdot (\tfrac{1}{2}\bar{\psi}\gamma_\mu \underset{\sim}{\tau}\psi + \underset{\sim}{\pi}\times\partial_\mu\underset{\sim}{\pi})\ \ldots \qquad . \tag{2}$$

The scalar interaction $(0^+,0)$ is represented by a broad resonance believed to represent two-pion exchange. In the static approximation, it contributes an attractive Yukawa potential. The vector meson ω_μ on the other hand contributes a repulsive interaction. These two meson exchanges can account for the saturation of nuclear matter. A model based on the first two chargeless mesons, the σ and ω_μ, was introduced many years ago by Johnson and Teller[10] and by Duerr.[11] It has been revived and extensively investigated by Walecka and collaborators,[12,13] who also examine the properties of finite nuclei.[14] We shall refer to a lagrangian that includes the σ and ω_μ as the standard Walecka model.

The lagrangian densities for the fields are

$$\mathcal{L}_{Dirac} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \qquad , \tag{3a}$$

$$\mathcal{L}_\sigma = \tfrac{1}{2}(\partial_\mu\sigma\partial^\mu\sigma - m_\sigma^2\sigma^2) - U(\sigma) \qquad , \tag{b}$$

$$\mathcal{L}_\omega = -\tfrac{1}{4}\omega_{\mu\nu}\omega^{\mu\nu} + \tfrac{1}{2}m_\omega^2\,\omega_\mu\omega^\mu \qquad , \tag{c}$$

$$\mathcal{L}_\pi = \tfrac{1}{2}(\partial_\mu \underset{\sim}{\pi} \cdot \partial^\mu \underset{\sim}{\pi} - m_\pi^2 \underset{\sim}{\pi} \cdot \underset{\sim}{\pi}) \quad , \tag{3d}$$

$$\mathcal{L}_\rho = -\tfrac{1}{4} \underset{\sim}{\rho}_{\mu\nu} \cdot \underset{\sim}{\rho}^{\mu\nu} + \tfrac{1}{2} m_\rho^2 \underset{\sim}{\rho}_\mu \cdot \underset{\sim}{\rho}^\mu \quad , \tag{e}$$

where we use the standard notation for Bjorken and Drell,[15] and

$$\omega_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu \quad . \tag{4}$$

The Euler-Lagrange equations based on the above lagrangian yield the following coupled field equations

$$(i\partial\!\!\!/ - m + g_s \sigma - g_\omega \omega\!\!\!/ - g_\rho \underset{\sim}{\rho\!\!\!/} \cdot \frac{\underset{\sim}{\tau}}{2} - g_\pi \gamma_5 \, \partial\!\!\!/ \underset{\sim}{\pi} \cdot \underset{\sim}{\tau})\psi = 0 \quad , \tag{5a}$$

$$(\Box + m_\sigma^2)\sigma = g_\sigma \bar{\psi}\psi - \frac{dU}{d\sigma} \quad , \tag{b}$$

$$(\Box + m_\pi^2)\underset{\sim}{\pi} = g_\pi \, \partial_\mu(\bar{\psi}\gamma_5 \gamma^\mu \underset{\sim}{\tau} \psi) + 2 g_\rho \underset{\sim}{\rho}^\mu \times \partial_\mu \underset{\sim}{\pi} \quad , \tag{c}$$

$$(\Box + m_\omega^2)\omega_\mu - \partial_\mu \partial^\nu \omega_\nu = g_\omega \bar{\psi}\gamma_\mu \psi \quad , \tag{d}$$

$$(\Box + m_\rho^2)\underset{\sim}{\rho}_\mu - \partial_\mu \partial^\nu \underset{\sim}{\rho}_\nu = g_\rho \{\tfrac{1}{2}\bar{\psi}\gamma_\mu \underset{\sim}{\tau} \psi + \underset{\sim}{\pi} \times \partial_\mu \underset{\sim}{\pi}\} \quad . \tag{e}$$

In the σ equation we have included a potential term U, which will be discussed later.

The above equations are intractible in their present form. However, they clearly show the connection between a finite ground state expectation value for a nucleon current, Eq. (1), with the existence of a finite amplitude for the corresponding field. For example, $\langle\bar{\psi}\psi\rangle \neq 0$ implies $\langle\sigma\rangle \neq 0$. In the normal state of symmetric nuclear matter, the σ and ω_0 have finite expectations values $\langle\sigma\rangle \neq 0$, $\langle\omega_0\rangle \neq 0$, and all other field expectations vanish. The other point of displaying the field equations (5) is to emphasize the highly non-linear character of the equations. The source currents are implicit functions of all the fields through the coupling of the nucleons to these fields, Eq. (5a). Thus it is quite possible that the field equations can be satisfied by several distinct sets of field configurations: $\{\langle\sigma\rangle_i, \langle\omega\rangle_i, \langle\pi\rangle_i, \ldots\}$ $i = 1,2,\ldots$.

The study of a fully developed pion condensate involves finding the conditions, if any, under which a finite $\langle\pi(x)\rangle$ is a solution to Eqs. (5).

B. Critique of Mean Field Approximation

We will solve the system of equations (5) by the mean field approximation. This is equivalent to the Hartree approximation. The ground state wavefunction is assumed to be a single Slater determinant ϕ_0, composed of quasi-particle wavefunctions that satisfy Eq. (5a), with all meson field operators replaced by their ground state expectation values,

$$\sigma \rightarrow \langle\sigma\rangle \, , \qquad \omega_\mu \rightarrow \langle\omega_\mu\rangle \, , \qquad \text{etc.} \tag{6}$$

These expectation values are computed in turn from Eqs. (5b-5e) by calculating the nucleon current operators appearing on the right-hand sides as the ground state expectations

$$\bar{\psi}\Gamma\psi \rightarrow \langle\bar{\psi}\Gamma\psi\rangle \equiv \langle\phi_0|\bar{\psi}\Gamma\psi|\phi_0\rangle \quad . \tag{7}$$

The system of equations (5) thus reduce to a set of transcendental equations in the unknown mean field configurations $\langle\sigma\rangle$, $\langle\omega_\mu\rangle$, etc.

This approximation has already been used extensively in the literature, both for the normal state of matter based on a lagrangian similar to ours, and for the pion condensed phase based on a chiral lagrangian.[1,4,5,16] Nevertheless, a justification for its use is in order.

The use of the mean field approximation to discuss the pion condensed state, a possible phase of matter lying in energy near the normal state, is analogous to the use of the shell model to calculate the nuclear spectrum near the Fermi level. Neither attempts to calculate the absolute energy using the fundamental coupling constants. To do so would require a theory with an accuracy on the order of 1 MeV on the scale of the total energy, on the order of GeV. Instead, our lagrangian, Eqs. (2)-(3), when used in the mean field approximation, is an effective theory with coupling constants adjusted to the ground state properties. The theory is then used to make moderate extrapolations from this point. Having thus determined the coupling constants it does not make sense to sum higher order diagrams since their contributions are implicitly built into the normalized coupling constants. Moreover, Chin has shown that while the coupling constants which lead to the correct saturation of nuclear matter are sensitive to the particular class of diagrams incorporated into the effective theory, the density dependence of the equation of state is not.[13] The higher order diagrams are found to vary only slowly with density. Thus the variation of the ground state energy with density is mainly determined by the mean field, and only the absolute scale is sensitive to all the diagrams. The corollary of this is that the effective coupling constants for the mean field theory are essentially density independent, over a moderate range of density.

The pion condensate phase corresponds to a long-range correlation (in the spin-isospin density) and so is susceptible to description by a mean field theory. However, the effective coupling constant g_π cannot be determined from the ground state properties, since the pion field $\langle\pi\rangle$ is either very small or vanishes in the ground state.[17] Other investigations, using the propagator approach, show that there is a considerable renormalization of g_π due to short-range correlations.[18] In addition, the renormalization will depend on whether the Δ resonance and finite-size form factors are incorporated explicitly into the mean field equations or not. In Section III we show how these effects can be incorporated as an effective g_π in our theory.

A description in the mean field approximation of the equation of state and any pion condensate phase that may develop as a function of density can be contrasted with the traditional approach to the many-body problem through the Schrodinger equation solved in the Breuckner or e^S theory. Here one uses potentials, sometimes of a form suggested by the field theory of nuclear forces, with constants adjusted to describe free-space scattering. These approaches then relate the ground state energy to the basic forces. They are powerful in the class of diagrams that they sum, and lead to valuable insights. However, they do represent an extreme extrapolation from the free-space interaction to the total energy of a nucleus. Since the spectrum of a nucleus or the pion condensate energy, which is sensitive to the spectrum, is a small fraction of the total energy, then these energy differences will likely lie within the error of the method and of the accuracy with which the potentials can be determined from scattering data. This situation is further aggravated by the fact that the non-relativistic approximation is inherent to any approach based on the Schrodinger equation. In addition, they are very cumbersome formalisms. The mean field approximation therefore has an advantage in providing essentially analytic insight into the physics of high (ρ,T) nuclear matter.

C. Self-Consistent Equations for the Mean Fields

The mean field approximation to the equations of motion, Eqs. (5), was described above. All meson fields and source currents are replaced by their ground state expectation values. In infinite homogeneous matter $\langle\bar{\psi}(x)\psi(x)\rangle$ and $\langle\bar{\psi}(x)\gamma_\mu\psi(x)\rangle$ are independent of x. The scalar and vector fields σ and ω_μ are therefore constants, and their mean values, $\bar{\sigma} \equiv \langle\sigma\rangle$ and $\bar{\omega}_\mu \equiv \langle\omega_\mu\rangle$, can be read from the equations of motion, Eqs. (5b,c). The rho meson plays no role in symmetric nuclear matter and so is not considered further. The equations for the mean fields corresponding to (5) are therefore

$$\left(i\partial\!\!\!/ - g_\omega\bar{\omega}\!\!\!/ - (m - g_\sigma\bar{\sigma}) - g_\pi\gamma_5\,\underset{\sim}{\tau}\cdot\partial\!\!\!/\langle\underset{\sim}{\pi}(x)\rangle\right)\psi(x) = 0 \quad , \qquad \text{(8a)}$$

$$m_\sigma^2\bar{\sigma} = g_\sigma\langle\bar{\psi}\psi\rangle - \left\langle\frac{dU}{d\sigma}\right\rangle \quad , \qquad \text{b)}$$

$$m_\omega^2\bar{\omega}_\mu = g_\omega\langle\bar{\psi}\gamma_\mu\psi\rangle \quad , \qquad \text{c)}$$

$$(\Box + m_\pi^2)\langle\underset{\sim}{\pi}(x)\rangle = g_\pi\,\partial^\mu\langle\bar{\psi}(x)\gamma_5\gamma_\mu\underset{\sim}{\tau}\psi(x)\rangle \quad . \qquad \text{d)}$$

Non-linear scalar field interactions are incorporated as Boguta and Bodmer,[14] through the potential density

$$U(\sigma) = \left(\tfrac{1}{3}\,bm + \tfrac{1}{4}\,c\,g_s\bar{\sigma}\right)\left(g_s\bar{\sigma}\right)^3 \quad . \tag{9}$$

The field configuration corresponding to the normal phase of symmetric nuclear matter is $\{\sigma \neq 0,\ \bar{\omega}_\mu = \delta_{\mu o}\bar{\omega}_o,\ \langle\pi\rangle = 0,\ \bar{\rho}_\mu = 0\}$. This is the configuration investigated by Walecka,[12] Chin,[13] Boguta and Bodmer.[14] Since, however, the equations (8) are a non-linear system of transcendental equations in the mean fields, the system may have another (abnormal) solution. As we stated earlier, our goal is to investigate whether a phase having a finite pion amplitude is also a solution, when the constants of the theory are chosen to represent correctly the bulk properties of nuclear matter.

The pion field must be allowed to have a space-time dependence because of the importance of the p-wave interaction. The pseudo-vector coupling ($\gamma_\mu\gamma_5$) between pions and nucleons is chosen to avoid the unphysical s-wave interaction of the pseudo-scalar couplings (γ_5). We investigate the class of solutions

$$\begin{aligned} \langle\pi(x)\rangle &= \bar{\pi}\left(\underset{\sim}{u}\cos kx + \underset{\sim}{v}\times\underset{\sim}{u}\sin kx\right) , \qquad (\underset{\sim}{v}\cdot\underset{\sim}{u} = 0) \\ kx &= k_o x_o - \underset{\sim}{k}\cdot\underset{\sim}{x} \end{aligned} \tag{10}$$

where $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are orthonormal vectors in isospin space. The particular choice $\underset{\sim}{u} = (1,0,0)$, $\underset{\sim}{v} = (0,0,1)$ corresponds to the charged running wave case

$$\langle\pi_\pm(x)\rangle = \frac{1}{\sqrt{2}}\,\bar{\pi}\,e^{\pm ikx} \quad , \qquad \langle\pi_o\rangle = 0 \quad , \tag{11}$$

that has been investigated in the σ-model.[5] In symmetric nuclear matter, the orientation of $\underset{\sim}{u}$ and $\underset{\sim}{v}$ is immaterial since there is no preferred direction in isospin space. All such solutions (10) are therefore degenerate in energy for symmetric matter.

The Dirac equation (8a) as it stands depends explicitly on space-time through the pion field. However this dependence can be transformed into a trivial phase factor, which is what reduces the system to a tractable one. Consider the space-time dependent rotation in isospin space

$$R_v(kx) = e^{-\frac{1}{2} i (kx) \underset{\sim}{\tau} \cdot \underset{\sim}{v}} \quad . \tag{12}$$

One can then show that (10) can be rewritten

$$\underset{\sim}{\tau} \cdot \langle \underset{\sim}{\pi}(x) \rangle = \bar{\pi}\, R_v(kx) \underset{\sim}{\tau} \cdot \underset{\sim}{u}\, R_v^{+}(kx) \quad , \tag{13}$$

and that the pion term in the Dirac equation is

$$\underset{\sim}{\tau} \cdot \not{\partial} \langle \underset{\sim}{\pi}(x) \rangle = \bar{\pi} \not{k}\, R_v(kx) \underset{\sim}{\tau} \cdot \underset{\sim}{v} \times \underset{\sim}{u}\, R_v^{+}(kx) \quad . \tag{14}$$

Thus by making the local isospin gauge transformation on the Dirac field

$$\psi_v(x) = R_v^{+}(kx)\, \psi(x) \quad , \tag{15}$$

the Dirac equation (8a) reduces to a space-time independent equation for $\psi_v(x)$,

$$[i\not{\partial} - g_\omega \bar{\not{\omega}} - (m - g_\sigma \bar{\sigma}) + \not{k}\, \underset{\sim}{\tau} \cdot (\tfrac{1}{2} \underset{\sim}{v} + g_\pi \bar{\pi} \gamma_5\, \underset{\sim}{v} \times \underset{\sim}{u})] \psi_v(x) = 0 \ . \tag{16}$$

Now we verify that the space-time dependence assumed for the pion field is a self-consistent solution of Eq. (8d). Since the Dirac equation for the transformed field, (16), has no x-dependent terms in it, and we are concerned with homogeneous nuclear matter, then momentum eigenstates are solutions of Eq. (16). In this case, $\partial_\mu \bar{\psi}$ and $\partial_\mu \psi$ are proportional to $\pm ip_\mu$. Therefore the source current in (8d) is

$$\partial^\mu <\bar{\psi}\gamma_5 \; \gamma_\mu \; \underset{\sim}{\tau} \; \psi> \; = \; <\bar{\psi}_v \gamma_5 \gamma_\mu \; \partial^\mu \; (R_v^+ \; \underset{\sim}{\tau} \; R_v)\psi_v> \tag{17}$$

Since

$$\partial_\mu \; R_v^+ \; \underset{\sim}{\tau} \; R_v \; = \; k_\mu \underset{\sim}{v} \times (\underset{\sim}{\tau} \cos kx - \underset{\sim}{\tau} \times \underset{\sim}{v} \sin kx) \tag{18}$$

we deduce from (8d) and (10) that $\underset{\sim}{u}$ must be orthogonal to both $\underset{\sim}{v}$ and the expectation value of $\bar{\psi}\gamma_5 \not{k} \; \underset{\sim}{\tau}\,\psi$. In that case,

$$(-k_o^2 + \underset{\sim}{k}^2 + m_\pi^2)\bar{\pi} \; = \; -g_\pi \; <\psi_v \; \gamma_5 \; \not{k} \; \underset{\sim}{\tau} \cdot \underset{\sim}{v} \times \underset{\sim}{u}\,\psi_v> \; . \tag{19}$$

Thus the pion field (10) is a solution of (8d) if its amplitude $\bar{\pi}$ is a solution of (19), where the right side is an implicit function of all the meson fields. One can infer from a general theorem given in Ref. 16, and as we show explicitly in appendix A, the space-like part of $\bar{\omega}_\mu$ vanish in the ground state. On the other hand, the time-like component of the vector field is fixed by the nuclear density from (8c),

$$m_\omega^2 \bar{\omega}_o \; = \; g_\omega \; <\psi_v^+ \; \psi_v> \; = \; g_\omega \; <\psi^+ \psi> \; = \; g_\omega \rho \; . \tag{20}$$

D. <u>Eigenvalues of the Dirac Equation and the Fermion Propagator</u>

Since the transformed nucleon fields ψ_v satisfy an equation which is independent of x, momentum eigenstates are solutions of (16),

$$\psi_v \; = \; U(p) \; e^{-ipx} \; , \tag{21}$$

where U(p) is an eight-component spinor for protons and neutrons which satisfies

$$[\not p - g_\omega \bar{\not\omega} - (m - g_s\bar\sigma) + \not k \underset{\sim}{\tau} \cdot (\tfrac{1}{2} v + g_\pi \gamma_5 \underset{\sim}{v} \times \underset{\sim}{u})] U(p) = 0 \quad . \tag{22}$$

The notation can be simplified by writing

$$p_\mu - g_\omega \bar\omega_\mu = P_\mu \quad , \tag{23a}$$

$$m - g_s \bar\sigma = m^* \quad , \tag{b)}$$

Then,

$$[\not P - m^* + \not k \underset{\sim}{\tau} \cdot (\tfrac{1}{2} \underset{\sim}{v} + g_\pi \bar\pi \gamma_5 \underset{\sim}{v} \times \underset{\sim}{u})] U(p) = 0 \quad . \tag{24}$$

As noted, $\bar\omega_\mu$ shifts the origin of four-momentum. The scalar field gives the nucleons an effective mass m*. This is the source of binding in the Walecka model.[12] The last term in (24) couples the neutron and proton components.

The eigenvalues can be found in the usual way by rationalizing the Dirac operator. Whereas multiplication of the usual Dirac equation $(\not p - m)u = 0$ by $(\not p + m)$ accomplishes this, yielding $p_o = \sqrt{\underset{\sim}{p}^2 + m^2}$, the operator needed to rationalize the operator in Eq. (24) is more complicated. The inverse of this operator is the propagator in momentum space. We find it to be

$$\begin{aligned} S_v(p) &= [\not P - m^* + \not k \underset{\sim}{\tau} \cdot (\tfrac{1}{2} \underset{\sim}{v} + g_\pi \bar\pi \gamma_5 \underset{\sim}{v} \times \underset{\sim}{u})]^{-1} \\ &= \frac{1}{D(p)} \left\{ (PP) - \varepsilon^2 - (Pk) \underset{\sim}{\tau} \cdot \underset{\sim}{v} - 2 g_\pi \bar\pi [(Pk) + m^* \not k] \gamma_5 \underset{\sim}{\tau} \cdot \underset{\sim}{v} \times \underset{\sim}{u} \right\} \\ &\times \left\{ \not P + m^* + \not k \underset{\sim}{\tau} \cdot [\tfrac{1}{2} \underset{\sim}{v} - g_\pi \bar\pi \gamma_5 \underset{\sim}{v} \times \underset{\sim}{u}] \right\} \quad , \end{aligned} \tag{25}$$

where

$$D(p_o,\underset{\sim}{p}) \equiv D(p) = \left((PP)-\varepsilon^2)\right)^2 - (Pk)^2 - 4g_\pi^2\,\bar{\pi}^2[(Pk)^2 - m^{*2}(kk)] \quad , \tag{26}$$

and

$$\varepsilon^2 = m^{*2} - \left(\tfrac{1}{4} + g_\pi^2\,\bar{\pi}^2\right)(kk) \quad , \tag{27}$$

$$(Pk) = P_o k_o - \underset{\sim}{P}\cdot\underset{\sim}{k} \quad , \tag{28}$$

where recall that $P_\mu = p_\mu - g_\omega\bar{\omega}_\mu$. The numerator of S_V is the operator that rationalizes the Dirac equation, yielding

$$D(p)\;U(p) = 0 \quad . \tag{29}$$

The quasiparticle spectrum $p_o = \omega(\underset{\sim}{p})$ is therefore given by solutions of

$$D\left(\omega(\underset{\sim}{p}),\underset{\sim}{p}\right) = 0 \quad . \tag{30}$$

It is a fourth order equation in P_o^4, P_o^2, P_o. However, we show later that for symmetric nuclear matter we must choose,

$$k_o = 0 \leftrightarrow \rho_N = \rho_Z \quad . \tag{31}$$

In this case we readily find the positive zeros of (30) to be

$$P_o = E_\pm = +\sqrt{\underset{\sim}{P}^2 + \varepsilon^2 \pm \Delta^2} \quad , \tag{32}$$

and corresponding negative solutions. Here

$$\Delta^2 = k\left\{(1+4g_\pi^2\,\bar{\pi}^2)\,P_\parallel^2 + (2g_\pi\,\bar{\pi}\,m^*)^2\right\}^{\frac{1}{2}}$$

$$k = |\underset{\sim}{k}| \quad , \qquad P_\parallel = \frac{\underset{\sim}{P}\cdot\underset{\sim}{k}}{k} \quad , \tag{33}$$

whence the Fermion spectrum is

$$\omega(\underset{\sim}{p}) = p_o = g_\omega \bar{\omega}_o + E_\pm \quad . \tag{34}$$

This result is analogous to the Walecka theory, where the Fermion spectrum is given by $g_\omega \bar{\omega}_o + \sqrt{p^2 + m^{*2}}$, which is spherically symmetric in momentum. However the situation here is more complicated. Due to the pion nucleon interaction, the Fermi surface is distorted. The spherical symmetry is destroyed, the new symmetry being cylindrical about the pion momentum $\underset{\sim}{k}$. Momentum $\underset{\sim}{P} \approx \pm \underset{\sim}{k}/2$ has lower energy compared to the case with no pions.

Two cases can be distinguished depending on the relative magnitude of k. In Fig. 1 the two positive energy solutions $E_\pm$ are plotted in the $P_\parallel$ plane. Depending on where the Fermi energy falls in this figure, only E_- or both E_- and E_+ will be occupied, and the region of momentum space occupied may be either connected or disconnected regions.

E. Evaluation of the Source Currents

In Section C the equations for the mean meson fields were derived. In this section we will derive explicit expressions for the source currents required, namely,

$$J_\Gamma = \langle \bar{\psi}_v \Gamma \psi_v \rangle$$
$$\Gamma = 1, \quad \gamma_\mu , \quad \gamma_5 \not{k}\, \underset{\sim}{\tau} \cdot \underset{\sim}{v} \times \underset{\sim}{u} \quad , \tag{35}$$

and thus complete the mathematical description of the problem.

The Dirac matrix equation (8a) which depends on space-time was transformed to a constant matrix equation of dimension 8, given by Eq.(22).

Therefore, the problem of evaluating the currents is an algebraic one. In principle the eigenspinors could be explicitly constructed. However, unlike the four-component free Dirac equation, this would be non-trivial. Instead we use the very powerful propagator technique to evaluate the sources. The spinors themselves are of no particular interest.

In general, the ground state expectation value of an operator Γ is given by[19]

$$<\bar{\psi}_v \Gamma \psi_v> \;=\; -i \lim_{\underset{\sim}{x}' \to \underset{\sim}{x}} \; \lim_{t' \to t^+} \; \mathrm{tr}\, \Gamma S_v(x - x') \tag{36}$$

where $S_v(x - x')$ is the coordinate space representation of the propagator of Eq. (16) or (25). In Appendix A the proof is sketched and we show how to reduce this to an explicit integral over momentum states.

$$<\bar{\psi}_v \Gamma \psi_v> \;=\; \mathrm{tr}\left[\Gamma \int \frac{d^3p}{(2\pi)^3} \sum_{p_o = \omega(\underset{\sim}{p})} \mathrm{Res} \;\; S_v(p_o, \underset{\sim}{p})\right] \quad . \tag{37}$$

The poles of the momentum representation of $S_v(p_o, \underset{\sim}{p})$ and the eigenvalue spectrum $\omega(\underset{\sim}{p})$ were given in Section D. The integration is carried out up to the Fermi surface specified by those momenta satisfying

$$E_\pm(p_\parallel, p_\perp) \;=\; E_F \tag{38}$$

where $E_\pm$ was given by Eq. (32) and $p_\parallel$ and $p_\perp$ are the momentum components parallel and perpendicular to $\underset{\sim}{k}$.

In Eq. (25) we have an explicit expression for our propagator. Denoting the numerator as N(p), we write it

$$S_v(p) \;=\; \frac{N(p_o, \underset{\sim}{p})}{D(p_o, \underset{\sim}{p})} \quad . \tag{39}$$

The denominator is given by Eq. (26) which for symmetric nuclear matter, Eq. (31), can be written in terms of the eigenvalues $E_\pm$ of Eq. (32), as

$$D(p_o,\underset{\sim}{p}) = (P_o - E_+)(P_o + E_+)(P_o - E_o)(P_o + E_-) \quad . \tag{40}$$

(Recall that $P_o = p_o - g_\omega \bar{\omega}_o$), from which we see that the denominators of the residues at $P_o = E_\pm$ are $\pm 4\Delta^2 E_\pm$. Hence,

$$<\bar{\psi}_v \Gamma \psi_v> = \int \frac{d^3p}{(2\pi)^3} \frac{1}{4\Delta^2} \left\{ \frac{\theta(E_F - E_+(\underset{\sim}{p}))}{E_+(\underset{\sim}{p})} (\mathrm{tr}\,\Gamma\, N(p))_{P_o=E_+} \right.$$

$$\left. - \frac{\theta(E_\mp - E_-(\underset{\sim}{p}))}{E_-(\underset{\sim}{p})} (\mathrm{tr}\,\Gamma\, N(p))_{P_o=E_-} \right\} \quad . \tag{41}$$

where $\theta(x)$ is unity for positive x and zero otherwise. The trace is a double trace over spinor and isospin space having dimensionality 4 and 2 respectively. The results of the trace evaluations are listed in the Appendix A.

The resulting evaluation of the source currents yields the following explicit equations, valid for symmetric matter ($k_o \equiv 0$):

$$(k^2 + m_\pi^2)\bar{\pi} = -2g_\pi \bar{\pi} k^2 \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{\theta_-}{E_-} \left(1 - \frac{2}{\Delta^2} (m^{*2} + P_\parallel^2)\right) \right.$$

$$\left. + \frac{\theta_+}{E_+} \left(1 + \frac{2}{\Delta^2} (m^{*2} + P_\parallel^2)\right) \right\} \tag{42a}$$

$$\bar{m}_\sigma^2 \bar{\sigma} = -\frac{dU}{d\bar{\sigma}} + 2g_\sigma(m - g_\sigma\bar{\sigma}) \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{\theta_-}{E_-} \left(1 - \frac{2g_\pi^2 \bar{\pi}^2 k^2}{\Delta^2}\right) \right.$$

$$\left. + \frac{\theta_+}{E_+} \left(1 + \frac{2g_\pi^2 \bar{\pi}^2 k^2}{\Delta^2}\right) \right\} \tag{42b}$$

$$\bar{\omega}_k \equiv 0 \quad , \qquad k = 1,2,3 \quad , \tag{42c}$$

$$\bar{\omega}_o = \frac{g_\omega}{m_\omega^2} \rho \quad , \tag{d)}$$

$$\rho = \langle\psi^+\psi\rangle = 2\int \frac{d^3p}{(2\pi)^3} (\theta_- + \theta_+) \quad , \tag{e)}$$

$$q = \langle\psi^+ \left(\frac{1+\tau_3}{2}\right) \psi\rangle = \tfrac{1}{2}\rho \quad , \tag{f)}$$

$$\varepsilon = 2\int \frac{d^3p}{(2\pi)^3} \left\{\theta_-(E_- + g_\omega \bar{\omega}_o) + \theta_+(E_+ + g_\omega \bar{\omega}_o)\right\}$$

$$+ \tfrac{1}{2}(\underset{\sim}{k}^2 + m_\pi^2)\bar{\pi}^2 - \tfrac{1}{2} m_\omega^2 \omega_o^2 + \tfrac{1}{2} m_\sigma^2 \bar{\sigma}^2 + U(\bar{\sigma}) \quad . \tag{g)}$$

We use the notation

$$\theta_\pm = \theta\left(E_F - E_\pm(\underset{\sim}{p})\right) \quad . \tag{43}$$

The first two of these equations define the self-consistency conditions on $\bar{\pi}$ and $\bar{\sigma}$. We also find that the space-like part of the vector field vanishes. The nucleon density is denoted by ρ, and the charge density by q, which as asserted earlier, is $\frac{1}{2}\rho$ for $k_o = 0$. The energy density is denoted by ε. All the integrals can be reduced to one-dimensional over $p_\parallel$, as shown in Appendix A.

These equations, together with the definitions of the quantities appearing in them, define our entire problem for symmetric nuclear matter.

The numerical procedure is to fix E_F, to solve Eq. (42a,b) for simultaneous solutions $\bar{\pi}$ and $\bar{\sigma}$, and then to calculate the corresponding density and energy density.

III. CONNECTION BETWEEN PION PROPAGATOR AND MEAN FIELD APPROACHES

As discussed in the introduction, the mean field approach allows us to investigate the fully developed pion condensate, whereas the propagator approach permits a calculation of only the critical density at which the new phase appears. However the effect of short-range correlations, the Δ-resonance, and finite-size form factors are easier to incorporate in the pion propagator (RPA) approach. Fortunately in the $\bar{\pi} \to 0$ limit of the mean field theory the two approaches can be related. The pion nucleon coupling constant g_π of the mean field approach can then be evaluated as an effective coupling which incorporates the above mentioned effects. This connection we now establish.

The pion propagator approach, proposed by Migdal and refined by Weise and Brown searches for the critical density for pion condensation by looking for the Goldstone modes in the spin-isospin $(0^-,1)$ channel.[1] This amounts to finding the density and pion wave number k at which the pion propagator $\Delta(\omega(\underset{\sim}{k}),\underset{\sim}{k})$ has a singularity for $\omega(\underset{\sim}{k}) = 0$ (Goldstone mode).

The pion propagator in the nuclear medium has the form

$$\Delta(\omega,\underset{\sim}{k}) = [\omega^2 - \underset{\sim}{k}^2 - m_\pi^2 - \Pi(\omega,\underset{\sim}{k})]^{-1} \quad . \tag{44}$$

Without Π this would be the free pion propagator. $\Pi(\omega,\underset{\sim}{k})$ is the proper self-energy or polarization operator and is the object that we can relate in the two approaches. The pion-like excitation is degenerate with the ground state when

$$\omega^2(\underset{\sim}{k}) = k^2 + m_\pi^2 + \Pi(0,\underset{\sim}{k}) = 0 \quad . \tag{45}$$

In actual calculations using this approach, Π is approximated by the pion self-energy due to the strong p-wave πNN and $\pi N\Delta$ vertices, together with an effective short-range interaction g' between particle-hole pairs. This leads to the following structure of Π in symmetric nuclear matter[1]

$$\Pi = \frac{\Pi_N + \Pi_\Delta}{1 - g' \frac{1}{k^2} (\Pi_N + \Pi_\Delta)} \qquad (46)$$

where Π_N is the nucleon particle-hole propagator (Lindhard function)

$$\Pi_N(0,\underset{\sim}{k}) \approx -f_\pi^2 k^2 F_{\pi N}^2(k)\; 2m^* p_F/\pi^2 \qquad (47)$$

with $f_\pi \approx 1/m_\pi$, m^* is the effective nucleon mass, p_F is the Fermi momentum, and $F_{\pi N}$ is the πNN form factor. The Δ particle-nucleon hole propagator Π_Δ is given by

$$\Pi_\Delta(0,\underset{\sim}{k}) \approx -f_\Delta^2 k^2 F_{\pi\Delta}^2(k) \frac{1}{\omega_\Delta} \frac{p_F^3}{3\pi^2} \quad . \qquad (48)$$

where $\omega_\Delta = 2.4\ m_\pi$, $f_\Delta^2 \approx 5/m_\pi^2$, and $F_{\pi\Delta}$ is the $\pi N\Delta$ form factor. Estimation for the value of g' varies between 0.5 ± 0.2 and is the subject of considerable theoretical controversy.[17]

The functional forms of Π_N and Π_Δ are easily understood as follows: Both Π_N and Π_Δ describe the amplitude for the pion to create a virtual particle and hole excitation in the medium. Therefore, the structure of both self energies is the familiar perturbation form

$$\Pi_i(\omega,\underset{\sim}{k}) = \sum_{\underset{\sim}{q}} \frac{\left|\langle\pi|\mathcal{H}_{int}|ph\rangle\right|^2}{\omega + \omega_h(\underset{\sim}{q}) - \omega_p(\underset{\sim}{q}+\underset{\sim}{k})} \quad , \qquad (49)$$

where $\omega_h(\underset{\sim}{q})$ is the energy of the initial nucleon (hole) of momentum $\underset{\sim}{q}$ that absorbs the pion with four momentum $(\omega,\underset{\sim}{k})$ and $\omega_p(\underset{\sim}{q}+\underset{\sim}{k})$ is the energy of the intermediate particle with momentum $\underset{\sim}{q}+\underset{\sim}{k}$. For Π_N the intermediate particle is a nucleon so that $\omega_h - \omega_p = -\underset{\sim}{q}\cdot\underset{\sim}{k}/m^*$. For Π_Δ the intermediate particle is a Δ_{33} resonance, so that $\omega_h - \omega_p \approx \omega_\Delta \approx 2.4\ m_\pi$, almost independent of $\underset{\sim}{k}$ and $\underset{\sim}{q}$. Thus, the energy denominators for $\omega = 0$ are very different for Π_N and Π_Δ.

Because both πNN and $\pi N\Delta$ interactions are p-wave, the numerator $|\langle\mathcal{H}_{int}\rangle|^2 \propto k^2$. However, for the nuclear intermediate states, the Pauli principle blocks occupied states and $|\langle\pi|\mathcal{H}_{int}|N\bar{N}\rangle|^2 \propto k^2\, n(\underset{\sim}{q})\,(1-n(\underset{\sim}{k}+\underset{\sim}{q}))$, while no Pauli blocking occurs for Δ and $|\langle\pi|\mathcal{H}_{int}|\Delta\bar{N}\rangle|^2 \propto k^2\, n(\underset{\sim}{q})$. Here $n(\underset{\sim}{q})$ is the Fermi distribution of occupied states. Because Π_Δ does not involve Pauli blocking and the energy denominator is insensitive to $\underset{\sim}{q}$, we can simply sum over $\underset{\sim}{q}$ giving $\Pi_\Delta \propto k^2\rho/\omega_\Delta$ as in Eq. (48). On the other hand, Pauli blocking severely limits the sum over $\underset{\sim}{q}$ for Π_N. As $k \to 0$ only those momenta $\underset{\sim}{q}$ inside the Fermi sphere that lie within a half shell of radius p_F and thickness k and oriented such that $\underset{\sim}{q}\cdot\underset{\sim}{k} > 0$ satisfy $n(\underset{\sim}{q})\big(1-n(\underset{\sim}{q}+\underset{\sim}{k})\big) \neq 0$. The number of states in that shell is $\propto p_F^2 k$. The particle-hole energy denominator is then $\propto p_F k/m^*$, so finally we obtain $\Pi_N \propto k^2 m^* p_F$ as in Eq. (47). Even for $k \sim p_F$, this form for Π_N holds because then $\omega_p - \omega_k \propto p_F^2/m^*$, while the sum over $\underset{\sim}{q}$ gives p_F^3 in the numerator. Therefore, we see that the forms of Π_N and Π_Δ in Eqs. (47) and (48) are easily understood.

To get a feeling for the magnitude of the self energies involved in Eqs. (46)-(48), consider Π_N, Π_Δ and Π for $\rho = \rho_0$, $p_F \approx 1.8\ m_\pi$, $g' = 0.5$, $m^* = m_N$, and for pion momentum $k \approx 2m_\pi$. Setting $F_{\pi N} = F_{\pi\Delta} = 1$, we find

$\Pi_N(0,2m_\pi) \approx -10\ m_\pi^2$ and $\Pi_\Delta \approx -2m_\pi^2$. Thus, the numerator of Eq. (46) is $\approx -12\ m_\pi^2$, while the denominator is ≈ 2.5, so that $\Pi(0,2m_\pi) \approx -5\ m_\pi^2$. So, we see that Π is large and cancels the kinetic energy $k^2 + m_\pi^2 = 5\,m_\pi^2$ at about normal densities.

Now consider the effect of finite form factors. Conventionally, $F_{\pi N}(k) = F_{\pi\Delta}(k) = (\Lambda^2 - m_\pi^2)/(\Lambda^2 + k^2) \approx 1 - (k^2 + m_\pi^2)/\Lambda^2$ is described by a monopole form factor with $\Lambda \approx 1$ GeV. Note, however, that this large value of Λ does not require an assumption that the quark bags are small. In Ref. 20, the form factor for the intermediate quark bag of radius $R_B = 0.72$ fm was computed as $F_{\pi N}(k) = F_{\pi\Delta}(k) = 3j_1(kR_B)/kR_B \approx 1 - (kR_B)^2/10$. For $k = 2m_\pi$, this in fact yields the same result as the monopole form factor with the large $\Lambda = \sqrt{12.5}\,/\,R_B = 970$ MeV, giving $F_{\pi N} = F_{\pi\Delta} \approx 0.9$. The net effect is to reduce $\Pi_N + \Pi_\Delta$ by about 20% to a value $\approx -10\ m_\pi^2$ at normal density. For $m^* < m_N$, this value is further reduced by a factor m^*/m_N.

We can conveniently summarize the combined influence of Π_Δ, g', and $F_{\pi N}$ on the self energy, Π, in Eq. (46) by defining an effective πNN coupling constant via

$$\Pi \equiv \left(\frac{f_{eff}}{f_\pi}\right)^2 \Pi_N = -f_{eff}^2\ k^2\, 2m^*\, p_F/\pi^2 \quad . \tag{50}$$

Comparing it with Eqs. (46)-(48), this effective coupling constant is given approximately by

$$f_{eff}^2(k,p_F) = f_\pi^2\ \frac{|F_{\pi N}(k)|^2\left[1 + \frac{1}{6}\ \frac{f_\Delta^2}{f_\pi^2}\ \frac{p_F^2}{m^*\omega_\Delta}\right]}{1 + g'|F_{\pi N}(k)|^2\ f_\pi^2\ \frac{2m^*}{\pi^2}\ p_F\left[1 + \frac{1}{6}\ \frac{f_\Delta^2}{f_\pi^2}\ \frac{p_F^2}{m^*\omega_\Delta}\right]} \quad . \tag{51}$$

Although the form of this "constant" certainly does not look constant as a function of p_F, Table 1 shows that f_{eff}^2 is actually very insensitive to m^* and $\rho = 2p_F^3/3\pi^2$. The remarkable property of Eq. (51) is that in the density range ρ_0 to $3\rho_0$, where we expect m^* to vary between $(1.0 \sim 0.7)m$, f_{eff}^2 varies by only ~10%.

In the Appendix we show that the relativistic mean field equations reduce to the relativistic RPA equations when the pion field strength vanishes. In particular, Eq. (8d) reduces, in the limit $\langle\pi(x)\rangle \to 0$, to

$$\left(-\underset{\sim}{k}^2 + m_\pi^2 + \Pi_{rel}(k)\right)\bar{\pi} = 0 \quad , \tag{52}$$

where Π_{rel} is the relativistic Lindhard function

$$\Pi_{rel}(k) = -2i\, g_\pi^2 \int \frac{d^4p}{(2\pi)^4}\, \mathrm{tr}\left(\not{k}\gamma_5\, S_0(p)\, \not{k}\gamma_5\, S_0(p+k)\right). \tag{53}$$

The contour of integration for p_0 is specified in Appendix A. Here S_0 is the nucleon propagator given by Eq. (25) when $\bar{\pi} = k = 0$. When $p_F \ll m^*$, Π_{rel} reduces to the non-relativistic Lindhard function above (with $F_{\pi N} = 1$). Recoil and relativistic kinematics lead to corrections

TABLE 1. Effective πNN coupling f_{eff} in units of fm for $k = 2m_\pi$, $g' = 0.5$, $|F_{\pi N}(k)|^2 = 0.81$ incorporating correlations, Δ production and form factors as a function of m^* and density from Eq. (51). Compare f_{eff} to $f_\pi = 1.41$ fm in free space.

m^*/m_N	ρ =	ρ_0	$2\rho_0$	$3\rho_0$
1.0	f_{eff} =	0.93	0.89	0.86
0.75		1.01	0.98	0.96
0.50		1.14	1.12	1.11

of the orders of $(p_F/m^*)^2$ and $(k/m^*)^2 \sim$ 10-20% to the non-relativistic self-energy. In any case, Eq. (50) and (53) provide the link between g_π and f_{eff} with uncertainties as quoted of 10-20% in the link as well as uncertainties in f_{eff} due to the imprecision with which g' and form factors are known.

We note that the idea of using an effective coupling to incorporate Δ and correlation effects was first pointed out in Ref. 18 in connection with neutron matter.

A last point we stress in connection with the pion self energy is the near proportionality of Π to the effective nucleon mass, $m^*(p_F)$. This means that the driving force for condensation depends sensitively on the details of the single particle-hole excitation spectrum and hence on nuclear structure physics.[21] Therefore, pion condensation calculations should only be carried out with models consistent with known nuclear properties. An example of a model not consistent with nuclear properties is the chiral model[4,5] where $m^* = m$. Since the effective mass, $m^* = m - g_\sigma\bar{\sigma}$ is less than the nucleon mass, the chiral model can be expected to overestimate the condensate energy. We note at this point that the chiral σ-model has not been solved in a fully self-consistent manner. It is self-consistent in the "chiral angle" but not in the "chiral radius."

IV. RESULTS

A. Determination of the Parameters

The equation of state of nuclear matter, $\varepsilon(\rho)$, is obtained in the theory by finding the self-consistent $\bar{\pi}$ and $\bar{\sigma}$ fields which are simultaneous solutions of Eqs. (42a-b), for a chosen Fermi energy, E_F, and then calculating the corresponding density and energy density, Eq. (42e,g). The pion wave number k is then varied to achieve a minimum energy. We note that $\bar{\pi} \equiv 0$, the normal state, is always a self-consistent solution. For given coupling constants the equations do not necessarily have a solution with finite pion amplitude. In particular, as we have discussed, our goal is to find whether such an abnormal state is compatible with the known bulk properties of nuclear matter. These properties are the saturation binding,[9] density,[9] and compressibility of symmetric nuclear matter.

$$E/A = \varepsilon/\rho - m = -15.96 \text{ MeV} , \qquad (54a)$$

$$\rho_0 = 0.145 \text{ fm}^{-3} , \qquad b)$$

$$K = 9\rho_0^2 \frac{\partial^2}{\partial\rho^2} (E/A) = 200\text{–}300 \text{ MeV} . \qquad c)$$

In addition to these constants, based in part on the results of Breuckner calculations, is the prejudice that the binding should reduce to zero somewhere in the range $2\rho_0 - 3\rho_0$. These four properties of the normal state determine for us the four parameters,

$$g_\sigma/m_\sigma , \quad g_\omega/m_\omega , \quad b , \quad c . \qquad (55)$$

(It can be verified that Eq. (42) depends on the ratios shown and not separately on coupling constants and masses of the scalar and vector meson.)

The normal state ($\pi \equiv 0$) does not depend on the fifth parameter of the theory, g_π. The free space πN coupling $g_\pi = 1.41$ fm gives the correct p-wave scattering length. However, as discussed in Section III, we can incorporate in our mean field theory the effects of short-range correlations, Δ resonance, and finite-size form factors by renormalizing g_π in the specific way described. These effects yield an effective coupling that is remarkably density independent. Within the many uncertainties discussed, they reduce g_π from the free-space value to about 1 fm, depending on the effective mass m^* (see Table 1).

B. The Normal State

In Figs. 2 and 3 we show the calculated equation of state for the normal ground state $\bar{\pi} \equiv 0$ corresponding respectively to two sets of parameters, Eq. (55), that have the three bulk properties, Eq. (54), and differ in the density at which the binding vanishes. We shall call these the stiffer and softer equations of state, referring to their high density behavior, though we stress that at the saturation point they have the same curvature, described by $K = 280$ MeV.

The difference between the equations of state in Figs. 2 and 3 is due mainly to the difference between effective mass, $m^*(\rho)$, for each case. In Fig. 4 we show $m^*(\rho)$ corresponding to each case.

In our theory, the $m^*(\rho)$ is controlled indirectly by the nonlinear σ interactions in Eq. (9). For $b = c = 0$, corresponding to the Walecka model, $m^*(\rho)$ is strongly density dependent and has a rather small value, ~ 0.6 m. Associated with this small m^* is a rather high compressibility $K \sim 500$ MeV, which follows from the large values g_σ/m_σ and g_v/m_v needed to reproduce the binding and saturation density. By introducing the nonlinear interactions of Eq. (9), which are equivalent to three and four body forces, we can lower g_σ/m_σ and g_v/m_v. Since g_v/m_v controls the magnitude of m^*[14] while g_σ/m_σ controls the density dependence of m^*, by lowering g_σ/m_σ and g_v/m_v we increase the magnitude of m^* and reduce its density dependence. This is what we find in Fig. 3. The larger the values of b and c, the smaller are the values of g_σ/m_σ and g_v/m_v needed to reproduce the saturation properties, and consequently $m^*(\rho)$ is larger and less density dependent. Also, by lowering g_σ/m_σ and g_v/m_v, we see from Figs. 2 and 3 that the equation of state becomes softer. Therefore, there is a correlation between larger three and four-body forces, a larger and less density dependent $m^*(\rho)$, a softer equation of state, a lower critical density, and greater condensation energy.

C. The Pion Condensate State

Turning now to abnormal states, for sufficiently large g_π, there is a second solution to the self-consistent equations (42), the pion condensate, that lowers the ground state energy. We show such solutions for several values of g_π in both the stiffer and softer case. As g_π is lowered from the vacuum value of 1.41 fm, simulating the correlations,

Δ and form factors, the critical density increases and the gain in binding due to condensation decreases. The dependence on g_π is quite different in the two cases shown in Figs. 2 and 3. At corresponding values of g_π, the condensate is much stronger for the softer (at high density) equation of state. Indeed, for Fig. 2, with the smaller non-linear σ potential (b,c), condensate solution exists only for $g_\pi >$ 1.18 fm. Since, according to our discussion of Section III, we believe that the renormalized coupling constant should be $g_\pi \approx 1$ fm, we conclude that for the stiffer equation of state, no pion condensate can exist. In the other case with larger non-linear terms leading to a softer equation of state, the critical density is merely shifted to higher values as g_π is reduced, and at the effective value of 1 fm lies at about $2\rho_0$.

We conclude that pion condensation in symmetric nuclear matter can be made consistent with the bulk properties of nuclei if the effective mass is large ($m^* \gtrsim 0.9$), which in our theory occurs when sufficiently large three and four-body forces exist. To gauge how large these forces are in the case of Fig. 3, we can compute from Eq. (9) the net contribution of these many body forces to the energy per nucleon at normal density. For these parameters of Fig. 3 this is

$$\left(\frac{E}{A}\right)_{3,4\ \text{body}} = \frac{U(\sigma)}{\rho_0} = -46\ \text{MeV} \quad , \tag{56}$$

which is quite large. However, slight variations of b,c yielding nearly identical results to Fig. 3 can result in a factor of 10 smaller value.

Next, we contrast our results to calculations using the chiral model[5] where m* is fixed to be m.. In Fig. 5 we compare the condensation energy, the difference in energy between the normal and condensed states, calculated in Ref. 5, to those obtained from Figs. 2 and 3 for g_π chosen to give approximately the same critical density. As expected, the chiral model with the much larger driving force tends to give a much higher condensation energy that increases rapidly with density. We conclude that self-consistency and compatibility with the bulk nuclear properties are very strong constraints on the existence and persistence of the condensate phase.

Finally, we want to know the expected magnitude of the pion field for various g_π. Typically, $\bar{\pi}$ in Eq. (10) turns out to be on the order of 0.1 m_π. In order to get a feeling for this number, we should compare the amplitude of spin-isospin oscillations to the normal baryon density. From Eq. (19), we can convert $\bar{\pi}$ into a magnitude of $\langle \bar{\psi}\gamma_5 \gamma_3 (\underset{\sim}{\tau}\cdot\underset{\sim}{u})\psi \rangle$ for $\underset{\sim}{k} = 2m_\pi \hat{e}_3$ as

$$\langle \bar{\psi} \gamma_5\gamma_3 \underset{\sim}{\tau}\cdot \underset{\sim}{v}\times\underset{\sim}{u}\psi\rangle = - \frac{(k^2+m_\pi^2)}{g_\pi k}\bar{\pi} , \qquad (57)$$

For $\bar{\pi} = 0.1\ m_\pi$, $g_\pi = 1/m_\pi$, the right-hand side is 0.25 m_π^3 which is about one-half the normal baryon density, $\rho_0 \approx 0.5\ m_\pi^3$. Thus, sizeable oscillations of the spin-isospin density are possible. It is important to note, however, that unlike the neutral condensate, the class of condensate solutions considered in Eq. (10) do not lead to density oscillations. The baryon density is uniform; only the spin-isospin density oscillates. In Fig. 6 we plot the ratio of the spin-isospin density $\langle\bar{\psi}\gamma_5\gamma_3\tau_3\psi\rangle$ to the baryon density $\langle\bar{\psi}\gamma_0\psi\rangle$ for a case with g_π = 1 fm in Fig. 3. Note that non-relativistically $\langle\bar{\psi}\gamma_5\gamma_3\tau_3\psi\rangle \approx \rho(p\uparrow) + \rho(n\downarrow) - \rho(p\downarrow) - \rho(n\uparrow)$ is the spin-isospin density, where $\rho(p\uparrow)$ is the density of protons with spin pointed along the z axis, etc. On the other hand, $\rho = \langle\bar{\psi}\gamma_0\psi\rangle \approx \rho(p\uparrow) + \rho(n\downarrow) + \rho(p\downarrow) + \rho(n\uparrow)$ is independent of x. The ratio $R_{53} = \langle\bar{\psi}\gamma_5\gamma_3\tau_3\psi\rangle/\langle\bar{\psi}\gamma_0\psi\rangle$ in Fig. 6a measures the magnitude of the spin-isospin density oscillations in the condensed state. We see that $R_{53} \approx 0.5$ for $\rho \gtrsim 2\rho_0$. The corresponding oscillations of the densities $\rho(p\uparrow) + \rho(n\downarrow)$ and $\rho(p\downarrow) + \rho(n\uparrow)$ are illustrated in Fig. 6b.

It is remarkable to note in Fig. 6a that although the condensate energy is very small, $\lesssim$ 3 MeV, the spin-isospin oscillations are about as large as they can possibly get. In fact, R_{53} only increases to 0.85 when $g_\pi = 1.41$, even though condensate energy is about 10 times larger. This has important consequences when considering dynamical effects of pionic instabilities. Clearly, the very slight softening of the equation of state, $E/A(\rho)$, due to condensation would have very little effect on the hydrodynamics of nuclear collisions. However, the large spin-isospin fluctuations can lead to critical scattering phenomena which we could hope to observe.[22]

D. The Non-Relativistic Approximation

We have solved a relativistic theory of nuclear matter in which Lorentz invariance was retained throughout. This offers the rare opportunity of assessing the importance of relativistic kinematics in a many-body theory. Therefore we have also solved the mean field equations in their non-relativistic limit. This limit is obtained when the effective mass m^* is regarded as much larger than typical kinetic energies $p_F^2/2m^*$, $k^2/2m^*$, etc. In that case the quasi-particle spectrum, Eq. (32), can be approximated by

$$E_{\pm} \approx m^* + \frac{1}{2m^*}\left\{\underset{\sim}{p}^2 - (\tfrac{1}{4} + g_\pi^2\pi^2)\,(kk) \pm \Delta^2\right\} \qquad (58)$$

where Δ^2 is still given by Eq. (33). The source currents are otherwise calculated as before in Eq. (42).

The numerical results are shown in Fig. 7, corresponding to the coupling constants of Fig. 3 (where the effective mass is the largest and therefore the relativistic kinematics least important). Comparing the normal state, calculated relativistically, and the non-relativistic approximation, we see that the binding is underestimated by about 5 MeV and the saturation point is shifted to slightly smaller density. The approximation is even worse for the condensate state. Of course, on the scale of the total energy density of ~940 MeV per nucleon at normal density, this is a small error. But that is precisely the point: A small error in the calculation of the total energy can be a large error in the binding energy. This might be a significant warning concerning the attempt to calculate nuclear binding energies in theories based on the Schrodinger equation.

V. SUMMARY

We have studied the pion condensate phase of symmetric nuclear matter, in the mean field approximation of a relativistic field theory, which is constrained to possess the known bulk properties of nuclear matter. It was shown how to incorporate the effects of short-range correlations, the Δ-resonance, and finite-size form factors into an effective coupling constant. It is essentially independent of density over the range considered. The relativistically covariant theory was solved self-consistently for the field configurations. Two cases were studied, corresponding to parameters of the lagrangian which in both cases yielded identical saturation properties, and differed only in their softness at higher density. This latitude was introduced to represent our ignorance of the equation of state away from the saturation point. Only one of these possessed a condensate solution for pion-nucleon coupling constant in the range of the expected effective coupling. This was the case where the equation of state was softer at higher density. Even in this case the condensate energy was very small, not exceeding 3 MeV for density up to $3\rho_0$. This is in sharp contrast with other studies based on the chiral σ-model. Since the normal state of the σ-model does not possess the saturation properties of nuclear matter, we believe that our estimate is more reliable. Thus self-consistency and compatibility with bulk nuclear properties are strong constraints on the existance, persistance and magnitude of the condensate phase. Despite the small condensate energy in our theory, the corresponding amplitude of the spin-isospin density was very significant, being about $\rho/2$. We conclude that a pion condensate is compatible with the known bulk properties of nuclear matter, at least within our theory.

Of broader interest to the many-body theory of the nucleus, we also evaluated the non-relativistic approximation of this covariant theory to assess the importance of relativistic kinematics. At normal nuclear density, this approximation introduces an error in the binding per nucleon, on the order of the binding itself. This might be regarded as an estimate of the error that is inherent in many-body theories that are based on the Schrodinger equation.

APPENDIX A:
PROPAGATOR METHOD OF EVALUATING EXPECTATION VALUES

We need ground state expectation values of various operators

$$\langle\Gamma\rangle = \int d^3x \; \langle\bar{\psi}(x)\;\Gamma(x)\;\psi(x)\rangle \quad . \qquad \text{(A1)}$$

Examine

$$\langle\bar{\psi}(x)\;\Gamma(\underset{\sim}{x})\;\psi(x)\rangle = \sum_{\alpha,\beta} \langle\bar{\psi}_\beta(x)\;\Gamma_{\beta\alpha}(x)\;\psi_\alpha(x)\rangle$$

$$= \lim_{x'\to x} \Gamma_{\beta\alpha}(x)\;\langle\bar{\psi}_\beta(x')\;\psi_\alpha(x)\rangle \quad , \qquad \text{(A2)}$$

where α,β are the spinor component labels. Now the Feynman propagator is defined by

$$i\;S_{\alpha\beta}(x-x') = \langle T\big(\psi_\alpha(x)\;\bar{\psi}_\beta(x')\big)\rangle$$

$$= \langle\psi_\alpha(x)\;\bar{\psi}_\beta(x')\;\theta(t-t') - \bar{\psi}_\beta(x')\,\psi_\alpha(x)\;\theta(t'-t)\rangle \; , \qquad \text{(A3)}$$

where T denotes the time-ordered product. Therefore,

$$\langle\bar{\psi}(x)\;\Gamma(\underset{\sim}{x})\;\psi(x)\rangle = -i \lim_{\underset{\sim}{x}'\to\underset{\sim}{x}} \;\lim_{t'\to t+} \Gamma_{\beta\alpha}(x)\;S_{\alpha\beta}(x-x')$$

$$= -i \lim_{x'\to x}\;\lim_{t'\to t+} \text{Trace}\;\Gamma(x)\;S(x-x') \quad . \qquad \text{(A4)}$$

This is a result derived in Fetter and Walecka.[19]

To learn the structure of S, expand the field operators

$$\psi(x) = \sum_s \int d^3p \Big(U(p,s)\;b(p,s)\;e^{-ipx} + V(p,s)\;d^+(p,s)\;e^{ipx}\Big) \qquad \text{(A5)}$$

where b and d^+ are destruction and creation operators respectively. The ground state of our system is schematically

$$> \;=\; b^+_{p_1} \dots b^+_{p_F} \,|0> \quad , \qquad (E_{p_F} = E_F) \; . \qquad \text{(A6)}$$

Because

$$< b^+(p,s)\; b(p',s')> \;=\; \delta_{pp'}\,\delta_{ss'}\;\theta(E_F - E_p) \quad , \qquad \text{(A7)}$$

then

$$<\bar{\psi}_\beta(x')\,\psi_\alpha(x)>\,\theta(t'-t) \;=\; \int d^3p\; \bar{U}_\beta U_\alpha\; e^{ip(x'-x)}\,\theta(t'-t)\;\theta(E_F - E_p) \; . \qquad \text{(A8)}$$

The infinite contribution from the filled negative energy sea has been subtracted away in (A8). Since we wish to avoid the explicit calculation of the eight-component spinors, U, we turn instead to the differential equation that is satisfied by S(x-x'), using (A8) to specify the boundary conditions. From Eq. (16), the propagator satisfies

$$[i\not\partial - g_\omega\bar{\omega} - m^* + \not k \underset{\sim}{\tau}\cdot(\tfrac{1}{2}\underset{\sim}{v} + g_\pi\bar{\pi}\,\gamma_5\,\underset{\sim}{v}\times\underset{\sim}{u})]\; S(x-x') \;=\; \delta^4(x-x') \; . \qquad \text{(A9)}$$

Write the Fourier transform of S as

$$S(x-x') \;=\; \int \frac{d^4p}{(2\pi)^4}\; e^{ip(x'-x)}\; S(p) \quad , \qquad \text{(A10)}$$

where it follows from (A9) that S(p) is given by Eq. (25). The contour of integration for the p_o integration, which yields the form (A8) is closed in the upper half plane and encloses only those singularities that correspond to occupied states.

These singularities are located at

$$p_o = E_{\mp}(\underset{\sim}{p}) + g_\omega \bar{\omega} \equiv \omega(\underset{\sim}{p}) \quad , \tag{A11}$$

as found in Eq. (34). For fixed $\underset{\sim}{p}$ the p_o integral yields

$$i \sum_{\omega(\underset{\sim}{p})} \left[e^{ip\,(x'-x)} \text{Res } S(p) \right]_{p_o = \omega(\underset{\sim}{p})} \theta(E_F - \omega(\underset{\sim}{p})) . \tag{A12}$$

Finally, if $\Gamma(x)$ is not a differential operator, the limits in (A4) can be taken immediately with the result

$$\langle \bar{\psi} \Gamma \psi \rangle = \text{tr } \Gamma \int \frac{d^3p}{(2\pi)^3} \sum_{\omega(\underset{\sim}{p})} \theta(E_F - \omega(\underset{\sim}{p})) \, [\text{Res } S(p)]_{p_o = \omega(\underset{\sim}{p})} \quad . \tag{A13}$$

If Γ is a differential operator, for example, the energy, $\gamma_o i \frac{\partial}{\partial t}$, it must be allowed to act before the limits are taken:

$$\langle \psi \, \gamma_o i \frac{\partial}{\partial t} \psi \rangle = \int \frac{d^3p}{(2\pi)^3} \sum_{\omega(\underset{\sim}{p})} \theta(E_F - \omega(\underset{\sim}{p})) \, \text{tr } \gamma_o [p_o \text{ Res } S(\underset{\sim}{p})]_{p_o = \omega(\underset{\sim}{p})} \quad . \tag{A14}$$

Using the above techniques, we obtain the expressions for all needed source currents in Eqs. (42a,b,e,g). These involve a three momentum integration, which may be reduced to a one-dimensional integral over the component of $\underset{\sim}{p}$ parallel to $\underset{\sim}{k}$ as follows: Observe that $E_\pm^2(p_\parallel, p_\perp) = E_\pm^2(p_\parallel, 0) + p_\perp^2$. For the π and σ source we need integrals of the form

$$\int \frac{d^3p}{(2\pi)^3} \frac{\theta_\pm}{E_\pm} f(p_\parallel) = \frac{1}{8\pi^2} \int_{-\infty}^{\infty} dp_\parallel \int_{-\infty}^{\infty} dp_\perp^2 \, \frac{\theta(E_F - E_\pm^2(p_\parallel, 0) - p_\perp^2)}{\sqrt{E_\pm^2(p_\parallel, 0) + p_\perp^2}} \, f(p_\parallel)$$

$$= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dp_\parallel \, [E_F - E_\pm(p_\parallel, 0)] \, f(p_\parallel) \; \theta(E_F - E_\pm(p_\parallel, 0)) \quad . \tag{A15}$$

Because of the theta function this is simply a one-dimensional integral over a finite interval, possibly two disconnected segments. To evaluate Eq. (A15) numerically it is convenient to remove a singular piece of the integrand. This singularity arises from the $1/\Delta^2$ behavior of $f(p_{\|})$ in Eqs. (54a,b). Since $\Delta^2 \to kp_{\|}$ as $\bar{\pi} \to 0$ and $\Delta^2 \to kp_{\|}(1+4g_{\pi}^2\bar{\pi}^2)^{\frac{1}{2}}$ as $m^* \to 0$, we see that in either limit the integration in Eq. (A15) diverges logarithmically if $p_{\|} = 0$ is allowed $(E_F > E_{\pm}(0,0))$. Thus, in particular for finite $\bar{\pi}$, the source integral behaves as $\log\bar{\pi}$ near $\bar{\pi} \to 0$. To extract this logarithmic behavior it is convenient to add and subtract $E_{\pm}(0,0)$ in Eq. (A15). Then the piece proportional to $E_F - E_{\pm}(0,0)$ can be evaluated in closed form and the remainder proportional to $E_{\pm}(0,0) - E_{\pm}(p_{\|},0)$ is free from singular behavior and can readily be evaluated with a few point Gaussian quadrature. To illustrate this procedure we evaluate the θ-piece of the pion integral in Eq. (54a)

$$
\begin{aligned}
I_{\pi}^{-} &= \int \frac{d^3p}{(2\pi)^3}\, \frac{\theta_-}{E_-}\left(1 - \frac{2}{\Delta^2}\,(m^{*2} + p_{\|}^2)\right) \\
&= \frac{1}{2\pi^2}\int_{x_1}^{x_2} dp_{\|}\left(E_F - E_-(0,0) + E_-(0,0) - E_-(p_{\|},0)\right)\left(1 - \frac{2}{\Delta^2}\,(m^{*2}+p_{\|}^2)\right) \\
&= \frac{1}{2\pi^2}\left\{(E_F - E_-(0,0))\left[(x_2 - x_1) - J(x_2) - J(x_1)\right]\right. \\
&\quad \left. + \int_{x_1}^{x_2} dp_{\|}\left(E_-(0,0) - E_-(p_{\|},0)\right)\left(1 - \frac{2}{\Delta^2}\,(m^{*2}+p_{\|}^2)\right)\right\} . \qquad \text{(A16)}
\end{aligned}
$$

The end points x_1 and x_2 are taken from the <u>positive</u> roots of the equation $E_-(p_{\|},0) = E_F$. If only one positive root exists then $x_1 = 0$ and x_2 is that root. The function $J(x)$ is

$$J(x) = \frac{2}{k}\left[\frac{m^{*2}}{\alpha^3}\ \ell n\left(\alpha x + \sqrt{\alpha^2x^2 + \beta^2}\right) + \frac{1}{2\alpha^2} \times \sqrt{\alpha^2x^2 + \beta^2}\right] \quad \text{(A17)}$$

with $\alpha = (1 + 4g_\pi^2\bar{\pi}^2)^{\frac{1}{2}}$, $\beta = 2g_\pi \bar{\pi} m^*$. Equation (A17) contains all the singular parts of the integral and shows explicitly that $J(0) \to \ell n\,\bar{\pi}$ as $\bar{\pi} \to 0$. The remaining integral in Eq. (A16) is done by Gaussian quadrature. The θ_+ part of the pion integral and the σ integrals in Eqs. (54a,b) are performed similarly.

For the density integral in Eq. (54e) we need

$$\int \frac{d^3p}{(2\pi)^3}\ \theta_\pm = \frac{1}{4\pi^2}\int_{x_1}^{x_2} dp_\parallel\ \left(E_F^2 - E_\pm^2(p_\parallel,0)\right) \quad , \quad \text{(A18)}$$

where x_1, x_2 are determined from the positive roots of $E_F = E_\pm(x,0)$. Although Eq. (A18) can be analytically done, a few point quadrature was used in practice. For the energy integral in Eq. (55), we need

$$\int \frac{d^3p}{(2\pi)^3}\ \theta_\pm E_\pm = \frac{1}{4\pi^2}\int_{x_2}^{x_1} dp_\parallel\ \frac{2}{3}\left(E_F^3 - E_\pm^3(p_\parallel,0)\right) \quad . \quad \text{(A19)}$$

As a final note, we show the trace results that are needed, according to Eqs. (35), (37), (39), (42f).

$$\mathrm{tr}(N) = 8m^*\left[P^2 - \varepsilon^2 + 2\ g_\pi{}^2\ \bar{\pi}^2(kk)\right] \quad \text{(A20)}$$

$$\mathrm{tr}(\gamma_\mu N) = 4\left[2(P^2 - \varepsilon^2)P^\nu - (1 + 4g_\pi{}^2\bar{\pi}^2)(Pk)k^\nu\right]g_{\mu\nu} \quad \text{(A21)}$$

$$\mathrm{tr}(\gamma_5 \not{k}\ \underset{\sim}{\tau}\cdot\underset{\sim}{v}\times\underset{\sim}{u}N) = 8g_\pi\bar{\pi}\left[2(Pk)^2 - 2m^{*2}(kk) - (P^2 - \varepsilon^2)(kk)\right] \quad \text{(A22)}$$

$$\mathrm{tr}(\gamma_\mu \tau_3 N) = 2\left[(P^2 - \varepsilon^2)k^\nu + 2(Pk)P^\nu\right] v_3 \, g_{\mu\nu} \tag{A23}$$

where $g_{\mu\nu}$ is the diagonal metric tensor (1, -1, -1, -1) and v_3 is the component of $\underset{\sim}{v}$ along the 3-axis. These results are obtained using the trace theorems in Ref. 15.

From the last of these we can prove that $k_0 = 0$ corresponds to symmetric nuclear matter Eqs. (31), (42f). The $\mu = 0$ component of (A23) is required for the evaluation of the charge density, Eq. (42f).

$$\langle \bar{\psi} \gamma_0 \tau_3 \psi \rangle \propto -P_0 \int d^3p \; \underset{\sim}{P} \cdot \underset{\sim}{k} \qquad , \text{for } k_0 = 0 \tag{A24}$$

In Sec. II.D we proved that when $k_0 = 0$, the eigenvalue spectrum $\omega(\underset{\sim}{p})$ is symmetric about the $P_{\|} = 0$ plane (Fig. 1). Therefore making the change of integration variable to $\underset{\sim}{P}$, by Eq. (23a), we see that any integral such as (A24) containing odd powers of $\underset{\sim}{P}$ must vanish. Thus Eq. (42f) follows. Turning to Eq. (A21) we see that when $k_0 = 0$, the space-like components, $\mu = 1,2,3$, are linear in $\underset{\sim}{P}$ and so the source, Eq. (8c) of the $\bar{\omega}_i$ field vanishes also. We have proven Eq. (42c).

APPENDIX B:

Here we want to derive the connection between the propagator and MFA approaches. Such a connection was discussed by Baym[16] We present an alternate and more explicit derivation here.

We start from the pion mean field equation, Eq. (8d),

$$(\Box + m^2) \langle \underset{\sim}{\pi}(x) \rangle = \underset{\sim}{J}_\pi(x) \tag{B1}$$

where

$$\underset{\sim}{J}_\pi(x) = g_\pi \, \partial^\mu \langle \bar{\psi}(x) \gamma_5 \gamma_\mu \underset{\sim}{\tau} \psi(x) \rangle \quad , \tag{B2}$$

which depends implicitly on $\langle \underset{\sim}{\pi}(x) \rangle$. In terms of the full propagator $S_\pi(x,y)$, which satisfies Eq. (16) with $\delta^4(x-y)$ replacing 0 on the right-hand side, we can write as in Eq. (A4)

$$\underset{\sim}{J}_\pi(x) = -ig_\pi \, \partial^\mu \, \mathrm{Tr} \, \{ \gamma_5 \gamma_\mu \underset{\sim}{\tau} \, S_\pi(x,x^+) \} \tag{B3}$$

where $x^+ = (\underset{\sim}{x}, t+0)$. We now expand $J_\pi(x)$ to first order in $\langle \underset{\sim}{\pi}(y) \rangle$:

$$\underset{\sim}{J}_\pi(x) = \underset{\sim}{J}_o(x) + \int d^4y \left[\frac{\delta \underset{\sim}{J}_\pi(x)}{\delta \langle \underset{\sim}{\pi}(y) \rangle} \right]_{\langle\pi\rangle \to 0} \langle \underset{\sim}{\pi}(y) \rangle + 0(\langle\pi\rangle^2) \; . \tag{B4}$$

Defining

$$\Pi_{ij}(x,y) = - \lim_{\langle \underset{\sim}{\pi}(y) \rangle \to 0} \left\{ \frac{\delta \, J_\pi^i(x)}{\delta \langle \pi_j(y) \rangle} \right\} , \tag{B5}$$

and noting that the divergence of the axial current is zero when

$\langle \underset{\sim}{\pi} \rangle = 0$ $(\underset{\sim}{J}_o = 0)$, we can rewrite Eq. (A.1) to first order in $\langle \underset{\sim}{\pi}(x) \rangle$ as

$$(\Box + m_\pi^2) \langle \pi_i(x) \rangle = -\int d^4y \, \Pi_{ij}(x,y) \langle \pi_j(y) \rangle \quad . \tag{B6}$$

In symmetric homogeneous nuclear matter $\Pi_{ij}(x,y) = \delta_{ij} \Pi(x-y)$, and

$$(\Box + m_\pi^2) \langle \pi_i(x) \rangle = -\int d^4y \, \Pi(x-y) \langle \pi_i(y) \rangle \quad . \tag{B7}$$

Expanding $\langle \pi_i(x) \rangle$ in Fourier components, we get finally

$$(-k^2 + m_\pi^2) \, \pi_i(k) = -\Pi(k) \, \pi_i(k) \quad . \tag{B8}$$

Taking, however, the Fourier transform of Eq. (B1) shows that the right-hand side is just $J_i(k)$. Therefore,

$$\Pi(k) = -\lim_{\pi_i(k) \to 0} \left\{ \frac{J_i(k)}{\pi_i(k)} \right\} \quad . \tag{B9}$$

In order to compute $\Pi_{ij}(x,y)$ or $\Pi(k)$ explicitly from Eq. (A.5), we can start from the integral equation for $S_\pi(x,y)$:

$$S_\pi(x,y) = S_o(x,y) + \int d^4z \, S_o(x,z) \, g_\pi \, \gamma_5 \gamma^\mu \frac{\partial}{\partial z^\mu} \langle \pi_i(z) \rangle \, \tau_i S_\pi(z,y) \quad . \tag{B10}$$

where $S_o(x,y)$ is the nucleon propagator when $\langle \pi \rangle = 0$. First integrate by parts to transfer the $\partial/\partial z^\mu$ away from the π field. Then recall that

$$\frac{\delta \langle \pi_i(x) \rangle}{\delta \langle \pi_j(y) \rangle} = \delta_{ij} \, \delta^4(x-y) \quad .$$

Thus,

$$\lim_{\langle\pi_j(y)\rangle\to 0} \frac{\delta S_\pi(x,z)}{\delta\langle\pi_j(y)\rangle} = -\frac{\partial}{\partial y^\nu}\left\{S_o(x,y)\ g_\pi \gamma_5 \gamma^\mu \tau_j S_o(y,z)\right\} \ . \tag{B12}$$

Using Eqs. (A.3) and (A.5), we obtain finally,

$$\Pi_{ij}(x,y) = -ig_\pi^2 \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \mathrm{Tr}(\gamma_5 \gamma^\mu \tau_i S_o(x,y)\ \gamma_5 \gamma^\nu \tau_j S_o(y,x)) \ , \tag{B13}$$

and therefore in symmetric homogeneous nuclear matter,

$$\Pi(k) = -2ig_\pi^2 \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left(\gamma_5 \not{k} S_o(p)\ \gamma_5 \not{k} S_o(p+k)\right) \ . \tag{B14}$$

In the non-relativistic limit ($p_F << m^*$), $S_o(p) \to (2m^*)^{-1}(p_o - g_v V_o - p^2/2m^* \mp i\varepsilon\ \theta(p_F - p))^{-1}$, Trace $\to -8|\underset{\sim}{k}|^2$ and $\Pi(k)$ reduces to the familiar Lindhard function (see, e.g., Ref. 7).

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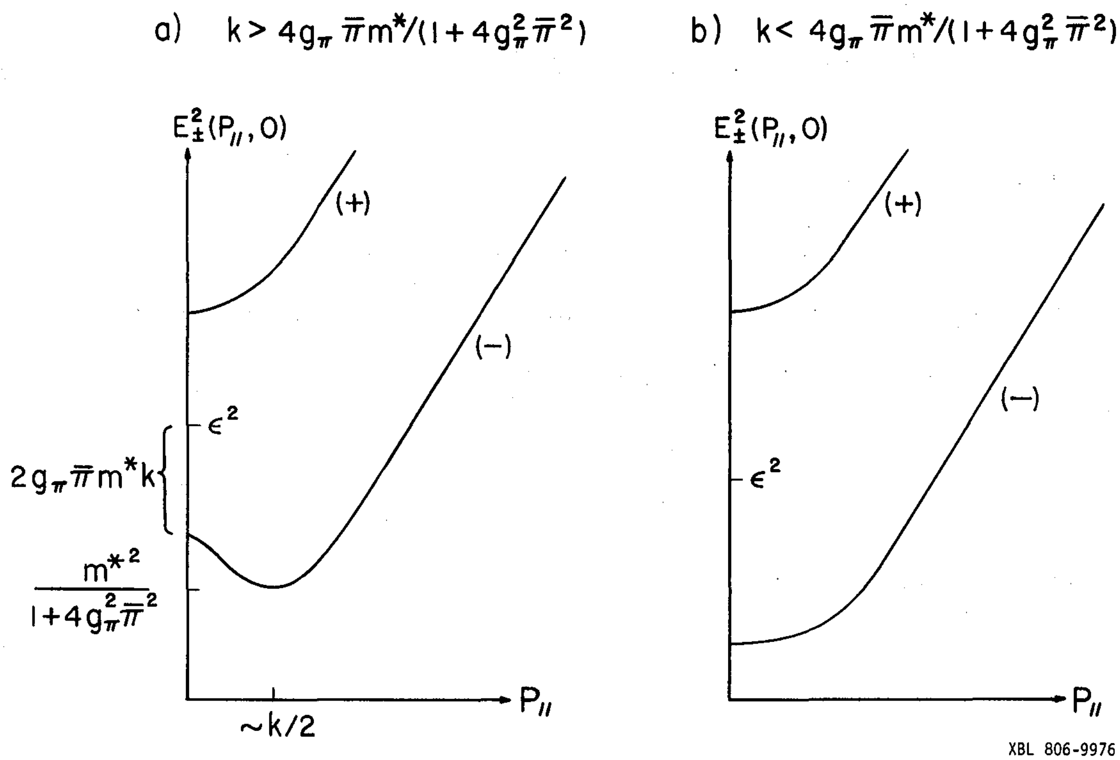
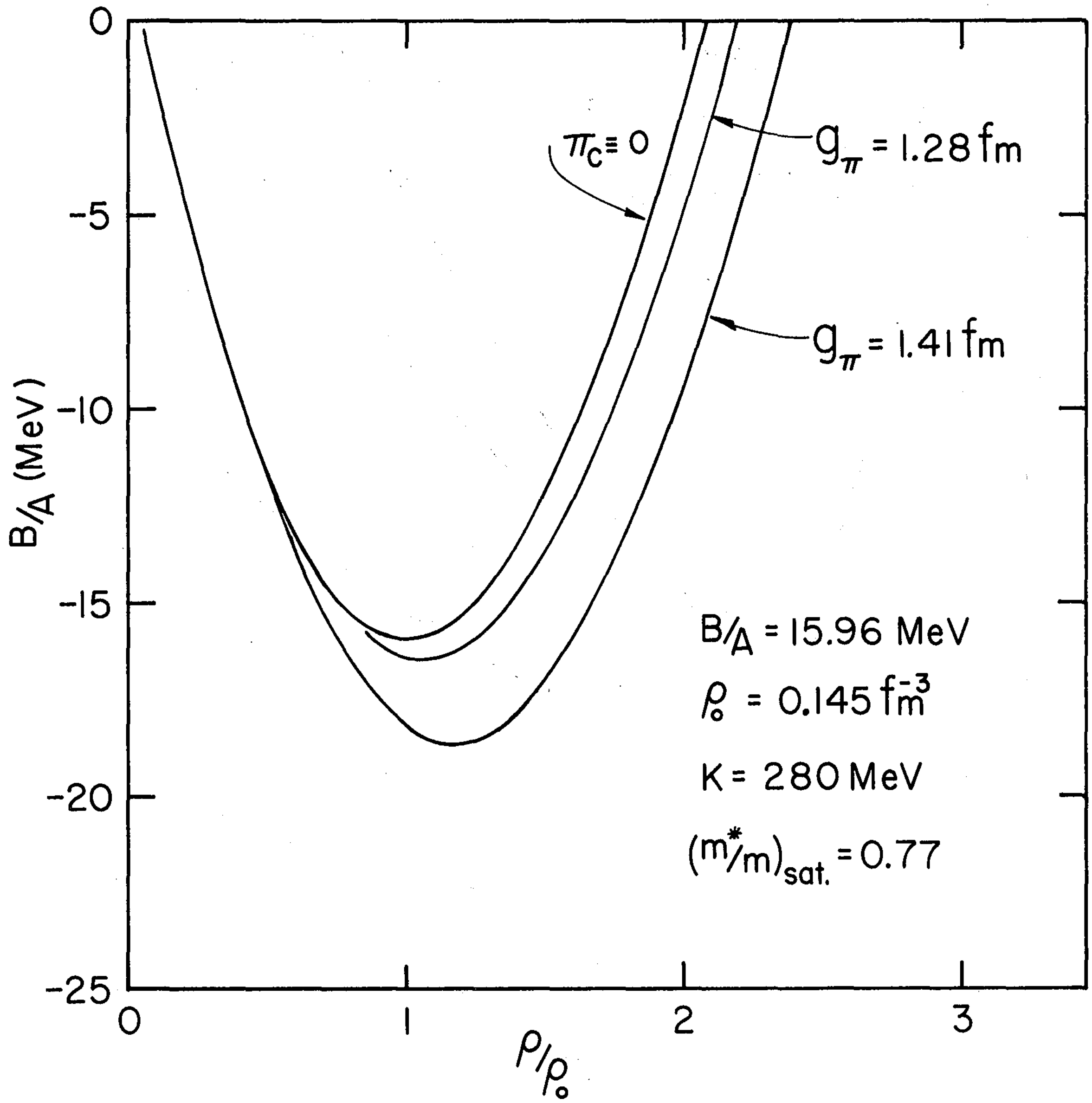


Fig. 1. Eigenvalues of the Dirac equation in the $P_{\|}$ plane, depending on the magnitude of k as shown. There is symmetry $P_{\|} \leftrightarrow -P_{\|}$. If E_F^2 is less than $\varepsilon^2 - 2g_\pi \bar{\pi} m^* k$ for case a, then the range of $P_{\|}$ is two segments which do not include zero.



XBL 7912-5246

Fig. 2. Binding energy per nucleon as a function of density. The $\pi_c = 0$ solution is the normal (non-condensed) state. Self-consistent condensate solutions exist for $g_\pi > 1.18$ fm. Two examples are shown. Parameters are $g_\sigma/m_\sigma = 15/m$, $g_\omega/m_\omega = 11/m$, $b = 0.004$, $c = 0.008$, where the nucleon mass is $m = 4.77\ \mathrm{fm}^{-1}$. Pion momentum k that minimizes energy is $k = 1.5\ \mathrm{fm}^{-1}$. Effective mass at saturation is indicated on the figure.

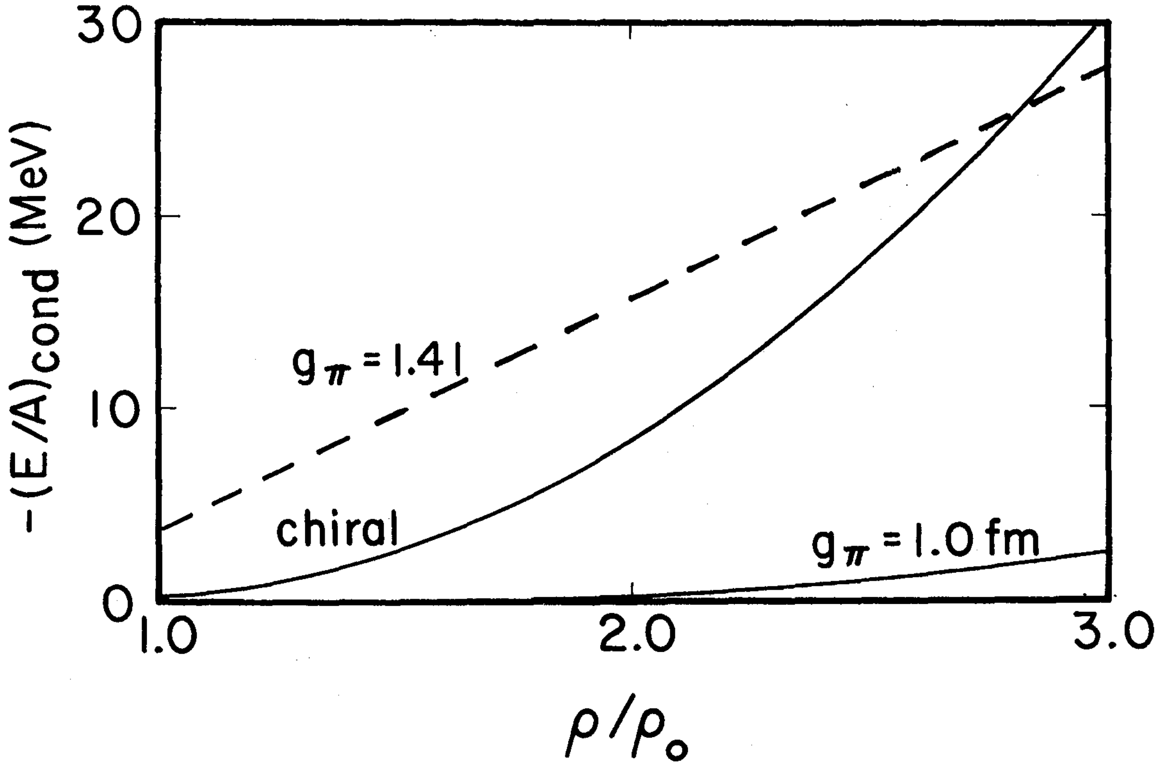


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Fig. 5. The condensation energy from Fig. 3 for g_π = 1 fm compared to the prediction of the chiral σ-model (Ref. 5).

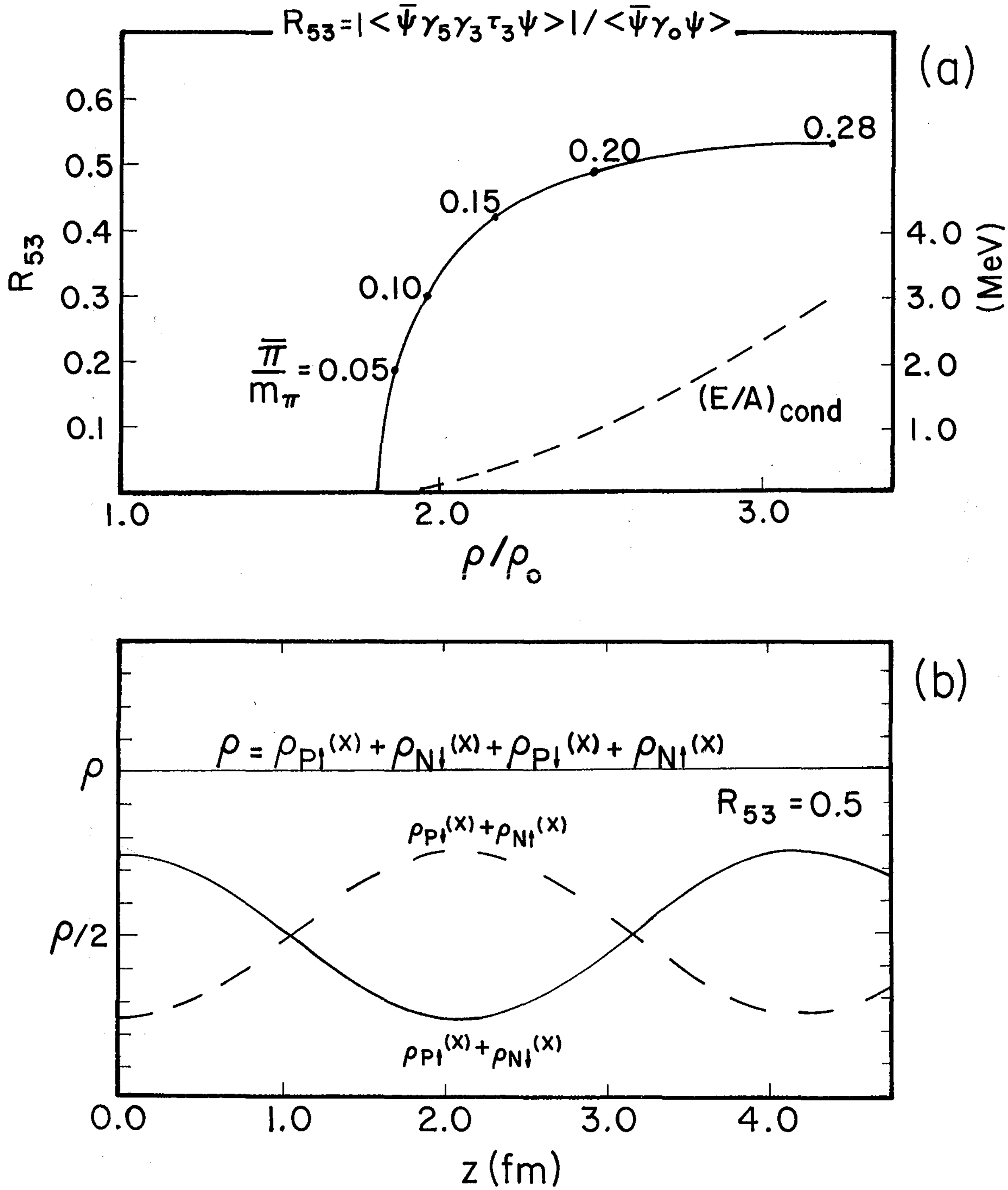

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Fig. 6. In (a) the amplitude R_{53} of the spin-isospin density oscillations in units of the baryon density for the $g_\pi = 1$ fm case of Fig. 3. Also indicated are values of the condensate field $\bar{\pi}$ and the condensate energy. Part (b) illustrates magnitude of the spin-isospin oscillations for $R_{53} = 0.5$ as a function of coordinate parallel to condensate momentum $\underset{\sim}{k}$. p↑ means proton with spin up.

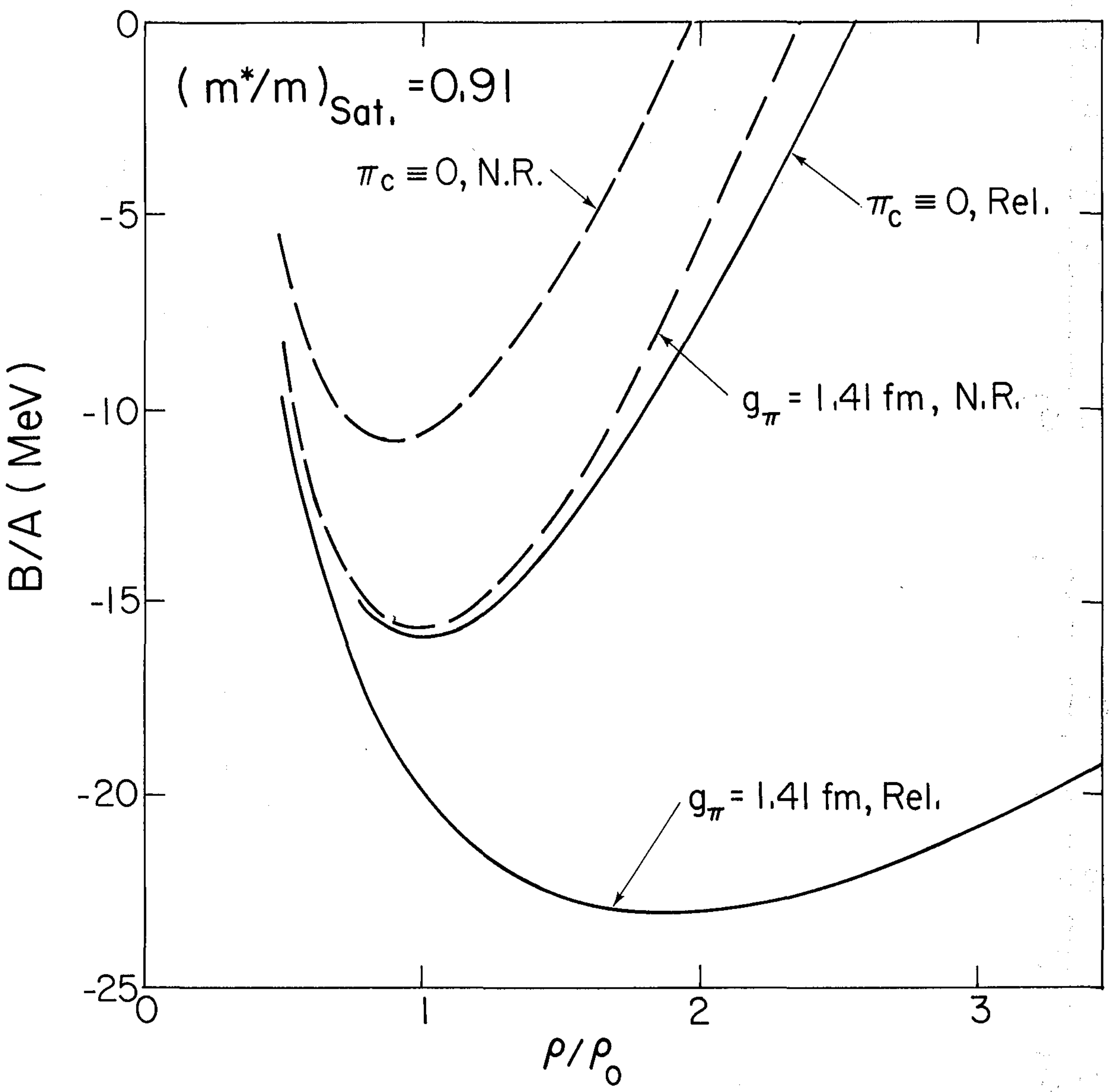

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Fig. 7. Self-consistent solutions of the theory for the non-relativistic approximation are compared with the relativistic results of Fig. 3 for the normal and condensed state.