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Reverse Engineering ADHM Construction from Non-Commutative Instantons

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Abstract

We study the non-commutative instanton solution proposed in hep-th/0009142 and obtain the spectrum of small oscillations. The spectrum thus obtained is in exact agreement with the spectrum of stringy excitations in a configuration of point like D0 branes sitting on top of D4-branes with a uniform magnetic field turned on in the world-volume of the D4-branes in the Seiberg-Witten decoupling limit. This provides further evidence for the solution of hep-th/0009142 and also enables us recover the ADHM data from the 0-4 string spectrum. Generalizations to higher co-dimension solitons are also discussed.

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1 Introduction

Non-commutative field theories have been the source of many interesting new physical insights. One of the many fascinating developments in this area has been the discovery of non-trivial solutions to the classical equations of motion [1]. A striking feature is that the non-commutative deformation permits solitonic solutions in theories which have no such in the commutative version. The study of non-trivial solutions in non-commutative field theories was initiated in [1], and has been dealt by many authors in differing contexts [3]-[31]; see [22, 24] for reviews on the subject and an extensive list of references.

We shall in the present work be interested mainly in the properties of co-dimension four solitons *i.e.*, instantons in non-commutative gauge theories. In the pioneering work of Nekrasov and Schwarz [2], a simple deformation of the classic ADHM construction was shown to lead to the non-commutative instanton. An interesting feature of this instanton is the resolution of the small instanton singularity in the moduli space. A proof of this was presented in [32], where from the sigma model description of the ADHM construction [33], it was argued that the moduli space of self-dual non-commutative instantons depends only on the anti self-dual part of the non-commutativity parameter, Θ . This implies that for self-dual Θ the moduli space has a small instanton singularity. The fact may be understood by noting the absence of a supersymmetric bound state between a localized Dp brane and a $D(p+4)$ brane in the presence of a NS-NS two-form field, unless the B -field is self-dual.

In the present work we will provide a more direct evidence for the above chain of ideas. Our starting point will be the explicit construction of self-dual instantons with self-dual non-commutativity [12]². We study the spectrum of small fluctuations about the instanton solution and reproduce the spectrum that one would obtain from a conformal field theory analysis for open strings in the Seiberg-Witten decoupling limit [32]. This would in principle amount to “re-deriving” the ADHM construction from the given explicit solution to the Yang-Mills equations of motion. It bears mentioning that this characterization of the instanton is a close cousin of the matrix theory description of the Dp - $D(p+4)$ bound state which was studied in [34]. The solution of [12] is particularly interesting for it is easily generalized to constructing higher co-dimension solutions and one can carry out an analogous exercise in these cases too.

Knowledge of the fluctuation spectrum for generic values of non-commutativity, where the solution is unstable, can be used to study the issue of tachyon condensation in the system. In [12] the study of tachyon potential was carried out for the co-dimension 2 soliton. The tachyon potential in the $D0$ - $D4$ story was studied in [36] from the string field theory perspective. One can use the present analysis to compute

²For recent work on non-commutative instantons see [14, 15, 28]

the exact tachyon potential as in the case of the co-dimension two soliton. Given the generalization of the solution to higher co-dimension solitons it would provide us with an handle towards understanding the physics governing the newly found supersymmetric bound states of $D0$ -branes with $D6$ -branes and $D8$ -branes [37, 38, 39, 40, 41].

The organization of the paper is as follows. In section 2 we review the construction of [12] and set forth our notation. We present the spectrum of fluctuations in section 3 and discuss the implications thereof. In section 4 higher co-dimension solitons are discussed. Some of the algebra is relegated to the appendices to make for a more coherent discussion.

2 The instanton in non-commutative Yang-Mills

In this section we review the basics of non-commutative gauge theory. In section 2.1 we write down the Lagrangian of non-commutative gauge theory coupled to adjoint scalar fields in the operator language following [42] and present the equations of motion. In the subsequent subsection we review the instanton solution of [12]. For most part of the discussion we shall focus on solutions with only the gauge field excited and the scalars shall be set to their vacuum value.

2.1 Non-commutative Yang-Mills: A Review

We would like to consider $4 + 1$ dimensional non-commutative Yang-Mills (NCYM) (we shall mainly focus on the $U(1)$ case) theory with 5 adjoint scalars, to mimic the bosonic field content of the low-energy effective world-volume theory of $D4$ -branes with a constant background magnetic field. The classical action in temporal gauge is given as

$$S = -\frac{1}{4g_{YM}^2} \int d^5x \left(F_{\mu\nu} * F^{\mu\nu} + 2 \sum_{i=1}^5 D_\mu \phi^i * D^\mu \phi^i + \sum_{i=1}^5 [\phi^i, \phi^j]_* [\phi^i, \phi^j]_* \right) \quad (1)$$

We are on a non-commutative \mathbf{R}^4 with a non-commutativity parameter Θ given by the block-diagonal form; $\Theta^{\mu\nu} = (\theta_1 \epsilon, \theta_2 \epsilon)$, (where ϵ is the anti-symmetric 2×2 matrix with $\epsilon^{12} = 1$). We introduce complex coordinates z^m , obeying the commutation relation

$$[z^m, \bar{z}^{\bar{n}}] = i\Theta^{m\bar{n}}. \quad (2)$$

We can exploit the relation between the algebra of functions on non-commutative \mathbf{R}^4 and the algebra of operators in the Hilbert space of a particle in 2-spatial dimensions by defining ladder operators *c.f.* [42],

$$\begin{aligned}
a_{\bar{m}} &= -i \Theta_{\bar{m}n}^{-1} z^n; & a_m^\dagger &= i \Theta_{m\bar{n}}^{-1} \bar{z}^{\bar{n}}, \\
[a_m^\dagger, a_{\bar{n}}] &= -i \Theta_{m\bar{n}}^{-1}.
\end{aligned} \tag{3}$$

To recast the NCYM action (1) in the operator language we parameterize the gauge field in terms of an operator in the Hilbert space C_m and use the fact that translations can be implemented by taking commutators with respect to the ladder operators. To wit,

$$\begin{aligned}
C_m &= -iA_m + a_m^\dagger; & C_{\bar{m}} &= iA_{\bar{m}} + a_{\bar{m}}, \\
F_{m\bar{n}} &= i[C_m, C_{\bar{n}}] - \Theta_{m\bar{n}}^{-1}, \\
D_m\phi &= -[C_m, \phi]; & D_{\bar{m}}\phi &= [C_{\bar{m}}, \phi].
\end{aligned} \tag{4}$$

Finally we arrive at the action

$$\begin{aligned}
S &= -\frac{4\pi^2 \text{Pf}(\Theta)}{4g_{YM}^2} \int dt \text{Tr} \left\{ -\partial_t \bar{C}_{\bar{m}} \partial_t C_m - \sum_{i=1}^5 \frac{1}{2} \partial_t \phi^i \partial_t \phi^i \right. \\
&\quad \left. - 4 \left(i[C_m, \bar{C}_{\bar{n}}] - \Theta_{m\bar{n}}^{-1} \right)^2 + 8 \left(i[C_m, C_n] \right) \left(i[\bar{C}_{\bar{m}}, \bar{C}_{\bar{n}}] \right) \right. \\
&\quad \left. + \sum_{i=1}^5 [C_m, \phi^i] [\phi^i, \bar{C}_{\bar{m}}] + \frac{1}{4} \sum_{i,j=1}^5 [\phi^i, \phi^j] [\phi^i, \phi^j] \right\}.
\end{aligned} \tag{5}$$

The equations of motion resulting from the variation of the action (5) are given as

$$\begin{aligned}
\partial_t^2 C_m &= [C_n, [C_m, \bar{C}_{\bar{n}}]] + \sum_{i=1}^5 [\phi^i, [C_m, \phi^i]] \\
\partial_t^2 \phi^i &= [C_m, [\phi^i, \bar{C}_{\bar{m}}]] + [\bar{C}_{\bar{m}}, [\phi^i, C_m]] + \sum_{j=1}^5 [\phi^j, [\phi^i, \phi^j]].
\end{aligned} \tag{6}$$

In addition to the equations of motion we also need to impose the Gauss Law constraint to pick out the physical states and this reads,

$$[\bar{C}_{\bar{m}}, \partial_t C_m] + [C_m, \partial_t \bar{C}_{\bar{m}}] + \sum_{i=1}^5 [\phi^i, \partial_t \phi^i] = 0. \tag{7}$$

2.2 The instanton

A static solution to the equations (6) was found in [12]. This solution carries a single unit of 1st Pontrjagin charge and is given as

$$C_m = T^\dagger a_m^\dagger T; \quad \phi_i = 0. \tag{8}$$

Here T is an operator obeying,

$$TT^\dagger = 1, \quad T^\dagger T = 1 - P_0, \quad P_0 T^\dagger = T P_0 = 0. \quad (9)$$

P_0 is the projection operator onto the ground state $|0, 0\rangle$, of the two-particle system, *i.e.*, it projects onto lowest radial wavefunction in both complex directions.

The field strength for this configuration evaluates to $F_{m\bar{n}} = -\Theta_{m\bar{n}}^{-1} P_0$ (note that the vacuum with zero field strength in this notation is $C_m = a_m^\dagger$), and this implies that the first Pontrjagin class of the solution is ± 1 depending on whether $\Theta_{\mu\nu}$ is self-dual or anti-self-dual. Evaluating the energy of the solution one finds $S = 2\pi^2 g_{YM}^2 \sqrt{\det\Theta} \Theta_{m\bar{n}}^{-1} (\Theta^{-1})^{m\bar{n}}$, which saturates the BPS bound iff Θ is self-dual or anti-self dual. We shall consider the case of instanton number one solution and hence we shall be interested in particular at the situation with self-dual Θ . Based on the energetics and the charge of the solution it was conjectured in [12] that the solution (8) corresponds to an anti-zero brane localized at a point on the four-brane³. Evidence for this was offered in [15] by directly solving the ADHM equations.

To complete the characterization of the solution we shall present an explicit representation of the operator T . In the standard number basis of states for a two-dimensional harmonic oscillator,

$$|n_1, n_2\rangle = \frac{(a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2}}{\sqrt{(n_1)! (n_2)!}} |0, 0\rangle, \quad (10)$$

we can define an integer ordering of states as follows:

$$|k = n_1 + \frac{(n_1 + n_2)(n_1 + n_2 + 1)}{2}\rangle \equiv |n_1, n_2\rangle. \quad (11)$$

T can then be represented as a shift operator with respect to this ordering, its matrix elements being given as

$$\langle k | T | l \rangle = \delta_{k, l-1}. \quad (12)$$

3 Classical Fluctuation analysis

The solution given in Eq (8) is supposed to be a co-dimension 4 soliton in a $4 + 1$ dimensional NCYM theory. In terms of a brane picture it should correspond to a localized $D0$ -brane on the world-volume of a $D4$ -brane with constant B -field. This configuration is generically not supersymmetric for arbitrary values of Θ , but for self-dual Θ we obtain a supersymmetric configuration [2, 32]. This is the special point in the moduli space of non-commutative instantons where the moduli space is the same

³We shall persist in referring to the system as that of a $D0$ -brane localized on a $D4$ -brane

as the commutative case, and the solution we have corresponds to this small instanton point.

The analysis of the string spectrum in this background was done in [32] and we refer to them for the basic results. It was found that the low lying modes in the decoupling limit are the standard massless modes of the 0–0 and 4–4 strings along with additional modes from the 0 – 4 with masses proportional to the non-commutativity parameter. Typically, in usual configurations of d-branes, taking the low energy limit leads to keeping only the massless modes of the strings. In the presence of the B -field however, the presence of an additional dimensionful parameter which is being scaled appropriately to preserve the non-commutativity, leads to additional set of modes which have masses related to the non-commutativity scale. In particular one has low lying modes with masses given by

$$\begin{aligned}
\frac{1}{2}E_{1,2(-)} &= \pm \left(\frac{1}{\theta_1} - \frac{1}{\theta_2} \right) \\
\frac{1}{2}E_{1+} &= \frac{3}{\theta_1} + \frac{1}{\theta_2} \\
\frac{1}{2}E_{2+} &= \frac{1}{\theta_1} + \frac{3}{\theta_2} \\
\frac{1}{2}E_i &= \frac{1}{\theta_1} + \frac{1}{\theta_2} \quad i = 5, \dots, 9.
\end{aligned} \tag{13}$$

The rest of the spectrum can be worked out by acting on the ground state with the oscillators.

The spectrum we present here is valid for generic values of the non-commutativity parameter. In particular note that there is a tachyonic mode (from $E_{1,2(-)}$ depending on the relative strengths of θ_1 and θ_2) which becomes massless along with another massive mode as Θ approaches the self-dual value. This is the indicator of restoration of supersymmetry as the solution becomes BPS for this special point [2, 32].

As mentioned, the system under consideration can be understood from a matrix theory point of view. In matrix theory one writes down classical brane configurations in terms of non-commuting matrices which satisfy the equations of motion. One builds a configuration of $D4$ -branes by having say, $[X_1, X_2] = i\theta_1$ and $[X_3, X_4] = i\theta_2$. This can be extended to include $D0$ -branes by letting the matrices have a block diagonal form with the upper block being given by the above form to make up the $D4$ -branes and the lower block being a diagonal matrix with the eigenvalues parametrising the position of the $D0$ -branes. This configuration was studied by Lifschytz [34], and the

fluctuation spectrum was found to be as follows;

$$\begin{aligned}
\frac{1}{2}E_{1(\pm)} &= \left(\frac{2n_1 + 1}{\theta_1} + \frac{2n_2 + 1}{\theta_2} \right) \pm \frac{2}{\theta_1} \\
\frac{1}{2}E_{2(\pm)} &= \left(\frac{2n_1 + 1}{\theta_1} + \frac{2n_2 + 1}{\theta_2} \right) \pm \frac{2}{\theta_2} \\
\frac{1}{2}E_i &= \left(\frac{2n_1 + 1}{\theta_1} + \frac{2n_2 + 1}{\theta_2} \right)
\end{aligned} \tag{14}$$

It bears mentioning that the matrix theory manipulations are closely related to the non-commutative gauge theory ones [42, 35], for, rewriting the non-commutative gauge theory in the operator language is akin to working with the dissolved $D0$ -branes in the world-volume of the $D4$ -branes. We shall exploit this correspondence to parameterize the fluctuations in a canonical fashion.

3.1 Scalar fluctuations

In the solution given by (8) the scalars are unexcited, which makes it easy for us to study the spectrum arising from their fluctuations. We shall detail the calculation for the case with the scalars unexcited and indicate the modifications arising when the scalars are given a non-trivial vev. As is clear from the geometric picture the scalars we are considering correspond to the directions transverse to the $D4$ -brane. This means that we shall be presently considering the situation when the $D0$ -brane is sitting right on top of the $D4$ -brane. In generic situation we could move the $D0$ -brane away giving additional contributions to the mass coming from the string having to stretch the extra distance.

Without loss of generality we can focus on the case of a single scalar. We choose to parameterize the fluctuation of the scalar $\delta\phi$ as (*c.f.*, [12])

$$\delta\phi = \chi + \psi + \bar{\psi} + T^\dagger \gamma T \tag{15}$$

where,

$$\begin{aligned}
\chi &= P_0 \delta\phi P_0, & \psi &= P_0 \delta\phi (1 - P_0), \\
T^\dagger \gamma T &= (1 - P_0) \delta\phi (1 - P_0).
\end{aligned} \tag{16}$$

The logic here is to separate the fluctuation into components such that the separation between the various string sectors is made manifest. From a matrix theory point of view this corresponds to the fact that the $D0$ and the $D4$ brane solutions are in different blocks along the diagonal and the off-diagonal piece is related to the ND strings of the $0 - 4$ sector. As already mentioned, the operator language used to write solutions in NCYM is closely related to the matrix formulation [35]. Denoting the complete Hilbert space by \mathcal{H} , let \mathcal{H}_0 be the subspace to which P_0 projects, and \mathcal{H}_\perp be the orthogonal

component *i.e.*, $\mathcal{H} = \mathcal{H}_0 \cup \mathcal{H}_\perp$. Then the analogy with the matrix description is made manifest by associating the \mathcal{H}_0 with the 0-brane block and the \mathcal{H}_\perp with the 4-brane block. So the 0-4 strings arise from the modes which are project from one side into \mathcal{H}_0 and from the other into \mathcal{H}_\perp .

Given this parameterization we can evaluate the scalar potential to quadratic order about the solution (8). The contributions to the potential at this order are going to come from the term $Tr[C_m, \phi][\phi, \bar{C}_{\bar{m}}]$. Plugging in the form suggested by (15), we find

$$\text{Tr}[C_m, \delta\phi][\delta\phi, \bar{C}_{\bar{m}}] = \text{Tr} \left(\left(C_{(0)m} \bar{C}_{(0)\bar{m}} + \bar{C}_{(0)\bar{m}} C_{(0)m} \right) \bar{\psi} \psi + [a_m^\dagger, \gamma][\gamma, a_{\bar{m}}] \right). \quad (17)$$

The field χ does not appear in the scalar potential at quadratic order. Hence a massless scalar from the 0 + 1 dimensional point of view. These are the fluctuation modes of the scalars on the $D0$ -brane in the directions transverse to the $D4$ -brane. This gels well with the intuition gained from comparison to matrix theory. The modes represented by γ have the right potential to be the transverse scalars on the world-volume of the $D4$ -brane. In particular the commutators with the creation-annihilation operators is exactly what is necessary to covariantize the derivatives. This leaves us with the fields ψ which are the 0 – 4 scalars. Their mass is given by the eigenvalues of the operator $\left(C_{(0)m} \bar{C}_{(0)\bar{m}} + \bar{C}_{(0)\bar{m}} C_{(0)m} \right)$. This is simple to evaluate in the integer ordered basis (11) for the Hilbert space. Expanding the field ψ ;

$$\psi = \sum_{k=0}^{\infty} \psi_k |0\rangle \langle k+1|, \quad (18)$$

we can write the relevant term in (17) as

$$\begin{aligned} & \sum_{k,l=0}^{\infty} \psi_k \bar{\psi}_l \langle k+1 | T^\dagger a_m^\dagger T T^\dagger a_{\bar{m}} T + T^\dagger a_{\bar{m}} T T^\dagger a_m^\dagger T | l+1 \rangle \\ &= \sum_{n_i=0}^{\infty} |\psi_{\{n_1 n_2\}}|^2 \langle n_1, n_2 | a_m^\dagger a_{\bar{m}} + a_{\bar{m}} a_m^\dagger | n_1, n_2 \rangle \\ &= \sum_{n_i=0}^{\infty} \left(\frac{2n_1 + 1}{\theta_1} + \frac{2n_2 + 1}{\theta_2} \right) |\psi_{\{j, m\}}|^2 \end{aligned} \quad (19)$$

In the above series of manipulations we have used the fact that T acts as a lowering operator in the basis (11). As a result of this we find that the summation extends over the whole Hilbert space of states and one can conveniently switch over from the basis (11) to the standard number basis for purposes of evaluating the matrix elements. From (19) we find that there is a low-lying mode $\psi_{\{0,0\}}$, of mass $\frac{1}{\theta_1} + \frac{1}{\theta_2}$ and over that we have a whole tower of massive modes. This matches perfectly with the CFT analysis of [32], given in Eq (13) and the matrix theory calculation of [34], Eq (14). Note that

in this analysis the Gauss law constraint (7) plays no role as the background solution (8) has the scalar field unexcited.

3.2 Gauge field fluctuations

The gauge field fluctuations can be analyzed in a fashion analogous to the scalar fluctuations. As before we decompose the fluctuations as

$$\begin{aligned} C_m &= C_{(0)m} + \delta C_m \\ \delta C_m &= A_m + W_m + \bar{Q}_m + T^\dagger D_m T, \end{aligned} \quad (20)$$

with,

$$\begin{aligned} A_m &= P_0 \delta C_m P_0, & W_m &= P_0 \delta C_m (1 - P_0), \\ \bar{Q}_m &= (1 - P_0) \delta C_m P_0, & T^\dagger D_m T &= (1 - P_0) \delta C_m (1 - P_0). \end{aligned} \quad (21)$$

The potential for the gauge field fluctuations comes from the term $\frac{1}{2} \left(i[C_m, \bar{C}_m] - \Theta_{m\bar{m}}^{-1} \right)^2$. Substituting the above form of the fluctuations we find the contribution to the potential to be

$$L(W, Q) + \frac{1}{2} \left([a_m^\dagger, \bar{D}] + [D, a_m] \right)^2. \quad (22)$$

The explicit form for $L(W, Q)$ is given in the appendix. One can find appropriate linear combinations of the fields $W_{1,2}$ and $Q_{\bar{1},\bar{2}}$, labeled U, V, X, Y in terms of which the Lagrangian $L(W, Q)$ is easily diagonalized.

As before with the case of the scalar fluctuations, from the absence of quadratic terms for the field A_m , we are led to conclude that these are the massless modes corresponding to the motion of the $D0$ -brane along the world-volume of the $D4$ -brane. The field D is the gauge field on the world-volume of the $D4$ -brane. To analyze the spectrum of the off-diagonal modes, W and Q we expand them in the harmonic oscillator basis; taking appropriate linear combinations (see Appendix for details) we find the spectrum,

$$\begin{aligned} \frac{1}{2} E_{\{n_1, n_2\}}(U) &= \frac{(2n_1 + 1)}{\theta_1} + \frac{(2n_2 + 1)}{\theta_2} + \frac{2}{\theta_1} \\ \frac{1}{2} E_{\{n_1, n_2\}}(V) &= \frac{(2n_1 + 1)}{\theta_1} + \frac{(2n_2 + 1)}{\theta_2} + \frac{2}{\theta_2} \\ \frac{1}{2} E_{\{n_1, n_2\}}(X) &= \frac{(2n_1 + 1)}{\theta_1} + \frac{(2n_2 + 1)}{\theta_2} - \frac{2}{\theta_1} \\ \frac{1}{2} E_{\{n_1, n_2\}}(Y) &= \frac{(2n_1 + 1)}{\theta_1} + \frac{(2n_2 + 1)}{\theta_2} - \frac{2}{\theta_2} \end{aligned} \quad (23)$$

This is indeed isomorphic to the spectrum given in (14). Apart from these modes we also have the $D0$ -brane gauge field, which arises from the component A_0 , of the gauge potential in the $D4$ -brane theory. In all one has the complete spectrum to be the

standard 0-0 and 4-4 string spectrum and the spectrum of the 0-4 strings as given in (19) and (23).

3.3 Relating to the ADHM construction

We have obtained the spectrum of fluctuations for a single point-like $D0$ -brane in a $4 + 1$ dimensional $U(1)$ non-commutative gauge theory. Generalizing the above to multi-instanton configurations is simply achieved by writing down the solution (8) with the operator T replaced by T^k , k being the number of instantons. The spectrum in this case works out just the same, from the gauge field fluctuations we get the 0-0 sector gauge fields which are in the adjoint of $U(k)$, and also scalars corresponding to motion of the $D0$ -brane along the four-brane which too are in the adjoint of $U(k)$. On the contrary the 0-4 strings are charged in the fundamental representation of $U(k)$. If we had considered a non-abelian generalization by having N $D4$ -branes, then we would have the 0-4 strings charged in the fundamental of $U(N)$. This is pretty much all the information we need to reconstruct the ADHM data by just following the chain of logic in section 5 of [32].

The low energy effective theory of the system is the quantum mechanics of the 0-4 strings and the 0-0 strings. The relevant modes are the adjoint scalars (the scalars which correspond to motion in directions transverse to the $D4$ -brane; the scalars in the directions tangential to the $D4$ -brane are Goldstone modes of the translational symmetry and decouple from the low energy dynamics), the gauge field from the 0-0 sector and the two low-lying modes $Q_{\bar{1}, \bar{2}\{0,0\}}$ with masses $\pm \left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right)$ from the 0-4 sector. The adjoint scalars are denoted as x and y and the fundamental scalars are p and q respectively. The tachyonic mode in fact serves to determine the strength of the FI coupling in this theory. The classical potential in this framework is given as

$$\text{Tr} \left\{ \left([x, x^\dagger] + [y, y^\dagger] + qa^\dagger - p^\dagger p - \zeta \right)^2 + | [x, y] + qp |^2 \right\}. \quad (24)$$

ζ is the FI coupling, determined from the existence of a tachyonic mode in the spectrum of the 0-4 strings at generic values of Θ . For self-dual Θ this vanishes. Hence although the locus $x = y = p = q = 0$ is not a solution to the minimum of the potential at generic Θ , it is indeed present when Θ is self-dual as then ζ vanishes. The ADHM equations are just the standard equations of motion for the quantum mechanics theory as is well known. Thus having verified that the spectrum of fluctuations about the instanton background (8) is indeed as predicted by a conformal field theory analysis, we have managed to recover all the ingredients essential to reverse engineer the ADHM construction.

4 Generalization to higher dimensions

Recently, it was shown by Witten [37] that one can in the presence of non-commutativity have supersymmetric bound states of $D0$ -branes with $D6$ -branes and $D8$ -branes ⁴. These systems were also discussed in [38, 39, 40, 41].

One can easily generalize the solution of [12] to obtain these higher co-dimension solitons. It is clear that in the case of a co-dimension $2n$ soliton ($n = 3, 4$), one needs to find the analog of the shift operator T in a n -particle Hilbert space. In the n -particle Hilbert space one may similarly introduce an integer ordering of the basis states and introduce a shift operator with respect to that ordering. As we saw in our analysis the explicit form for the operator was not quite essential in determining the spectrum of small fluctuations.

The scalar mass spectrum is given by the eigenvalues of the operator $(C_{(0)m}\bar{C}_{(0)\bar{m}} + \bar{C}_{(0)\bar{m}}C_{(0)m})$. This masses work out to be $2\left(\frac{(2n_1+1)}{\theta_1} + \frac{(2n_2+1)}{\theta_2} + \frac{(2n_3+1)}{\theta_3}\right)$ for the $D0$ - $D6$ case. To obtain the gauge field fluctuations one would have to do a little more work, but the end result is simple. In case of the $D0$ - $D6$ configuration we obtain,

$$\begin{aligned}\frac{1}{2}E_1^\pm &= \frac{(2n_1+1)}{\theta_1} + \frac{(2n_2+1)}{\theta_2} + \frac{(2n_3+1)}{\theta_3} \pm \frac{2}{\theta_1} \\ \frac{1}{2}E_2^\pm &= \frac{(2n_1+1)}{\theta_1} + \frac{(2n_2+1)}{\theta_2} + \frac{(2n_3+1)}{\theta_3} \pm \frac{2}{\theta_2} \\ \frac{1}{2}E_3^\pm &= \frac{(2n_1+1)}{\theta_1} + \frac{(2n_2+1)}{\theta_2} + \frac{(2n_3+1)}{\theta_3} \pm \frac{2}{\theta_3}\end{aligned}\tag{25}$$

and for the $D0$ - $D8$ we have

$$\begin{aligned}\frac{1}{2}E_1^\pm &= \frac{(2n_1+1)}{\theta_1} + \frac{(2n_2+1)}{\theta_2} + \frac{(2n_3+1)}{\theta_3} + \frac{(2n_4+1)}{\theta_4} \pm \frac{2}{\theta_1} \\ \frac{1}{2}E_2^\pm &= \frac{(2n_1+1)}{\theta_1} + \frac{(2n_2+1)}{\theta_2} + \frac{(2n_3+1)}{\theta_3} + \frac{(2n_4+1)}{\theta_4} \pm \frac{2}{\theta_2} \\ \frac{1}{2}E_3^\pm &= \frac{(2n_1+1)}{\theta_1} + \frac{(2n_2+1)}{\theta_2} + \frac{(2n_3+1)}{\theta_3} + \frac{(2n_4+1)}{\theta_4} \pm \frac{2}{\theta_3} \\ \frac{1}{2}E_4^\pm &= \frac{(2n_1+1)}{\theta_1} + \frac{(2n_2+1)}{\theta_2} + \frac{(2n_3+1)}{\theta_3} + \frac{(2n_4+1)}{\theta_4} \pm \frac{2}{\theta_4}\end{aligned}\tag{26}$$

Yet again from the knowledge of the fluctuation spectrum we can write down the low energy effective theory governing the dynamics of the bound state. It would be given

⁴In the absence of a B-field the $D0$ - $D6$ is non-supersymmetric, while the $D0$ - $D8$ admits a susy bound state. In the presence of non-commutativity there is a second supersymmetric branch of the $D0$ -brane with a $D8$ -brane.

by the quantum mechanics of the 0-0 strings interacting with the 0-6 or 0-8 strings, depending on the case of interest.

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Appendix A

In this appendix we present the details of the calculation pertaining to the gauge field fluctuations.

The potential for the gauge field fluctuations comes from the terms of the kind $\frac{1}{2} \left(i[C_m, \bar{C}_{\bar{n}}] - \Theta_{m\bar{n}}^{-1} \right)^2$. Substituting the form of the fluctuations as in (21) we get (modulo an overall factor of $\frac{\text{Pf}(\Theta)}{g_{YM}^2}$)

$$\frac{1}{2} \left([a_m^\dagger, \bar{D}] + [D, a_{\bar{m}}] \right)^2 + L(W, Q) \quad (27)$$

Expanding the off-diagonal fields in the mode expansion as in the scalar case (18)

$$\begin{aligned} W_m &= \sum_{k=0}^{\infty} W_{m(k)} |0\rangle \langle k+1|, \\ Q_{\bar{m}} &= \sum_{k=0}^{\infty} Q_{\bar{m}(k)} |0\rangle \langle k+1|. \end{aligned} \quad (28)$$

$L(W, Q)$ can be written as

$$\begin{aligned}
& \sum_{k,l=0}^{\infty} W_{1(k)} \bar{W}_{\bar{1}(l)} \langle k | a_{\bar{1}} a_1^\dagger + a_{\bar{2}} a_2^\dagger + a_2^\dagger a_{\bar{2}} | l \rangle + Q_{\bar{1}(k)} \bar{Q}_{1(l)} \langle k | a_1^\dagger a_{\bar{1}} + a_2^\dagger a_{\bar{2}} + a_{\bar{2}} a_2^\dagger | l \rangle \\
& + W_{2(k)} \bar{W}_{\bar{2}(l)} \langle k | a_{\bar{2}} a_2^\dagger + a_{\bar{1}} a_1^\dagger + a_1^\dagger a_{\bar{1}} | l \rangle + Q_{\bar{2}(k)} \bar{Q}_{2(l)} \langle k | a_2^\dagger a_{\bar{2}} + a_1^\dagger a_{\bar{1}} + a_{\bar{1}} a_1^\dagger | l \rangle \\
& - \left(W_{1(k)} \bar{Q}_{1(l)} \langle k | a_{\bar{1}} a_{\bar{1}} | l \rangle + W_{1(k)} \bar{Q}_{2(l)} \langle k | a_{\bar{2}} a_{\bar{2}} | l \rangle + c.c \right) \\
& - \left(W_{2(k)} \bar{Q}_{2(l)} \langle k | a_{\bar{2}} a_{\bar{2}} | l \rangle + W_{2(k)} \bar{Q}_{1(l)} \langle k | a_{\bar{1}} a_{\bar{1}} | l \rangle + c.c \right) \\
& - \left(W_{1(k)} \bar{W}_{\bar{2}(l)} \langle k | a_{\bar{1}} a_2^\dagger | l \rangle + Q_{\bar{1}(k)} \bar{Q}_{2(l)} \langle k | a_{\bar{2}} a_1^\dagger | l \rangle + c.c \right) \\
& + i \Theta_{m\bar{n}}^{-1} \left(\sum_{k=0}^{\infty} W_{m(k)} \bar{W}_{\bar{n}(k)} - Q_{\bar{n}(k)} \bar{Q}_{m(k)} \right).
\end{aligned} \tag{29}$$

We shall in the following set Θ to unity to avoid notational clutter. As before we only have to evaluate the matrix elements. We find

$$\begin{aligned}
L(W, Q) &= \sum_{n_i=0}^{\infty} (n_1 + 2n_2 + 3) W_{1\{n_1, n_2\}} \bar{W}_{\bar{1}\{n_1, n_2\}} + (2n_1 + n_2 + 3) W_{2\{n_1, n_2\}} \bar{W}_{\bar{2}\{n_1, n_2\}} \\
& + (n_1 + 2n_2) Q_{\bar{1}\{n_1, n_2\}} \bar{Q}_{1\{n_1, n_2\}} + (n_2 + 2n_1) Q_{\bar{2}\{n_1, n_2\}} \bar{Q}_{2\{n_1, n_2\}} \\
& - \left(\sqrt{n_2(n_1 + 1)} W_{1\{n_1, n_2\}} \bar{W}_{\bar{2}\{n_1+1, n_2-1\}} + \sqrt{n_1(n_2 + 1)} Q_{\bar{1}\{n_1, n_2\}} \bar{Q}_{1\{n_1-1, n_2+1\}} + c.c. \right) \\
& - \sqrt{(n_1 + 1)(n_2 + 1)} \left(W_{1\{n_1, n_2\}} \bar{Q}_{2\{n_1+1, n_2+1\}} + W_{2\{n_1, n_2\}} \bar{Q}_{1\{n_1+1, n_2+1\}} + c.c \right) \\
& - \sqrt{(n_1 + 1)(n_1 + 2)} \left(W_{1\{n_1, n_2\}} \bar{Q}_{1\{n_1+2, n_2\}} + c.c \right) \\
& - \sqrt{(n_2 + 1)(n_2 + 2)} \left(W_{2\{n_1, n_2\}} \bar{Q}_{2\{n_1, n_2+2\}} + c.c \right)
\end{aligned} \tag{30}$$

It is useful to introduce linear combinations of the fields W and Q ,

$$\begin{aligned}
U_{\{n_1, n_2\}} &= \sqrt{n_1 + 2n_2 + 3} W_{1\{n_1, n_2\}} - \sqrt{\frac{(n_1 + 1)(n_1 + 2)}{(n_1 + 2n_2 + 3)}} \bar{Q}_{1\{n_1+2, n_2\}} \\
&\quad - \sqrt{\frac{(n_1 + 1)(n_2 + 1)}{(n_1 + 2n_2 + 3)}} \bar{Q}_{2\{n_1+1, n_2+1\}} - \sqrt{\frac{n_2(n_1 + 1)}{(n_1 + 2n_2 + 3)}} \bar{W}_{2\{n_1+1, n_2-1\}} \\
V_{\{n_1, n_2\}} &= \sqrt{\frac{2(n_1 + n_2 + 2)}{(n_1 + 2n_2 + 4)}} (\sqrt{(n_1 + n_2 + 3)} W_{2\{n_1, n_2\}} - \sqrt{(n_2 + 1)(n_2 + 2)} \bar{Q}_{2\{n_1, n_2+2\}} \\
&\quad - \sqrt{(n_1 + 1)(n_2 + 1)} \bar{Q}_{1\{n_1+1, n_2+1\}}) \\
X_{\{n_1, n_2\}} &= \sqrt{\frac{2(n_1 + n_2)}{(n_1 + n_2 + 1)}} (\sqrt{n_2 + 1} Q_{1\{n_1, n_2\}} - \sqrt{n_1} Q_{\{n_1-1, n_2+1\}}) \\
Y_{\{n_1, n_2\}} &= Q_{2\{n_1, n_2\}}
\end{aligned} \tag{31}$$

leading to a simple form for the Lagrangian;

$$\begin{aligned}
L(W, Q) &= (2n_1 + 2n_2 + 4) |U_{\{n_1, n_2\}}|^2 + (2n_1 + 2n_2 + 4) |V_{\{n_1, n_2\}}|^2 \\
&\quad + (2n_1 + 2n_2) |X_{\{n_1, n_2\}}|^2 + (2n_1 + 2n_2) |Y_{\{n_1, n_2\}}|^2
\end{aligned} \tag{32}$$

It is sufficiently simple to reintroduce the appropriate powers of θ_1, θ_2 into the expressions for the masses and we end up with the result given in (23).

Howeve, there are a couple of subtleties that need to be addressed. From (30) we see that the modes $Q_{\bar{1}, \bar{2}\{0, 0\}}$ are already diagonal and have masses $\pm \left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right)$. The modes $Q_{\bar{1}, \bar{2}\{1, 0\}}$ and $Q_{\bar{1}, \bar{2}\{0, 1\}}$ on the other hand only mix amongst themselves, but they too have the right masses to fit into the general scheme given in (23). One other issue to worry about is that of the linear combinations orthogonal to the ones introduced in (31). If these modes were physical then our correspondence would be destroyed by the presence of a large number of massless modes. Fortunately for us this is not the case, these modes can be shown to be pure gauge and hence are unphysical. This is easily seen by write out the Gauss law constraint (7) in terms of the modes introduced and finding that the aforementioned modes have vanishing time derivatives.

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