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Markov processes and variational problems

by

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Professor James W. Pitman

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Abstract

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In this thesis, we study the statistical properties of non-linear transforms of Markov processes. These transforms are defined via variational formulas, and arise in various fields such as statistics, mathematical finance, convex analysis, statistical mechanics and hydrodynamic turbulence. In particular, we will focus on two sets of problems. The first problem that is addressed in Chapter 2, concerns the study of the Lipschitz minorant of the sample paths of a Lévy process. The study of this minorant was initiated by Abramson and Evans, but here we shed a light on its excursion structure away from its contact set. When the Lévy process is a Brownian motion with drift, an explicit path decomposition of these excursions is given, together with the decomposition of the semimartingales in the progressive enlargement of the canonical filtration by the first positive point in the contact set. In the second set of problems, we will consider physical solutions (also called entropy solutions) to scalar conservation laws (or equivalently Hamilton-Jacobi equations) with random initial data. In Chapter 3, we investigate the distribution of the solutions at later times in the one-dimensional case and when the initial data is a Brownian white noise for any general convex Hamiltonian. This settles a conjecture of Menon and Srinivasan and extends Groeneboom's result for the Burgers case. In Chapter 4, we consider the higher dimensional case and focus in particular on the planar case. We construct a family of random convex piecewise linear functions which are the dimension 2 analogue of the anti-derivative of pure-jump Markov processes. At the heart of this construction is a novel class of kinetic equations. The invariance of this class of processes under the flow of Hamilton-Jacobi equations is also discussed.

To my parents, my grandmother, and my late grandfather

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Chapter 1

Introduction

1.1 Minorants of stochastic processes

For a class $\mathcal{C}$ of real-valued functions that are defined on a fixed interval I and an arbitrary function f, we define *the greatest* $\mathcal{C}$*-minorant* of f to be the function c defined pointwise as follows

$$c(x) = \sup_{g \in \mathcal{C}, g \leq f} g(x), \ x \in I$$

should the set $\{g \in \mathcal{C}, g(u) \leq f(u) \ \forall u \in I\}$ be non-empty. Fix $\alpha > 0$, when $\mathcal{C}$ is the set of convex (resp. α-Lipschitz) functions, we call c *the greatest convex (resp.* α*-Lipschitz) minorant* of f. For these two cases, it is not hard to show that $c \in \mathcal{C}$, as convex and Lipschitz functions are stable under the pointwise supremum operation.

The goal of this section is to give a historical exposition of the study of convex and Lipschitz minorants of various continuous-time stochastic processes.

1.1.1 Convex minorants

There is a rich literature on convex minorants of stochastic processes such as random walks, Brownian motion and Lévy processes. In the one-dimensional case, these convex minorants are exactly the convex hulls of the corresponding graphs, and as such are defined via a double variational formula. Indeed, the Legendre transform (also called the convex dual) of a function h defined on the real line is the function h^* given by the formula

$$h^*(x) = \sup_y (xy - h(y))$$

The convex minorant of h is then equal to h^{**}. The fact that convex minorants do arise in contexts such as statistics is not surprising, as it is very common in that field to encounter optimization problems involving samples perturbed by independent Gaussian noise. Indeed, one such example is that of isotonic regression (see [59]) where the distribution of the convex minorant of a random walk is related to the isotonic Least Squares Estimator (LSE). In this thesis, we will only focus on the continuous-time analogues of random walks, the so-called Lévy processes. A particular focus will be given to the Brownian motion, which even though it belongs to the class of Lévy processes, has a special structure that allows us to have more explicit formulas.

Brownian motion

Let $(B_t)_{t\geq 0}$ be a standard linear Brownian motion on $[0,\infty)$ started at the origin. The largest convex minorant (or equivalently the smallest concave majorant) of $(B_t)_{t\geq 0}$ has been first considered by Groeneboom in [26]. A crucial element in this study is the argmax process in the Legendre transform formula of B^* (where B^* is the convex dual of B). The link between this process and the concave majorant of the Brownian motion will be made clearer later on. To be more precise, let us define the process $(\sigma(a))_{a>0}$ as

$$\sigma(a) := \sup\{t \geq 0 : B_t - at = \sup_{s\geq 0}(B_s - as)\} \tag{1.1}$$

that is, $\sigma(a)$ is the last time (and with probability one, the only time) the Brownian motion with drift $(B_t - at)_{t\geq 0}$ attains its maximum. We then define $\tau(a) = \sigma(\frac{1}{a})$ and $\tau(0) = 0$. The main result of [26] is that the process $\{\tau(a) : a \geq 0\}$ is pure-jump with independent nonstationary increments and increasing paths. More precisely, Groeneboom has proved the following result

Theorem 1.1.1 ([26]) *The process τ is a pure-jump process with independent nonstationary increments and right-continuous increasing paths. The process τ has the following representation*

$$\tau(a) = \int_0^\infty l\eta([0,a]\times dl) \quad ,a>0 \tag{1.2}$$

where $\eta(\mathrm{da}\times \mathrm{dl})$ is a Poisson measure with mean $n(da\times dl) = \frac{1}{a^2\sqrt{l}}\phi\left(\frac{\sqrt{l}}{a}\right) da\, dl$ for $a,l>0$ and $\phi(x) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2})$, $x\in\mathbb{R}$.
The marginal density of $\tau(a)$ has Laplace transform

$$\mathbb{E}\left[\exp(-\lambda\tau(a))\right] = \frac{2}{1+\sqrt{2\lambda a^2+1}}, \quad a\geq 0, \lambda>0 \tag{1.3}$$

and $\tau(b)-\tau(a)$ has Laplace transform

$$\mathbb{E}[\exp(-\lambda(\tau(b)-\tau(a)))] = \frac{1+\sqrt{2\lambda a^2+1}}{1+\sqrt{2\lambda b^2+1}} \tag{1.4}$$

Furthermore, the number of jumps of τ in an interval (a,b), $0<a<b<\infty$, has a Poisson distribution with mean $\log\left(\frac{b}{a}\right)$.

The proof of this theorem relies mainly on the celebrated Williams path decomposition of the Brownian motion with drift at its maximum (see [65]). In a nutshell, this decomposition says that the pre-maximum process $(B_t - at, 0\leq t\leq \sigma(a))$ and post-maximum processes $(B_t - at, t\geq \sigma(a))$ are conditionally independent given $(\sigma(a), B_{\sigma(a)})$. Moreover, we have an explicit knowledge on the conditional distribution of each of theses processes together

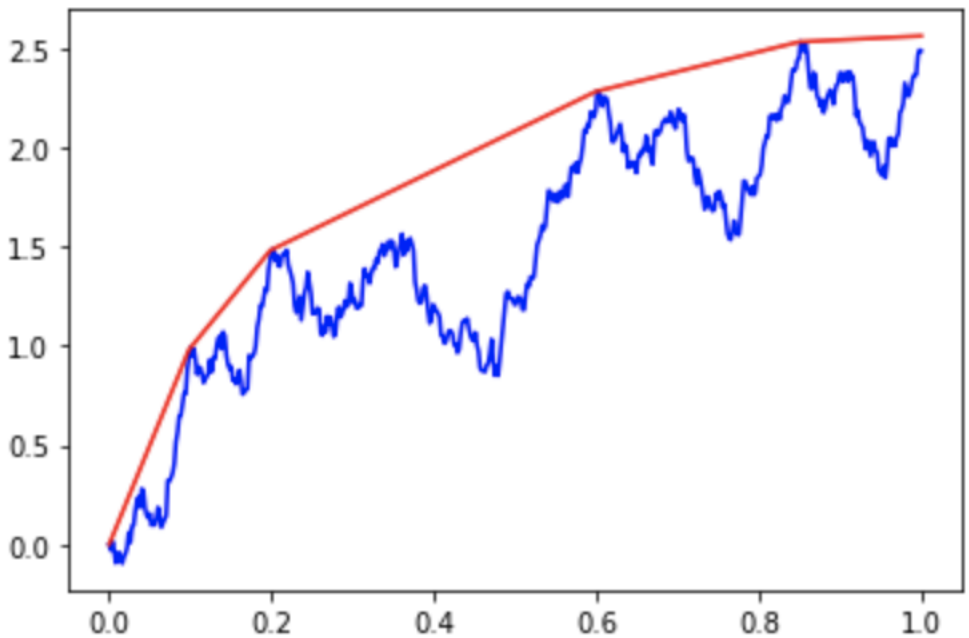

Figure 1.1: Approximation of the Brownian motion path and its least concave majorant

with the joint distribution of $(\sigma(a), B_{\sigma(a)})$. The precise statement of this theorem is given in Chapter 2, Theorem 2.5.1.

We call the process τ the *slope process*. We now relate this process to the concave majorant of B that we denote $(K_t)_{t\geq 0}$. Let

$$\mathcal{H} := \{(r, \Delta\tau_r := \tau_r - \tau_{r-}); \Delta\tau_r > 0\}$$

be the point process of jumps of τ. By Theorem 1.1.1, the process $\mathcal{H}$ is a Poisson point process with intensity measure n. The concave majorant K is now simply the concatenation of the increasing line segments with slope $\frac{1}{r}$ and duration $\Delta\tau_r$ for $(r, \Delta\tau_r) \in \mathcal{H}$. The joint law of K and B is described in the following theorem which is also due to Groeneboom.

Theorem 1.1.2 ([26]) *The standard linear Brownian motion B can be decomposed into the process τ and independent Brownian excursions. More precisely, conditionnally on the process τ, the vertical distance of the Brownian motion to the concave majorant $(K_t - B_t)_{t\geq 0}$ is a succession of independent Brownian excursions, i.e for any measurable enumeration $(T_i)_{i\in\mathbb{Z}}$ of the jump times of τ, depending only on the process τ, the process $(K_t - B_t)_{T_i\leq t\leq T_{i+1}}$ has the same distribution as $(\sqrt{T_{i+1} - T_i}\mathbf{e}_i\left(\frac{t-T_i}{T_{i+1}-T_i}\right))_{T_i\leq t\leq T_{i+1}}$ where $\mathbf{e}_i$ is a standard Brownian excursion on $[0, 1]$, and these excursions are independent as i varies.*

Figure 1.1 shows the path of a Brownian motion and its concave majorant. One of the consequences of this theorem, is the derivation of the law of the concave majorant at a fixed time $t > 0$. Before stating this proposition, we will introduce a little bit of notation.

Notation 1.1.3 *For $t > 0$, let G_t (resp. D_t) be the left-hand (resp. right-hand) vertices of the segment of K straddling t. For fixed $t > 0$, we almost surely have $0 < G_t < t < D_t$. Define $I_t = K_t - tK'_t$ to be the intercept at 0 of the line extending this segment (K' here is the right-hand derivative of K). As the process (K, B) enjoys Brownian scaling, we can restrict the discussion to time $t = 1$.*

For $\mu, y \in \mathbb{R}$, let $f_{\mu,y}$ be the density of the inverse Gaussian distribution, given by the formula

$$f_{\mu,y}(t) = \frac{y}{\sqrt{2\pi t^3}} \exp\left(-\frac{(y-\mu t)^2}{2t}\right), \quad t > 0$$

Similarly, we define the size-biased inverse Gaussian distribution, which density is given by

$$f^*_{\mu,y}(t) = \frac{\mu}{y} t f_{\mu,y}(t)$$

We are now ready to state the next proposition which is mainly due to Groeneboom. However, it was put on a different form and given a much simpler proof by Pitman and the author in [42].

Proposition 1.1.4 ([42]) *The density function of $(K'_1, I_1, K_1 - B_1, \frac{1}{G_1}, D_1)$ is*

$$f_5(a,b,y,v,w) = \sqrt{\frac{2}{\pi^3 (v-1)^3 (w-1)^3}}\, ab(wv-1) \times \exp\left(-\frac{1}{2}\left(b^2 w + 2ab + a^2 v + y^2 \frac{wv-1}{(v-1)(w-1)}\right)\right) \mathbb{1}_{\{w,v>1\}} \mathbb{1}_{\{a,b,y>0\}}$$

In particular, the following marginals take simpler forms.

- *The joint density of $(K\prime 1, I_1, K_1 - B_1)$ at (a,b,y) is*

  $$f_3(a,b,y) = 4y(a+b+y)\phi(a+b+y)\mathbb{1}_{\{a,b,y>0\}}$$

- *The conditional density of $D_1 - 1$ at t given $(K'_1 = a, I_1 = b, K_1 - B_1 = y)$ is given by*

  $$g(t) = \frac{a}{a+b+y} f_{a,y}(t) + \frac{b+y}{a+b+y} f^*_{a,y}(t)$$

  where $f_{a,y}$ and $f^*_{a,y}$ are respectively the inverse Gaussian and size-biased inverse Gaussian densities with parameters (a, y).

Beyond Groeneboom's results which cover the case of a Brownian motion over the half real-line, more results have been obtained in the case of a finite interval and have been extended to other variants of the Brownian path (such as the Brownian bridge, Brownian meander etc..). These results are due to Pitman and Ross in [47], we compile them below. The first result concerns a Poissonian description of the convex minorant.

Theorem 1.1.5 ([47]) *Let Γ_1 be an exponential random variable with rate one. The lengths x and slopes s of the faces of the convex minorant of a Brownian motion on $[0, \Gamma_1]$ form a Poisson point process on $\mathbb{R}^+ \times \mathbb{R}$ with intensity measure*

$$\frac{\exp\left(-\frac{x}{2}(2+s^2)\right)}{\sqrt{2\pi x}} ds dx, \quad x \geq 0, s \in \mathbb{R}$$

The second description is a sequential Markovian one. Before stating the result, we make first the following definition

Definition 1.1.6 *We say that a sequence of random variables $(\tau_n, \rho_n)_{n\geq 0}$ satisfies the (τ, ρ)-recursion if for all $n \geq 0$:*

$$\rho_{n+1} = U_n \rho_n$$

and

$$\tau_{n+1} = \frac{\tau_n \rho_{n+1}^2}{\tau_n Z_{n+1}^2 + \rho_{n+1}^2}$$

for the two independent sequences of i.i.d uniform $(0,1)$ variables U_n and i.i.d squares of standard normal random variables Z_n^2, both independent of (τ_0, ρ_0).

Theorem 1.1.7 ([47]) *Let $(X(v), 0 \leq v \leq t)$ be one of the following:*

- *a $\mathrm{BES}(3)$ bridge from $(0,0)$ to (t,r) for $r > 0$.*
- *a $\mathrm{BES}^0(3)$ process.*
- *a Brownian meander of length t.*

Let $(\underline{C}(v), 0 \leq v \leq t)$ be the convex minorant of X. The vertices of $(\underline{C}(v), 0 \leq v \leq t)$ occur at times $0 = V_0 < V_1 < V_2 < \cdots$ with $\lim_n V_n = t$. Let $\tau_n := t - V_n$ so $\tau_0 = t > \tau_1 > \tau_2 > \cdots$ with $\lim_n \tau_n = 0$. Let $\rho_0 = X(t)$ and for $n \geq 1$ let $\rho_0 - \rho_n$ denote the intercept at time t of the line extending the segment of the convex minorant of X on the interval (V_{n-1}, V_n), so that

$$\rho_0 - \rho_n = \frac{\underline{C}(V_n) - \underline{C}(V_{n-1})}{V_n - V_{n-1}}(t - V_n) + \underline{C}(V_n)$$

The convex minorant $\underline{C}$ of X is uniquely determined by the sequence (τ_n, ρ_n) for $n = 1, 2, \ldots$ which satisfies the (τ, ρ)-recursion with

$$\rho_0 = X(t) \text{ and } \tau_0 = t$$

Moreover, conditionally given $(\underline{C}(v), 0 \leq v \leq t)$ the process $(X(v) - \underline{C}(v), 0 \leq v \leq t)$ is a concatenantion of independent Brownian excursions of length $\tau_{n-1} - \tau_n$ for $n \geq 1$.

Remark 1.1.8 *From the theorem above, one can deduce a description of the convex minorant of a Brownian motion on a finite interval through Denisov's decomposition at the minimum. Denisov's decomposition of the Brownian motion on a fixed interval states that conditionally on its argmin, the pre and post minimum processes are independent Brownian meanders of appropriate lengths.*

Lévy processes

We recall that a Lévy process $(X_t)_{t\geq 0}$ is a real-valued process that has right-continuous paths with left hand limits, and independent stationary increments. The treatment of the convex minorants of Lévy processes was mostly done in the work of Pitman and Bravo in [49] under the assumption that for every $t > 0$, X_t has a continuous distribution (i.e that for all $x \in \mathbb{R}$ we have $\mathbb{P}(X_t = x) = 0$). This assumption was alleviated in the recent work [16]. The first proposition uncovers a structural property that we have seen previously in the work of Groeneboom.

Proposition 1.1.9 ([49]) *Let X be a Lévy process with continuous distributions and C its convex minorant on $[0,t]$. The following conditions hold almost surely:*

- *The open set $\mathcal{O} := \{s \in (0,t) : C_s < X_s \wedge X_{s-}\}$ has Lebesgue measure t.*
- *For every component interval (g,d) of $\mathcal{O}$, the jumps that X might have at g and d have the same sign. When X has unbounded variation on finite intervals, both jumps are zero.*
- *If (g_1,d_1) and (g_2,d_2) are different component intervals of $\mathcal{O}$, then their slopes differ*

$$\frac{C_{d_1} - C_{g_1}}{d_1 - g_1} \neq \frac{C_{d_2} - C_{g_2}}{d_2 - g_2}$$

We call each connected component (g,d) of $\mathcal{O}$ an *excursion interval*. Associated with each excursion interval (g,d) are the *vertices* g and d, the *length* $d-g$, the increment $C_d - C_g$ and the slope $(C_d - C_g)/(d-g)$. The result belows gives a sequential description of these excursions.

Theorem 1.1.10 ([49]) *Let (U_i) be a sequence of uniform random variables on $(0,t)$ independent of the Lévy process X which has continuous distributions. Let $(g_1,d_1),(g_2,d_2),\dots$ be the sequence of distinct excursion intervals which are successively discovered by the sequence (U_i). Consider another i.i.d sequence (V_i) of uniform random variables on $(0,1)$ independent of X, and construct the associatd uniform stick-breaking process L by*

$$L_1 = tV_1 \quad and \quad for\ i \geq 1 \quad L_{i+1} = V_{i+1}(t - S_i).$$

where

$$S_0 = 0 \quad and \quad for\ i \geq 1 \quad S_i = L_1 + \cdots + L_i$$

The following equality in distribution holds:

$$((d_i - g_i, C_{d_i} - C_{g_i}), i \geq 1) \stackrel{d}{=} ((L_i, X_{S_i} - X_{S_{i-1}}), i \geq 1)$$

Integrating this last result, we obtain a description of the convex minorant on the random interval $[0,T_\theta]$ where T_θ is an exponential random variable of paramter θ independent of X.

Corollary 1.1.11 ([49]) *Let T be an exponential of parameter θ and independent of the Lévy process X. Let Ξ_T be the point process with atoms at lengths and increments of excursions intervals of the convex minorant of X on $[0,T]$. Then Ξ_T is a Poisson point process with intensity*

$$\mu_\theta(dt,dx) = e^{-\theta t}\frac{dt}{t}\mathbb{P}(X_t \in dx)$$

From Theorem 1.1.10, we can also derive the behavior of the convex minorant of X on $[0,\infty)$ as described by Groeneboom in the case of the Brownian motion.

Corollary 1.1.12 ([49]) *The quantity* $l = \liminf_{t\to\infty} \frac{X_t}{t}$ *belongs to* $(-\infty, \infty]$ *and is almost surely constant if and only if the convex minorant of* X *on* $[0, \infty)$ *is almost surely finite. In this case, and under the assumption that* X *has continuous distributions,* Ξ_∞ *is a Poisson point process with intensity*

$$\mu_\infty(dt, dx) = \frac{\mathbb{1}_{\{x<lt\}}}{t} \mathbb{P}(X_t \in dx)dt$$

The special case of Cauchy process

When X is a Cauchy process, i.e the distribution of X_1 is given by

$$F(x) := \mathbb{P}(X_1 \leq x) = \frac{1}{2} + \frac{\arctan(x)}{\pi}$$

the convex minorant C of X enjoys some special properties. Indeed if we denote by C' the right-hand derivative of C, and introduce its right-continuous inverse

$$I_x = \inf\{t \geq 0 : C'_t > x\} \text{ for } x \in \mathbb{R}$$

Let γ be a Gamma subordinator, i.e a Lévy process with Laplace exponent given by

$$\mathbb{E}[e^{-q\gamma_t}] = \left(\frac{1}{1+q}\right)^t$$

we have the following results due to Bertoin in [11] and Pitman-Bravo in [49]

Theorem 1.1.13 ([11],[49])

- *The symmetric Cauchy process is characterized by the independence of lengths and slopes of excursions intervals on* $[0, 1]$*.*
- $(I_x, x \in \mathbb{R})$ *have the same law as* $(\frac{\gamma_{F(x)}}{\gamma_1}, x \in \mathbb{R})$*.*
- *The process* $(C'_s, 0 < s < 1)$ *is continuous and has the same law as* $(-\cot(\pi L(s\gamma_1)), 0 < s < 1)$*, where* L *is the right-continuous inverse of* γ*.*

1.1.2 Lipschitz minorants

The study of the Lipschitz minorants of Lévy processes and Brownian motion has been introduced by Evans and Abramson in the work [2]. For a function f, and $\alpha > 0$, its α-Lipschitz minorant exists if and only if f is bounded from below on compact intervals and satisfies $\liminf_{t\to-\infty} f(t) - \alpha t > -\infty$ and $\liminf_{t\to+\infty} f(t) + \alpha t > -\infty$. If these conditions are verified, we denote by m the greatest α-Lipschitz minorant of f, that is given by the formula

$$m(t) = \inf\{f(s) + \alpha|t - s|: s \in \mathbb{R}\}$$

Abramson and Evans investigated the process $(M_t)_{t\in\mathbb{R}}$ that is the α-Lipschitz minorant of a real-valued Lévy process $(X_t)_{t\in\mathbb{R}}$. Let σ be the Brownian exponent of X and Π its Lévy measure. In the case where X has bounded variation, let d be its drift (i.e the slope of its linear continuous part). This first proposition gives necessary and sufficient conditions for the existence of the Lipschitz minorant of a Lévy process.

Proposition 1.1.14 ([2]) *Let X be a Lévy process. The α-Lipschitz minorant of X exists almost surely if and only if $\sigma = 0$, $\Pi = 0$ and $|d| = \alpha$ (or equivalently if $X_t = \alpha t$ for all $t \in \mathbb{R}$, or $X_t = -\alpha t$ for all $t \in \mathbb{R}$), or $\mathbb{E}[|X_1|] < \infty$ and $|\mathbb{E}[X_1]| < \alpha$.*

We assume from now on that X verifies these properties. The first striking result from the study of the joint law of $(M_t, X_t)_{t\in\mathbb{R}}$ is the *regenerative property* of the the contact set

$$\mathcal{Z} := \{t \in \mathbb{R} : M_t = X_t \wedge X_{t-}\}$$

The *regenerative property* of a closed set (also sometimes called *Markov sets*) has many equivalent definitions. We will give here the defition of Fitzsimmons and Taksar in [22]. For a more general concept of *space-time regenerative systems*, we refer the reader to Chapter 2.

Definition 1.1.15 *Let Ω^0 denote the class of closed subsets of $\mathbb{R}$. For $t \in \mathbb{R}$ and $\omega^0 \in \Omega^0$, define*

$$d_t(\omega^0) := \inf\{s > t : s \in \omega^0\}, \quad r_t(\omega^0) = d_t(\omega^0) - t$$

and

$$\tau_t(\omega^0) = \mathbf{cl}((\omega^0 - t) \cap (0, \infty))$$

We define the filtration $\mathcal{G}_t^0 := \sigma\{r_s : s \leq t\}$, and $\mathcal{G}^0 = \sigma\{r_s : s \in \mathbb{R}\}$. A random set is a measurable mapping S from any probability space $(\Omega, \mathcal{F})$ to $(\Omega^0, \mathcal{G}^0)$.
A probability measure $\mathbb{Q}$ on $(\Omega^0, \mathcal{G}^0)$ is regenerative with regeneration law $\mathbb{Q}^0$ if

(i) $\mathbb{Q}\{d_t = \infty\} = 0$ *for all* $t \in \mathbb{R}$.

(ii) For all $t \in \mathbb{R}$ and for all $\mathcal{G}^0$-measurable nonnegative function F,

$$\mathbb{Q}[F(\tau_{d_t})|\mathcal{G}_{t+}^0] = \mathbb{Q}^0[F]$$

where we write $\mathbb{Q}[\cdot]$ and $\mathbb{Q}^0[\cdot]$ for expectations with respect to $\mathbb{Q}$ and $\mathbb{Q}^0$. A random set S defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a regenerative set if the push-forward of $\mathbb{P}$ by the map S (that is, the distribution of S) is a regenerative probability measure

Remark 1.1.16 *Every regenerative set S is associated with a subordinator $(Y_t)_{t\geq 0}$ such that the range of this subordinator coincides with S. This subordinator is characterized by its drift δ and its jump measure Λ. The drift is related to the Lebesgue measure of S (it is positive or zero wether the Lebesgue measure is zero or infinite). When the set is stationary (i.e S has the same distribution as $u + S$ for all $u \in \mathbb{R}$, then it turns out that $\int_{\mathbb{R}^+} y\Lambda(dy) < \infty$.*

One of the main theorems of [2] is the following

Theorem 1.1.17 *The random (closed) set $\mathcal{Z}$ is stationary and regenerative.*

As a consequence of this result, the study of the set $\mathcal{Z}$ is related to the study of the corresponding subordinator $(Y_t)_{t\geq 0}$. Note that Y is unique up to a speed factor. The result below sheds a light on the Lebesgue measure of $\mathcal{Z}$ (equivalently, the drift δ of Y).

Theorem 1.1.18 ([2]) *If $\sigma = 0$, $\Pi(\mathbb{R}) < \infty$, and $|d| = \alpha$, then the Lebesgue measure of $\mathcal{Z}$ is almost surely infinite. If X is not of this form, then the Lebesgue measure of $\mathcal{Z}$ is almost surely zero if and only if zero is regular for the interval $(-\infty, 0]$ for at least one of the Lǘy processes $(X_t + \alpha t)_{t\geq 0}$ and $(X_t + \alpha t)_{t\geq 0}$.*

The proof of these results relies mostly on the fluctuation theory of the Lévy processes and Rogozin's integral criterion of regularity. Another refined aspect of $\mathcal{Z}$ is when the Lebesgue measure is zero, we can ask the question as to if $\mathcal{Z}$ is discrete or dense (also called a *perfect* set). This dichotomy is related to whether the Lévy measure of Y has a finite or infinite mass. The theorem below answers this question (we also recall this result in Chapter 2).

Theorem 1.1.19 ([2]) *Let X be a Lévy process that admits an α-Lipschitz minorant and such that $\Pi(\mathbb{R}) = \infty$, then $\Lambda(\mathbb{R}^+) < \infty$ if and only if*

$$\int_0^1 t^{-1}\mathbb{P}(X_t \in [-\alpha t, \alpha t])dt < \infty$$

When X has unbounded variation, one can say more about the measure Λ, in particular we are able to determine its Laplace transform in terms of the distribution of X under some additional mild assumptions. This is stated in the following theorem

Proposition 1.1.20 ([2]) *Let X be a Lévy process with paths of unbounded variation and that admits an α-Lipschitz minorant almost surely. Suppose further that X_t has an absolutely continuous distribution for all $t \neq 0$, and that the densities of the random variables $\inf_{t\geq 0}(X_t + \alpha t)$ and $\inf_{t\geq 0}(X_{-t} + \alpha t)$ are square integrable. Then, $\delta = 0$ and Λ is characterized by*

$$\begin{aligned}\frac{\int_{\mathbb{R}^+}(1-e^{-\theta x})\Lambda(dx)}{\int_{\mathbb{R}^+} x\Lambda(dx)} = 4\pi\alpha \int_{-\infty}^{\infty} \Bigg\{ &\exp\Bigg(\int_0^\infty t^{-1}\mathbb{E}\left[(e^{izX_t - iz\alpha t} - 1)\mathbf{1}\{X_t \geq \alpha t\} + \right. \\ &\left.(e^{izX_t + iz\alpha t} - 1)\mathbf{1}\{X_t \leq -\alpha t\}\right] dt \Bigg)\Bigg] \\ &- \exp\Bigg(\int_0^\infty t^{-1}\mathbb{E}\left[(e^{-\theta t + izX_t - iz\alpha t} - 1)\mathbf{1}\{X_t \geq \alpha t\} + \right. \\ &\left.(e^{-\theta t + izX_t + iz\alpha t} - 1)\mathbf{1}\{X_t \leq -\alpha t\}\right] dt \Bigg)\Bigg]\Bigg\} dz\end{aligned}$$

for $\theta \geq 0$, and, moreover $\Lambda(\mathbb{R}^+) < \infty$.

When the conditions above for X are not satisfied, one can approximate X by $X^\epsilon = X + \epsilon B$ where B is a two-sided linear Brownian motion independent of X. The Lévy processes X^ϵ do satisfy the conditions of the above proposition, and hence one can get the Laplace transform of Λ by approximation.

The study of the Lipschitz minorant in the special case of a Brownian motion with drift has also been discussed in [2], in particular the authors were able to derive the law of several special points of the Lipschitz minorant, such as the first positive point in the contact set after 0 denoted by D, and the maximal height of the Lipschitz minorant in the interval $[D, G]$, where G is the last negative point in $\mathcal{Z}$. These results have been extended in the work [21], that is the subject of Chapter 2.

1.2 Hamilton-Jacobi equations and scalar conservation laws

1.2.1 The deterministic picture

Momentum only Hamilton-Jacobi equations takes the form:

$$\begin{cases} u_t + H(Du) = 0 \text{ in } \mathbb{R}^d \times (0, \infty) \\ u = g \text{ on } \mathbb{R}^d \times \{0\} \end{cases} \tag{1.5}$$

Here $u : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ is the *unknown*. We denote the variables by $(x, t) \in \mathbb{R}^d \times [0, \infty)$. We refer to $x \in \mathbb{R}^d$ as the *space* variable and $t \geq 0$ as the *time* variable. $Du = D_x u = (u_{x_1}, \ldots, u_{x_d})$ is the space gradient. The *Hamiltonian* $H : \mathbb{R}^d \to \mathbb{R}$ and the initial condition $g : \mathbb{R}^d \to \mathbb{R}$ are given. We assume from now that H is convex, in C^2 and has superlinear growth at infinity.

It is a classical fact in the PDE literature that the method of characteristics fails globally for Hamilton-Jacobi equations. Indeed, the function u is shown to be constant along any characteristic, however for a wide class of initial conditions g, different characteristics may cross in short times, which leads to a contradiction regarding the existence of global smooth solutions. A smooth solution to the above PDE would then only exist locally, and thus a theory of weak or generalized solutions is indeed needed for well-posedeness purposes. We introduce first the *Hopf-Lax* formula, that is derived as a conjectural candidate to Hamilton-Jacobi PDE using formal computations from the least action principle.

Definition 1.2.1 *The Hopf-Lax formula associated with the Hamiltonian H and the initial condition g is the function $u(x, t)$ given by*

$$u(x, t) = \min_{y \in \mathbb{R}^d} \left\{ g(y) + tL\left(\frac{x - y}{t}\right) \right\} \tag{1.6}$$

where $L(q) = \sup_{p \in \mathbb{R}^d} (p \cdot q - H(p))$ is the Legendre transform of H.

The *Hopf-Lax* formula solves the Hamilton-Jacobi PDE in the following sense

Proposition 1.2.2 ([20]) *The function u defined by the Hopf-Lax formula in (3.3) is Lipschitz continuous, differentiable a.e in $\mathbb{R}^d \times (0, \infty)$ and solves the initial value problem*

$$\begin{cases} u_t + H(Du) = 0 \ \textit{a.e. in } \mathbb{R}^d \times (0, \infty) \\ u = g \ \textit{on } \mathbb{R}^d \times \{0\} \end{cases} \tag{1.7}$$

The Hopf-Lax solution is also unique under the additional assumption of semi-concavity. Indeed, it is the unique a.e solution to Hamilton-Jacobi PDE that satisfies the following statement:

$$\text{There exists } C \geq 0 \text{ that for all } t > 0 \text{ and } x, z \in \mathbb{R}^d$$
$$u(x+z,t) - 2u(x,t) + u(x-z,t) \leq C\left(1 + \frac{1}{t}\right)|z|^2$$

When the space dimension d is equal to one, it is more convenient to keep track of the inclination of u that we denote $\rho = u_x$. This leads to considering the initial-value problem called the *scalar conservation law* given by

$$\begin{cases} \rho_t + H(\rho)_x = 0 \text{ in } \mathbb{R} \times (0, \infty) \\ \rho = h \text{ on } \mathbb{R}^d \times \{0\} \end{cases} \tag{1.8}$$

where h is a bounded function. Similarly to the Hamilton-Jacobi PDE, the method of characteristics also fails for the scalar conservation law equation, and so we are in need of a notion of *weak solution*. We give it below

Definition 1.2.3 *We say that $\rho \in L^\infty(\mathbb{R} \times (0, \infty))$ is an integral solution to (3.1.8) if for any smooth function v with compact support, we have that*

$$\int_0^\infty \int_{-\infty}^\infty (\rho v_t + H(\rho) v_x)\, dx dt + \int_{-\infty}^\infty h v(x,0) dx = 0$$

By using the *Hopf-Lax* formula for the Hamilton-Jacobi equation and the loose connection $\rho = u_x$ to scalar conservation laws, one can derive a formula under the name of *Lax-Oleineik* that is a candidate for an integral solution to (3.1.8). The following theorem confirms this intuition

Theorem 1.2.4 ([20]) *Assume that $H : \mathbb{R} \to \mathbb{R}$ is smooth, uniformly convex and $h \in L^\infty(\mathbb{R})$. Let $g(x) := \int_0^x h(y)dy$ for $x \in \mathbb{R}$.*

- *For each time $t > 0$ there exists for all but a countably many values of $x \in \mathbb{R}$ a unique point $y(x,t)$ such that*

$$\min_{y \in \mathbb{R}} \left\{ g(y) + tL\left(\frac{x-y}{t}\right) \right\} = g(y(x,t)) + tL\left(\frac{x - y(x,t)}{t}\right)$$

- *The mapping $x \to y(x,t)$ is nondecreasing*
- *For each time $t > 0$, define the function $\rho(x,t)$ by*

$$\rho(x,t) = (H')^{-1}\left(\frac{x - y(x,t)}{t}\right)$$

The function ρ is an integral solution to (3.1.8).

- *The function ρ above is the unique integral solution to (3.1.8) under the additional entropy condition, that is there exists $C \geq 0$ such that for all $t > 0$, and $x, z \in \mathbb{R}$ with $z > 0$, we have*

$$\rho(x+z,t) - \rho(x,t) \leq \frac{C}{t} z$$

As a brief example of a *scalar conservation law* solution, we will discuss ***Riemann***'s problem:

Assume that the initial condition h is piecewise-constant

$$h(x) = \begin{cases} \rho_l & \text{if} \quad x < 0 \\ \rho_r & \text{if} \quad x > 0 \end{cases} \tag{1.9}$$

The following theorem gives the unique *entropy* solution to the *scalar conservation law* with the above initial condition

Theorem 1.2.5 ([20])

- *If $\rho_l > \rho_r$, the unique entropy solution to Riemann's problem is*

$$\rho(x,t) = \begin{cases} \rho_l & \text{if} \quad \frac{x}{t} < \sigma \\ \rho_r & \text{if} \quad \frac{x}{t} > \sigma \end{cases} \tag{1.10}$$

  where

$$\sigma := \frac{H(\rho_l) - H(\rho_r)}{\rho_l - \rho_r}$$

- *If $\rho_l < \rho_r$, the unique entropy solution to Riemann's problem is*

$$\rho(x,t) = \begin{cases} \rho_l & \text{if} \quad \frac{x}{t} < H'(\rho_l) \\ (H')^{-1}\left(\frac{x}{t}\right) & \text{if} \quad H'(\rho_l) < \frac{x}{t} < H'(\rho_r) \\ \rho_r & \text{if} \quad \frac{x}{t} > H'(\rho_r) \end{cases} \tag{1.11}$$

Remark 1.2.6 *In the first case, the states ρ_l and ρ_r are separated by a shock wave of constant speed σ. In the second case, the states ρ_l and ρ_r are separated by a rarefaction wave and in this case the solution ρ is continuous in space for every time $t > 0$.*

1.2.2 Scalar conservation laws with random initial data

When the Hamiltonian H is quadratic (i.e $H(p) = p^2$), the corresponding *scalar conservation law* is referred to as Burgers equation. Burgers introduced this equation as a crude model of hydrodynamic turbulence, and considered it with stochastic initial data with an aim to understand the statistical moments and correlations of the entropy solution $\rho(x,t)$. The Burgers equation with random initial condition thus received a wide interest from mathematical physicists, however identifying the distribution of its solution at later times remained a challenge, especially for the case of Gaussian white noise initial data. Burgers himself managed to link this distribution with the solution to some boundary problems, but very little was known about the exact form of these solutions. This remained the case, until the work of Groeneboom in [25], who managed to completely determine the law of the entropy solution to Burgers equation for Brownian white noise initial data. Groeneboom's original problem was related to the global behavior of isotonic regressors and its link with the Brownian motion with parabolic drift. However, it turns out that these two problems are related via the variational formula of *Hopf-Lax*. We give an extensive overview of his result in Chapter 3, mainly Theorem 3.1.1.

After Groeneboom's results, much more progress has been made in this area (see [58],[4],[5]). In 1994, Carraro and Duchon in [15] realized that Lévy processes were well-suited as initial conditions to Burgers equation due to their spatial and temporal homogeneity. They defined a notion of *statistical solutions* which is similar in flavor to integral solutions in the PDE's world. In brief, a *statistical solution* to Burgers equation is a family of measures $(\mu_t)_{t\geq 0}$ such that

$$\frac{\partial}{\partial t}\hat{\mu}_t(v) = i\int_E \left\langle \frac{1}{2}u^2, v' \right\rangle \exp\left(i\langle u, v\rangle\right) d\mu_t(u)$$

for all v smooth with compact support and where E is the space of real-valued càdlàg functions. This statistical solution approach was further developed by the authors and Chabanol and Duchon in [14] and [17]. However there is a drawback: given a (random) entropy solution $\rho(x,t)$ to the inviscid Burgers' equation, the law of $\rho(\cdot,t)$ is a statistical solution, but it is not always the case that a statistical solution yields a entropy solution.

The next big result in this area was achieved by Bertoin in 1998 in his work [12], who proved a remarkable closure theorem for Lévy initial data. The full statement of this result is deferred to Chapter 3 in Theorem 3.1.3. In summary, he proved that when the initial condition is a Lévy process with only upward jumps, the entropy solution to Burgers equation remains Lévy at later times. Furthermore, the corresponding Laplace exponents themselves solve a Burgers equation. In 2007, Menon and Pego in [35] used the Lévy-Khintchine representation for the Laplace exponents and observed that this evolution according to Burgers equation corresponds to a Smoluchowski coagulation equation, with additive collision kernel, for the jump measure of the Lévy process $y(\cdot,t)$. The jumps of $y(\cdot,t)$ correspond to shocks in the solution $\rho(\cdot,t)$.

It is natural to wonder whether this complete integrability structure is *intrinsic* to Burgers equation through Markov processes or is there a similar phenomenon for general scalar conservation laws with any general Hamiltonian H. Note that the problem of Lipschitz minorants that we have described in the section above, corresponds to the *Hopf-Lax* formula when the Hamiltonian H takes the form

$$H(p) = +\infty\mathbf{1}(|p| > \alpha)$$

For the general convex $H \in C^2$, very little could be said until the work of Menon and Srinivasan in [36]. There, it is shown that when the initial condition ξ is a spectrally negative strong Markov process, the backward Lagrangian process $y(\cdot,t)$ and the solution $\rho(\cdot,t)$ remain Markov for fixed $t > 0$, the latter again being spectrally negative. The argument uses the notion of last exit times, due to Getoor in [23], to verify the Markov property according to its bare definition. However, the same could not be immediately be drawn for the Feller property. The authors formulated a conjecture for general random initial data (see Chapter 3). This conjecture was partially solved by Kaspar and Rezakhanlou in [32] first in the case of pure-jump Markov processes and later in the case of piecewise-deterministic Markov processes in [31]. The full statement of the conjecture and the latter result of Kaspar and Rezakhanlou is deferred to Chapter 3. We state it here their result for pure-jump Markov processes.

Theorem 1.2.7 ([32]) *Consider a process $\xi = \xi(x)$ that is a pure-jump Markov process started at $\xi(0) = 0$ and evolving for $x > 0$ according to a rate kernel $g(\rho_-, d\rho_+)$. We assume that for some constant $P > 0$ that the kernel g is supported on*

$$\{(\rho_-, \rho_+) : 0 \leq \rho_- \leq \rho_+ \leq P\}$$

and has a total rate which is constant in ρ^-:

$$\lambda = \int g(\rho_-, d\rho_+)$$

for all $0 \leq \rho_- \leq P$. Assume that the Hamiltonian $H : [0, P] \to \mathbb{R}$ is smooth, convex, has nonnegative right-derivative at $p = 0$ and finite left-derivative at $p = P$. Let $\mathcal{K}_+$ (resp. $\mathcal{M}_+$) be the space of non-negative kernels on $[0, P]$ (resp. non-negative measures on $[0, P]$). Let ρ be the unique entropy solution to the scalar conservation law $\rho_t = H(\rho)_x$ with initial condition $\rho(x, 0) = \xi(x)$ for $x \in \mathbb{R}$, then for each fixed $t > 0$, $x \to \rho(x, t)$ has a $x = 0$ marginal given by $\ell(t, d\rho_0)$ and for $x > 0$ evolves as a pure-jump Markov process with kernel $f(t, \rho_-, d\rho_+)$, where

$$f : [0, \infty) \to \mathcal{K}_+$$
$$\ell : [0, \infty) \to \mathcal{M}_+$$

are the unique positive kernels satisfying the initial conditions

$$f(0, \rho_-, d\rho_+) = g(\rho_-, d\rho_+), \quad \ell(0, d\rho_0) = \delta_0(d\rho_0)$$

and solving in an integral sense the equations

$$f_t = \mathcal{L}^\kappa f, \quad \text{and} \quad \ell_t = \mathcal{L}^0 \ell$$

where the operators $\mathcal{L}^\kappa$ and $\mathcal{L}^0$ are as follows, with integration over ρ_* only:

$$\mathcal{L}^\kappa f(t, \rho_-, d\rho_+) = \int (H[\rho_*, \rho_+] - H[\rho_-, \rho_*]) f(t, \rho_-, d\rho_*) f(t, \rho_*, d\rho_+)$$
$$- \left[\int H[\rho_+, \rho_*] f(t, \rho_+, d\rho_*) - \int H[\rho_-, \rho_*] f(t, \rho_-, d\rho_* \right] f(t, \rho_-, d\rho_+)$$

and

$$\mathcal{L}^0 \ell(t, d\rho_0) = \int H[\rho_*, \rho_0] \ell(t, d\rho_*) f(t, \rho_*, d\rho_0) - \left[\int H[\rho_0, \rho_*] f(t, \rho_0, d\rho_*) \right] \ell(t, d\rho_0)$$

for all $t \geq 0$ *and for all* $0 \leq \rho_- \leq P$*, with* $H[y,z] = \frac{H(y)-H(z)}{y-z}$ *for all* $y \neq z$*. Furthermore, the total integrals are conserved*

$$\lambda = \int f(t, \rho_-, d\rho_+), \qquad \int \ell(t, d\rho_0) = 1$$

Finally, if

$$g(\rho_-, [0, \rho_-] \cup \{P\}) = 0$$

for all $0 \leq \rho_- < P$*, then the same property holds for* $f(t, \cdot, \cdot)$ *for all* $t > 0$*.*

The resolution of the conjecture of Menon and Srinivasan for the remaining case of white noise initial data was achieved by the author in [41]. These series of work ([36],[32],[31],[41]) have achieved the same level of understanding that so far was only restricted to Burgers equation to the wider class of scalar conservation laws with convex flux H.

We finish this introduction by pointing out that the case of Hamilton-Jacobi equations in higher dimensions $d \geq 2$ with random initial data has so far been untouched. The first result in this area was achieved by Rezakhanlou and the author in [43], where we go through a construction of an analogue of pure-jump Markov processes and then prove a closure theorem by showing that this class of measures remains invariant under the flow of the PDE. This is the content of Chapter 4 of this thesis.

Chapter 2

Excursions away from the Lipschitz minorant of a Lévy process

This chapter is based on the article [21] written in collaboration with *Steven N. Evans* that is published in *Annales de l'Institut Henri Poincaré (B), Probabilités et Statistiques.*

2.1 Introduction

Recall that a function $g : \mathbb{R} \to \mathbb{R}$ is α-Lipschitz for some $\alpha > 0$ if $|g(s) - g(t)| \leq \alpha|s - t|$ for all $s, t \in \mathbb{R}$. Given a function $f : \mathbb{R} \to \mathbb{R}$, we say that f *dominates* the α-Lipschitz function g if $g(t) \leq f(t)$ for all $t \in \mathbb{R}$. A necessary and sufficient condition that f dominates some α-Lipschitz function is that f is bounded below on compact intervals and satisfies $\liminf_{t\to-\infty} f(t) - \alpha t > -\infty$ and $\liminf_{t\to+\infty} f(t) + \alpha t > -\infty$. When the function f dominates some α-Lipschitz function there is an α-Lipschitz function m dominated by f such that $g(t) \leq m(t)$ for all $t \in \mathbb{R}$ for any α-Lipschitz function g dominated by f; we call m the *α-Lipschitz minorant* of f. The α-Lipschitz minorant is given concretely by

$$\begin{aligned} m(t) &= \sup\{h \in \mathbb{R} : h - \alpha|t - s| \leq f(s) \text{ for all } s \in \mathbb{R}\} \\ &= \inf\{f(s) + \alpha|t - s| : s \in \mathbb{R}\}. \end{aligned} \tag{2.1}$$

The purpose of this chapter is to continue the study of the α-Lipschitz minorants of the sample paths of a two-sided Lévy process begun in [2].

A two-sided Lévy process is a real-valued stochastic process indexed by the real numbers that has càdlàg paths, stationary independent increments, and takes the value 0 at time 0. The distribution of a two-sided Lévy process X is characterized by the Lévy-Khintchine formula $[e^{i\theta(X_t-X_s)}] = e^{-(t-s)\Psi(\theta)}$ for $\theta \in \mathbb{R}$ and $-\infty < s \leq t < \infty$, where

$$\Psi(\theta) = -ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x \mathbb{1}_{\{|x|\leq 1\}})\, \Pi(dx)$$

with $a \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$, and Π a σ-finite measure concentrated on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R}} (1 \wedge x^2)\, \Pi(dx) < \infty$ (see [9, 56] for information about (one-sided) Lévy processes — the two-sided

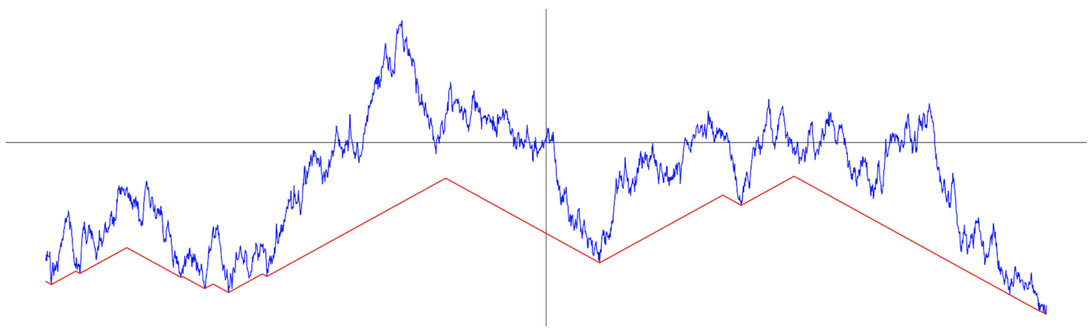

Figure 2.1: A typical Brownian motion sample path and its associated α-Lipschitz minorant.

case involves only trivial modifications). In order to avoid having to consider annoying, but trivial, special cases in what follows, we henceforth assume that X is not just deterministic linear drift $X_t = at$, $t \in \mathbb{R}$, for some $a \in \mathbb{R}$; that is, we assume that there is a non-trivial Brownian component ($\sigma > 0$) or a non-trivial jump component ($\Pi \neq 0$).

The sample paths of X have bounded variation almost surely if and only if $\sigma = 0$ and $\int_{\mathbb{R}}(1 \wedge |x|)\,\Pi(dx) < \infty$. In this case Ψ can be rewritten as

$$\Psi(\theta) = -id\theta + \int_{\mathbb{R}} (1 - e^{i\theta x})\,\Pi(dx).$$

We call $d \in \mathbb{R}$ the drift coefficient.

We now recall a few facts about the α-Lipschitz minorants of the sample paths of X from [2].

Either the α-Lipschitz minorant exists for almost all sample paths of X or it fails to exist for almost all sample paths of X. A necessary and sufficient condition for the α-Lipschitz minorant to exist for almost all sample paths is that $\mathbb{E}[|X_1|] < \infty$ and $|\mathbb{E}[X_1]| < \alpha$. We assume from now on that this condition holds and denote the corresponding minorant process by $(M_t)_{t\in\mathbb{R}}$. Figure 2.1 shows an example of a typical Brownian motion sample path and its associated α-Lipschitz minorant.

Set $\mathcal{Z} := \{t \in \mathbb{R} : M_t = X_t \wedge X_{t-}\}$. We call $\mathcal{Z}$ the *contact set*. The random closed set $\mathcal{Z}$ is non-empty, stationary, and regenerative in the sense of [22] (see Definition 2.2.1 below for a re-statement of the definition). Such a random closed set either has infinite Lebesgue measure almost surely or zero Lebesgue measure almost surely.

- If the sample paths of X have unbounded variation almost surely, then $\mathcal{Z}$ has zero Lebesgue measure almost surely.

- If X has sample paths of bounded variation and $|d| > \alpha$, then $\mathcal{Z}$ has zero Lebesgue measure almost surely.

- If X has sample paths of bounded variation and $|d|< \alpha$, then $\mathcal{Z}$ has infinite Lebesgue measure almost surely.
- If X has sample paths of bounded variation and $|d|= \alpha$, then whether the Lebesgue measure of $\mathcal{Z}$ is infinite or zero is determined by an integral condition involving the Lévy measure Π that we omit. In particular, if $\sigma = 0$, $\Pi(\mathbb{R}) < \infty$, and $|d|= \alpha$, then the Lebesgue measure of $\mathcal{Z}$ is almost surely infinite.

If $\mathcal{Z}$ has zero Lebesgue measure, then $\mathcal{Z}$ is either almost surely a discrete set or almost surely a perfect set with empty interior.

- If $\sigma > 0$, then $\mathcal{Z}$ is almost surely discrete.
- If $\sigma = 0$ and $\Pi(\mathbb{R}) = \infty$, then $\mathcal{Z}$ is almost surely discrete if and only if

$$\int_0^1 t^{-1}\mathbb{P}\{X_t \in [-\alpha t, \alpha t]\}\, dt < \infty.$$

- If $\sigma = 0$ and $\Pi(\mathbb{R}) < \infty$, then $\mathcal{Z}$ is almost surely discrete if and only if $|d|> \alpha$.

The outline of the remainder of the chapter is as follows.

In Section 2.2 we show that the pair $((X_t)_{t\in\mathbb{R}}, \mathcal{Z})$ is a *space-time regenerative system* in the sense that if $D_t := \inf\{s \geq t : s \in \mathcal{Z}\}$ for any $t \in \mathbb{R}$, then $((X_{D_t+u} - X_{D_t})_{u\geq 0}, \mathcal{Z}\cap[D_t, \infty) - D_t)$ is independent of $((X_u)_{u\leq D_t}, \mathcal{Z}\cap(-\infty, D_t])$ with a distribution that does not depend on $t \in \mathbb{R}$. It follows that if $\mathcal{Z}$ is discrete, we write $0 < T_1 < T_2 < \dots$ for the successive positive elements of $\mathcal{Z}$, and we set $Y^n = (X_{T_n+t} - X_{T_n},\ 0 \leq t \leq T_{n+1} - T_n)$, $n \in \mathbb{N} := \{1, 2, \dots\}$, for the corresponding sequence of excursions away from the contact set, then these excursions are independent and identically distributed. When $\mathcal{Z}$ is not discrete there is a "local time" on $\mathcal{Z} \cap [0, \infty)$ and we give a description of the corresponding excursions away from the contact set as the pobZ of a Poisson point process that is analogous to Itô's description of the excursions of a Markov process away from a regular point.

Because $((X_t)_{t\in\mathbb{R}}, \mathcal{Z})$ is stationary, the key to establishing the space-time regenerative property is to show that if $D := D_0$ is the first positive point in $\mathcal{Z}$, then $((X_{D+t} - X_D)_{t\geq 0}, \mathcal{Z} \cap [D, \infty) - D)$ is independent of $((X_t)_{t\leq D}, \mathcal{Z} \cap (-\infty, D])$. This is nontrivial because D is most definitely not a stopping time for the canonical filtration of X and so we can't just apply the strong Markov property. We derive the claimed fact in Section 2.3 using a result from [37] on the path decomposition of a real-valued Markov process at the time it achieves its global minimum. This result in turn is based on general last-exit decompositions from [23].

When the contact set is discrete we obtain some information about the excursion away from the α-Lipschitz minorant that contains the time zero in Section 2.4 using ideas from [61]. If G is the last contact time before zero and D, as above, is the first contact time after zero, we show that $\frac{D}{D-G}$ is independent of $(X_t - X_G,\ G \leq t < D)$ and uniformly distributed on $[0, 1]$. This observation allows us to describe the finite-dimensional distributions of $(X_t,\ G \leq t < D)$ in terms of those of $(X_t,\ 0 \leq t < D)$, and we are able to determine the latter explicitly. The argument here is based on a generalization of the fact that if V is a nonnegative random variable, U is uniformly distributed on $[0, 1]$, and U and V are independent, then it is possible to express the distribution of V in terms of that of UV.

As before, write Y_n, $n \in \mathbb{N}$, for the independent, identically distributed sequence of excursions away from the contact set that occur at positive times in the case where the contact set is discrete. When X is Brownian motion with drift β, where $|\beta|< \alpha$ in order for the α-Lipschitz minorant to exist, we establish a path decomposition description for the common distribution of the Y_n in Section 2.5. Using this path decomposition we can determine the distributions of quantities such as the length $T_{n+1} - T_n$ and the the final value $X_{T_{n+1}} - X_{T_n}$. Moreover, if we write Y_0 for the excursion straddling time zero, then we have the "size-biasing" relationship $\mathbb{E}[f(Y_0)] = \mathbb{E}[f(Y_n)(T_{n+1} - T_n)]/\mathbb{E}[T_{n+1} - T_n]$, $n \in \mathbb{N}$, for nonnegative measurable functions f, and this allows us to recover information about the distribution of Y_0 from a knowledge of the common distribution of the "generic" excursions Y_n, $n \in \mathbb{N}$.

As we noted above, the random time D is not a stopping time for the canonical filtration of X. In Section 2.6 we investigate the filtration obtained by enlarging the Brownian filtration in such way that D becomes a stopping time. Martingales for the Brownian filtration become semimartingales in the enlarged filtration and we are able to describe their canonical semimartingale decompositions quite explicitly.

The chapter finishes with two auxiliary sections. Section 2.7 contains some (deterministic) results about the α-Lipschitz minorant construction that are used throughout the chapter. Section 2.8 details two general lemmas about random times for Lévy processes that are used in Section 2.3 and Section 2.6.

2.2 Space-time regenerative systems

Let $\Omega^{\leftrightarrow}$ (resp. $\Omega^{\rightarrow}$) denote the space of càdlàg $\mathbb{R}$-valued paths indexed by $\mathbb{R}$ (resp. $\mathbb{R}_+$). For $t \in \mathbb{R}$, define $\tau_t : \Omega^{\leftrightarrow} \to \Omega^{\rightarrow}$ by

$$(\tau_t(\omega^{\leftrightarrow}))_s := \omega^{\leftrightarrow}_{t+s} - \omega^{\leftrightarrow}_t, s \geq 0.$$

For $t \in \mathbb{R}$ define $x_t : \Omega^{\leftrightarrow} \to \mathbb{R}$ by

$$x_t(\omega^{\leftrightarrow}) := \omega^{\leftrightarrow}_t.$$

For $t \in \mathbb{R}$, define $k_t : \Omega^{\leftrightarrow} \to \Omega^{\leftrightarrow}$ by

$$(k_t(\omega^{\leftrightarrow}))_s := \begin{cases} \omega^{\leftrightarrow}_s, & \text{if } s \leq t, \\ \omega^{\leftrightarrow}_t, & \text{if } s > t. \end{cases}$$

Let $\tilde{\Omega}^{\leftrightarrow}$ (resp. $\tilde{\Omega}^{\rightarrow}$) denote the class of closed subsets of $\mathbb{R}$ (resp. $\mathbb{R}_+$). For $t \in \mathbb{R}$ define $\tilde{\tau}_t : \tilde{\Omega}^{\leftrightarrow} \to \tilde{\Omega}^{\rightarrow}$ by

$$\tilde{\tau}_t(\tilde{\omega}^{\leftrightarrow}) := \{s - t : s \in \tilde{\omega}^{\leftrightarrow} \cap [t, \infty)\}.$$

For $t \in \mathbb{R}$ define $d_t : \tilde{\Omega}^{\leftrightarrow} \to \mathbb{R} \cup \{+\infty\}$ by

$$d_t(\tilde{\omega}^{\leftrightarrow}) := \inf\{s > t : s \in \tilde{\omega}^{\leftrightarrow}\}$$

and $r_t : \tilde{\Omega}^{\leftrightarrow} \to \mathbb{R}_+ \cup \{+\infty\}$ by

$$r_t(\tilde{\omega}^{\leftrightarrow}) := d_t(\tilde{\omega}^{\leftrightarrow}) - t.$$

With a slight abuse of notation, also use d_t and r_t, $t \in \mathbb{R}_+$, to denote the analogously defined maps from $\tilde{\Omega}^{\rightarrow}$ to $\mathbb{R}_+ \cup \{+\infty\}$.

Put $\bar{\Omega}^{\leftrightarrow} := \Omega^{\leftrightarrow} \times \tilde{\Omega}^{\leftrightarrow}$ and $\bar{\Omega}^{\rightarrow} := \Omega^{\rightarrow} \times \tilde{\Omega}^{\rightarrow}$. Define $\bar{\tau}_t : \bar{\Omega}^{\leftrightarrow} \to \bar{\Omega}^{\rightarrow}$ by

$$\bar{\tau}_t(\omega^{\leftrightarrow}, \tilde{\omega}^{\leftrightarrow}) := (\tau_t(\omega^{\leftrightarrow}), \tilde{\tau}_t(\tilde{\omega}^{\leftrightarrow})).$$

Define $\bar{d}_t : \bar{\Omega}^{\leftrightarrow} \to \mathbb{R} \cup \{+\infty\}$ by

$$\bar{d}_t(\omega^{\leftrightarrow}, \tilde{\omega}^{\leftrightarrow}) := d_t(\tilde{\omega}^{\leftrightarrow}).$$

Finally, for $t \in \mathbb{R}$ define the following σ-fields on $\bar{\Omega}^{\leftrightarrow}$:

$$\bar{\mathcal{G}}_t^{\leftrightarrow} := \sigma\{\bar{d}_s, k_{\bar{d}_s}, s \le t\}$$

and

$$\bar{\mathcal{G}}^{\leftrightarrow} := \sigma\{\bar{d}_s, k_{\bar{d}_s}, s \in \mathbb{R}\}.$$

Define $\bar{\mathcal{G}}_t^{\rightarrow}$ and $\bar{\mathcal{G}}^{\rightarrow}$ analogously.

Definition 2.2.1 *Let $\bar{\mathbb{Q}}^{\leftrightarrow}$ (resp. $\bar{\mathbb{Q}}^{\rightarrow}$) be a probability measure on $(\bar{\Omega}^{\leftrightarrow}, \bar{\mathcal{G}}^{\leftrightarrow})$ (resp. $(\bar{\Omega}^{\rightarrow}, \bar{\mathcal{G}}^{\rightarrow})$). Then $\bar{\mathbb{Q}}^{\leftrightarrow}$ is space-time regenerative with regeneration law $\bar{\mathbb{Q}}^{\rightarrow}$ if*

(i) $\bar{\mathbb{Q}}^{\leftrightarrow}\{\bar{d}_t = +\infty\} = 0$, *for all* $t \in \mathbb{R}$;

(ii) for all $t \in \mathbb{R}$ and for all $\bar{\mathcal{G}}^{\rightarrow}$-measurable nonnegative functions F,

$$\bar{\mathbb{Q}}^{\leftrightarrow}\left[F(\bar{\tau}_{\bar{d}_t}) \,|\, \bar{\mathcal{G}}_{t+}\right] = \bar{\mathbb{Q}}^{\rightarrow}[F],$$

where we write $\bar{\mathbb{Q}}^{\leftrightarrow}[\cdot]$ and $\bar{\mathbb{Q}}^{\rightarrow}[\cdot]$ for expectations with respect to $\bar{\mathbb{Q}}^{\leftrightarrow}$ and $\bar{\mathbb{Q}}^{\rightarrow}$.

Remark 2.2.2 *Suppose that the probability measure $\bar{\mathbb{Q}}^{\leftrightarrow}$ on $(\bar{\Omega}^{\leftrightarrow}, \bar{\mathcal{G}}^{\leftrightarrow})$ is stationary; that is, that under $\bar{\mathbb{Q}}^{\leftrightarrow}$ the process $(\omega^{\leftrightarrow}, \tilde{\omega}^{\leftrightarrow}) \mapsto (x_t(\omega^{\leftrightarrow}), r_t(\tilde{\omega}^{\leftrightarrow}))_{t\in\mathbb{R}}$ has the same distribution as the process $(\omega^{\leftrightarrow}, \tilde{\omega}^{\leftrightarrow}) \mapsto (x_{s+t}(\omega^{\leftrightarrow}) - x_s(\omega^{\leftrightarrow}), r_{s+t}(\tilde{\omega}^{\leftrightarrow}))_{t\in\mathbb{R}}$ for all $s \in \mathbb{R}$. Then, in order to check conditions (i) and (ii) of Definition 2.2.1, it suffices to check them for the case $t = 0$.*

Theorem 2.2.3 *(i) In order to check that the probability measure $\bar{\mathbb{Q}}^{\leftrightarrow}$ on $(\bar{\Omega}^{\leftrightarrow}, \bar{\mathcal{G}}^{\leftrightarrow})$ is space-time regenerative with the probability measure $\bar{\mathbb{Q}}^{\rightarrow}$ on $(\bar{\Omega}^{\rightarrow}, \bar{\mathcal{G}}^{\rightarrow})$ as regeneration law, it suffices to check*

(a) $\bar{\mathbb{Q}}^{\leftrightarrow}\{\bar{d}_t = +\infty\} = 0$, *for all* $t \in \mathbb{R}$;

(b) for all $t \in \mathbb{R}$ and for all $\bar{\mathcal{G}}^{\rightarrow}$-measurable nonnegative functions F,

$$\bar{\mathbb{Q}}^{\leftrightarrow}\left[F(\bar{\tau}_{\bar{d}_t}) \,|\, \bar{\mathcal{G}}_t\right] = \bar{\mathbb{Q}}^{\rightarrow}[F].$$

(ii) Suppose that the probability measure $\bar{\mathbb{Q}}^{\leftrightarrow}$ on $(\bar{\Omega}^{\leftrightarrow}, \bar{\mathcal{G}}^{\leftrightarrow})$ is space-time regenerative with the probability measure $\bar{\mathbb{Q}}^{\rightarrow}$ on $(\bar{\Omega}^{\rightarrow}, \bar{\mathcal{G}}^{\rightarrow})$ as regeneration law and that T is an almost surely finite $(\bar{\mathcal{G}}_{t+}^{\leftrightarrow})_{t\in\mathbb{R}}$-stopping time. Then for all $\bar{\mathcal{G}}^{\rightarrow}$-measurable nonnegative functions F

$$\bar{\mathbb{Q}}^{\leftrightarrow}\left[F(\bar{\tau}_{\bar{d}_T}) \,|\, \bar{\mathcal{G}}_{T+}\right] = \bar{\mathbb{Q}}^{\rightarrow}[F].$$

Proof (i) Fix $t \in \mathbb{R}$. For $n \in \mathbb{N}$ set $t_n := t + 2^{-n}$. Consider $F : \bar{\Omega}^{\rightarrow} \to \mathbb{R}_+$ of the form

$$F((\omega^{\rightarrow}, \tilde{\omega}^{\rightarrow})) = f(\omega^{\rightarrow}_{s_1}, \ldots, \omega^{\rightarrow}_{s_\ell}, r_{s_1}(\tilde{\omega}^{\rightarrow}), \ldots, r_{s_\ell}(\tilde{\omega}^{\rightarrow}))$$

for some $0 \leq s_1 < s_2 < \ldots < s_\ell$ and bounded, continuous function $f : \mathbb{R}^\ell \times (\mathbb{R} \cup \{+\infty\})^\ell \to \mathbb{R}_+$. For such an F we have

$$\lim_{n\to\infty} F(\bar{\tau}_{\bar{d}_{t_n}}(\bar{\omega}^{\leftrightarrow})) = F(\bar{\tau}_{\bar{d}_t}(\bar{\omega}^{\leftrightarrow}))$$

for all $\bar{\omega}^{\leftrightarrow} \in \bar{\Omega}^{\leftrightarrow}$ and it suffices by a monotone class argument to show that

$$\bar{\mathbb{Q}}^{\leftrightarrow}\left[F(\bar{\tau}_{\bar{d}_{t_n}}) \,|\, \bar{\mathcal{G}}^{\leftrightarrow}_{t+}\right] = \bar{\mathbb{Q}}^{\rightarrow}[F]$$

for all $n \in \mathbb{N}$. This, however, is clear because $\bar{\mathcal{G}}^{\leftrightarrow}_{t+} \subseteq \bar{\mathcal{G}}^{\leftrightarrow}_{t_n}$ and

$$\bar{\mathbb{Q}}^{\leftrightarrow}\left[F(\bar{\tau}_{\bar{d}_{t_n}}) \,|\, \bar{\mathcal{G}}^{\leftrightarrow}_{t_n}\right] = \bar{\mathbb{Q}}^{\rightarrow}[F]$$

by assumption.
(ii) For $n \in \mathbb{N}$ define a $(\bar{\mathcal{G}}^{\leftrightarrow}_t)_{t\in\mathbb{R}}$-stopping time T_n by declaring that $T_n := \frac{k}{2^n}$ when $T \in [\frac{k-1}{2^n}, \frac{k}{2^n})$, $k \in \mathbb{Z}$. Let F be as in the proof of part (i). For such an F we have

$$\lim_{n\to\infty} F(\bar{\tau}_{\bar{d}_{T_n}}(\bar{\omega}^{\leftrightarrow})) = F(\bar{\tau}_{\bar{d}_T}(\bar{\omega}^{\leftrightarrow}))$$

for all $\bar{\omega}^{\leftrightarrow} \in \bar{\Omega}^{\leftrightarrow}$ and it suffices by a monotone class argument to show that

$$\bar{\mathbb{Q}}^{\leftrightarrow}\left[F(\bar{\tau}_{\bar{d}_{T_n}}) \,|\, \bar{\mathcal{G}}^{\leftrightarrow}_{T+}\right] = \bar{\mathbb{Q}}^{\rightarrow}[F]$$

for all $n \in \mathbb{N}$. Since $\bar{\mathcal{G}}_{T+} \subseteq \bar{\mathcal{G}}_{T_n+}$ for all $n \in \mathbb{N}$, it further suffices to show that

$$\bar{\mathbb{Q}}^{\leftrightarrow}\left[F(\bar{\tau}_{\bar{d}_{T_n}}) \,|\, \bar{\mathcal{G}}^{\leftrightarrow}_{T_n+}\right] = \bar{\mathbb{Q}}^{\rightarrow}[F].$$

Fix $n \in \mathbb{N}$ and suppose that G is a nonnegative $\bar{\mathcal{G}}^{\leftrightarrow}_{T_n+}$-measurable random variable. We have

$$\begin{aligned}
\bar{\mathbb{Q}}^{\leftrightarrow}\left[F(\bar{\tau}_{\bar{d}_{T_n}})\, G\right] &= \sum_{k\in\mathbb{Z}} \bar{\mathbb{Q}}^{\leftrightarrow}\left[F(\bar{\tau}_{\bar{d}_{T_n}})\, G\, \mathbb{1}\left\{T_n = \frac{k}{2^n}\right\}\right] \\
&= \sum_{k\in\mathbb{Z}} \bar{\mathbb{Q}}^{\leftrightarrow}\left[F(\bar{\tau}_{\bar{d}_{\frac{k}{2^n}}})\, G\, \mathbb{1}\left\{T_n = \frac{k}{2^n}\right\}\right] \\
&= \bar{\mathbb{Q}}^{\rightarrow}[F] \sum_{k\in\mathbb{Z}} \bar{\mathbb{Q}}^{\leftrightarrow}\left[G\, \mathbb{1}\left\{T_n = \frac{k}{2^n}\right\}\right] \\
&= \bar{\mathbb{Q}}^{\rightarrow}[F]\, \bar{\mathbb{Q}}^{\leftrightarrow}[G],
\end{aligned}$$

where in the penultimate equality we used the fact that $G\,\mathbb{1}\{T_n = \frac{k}{2^n}\}$ is $\bar{\mathcal{G}}^{\leftrightarrow}_{\frac{k}{2^n}}$-measurable (see, for example, [30, Lemma 7.1(ii)]). This completes the proof. □

Theorem 2.2.4 *Suppose for the Lévy process $(X_t + \alpha t)_{t\in\mathbb{R}}$ that 0 is regular for $(0,\infty)$. Then the distribution of $((X_t)_{t\in\mathbb{R}}, \mathcal{Z})$ is space-time regenerative.*

Proof. Use Theorem 2.3.6 below, Remark 2.2.2, and part (i) of Theorem 2.2.3. □

Remark 2.2.5 *If* 0 *is not regular for* $(0,\infty)$ *for the Lévy process* $(X_t+\alpha t)_{t\in\mathbb{R}}$*, then* 0 *is regular for* $(-\infty,0)$ *for the Lévy process* $(X_t-\alpha t)_{t\in\mathbb{R}}$*. Equivalently, if* 0 *is not regular for* $(0,\infty)$ *for the Lévy process* $(X_t+\alpha t)_{t\in\mathbb{R}}$*, then* 0 *is regular for* $(0,\infty)$ *for the Lévy process* $(-X_t+\alpha t)_{t\in\mathbb{R}}$ *and hence for the Lévy process* $(X_{-t-}+\alpha t)_{t\in\mathbb{R}}$*. Thus, either the distribution of* $((X_t)_{t\in\mathbb{R}},\mathcal{Z})$ *is space-time regenerative or the distribution of* $((X_{-t-})_{t\in\mathbb{R}},\mathcal{Z})$ *is space-time regenerative.*

Write $\tilde{\pi}^{\leftrightarrow}:\bar{\Omega}^{\leftrightarrow}\to\tilde{\Omega}^{\leftrightarrow}$ for the projection $\bar{\omega}^{\leftrightarrow}=(\omega^{\leftrightarrow},\tilde{\omega}^{\leftrightarrow})\mapsto\tilde{\omega}^{\leftrightarrow}$. Define $\tilde{\pi}^{\rightarrow}:\bar{\Omega}^{\rightarrow}\to\tilde{\Omega}^{\rightarrow}$ similarly. If $\bar{\mathbb{Q}}^{\leftrightarrow}$ is space-time regenerative with regeneration law $\bar{\mathbb{Q}}^{\rightarrow}$, then, in the sense of [22], the push-forward of $\bar{\mathbb{Q}}^{\leftrightarrow}$ by $\tilde{\pi}^{\leftrightarrow}$ is regenerative with regeneration law the push-forward of $\bar{\mathbb{Q}}^{\rightarrow}$ by $\tilde{\pi}^{\rightarrow}$. It follows that $\bar{\mathbb{Q}}^{\rightarrow}\{(\omega^{\rightarrow},\tilde{\omega}^{\rightarrow}):\tilde{\omega}^{\rightarrow}\text{ is discrete}\}$ is either 1 or 0.

Suppose that the probability in question is 1. Define $(\bar{\mathcal{G}}_t^{\leftrightarrow})_{t\in\mathbb{R}}$-stopping times $T_1,T_2,\dots$ with $0<T_1<T_2<\dots$ almost surely by

$$T_1:=\bar{d}_0$$

and

$$T_{n+1}(\bar{\omega}^{\leftrightarrow}):=\bar{d}_{T_n(\bar{\omega}^{\leftrightarrow})}(\bar{\omega}^{\leftrightarrow})=T_n(\bar{\omega}^{\leftrightarrow})+\bar{d}_0\circ\bar{\theta}_{T_n(\bar{\omega}^{\leftrightarrow})}(\bar{\omega}^{\leftrightarrow}),n\in\mathbb{N},$$

where $\bar{\theta}_t:\bar{\Omega}^{\leftrightarrow}\to\bar{\Omega}^{\leftrightarrow}$, $t\in\mathbb{R}$, are the shift maps given by $\bar{\theta}(\omega^{\leftrightarrow},\tilde{\omega}^{\leftrightarrow})=((\omega^{\leftrightarrow}_{t+u})_{u\in\mathbb{R}},\tilde{\omega}^{\leftrightarrow}-t)$. Let ∂ be an isolated cemetery state adjoined to $\mathbb{R}$. Define càdlàg $\mathbb{R}\cup\{\partial\}$-valued processes $Y^n=(Y_t^n)_{t\in\mathbb{R}_+}$, $n\in\mathbb{N}$, by

$$Y_t^n(\omega^{\leftrightarrow},\tilde{\omega}^{\leftrightarrow}):=\begin{cases}\pi^{\rightarrow}\circ\tau_{\bar{T}_n(\omega^{\leftrightarrow},\tilde{\omega}^{\leftrightarrow})}(\omega^{\leftrightarrow},\tilde{\omega}^{\leftrightarrow})_t, & 0\le t<\bar{d}_0\circ\bar{\theta}_{T_n(\bar{\omega}^{\leftrightarrow})}(\bar{\omega}^{\leftrightarrow})=\zeta_n,\\ \partial, & t\ge\bar{d}_0\circ\bar{\theta}_{T_n(\bar{\omega}^{\leftrightarrow})}(\bar{\omega}^{\leftrightarrow})=\zeta_n,\end{cases}$$

where $\pi^{\rightarrow}:\bar{\Omega}^{\rightarrow}\to\Omega^{\rightarrow}$ is the projection $(\omega^{\rightarrow},\tilde{\omega}^{\rightarrow})\mapsto\omega^{\rightarrow}$. Then, under $\bar{\mathbb{Q}}^{\leftrightarrow}$, the sequence Y^n, $n\in\mathbb{N}$, is independent and identically distributed.

The path of of each Y_n lies in the set $\Omega^{0,\partial}$ consisting of càdlàg functions $f:\mathbb{R}_+\to\mathbb{R}\cup\{\partial\}$ such that $f(0)=0$, $0<\inf\{s\ge0:f(s)=\partial\}<\infty$, and $f(t)=\partial$ for all $t\ge\inf\{s\ge0:f(s)=\partial\}$.

When the probability in question is 0 there is a local time on our regenerative set and we can construct a Poisson random measure on the set $\mathbb{R}\times\Omega^{0,\partial}$ that records the excursions away from the contact set and the order in which they occur. We use the following theorem which is a restatement of [24, Corollary 3.1].

Theorem 2.2.6 *Let* $(O_k)_{k\in\mathbb{N}}$ *be an increasing family of measurable sets in a measurable space* $(O,\mathcal{O})$ *such that* $O=\bigcup_{k\in\mathbb{N}}O_k$*. Let* $\mathbf{V}$ *be an* O*-valued point process; that is,* $\mathbf{V}=(\mathbf{V}_t)_{t\ge0}$ *is a stochastic process with values in* $O\cup\{\dagger\}$ *for some adjoined point* $\dagger$ *such that* $\{t\ge0:\mathbf{V}_t\ne\dagger\}$ *is almost surely countable. Suppose that* $\{t\ge0:\mathbf{V}_t\in O_k\}$ *is almost surely discrete and unbounded for all* $k\in\mathbb{N}$ *while* $\{t\ge0:\mathbf{V}_t\in O\}$ *is almost surely not discrete. Suppose that the sequence* $\{\mathbf{V}_t:\mathbf{V}_t\in O_k\}$ *is independent and identically distributed for each* $k\in\mathbb{N}$*. For* $k\in\mathbb{N}$ *define* $(N_k(t))_{t\ge0}$ *by setting* $N_k(t)=\#\{0\le u\le t:\mathbf{V}_u\in O_k\}$ *for* $t\ge0$*.*

For $k \in \mathbb{N}$ set $T_k = \inf\{t > 0 : \mathbf{V}_t \in O_k\}$ and put $p_k = \mathbb{P}\{\mathbf{V}_{T_k} \in O_1\}$. Then, for almost all $\omega \in \Omega$, uniformly for bounded $t \geq 0$,

$$\lim_{k \to \infty} p_k N_k(t, \omega) = L(t, \omega),$$

where $L(t, \omega)$ is continuous and nondecreasing in $t \geq 0$, and strictly increasing on $\{t \geq 0 : \mathbf{V}(t, \omega) \neq \dagger\}$. For such ω, set

$$\mathbf{V}^*(s, \omega) = \begin{cases} V(t, \omega), & \text{if } s = L(t, \omega), \\ \dagger, & \text{otherwise.} \end{cases}$$

Then $\mathbf{V}^*$ is a homogeneous Poisson point process; that is, the random measure that puts mass 1 at each point $(t, f) \in \mathbb{R}_+ \times O$ such that $\mathbf{V}^*_t = f$ is a Poisson random measure with intensity of the form $\lambda \otimes \nu$, where λ is Lebesgue measure on $\mathbb{R}_+$ and ν is a σ-finite measure on $(O, \mathcal{O})$. Moreover, for almost all $\omega \in \Omega$ for all t with $\mathbf{V}(t, \omega) \neq \dagger$, $\mathbf{V}(t, \omega) = \mathbf{V}^*(L(t, \omega), \omega)$.

We can apply this theorem if we consider O to be the space $\Omega^{0,\partial}$ of càdlàg paths that vanish at the origin and have finite lifetimes, and take O_k to be the subspace of paths with lifetime at least $\frac{1}{k}$. We define $\mathbf{V}$ to be the point process of the excursions such that, for every $t \geq 0$, $\mathbf{V}_t$ is equal to the excursion whose right end point is t, with the convention that $\mathbf{V}_t = \dagger$ if t is not the right end point of an excursion. In the case where $\mathcal{Z}$ is not discrete, all the conditions of Theorem 2.2.6 can readily be checked and we obtain a time-changed Poisson point process.

2.3 The process after the first positive point in the contact set

Notation 2.3.1 *For $t \in \mathbb{R}$ set $G_t := \sup(\mathcal{Z} \cap (-\infty, t))$ and $D_t := \inf(\mathcal{Z} \cap (t, +\infty))$. Put $G := G_0$ and $D := D_0$.*

Remark 2.3.2 *We have from Lemma 2.7.3 that*

$$D = \inf\{t \geq S : X_t \wedge X_{t-} + \alpha t = \inf\{X_u + \alpha u : u \geq S\}\},$$

where

$$S = S_0 := \inf\{s > 0 : X_s \wedge X_{s-} - \alpha s \leq \inf\{X_u - \alpha u : u \leq 0\}\}$$

because almost surely $X_S \leq X_{S-}$. The latter result was shown in the proof of [2, Theorem 2.6].

Notation 2.3.3 *For $t \in \mathbb{R}$, put $\mathcal{F}_t = \bigcap_{\epsilon > 0} \sigma\{X_s : -\infty < s \leq t + \epsilon\}$. Define the σ-field $\mathcal{F}_U$ for any nonnegative random time U to be the σ-field generated by all the random variables of the form ξ_U where $(\xi_t)_{t \in \mathbb{R}}$ is an optional process with respect to the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}}$. Similarly, define $\mathcal{F}_{U-}$ to be the σ-field generated by all the random variables of the form ξ_U where $(\xi_t)_{t \in \mathbb{R}}$ is now a previsible process with respect to the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}}$.*

Notation 2.3.4 *Let $\tilde{X} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{X}_t, \tilde{\theta}_t, \tilde{\mathbb{P}}^x)$ be a Hunt process such that the distribution of $\tilde{X}$ under $\tilde{\mathbb{P}}^x$ is that of $(x + X_t + \alpha t)_{t\geq 0}$. Put $\breve{T} := \inf\{t > 0 : \tilde{X}_s \wedge \tilde{X}_{s-} < 0\}$, and for $t \geq 0$ and $x, y > 0$ put*

$$\breve{H}_t(x, dy) := \tilde{\mathbb{P}}^x\{\tilde{X}_t \in dy, t < \breve{T}\} \frac{\tilde{\mathbb{P}}^y\{\breve{T} = \infty\}}{\tilde{\mathbb{P}}^x\{\breve{T} = \infty\}}.$$

Remark 2.3.5 *Write $\mathbb{P}^{\uparrow}_x$, $x > 0$, for the distribution on the Skorokhod space of càdlàg $[0, \infty)$-valued paths of a Markov process with transition functions $(\breve{H}_t)_{t\geq 0}$. It is shown in [18, Theorem 2] that if 0 is regular for $(0, \infty)$ for the Markov process $\tilde{X}$, then $\mathbb{P}^{\uparrow}_x$ converges weakly as $x \downarrow 0$ to a distribution $\mathbb{P}^{\uparrow}$ that assigns all of its mass to paths with initial value 0. The distribution $\mathbb{P}^{\uparrow}$ is the distribution of a Markov process on $[0, \infty)$ with an enlarged semigroup which, with a slight abuse of notation, we denote $\breve{H}_t(x, dy)$, $t \geq 0$, $x, y \geq 0$ for suitably defined $\breve{H}_t(0, \cdot)$, $t \geq 0$. We interpret (the enlarged semigroup) $(\breve{H}_t)_{t\geq 0}$ as the semigroup of the Markov process $\tilde{X}$ conditioned to stay positive. The semigroup $(\breve{H}_t)_{t\geq 0}$ is Feller on $C_0([0, \infty))$, the space of continuous functions on $[0, \infty)$ vanishing at infinity.*

Theorem 2.3.6 *Suppose that 0 is regular for $(0, \infty)$ for the Markov process $\tilde{X}$. Then the process $(X_{t+D} - X_D)_{t\geq 0}$ is independent of $(X_t, -\infty < t \leq D)$. Moreover, the process $(X_{t+D} - X_D + \alpha t)_{t\geq 0}$ is Markovian and has the distribution $\mathbb{P}^{\uparrow}$ of the process $(X_t + \alpha t)_{t\geq 0}$ conditioned to stay positive.*

Proof. Because $(X_t)_{t\in\mathbb{R}}$ is a two-sided Lévy process and S is a stopping time, the process $\breve{X} := (X_{t+S} - X_S + \alpha t)_{t\geq 0}$ is, by the strong Markov property, independent of $\mathcal{F}_S$ and has the same distribution as the process $\tilde{X}$ under $\tilde{\mathbb{P}}^0$.

Suppose that $\tilde{X}$ is not a compound Poisson process with drift (that is, that either $\sigma > 0$ or $\Pi(\mathbb{R}) = \infty$). Then, by [38, Proposition 2.2], the set $\{t \geq 0 : \tilde{X}_t \wedge \tilde{X}_{t-} = \inf\{\tilde{X}_s : s \geq 0\}\}$ consists $\tilde{\mathbb{P}}^x$-almost surely of a single point $\tilde{T}$ for all $x \in \mathbb{R}$. If $\tilde{X}$ is a compound Poisson process with drift, then, by the assumption that 0 is regular for $(0, \infty)$, the drift must be strictly positive and in this case an easy argument based on the fact that the times between jumps are independent exponential random variables shows that the conclusion of the previous sentence still holds. Consequently, in either case the set $\{t \geq 0 : \breve{X}_t \wedge \breve{X}_{t-} = \inf\{\breve{X}_s : s \geq 0\}\}$ also consists almost surely of a single point $\breve{T}$. From Remark 2.3.2 we have $D = S + \breve{T}$.

Because 0 is regular for $(0, \infty)$ for the Markov process $\tilde{X}$, it follows from that [38, Proposition 2.4] $\tilde{X}_{\tilde{T}} = \inf\{\tilde{X}_s : s \geq 0\}$. (That result is given under a blanket assumption that the Lévy process in question is not a compound Poisson process with drift but it is straightforward to see that the same argument applies to give the result for the latter class of Lévy processes.)

The sole theorem in [37] gives that the process $(\tilde{X}_{\tilde{T}+t})_{t\geq 0}$ is independent of $\tilde{\mathcal{F}}_{\tilde{T}}$ given $\tilde{X}_{\tilde{T}}$. Moreover, there exists a family of entrance laws $(Q_t(x; \cdot))_{t\geq 0}$ for each $x \in \mathbb{R}$ and a family of transition functions $(H_t(x; \cdot, \cdot))_{t\geq 0}$ for each $x \in \mathbb{R}$ such that

$$\tilde{\mathbb{P}}^x\{\tilde{X}_{t+\tilde{T}} \in A \,|\, \tilde{\mathcal{F}}_{\tilde{T}}\} = Q_t(\tilde{X}_{\tilde{T}}; A), t \geq 0,$$

and

$$\tilde{\mathbb{P}}^x\{\tilde{X}_{t+\tilde{T}} \in A \,|\, \tilde{\mathcal{F}}_{\tilde{T}+s}\} = H_{t-s}(\tilde{X}_{\tilde{T}}; \tilde{X}_{\tilde{T}+s}, A), 0 < s < t.$$

Using the fact that the processes $(x+\tilde{X}_{t+\tilde{T}})_{t\geq 0}$ under $\tilde{\mathbb{P}}^0$ and $(\tilde{X}_{t+\tilde{T}})_{t\geq 0}$ under $\tilde{\mathbb{P}}^x$ have the same law, it follows that $Q_t(x; x+A) = Q_t(0, A)$ and $H_t(x; x+y, x+A) = H_t(0; y, A)$. Thus the process $(\tilde{X}_{t+\tilde{T}} - \tilde{X}_{\tilde{T}})_{t\geq 0}$ is independent of $\tilde{\mathcal{F}}_{\tilde{T}}$ and, moreover, this process is Markovian with the entrance law $A \mapsto Q_t(0; A) =: \check{Q}_t(A)$, $t \geq 0$, and transition functions $(y, A) \mapsto H_t(0; y, A) = \check{H}_t(y, A)$, $t \geq 0$. Applying Lemma 2.8.2 we get that $(X_{t+D} - X_D + \alpha t)_{t\geq 0} = (\check{X}_{t+\check{T}} - \check{X}_{\check{T}})_{t\geq 0}$ is independent of $\mathcal{F}_{D-} \vee \sigma\{X_D\}$ and Markovian with transition functions $(\check{H}_t)_{t\geq 0}$ and entrance laws $(\check{Q}_t)_{t\geq 0}$.

By the right-continuity of $(\tilde{X}_{t+\tilde{T}} - \tilde{X}_{\tilde{T}})_{t\geq 0}$, the probability measures $\check{Q}_t$ converge weakly to the point mass at 0 as $t \downarrow 0$. It follows from the Feller property of the semigroup $(\check{H}_t)_{t\geq 0}$ noted in Remark 2.3.5 that for $f \in C_0([0,\infty))$

$$\check{Q}_t f = \check{Q}_s \check{H}_{t-s} f = \lim_{s\downarrow 0} \check{Q}_s \check{H}_{t-s} f = \check{H}_t f(0),$$

so that $\check{Q}_t = \check{H}_t(0, \cdot)$ and hence $(X_{t+D} - X_D + \alpha t)_{t\geq 0} = (\check{X}_{t+\check{T}} - \check{X}_{\check{T}})_{t\geq 0}$ has distribution $\mathbb{P}^{\uparrow}$.

Introduce the killed process $(\bar{X}_t)_{t\in\mathbb{R}}$ defined by

$$\bar{X}_t := \begin{cases} X_t, & t < D, \\ X_D, & t = D, \\ \partial, & t > D, \end{cases}$$

where ∂ is an adjoined isolated point. To complete the proof, it suffices to show that $\sigma\{X_t, -\infty < t \leq D\} \equiv \sigma\{\bar{X}_t, t \in \mathbb{R}\} \subseteq \mathcal{F}_{D-} \vee \sigma\{X_D\}$; that is, that $\bar{X}_t$ is $\mathcal{F}_{D-} \vee \sigma\{X_D\}$-measurable for all $t \in \mathbb{R}$. For all $u \in \mathbb{R}$ the process $(\mathbb{1}_{\{s>u\}})_{s\in\mathbb{R}}$ is left-continuous, right-limited, and $(\mathcal{F}_s)_{s\in\mathbb{R}}$-adapted. Therefore, for all $u \in \mathbb{R}$, the random variable $\mathbb{1}_{\{D>u\}}$ is $\mathcal{F}_{D-}$-measurable and so the random variable D is $\mathcal{F}_{D-}$-measurable. In particular, the event $\{t > D\}$ is $\mathcal{F}_{D-}$-measurable. Next, the process $(X_t \mathbb{1}_{t<s})_{s\in\mathbb{R}}$ is also left-continuous, right-limited, and $(\mathcal{F}_s)_{s\in\mathbb{R}}$-adapted and hence the random variable $X_t \mathbb{1}_{\{t<D\}}$ is $\mathcal{F}_{D-}$-measurable. Consequently, for any Borel subset $A \subseteq \mathbb{R} \cup \{\partial\}$ we have

$$\begin{aligned} &\{\bar{X}_t \in A\} \\ &= (\{\bar{X}_t \in A\} \cap \{t < D\}) \cup (\{\bar{X}_t \in A\} \cap \{t = D\}) \cup (\{\bar{X}_t \in A\} \cap \{t > D\}) \\ &= \begin{cases} (\{X_t \mathbb{1}_{\{t<D\}} \in A\} \cap \{t < D\}) \cup (\{X_D \in A\} \cap \{t = D\}) \cup \{t > D\}, & \partial \in A, \\ (\{X_t \mathbb{1}_{\{t<D\}} \in A\} \cap \{t < D\}) \cup (\{X_D \in A\} \cap \{t = D\}), & \partial \notin A, \end{cases} \\ &\in \mathcal{F}_{D-} \vee \sigma\{X_D\}, \end{aligned}$$

as claimed.

2.4 The excursion straddling zero

In this section we focus on the excursion away from the contact set that straddles the time zero; that is, the piece of the path of X between the times G and D of Notation 2.3.1.

The following proposition gives an explicit path decomposition for, and hence the distribution of, the process $(X_u,\ 0 \leq u \leq D)$.

Proposition 2.4.1 *Set*
$$I^- := \inf\{X_u - \alpha u : u \le 0\}.$$
Consider the following independent random objects :

- *A random variable* Γ *with the same distribution as* I^-*,*
- $(X'_t)_{t\ge 0}$ *and* $(X''_t)_{t\ge 0}$ *two independent copies of* $(X_t)_{t\ge 0}$*.*

Define the process $(\mathfrak{Z}_t)_{t\ge 0}$ *by*

$$\mathfrak{Z}_t := \begin{cases} X'_t, & 0 \le t \le T'_\Gamma, \\ X''_{t-T'_\Gamma} + X'_{T'_\Gamma}, & T'_\Gamma \le t \le T'_\Gamma + \tilde{T}'', \\ \partial, & t > T'_\Gamma + \tilde{T}'', \end{cases}$$

where
$$T'_\Gamma := \inf\{t \ge 0 : X'_t \wedge X'_{t-} - \alpha t \le \Gamma\}$$
and
$$\tilde{T}'' := \inf\{t \ge 0 : X''_t \wedge X''_{t-} + \alpha t = \inf\{X''_u + \alpha u : u \ge 0\}\}.$$
Then,
$$(X_t,\, 0 \le t \le D) \stackrel{d}{=} (\mathfrak{Z}_t,\, 0 \le t \le T'_\Gamma + \tilde{T}'').$$

Proof. The path decomposition follows from the construction of the pobZ S and D in Remark 2.3.2. The proof is left to the reader. □

Now that we have the distribution of the path of X on $[0, D]$, let us extend it to the whole interval $[G, D]$. First of all, we will prove that the random variable $U = \frac{-G}{D-G}$ is independent of the straddling excursion $(X_{t+G} - X_G,\, 0 \le t \le D - G)$ and has a uniform distribution on the interval $(0, 1]$.

Our approach here uses ideas from [61, Chapter 8] but with a modification of the particular shift operator considered there; see also [6] for a framework with general shift operators that encompasses the setting we work in. There is a large literature in this area of general Palm theory that is surveyed in [61, 6] but we mention [39, 46] as being of particular relevance.

We will prove general results for the path space $(H, \mathcal{H})$ and sequence space $(L, \mathcal{L})$ defined by
$$H := \{(z_t)_{t\in\mathbb{R}} : z \text{ is real-valued and càdlàg with } z(0) = 0\}$$
and
$$L := \{(s_k)_{k\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}} : -\infty \leftarrow \cdots < s_{-1} < 0 \le s_0 < s_1 < \ldots \to \infty\}.$$
We take $\mathcal{H}$ to be the σ-field on H that makes all of the maps $z \mapsto z_t$, $t \in \mathbb{R}$, measurable, and $\mathcal{L}$ to be the trace of the product σ-field on L.

For $t \in \mathbb{R}$ define the *shift* $\theta_t : H \times L \to H \times L$ by

$$\theta_t((z_s)_{s\in\mathbb{R}}, (s_k)_{k\in\mathbb{Z}}) = ((z_{t+s} - z_t)_{s\in\mathbb{R}}, (s_{n_t+k} - t)_{k\in\mathbb{Z}})$$

where $n_t = n$ if and only if $t \in [s_{n-1}, s_n)$. The family $(\theta_t)_{t\in\mathbb{R}}$ is measurable in the sense that the mapping

$$((z_s)_{s\in\mathbb{R}}, (s_k)_{k\in\mathbb{Z}}, t) \in H \times L \times \mathbb{R} \mapsto \theta_t((z_s)_{s\in\mathbb{R}}, (s_k)_{k\in\mathbb{Z}}) \in H \times L$$

is $\mathcal{H} \otimes \mathcal{L} \otimes \mathcal{B}/\mathcal{H} \otimes \mathcal{L}$ measurable, where $\mathcal{B}$ is the Borel σ-field on $\mathbb{R}$.

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a random pair (K, P) that take values on $H \times L$. We assume furthermore that (K, P) is *space-homogeneous stationary* in the sense that

$$\theta_s(K, P) \stackrel{d}{=} (K, P), \text{ for all } s \in \mathbb{R}.$$

Remark 2.4.2 *When $\mathcal{Z}$ is discrete, the space-time regenerative system $((X_t)_{t\in\mathbb{R}}, \mathcal{Z})$ is obviously space-homogeneous stationary due to the two facts that for any $s \in \mathbb{R}$ we have $(X_{t+s} - X_s)_{t\in\mathbb{R}} \stackrel{d}{=} (X_t)_{t\in\mathbb{R}}$ and that the contact set for $(X_{t+s} - X_s)_{t\in\mathbb{R}}$ is, by Lemma 2.7.1, just $\mathcal{Z} - s$.*

Definition 2.4.3

- *Write l_n for the n^{th} cycle length defined by $l_n = P_n - P_{n-1}$.*
- *For $t \in \mathbb{R}$, put $N_t = n$ for $t \in [P_{n-1}, P_n)$.*
- *Define the relative position of t in $[P_{N_t-1}, P_{N_t})$ by $U_t := \frac{t-P_{N_t-1}}{P_{N_t}}$.*
- *Define the random variable (K°, P°) by*

$$(K^\circ, P^\circ) = \theta_{P_0}(K, P) = ((K_{t+P_0} - K_{P_0})_{t\in\mathbb{R}}, (P_k - P_0)_{k\in\mathbb{Z}}).$$

The following are two important features of the family $(\theta_t)_{t\in\mathbb{R}}$ that are useful in proving results analogous to those in [61, Chapter 8, Section 3].

Proposition 2.4.4 *The family of shifts $(\theta_t)_{t\in\mathbb{R}}$ enjoys the two following properties.*

(i) The family $(\theta_t)_{t\in\mathbb{R}}$ is semigroup; that is, for every $t, s \in \mathbb{R} : \theta_t \circ \theta_s = \theta_{t+s}$.

(ii) For all $s \in \mathbb{R}$ and $(K, P) \in H \times L$ we have $\theta_s(K, P)^\circ = \theta_{N_s}(K, P)$.

Proof. For all $t, s \in \mathbb{R}$, and $(K, P) \in H \times L$

$$\begin{aligned}\text{Proj}_H[(K_t \circ \theta_s(K, P))]_u &= \theta_s(K)_{u+t} - \theta_s(K)_t \\ &= (K_{u+t+s} - K_s) - (K_{t+s} - K_s) \\ &= K_{u+t+s} - K_{t+s} \\ &= (\theta_{t+s}(K))_u \\ &= \text{Proj}_H[\theta_{t+s}(K, P)]_u\end{aligned}$$

where Proj_H is the projection from $H \times L$ to H. The proof for the action of the shift on the sequence component is given in [61, Chapter 8, Section 2].

We prove (ii) in a similar manner. We have

$$\begin{aligned}
\theta_s(K,P)^\circ &= ((\theta_s(K)_{t+\theta_s(P)_0} - \theta_s(K)_{\theta_s(P)_0})_{t\in\mathbb{R}}, (P_{N_s+k} - P_{N_s})_{k\in\mathbb{Z}}) \\
&= (((K_{t+s+\theta_s(P)_0} - K_s) - (K_{\theta_s(P)_0+s} - K_s))_{t\in\mathbb{R}}, (P_{N_s+k} - P_{N_s})_{k\in\mathbb{Z}}) \\
&= ((K_{t+s+P_{N_s}-s} - K_{s+P_{N_s}-s})_{t\in\mathbb{R}}, (P_{N_s+k} - P_{N_s})_{k\in\mathbb{Z}}) \\
&= ((K_{t+P_{N_s}} - K_{P_{N_s}}, (P_{N_s+k} - P_{N_s})_{k\in\mathbb{Z}}) \\
&= \theta_{P_{N_s}}(K,P)
\end{aligned}$$

□

We state now a theorem that is analogous to parts of [61, Chapter 8, Theorem 3.1]. The proof uses the same key ideas as that result and just exploits the two properties of the family of shifts laid out in Proposition 2.4.4.

Theorem 2.4.5 *The random variable U_0 is uniform on $[0,1)$ and is independent of (K°, P°). Also,*

$$\mathbb{E}\left[\sum_{k=1}^{N_t} f(\theta_{P_k}(K,P))\right] = t\mathbb{E}\left[\frac{f(K^\circ,P^\circ)}{l_0}\right].$$

*Proof.*Consider a nonnegative Borel function g on $[0,1)$ and a nonnegative $\mathcal{H}\otimes\mathcal{L}$-measurable function f. To establish both claims of the theorem, it suffices to prove that

$$t\mathbb{E}\left[g(U_0)\frac{f(K^\circ,P^\circ)}{l_0}\right] = \left(\int_0^1 g(x)\,dx\right)\mathbb{E}\left[\sum_{k=1}^{N_t} f(\theta_{P_k}(K,P))\right]. \tag{2.2}$$

By stationarity, the left-hand side of equation (2.2) is

$$t\mathbb{E}\left[g(U_0)\frac{f(K^\circ,P^\circ)}{l_0}\right] = \int_0^t \mathbb{E}\left[\theta_s\left[g(U_0)\frac{f(K^\circ,P^\circ)}{l_0}\right]\right]\,ds$$

Because $\theta_s(U_0) = U_s$ and $\theta_s(l_0) = l_{N_s}$, and using the fact that $\theta_s(K,P)^\circ = \theta_{N_s}(K,P)$. We have

$$\begin{aligned}
t\mathbb{E}\left[g(U_0)\frac{f(K^\circ,P^\circ)}{X_0}\right] &= \int_0^t \mathbb{E}\left[g(U_s)\frac{f(\theta_{P_{N_s}}(K,P)}{l_{N_s}}\right]\,ds \\
&= \mathbb{E}\left[\sum_{k=1}^{N_t}\int_{P_{k-1}}^{P_k} \frac{g(U_s)f(\theta_{P_{N_s}}(K,P))}{l_{N_s}}\,ds\right] \\
&+ \mathbb{E}\left[\int_0^{P_0} g(U_s)\frac{f(\theta_{P_{N_s}}(K,P)}{l_{N_s}}ds\right] \\
&- \mathbb{E}\left[\int_t^{P_{N_t}} g(U_s)\frac{f(\theta_{P_{N_s}}(K,P)}{l_{N_s}}ds\right] \\
&= \mathbb{E}\left[\sum_{k=1}^{N_t} f(\theta_k(K,P))\int_{P_{k-1}}^{P_k}\frac{g(U_s)}{l_k}\,ds\right].
\end{aligned}$$

It follows from stationarity that

$$\mathbb{E}\left[\int_0^{P_0} g(U_s)\frac{f(\theta_{P_{N_s}}(K,P)}{l_{N_s}}ds\right] = \mathbb{E}\left[\int_t^{P_{N_t}} g(U_s)\frac{f(\theta_{P_{N_s}}(K,P)}{l_{N_s}}ds\right].$$

A change of variable in the integral shows that

$$\int_{P_{k-1}}^{P_k} \frac{g(U_s)}{l_k}\,ds = \int_0^{l_k} \frac{g(\frac{s}{l_k})}{l_k}\,ds = \int_0^1 g(x)\,dx,$$

and this proves the claim (2.2). □

The next result is the analogue of the identity [61, Chapter 8, (4.5)] with $g \equiv 1$.

Corollary 2.4.6 *For any nonnegative $\mathcal{H}\otimes\mathcal{L}$- measurable function f and every $n \in \mathbb{Z}$, we have*

$$\mathbb{E}\left[\frac{f(\theta_{P_n}(K,P))}{l_0}\right] = \mathbb{E}\left[\frac{f(K^\circ,P^\circ)}{l_0}\right].$$

Proof. It suffices to consider the case when f is bounded by a constant A. Applying Theorem 2.2 with the function f replaced by $f\circ\theta_{P_n}$, we have, for all $t \geq 0$,

$$\begin{aligned}
t\mathbb{E}\left[\frac{f(\theta_{P_n}(K,P))}{l_0}\right] &= \mathbb{E}\left[\sum_{k=1}^{N_t} f(\theta_{P_{k+n}}(K,P))\right] \\
&= \mathbb{E}\left[\sum_{k=1}^{N_t} f(\theta_{P_k}(K,P))\right] - \mathbb{E}\left[\sum_{k=1}^{n} f(\theta_{P_k}(K,P))\right] \\
&+ \mathbb{E}\left[\sum_{k=N_t+1}^{N_t+n} f(\theta_{P_k}(K,P))\right] \\
&= t\mathbb{E}\left[\frac{f(K^\circ,P^\circ)}{l_0}\right] - \mathbb{E}\left[\sum_{k=1}^{n} f(\theta_{P_k}(K,P))\right] \\
&+ \mathbb{E}\left[\sum_{k=N_t+1}^{N_t+n} f(\theta_{P_k}(K,P))\right].
\end{aligned}$$

Hence

$$\left|\mathbb{E}\left[\frac{f(\theta_{P_n}(K,P))}{l_0}\right] - \mathbb{E}\left[\frac{f(K^\circ,P^\circ)}{l_0}\right]\right| \leq \frac{An}{t}.$$

Letting $t \to \infty$ finishes the proof. □

Of particular interest to us in our Lévy process setting is the case where the sequence $((K_{t+P_{n-1}} - K_{P_{n-1}},\ 0 \leq t < l_n),\ l_n)_{n\in\mathbb{Z}}$ is independent, in which case the sequence $((K_{t+P_{n-1}} - K_{P_{n-1}},\ 0 \leq t < l_n),\ l_n)_{n\neq 0}$ is independent and identically distributed (cf. [61, Chapter 8, Remark 4.1]). Part (i) of the following result is a straightforward consequence of Corollary 2.4.6. Part (ii) is immediate from part (i). We omit the proofs.

Corollary 2.4.7 *Suppose that the sequence* $((K_{t+P_{n-1}} - K_{P_{n-1}},\, 0 \le t < l_n),\, l_n)_{n\in\mathbb{Z}}$ *is independent*

(i) For a nonnegative measurable function f*,*

$$\begin{aligned}&\mathbb{E}\left[f((K_{t+P_0} - K_{P_0},\, 0 \le t < l_1),\, l_1)\right]\\ &= \mathbb{E}\left[\frac{1}{l_0}\right]^{-1} \mathbb{E}\left[f((K_{t+P_{-1}} - K_{P_{-1}},\, 0 \le t < l_0),\, l_0)\frac{1}{l_0}\right].\end{aligned}$$

(ii) For a nonnegative measurable function f*,*

$$\begin{aligned}&\mathbb{E}\left[f((K_{t+P_{-1}} - K_{P_{-1}},\, 0 \le t < l_0),\, l_0)\right]\\ &= \mathbb{E}[l_1]^{-1}\mathbb{E}\left[f((K_{t+P_0} - K_{P_0},\, 0 \le t < l_1),\, l_1)l_1\right].\end{aligned}$$

We return to our Lévy process set-up and assume that $\mathcal{Z}$ is discrete. The pair $((X_t)_{t\in\mathbb{R}}, \mathcal{Z})$ is space-homogeneous stationary and hence, by Theorem 2.4.5, the random variable $U_0 = \frac{-G}{D-G}$ is uniform on $[0,1)$ and independent of the process $(X_{t+D} - X_D)_{t\in\mathbb{R}}$. Put

$$V_t := \begin{cases} X_D - X_{(D-t)-}, & 0 \le t < D - G =: \zeta_V, \\ \partial, & t \ge D - G. \end{cases}$$

It is easy to check that $D - G$ is the first positive point of the contact set of the process $(X_D - X_{(D-t)-})_{t\in\mathbb{R}}$ and so the process $(V_t)_{t\in\mathbb{R}}$ can be written as

$$V = F(X_{D+\cdot} - X_D),$$

where F is a measurable function from the space of càdlàg functions on the real line to the space of càdlàg functions on the positive real line. Hence the random variable $U = 1 - U_0$ is independent of $(V_t)_{t\ge 0}$. We have already observed that we know the distribution of the process

$$W_t := \begin{cases} V_t, & 0 \le t < D = U(D-G) =: \zeta_W, \\ \partial, & t \ge D = U(D-G). \end{cases}$$

We now show that it is possible to derive the distribution of V from that of W.

Corollary 2.4.8 *Recall that* ζ_V *(resp.* ζ_W*) is the lifetime of the process* $(V_t)_{t\ge0}$ *(resp.* $(W_t)_{t\ge0}$*). For bounded, measurable functions* $f_1,\ldots,f_n$ *that take the value* 0 *at* ∂ *and times* $0 \le t_1 < \ldots < t_n < t < \infty$*,*

$$\begin{aligned}\mathbb{E}[f_1(V_{t_1})\cdots f_n(V_{t_n})\,\mathbb{1}\{t<\zeta_V\}] &= \mathbb{E}[f_1(W_{t_1})\cdots f_n(W_{t_n})\,\mathbb{1}\{t<\zeta_W\}]\\ &\quad + t\frac{\mathbb{E}[f_1(W_{t_1})\cdots f_n(W_{t_n})\,\mathbb{1}\{\zeta_W\in dt\}]}{dt}.\end{aligned}$$

Proof. Observe that

$$\begin{aligned}
&\mathbb{E}[f_1(W_{t_1})\cdots f_n(W_{t_n})\,\mathbb{1}\{t<\zeta_W\}]\\
&=\mathbb{E}[f_1(V_{t_1})\cdots f_n(V_{t_n})\,\mathbb{1}\{t<U\zeta_V\}]\\
&=\int_0^1 \mathbb{E}[f_1(V_{t_1})\cdots f_n(V_{t_n})\,\mathbb{1}\{t/u<\zeta_V\}]\,du\\
&=\int_1^\infty \mathbb{E}[f_1(V_{t_1})\cdots f_n(V_{t_n})\,\mathbb{1}\{ts<\zeta_V\}]\frac{1}{s^2}\,ds\\
&=\int_t^\infty \mathbb{E}[f_1(V_{t_1})\cdots f_n(V_{t_n})\,\mathbb{1}\{r<\zeta_V\}]\frac{t^2}{r^2}\frac{1}{t}\,dr\\
&=t\int_t^\infty \mathbb{E}[f_1(V_{t_1})\cdots f_n(V_{t_n})\,\mathbb{1}\{r<\zeta_V\}]\frac{1}{r^2}\,dr,
\end{aligned}$$

so that

$$\int_t^\infty \mathbb{E}[f_1(V_{t_1})\cdots f_n(V_{t_n})\,\mathbb{1}\{r<\zeta_V\}]\frac{1}{r^2}\,dr=\frac{1}{t}\mathbb{E}[f_1(W_{t_1})\cdots f_n(W_{t_n})\,\mathbb{1}\{t<\zeta_W\}].$$

Differentiating both sides with respect to t and rearranging gives the result. □

Remark 2.4.9 *(i) The proof of Corollary 2.4.8 is similar to that of [60, Appendix, Proposition 3.12] which gives an analytic link between the distributions of Z and UZ where Z and U are independent nonnegative random variables with U uniform on $[0,1]$.*
(ii) We gave above a way to find the distribution of the excursion straddling zero. To determine the distribution of $(X_t,\,G\le t\le D)$ we generate the process V according to the distribution described above with lifetime ζ_V, and then take an independent random variable U uniform on $[0,1]$, then we have the equality of distributions

$$(X_t,\,G\le t\le D)\stackrel{d}{=}(V_{t+U\zeta_V}-V_{U\zeta_V},\,-U\zeta_V\le t\le(1-U)\zeta_V).$$

(iii) As a particular consequence of the Theorem 2.4.5, we can find the distribution of the straddling excursion length $D-G$ if we know the distribution of the right-hand endpoint D. See [2, Remark 8.2] where the relevant calculations are carried out to find the Laplace transform $D-G$.

From now on, we consider the generic excursions (that is, all the excursions that start after time D or finish before G). These excursions are independent and identically distributed and independent of the excursion straddling zero between G and D. In the next section we give a description of the common distribution of the generic excursions in the case of the Brownian motion with drift.

2.5 A generic excursion for Brownian motion with drift

Suppose in this section that X is two-sided Brownian motion with drift β such that $|\beta|<\alpha$; that is, $X=(B_t+\beta t)_{t\in\mathbb{R}}$, where B is a standard linear Brownian motion.

We recall the Williams path decomposition for Brownian motion with drift (see, for example, [55, Chapter VI, Theorem 55.9]).

Theorem 2.5.1 *Let $\mu > 0$. On some probability space take the following three independent random elements:*

- $(B_t^{(-\mu)},\, t \geq 0)$ *a BM with drift $-\mu$;*
- $(R_t^{(\mu)}, t \geq 0)$ *a diffusion that is solution of the following SDE*

$$dR_t^{(\mu)} = dB_t + \mu \coth(\mu R_t^{(\mu)})dt, R_0^{(\mu)} = 0,$$

  where B is a standard Brownian motion;

- γ *an exponential r.v with rate 2μ.*

Set $\tau = \inf\{t \geq 0 : B_t^{(-\mu)} = -\gamma\}$ and

$$H_t = \begin{cases} B_t^{(-\mu)}, & 0 \leq t \leq \tau, \\ R_{t-\tau}^{(\mu)} - \gamma, & t \geq \tau. \end{cases}$$

Then, $(H_t)_{t\geq 0}$ is a Brownian motion with drift μ.

Remark 2.5.2 *The diffusion $R^{(\mu)}$ is called a 3-dimensional Bessel process with drift μ and is denoted* $\mathrm{BES}(3, \mu)$*. We may use a superscript to refer to the starting position of this process, when there is no superscript it implicitly means we start at zero. This process has the same distribution as the radial part of a 3-dimensional Brownian motion with drift of magnitude μ [54, Section 3]. This process may be thought of as a Brownian motion with drift μ conditioned to stay positive.*

We give some results about Bessel processes that will be useful later in our proofs. The first result is a last exit decomposition of a Bessel process presented in [50, Chapter 6, Proposition 3.9].

Proposition 2.5.3 *Let ρ be $\mathrm{BES}^x(3)$; that is, ρ is a 3-dimensional Bessel process started at $x \geq 0$. Let T be an a.s finite stopping time with respect to the filtration $\mathcal{F}^{\rho,J} := (\sigma\{\rho_s, J_s, 0 \leq s \leq t\})_{t\geq 0}$, where $J_t := \inf_{s\geq t} \rho_s$ is the future infinimum of ρ and such that $\rho_T = J_T$. Then $(\rho_{T+t} - \rho_T)_{t\geq 0}$ is a $\mathrm{BES}^0(3)$ that is independent of $(\rho_t,\, 0 \leq t \leq T)$. In particular if*

$$g_{x,y} := \sup\{t \geq 0 : \rho_t = y\} = \inf\{t \geq 0 : \rho_t = J_t = y\}, y \geq x,$$

then $(\rho_{t+g_{x,y}} - y)_{t\geq 0}$ is a $\mathrm{BES}^0(3)$ independent of $(\rho_t,\, 0 \leq t \leq g_{x,y})$.

The next result relates the time-reversed Bessel process and the Brownian motion. It is from [50, Chapter 7, Corollary 4.6].

Proposition 2.5.4 *Let $b > 0$, ρ be a $\mathrm{BES}^0(3)$, and B be a standard linear Brownian motion. We have the equality of distributions*

$$(\rho_{L_b - t},\, 0 \leq t \leq L_b) \stackrel{d}{=} (b - B_t,\, 0 \leq t \leq T_b),$$

where $L_b := \sup\{t \geq 0 : \rho_t = b\}$ is the last passage time of ρ at the level b and $T_b := \inf\{t \geq 0 : B_t = b\}$ is the first hitting time of the Brownian motion B started at zero to b. In particular,

$$L_b \stackrel{d}{=} T_b.$$

The final result we will need is a path decomposition of a 3-dimensional Bessel process with drift started at a positive initial state when it hits its ultimate minimum. We don't know a reference for this result, so we give its proof for the sake of completeness.

Theorem 2.5.5 *Let $b, \mu > 0$. Consider the following three independent random elements :*

- *a random variable g with density proportional to $e^{2\mu x}$ supported on $[0, b]$;*
- *a Brownian motion $(B_t^{(b,-\mu)})_{t\geq 0}$ with drift $-\mu$ started at b;*
- *a 3-dimensional Bessel process $(R_t^{(\mu)})_{t\geq 0}$ with drift μ started at zero.*

Define the process :

$$R_t^{(b,\mu)} = \begin{cases} B_t^{(b,-\mu)}, & 0 \leq t \leq \tilde{T}_g, \\ g + R^{(\mu)}_{t-\tilde{T}_g}, & t \geq \tilde{T}_g. \end{cases}$$

where $\tilde{T}_g := \inf\{t \geq 0 : B_t^{(b,-\mu)} = g\}$.

Then, $R^{(b,\mu)} \stackrel{d}{=} \mathrm{BES}^b(3,\mu)$; that is, $R^{(b,\mu)}$ is a 3-dimensional Bessel process with drift μ started at b.

Proof. The distribution of a 3-dimensional Bessel process with drift μ and started at $b > 0$ is the conditional distribution of a Brownian motion with drift μ started at b conditioned to stay positive (see the Remarks at the end of [54, Section 3]). The event we condition on has a positive probability, so it is just the usual naive conditioning

$$(b - \mathrm{BM}^0(-\mu)) \;\Big|\; \left\{\sup_{t\geq 0} \mathrm{BM}^0(-\mu)_t \leq b\right\} \stackrel{d}{=} \mathrm{BES}^b(3,\mu),$$

where $\mathrm{BM}^0(-\mu)$ is a Brownian motion with drift $-\mu$ and started at zero. The theorem is then just an application of the Williams path decomposition Theorem 2.5.1. □

Recall that in this section $(X_t)_{t\in\mathbb{R}}$ is a Brownian motion with drift β. The discussion in Theorem 2.3.6 and the Williams path decomposition Theorem 2.5.1 shows that $(X_{t+D} - X_D + \alpha t)_{t\geq 0}$ has the same distribution as $(B_t + (\alpha+\beta)t)_{t\in\mathbb{R}}$ conditioned to stay positive. Thus,

$$(X_{t+D} - X_D,\, t \geq 0) = (R_t^{(\alpha+\beta)} - \alpha t,\, t \geq 0),$$

where $R^{(\alpha+\beta)} \stackrel{d}{=} \mathrm{BES}(3, \alpha+\beta)$. We aim now to provide a path decomposition of the first positive generic excursion away from the contact set (and thus all generic excursions), that is the path of $(Z_t)_{t\geq 0} := (X_{t+D} - X_D)_{t\geq 0} = (X_{t+D_0} - X_{D_0})_{t\geq 0}$ until it hits the first contact point $D_{D_0} - D_0$.

Notation 2.5.6 *Using Lemma 2.7.4, let us define the following times that are the analogues of* $\boldsymbol{s}$ *and* $\boldsymbol{d}$ *for this generic excursion.*

$$\mathfrak{T} := \inf\{t > 0 : Z_t - \alpha t \leq 0\} = \inf\left\{t > 0 : \frac{R_t^{(\alpha+\beta)}}{t} = 2\alpha\right\}$$

and

$$\zeta := \inf\{t \geq \mathfrak{T} : Z_t + \alpha t = \inf\{Z_u + \alpha u : u \geq \mathfrak{T}\}\}$$
$$= \inf\{t \geq \mathfrak{T} : R_t^{(\alpha+\beta)} = \inf\{R_u^{(\alpha+\beta)} : u \geq \mathfrak{T}\}\}.$$

The following theorem is a path decomposition of a generic excursion away from the contact set.

Theorem 2.5.7 *Consider the following independent random elements:*

- *a pair of random variables* $(\tau, \hat{\gamma})$ *with the joint density*

$$f_{\tau,\hat{\gamma}}(t,x) = \frac{\exp\left(-\frac{(\alpha-\beta)^2 t}{2} - 2(\alpha+\beta)x\right)}{\sqrt{2\pi t^3}} \mathbb{1}_{0\leq x\leq 2\alpha t}, \; t > 0, \tag{2.3}$$

- *a standard Brownian excursion* $\mathbf{e}$ *on* $[0,1]$*,*
- *a linear Brownian motion* $(\tilde{B}_t^{-(\alpha+\beta)})_{t\geq 0}$ *with drift* $-(\alpha+\beta)$*.*

Define the process

$$\mathfrak{E}_t = \begin{cases} \sqrt{\tau}\mathbf{e}(\frac{t}{\tau}) + 2\alpha t, & 0 \leq t \leq \tau, \\ 2\alpha\tau + \tilde{B}_{t-\tau}^{-(\alpha+\beta)}, & \tau \leq t \leq \tau + \tilde{T}_{\hat{\gamma}}, \end{cases}$$

where $\tilde{T}_{\hat{\gamma}} := \inf\{t \geq 0 : \tilde{B}_t^{-(\alpha+\beta)} = -\hat{\gamma}\}$*. Then,*

$$(X_{t+D} - X_D + \alpha t,\, 0 \leq t \leq \zeta) \stackrel{d}{=} (\mathfrak{E}_t,\, 0 \leq t \leq \tau + \tilde{T}_{\hat{\gamma}}).$$

Proof. Let us first find the distribution of the path of $R^{(\alpha+\beta)}$ on $[0, \mathfrak{T}]$. As $\mathfrak{T}$ is a stopping time (with respect to the filtration generated by $R^{(\alpha+\beta)}$) and $R^{(\alpha+\beta)}$ is a time-homogeneous strong Markov process, we have that conditionally on the event $\{\mathfrak{T} = T\} = \{R_{\mathfrak{T}}^{(\alpha+\beta)} = 2\alpha T\}$, the processes $\{R_u^{(\alpha+\beta)} : 0 \leq u \leq \mathfrak{T}\}$ and $\{R_u^{(\alpha+\beta)} : u \geq \mathfrak{T}\}$ are independent. Now define $(Y_t)_{t>0}$ by

$$Y_t := t R_{\frac{1}{t}}^{(\alpha+\beta)}, \; t > 0.$$

By the time-inversion property of Brownian motion, Y is a $\text{BES}^{\alpha+\beta}(3)$; that is, Y is a 3-dimensional Bessel process started at $\alpha+\beta$ (with no drift). The stopping time $\mathfrak{T}$ can be expressed as

$$\mathfrak{T} = \frac{1}{\sup\{t \geq 0 : Y_t = 2\alpha\}} \stackrel{d}{=} \frac{1}{g_{\alpha+\beta, 2\alpha}} \tag{2.4}$$

Hence, by applying Proposition 2.5.3 to our process Y, we find that

$$(G_t,\, t \geq 0) := (Y_{t+\frac{1}{\mathfrak{T}}} - 2\alpha,\, t \geq 0)$$

is a $\text{BES}^0(3)$ independent from $\sigma\{Y_u : u \leq \frac{1}{\mathfrak{T}}\} = \sigma\{R_u^{(\alpha+\beta)} : u \geq \mathfrak{T}\}$. Now, conditionally on $\{\mathfrak{T} = T\}$, we have

$$(R_u^{(\alpha+\beta)},\, 0 \leq u \leq T) = (u(G_{\frac{1}{u}-\frac{1}{T}} + 2\alpha),\, 0 \leq u \leq T)$$
$$= (uG_{\frac{T-u}{uT}} + 2\alpha u,\, 0 \leq u \leq T).$$

However, it is known that $(uG_{\frac{T-u}{uT}},\, 0 \le u \le T)$ is just a Brownian excursion of length T (that is a 3-dimensional Bessel bridge between $(0,0)$ and $(T,0)$). This can easily be seen from the same time transformation that maps Brownian motions to Brownian bridges); for a reference to this path transformation, see [p 226][45]. Hence, given $\{\mathfrak{T} = T\}$,

$$(W_u,\, 0 \le u \le T) = (\mathbf{e}_T(u) + \alpha u, 0 \le u \le T) \stackrel{d}{=} \left(\sqrt{T}\mathbf{e}\left(\frac{u}{T}\right) + \alpha u,\, 0 \le u \le T\right),$$

where $\mathbf{e}_T$ is a Brownian excursion on $[0,T]$, and $\mathbf{e}$ is a standard Brownian excursion on $[0,1]$ obtained by Brownian scaling.

Now let us move to the second fragment of our path; that is, the process W on $[\mathfrak{T}, \zeta]$. Because of the fact that $\mathfrak{T}$ is a stopping time and $R^{(\alpha+\beta)}$ is a strong Markov process, conditionally on $\{\mathfrak{T} = T\}$, the process $(R^{(\alpha+\beta)}_{t+\mathfrak{T}},\, 0 \le t \le \zeta - \mathfrak{T})$ is just a $\mathrm{BES}^{2\alpha T}(3, \alpha+\beta)$ stopped at the time it hits its ultimate minimum. Hence, by applying Theorem 2.5.5,

$$(R^{(\alpha+\beta)}_{t+\mathfrak{T}},\, 0 \le t \le \zeta - \mathfrak{T}) \stackrel{d}{=} (\tilde{B}^{(2\alpha T, -(\alpha+\beta))}_t,\, 0 \le t \le \tilde{T}_\gamma),$$

where $\tilde{B}^{(2\alpha T, -(\alpha+\beta))}$ is a standard Brownian motion with drift $-(\alpha+\beta)$ started at $2\alpha T$ and $\tilde{T}_\gamma := \inf\{t \ge 0 : \tilde{B}^{(2\alpha T, -(\alpha+\beta))}_t = \gamma\}$, and γ is independent of $\tilde{B}^{(2\alpha T, -(\alpha+\beta))}$ with density on $[0, 2\alpha T]$ proportional to $x \mapsto e^{2(\alpha+\beta)x}$. Finally by setting $\hat{\gamma} = 2\alpha\mathfrak{T} - \gamma$, it suffices to prove that $(\mathfrak{T}, \hat{\gamma})$ has the joint density in (2.3) to finish our proof.

We know that the conditional density of $\hat{\gamma}$ given $\{\mathfrak{T} = t\}$ is proportional to $x \mapsto e^{-2(\alpha+\beta)x}$ restricted to $[0, 2\alpha t]$. That is,

$$f_{\hat{\gamma}|\mathfrak{T}=t}(x) = \frac{2(\alpha+\beta)e^{-2(\alpha+\beta)x}}{1 - e^{-4(\alpha+\beta)\alpha t}} \mathbb{1}_{0 \le x \le 2\alpha t}. \tag{2.5}$$

To finish, let us find the distribution of $\mathfrak{T}$. Recall from (2.4) that we have

$$\mathfrak{T} \stackrel{d}{=} \frac{1}{g_{\alpha+\beta, 2\alpha}}.$$

Now $g_{\alpha+\beta,2\alpha}$ is the last time a 3-dimensional Bessel process started at $\alpha+\beta$ visits the state 2α. Consider $(\tilde{Y}_t)_{t\ge 0}$ a $\mathrm{BES}^0(3)$, and let $H_{\alpha+\beta} := \inf\{t \ge 0 : \tilde{Y}_t = \alpha+\beta\}$ be the first hitting time of $\alpha+\beta$. Then, by the strong Markov property at time $H_{\alpha+\beta}$, we have

$$L_{2\alpha} \stackrel{d}{=} H_{\alpha+\beta} + g_{\alpha+\beta,2\alpha},$$

where $g_{\alpha+\beta,2\alpha}$ and $H_{\alpha+\beta}$ are independent, and $L_{2\alpha}$ is the last time $\tilde{Y}$ visits 2α. Hence we get the Laplace transform of $g_{\alpha+\beta,2\alpha}$ is

$$\mathbb{E}[\exp(-\lambda g_{\alpha+\beta,2\alpha})] = \frac{\mathbb{E}[\exp(-\lambda L_{2\alpha})]}{\mathbb{E}[\exp(-\lambda H_{\alpha+\beta})]}.$$

Using Proposition 2.5.4, we know with the same notation that $L_{2\alpha} \stackrel{d}{=} T_{2\alpha}$. Thus,

$$\mathbb{E}[\exp(-\lambda T_{2\alpha})] = \exp(-2\alpha\sqrt{2\lambda}).$$

On the other hand, we obtain the Laplace transform of $H_{\alpha+\beta}$ from [13, equation 2.1.4, p463], namely,

$$\mathbb{E}[\exp(-\lambda H_{\alpha+\beta})] = \frac{(\alpha+\beta)\sqrt{2\lambda}}{\sinh((\alpha+\beta)\sqrt{2\lambda})}.$$

Thus,

$$\mathbb{E}[\exp(-\lambda g_{\alpha+\beta,2\alpha})] = \frac{e^{-2\alpha\sqrt{2\lambda}}\sinh((\alpha+\beta)\sqrt{2\lambda})}{(\alpha+\beta)\sqrt{2\lambda}}.$$

Inverting this Laplace transform, we get the density of $g_{\alpha+\beta,2\alpha}$; that is,

$$f_{g_{\alpha+\beta,2\alpha}}(t) = \frac{e^{-\frac{(\alpha-\beta)^2}{2t}} - e^{-\frac{(3\alpha+\beta)^2}{2t}}}{2(\alpha+\beta)\sqrt{2\pi t}}.$$

The density of $\mathfrak{T}$ is thus

$$f_{\mathfrak{T}}(t) = \frac{1}{t^2} f_{g_{\alpha+\beta,2\alpha}}\left(\frac{1}{t}\right) = \frac{e^{-\frac{(\alpha-\beta)^2 t}{2}} - e^{-\frac{(3\alpha+\beta)^2 t}{2}}}{2(\alpha+\beta)\sqrt{2\pi t^3}} \mathbb{1}_{t>0}. \tag{2.6}$$

Multiplying the (2.6) and (2.5) gives the desired equality. □

Now we have an explicit path decomposition of a generic excursion and we know the expression of the α-Lipschitz minorant on the same interval in terms of the locations of the excursion at its end-points using Lemma 2.7.5. It is interesting to identify the distributions of the most important features such as:

- the lifetime ζ of the excursion;
- the time L at which the α-Lipschitz minorant of the excursion attains its maximal value;
- the final value Z_ζ of the excursion

– see Figure 2.2.

Using the notation from Theorem 2.5.7 and from Lemma 2.7.5 we have the following expressions

$$\begin{aligned} \zeta &= \tau + \tilde{T}_{\hat{\gamma}}, \\ L &= \tau - \frac{\hat{\gamma}}{2\alpha}, \\ \zeta - L &= \tilde{T}_{\hat{\gamma}} + \frac{\hat{\gamma}}{2\alpha}, \\ Z_\zeta &= \alpha\left(\tau - \tilde{T}_{\hat{\gamma}} - \frac{\hat{\gamma}}{\alpha}\right). \end{aligned}$$

Proposition 2.5.8 *(i) The joint Laplace transform of $(\zeta, L, \zeta - L, Z_\zeta)$ is*

$$\begin{aligned} &\mathbb{E}[\exp(-(\rho_1\zeta + \rho_2 L + \rho_3(\zeta - L) + \rho_4 Z_\zeta))] \\ &= \frac{4\alpha}{2\alpha + \sqrt{2(\rho_1+\rho_3-\alpha\rho_4)+(\alpha+\beta)^2} + \sqrt{2(\rho_1+\rho_2+\alpha\rho_4)+(\alpha-\beta)^2}}. \end{aligned}$$

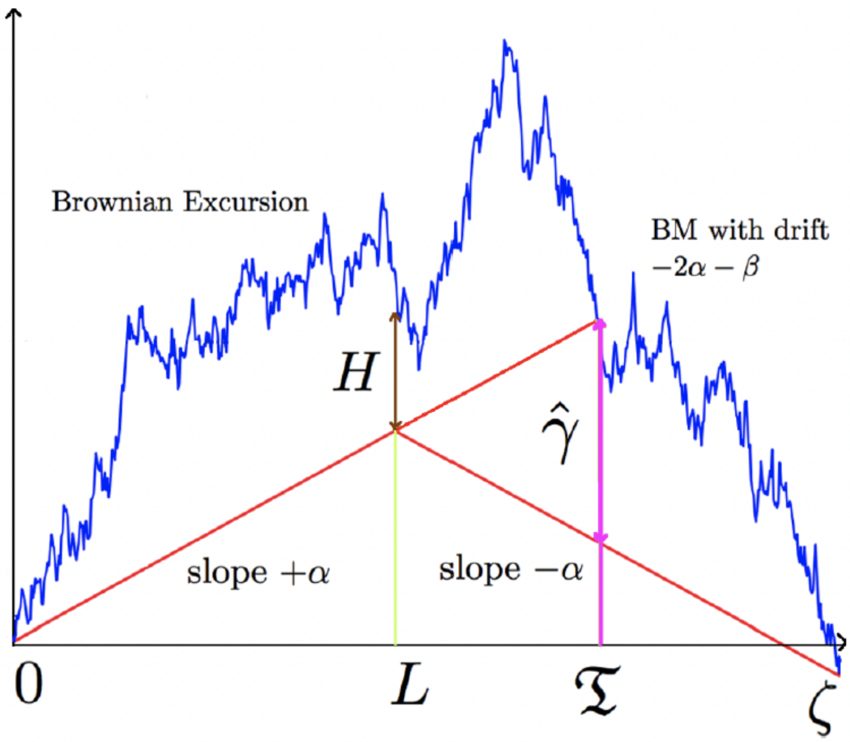

Figure 2.2: A generic excursion away from the contact set.

(ii) *The Laplace transform of the excursion length ζ is*

$$\mathbb{E}[\exp(-\lambda\zeta)] = \frac{4\alpha}{2\alpha + \sqrt{2\lambda + (\alpha+\beta)^2} + \sqrt{2\lambda + (\alpha-\beta)^2}}.$$

In particular, for $\beta = 0$ the probability density of ζ is

$$l \mapsto 2\alpha \frac{e^{-\frac{\alpha^2 l}{2}}}{\sqrt{2\pi l}} - 2\alpha^2 \overline{\Phi}(\alpha\sqrt{l})$$

where $\overline{\Phi}(x) := \int_x^\infty \frac{e^{-u^2/2}}{\sqrt{2\pi}}\, du$.

(iii) *The Laplace transform of the time L to the peak of the minorant during the excursion is*

$$[\exp(-\lambda L)] = \frac{4\alpha}{3\alpha + \beta + \sqrt{2\lambda + (\alpha-\beta)^2}}$$

The corresponding density is

$$l \mapsto 4\alpha \frac{e^{-\frac{(\alpha-\beta)^2 l}{2}}}{\sqrt{2\pi l}} - 4\alpha(3\alpha+\beta)e^{4\alpha(\alpha+\beta)l}\overline{\Phi}(\sqrt{l}(3\alpha+\beta).$$

(iv) *The Laplace transform of the time $\zeta - L$ after the peak of the minorant during the excursion is*

$$[\exp(-\lambda(\zeta - L))] = \frac{4\alpha}{3\alpha - \beta + \sqrt{2\lambda + (\alpha+\beta)^2}}$$

The corresponding density is

$$l \mapsto 4\alpha \frac{e^{-\frac{(\alpha+\beta)^2 l}{2}}}{\sqrt{2\pi l}} - 4\alpha(3\alpha-\beta)e^{4\alpha(\alpha-\beta)l}\overline{\Phi}(\sqrt{l}(3\alpha-\beta).$$

(v) The Laplace transform of Z_ζ, the final value of the excursion, is

$$\mathbb{E}[\exp(-\lambda Z_\zeta)] = \frac{4\alpha}{2\alpha + \sqrt{(\alpha+\beta)^2 - 2\lambda\alpha} + \sqrt{(\alpha-\beta)^2 + 2\lambda\alpha}}.$$

We give the proof of Proposition 2.5.8 below after some preparatory results. We first recall a result about the distribution of the first hitting time of a Brownian motion with drift. [13, equations 2.0.1 & 2.0.2, page 295].

Lemma 2.5.9 *Let $(B_t^{(\mu)})_{t\geq 0}$ a Brownian motion with drift $\mu > 0$ started at zero. Let $y > 0$ and define $T_{\mu,y} := \inf\{t \geq 0 : B_t^{(\mu)} = y\}$. The density function of $T_{\mu,y}$ is*

$$f_{T_{\mu,y}}(t) = \frac{y}{\sqrt{2\pi t^3}} \exp\left(-\frac{(y-\mu t)^2}{2t}\right)$$

and its Laplace transform is

$$\mathbb{E}[\exp(-\lambda T_{\mu,y})] = e^{-y(\sqrt{2\lambda+\mu^2}-\mu)}.$$

The following simple lemma is well-known and follows readily from the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Lemma 2.5.10 *For $a, b > 0$,*

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{\sqrt{2\pi t^3}}\, dt = \sqrt{2b} - \sqrt{2a}.$$

We now give the proof of Proposition 2.5.8.

Proof. We claim that

$$\begin{aligned}
&\mathbb{E}[\exp(-\lambda_1\tau - \lambda_2\hat{\gamma} - \lambda_3\tilde{T}_{\hat{\gamma}})] \\
&= \frac{4\alpha}{\sqrt{2(\lambda_1-\lambda_3) + 4\alpha\lambda_2 + (2\alpha + \sqrt{2\lambda_3 + (\alpha+\beta)^2})^2} + \sqrt{2\lambda_1 + (\alpha-\beta)^2}}.
\end{aligned} \tag{2.7}$$

The stated equation for $\mathbb{E}[\exp(-(\rho_1\zeta + \rho_2 L + \rho_3(\zeta - L) + \rho_4 Z_\zeta))]$ then follows by noting that

$$\begin{aligned}
&\rho_1\zeta + \rho_2 L + \rho_3(\zeta - L) + \rho_4 Z_\zeta \\
&= \rho_1(\tau + \tilde{T}_{\hat{\gamma}}) + \rho_2\left(\tau - \frac{\hat{\gamma}}{2\alpha}\right) + \rho_3\left(\tilde{T}_{\hat{\gamma}} + \frac{\hat{\gamma}}{2\alpha}\right) + \rho_4\alpha\left(\tau - \tilde{T}_{\hat{\gamma}} - \frac{\hat{\gamma}}{\alpha}\right) \\
&= (\rho_1 + \rho_2 + \alpha\rho_4)\tau + \left(-\frac{\rho_2}{2\alpha} + \frac{\rho_3}{2\alpha} - \rho_4\right)\hat{\gamma} + (\rho_1 + \rho_3 - \alpha\rho_4)\tilde{T}_{\hat{\gamma}}.
\end{aligned}$$

The Laplace transforms for the individual random variables follow by specialization and the claimed expressions for densities then follow from standard inversion formulas.

Rather than deriving (2.7) we will instead derive directly the Laplace transform of ζ. This illustrates the method of proof with less notational overhead. We have

$$\begin{aligned}
&\mathbb{E}[\exp(-\lambda\zeta)] \\
&= \mathbb{E}[e^{-\lambda\tau}\mathbb{E}[e^{-\lambda\tilde{T}_{\hat{\gamma}}}|\tau,\hat{\gamma}]] \\
&= \mathbb{E}[e^{-\lambda\tau}e^{-\hat{\gamma}(\sqrt{2\lambda+(\alpha+\beta)^2}-(\alpha+\beta))}] \\
&= \int_0^\infty\int_0^{2\alpha t} e^{-\lambda t}e^{-x(\sqrt{2\lambda+(\alpha+\beta)^2}-(\alpha+\beta))}\frac{\exp\left(-\frac{(\alpha-\beta)^2 t}{2}-2(\alpha+\beta)x\right)}{\sqrt{2\pi t^3}}\,dt\,dx \\
&= \frac{1}{\sqrt{2\lambda+(\alpha+\beta)^2}+\alpha+\beta}\int_0^\infty \frac{e^{-(\lambda+\frac{(\alpha-\beta)^2}{2})t}(1-e^{-2\alpha t(\sqrt{2\lambda+(\alpha+\beta)^2}+\alpha+\beta)})}{\sqrt{2\pi t^3}}\,dt \\
&= \frac{1}{\sqrt{2\lambda+(\alpha+\beta)^2}+\alpha+\beta}\int_0^\infty \frac{e^{-at}-e^{-bt}}{\sqrt{2\pi t^3}}dt
\end{aligned}$$

for

$$a=\lambda+\frac{(\alpha-\beta)^2}{2}, b=\lambda+\frac{(\alpha-\beta)^2}{2}+2\alpha(\sqrt{2\lambda+(\alpha+\beta)^2}+\alpha+\beta).$$

A little algebra shows that

$$a=\frac{1}{2}(2\lambda+(\alpha-\beta)^2),\quad b=\frac{1}{2}(\sqrt{2\lambda+(\alpha+\beta)^2}+2\alpha)^2.$$

Hence, using Lemma 2.5.10, we get that

$$\mathbb{E}[\exp(-\lambda\zeta)]=\frac{2\alpha+\sqrt{2\lambda+(\alpha+\beta)^2}-\sqrt{2\lambda+(\alpha-\beta)^2}}{\sqrt{2\lambda+(\alpha+\beta)^2}+\alpha+\beta}.$$

After multiplying top and bottom by the conjugate this has following simple form

$$\mathbb{E}[\exp(-\lambda\zeta)]=\frac{4\alpha}{2\alpha+\sqrt{2\lambda+(\alpha+\beta)^2}+\sqrt{2\lambda+(\alpha-\beta)^2}}.$$

□

Remark 2.5.11 *(i) Write* $H := Z_L - M_L = \sqrt{\tau}\mathbf{e}(\frac{L}{\tau})$ *for the difference between the Brownian motion and its minorant at time* L *— see Figure 2.2. We can get an explicit description for the distribution of this random variable, though computing either its Laplace transform or density seems tedious. Indeed, we know that for every* $0 \le u \le 1$*, we have that* $\mathbf{e}(u) \stackrel{d}{=} \sqrt{u(1-u)}\chi_3$*, where* $\chi_3^2 \stackrel{d}{=} Q_1^2+Q_2^2+Q_3^2$ *for* Q_1, Q_2, Q_3 *three independent standard Gaussian random variables. Hence,*

$$H \stackrel{d}{=} \sqrt{L\left(1-\frac{L}{\tau}\right)}\chi_3 = \sqrt{L\left(\frac{\hat{\gamma}}{2\alpha}\right)}\chi_3 = \tau\sqrt{\mathfrak{U}(1-\mathfrak{U})}\chi_3.$$

where $\mathfrak{U} := \frac{\hat{\gamma}}{2\alpha\tau}$. *Using the density in Theorem 2.5.7 and a change of variable gives that the joint density of* $(\tau, \mathfrak{U})$ *at the point* $(t, u) \in (0, \infty) \times [0, 1]$ *is*

$$f_{\tau,\mathfrak{U}}(t,u) = \frac{2\alpha}{\sqrt{2\pi t}} \exp\left(-\frac{(\alpha-\beta)^2}{2}t - \frac{\alpha+\beta}{\alpha}tu\right)$$

and χ_3 *independent of* $(\tau, \mathfrak{U})$.

(ii) Set $\Psi(\rho_1, \rho_2, \rho_3, \rho_4; \alpha, \beta) = \mathbb{E}[\exp(-(\rho_1\zeta + \rho_2 L + \rho_3(\zeta - L) + \rho_4 Z_\zeta))]$. *From the time-reversal symmetry* $(B_t)_{t\in\mathbb{R}} \stackrel{d}{=} (B_{-t})_{t\in\mathbb{R}}$, *we expect that*

$$\Psi(\rho_1, \rho_2, \rho_3, \rho_4; \alpha, \beta) = \Psi(\rho_1, \rho_3, \rho_2, -\rho_4; \alpha, -\beta),$$

and this is indeed the case. This symmetry is somewhat surprising, as it is certainly not apparent from our path decomposition. Similarly, from the Brownian scaling $(c^{-1}B_{c^2t})_{t\in\mathbb{R}} \stackrel{d}{=} (B_t)_{t\in\mathbb{R}}$, $c > 0$, *we expect that*

$$\Psi(\rho_1, \rho_2, \rho_3, \rho_4; \alpha, \beta) = \Psi(c^2\rho_1, c^2\rho_2, c^2\rho_3, c^2\rho_4; c\alpha, c\beta),$$

and this also holds.

(iii) It follows from the proposition that

$$\mathbb{E}[\zeta] = -\frac{d}{d\lambda}\mathbb{E}[\exp(-\lambda\zeta)]|_{\lambda=0} = \frac{1}{2(\alpha^2 - \beta^2)}.$$

Similarly,

$$\mathbb{E}[L] = \frac{1}{4\alpha(\alpha-\beta)},$$

$$\mathbb{E}[\zeta - L] = \frac{1}{4\alpha(\alpha+\beta)},$$

and

$$\mathbb{E}[Z_\zeta] = \frac{\beta}{2(\alpha^2 - \beta^2)}.$$

Note that since $\lim_{t\to\infty}(B_t + \beta t)/t = \beta$ *almost surely, we expect* $\mathbb{E}[Z_\zeta] = \beta\mathbb{E}[\zeta]$ *by a renewal–reward argument.*

(iv) The results of this section advance the study of the excursion straddling zero in the case of the Brownian motion with drift carried out in [2, Section 8]. Indeed, the previous study only determined the four-dimensional distribution $(G, D, T, \tilde{H})$, *where* $T := \operatorname{argmax}\{M_t : G \le t \le D\}$ *and* $\tilde{H} := X_T - M_T$. *Our approach here gives the distribution of the whole path of a generic excursion. Let us define*

$$W^{\text{straddle}} := (X_{t+G} - X_G,\, 0 \le t \le D - G)$$

and

$$W^{\text{generic}} := (X_{t+D} - X_D,\, 0 \le t \le \zeta).$$

By Corollary 2.4.7, we have

$$\mathbb{E}\left[F(W^{\text{straddle}})\right] = \mathbb{E}[\zeta]^{-1}\mathbb{E}\left[\zeta F(W^{\text{generic}})\right]$$

Because we know the distribution of W^{generic}*, the distribution of the straddling excursion can be recovered. In particular, the distribution of* $D-G$ *is just the size-biasing of the distribution of* ζ*; that is,* $E[f(D-G)] = \mathbb{E}[\zeta]^{-1}\mathbb{E}[\zeta f(\zeta)]$ *for any nonnegative measurable function* f*. For example, the joint Laplace transform of the analogues of* $(\zeta, L, \zeta - L, Z_\zeta)$ *for the straddling excursion is*

$$\frac{-\frac{d}{d\rho_1}\Psi(\rho_1, \rho_2, \rho_3, \rho_4)}{\mathbb{E}[\zeta]}.$$

Finally, if we denote by Λ *the Lévy measure of the subordinator associated with the regenerative set* $\mathcal{Z}$*, then it has the density given by the following formula*

$$\frac{\Lambda(dx)}{\Lambda(\mathbb{R}+)} = \left(2\alpha \frac{e^{-\frac{\alpha^2 x}{2}}}{\sqrt{2\pi x}} - 2\alpha^2 \overline{\Phi}(\alpha\sqrt{x})\right) dx$$

(recall that Λ *is only defined up to a multiplicative constant).*

2.6 Enlargement of the Brownian filtration

In this section, the Lévy process $(X_t)_{t\in\mathbb{R}}$ is the standard two-sided linear Brownian motion. Set

$$\overline{\mathcal{F}}_t := \sigma\{X_u : u \le t\} \vee \{\text{the null sets of } \mathbb{P}\},\ t \in \mathbb{R}.$$

From [50, Chapter 3, Proposition 2.10] $(\overline{\mathcal{F}}_t)_{t\in\mathbb{R}}$ is then right-continuous and $(X_t)_{t\in\mathbb{R}}$ is a $(\overline{\mathcal{F}}_t)_{t\in\mathbb{R}}$-two-sided linear standard Brownian motion. We denote $(M_t)_{t\in\mathbb{R}}$ the α-Lipschitz minorant of X and we let D be defined, as above, by

$$D := \inf\{t \ge 0 : X_t = M_t\}.$$

By Lemma 2.7.3, the random time D can be constructed as follows. Consider first the stopping time S given by

$$S = \inf\{t > 0 : X_t - \alpha t = \inf\{X_u - \alpha u : u \le 0\}\}.$$

Then

$$D = \inf\{t \ge S : X_t + \alpha t = \inf\{X_u + \alpha u : u \ge S\}\}.$$

Thus, if we introduce the one-sided Brownian motion $\check{X} = (X_{t+S} - X_S)_{t\ge 0}$ which is independent of $\overline{\mathcal{F}}_S$, and we let $\check{T}$ be the time at which the process $(\check{X}_t + \alpha t)_{t\ge 0}$ hits its ultimate infimum (this point is almost surely unique), then

$$D = S + \check{T}.$$

As we have seen previously, the random time D is not a stopping time. However, D is an honest time in the sense of the following definition, that is present in Martin Barlow's work in [7].

Definition 2.6.1 *Let* L *be a random variable with values in* $[0, \infty]$*,* L *is said to be* honest *with respect to the filtration* $(\overline{\mathcal{F}}_t)_{t\in\mathbb{R}}$ *if, for every* $t \ge 0$*, there exists an* $\overline{\mathcal{F}}_t$*-measurable random variable* L_t *such that on the set* $\{L < t\}$ *we have* $L = L_t$*.*

Lemma 2.6.2 *The random time D is an honest time. Moreover, if T is a stopping time, then $\mathbb{P}\{D = T\} = 0$.*

Proof. We can write D on the event $\{D < a\}$ as

$$\begin{aligned} D\mathbb{1}_{D<a} &= S\mathbb{1}_{\{S<a\}} + \inf\{t \geq 0 : X_{t+S\mathbb{1}_{\{S<a\}}} - X_{S\mathbb{1}_{\{S<a\}}} + \alpha t \\ &= \inf\{X_t + \alpha t : S\mathbb{1}_{\{S<a\}} \leq t \leq a\}\}. \end{aligned}$$

The right-hand side is $\overline{\mathcal{F}}_a$-measurable and hence D is an honest time. Also, $\mathbb{P}\{D = T\} = 0$ for any stopping time T because $\mathbb{P}\{X_{D+t} > X_D - \alpha t, \forall t > 0\} = 1$ whereas $\mathbb{P}(\bigcap_{\epsilon>0}\{\exists 0 < t < \epsilon, X_{T+t} < X_T - \alpha t\}) = 1$. □

We introduce now a larger filtration that is the smallest filtration containing $(\overline{\mathcal{F}}_t)_{t\in\mathbb{R}}$ that makes D a stopping time.

Notation 2.6.3 *For $t \in \mathbb{R}$, set*

$$\overline{\mathcal{F}}_t^D := \bigcap_{\epsilon>0}(\overline{\mathcal{F}}_{t+\epsilon} \vee \sigma(D \wedge (t+\epsilon))).$$

Remark 2.6.4 *For honest times D,*

$$\overline{\mathcal{F}}_t^D = \{A \in \overline{\mathcal{F}}_\infty : \exists A_t, B_t \in \overline{\mathcal{F}}_t,\ A = (A_t \cap \{D > t\}) \cup (B_t \cap \{D \leq t\})\}$$

– see [29, Chapter 5].

Our goal now is to verify that every $(\overline{\mathcal{F}}_t)_{t\geq 0}$-semimartingale remains a $(\overline{\mathcal{F}}_t^D)_{t\geq 0}$-semimartingale, and to give a formula for the canonical semimartingale decomposition in the larger filtration.

Definition 2.6.5 *For any random time ρ, we call the $(\overline{\mathcal{F}}_t)_{t\geq 0}$-supermartingale defined by*

$$Z_t^\rho = \mathbb{P}[\rho > t \,|\, \overline{\mathcal{F}}_t]$$

the Azéma supermartingale *associated with ρ. We choose versions of the conditional expectations so that this process is càdlàg.*

We recall the following result from [8, Theorem A].

Theorem 2.6.6 *Let L be an honest time. A $(\overline{\mathcal{F}}_t)_{t\geq 0}$ local martingale $(\mathfrak{M}_t)_{t\geq 0}$ is a semimartingale in the larger filtration $(\overline{\mathcal{F}}_t^L)_{t\geq 0}$ and decomposes as*

$$\mathfrak{M}_t = \tilde{\mathfrak{M}}_t + \int_0^{t\wedge L} \frac{d\langle \mathfrak{M}, Z^L\rangle_s}{Z_{s-}^L} - \int_L^t \frac{d\langle \mathfrak{M}, Z^L\rangle_s}{1 - Z_{s-}^L},$$

where $(\tilde{\mathfrak{M}}_t)_{t\geq 0}$ is a $((\overline{\mathcal{F}}_t^L)_{t\geq 0}, \mathbb{P})$-local martingale.

It remains to find an explicit formula for Z_t^D. Define a decreasing sequence of stopping times $(S_n)_{n\geq 0}$ that converges almost surely to S by

$$S_n := \sum_{k=0}^{\infty} \frac{k+1}{2^n} \mathbb{1}_{\{\frac{k}{2^n} \leq S < \frac{k+1}{2^n}\}}.$$

Define the random times $(\check{T}_n)_{n\geq 0}$ by

$$\check{T}_n = \sup\{t \geq 0 : X_{t+S_n} - X_{S_n} + \alpha t = \inf\{X_{u+S_n} - X_{S_n} + \alpha u, u \geq 0\}\}.$$

Note that $\check{T}_n \underset{n\to\infty}{\to} \check{T}$ almost surely because $\check{T}_n = \operatorname{argmin}\{X_{u+S} - X_S + \alpha u : u \geq S_n - S\} + S - S_n$ and $\check{T} > 0$ with probability 1. Hence,

$$\begin{aligned}
Z_t^D &= \mathbb{P}\{D > t \,|\, \overline{\mathcal{F}}_t\} = \mathbb{P}\{S + \check{T} > t \,|\, \overline{\mathcal{F}}_t\} \\
&= \lim_{n\to\infty} \mathbb{P}\{\check{T}_n + S_n > t \,|\, \overline{\mathcal{F}}_t\} \\
&= \lim_{n\to\infty} \mathbb{1}_{\{S_n \geq t\}} + \mathbb{P}\{\check{T}_n > t - S_n,\, S_n \leq t \,|\, \overline{\mathcal{F}}_t\} \\
&= \lim_{n\to\infty} \mathbb{1}_{\{S_n \geq t\}} + \sum_{k=0,\frac{k+1}{2^n}\leq t} \mathbb{P}\left\{\check{T}_n > t - \frac{k+1}{2^n},\, S_n = \frac{k+1}{2^n} \,|\, \overline{\mathcal{F}}_t\right\} \\
&= \lim_{n\to\infty} \mathbb{1}_{\{S_n \geq t\}} + \sum_{k=0,\frac{k+1}{2^n}\leq t} \mathbb{P}\left\{\check{T}_n > t - \frac{k+1}{2^n} \,|\, \overline{\mathcal{F}}_t\right\} \mathbb{1}_{\{S_n = \frac{k+1}{2^n}\}}
\end{aligned}$$

If we apply Lemma 2.8.1 for $\mathfrak{R} := S_n$ and $\mathfrak{X} := \mathbb{1}_{\{\check{T}_n > t - \frac{k+1}{2^n}\}}$ we get

$$Z_t^D = \lim_{n\to\infty} \mathbb{1}_{\{S_n \geq t\}} + \sum_{k=0,\frac{k+1}{2^n}\leq t} \mathbb{P}\left\{\check{T}_n > t - \frac{k+1}{2^n} \,|\, \check{\mathcal{F}}^{(n)}_{t-\frac{k+1}{2^n}}\right\} \mathbb{1}_{\{S_n = \frac{k+1}{2^n}\}}$$

where $(\check{\mathcal{F}}_t^{(n)})_{t\geq 0} = (\bigcap_{\epsilon>0} \sigma\{X_{u+S_n} - X_{S_n} : 0 \leq u \leq t+\epsilon\})_{t\geq 0}$. Now we use the following theorem from [40, Theorem 8.22] .

Proposition 2.6.7 *Let $(N_t)_{t\geq 0}$ be a continuous local martingale such that $N_0 = 1$ and $\lim_{t\to\infty} N_t = 0$. Let $S_t = \sup_{s\leq t} N_s$. Set*

$$g := \sup\{t \geq 0 : N_t = S_\infty\} = \sup\{t \geq 0 : N_t = S_t\}$$

Then, the Azéma supermartingale associated with the honest time g is given by

$$Z_t^g = \mathbb{P}\{g > t \,|\, \mathcal{F}_t\} = \frac{N_t}{S_t}.$$

We apply Proposition 2.6.7 to our case for $g := \check{T}_n$ and the filtration $(\check{\mathcal{F}}_t^{(n)})_{t\geq 0} = (\bigcap_{\epsilon>0} \sigma\{X_{u+S_n} - X_{S_n} : 0 \leq u \leq t+\epsilon\})_{t\geq 0}$.

By definition, we have $\check{T}_n = \sup\{t \geq 0 : \check{X}_t^{(n)} + \alpha t = \inf\{\check{X}_u^{(n)} + \alpha u : u \geq 0\}\}$, where $\check{X}_t^{(n)} = X_{t+S_n} - X_{S_n}$. Set

$$N_t = \exp(-2\alpha(\check{X}_t^{(n)} + \alpha t)).$$

The process N is clearly a local martingale that verifies the conditions of the last proposition and we also have

$$\check{T}_n = \sup\{t \geq 0 : N_t = \sup_{s\geq 0} N_s\}.$$

Hence

$$\mathbb{P}\{\check{T}_n > t \,|\, \check{\mathcal{F}}_t^{(n)}\} = \exp\left(-2\alpha(\check{X}_t^{(n)} + \alpha t) + 2\alpha(\inf_{s\leq t}(\check{X}_s^{(n)} + \alpha s))\right).$$

Finally, we get the expression of the Azéma supermartingale associated with D as

$$Z_t^D = \lim_{n\to\infty} \mathbb{1}_{\{S_n\geq t\}} + \sum_{k=0,\frac{k+1}{2^n}\leq t}^{\infty} \exp\left(-2\alpha\left(\check{X}^{(n)}_{t-\frac{k+1}{2^n}} + \alpha\left(t - \frac{k+1}{2^n}\right)\right) + 2\alpha \inf_{0\leq s\leq t-\frac{k+1}{2^n}} \left(\check{X}_s^{(n)} + \alpha s\right)\right) \mathbb{1}_{\{S_n = \frac{k+1}{2^n}\}}$$

That is,

$$Z_t^D = \lim_{n\to\infty} \mathbb{1}_{\{S_n\geq t\}} + \left[\exp\left(-2\alpha(X_t + \alpha(t - S_n)) + 2\alpha\left(\inf_{s\leq t-S_n}(X_{s+S_n} + \alpha s)\right)\right]\mathbb{1}_{\{S_n<t\}}\right.$$

Thus, by sending $n\to\infty$, we get that

$$Z_t^D = \mathbb{1}_{\{S\geq t\}} + \left[\exp\left(-2\alpha(\check{X}_{t-S} + \alpha(t-S)\right) + 2\alpha\left(\inf_{s\leq t-S}(\check{X}_s + \alpha s)\right)\right]\mathbb{1}_{\{S<t\}}.$$

Now, using Theorem 2.6.6, every $(\mathfrak{M}_t)_{t\geq 0}$ $(\overline{\mathcal{F}}_t)_{t\geq 0}$-local martingale is a $(\overline{\mathcal{F}}_t^D)_{t\geq 0}$-semimartingale and decomposes as follows

$$\mathfrak{M}_t = \tilde{\mathfrak{M}}_t + \int_0^{t\wedge D} \frac{d\langle \mathfrak{M}, Z^D\rangle_s}{Z_s^D} - \int_D^t \frac{d\langle \mathfrak{M}, Z^D\rangle_s}{1 - Z_s^D},$$

where $(\tilde{\mathfrak{M}}_t)_{t\geq 0}$ denotes a $((\overline{\mathcal{F}}_t^D), \mathbb{P})$-local martingale.

We develop further the expression of Z^D to get an explicit integral representation of its local martingale part.

Lemma 2.6.8 *Let B be a standard Brownian motion and $\alpha > 0$. Define the process $(\mathfrak{H}_t)_{t\geq 0}$ by*

$$\mathfrak{H}_t = \exp\left(-2\alpha\left[(B_t + \alpha t) - \inf_{s\leq t}(B_s + \alpha s)\right]\right)$$

Put $I_t = \inf_{s\leq t}(B_s + \alpha s)$. Then,

$$\mathfrak{H}_t = 1 - 2\alpha\int_0^t \mathfrak{H}_u \, dB_u + 2\alpha I_t.$$

Proof. Applying Itô's formula on the semimartingale $\mathfrak{H}_t = F(B_t + \alpha t, I_t)$, where $F(x, y) = \exp(2\alpha(y - x))$, gives

$$\begin{aligned}
d\mathfrak{H}_t &= -2\alpha\mathfrak{H}_t dB_t - 2\alpha^2\mathfrak{H}_t dt + 2\alpha\mathfrak{H}_t dI_t + \frac{1}{2}(4\alpha^2)\mathfrak{H}_t dt \\
&= -2\alpha\mathfrak{H}_t dB_t + 2\alpha\mathfrak{H}_t dI_t \\
d\mathfrak{H}_t &= -2\alpha\mathfrak{H}_t dB_t + 2\alpha dI_t
\end{aligned}$$

The last line follows from the fact that the measure dI_t is carried on the set $\{t : B_t + \alpha t = I_t\} = \{t : \mathfrak{H}_t = 1\}$. □

Substituting formula from Lemma 2.6.8 into the expression for Z^D we get that

$$Z_t^D = \mathbb{1}_{\{S\geq t\}} + \mathbb{1}_{\{S<t\}}(1 - 2\alpha \int_0^{t-S} \exp(-2\alpha(\check{X}_u + \alpha u) + 2\alpha(\inf_{s\leq u}(\check{X}_s + \alpha s))\, d\check{X}_u + 2\alpha \inf_{s\leq t-S}(\check{X}_s + \alpha s)).$$

This can also be written as

$$Z_t^D = 1 + 2\alpha \mathbb{1}_{\{S<t\}} \inf_{s\leq t-S}(\check{X}_s + \alpha s) - 2\alpha \int_0^{(t-S)\vee 0} \exp\left(-2\alpha(\check{X}_u + \alpha u) + 2\alpha\left(\inf_{s\leq u}(\check{X}_s + \alpha s)\right)\right) d\check{X}_u.$$

Put $H_u := \exp(-2\alpha(\check{X}_u + \alpha u) + 2\alpha(\inf_{s\leq u}(\check{X}_s + \alpha s)))$. We want to write the integral $\int_0^{(t-S)\vee 0} H_u d\check{X}_u$ as a stochastic integral with respect to the original Brownian motion X. For that we consider the time-change $(C_t,\, t \geq 0)$ defined by $C_t := t + S$. It is clear that this a family of stopping times such that the maps $s \mapsto C_s$ are almost surely increasing and continuous. Using [50, Chapter V, Proposition 1.5], we get that for every bounded $(\overline{\mathcal{F}}_t)_{t\geq 0}$-progressively measurable process $(H_t)_{t\geq 0}$ we have

$$\int_{C_0}^{C_t} H_u\, dX_u = \int_0^t H_{C_u}\, dX_{C_u}.$$

In our case this becomes

$$\int_S^{t+S} H_{u-S}\, dX_u = \int_0^t H_u\, d\check{X}_u.$$

Hence,

$$\int_0^{(t-S)\vee 0} H_u\, d\check{X}_u = \int_S^{(t-S)\vee 0+S} H_{u-S}\; dX_u = \int_0^t \mathbb{1}_{u\geq S} H_{u-S}\, dX_u.$$

Finally,

$$Z_t^D = 1 + 2\alpha \mathbb{1}_{\{S<t\}} \inf_{s\leq t-S}(\check{X}_s + \alpha s) - 2\alpha \int_0^t A_u\, dX_u,$$

where

$$A_u = \mathbb{1}_{u\geq S} H_{u-S} = \mathbb{1}_{u\geq S} \exp\left(-2\alpha(\check{X}_{u-S} + \alpha u) + 2\alpha\left(\inf_{s\leq u-S}(\check{X}_s + \alpha s)\right)\right);$$

that is,

$$A_u = \mathbb{1}_{u\geq S} \exp\left(-2\alpha(X_u + \alpha u) + 2\alpha\left(\inf_{S\leq s\leq u}(X_s + \alpha s)\right)\right).$$

The process $t \mapsto 2\alpha \mathbb{1}_{\{S<t\}} \inf_{s \le t-S}(\check{X}_s + \alpha s))$ is decreasing and so the $(\overline{\mathcal{F}}_t)_{t\ge0}$-local martingale part of Z^D is equal to

$$-2\alpha \int_0^t A_u \, dX_u.$$

From the integral representation of martingales with respect to the Brownian filtration (see [50, Chapter 5, Theorem 3.4], every bounded $(\overline{\mathcal{F}}_t)_{t\ge0}$-martingale $(\mathfrak{M}_t)_{t\ge0}$ can be written as

$$\mathfrak{M}_t = C + \int_0^t \mu_s \, dX_s.$$

Such a process decomposes as a $(\overline{\mathcal{F}}_t^D)_{t\ge0}$-semimartingale in the following way

$$\mathfrak{M}_t = \tilde{\mathfrak{M}}_t - 2\alpha \int_0^{t\wedge D} \frac{\mu_s A_s \, ds}{Z_s^D} + 2\alpha \int_D^t \frac{\mu_s A_s \, ds}{1 - Z_s^D},$$

where $(\tilde{\mathfrak{M}}_t)_{t\ge0}$ is a $((\overline{\mathcal{F}}_t^D)_{t\ge0}, \mathbb{P})$-local martingale.

2.7 General facts about the α-Lipschitz minorant

Recall that a function $f : \mathbb{R} \mapsto \mathbb{R}$ admits an α-Lipschitz minorant m if and only if f is bounded below on compact sets, $\liminf_{t\to-\infty} f(t) - \alpha t > -\infty$, and $\liminf_{t\to+\infty} f(t) + \alpha t > -\infty$. In this case,

$$m(t) = \inf\{f(s) + \alpha|t-s| \colon s \in \mathbb{R}\}, t \in \mathbb{R}. \tag{2.8}$$

The following result is obvious from (2.8).

Lemma 2.7.1 *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function with an α-Lipschitz minorant. For $x, s \in \mathbb{R}$, define $g : \mathbb{R} \to \mathbb{R}$ by $g = x + f(s + \cdot)$. Write m_f and m_g for the respective α-Lipschitz minorants of f and g. Then $m_g = x + m_f(s + \cdot)$.*

The next result is a consequence of [2, Corollary 9.2] and Lemma 2.7.1, but we include a proof for the sake of completeness.

Lemma 2.7.2 *Consider a function $f : \mathbb{R} \to \mathbb{R}$ for which the α-Lipschitz minorant m exists. Fix $a \in \mathbb{R}$ such that $m(a) = f(a)$. Define $f^{\to} : \mathbb{R} \to \mathbb{R}$ by*

$$f^{\to}(t) = \begin{cases} f(a) + \alpha(t-a), & t \le a, \\ f(t), & t > a. \end{cases}$$

Denote the α-Lipschitz minorant of $f^{\to}$ by $m^{\to}$. Then $m(t) = m^{\to}(t)$ for all $t \ge a$.

Proof. From the expression of $m^{\to}$ we have for every $t \ge a$

$$m^{\to}(t) = \inf\{f(s) + \alpha|t-s| \colon s > a\} \wedge (m(a) + \alpha(t-a)).$$

Note that

$$\begin{aligned} m(a) + \alpha(t-a) &= \inf\{f(s) + \alpha|s-a| \colon s \in \mathbb{R}\} + \alpha(t-a) \\ &\le \inf\{f(s) + \alpha|s-a| \colon s \le a\} + \alpha(t-a) \\ &= \inf\{f(s) + \alpha(a-s) + \alpha(t-a) : s \le a\} \\ &= \inf\{f(s) + \alpha|t-s| \colon s \le a\} \end{aligned}$$

and so $m^{\rightarrow}(t) \le m(t)$ for $t \ge a$.

For the reverse inequality, it suffices to prove that

$$\inf\{f(s) + \alpha|t-s| \colon s \in \mathbb{R}\} \le m(a) + \alpha(t-a), t \ge a.$$

By definition, $m(a) \le f(s) + \alpha|s-a|$ for all $s \in \mathbb{R}$, and so, by the triangle inequality,

$$m(a) + \alpha(t-a) \ge f(s) + \alpha|s-a| + \alpha|a-t| \ge f(s) + \alpha|t-s|$$

for every $s \in \mathbb{R}$. □

The following result is [2, Lemma 9.4].

Lemma 2.7.3 *Let $f : \mathbb{R} \to \mathbb{R}$ be a càdlàg function with α-Lipschitz minorant $m : \mathbb{R} \to \mathbb{R}$. Set*

$$\mathbf{d} := \inf\{t > 0 : f(t) \wedge f(t-) = m(t)\},$$
$$\mathbf{s} := \inf\left\{t > 0 : f(t) \wedge f(t-) - \alpha t \le \inf\{f(u) - \alpha u : u \le 0\}\right\},$$

and

$$\mathbf{e} := \inf\left\{t \ge \mathbf{s} : f(t) \wedge f(t-) + \alpha(t-\mathbf{s}) = \inf\{f(u) + \alpha(u-\mathbf{s}) : u \ge \mathbf{s}\}\right\}.$$

Suppose that $f(\mathbf{s}) \le f(\mathbf{s}-)$. Then, $\mathbf{e} = \mathbf{d}$.

Let us also state here a simple expression of the time $\mathbf{s}$ when the time zero is a contact point.

Lemma 2.7.4 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with α-Lipschitz minorant $m : \mathbb{R} \to \mathbb{R}$, and suppose that we have $m(0) = f(0) = 0$, then $\mathbf{s}$ defined in Lemma 2.7.3 takes the following form*

$$\mathbf{s} = \inf\{t > 0 : f(t) = \alpha t\}.$$

Proof. This is straightforward, as

$$0 \ge \inf\{f(u) - \alpha u : u \le 0\} \ge \inf\{f(u) + \alpha|u| \colon u \in \mathbb{R}\} = m(0) = 0$$

because $f(0) = 0$. □

The following lemma describes the shape of the α-Lipschitz minorant between two consecutive points of the contact set. It is [2, Lemma 8.3].

Lemma 2.7.5 *Suppose that $f : \mathbb{R} \to \mathbb{R}$ that is a càdlàg with α-Lipschitz minorant $m : \mathbb{R} \to \mathbb{R}$. The set $\{t \in \mathbb{R} : m(t) = f(t) \wedge f(t-)\}$ is closed. If t'¡t" are such that $f(t') \wedge f(t'-) = m(t')$, $f(t'') \wedge f(t''-) = m(t'')$, and $f(t) \wedge f(t-) > m(t)$ for all $t' < t < t''$, then setting $t^* = (f(t'') \wedge f(t''-) - f(t') \wedge f(t'-) + \alpha(t'' + t'))/(2\alpha)$,*

$$m(t) = \begin{cases} f(t') \wedge f(t'-) + \alpha(t-t'), & t' \le t \le t^*, \\ f(t'') \wedge f(t'') + \alpha(t''-t), & , t^* \le t \le t''. \end{cases}$$

2.8 Two random time lemmas

We detail in this section two lemmas that we used previously in Section 2.3 and Section 2.6. We consider here $(X_t)_{t\in\mathbb{R}}$ to be two-sided Lévy process with $(\mathcal{F}_t)_{t\in\mathbb{R}}$ as its canonical right-continuous filtration; that is, $\mathcal{F}_t := \bigcap_{\epsilon>0} \sigma\{X_s, -\infty < s \le t+\epsilon\}$, $t \in \mathbb{R}$.

Lemma 2.8.1 *Let $\mathfrak{R}$ be a $(\mathcal{F}_t)_{t\in\mathbb{R}}$-stopping time that takes values in a countable subset of $\mathbb{R}$. Define the σ-fields $\check{\mathcal{F}}_t := \bigcap_{\epsilon>0} \sigma(\{X_{u+\mathfrak{R}} - X_{\mathfrak{R}} : 0 \le u \le t+\epsilon\})$, $t \ge 0$, and put $\check{\mathcal{F}}_\infty = \bigvee_{t\ge 0} \check{\mathcal{F}}_\infty$. For every random variable $\mathfrak{X}$ measurable with respect to $\check{\mathcal{F}}_\infty$ we have for every $t \in \mathbb{R}$ and $r \le t$ that almost surely*

$$\mathbb{E}\left[\mathfrak{X} \,|\, \mathcal{F}_t\right] \mathbb{1}_{\{\mathfrak{R}=r\}} = \mathbb{E}\left[\mathfrak{X}\mathbb{1}_{\{\mathfrak{R}=r\}} \,|\, \mathcal{F}_t\right] = \mathbb{E}\left[\mathfrak{X} \,|\, \check{\mathcal{F}}_{t-r}\right] \mathbb{1}_{\{\mathfrak{R}=r\}}.$$

Proof. The first equality is trivial because the event $\{\mathfrak{R} = r\}$ is $\mathcal{F}_r$-measurable and hence $\mathcal{F}_t$-measurable. We therefore need only prove the second equality.

By a monotone class argument, it suffices to show that the second inequality holds for $\mathfrak{X} = \prod_{i=1}^n f_i(X_{u_i+\mathfrak{R}} - X_{\mathfrak{R}})$, where $0 \le u_1 < u_2 < \ldots < u_n$ and $f_1, \ldots, f_n$ are nonnegative Borel functions.

We have for any $\mathcal{F}_t$-measurable nonnegative random variable A_t that

$$\begin{aligned}
\mathbb{E}\left[A_t \mathbb{1}_{\{\mathfrak{R}=r\}} \prod_{i=1}^n (f_i(X_{u_i+\mathfrak{R}} - X_{\mathfrak{R}}))\right] &= \mathbb{E}\left[A_t \mathbb{1}_{\{\mathfrak{R}=r\}} \prod_{i=1}^n (f_i(X_{u_i+r} - X_r))\right] \\
&= \mathbb{E}\Bigg[A_t \mathbb{1}_{\{\mathfrak{R}=r\}} \prod_{u_i<t-r} (f_i(X_{u_i+r} - X_r)) \\
&\times \mathbb{E}\Bigg[\prod_{u_i \ge t-r} (f_i(X_{u_i+r} - X_r)) \,|\, \mathcal{F}_t\Bigg]\Bigg] \\
&= \mathbb{E}\Bigg[A_t \mathbb{1}_{\{\mathfrak{R}=r\}} \prod_{u_i<t-r} (f_i(X_{u_i+r} - X_r)) \\
&\times \mathbb{E}\Bigg[\prod_{u_i \ge t-r} (f_i(X_{u_i+r} - X_t + X_{(t-r)+r} - X_r)) \,|\, \mathcal{F}_t\Bigg]\Bigg].
\end{aligned}$$

Using the independence and stationarity of the increments of the Lévy process X gives

$$\begin{aligned}
\mathbb{E}\left[A_t \mathbb{1}_{\{\mathfrak{R}=r\}} \prod_{i=1}^n (f_i(X_{u_i+\mathfrak{R}} - X_{\mathfrak{R}}))\right] &= \mathbb{E}\Bigg[A_t \mathbb{1}_{\{\mathfrak{R}=r\}} \prod_{u_i<t-r} (f_i(X_{u_i+r} - X_r)) \\
&\times \prod_{u_i \ge t-r}^{n} (g_i(X_{(t-r)+r} - X_r))\Bigg]
\end{aligned}$$

for $g_i := \mathbb{E}\left[f_i(X_{u_i+r-t}+\cdot)\right]$. Thus,

$$\mathbb{E}\left[A_t \mathbb{1}_{\{\mathfrak{R}=r\}} \prod_{i=1}^{n} (f_i(X_{u_i+\mathfrak{R}} - X_{\mathfrak{R}}))\right] = \mathbb{E}\Bigg[A_t \mathbb{1}_{\{\mathfrak{R}=r\}} \prod_{u_i<t-r} (f_i(X_{u_i+\mathfrak{R}} - X_{\mathfrak{R}})) \times \prod_{u_i \geq t-r} (g_i(X_{(t-r)+\mathfrak{R}} - X_{\mathfrak{R}}))\Bigg].$$

Because the process $(X_{u+\mathfrak{R}} - X_{\mathfrak{R}})_{u\geq 0}$ is itself a Lévy process with respect to the filtration $(\check{\mathcal{F}}_t)_{t\geq 0}$ and it has the same distribution as $(X_t)_{t\geq 0}$, we have

$$\mathbb{E}\left[\prod_{i=1}^{n} (f_i(X_{u_i+\mathfrak{R}} - X_{\mathfrak{R}})) \,|\, \check{\mathcal{F}}_{t-r}\right] = \prod_{u_i<t-r} (f_i(X_{u_i+\mathfrak{R}} - X_{\mathfrak{R}})) \prod_{u_i\geq t-r} (g_i(X_{(t-r)+\mathfrak{R}} - X_{\mathfrak{R}}))$$

Thus we finally get the desired equality

$$\mathbb{E}\left[A_t \mathbb{1}_{\{\mathfrak{R}=r\}} \prod_{i=1}^{n} (f_i(X_{u_i+\mathfrak{R}} - X_{\mathfrak{R}}))\right] = \mathbb{E}\left[A_t \mathbb{1}_{\{\mathfrak{R}=r\}} \mathbb{E}\left[\prod_{i=1}^{n} (f_i(X_{u_i+\mathfrak{R}} - X_{\mathfrak{R}})) \,|\, \check{\mathcal{F}}_{t-r}\right]\right]$$

□

Lemma 2.8.2 *Suppose that almost surely* $\lim_{t\to\infty} X_t = \infty$ *and that zero is regular for* $(0,\infty)$ *for the process* $(X_t)_{t\in\mathbb{R}}$. *Let* $\mathfrak{R}$ *be an almost surely finite* $(\mathcal{F}_t)_{t\in\mathbb{R}}$*-stopping time. Put* $(\check{X}_t)_{t\geq 0} := (X_{t+\mathfrak{R}} - X_{\mathfrak{R}})_{t\geq 0}$. *Consider the random time* $\mathfrak{L} := \sup\{t \geq 0 : \check{X}_t \wedge \check{X}_{t-} = \inf\{\check{X}_u : u \geq 0\}\}$. *Then, setting* $\mathfrak{D} := \mathfrak{R} + \mathfrak{L}$, *the* σ*-field* $\sigma\{\check{X}_{t+\mathfrak{L}} - \check{X}_{\mathfrak{L}} : t \geq 0\}$ *is independent of the* σ*-field* $\mathcal{F}_{\mathfrak{D}-} \vee \sigma\{X_{\mathfrak{D}}\} = \mathcal{F}_{\mathfrak{D}-}$.

Proof. We begin with an observation. Define the σ-fields $\check{\mathcal{F}}_t := \bigcap_{\epsilon>0} \sigma(\{X_{s+\mathfrak{R}} - X_{\mathfrak{R}} : 0 \leq s \leq t+\epsilon\})$, $t \geq 0$, and put $\check{\mathcal{F}}_\infty = \bigvee_{t\geq 0} \check{\mathcal{F}}_\infty$. It follows from the part of the proof of Theorem 2.3.6 which comes before we employ the current lemma that $\sigma\{\check{X}_{t+\mathfrak{L}} - \check{X}_{\mathfrak{L}} : t \geq 0\}$ is independent of

$$\check{\mathcal{F}}_{\mathfrak{L}} := \sigma\{\check{\xi}_{\mathfrak{L}} : (\check{\xi}_t)_{t\geq 0} \text{ is an optional process with respect to the filtration } (\check{\mathcal{F}}_t)_{t\geq 0}\}.$$

Returning to the statement of the lemma, and by noticing that $X_D = X_{\mathfrak{R}} + \check{X}_{\mathfrak{L}}$, it suffices to prove for any bounded, nonnegative $\sigma\{\check{X}_{t+\mathfrak{L}} - \check{X}_{\mathfrak{L}} : t \geq 0\}$-measurable random variable $\mathfrak{Y}$, any bounded, nonnegative, continuous functions $g^1, \ldots, g^n, h^1, h^2$, and any previsible processes $\xi^1, \ldots, \xi^n$ with respect to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$ that

$$\mathbb{E}\left[\mathfrak{Y} \prod_{i=1}^{n} g^i(\xi^i_{\mathfrak{D}}) h^1(X_{\mathfrak{R}}) h^2(\check{X}_{\mathfrak{L}})\right] = \mathbb{E}[\mathfrak{Y}]\mathbb{E}\left[\prod_{i=1}^{n} g^i(\xi^i_{\mathfrak{D}}) h^1(X_{\mathfrak{R}}) h^2(\check{X}_{\mathfrak{L}})\right]. \tag{2.9}$$

However, $(\prod_{i=1}^{n} g^i(\xi^i_t))_{t\in\mathbb{R}}$ is itself a previsible process, so it suffices for (2.9) to prove for any bounded, nonnegative $\sigma\{\check{X}_{t+\mathfrak{L}} - \check{X}_{\mathfrak{L}} : t \geq 0\}$-measurable random variable $\mathfrak{Y}$, any bounded, nonnegative process ξ that is previsible with respect to filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$, and any bounded, nonnegative, continuous functions h^1, h^2 that

$$\mathbb{E}[\mathfrak{Y}\xi_{\mathfrak{D}} h^1(X_{\mathfrak{R}})h^2(\check{X}_{\mathfrak{L}})] = \mathbb{E}[\mathfrak{Y}]\mathbb{E}[\xi_{\mathfrak{D}} h^1(X_{\mathfrak{R}})h^2(\check{X}_{\mathfrak{L}})]. \tag{2.10}$$

A stochastic process viewed as a map from $\Omega \times \mathbb{R}$ to $\mathbb{R}$ is previsible with respect to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$ if and only if it is measurable with respect to the σ-field generated by the maps $(\omega, t) \mapsto \mathbb{1}_{t>T(\omega)}$, where T ranges through the set of $\mathbb{R}\cup\{+\infty\}$-valued $(\mathcal{F}_t)_{t\in\mathbb{R}}$-stopping times (see [55, Chapter IV, Corollary 6.9] for the analogous fact about previsible processes indexed by $(0, \infty)$). Also, note that the collection of the sets $\mathcal{A} = \{\{(\omega, t) : t > T(\omega)\} : T$ is a stopping time$\}$ is a π-system because the minimum of two stopping times is a stopping time. Hence, to establish (2.10), it suffices by a monotone class argument to show for any $\mathbb{R} \cup \{+\infty\}$ -valued $(\mathcal{F}_t)_{t\in\mathbb{R}}$-stopping time T that

$$\mathbb{E}[\mathfrak{Y}\mathbb{1}_{\{\mathfrak{D}>T\}} h^1(X_{\mathfrak{R}})h^2(\check{X}_{\mathfrak{L}})] = \mathbb{E}[\mathfrak{Y}]\mathbb{E}[\mathbb{1}_{\{\mathfrak{D}>T\}} h^1(X_{\mathfrak{R}})h^2(\check{X}_{\mathfrak{L}})]. \tag{2.11}$$

Because we have that $\mathbb{1}_{\{D>T\}} = \lim_{n\to\infty} \mathbb{1}_{\{D>T\wedge n\}}$, it further suffices to check (2.11) for T an $\mathbb{R}$-valued $(\mathcal{F}_t)_{t\in\mathbb{R}}$-stopping time.

Consider (2.11) in the special case when $\mathfrak{R}$ and T take values in the countable set $\{r_k := \frac{k}{2^n}, k \in \mathbb{Z}\}$. We then have

$$\begin{aligned}
\mathbb{E}[\mathfrak{Y}\mathbb{1}_{\{\mathfrak{D}>T\}} h^1(X_{\mathfrak{R}})h^2(\check{X}_{\mathfrak{L}})] &= \mathbb{E}[\mathfrak{Y}\mathbb{1}_{\{\mathfrak{R}+\mathfrak{L}>T\}} h^1(X_{\mathfrak{R}})h^2(\check{X}_{\mathfrak{L}})] \\
&= \sum_{k,l\in\mathbb{Z}} \mathbb{E}[\mathfrak{Y}\mathbb{1}_{\{\mathfrak{L}>r_l-r_k\}} h^1(X_{r_k})h^2(\check{X}_{\mathfrak{L}})\mathbb{1}_{\{\mathfrak{R}=r_k, T=r_l\}}] \\
&= \sum_{l<k} \mathbb{E}[\mathfrak{Y}\mathbb{1}_{\{\mathfrak{R}=r_k, T=r_l\}} h^1(X_{r_k})h^2(\check{X}_{\mathfrak{L}})] \\
&+ \sum_{k\le l} \mathbb{E}[\mathfrak{Y}\mathbb{1}_{\{\mathfrak{L}>r_l-r_k\}}\mathbb{1}_{\{\mathfrak{R}=r_k, T=r_l\}} h^1(X_{r_k})h^2(\check{X}_{\mathfrak{L}})] \\
&= \sum_{l<k} \mathbb{E}[\mathfrak{Y}]\mathbb{E}[\mathbb{1}_{\{\mathfrak{R}=r_k, T=r_l\}} h^1(X_{r_k})h^2(\check{X}_{\mathfrak{L}})] \\
&+ \sum_{k\le l} \mathbb{E}[\mathbb{E}[\mathfrak{Y}\mathbb{1}_{\{\mathfrak{L}>r_l-r_k\}} h^2(\check{X}_{\mathfrak{L}}) \,|\, \mathcal{F}_{r_l}]\mathbb{1}_{\{\mathfrak{R}=r_k, T=r_l\}} h^1(X_{r_k})]
\end{aligned}$$

By applying Lemma 2.8.1 for $\mathfrak{X} = \mathfrak{Y}\mathbb{1}_{\{\mathfrak{L}>r_l-r_k\}} h^2(\check{X}_{\mathfrak{L}})$, we have, for $k \le l$, that

$$\mathbb{E}[\mathfrak{Y}\mathbb{1}_{\{\mathfrak{L}>r_l-r_k\}} h^2(\check{X}_{\mathfrak{L}})|\mathcal{F}_{r_l}]\mathbb{1}_{\{\mathfrak{R}=r_k, T=r_l\}} = \mathbb{E}[\mathfrak{Y}\mathbb{1}_{\{\mathfrak{L}>r_l-r_k\}} h^2(\check{X}_{\mathfrak{L}})|\check{\mathcal{F}}_{r_l-r_k}]\mathbb{1}_{\{\mathfrak{R}=r_k, T=r_l\}}.$$

Moreover, if we let $\check{A}$ to be an event in $\check{\mathcal{F}}_{r_l-r_k}$, then

$$\mathbb{E}[\mathfrak{Y}\mathbb{1}_{\{\mathfrak{L}>r_l-r_k\}\cap\check{A}} h^2(\check{X}_{\mathfrak{L}})] = \mathbb{E}[\mathfrak{Y}]\mathbb{E}[\mathbb{1}_{\{\mathfrak{L}>r_l-r_k\}\cap\check{A}} h^2(\check{X}_{\mathfrak{L}})],$$

because the process $(\check{\xi})_{t\ge0} = (\mathbb{1}_{\{t>r_l-r_k\}\cap\check{A}} h^2(\check{X}_t))_{t\ge0}$ is clearly an $(\check{\mathcal{F}}_t)_{t\ge0}$-optional process (as it is the product of the left-continuous, right-limited $(\check{\mathcal{F}}_t)_{t\ge0}$-adapted process $(\mathbb{1}_{\{t>r_l-r_k\}\cap\check{A}})_{t\ge0}$ and the càdlàg $(\check{\mathcal{F}}_t)_{t\ge0}$-adapted process $(h^2(\check{X}_t)_{t\ge0})$). Hence

$$\mathbb{E}[\mathfrak{Y}\mathbb{1}_{\{\mathfrak{L}>r_l-r_k\}}|\check{\mathcal{F}}_{r_l-r_k}] = \mathbb{E}[\mathfrak{Y}]\mathbb{E}[\mathbb{1}_{\{\mathfrak{L}\ge r_l-r_k\}} h^2(\check{X}_{\mathfrak{L}})|\check{\mathcal{F}}_{r_l-r_k}].$$

Substituting in this equality gives

$$\begin{aligned}\mathbb{E}[\mathfrak{Y}\mathbb{1}_{\{\mathfrak{D}>T\}}h^1(X_\mathfrak{R})h^2(\check{X}_\mathfrak{L})] &= \sum_{l<k}\mathbb{E}[\mathfrak{Y}]\mathbb{E}[\mathbb{1}_{\{\mathfrak{R}=r_k, T=r_l\}}h^1(X_\mathfrak{R})h^2(\check{X}_\mathfrak{L})] \\ &+ \sum_{k\le l}\mathbb{E}[\mathfrak{Y}]\mathbb{E}[\mathbb{1}_{\{\mathfrak{L}>r_l-r_k,\, R=r_k,\, T=r_l\}}h^1(X_\mathfrak{R})h^2(\check{X}_\mathfrak{L})] \\ &= \mathbb{E}[\mathfrak{Y}]\mathbb{E}[\mathbb{1}_{\{\mathfrak{L}>T-\mathfrak{R}\}}h^1(X_\mathfrak{R})h^2(\check{X}_\mathfrak{L})] \\ &= \mathbb{E}[\mathfrak{Y}]\mathbb{E}[\mathbb{1}_{\{\mathfrak{D}>T\}}h^1(X_\mathfrak{R})h^2(\check{X}_\mathfrak{L})].\end{aligned}$$

We have thus proved (2.11) when $\mathfrak{R}$ and T both take values in the set $\{r_k := \frac{k}{2^n}, k \in \mathbb{Z}\}$. Suppose now that T is an arbitrary $\mathbb{R}$-valued stopping time but that $\mathfrak{R}$ still takes values in $\{r_k := \frac{k}{2^n}, k \in \mathbb{Z}\}$. For $m \in \mathbb{N}$ set $T_m := \frac{k}{2^m}$ when $\frac{k-1}{2^m} < T \le \frac{k}{2^m}$, $k \in \mathbb{Z}$. Thus $(T_m)_{m\in\mathbb{N}}$ is a decreasing sequence of $(\mathcal{F}_t)_{t\in\mathbb{R}}$-stopping times converging to T. Taking (2.11) with T replaced by T_m and letting $m \to \infty$ we get (2.11) for $\mathfrak{R}$ taking values in the set $\{r_k := \frac{k}{2^n}, k \in \mathbb{Z}\}$ and general $\mathbb{R}$-valued T.

We now want to extend to the completely general case of (2.11). Put $(\check{X}^\mathfrak{R}_t)_{t\ge 0} := (X_{t+\mathfrak{R}} - X_\mathfrak{R})_{t\ge 0}$. Denote the corresponding random variables $\mathfrak{L}$, $\mathfrak{Y}$, and $\mathfrak{D}$ by $\mathfrak{L}^\mathfrak{R}$,$\mathfrak{Y}^\mathfrak{R}$, and $\mathfrak{D}^\mathfrak{R}$, respectively. Recalling that $\mathfrak{Y}^\mathfrak{R}$ is an arbitrary bounded, nonnegative random variable measurable with respect to $\sigma\{\check{X}^\mathfrak{R}_{t+\mathfrak{L}^\mathfrak{R}} - \check{X}^\mathfrak{R}_{\mathfrak{L}^\mathfrak{R}}, t \ge 0\}$, it suffices by a monotone class argument it suffices to show (2.11) in the special case where

$$\mathfrak{Y}^\mathfrak{R} = \prod_{i=1}^m f^i(\check{X}^\mathfrak{R}_{t_i+\mathfrak{L}^\mathfrak{R}} - \check{X}^\mathfrak{R}_{\mathfrak{L}^\mathfrak{R}}) = \prod_{i=1}^m f^i(X_{t_i+\mathfrak{L}^\mathfrak{R}+\mathfrak{R}} - X_{\mathfrak{L}^\mathfrak{R}+\mathfrak{R}})$$

for f^i , $i = 1, \ldots, m$, bounded, nonnegative, continuous functions and $0 \le t_1 < \cdots < t_m$.

For $n \in \mathbb{N}$ set $\mathfrak{R}_n := \frac{k}{2^n}$ when $\frac{k-1}{2^n} < \mathfrak{R} \le \frac{k}{2^n}$, $k \in \mathbb{Z}$. Thus $(\mathfrak{R}_n)_{n\in\mathbb{N}}$ is a decreasing sequence of $(\mathcal{F}_t)_{t\in\mathbb{R}}$-stopping times converging to $\mathfrak{R}$. Note that

$$\mathfrak{L}^{\mathfrak{R}_\mathfrak{n}} = \operatorname{argmin}\{X_{u+\mathfrak{R}} - X_\mathfrak{R} : u \ge \mathfrak{R}_\mathfrak{n} - \mathfrak{R}\} + \mathfrak{R} - \mathfrak{R}_n.$$

Thus, if $\mathfrak{L}^\mathfrak{R} = 0$, then $\mathfrak{D}^{\mathfrak{R}_\mathfrak{n}} \downarrow \mathfrak{D}^\mathfrak{R}$ by the right-continuity of the sample paths of X. On the other hand, if $\mathfrak{L}^\mathfrak{R} > 0$, then, for n large enough, we have that $\mathfrak{D}^{\mathfrak{R}_\mathfrak{n}} = \mathfrak{D}^\mathfrak{R}$. Hence, by applying the special case of (2.11) for the stopping times $\mathfrak{R}_n$ taking discrete values, and using the fact that X has càdlàg paths we get

$$\begin{aligned}
&\mathbb{E}[\mathfrak{Y}^{\mathfrak{R}}\mathbb{1}_{\{\mathfrak{D}^{\mathfrak{R}}>T\}}h^1(X_{\mathfrak{R}})h^2(\check{X}^{\mathfrak{R}}_{\mathfrak{L}^{\mathfrak{R}}})] \\
&\quad = \mathbb{E}\left[\prod_{i=1}^{m} f^i(X_{t_i+\mathfrak{D}^{\mathfrak{R}}} - X_{\mathfrak{D}^{\mathfrak{R}}})\mathbb{1}_{\{\mathfrak{D}^{\mathfrak{R}}>T\}}h^1(X_{\mathfrak{R}})h^2(X_{\mathfrak{D}^{\mathfrak{R}}} - X_{\mathfrak{R}})\right] \\
&\quad = \lim_{n\to\infty} \mathbb{E}\left[\prod_{i=1}^{m} f^i(X_{t_i+\mathfrak{D}^{\mathfrak{R}_n}} - X_{\mathfrak{D}^{\mathfrak{R}_n}})\mathbb{1}_{\{\mathfrak{D}^{\mathfrak{R}_n}>T\}}h^1(X_{\mathfrak{R}_n})h^2(X_{\mathfrak{D}^{\mathfrak{R}_n}} - X_{\mathfrak{R}_n})\right] \\
&\quad = \lim_{n\to\infty} \mathbb{E}\left[\prod_{i=1}^{m} f^i(X_{t_i+\mathfrak{D}^{\mathfrak{R}_n}} - X_{\mathfrak{D}^{\mathfrak{R}_n}})\right] \mathbb{E}[\mathbb{1}_{\{\mathfrak{D}^{\mathfrak{R}_n}>T\}}h^1(X_{\mathfrak{R}_n})h^2(X_{\mathfrak{D}^{\mathfrak{R}_n}} - X_{\mathfrak{R}_n})] \\
&\quad = \mathbb{E}\left[\prod_{i=1}^{m} f^i(X_{t_i+\mathfrak{D}^{\mathfrak{R}}} - X_{\mathfrak{D}^{\mathfrak{R}}})\right] \mathbb{E}[\mathbb{1}_{\{\mathfrak{D}^{\mathfrak{R}}>T\}}h^1(X_{\mathfrak{R}})h^2(X_{\mathfrak{D}^{\mathfrak{R}}} - X_{\mathfrak{R}})] \\
&\quad = \mathbb{E}[\mathfrak{Y}^{\mathfrak{R}}]\mathbb{E}[\mathbb{1}_{\{\mathfrak{D}^{\mathfrak{R}}>T\}}h^1(X_{\mathfrak{R}})h^2(X_{\mathfrak{D}^{\mathfrak{R}}} - X_{\mathfrak{R}})].
\end{aligned}$$

It remains to show that $\mathcal{F}_{\mathfrak{D}-} \vee \sigma\{X_{\mathfrak{D}}\} = \mathcal{F}_{\mathfrak{D}-}$. This, however, is a consequence of [53, Corollary 1(ii)] from which it follows that $X_{\mathfrak{D}} = X_{\mathfrak{D}-}$ when zero is regular for $(0, \infty)$ for the process $(X_t)_{t\in\mathbb{R}}$.

□

Chapter 3

Scalar conservation laws with white noise initial data

This chapter is based on the article [41] that is published in *Probability Theory and Related Fields*.

3.1 Introduction

We are interested in the following conservation law problem

$$\begin{cases} \rho_t = (H(\rho))_x \ , \text{ for } \quad t > 0, x \in \mathbb{R} \\ \rho(x,0) = \xi(x) \ , \quad x \in \mathbb{R} \end{cases} \tag{3.1}$$

where H is a C^2 strictly convex function with superlinear growth at infinity and ξ is a white noise derived from a standard linear Brownian motion. A question of interest is to describe the law of the process $\rho(\cdot,t)$ at any given time $t > 0$.

3.1.1 Background

There is a straightforward link between the scalar conservation law and the Hamilton-Jacobi PDE that was expanded in the introduction of the thesis. We will briefly recall some of these deterministic facts here below. If one defines the spatial anti-derivative of $\rho(\cdot,t)$ for any fixed t by

$$u(x,t) = \int_{-\infty}^{x} \rho(y,t)dy$$

and the potential

$$U_0(x) = \int_{-\infty}^{x} \xi(y)dy$$

then u solves the Hamilton-Jacobi PDE

$$u_t = H(u_x) \tag{3.2}$$

and is determined by the Hopf-Lax variational formula (see [20][Theorem 4, Chapter 3.3])

$$u(x,t) = \sup_{y\in\mathbb{R}} \left(U_0(y) - tL\left(\frac{y-x}{t}\right)\right) \tag{3.3}$$

where L is the Legendre transform of H defined as

$$L(q) = \max_{p\in\mathbb{R}} (qp - H(p))$$

The rightmost maximizer $y(x,t)$ in the equation (3.3) is called the backward Lagrangian, and is directly linked to the entropy solution ρ of the scalar conservation law (3.1) by the Lax-Oleinik formula (see [20][Theorem 1, Chapter 3.4])

$$\rho(x,t) = (H')^{-1}\left(\frac{y(x,t)-x}{t}\right) = L'\left(\frac{y(x,t)-x}{t}\right)$$

The reader may be familiar with this other form of the Hamilton-Jacobi PDE

$$u_t + H(u_x) = 0 \tag{3.4}$$

If we denote by u a solution of (3.4), then it is easy to see that $\tilde{u}(x,t) := -u(x,t)$ verifies $\tilde{u}_t = \tilde{H}(\tilde{u}_x)$ for the Hamiltonian $\tilde{H}(\rho) = H(-\rho)$. We will thus only restrict ourselves to the version of the scalar conservation law in (3.1).
When the Hamiltonian H takes the simple form $H(\rho) = \frac{\rho^2}{2}$, the scalar conservation law (3.1) is called Burgers equation and is written $\rho_t = \rho\rho_x$. The Lax-Oleinik formula simplifies to

$$\rho(x,t) = \frac{y(x,t)-x}{t} \tag{3.5}$$

The Burgers equation has seen an extensive interest when the initial data $\rho(\cdot,0)$ is random in the context of Burgers turbulence. We will present hereby the most relevant results in this area.

3.1.2 Burgers equation when $\rho(\cdot,0)$ is a Brownian white noise

This is the case when the initial potential U_0 is expressed as

$$U_0(x) = \sigma B(x), \quad x\in\mathbb{R} \tag{3.6}$$

where $\sigma > 0$ is a diffusion factor and B is a two-sided standard linear Brownian motion. In a remarkable paper [25] with the aim of studying the global behavior of isotonic estimators, Groeneboom completely determined the statistics of the process

$$\left(V(a) := \sup\left\{x\in\mathbb{R} : B(x) - (x-a)^2 \text{ is maximal}\right\}, a\in\mathbb{R}\right)$$

He showed that this process is pure-jump with jump kernels expressed in terms of Airy functions. By the Hopf-Lax formula and (3.5), this process is related to the solution of the Burgers equation with Brownian white noise initial data.

More precisely, let $\rho_\sigma(x,t)$ be the entropy solution of Burgers equation when the initial potential is determined by (3.6). Since in the Burgers case the Hamiltonian enjoys the same scaling as the Brownian motion. It follows that for every $t>0$, the process $(\rho_\sigma(x,t), x\in\mathbb{R})$ has the same law as $(\sigma^{\frac{2}{3}}t^{-\frac{1}{3}}\rho_1(x((\sigma t)^{-\frac{2}{3}},1), x\in\mathbb{R})$. The following theorem gives a precise description of the law of the entropy solution at time $t=1$.

Theorem 3.1.1 ([25]) *The process $(\rho_{\frac{1}{\sqrt{2}}}(x,1), x\in\mathbb{R})$ is a stationary piecewise-linear Markov process with generator $\mathcal{A}$ acting on a test function $\varphi\in C_c^\infty(\mathbb{R})$ as*

$$\mathcal{A}\varphi(y) = -\varphi'(y) + \int_y^\infty (\varphi(z)-\varphi(y))n(y,z)dz$$

The jump density n is given by the formula

$$n(y,z) = \frac{J(z)}{J(y)}K(z-y) \quad , \quad z>y$$

where J and Z are positive functions defined on the line and positive half-line respectively, whose Laplace transforms

$$j(q) = \int_{-\infty}^\infty e^{qy}J(y)dy, \qquad k(q) = \int_0^\infty e^{-qy}K(y)dy$$

are meromorphic functions on $\mathbb{C}$ given by

$$j(q) = \frac{1}{\mathrm{Ai}(q)}, \qquad k(q) = -2\frac{d^2}{dq^2}\log\mathrm{Ai}(q)$$

where Ai *denotes the first Airy function.*

Remark 3.1.2 *For general $t>0$, the process $(\rho_{\frac{1}{\sqrt{2}}}(x,t), x\in\mathbb{R})$ is also a stationary piecewise-linear Markov process with generator*

$$\mathcal{A}^t\varphi(y) = -\frac{1}{t}\varphi'(y) + \int_y^\infty t^{-\frac{1}{3}}n(yt^{\frac{1}{3}}, zt^{\frac{1}{3}})(\varphi(z)-\varphi(y))dz$$

In particular, the linear pieces have slope $-\frac{1}{t}$.

3.1.3 Burgers equation when $\rho(\cdot,0)$ is a spectrally negative Lévy process

A Lévy process $(X_t)_{t\in\mathbb{R}}$ is a process with stationary independent increments and such that $X_0=0$. By spectrally negative Lévy process, we mean a process that has only downward jumps. For the Burgers equation, Bertoin in [12] proved a remarkable closure theorem for this class of initial data. We quote here his result.

Theorem 3.1.3 ([12]) *Consider Burgers equation of the form $\rho_t + \rho\rho_x = 0$ with initial data $\xi(x)$ which is a spectrally negative Lévy process for $x \geq 0$ and $\xi(x) = 0$ for $x < 0$. Assume that the expected value of $\xi(1)$ is positive. Then for each fixed $t > 0$, the backward Lagrangian $y(x,t)$ has the property that $(y(x,t) - y(0,t))_{x\geq 0}$ is independent of $y(0,t)$ and is in the parameter x a subordinator, i.e. a nondecreasing Lévy process. Its distribution is that of the first passage process*

$$x \mapsto \inf\{z \geq 0 : t\xi(z) + z > x\}$$

Furthermore, if we denote by $\psi(s)$ and $\Theta(t,s)$ $(s \geq 0)$ respectively the Laplace exponents of $\xi(x)$ and $y(x,t) - y(x,0)$,

$$\mathbb{E}[\exp(s\xi(x))] = \exp(x\psi(s))$$
$$\mathbb{E}[\exp(s(y(x,t) - y(0,t)))] = \exp(x\Theta(t,s))$$

then we have the functional identity

$$\psi(t\Theta(t,s)) + \Theta(t,s) = s$$

Moreover, the process $(\rho(x,t) - \rho(0,t))_{x\geq 0}$ is a Lévy process, and its Laplace exponent $\psi(t,q)$ verifies the Burgers equation

$$\psi_t + \psi\psi_q = 0 \tag{3.7}$$

Remark 3.1.4 *This theorem is remarkable in the sense that it provides an infinite-dimensional, nonlinear dynamical system which preserves the independence and homogeneity properties of its random initial configuration. Moreover, it was observed in [35] that the evolution according to Burgers equation of the Laplace exponents in (3.7) corresponds to a Smoluchowski coagulation equation [64] with additive rate which determines the jump statistics. This connection is simply due to the Lévy-Khintchine representation of Laplace exponents.*

3.1.4 Scalar conservation law with general Hamiltonian H

A natural question that arises is if the previous phenomenon (the entropy solution at later times having a simple form that can be explicitly described) is intrinsic to the Burgers equation or if the same holds for scalar conservation laws with general Hamiltonians H. In an attempt to answer this question, Menon and Srinivasan in [36] proved that when the initial condition ξ is a spectrally positive strong Markov process, then the entropy solution of (3.1) at later times remains Markov and spectrally positive. However, it is not as clear whether the Feller property is preserved through time. The following conjecture was stated in that paper, together with different heuristic but convincing ways to see why that must be true.

Conjecture. If the initial data ξ of the scalar conservation law in (3.1) is either

1. A *white noise* derived from a spectrally positive Lévy process.
2. A stationary spectrally positive Feller process with bounded variation.

then the solution $\rho(\cdot,t)$ for any fixed time $t>0$ is a stationary spectrally positive Feller process with bounded variation. Moreover, its jump kernel and drift verify an integro-differential equation.

Remark 3.1.5 *By a result of Courrège (see [3][Theorem 3.5.3]), the generator $\mathcal{A}$ of any spectrally positive Feller process with bounded variation takes the form*

$$\mathcal{A}\varphi(y) = b(y)\varphi'(y) + \int_y^\infty (\varphi(z)-\varphi(y))n(y,dz)$$

given that $C_c^\infty(\mathbb{R}) \subset \mathcal{D}(\mathcal{A})$ ($C_c^\infty(\mathbb{R})$ is the space of infinitely differentiable functions with compact support and $\mathcal{D}(\mathcal{A})$ is the domain of the generator $\mathcal{A}$). Moreover the kernel n verifies the integrability condition : $\int_y^\infty (1\wedge|y-z|^2)n(y,dz)<\infty$.

A variant[1] of the second part of this conjecture when the initial data is a piecewise-deterministic spectrally positive Feller process was recently proved by Kaspar and Rezakhanlou in [32] and [31]. We give below an explicit exposition of their result together with the exact form of the integro-differential equation verified by the drift and the jump kernel. This equation (3.8) was formally derived by Menon and Srinivasan in [36] and shown to be equivalent to the following Lax equation

$$\partial_t\mathcal{A} = [\mathcal{A},\mathcal{B}] = \mathcal{A}\mathcal{B}-\mathcal{B}\mathcal{A}$$

where $\mathcal{A}^t$ is the generator of $x\mapsto\rho(x,t)$ and $\mathcal{B}^t$ is the generator of $t\mapsto\rho(x,t)$. We give explicit formulas for these generators below in the statement of Theorem 3.1.7.

Notation 3.1.6 *We write $\mathcal{M}_1$ for the set of probability measures on the real line, and*

$$[H]_{y,z} = \frac{H(y)-H(z)}{y-z} \text{ for } y\neq z$$

Theorem 3.1.7 ([31]) *Assume that the initial data $\rho^0=\rho^0(x)$ is zero of $x<0$ and is a Markov process for $x\geq 0$ that starts at $\rho^0(0)=0$. More precisely, its infinitesimal generator $\mathcal{A}^0$ has the form*

$$\mathcal{A}^0\varphi(\rho_-) = b^0(\rho_-)\varphi'(\rho_-) + \int_{\rho_-}^\infty (\varphi(\rho_+)-\varphi(\rho_-))f^0(\rho_-,\rho_+)d\rho_+$$

Furthermore, assume that

1. *The rate kernel $f^0(p-,p_+)$ is C^1 and is supported on*

$$\{(p_-,p_+): P_-\leq p_-\leq p_+\leq P_+\}$$

for some constants $P_\pm$.

[1] Under some mild conditions on the Hamiltonian H, and a slight modification of the nature of the initial data.

2. *The Hamiltonian function* $H : [P_-, P_+] \to \mathbb{R}$ *is* C^2*, convex, has positive right-derivative at* $p = P_-$ *and finite left-derivative at* $p = P_+$.

3. *The initial drift* b^0 *is* C^1 *and satisfies* $b^0 \le 0$ *with* $b^0(\rho) = 0$ *whenever* $\rho \notin [P_-, P_+]$.

Then for each fixed $t > 0$*, the process* $x \mapsto \rho(x,t)$ *(where* ρ *is a solution of (3.1)) has* $x = 0$ *marginal given by* $\ell^0(d\rho_0, t)$ *where* $\ell^0 : [0,\infty) \to \mathcal{M}_1$ *is the unique function such that* $\ell^0(d\rho, 0) = \delta_0(d\rho)$ *and*

$$\frac{d\ell^0(d\rho,t)}{dt} = (\mathcal{B}^{t*}\ell^0(\cdot,t))(d\rho,t)$$

where $\mathcal{B}^{t*}$ *is the adjoint operator of* $\mathcal{B}^t$*, that acts on measures with*

$$\mathcal{B}^t\varphi(\rho_-) = -H'(\rho_-)b(\rho_-,t)\varphi'(\rho_-) - \int_{\rho_-}^{\infty} [H]_{\rho_-,\rho_+}(\varphi(\rho_+) - \varphi(\rho_-))f(\rho_-,\rho_+,t)d\rho_+$$

for any test function φ*. Moreover the process* $x \mapsto \rho(x,t)$ *evolves for* $0 < x < \infty$ *according to a Markov process with generator* $\mathcal{A}^t$ *given by*

$$\mathcal{A}^t\varphi(\rho_-) = b(\rho_-,t)\varphi'(\rho_-) + \int_{\rho_-}^{\infty} (\varphi(\rho_+) - \varphi(\rho_-))f(\rho_-,\rho_+,t)d\rho_+$$

Here b *and* f *are obtained from their initial conditions*

$$b(\rho,0) = b^0(\rho), \qquad f(\rho_-,\rho_+,0) = f^0(\rho_-,\rho_+)$$

b *solves the ODE with parameter*

$$\partial_t b(\rho,t) = H''(\rho)b(\rho,t)^2$$

and f *solves the following Boltzmann-like kinetic equation*

$$\partial_t f(\rho_-,\rho_-,t) = Q(f,f) + C(f) + \partial_{\rho_-}(fV_{\rho_-}(\rho_-,\rho_+,t)) + \partial_{\rho_+}(fV_{\rho_+}(\rho_-,\rho_+,t)) \tag{3.8}$$

where the velocities V_{ρ_-} *and* V_{ρ_+} *are given by*

$$\begin{aligned} V_{\rho_-}(\rho_-,\rho_+,t) &= ([H]_{\rho_-,\rho_+} - H'(\rho_-))b(\rho_-,t) \\ V_{\rho_+}(\rho_-,\rho_+,t) &= ([H]_{\rho_-,\rho_+} - H'(\rho_+))b(\rho_+,t) \end{aligned}$$

the coagulation-like collision kernel Q *is*

$$\begin{aligned} Q(f,f)(\rho_-,\rho_+,t) = &\int_{\rho_-}^{\rho_+} ([H]_{\rho_*,\rho_+} - [H]_{\rho_-,\rho_*})f(\rho_-,\rho_*,t)f(\rho_*,\rho_+,t)d\rho_* \\ &- \int_{\rho_+}^{\infty} ([H]_{\rho_-,\rho_+} - [H]_{\rho_+,\rho_*})f(\rho_-,\rho_+,t)f(\rho_+,\rho_*,t)d\rho_* \\ &- \int_{\rho_-}^{\infty} ([H]_{\rho_-,\rho_*} - [H]_{\rho_-,\rho_+})f(\rho_-,\rho_+,t)f(\rho_-,\rho_*,t)d\rho_* \end{aligned}$$

and the linear operator C *is given by*

$$C(f)(\rho_-,\rho+) = f(\rho_-,\rho_+)(b(\rho_-,t)H''(\rho_-) - ([H]_{\rho_-,\rho_+} - H'(\rho_-))\partial_{\rho_-}b(\rho_-,t))$$

The purpose of this chapter is to prove the first part of the conjecture when the initial data ξ is a Brownian *white noise* and thus extend the results of Groeneboom [25] in the Burgers case. We show that at any fixed time $t > 0$, the solution $\rho(\cdot, t)$ is a stationary piecewise-smooth Feller process and we give an explicit description of its generator. This result proves the complete integrability of scalar conservation laws for this class of initial data and moves away from the unnatural emphasis on Burgers equation. Our method as will be seen by the reader can be extended when the white noise is derived from a spectrally positive Lévy process with non-zero Brownian exponent. Our shortcoming in this case will be not having explicit formulas for the jump kernel. We also show that the structure of shocks of Burgers turbulence holds for the general scalar conservation law under the assumption of rough initial data.

Since the entropy solution is expressed via the Lax-Oleinik variational formula. It is natural to study the law of the process Ψ^ϕ defined as

$$\Psi^\phi(x) = \sup\left\{ y \in \mathbb{R} : U_0(y) - \phi(y-x) = \max_{z\in\mathbb{R}} \left(U_0(z) - \phi(z-x)\right)\right\}, \quad x \in \mathbb{R} \tag{3.9}$$

where U_0 is a spectrally positive Lévy process and ϕ is a C^2 strictly convex function with superlinear growth, such that $U_0(y) = o(\phi(y))$[2] for $|y| \to \infty$. The relationship between the process Ψ^ϕ and the entropy solution $\rho(\cdot, t)$ of (3.1) is the following

$$\rho(x,t) = L'\left(\frac{\Psi^{tL(\frac{\cdot}{t})}(x) - x}{t}\right)$$

Our chapter is organized as follows

1. In Section 3.2, we give some preliminary results on the process Ψ^ϕ when U_0 is a spectrally positive Lévy process such as its Markovian property.

2. In Section 3.3, we will focus on the case where U_0 is a two-sided Brownian motion and show that the process Ψ^ϕ is pure jump, following similar ideas used by Groeneboom in [25]. The main ingredient being the path decomposition of Markov processes when they reach their ultimate maximum. This result implies that the Brownian motion U_0 has excursions below the sequence of convex functions $(x \mapsto \phi(x - x_n))_{n\in\mathbb{N}}$ where $(x_n)_{n\in\mathbb{N}}$ are the jump times of the process Ψ^ϕ (which is a discrete set by a result of Section 3.5). However, the justification of many manipulations used in [25] rely on the regularity and asymptotic properties of Airy functions at infinity, as those arise naturally in the expressions of transition densities used throughout the study of the Brownian motion with parabolic drift. Unfortunately, those special functions are intrinsic to this special case as we will explain later, and one do not have similar expressions in the general case.

3. In Section 3.4, we circumvent this difficulty by using a more analytic approach to prove the smoothness and integrability of the densities that were used in Section 3.3.

[2] We write $f = o(g)$ if $\lim \frac{f}{g} = 0$ and $f = O(g)$ if $\frac{f}{g}$ is bounded.

Moreover, via Girsanov theorem we manage to express explicitly the jump kernel of the process Ψ^{ϕ} in terms of the distribution of Brownian excursion areas. Along the way, we find the joint density of the maximum and its location of the process $(W(z)-\phi(z))_{z\in\mathbb{R}}$ where W is a two-sided Brownian motion. In particular, the density of $\text{argmax}_{z\in\mathbb{R}}(W(z)-\phi(z))$ enjoys a simple expression similar to Chernoff distribution for the parabolic drift.

4. Finally, in Section 3.5 we give a sufficient condition on the Lévy process U_0 for the process Ψ^{ϕ} to have discrete range (with the convention that a set is discrete if it is countable with no accumulation points). As a consequence, this implies that the structure of shocks of the entropy solution $\rho(\cdot,t)$ is discrete for any time $t>0$ when the initial data belongs to the large class of *abrupt* Lévy processes introduced by Vigon in [63], this result generalizes the findings of Bertoin [10] and Abramson [1] when U_0 is spectrally positive.

We give here our main results

Theorem 3.1.8 *Suppose that the initial potential U_0 is a two-sided Brownian motion and let ρ be the solution of the scalar conservation law $\rho_t = (H(\rho))_x$. Then for every fixed $t>0$, the process $x\mapsto\rho(x,t)$ is a stationary piecewise-smooth Feller process. Its generator is given by*

$$\mathcal{A}^t\varphi(\rho_-) = -\frac{\varphi'(\rho_-)}{tH''(\rho_-)} + \int_{\rho_-}^{\infty}(\varphi(\rho_+)-\varphi(\rho_-))n(\rho_-,\rho_+,t)d\rho_+$$

for any test function $\varphi\in C_c^{\infty}(\mathbb{R})$, where

$$n(\rho_-,\rho_+,t) = \frac{(\rho_+-\rho_-)K(\rho_-,\rho_+,t)}{\sqrt{2\pi t(H'(\rho_+)-H'(\rho_-))^3}}\,\frac{\rho_+ + \int_{\rho_+}^{\infty}\frac{H''(\rho)-K(\rho_+,\rho,t)}{\sqrt{2\pi t(H'(\rho)-H'(\rho_+))^3}}d\rho}{\rho_- + \int_{\rho_-}^{\infty}\frac{H''(\rho)-K(\rho_-,\rho,t)}{\sqrt{2\pi t(H'(\rho)-H'(\rho_-))^3}}d\rho} \tag{3.10}$$

for $\rho_-<\rho_+$, and

$$K(\rho_-,\rho_+,t) = H''(\rho_+)exp\left(-\frac{t}{2}\int_{\rho_-}^{\rho_+}\rho_*^2H''(\rho_*)d\rho_*\right)\mathbb{E}\left[exp\left(-\int_{\rho_-}^{\rho_+}\boldsymbol{e}(tH'(\rho_*))d\rho_*\right)\right]$$

where $\boldsymbol{e}$ is a Brownian excursion on the interval $[tH'(\rho_-),tH'(\rho_+)]$.

Remark 3.1.9

1. *The profile of the solution at any fixed time $t>0$ is a concatenation of smooth pieces that evolve as solutions of ODEs with vector field (or drift) $b(\rho,t):=-\frac{1}{tH''(\rho)}$ and are interrupted by stochastic upward jumps distributed via the jump kernel $n(\cdot,\cdot,t)$. We prove in Section 3.5 that in the Brownian white noise case, under mild assumptions on the Hamiltonian H, the set of jump times is discrete, i.e. : there are only a finite number of jumps on any given compact interval.*

2. *For any $\epsilon > 0$, the profile of $x \mapsto \rho(x,\epsilon)$ is a piecewise-deterministic Markov process and belongs to the class of initial data considered in the second part of the Conjecture. A consequence of this observation would be that the kernel $(\rho_-, \rho_+, t) \mapsto n(\rho_-, \rho_+, t)$ in the expression (3.10) verifies the kinetic equation (3.8). However, Theorem 3.1.7 only considers a variant of the original statement of the conjecture as it forces the initial data to be flat on the negative real-line (whereas here we deal with a stationary process) and restricts the range of ρ^0 on a compact interval $[P_-, P_+]$. These technical modifications arise from the very challenging proof of existence and uniqueness of a classical solution to (3.8) under general assumptions. Verifying that the kernel n in the Brownian white noise case is a solution to the kinetic equation (3.8) from the explicit expression (3.10) seems also inaccessible at the present due to the complicated term involving the Brownian excursion. This verification was done for the Burgers case by Menon and Srinivasan in [36][Section 6] through many non-trivial calculations, but relied extensively on the connection with Airy functions and an associated Painlevé property.*

The following result is a consequence of our study of the process Ψ^ϕ. It gives an explicit formula for the density of the random variable $\text{argmax}_{\omega\in\mathbb{R}}(W(\omega) - \phi(\omega))$ where W is a two-sided Brownian motion. From results of Section 3.4, we also have access to the joint distribution of

$$(\text{argmax}_{\omega\in\mathbb{R}}(W(\omega) - \phi(\omega)), \max_{\omega\in\mathbb{R}}(W(\omega) - \phi(\omega)))$$

but we omit it here because the expression is quite large.

Theorem 3.1.10 *Let ω_M be the location of the maximum of the process $(S(\omega) = W(\omega) - \phi(\omega))_{\omega\in\mathbb{R}}$ where W is a two-sided Brownian motion, its density is equal to*

$$\frac{\mathbb{P}[\omega_M \in dt]}{dt} = \frac{1}{2} f^\phi(t) f^{\phi(-\cdot)}(-t)$$

for any $t \in \mathbb{R}$, and where

$$f^\phi(t) = \phi'(t) + \int_0^\infty \frac{1 - p^\phi(t,u)}{\sqrt{2\pi u^3}} du$$

with

$$p^\phi(t,u) = exp\left(-\frac{1}{2}\int_t^{t+u} \phi'(z)^2 dz\right) \mathbb{E}\left[exp\left(-\int_t^{t+u} \phi''(z)\boldsymbol{e}(z)dz\right)\right] \text{ for } u > 0$$

where $\boldsymbol{e}$ is a Brownian excursion on $[t, t+u]$.

Remark 3.1.11 *In the parabolic drift case (Chernoff distribution), the term ϕ'' is constant and the Laplace transform of a standard Brownian excursion area is known to be expressed via Airy functions. We will develop on the connection between the formulas found by Groeneboom in [25] and ours at the end of Section 3.4. Also, we refer the reader to the survey [28] for a more detailed exposition on the distribution and Laplace transform of various Brownian paths areas.*

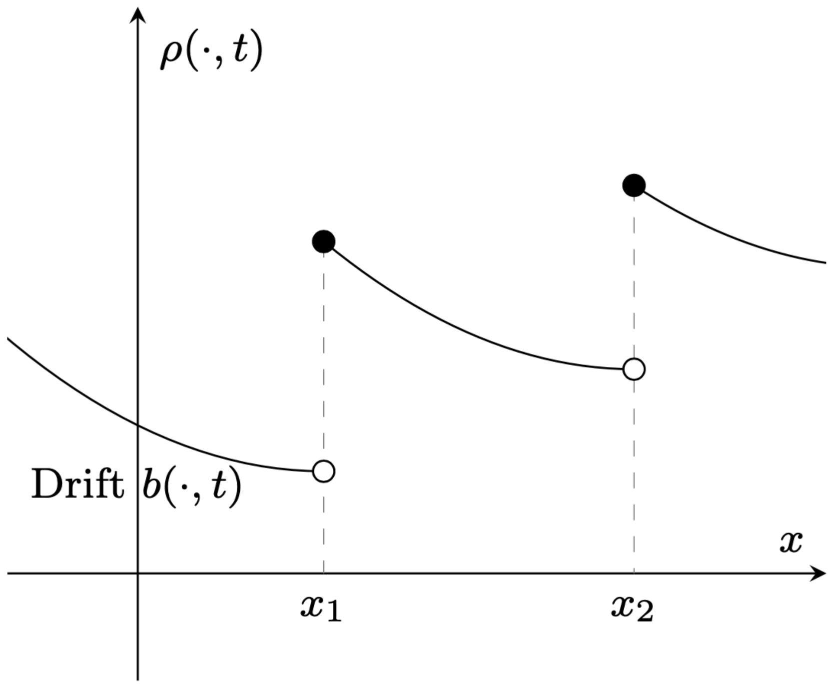

Figure 3.1: The typical profile of the entropy solution at a given time $t > 0$.

We define now a class of rough Lévy processes called *abrupt* that were introduced by Vigon in [63].

Definition 3.1.12 *A Lévy process $(X_t)_{t\in\mathbb{R}}$ is said to be abrupt if its paths have unbounded variation and almost surely for all local maxima m of X we have*

$$\liminf_{h\downarrow 0} \frac{1}{h}(X_{m-h} - X_{m-}) = +\infty \text{ and } \limsup_{h\downarrow 0} \frac{1}{h}(X_{m+h} - X_m) = -\infty$$

Remark 3.1.13 *A Lévy process X with paths of unbounded variation is abrupt if and only if*

$$\int_0^1 t^{-1}\mathbb{P}\left[X_t \in [at, bt]\right] dt < \infty, \quad \forall a < b$$

Examples of abrupt Lévy processes include stable processes with index $\alpha \in (1,2]$ and any process with non-zero Brownian exponent.

Our last main result determines the structure of shocks of the scalar conservation law when the initial data is a white noise derived from an *abrupt* Lévy process.

Theorem 3.1.14 *Assume that the Lévy process U_0 is spectrally positive, abrupt and is such that $U_0(y) = O(|y|)$ for $|y|\to\infty$, then the set*

$$\mathcal{L}^t = \{y \in \mathbb{R} : y = y(x,t) \text{ or } y = y(x-,t) \text{ for some } x \in \mathbb{R}\}$$

is almost surely discrete for any fixed time $t > 0$. We say then that the structure of shocks of the entropy solution $\rho(\cdot, t)$ is discrete.

Remark 3.1.15 *From a point of view of hydrodynamic turbulence, a discontinuity of the entropy solution $\rho(\cdot, t)$ at position x means the presence of a cluster of particles at this location at time t. Those clusters interact with each other via inelastic shocks, and the cluster at location x and at time t contains all the particles that were initially located in $[y(x-, t), y(x, t))$. Our result shows that at any given time $t > 0$, the set of clusters is discrete. When the initial data is a Lévy white noise, we can picture that there are infinitely many particles initially scattered everywhere with i.i.d velocities. Therefore, when we assume that this initial profile is rough (as it is the case when the potential U_0 is an abrupt Lévy process), this turbulence forces all the particles to aggregate in heavy disjoint lumps instantaneously for any time $t > 0$.*

3.2 Preliminaries

Notation 3.2.1 *We will use the notation $\text{argmax}^+ f$ to denote the rightmost maximizer of a function f (i.e. : the last time at which a function f reaches its maximum).*

Menon and Srinivisan proved in their paper [36] a closure theorem for white noise initial data for the scalar conservation law solutions. They showed that if initially the potential U_0 is spectrally positive with independent increments then $\rho(\cdot, t)$ is a spectrally positive Markov process for any fixed $t > 0$. The proof of this statement follows from standard use of path decomposition of strong Markov processes at their ultimate maximum. The same holds for our process Ψ^ϕ. Precisely, we have the following theorem for which we give the proof for the sake of completeness.

Theorem 3.2.2 *Assume that U_0 is a spectrally positive Lévy process, then the process Ψ^ϕ is a non-decreasing Markov process. Moreover for any $a \in \mathbb{R}$, the process $\Psi^\phi(. + a) - a$ has the same distribution as Ψ^ϕ.*

Proof. For $x_1 \leq x_2$ and $y \leq \Psi^\phi(x_1)$, we have that

$$\begin{aligned} U_0(\Psi^\phi(x_1)) - U_0(y) &\geq \phi(\Psi^\phi(x_1) - x_1) - \phi(y - x_1) \\ &\geq \phi(\Psi^\phi(x_1) - x_2) - \phi(y - x_2) \end{aligned}$$

By the convexity of ϕ, and hence $\Psi^\phi(x_1) \leq \Psi^\phi(x_2)$. Also, by definition Ψ^ϕ is a càdlàg process (right continuous with left hand limits). Take $h > 0$, then

$$\Psi^\phi(x + h) = \Psi^\phi(x) + \text{argmax}^+_{y \geq 0} \left(U_0(y + \Psi^\phi(x)) - \phi(y + \Psi^\phi(x) - (x + h))\right) \tag{3.11}$$

The process $U^x(y) := U_0(y) - \phi(y - x)$ is clearly Markov. By Millar's theorem of path decomposition of Markov processes when they reach their ultimate maximum (see [44]), the process $(U^x(y + \Psi^\phi(x))_{y \geq 0}$ is independent of $(U^x(y))_{y \leq \Psi^\phi(x)}$ given $(\Psi^\phi(x), U^x(\Psi^\phi(x)))$ (because of the upward jumps of U_0, the maximum is attained at the right hand limit). Moreover, because of the independence of the increments of U^x, the process $(U^x(y + \Psi^\phi(x)) - U^x(\Psi^\phi(x)))_{y \geq 0}$

is independent of $(U^x(y))_{y\le\Psi^\phi(x)}$ given $\Psi^\phi(x)$. Now it suffices to see that $(\Psi^\phi(y))_{y\le x}$ only depends on the pre-maximum process $(U^x(y))_{y\le\Psi^\phi(x)}$ because of the monotonicity of Ψ^ϕ, this fact alongside the equation (3.11) gives the Markov property of the process Ψ^ϕ. The last statement follows easily from the stationarity of increments of U_0. □

Remark 3.2.3 *Notice that except in the last statement, the stationarity of increments was not used in the proof of the Markovian property of the process Ψ^ϕ.*

3.3 The process Ψ^ϕ in the Brownian case

In this section, we assume that $W := U_0$ is a two-sided Brownian motion. We proved in the previous section that the process Ψ^ϕ is Markov and enjoys a space-time shifted stationarity property. Hence, we shall only determine its transition function at time zero and consequently the form of its generator at this time. In this section we will differentiate and switch the order of integrals and differentiations without any justification, as Section 3.4 is devoted to take care of all those technicalities.

Notation 3.3.1 *In the sequel, we will deal with functions of the form $f(s,x,t,y)$ where t and s play the role of temporal variables, and x and y that of spatial variables. Without confusion, the notation $\partial_x f(s,x,t,y)$ (resp. $\partial_y f(s,x,t,y)$) refer to the partial derivative of f with respect to the second variable (resp. fourth variable).*

We state here the first result regarding the transition function of the process Ψ^ϕ.

Theorem 3.3.2 *Let $h>0$ and $\omega_1<\omega_2$ be two real numbers. Then we have that*

$$\mathbb{P}[\Psi^\phi(h)\in d\omega_2|\Psi^\phi(0)=\omega_1]=\mathbb{P}[argmax^+_{\omega\ge\omega_1}X^h(\omega)\in d\omega_2]$$

where $X^h(\omega):=S^\downarrow(\omega)+r^h(\omega)$ and

- *$(S^\downarrow(\omega))_{\omega\ge\omega_1}$ is the Markov process $(S(\omega):=W(\omega)-\phi(\omega))_{\omega\ge\omega_1}$ started at zero and Doob-conditioned to stay negative (i.e to hit $-\infty$ before 0). Precisely, its transition function is given by*

$$\mathbb{P}[S^\downarrow(t)\in dy|S^\downarrow(s)=x]=\frac{\mathbb{P}[\tau_0=\infty|S(t)=y]}{\mathbb{P}[\tau_0=\infty|S(s)=x]}f(s,x,t,y)dy \tag{3.12}$$

  for $t>s>\omega_1$ and $x,y<0$, and where τ_0 is the first hitting time of zero of the process S. The function f is the transition density of the process S killed at zero, at time t and state y, formally defined as

$$\mathbb{P}[S(t)\in dy,\max_{s\le u\le t}S(u)<0|S(s)=x]=f(s,x,t,y)dy$$

  Moreover, the entrance law of $S^\downarrow$ is given by

$$\mathbb{P}[S^\downarrow(t)\in dy]=\frac{\mathbb{P}[\tau_0=\infty|S(t)=y]}{\partial_x\mathbb{P}[\tau_0=\infty|S(s)=x]_{|x=0}}\partial_x f(\omega_1,0,t,y)dy \tag{3.13}$$

- *The function r^h is defined as $r^h(\omega) = \phi(\omega) - \phi(\omega - h) + c$ where c is a constant such that $r^h(\omega_1) = 0$.*

Proof. We have that

$$\begin{aligned}\mathbb{P}[\Psi^\phi(h) \in d\omega_2 | \Psi^\phi(0) = \omega_1] &= \mathbb{P}[\text{argmax}^+_{\omega \geq \omega_1}(W(\omega) - \phi(\omega - h)) \in d\omega_2 | \Psi^\phi(0) = \omega_1] \\ &= \mathbb{P}[\text{argmax}^+_{\omega \geq \omega_1}(S(\omega) - S(\omega_1) + r^h(\omega)) \in d\omega_2 | \text{argmax}^+ S(\omega) = \omega_1]\end{aligned}$$

Now, using Millar path decomposition of Markov processes when they reach their ultimate maximum, the expression of the transition densities of the post-maximum process in [44][Equation 9] on the process S, and the spatial homogeneity of the Brownian motion (and thus of S), we get (3.12). To get the entrance law it suffices to send s to ω_1 and x to zero. □

Let us now introduce some notation to keep our formulas compact.

Notation 3.3.3 *Denote by*

$$J(s,x) = \mathbb{P}[\tau_0 = \infty | S(s) = x] = \mathbb{P}[S(u) < 0 \text{ for all } u \geq s | S(s) = x], \quad x < 0$$

and define

$$j(s,x) = \frac{\partial}{\partial x} J(s,x), \quad s \in \mathbb{R}, \quad x < 0$$

$$j(s) = \lim_{x \uparrow 0} \frac{\partial}{\partial x} J(s,x), \quad s \in \mathbb{R}$$

Also denote

$$\Phi(s,x,\omega) = \frac{\mathbb{P}[\tau_0 \in d\omega | S(s) = x]}{d\omega}, \quad s < \omega, \quad x \in \mathbb{R}$$

Furthermore, let $\tilde{S}$ be the process defined as $(\tilde{S}(\omega) := W(\omega) - \phi(-\omega))_{\omega \in \mathbb{R}}$. We define $\tilde{f}$ and $\tilde{\Phi}$ analogously.

With this notation, the entrance law of the process $S^\downarrow$ is expressed as

$$\mathbb{P}[S^\downarrow(t) \in dy] = \frac{J(t,y)}{j(s)} \partial_x f(\omega_1, 0, t, y) dy, \quad t > \omega_1 \text{ and } y < 0 \tag{3.14}$$

The next result will allow us to recover the transition function of the process Ψ^ϕ.

Theorem 3.3.4 *Let $\omega_1 < \omega_2$ and $x^* \in (0, r^h(\omega_2))$. Define $\omega^* \in (\omega_1, \omega_2)$ to be the unique point such that $r_h(\omega^*) = x^*$ (such a time exists because of the strict convexity of ϕ that makes r_h strictly increasing). Then we have that*

$$\frac{\mathbb{P}[\text{argmax}^+_{\omega \geq \omega_1} X^h(\omega) \in d\omega_2, \max_{\omega \geq \omega_1} X^h(\omega) \in dx^*]}{d\omega_2 dx^*} =$$

$$2 \int_{-\infty}^{x_*} \frac{j(\omega_2 - h)}{j(\omega_1)} \Phi(\omega^* - h, y - x^*, \omega_2 - h) \tilde{\Phi}(-\omega^*, y - x^*, -\omega_1) dy$$

Before proving this theorem, we will state a lemma that links the joint distribution of the maximum of the diffusion S and its location with the functionals f and J .

Lemma 3.3.5 *Let M and ω_M be respectively the maximum of the process $(S(\omega))_{\omega\geq s}$ and its location, we have then that*

$$\frac{\mathbb{P}[\omega_M \in dt, M \in dz|S(s) = x]}{dtdz} = \frac{1}{2}j(t)\partial_y f(s, x-z, t, 0) = -j(t)\Phi(s, x-z, t) \tag{3.15}$$

Proof. We have by the Markov property that

$$\begin{aligned}\mathbb{P}[\omega_M > t, M \in dz|S(s) = x] &= \mathbb{P}[\max_{s\leq u\leq t} S(u) < z, \max_{u\geq t} S(u) \in dz|S(s) = x] \\ &= \int_{-\infty}^{z} f(s, x-z, t, y-z)\mathbb{P}[\max_{u\geq t} S(u) \in dz|S(t) = y]dy\end{aligned}$$

Now we see that

$$\mathbb{P}[\max_{u\geq t} S(u) \in dz|S(t) = y] = J(t, y-z-dz) - J(t, y-z) = -j(t, y-z)dz$$

Hence

$$\mathbb{P}[\omega_M > t, M \in dz|S(s) = x] = -\int_{-\infty}^{0} f(s, x-z, t, y)j(t, y)dydz$$

Thus

$$\frac{\mathbb{P}[\omega_M \in dt, M \in dz]}{dzdt} = \frac{\partial}{\partial t}\int_{-\infty}^{0} f(s, x-z, t, y)j(t, y)dy \tag{3.16}$$

Now, by Kolmogorov forward and backward equations on the diffusion S we have that

$$\partial_t f = \phi'(t)\partial_y f + \frac{1}{2}\partial_y^2 f$$

and

$$\partial_t j = \phi'(t)\partial_y j - \frac{1}{2}\partial_y^2 j$$

By interchanging the time partial derivative and the integral sign in (3.16), we find by integration by parts

$$\frac{\partial}{\partial t}\int_{-\infty}^{0} f(s, x-z, t, y)j(t, y)dy = \phi'(t)\left[fj\right]_{-\infty}^{0} + \frac{1}{2}\left[j\partial_y f - f\partial_y j\right]_{-\infty}^{0}$$

Now it suffices to see that f vanishes at both zero and infinity, from which the first equality follows. For the second equality, it suffices to see that

$$\mathbb{P}[\tau_0 > t|S(s) = x] = \int_{-\infty}^{0} f(s, x, t, y)dy$$

Differentiating with respect to time and using the Kolmogorov forward equation in the same fashion as was done before gives the result. □

Remark 3.3.6 *All these differentiations and integrations by parts are justified by the fact that f and j are sufficiently smooth and integrable away from $\{t = s\}$. This fact will be proved in the next section.*

Proof of Theorem 4.3.1 We have that

$$\mathbb{P}[\text{argmax}^+_{\omega\geq\omega_1} X^h(\omega) \in d\omega_2, \max_{\omega\geq\omega_1} X^h(\omega) \in dx^*] =$$
$$\int_{-\infty}^{x^*} \mathbb{P}[X^h(\omega^*) \in dy, \text{argmax}^+_{\omega\geq\omega_1} X^h(\omega) \in d\omega_2, \max_{\omega\geq\omega_1} X^h(\omega) \in dx^*]$$

Because for $\omega \in [\omega_1, \omega^*)$, we have that $X^h(\omega) \leq r_h(\omega) < x^*$, then by the Markov property we get that

$$\mathbb{P}[X^h(\omega^*) \in dy, \text{argmax}^+_{\omega\geq\omega_1} X^h(\omega) \in d\omega_2, \max_{\omega\geq\omega_1} X^h(\omega) \in dx^*] =$$
$$\mathbb{P}[X^h(\omega^*) \in dy]\mathbb{P}[\text{argmax}^+_{\omega\geq\omega^*} X^h(\omega) \in d\omega_2, \max_{\omega\geq\omega^*} X^h(\omega) \in dx^* | X^h(\omega^*) = y]$$

Let us focus first on the second term of this product. The law of the Markov process X^h is that of the process $S + r^h$ conditioned to stay below r_h. However, when X^h starts from the state $y < x^*$ at time ω^*, the event we condition on has positive probability and hence it is just the naive conditioning. Thus, we can write

$$\mathbb{P}\left[\text{argmax}^+_{\omega\geq\omega^*} X^h(\omega) \in d\omega_2, \max_{\omega\geq\omega^*} X^h(\omega) \in dx^* | X^h(\omega^*) = y\right] =$$
$$\mathbb{P}\Big[\text{argmax}^+_{\omega\geq\omega^*} S(\omega) + r^h(\omega) \in d\omega_2, \max_{\omega\geq\omega^*} S(\omega) + r^h(\omega) \in dx^*$$
$$| S(\omega^*) = y - x^*, S(\omega) \leq 0 \text{ for all } \omega \geq \omega^*]$$

This probability is equal to the ratio of this probability

$$\mathbb{P}_1 = \mathbb{P}\Big[\text{argmax}^+_{\omega\geq\omega^*} S(\omega) + r^h(\omega) \in d\omega_2, \max_{\omega\geq\omega^*} S(\omega) + r^h(\omega) \in dx^*,$$
$$S(\omega) \leq 0 \text{ for all } \omega \geq \omega^* | S(\omega^*) = y - x^*]$$

over the probability

$$\mathbb{P}_2 = \mathbb{P}\left[S(\omega) \leq 0 \text{ for all } \omega \geq \omega^* | S(\omega^*) = y - x^*\right]$$

For the first probability $\mathbb{P}_1$, notice that on the event that $\{\max_{\omega\geq\omega^*} S(\omega) + r^h(\omega) \in dx^*\}$, we always have that $S(\omega) \leq 0$ for all $\omega \geq \omega^*$, because $r^h(\omega) \geq x^*$ for $\omega \geq \omega^*$. Thus

$$\mathbb{P}_1 = \mathbb{P}\Big[\text{argmax}^+_{\omega\geq\omega^*} S(\omega) + r^h(\omega) \in d\omega_2, \max_{\omega\geq\omega^*} S(\omega) + r^h(\omega) \in dx^* \ | S(\omega^*) = y - x^*]$$

Now we have that

$$S(\omega) + r^h(\omega) = W(\omega) - \phi(\omega - h) + c \quad , \quad \omega \geq \omega^*$$

Hence

$$(S(\omega) + r^h(\omega) | S(\omega^*) = y - x^*)_{\omega \geq \omega^*} \stackrel{d}{=} (S(\omega - h) | S(\omega^* - h) = y)_{\omega \geq \omega^*}$$

Thus by using Lemma 3.3.5 for $s = \omega^* - h$ and $x = y - x^*$, we get that

$$\mathbb{P}_1 = -j(\omega_2 - h)\Phi(\omega^* - h, y - x^*, \omega_2 - h) d\omega_2 dx^*$$

Therefore

$$\frac{\mathbb{P}_1}{\mathbb{P}_2} = -\frac{j(\omega_2 - h)\Phi(\omega^* - h, y - x^*, \omega_2 - h)}{J(\omega^*, y - x^*)} d\omega_2 dx^* \tag{3.17}$$

Finally for the first term $\mathbb{P}[X^h(\omega^*) \in dy]$, we have that

$$\begin{aligned}
\mathbb{P}[X^h(\omega^*) \in dy] &= \mathbb{P}[S^{\downarrow}(\omega^*) \in d(y - r^h(\omega^*))] \\
&= \mathbb{P}[S^{\downarrow}(\omega^*) \in d(y - x^*)] \\
&= \frac{J(\omega^*, y - x^*)}{j(\omega_1)} \partial_x f(\omega_1, 0, \omega^*, y - x^*) dy
\end{aligned}$$

Now it is not hard to see that we have the following equality

$$\tilde{f}(s, x, t, y) = f(-t, y, -s, x) \tag{3.18}$$

This is true because both those functions verify the same PDE with the same boundary and growth conditions, by combining the backward and forward Kolmogorov equations. Hence

$$\partial_x f(s, x, t, y) = \partial_y \tilde{f}(-t, y, -s, x)$$

Hence, by Lemma 3.3.5

$$\begin{aligned}
\partial_x f(\omega_1, 0, \omega^*, y - x^*) &= \partial_y \tilde{f}(-\omega^*, y - x^*, -\omega_1, 0) \\
&= -2\tilde{\Phi}(-\omega^*, y - x^*, -\omega_1)
\end{aligned}$$

Thus

$$\mathbb{P}[X^h(\omega^*) \in dy | X^h(\omega_1) = 0] = -2\frac{J(\omega^*, y - x^*)}{j(\omega_1)} \tilde{\Phi}(-\omega^*, y - x^*, -\omega_1) dy \tag{3.19}$$

Multiplying equations (3.17) and (3.19) and integrating with respect to y on $(-\infty, x^*)$ gives the result. $\square$

We are ready now to state the main result of this section.

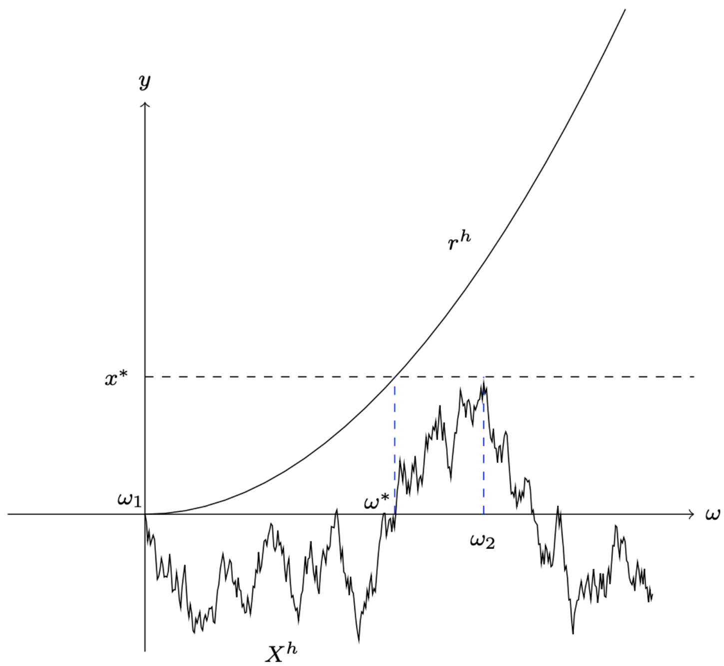

Figure 3.2: Path decomposition of X^h at its maximum

Theorem 3.3.7 *The transition function of the process Ψ^ϕ is given by*

$$\mathbb{P}[\Psi^\phi(h) \in d\omega_2 | \Psi^\phi(0) = \omega_1] = 2\frac{j(\omega_2 - h)}{j(\omega_1)} \int_{\omega_1}^{\omega_2} \int_{\infty}^{0} (r^h)'(\omega)\Phi(\omega - h, y, \omega_2 - h) \times\tilde{\Phi}(-\omega, y, -\omega_1)dyd\omega$$

Moreover, the process Ψ^ϕ is pure-jump and its generator at zero is given by its action on any test function $\varphi \in C_c^\infty(\mathbb{R})$

$$\mathcal{A}^\phi\varphi(y) = \int_y^\infty (\varphi(z) - \varphi(y))n^\phi(y,z)dz$$

where

$$n^\phi(y,z) = 2\frac{j(z)}{j(y)} \int_y^z \int_{-\infty}^0 \phi''(\omega)\Phi(\omega, x, z)\tilde{\Phi}(-\omega, x, -y)dxd\omega =: \frac{j(z)}{j(y)}K^\phi(y,z)$$

Proof. By integrating the formula in Theorem 4.3.1 with respect to x^* between and 0 and $r^h(\omega_2)$ (as X^h is pointwise at most r^h), we get that

$$\mathbb{P}[\text{argmax}^+_{\omega \geq \omega_1} X^h(\omega) \in d\omega_2] = 2\frac{j(\omega_2 - h)}{j(\omega_1)} \int_0^{r^h(\omega_2)} \int_{-\infty}^{x^*} \Phi(\omega^* - h, y - x^*, \omega_2 - h) \times \tilde{\Phi}(-\omega^*, y - x^*, -\omega_1)dydx^*$$

Now it suffices to do the change of variables $y' = y - x^*$ and $\omega = (r^h)^{-1}(x^*)$ to get the transition density. As for the generator part, it suffices to do the following Taylor expansion for $h \to 0$

$$(r^h)'(\omega) = \phi''(\omega)h + O(h^2)$$

□

Remark 3.3.8 *In the next section, we will greatly simplify this expression of the generator by giving explicit formulas of K^ϕ and j in Proposition 3.4.7 and Theorem 3.4.8 respectively.*

3.4 Regularity of the transition functions and explicit formulas

The goal of this section is to prove the regularity of the transition density $f(s, x, t, y)$ away from the line $\{t = s\}$, so that we can justify all the operations we did in the previous section and to deduce along the way explicit formulas for the jump kernel of the process Ψ^ϕ.

Processes such as the three-dimensional Bessel process, the three-dimensional Bessel bridges, and the Brownian motion killed at zero will be mentioned in some of the results of this section. We refer the unfamiliar reader to [50][Chapters 3,6,11] for basic facts about these processes.

The following proposition gives a closed formula for the density f.

Proposition 3.4.1 *Let $x, y < 0$ and $t > s$, the density f is given by the formula*

$$f(s,x,t,y) = G(s,x,t,y)exp\left(-\phi'(t)y + \phi'(s)x - \frac{1}{2}\int_s^t (\phi'(u))^2 du\right) \times \\ \mathbb{E}\left[exp\left(-\int_s^t \phi''(u)B(u)du\right)|B(s) = -x, B(t) = -y\right]$$

where B is a three-dimensional Bessel process, and G is the transition density function of the Brownian motion killed at zero, given explicitly by

$$G(s,x,t,y) = \frac{1}{\sqrt{2\pi(t-s)}}\left(e^{-\frac{(x-y)^2}{2(t-s)}} - e^{-\frac{(x+y)^2}{2(t-s)}}\right)$$

Proof. The process S can be expressed as

$$S(t) = W(t) - \phi(t) = W(t) - \int_s^t \phi'(u)du - \phi(s)$$

Thus by Girsanov theorem, S is a Brownian motion under the measure $\mathbb{Q}$ with Radon-Nikodym derivative given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}_{|\mathcal{F}_t} = \exp\left(\int_s^t \phi'(u)dW_u - \frac{1}{2}\int_s^t (\phi'(u))^2 du\right)$$

where $\mathcal{F}_t := \sigma\{W(u) : s \leq u \leq t\}$ is the canonical filtration of W. Thus for any function F we have that

$$\mathbb{E}[F(S(t))\mathbb{1}_{\{\max\limits_{s\leq u\leq t} S(u)<0\}}|S(s)=x] = \mathbb{E}[Z(t)F(W(t))\mathbb{1}_{\{\max\limits_{s\leq u\leq t} W(u)<0\}}|W(s)=x]$$

where

$$Z(t) := \exp\left(-\int_s^t \phi'(u)dW_u - \frac{1}{2}\int_s^t \phi'(u)^2 du\right)$$

In particular for $F = F_\epsilon := \frac{1}{2\epsilon}\mathbb{1}_{[y-\epsilon,y+\epsilon]}$, we have that

$$\begin{aligned} f(s,x,t,y) &= \lim_{\epsilon\to 0}\mathbb{E}[F_\epsilon(S(t))\mathbb{1}_{\{\max\limits_{s\leq u\leq t} S(u)<0\}}|S(s)=x] \\ &= \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_{y-\epsilon}^{y+\epsilon}\mathbb{E}[Z(t)\mathbb{1}_{\{W(t)\in dz,\ \max\limits_{s\leq u\leq t} W(u)<0\}}|W(s)=x] \end{aligned}$$

Now if we denote by W^∂ the Brownian motion killed at zero whose law is defined as

$$\mathbb{E}[F(W^\partial(t))|W^\partial(s)=x] = \mathbb{E}[F(W(t))\mathbb{1}_{\{\max_{s\leq u\leq t} W(u)<0\}}|W(s)=x]$$

Thus

$$\begin{aligned} f(s,x,t,y) &= \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_{y-\epsilon}^{y+\epsilon}\mathbb{E}[Z^\partial(t)|W^\partial(t)=y, W^\partial(s)=x]p^\partial_{t-s}(x,z)dz \\ &= p^\partial_{t-s}(x,y)\mathbb{E}[Z^\partial(t)|W^\partial(t)=y, W^\partial(s)=x] \end{aligned}$$

where Z^∂ is the same as Z with W replaced by W^∂, and $p^\partial_t(x,y)$ is the transition density function of the process W^∂. However it is a well-known fact that $p^\partial_{t-s}(x,y) = G(s,x,t,y)$, and the law of the Brownian motion killed at zero between s and t conditioned on its extreme values is the law of the reflection of a three-dimensional Bessel bridge between $(s,-x)$ and $(t,-y)$ (as our killed Brownian motion stays negative and the Bessel bridges are by definition positive). Finally, by using an integration by parts we have that

$$d(B(u)\phi'(u)) = \phi'(u)dB(u) + \phi''(u)B(u)du$$

Integrating between s and t, we get the desired result. □

Remark 3.4.2 *From the last proposition, one can readily see that for fixed s and x*

$$0 \leq f(s,x,t,y) \leq C(t)e^{-A(t)y^2} \text{ for all } y$$

where C and A are locally bounded, and A is locally bounded from below by a positive constant.

Let us now prove that f is smooth. First of all, one can extend f to the positive line as well by defining

$$f(s,x,t,y) = -f(s,x,t,-y), \quad y>0$$

Then f verifies in the distribution sense the following PDE (Kolmogorov forward equation)

$$\partial_t f - \frac{1}{2}\partial_y^2 f = \phi'(t)\partial_y f \text{ on } (t,y) \in (s,+\infty)\times\mathbb{R} \tag{3.20}$$

and with boundary conditions $f(s,x,s,\cdot) = \delta_x - \delta_{-x}$, and obviously $f(s,x,t,0) = 0$. Now, it is well-known that the function G that we defined in Proposition 3.4.1 verifies the heat equation

$$\partial_t G - \frac{1}{2}\partial_y^2 G = 0$$

with the same boundary conditions as f. Moreover, if one defines the function $\hat{G}$ as

$$\hat{G}(s,x,t,y) = \frac{1}{\sqrt{2\pi(t-s)}}e^{-\frac{(x-y)^2}{2(t-s)}}$$

it is also a solution for the heat equation but with boundary condition $\hat{G}(s,x,s,\cdot) = \delta_x$. Thus, in order to study the regularity properties of the solution to (3.20), one might use Duhamel's principle to get a representation formula for f. More precisely, we shall prove the following theorem

Theorem 3.4.3 *Fix $s,x \in \mathbb{R}$. There exists a function $h \in C([s,+\infty), L^1(\mathbb{R})\cap L^\infty(\mathbb{R}))$ (where here $L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$ is the space of continuous functions on the real line that are uniformly bounded and absolutely integrable), such that*

$$h(t,y) = \int_s^t \int_{\mathbb{R}} \phi'(u)\hat{G}(u,z,t,y)\partial_z G(s,x,u,z)dzdu - \int_s^t \int_{\mathbb{R}} \phi'(u)\partial_z\hat{G}(u,z,t,y)h(u,z)dzdu$$

Furthermore, h is smooth.

Proof. Let us fix $T > s$. Define the functional Ξ^T from $\mathcal{C}_T := C([s,T], L^1(\mathbb{R})\cap L^\infty(\mathbb{R}))$ into itself equipped with the norm

$$||h||_{\mathcal{C}_T} := \sup_{s\le t\le T} ||h(t)||_{L^\infty} + ||h(t)||_{L^1}$$

by

$$\Xi^T[h](t,y) = \int_s^t \int_{\mathbb{R}} \phi'(u)\hat{G}(u,z,t,y)\partial_z G(s,x,u,z)dzdu - \int_s^t \int_{\mathbb{R}} \phi'(u)\partial_z\hat{G}(u,z,t,y)h(u,z)dzdu$$

It is clear that Ξ^T sends $\mathcal{C}_T$ to itself due to the growth rate of the Green functions G and $\hat{G}$ at infinity in space. Moreover we have that for any two functions h and $\tilde{h}$ in $\mathcal{C}_T$

$$||\Xi^T[h](t,.) - \Xi^T[\tilde{h}](t,.)||_{L^1} \le \int_s^t |\phi'(u)|du \int_{\mathbb{R}} |h(u,z) - \tilde{h}(u,z)|dz \int_{\mathbb{R}} |\partial_z\hat{G}(u,z,t,y)|dy$$

Now we see that

$$\partial_z \hat{G}(u,z,t,y) = \frac{y-z}{\sqrt{2\pi(t-u)^3}} e^{-\frac{(y-z)^2}{2(t-u)}}$$

Hence

$$\int_{\mathbb{R}} |\partial_z \hat{G}(u,z,t,y)| dy \leq \frac{2}{\sqrt{2\pi(t-u)^3}} \int_0^\infty \omega e^{-\frac{\omega^2}{2(t-u)}} d\omega = \frac{2}{\sqrt{2\pi(t-u)}}$$

Thus

$$||\Xi^T[h](t,.) - \Xi^T[\tilde{h}](t,.)||_{L^1} \leq \frac{4\sqrt{T-s}\,\sup_{u\in[s,T]}|\phi'(u)|}{\sqrt{\pi}} ||h-\tilde{h}||_{\mathcal{C}_T}$$

A similar bound is found for the L^∞ norm. Thus, for T close enough to s, the operator Ξ^T becomes a contraction, and thus by Picard theorem, it admits a unique fixed point.
Now define

$$T^* = \sup\{T \geq s : \exists h \in \mathcal{C}_T \text{ such that } \Xi^T[h] = h\}$$

Suppose that $T^* < \infty$, then it is easy to see by Gronwall inequality that for any sequence $(t_m)_{m\in\mathbb{N}}$ such that $t_m \uparrow T^*$, the sequence $(h(t_m,\cdot))_{m\in\mathbb{N}}$ is Cauchy in $L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ and thus converge strongly to a unique limit that we denote $h(T^*,\cdot)$. This extension thus belongs to $\mathcal{C}_{T^*}$. However, for small $\epsilon > 0$, one can further extend the fixed point h to $\mathcal{C}_{T^*+\epsilon}$ by the same contraction argument. This contradicts the definition of T^*, and thus $T^* = \infty$ from which follow the existence of a global solution. The smoothness of h follows readily from that of the Green function $\hat{G}$ and the dominated convergence theorem. □

We are now ready to prove the following result

Theorem 3.4.4 *The function $f - G$ is everywhere smooth in the variables (t,y), in particular the function f is smooth away from $\{t = s\}$.*

Proof. Define the function q by

$$q(s,x,t,y) = h(t,y) + G(s,x,t,y)$$

where h is the global solution from Theorem 3.4.3. By integration by parts we have that

$$\begin{aligned} q(s,x,t,y) &= G(s,x,t,y) + \int_s^t \int_{\mathbb{R}} \phi'(u)\hat{G}(u,z,t,y)\partial_z G(s,x,u,z)dzdu \\ &+ \int_s^t \int_{\mathbb{R}} \phi'(u)\hat{G}(u,z,t,y)\partial_z h(u,z)dudz \\ &= G(s,x,t,y) + \int_s^\infty \int_{\mathbb{R}} \phi'(u)\mathbb{1}_{\{t\in(u,+\infty)\}}\hat{G}(u,z,t,y)\partial_z q(u,z)dudz \end{aligned}$$

Now it suffices to see that

$$(\partial_t - \frac{1}{2}\partial_y^2)(\mathbb{1}_{t\in(u,+\infty)}\hat{G}(u,z,t,y)) = \delta_0(t-u)\hat{G}(u,z,u,y) = \delta_0(t-u)\delta_0(y-z)$$

and thus the function q verifies the PDE (3.20) with the boundary conditions $q(s, x, s, \cdot) = \delta_x - \delta_{-x}$. The result now would follow if we can prove that $f = q$. Consider the function $v := f - q$, it verifies the PDE (3.20) with vanishing initial condition. The growth condition of v at infinity in space ensures that v can be viewed as a tempered distribution. By taking the Fourier transform in space in the PDE (3.20) we get that

$$\partial_t \hat{v}(t, k) = \left(-\frac{1}{2}k^2 + i\phi'(t)k\right)\hat{v}(t, k)$$

Thus

$$\partial_t(\hat{v}(t, k)e^{\frac{1}{2}k^2 t - i\phi(t)k}) = 0$$

which means that the distribution $\hat{v}(t, k)e^{\frac{1}{2}k^2 t - i\phi(t)k}$ is constant along the time variable t. Moreover, we also have that

$$\lim_{t \to s} v(t, \cdot) = 0$$

in the tempered distribution sense. Indeed for any φ in the Schwartz space $\mathcal{S}(\mathbb{R})$, if we denote by S^∂ is the diffusion S killed at zero we have that

$$\begin{aligned}\lim_{t \to s} \int_{\mathbb{R}} \varphi(y)v(t, y)dy = \lim_{t \to s} \Big[\big(&\mathbb{E}[\varphi(S^\partial(t))|S^\partial(s) = x] - \mathbb{E}[\varphi(W^\partial(t))|W^\partial(s) = x]\big) \\ &- \big(\mathbb{E}[\varphi(-S^\partial(t))|S^\partial(s) = x] - \mathbb{E}[\varphi(W^\partial(t))|W^\partial(s) = -x]\big) \\ &- \int_{\mathbb{R}} \varphi(y)h(t, y)dy\Big] = 0\end{aligned}$$

as $h(s, \cdot) = 0$ and by using the dominated convergence theorem. Thus by continuity of the Fourier transform, one deduces that v is zero everywhere, and hence $q = f$ as desired. $\square$

Let us introduce now a function that is going to play a fundamental role in our calculations. Define g by

$$g(s, x, t, y) = G(s, x, t, y)\mathbb{E}\left[\exp\left(-\int_s^t \phi''(u)B(u)du\right)|B(s) = -x, B(t) = -y\right] \tag{3.21}$$

for $x, y < 0$ and $t \geq s$, where B is a three-dimensional Bessel process. Because f is smooth away from $\{t = s\}$, the same holds for g. We have then the following lemma.

Lemma 3.4.5 *The function g verifies the following PDE*

$$\partial_t g = \frac{1}{2}\partial_y^2 g + \phi''(t)yg \tag{3.22}$$

for $(t, y) \in (s, +\infty) \times (-\infty, 0)$.

Proof. We can replace the Bessel process B by the Brownian motion killed at zero W^∂ in the expression of g in (3.21) for the same reasons we gave earlier. Now let $\varphi \in C_c^\infty((s,+\infty)\times(-\infty,0))$ be a test function. We apply Ito formula to the following semi-martingale

$$Y(t) = \varphi(t, W(t))\exp\left(\int_s^t \phi''(u)W(u)du\right)$$

where W is a Brownian motion started at x. We get then

$$\begin{aligned} dY(t) &= \partial_y\varphi(t,W(t))\exp\left(\int_s^t \phi''(u)W(u)du\right)dW(t) \\ &+\left(\partial_t\varphi(t,W(t)) + \frac{1}{2}\partial_y^2\varphi(t,W(t)) + \varphi(t,W(t))\phi''(t)W(t)\right)\exp\left(\int_s^t \phi''(u)W(u)du\right)dt \end{aligned}$$

We integrate between s and $t\wedge\tau_0$ (where τ_0 is the first hitting time of zero of W). As the first term is a bounded local martingale (and hence a true martingale), by taking the expectation we get that

$$\begin{aligned} \mathbb{E}\left[\varphi(t\wedge\tau_0, W(t\wedge\tau_0))\right] = \mathbb{E}[\int_s^{t\wedge\tau_0} &\left(\partial_t\varphi(u,W(u)) + \frac{1}{2}\partial_y^2\varphi(u,W(u))\right. \\ &\left.+\varphi(u,W(u))\phi''(u)W(u)\right)\exp\left(\int_s^u \phi''(\omega)W(\omega)d\omega\right)du\Big] \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{E}\left[\varphi(t\wedge\tau_0, W(t\wedge\tau_0))\right] &= \mathbb{E}[\int_s^t \mathbb{1}_{\{\max_{s\le z\le u} W(z)<0\}}\left(\partial_t\varphi(u,W(u)) + \frac{1}{2}\partial_y^2\varphi(u,W(u))\right. \\ &\left.+\varphi(u,W(u))\phi''(u)W(u)\right)\exp\left(\int_s^u \phi''(\omega)W(\omega)d\omega\right)du\Big] \\ &= \int_s^t \mathbb{E}\left[\left(\partial_t\varphi(u,W^\partial(u)) + \frac{1}{2}\partial_y^2\varphi(u,W^\partial(u))\right.\right. \\ &\left.\left.+\varphi(u,W^\partial(u))\phi''(u)W^\partial(u)\right)\exp\left(\int_s^u \phi''(\omega)W^\partial(\omega)d\omega\right)\right]du \end{aligned}$$

By sending $t\to\infty$ and conditioning on the value of $W^\partial(u)$, we get

$$\int_s^\infty \int_{-\infty}^0 \left(\partial_t\varphi(u,y) + \frac{1}{2}\partial_y^2\varphi(u,y) + \phi''(u)y\varphi(u,y)\right)g(u,y)dydu = 0$$

Thus we get the PDE in the distribution sense, but also in the classical sense because g is smooth on the interior of its domain. ☐

We give now an explicit formula for the functional Φ that was introduced in the previous section.

Proposition 3.4.6 *The function* Φ *can be expressed as*

$$\Phi(s,x,t) = \frac{-x}{\sqrt{2\pi(t-s)^3}} e^{-\frac{x^2}{2(t-s)}} exp\left(\phi'(s)x - \frac{1}{2}\int_s^t (\phi'(u))^2 du\right) \times \\ \mathbb{E}^{(s,-x)\to(t,0)}\left[exp\left(-\int_s^t \phi''(u)B^{br}(u)du\right)\right]$$

for $s < t$ *and* $x < 0$*.* B^{br} *here is a three-dimensional Bessel bridge from* $(s,-x)$ *to* $(t,0)$*.*

Proof. From Lemma 3.3.5, we have that

$$\Phi(s,x,t) = -\frac{1}{2}\partial_y f(s,x,t,0)$$

Since

$$f(s,x,t,y) = \exp\left(-\phi'(t)y + \phi'(s)x - \frac{1}{2}\int_s^t (\phi'(u))^2 du\right) g(s,x,t,y)$$

and

$$\partial_y g(s,x,t,0) = \lim_{y\uparrow 0} \partial_y G(s,x,t,y)\mathbb{E}\left[\exp\left(-\int_s^t \phi''(u)B(u)du\right)|B(s) = -x, B(t) = -y\right] \\ + lim_{y\uparrow 0} G(s,x,t,y)\partial_y \mathbb{E}\left[\exp\left(-\int_s^t \phi''(u)B(u)du\right)|B(s) = -x, B(t) = -y\right]$$

it suffices to prove that

$$\lim_{y\uparrow 0} \partial_y \mathbb{E}\left[\exp\left(-\int_s^t \phi''(u)B(u)du\right)|B(s) = -x, B(t) = -y\right] < \infty$$

as $G(s,x,t,0) = 0$. We have by Hopital's rule applied twice

$$\begin{aligned}\lim_{y\uparrow 0} \partial_y \mathbb{E}\left[\exp\left(-\int_s^t \phi''(u)B(u)du\right)|B(s) = -x, B(t) = -y\right] &= \lim_{y\uparrow 0} \frac{(\partial_y g)G - (\partial_y G)g}{G^2} \\ &= \lim_{y\uparrow 0} \frac{(\partial_y^2 g)G - (\partial_y^2 G)g}{2G\partial_y G} \\ &= \lim_{y\uparrow 0} \frac{\partial_y^2 g}{2\partial_y G} - \lim_{y\uparrow 0} \frac{(\partial_y^2 G)g}{2G\partial_y G} \\ &= -\lim_{y\uparrow 0} \frac{(\partial_y^2 G)g}{2G\partial_y G} \\ &= -\lim_{y\uparrow 0} \frac{(\partial_y^3 G)g + (\partial_y^2 G)\partial_y g}{2(\partial_y G)^2 + 2G\partial_y^2 G}\end{aligned}$$

$$\lim_{y\uparrow 0} \partial_y \mathbb{E}\left[\exp\left(-\int_s^t \phi''(u)B(u)du\right)|B(s) = -x, B(t) = -y\right] = 0$$

In the fourth line we used the fact that $\lim_{y\uparrow 0}\partial_y^2 g = 0$. This follows from the PDE (3.22) verified by g and the fact that $g(s,x,t,0) = 0$. Moreover because $\lim_{y\uparrow 0}\partial_y G \neq 0$, we can conclude that the limit is equal to zero in the penultimate equality.

To finish the proof, we refer to the fact that the weak limit of the law of the three-dimensional Bessel process conditioned to end at y when y goes to zero is that of the corresponding three-dimensional Bessel bridge, and thus the result follows from the expression of the Green function G. □

We are ready to give an explicit formula of the kernel K^ϕ.

Proposition 3.4.7 *The kernel K^ϕ has the following expression*

$$K^\phi(y,z) = \frac{\phi'(z)-\phi'(y)}{\sqrt{2\pi(z-y)^3}}\, exp\left(-\frac{1}{2}\int_y^z (\phi'(u))^2 du\right)\mathbb{E}\left[exp\left(-\int_y^z \phi''(u)\boldsymbol{e}(u)du\right)\right]$$

for $y \leq z$, where $(\boldsymbol{e}(u), y \leq u \leq z)$ is a Brownian excursion on $[y,z]$.

Proof. Recall that K^ϕ is given by

$$K^\phi(y,z) = 2\int_y^z\int_0^\infty \phi''(\omega)\Phi(\omega,-x,z)\tilde{\Phi}(-\omega,-x,-y)dxd\omega$$

Remember that $\tilde{\Phi}$ is the same as Φ with the function ϕ replaced by $\phi(-\cdot)$. Hence

$$\begin{aligned}\Phi(\omega,-x,z)\tilde{\Phi}(-\omega,-x,-y) = {} & \frac{x^2}{2\pi\sqrt{(z-\omega)^3(\omega-y)^3}}e^{-\frac{x^2}{2(z-\omega)}-\frac{x^2}{2(\omega-y)}}\times \\ & \exp\left(-\frac{1}{2}\int_\omega^z(\phi'(u))^2du - \frac{1}{2}\int_{-\omega}^{-y}(\phi'(-u))^2du\right)\times \\ & \mathbb{E}^{(\omega,x)\to(z,0)}\left[\exp\left(-\int_\omega^z\phi''(u)B^{br}(u)du\right)\right]\times \\ & \mathbb{E}^{(-\omega,x)\to(-y,0)}\left[\exp\left(-\int_{-\omega}^{-y}\phi''(-u)B^{br}(u)du\right)\right]\end{aligned}$$

Consider now a Brownian excursion $\mathbf{e}$ on $[y,z]$, conditionally on its value at $\omega \in [y,z]$, the two paths $(\mathbf{e}(u),\ \ y \leq u \leq \omega)$ and $(\mathbf{e}(u),\ \ \omega \leq u \leq z)$ are independent, and each path has the distribution of a three-dimensional Bessel bridge. Furthermore, because of the Brownian scaling we have that

$$(\mathbf{e}(u), y \leq u \leq z) \stackrel{d}{=} (\sqrt{y-z}\,\mathbf{e}^{\text{std}}\left(\frac{u-y}{z-y}\right), y \leq u \leq z) \tag{3.23}$$

where $(\mathbf{e}^{\text{std}}(u), 0 \leq u \leq 1)$ is a standard Brownian excursion. Thus, using the fact that

$$\mathbb{P}\left[\mathbf{e}^{\text{std}}(t)\in dx\right] = \frac{2x^2}{\sqrt{2\pi t^3(1-t)^3}}e^{-\frac{x^2}{2t(1-t)}}dx$$

then it follows that for $\omega \in [y, z]$

$$\mathbb{P}\left[\mathbf{e}(\omega) \in dx\right] = \frac{2x^2\sqrt{(z-y)^3}}{\sqrt{2\pi(z-\omega)^3(\omega-y)^3}} e^{-\frac{x^2}{2(z-\omega)} - \frac{x^2}{2(\omega-y)}} dx$$

Thus by the time-reversal property of the three-dimensional Bessel bridges we have that

$$\Phi(\omega, -x, z)\tilde{\Phi}(-\omega, -x, -y) = \frac{1}{\sqrt{2\pi(z-y)^3}} \mathbb{E}[\exp\left(-\int_y^z \phi''(u)\mathbf{e}(u)du\right)|\mathbf{e}(\omega) = x] \times \frac{\mathbb{P}[\mathbf{e}(\omega) \in dx]}{dx}$$

By integrating with respect to x and ω we get the desired result. ☐

The next theorem gives a closed formula for the function j.

Theorem 3.4.8 *Let $s \in \mathbb{R}$, define the function l^s on $(0, \infty)$ by*

$$l^s(u) = exp\left(-\frac{1}{2}\int_s^{s+u} \phi'(z)^2 dz\right) \mathbb{E}\left[exp\left(-\int_s^{s+u} \phi''(z)\boldsymbol{e}(z)dz\right)\right], \quad u > 0$$

where $\boldsymbol{e}$ is a Brownian excursion on $[s, s+u]$. Then

$$j(s) = -\phi'(s) + \int_0^\infty \frac{l^s(u) - 1}{\sqrt{2\pi u^3}} du$$

Proof. The function J is defined as

$$\begin{aligned}
J(s,x) &= \mathbb{P}[S(\omega) < 0 \text{ for all } \omega \geq s | S(s) = x] \\
&= \lim_{t\to\infty} \mathbb{P}[S(\omega) < 0 \text{ for all } s \leq \omega \leq t | S(s) = x] \\
&= \lim_{t\to\infty} \int_{-\infty}^0 f(s,x,t,y)dy \\
&= e^{\phi'(s)x} \lim_{t\to\infty} e^{-\frac{1}{2}\int_s^t (\phi'(u))^2 du} \int_{-\infty}^0 e^{-\phi'(t)y} g(s,x,t,y)dy \\
&= e^{\phi'(s)x} \lim_{t\to\infty} e^{-\frac{1}{2}\int_s^t (\phi'(u))^2 du} \int_0^\infty e^{\phi'(t)y} m(s,x,t,y)dy
\end{aligned}$$

where the function m is defined as

$$m(s,x,t,y) = g(s,x,t,-y)$$

It verifies the following PDE

$$\partial_t m = \frac{1}{2}\partial^2_{yy} m - \phi''(t) y m \qquad (3.24)$$

Because of the asymptotic behavior of g in space at infinity, we can define for every $\lambda \in \mathbb{R}$ the Laplace transform

$$\hat{m}(s,x,t,\lambda) = \int_0^\infty e^{\lambda y} m(s,x,t,y)dy$$

From the representation formula of the function h (and thus that of g) in the statement of Theorem 3.4.3 and the fast decay of the Green functions G and $\hat{G}$ in space, we can interchange the order of differentiation and integration for the Laplace transform $\hat{m}$, hence

$$\begin{aligned}\partial_t \hat{m} &= \int_0^\infty e^{\lambda y} \partial_t m(s,x,t,y) dy \\ &= \int_0^\infty e^{\lambda y} \left(\frac{1}{2} \partial_{yy}^2 m(s,x,t,y) - \phi''(t) y m(s,x,t,y) \right) dy \\ &= \frac{1}{2} \left[e^{\lambda y} \partial_y m(s,x,t,y) \right]_0^\infty + \frac{1}{2} \lambda^2 \hat{m}(s,x,t,\lambda) - \phi''(t) \partial_\lambda \hat{m}(s,x,t,\lambda) \\ &= \frac{1}{2} \lambda^2 \hat{m}(s,x,t,\lambda) - \phi''(t) \partial_\lambda \hat{m}(s,x,t,\lambda) - \frac{1}{2} \partial_y m(s,x,t,0)\end{aligned}$$

by integration by parts and using the fact that $m(s,x,t,0) = 0$. From the expression of g we deduce that

$$\begin{aligned}\partial_y m(s,x,t,0) = -\partial_y g(s,x,t,0) = \frac{-2x}{\sqrt{2\pi(t-s)^3}} e^{-\frac{x^2}{2(t-s)}} \times \\ \mathbb{E}\left[\exp\left(-\int_s^t \phi''(u) B(u) du \right) | B(s) = -x, B(t) = 0 \right] \\ = 2\Phi(s,x,t) \exp\left(-\phi'(s) x + \frac{1}{2} \int_s^t (\phi'(u))^2 du \right) =: -2\Upsilon(t)\end{aligned}$$

Since x and s are fixed for now, we will often omit them when writing out expressions where they do not vary. Thus, the PDE verified by $\hat{m}$ takes the form

$$\partial_t \hat{m} + \phi''(t) \partial_\lambda \hat{m} - \frac{1}{2} \lambda^2 \hat{m} - \Upsilon(t) = 0$$

This is a first order non-linear PDE that can be solved by the method of characteristics. If we denote the variables by $x_1 := t$ and $x_2 := \lambda$ and the value of the function $z = \hat{m}(x_1, x_2)$, the characteristic ODEs take the form

$$\begin{cases} \dot{x_1}(u) = 1 \\ \dot{x_2}(u) = \phi''(x_1(u)) \\ \dot{z}(u) = \frac{1}{2} x_2^2(u) z(u) + \Upsilon(x_1(u)) \end{cases}$$

We choose the initial conditions such that $x_1(u) = u$ and $x_2(u) = \phi'(u) + (\lambda - \phi'(t))$ for $u \geq s$. Hence

$$\dot{z}(u) = \frac{1}{2} (\phi'(u) + \lambda - \phi'(t))^2 z(u) + \Upsilon(u)$$

Introduce the function v^λ defined by

$$v^\lambda(u) = \exp\left(-\frac{1}{2} \int_s^u (\phi'(z) + \lambda - \phi'(t))^2 dz \right)$$

Then it is clear that

$$(\dot{v^\lambda z})(u) = v^\lambda(u) \Upsilon(u)$$

In order to avoid the singularity at $\{t = s\}$, we integrate thus between $s + \epsilon$ and t for $\epsilon > 0$ to get that

$$v^{\lambda}(t)z(t) - v^{\lambda}(s+\epsilon)z(s+\epsilon) = \int_{s+\epsilon}^{t} v^{\lambda}(u)\Upsilon(u)du$$

which is equivalent to

$$\hat{m}(s,x,t,\lambda)v^{\lambda}(t) - \hat{m}(s,x,s+\epsilon,\phi'(s+\epsilon)+\lambda-\phi'(t))v^{\lambda}(s+\epsilon) = \int_{s+\epsilon}^{t} v^{\lambda}(u)\Upsilon(u)du$$

By taking $\lambda = \phi'(t)$, we get

$$\hat{m}(s,x,t,\phi'(t))e^{-\frac{1}{2}\int_s^t \phi'(u)^2 du} - \hat{m}(s,x,s+\epsilon,\phi'(s+\epsilon))e^{-\frac{1}{2}\int_s^{s+\epsilon} \phi'(u)^2 du} = \int_{s+\epsilon}^{t} e^{-\frac{1}{2}\int_s^u \phi'(\omega)^2 d\omega}\Upsilon(u)du \tag{3.25}$$

As $J(s,x) = e^{\phi'(s)x} \lim_{t\to\infty} e^{-\frac{1}{2}\int_s^t (\phi'(u))^2 du}\hat{m}(s,x,t,\phi'(t))$. By sending t to ∞ in the expression (3.25), we have

$$J(s,x) = e^{\phi'(s)x}\left[\hat{m}(s,x,s+\epsilon,\phi'(s+\epsilon))e^{-\frac{1}{2}\int_s^{s+\epsilon}\phi'(u)^2 du} + \int_{s+\epsilon}^{\infty} e^{-\frac{1}{2}\int_s^u \phi'(\omega)^2 d\omega}\Upsilon(s,x,u)du\right]$$

It follows that

$$\begin{aligned} j(s) := \lim_{x\uparrow 0}\frac{\partial}{\partial x}J(s,x) = e^{-\frac{1}{2}\int_s^{s+\epsilon}\phi'(u)^2 du}\lim_{x\uparrow 0}\frac{\partial}{\partial x}\hat{m}(s,x,s+\epsilon,\phi'(s+\epsilon)) + \\ \int_{s+\epsilon}^{\infty} e^{-\frac{1}{2}\int_s^u \phi'(\omega)^2 d\omega}\lim_{x\uparrow 0}\frac{\partial}{\partial x}\Upsilon(s,x,u)du \end{aligned} \tag{3.26}$$

since $m(s,0,s+\epsilon,\cdot) = 0$, and we can interchange differentiation and the integral sign in the second term because we are away from the singularity line $\{t = s\}$. Now, we have that

$$\hat{m}(s,x,s+\epsilon,\phi'(s+\epsilon)) = \int_0^{\infty} e^{\phi'(s+\epsilon)y} m(s,x,s+\epsilon,y)dy$$

It is clear that m is smooth in the parameters (s,x) as well. Our analysis of regularity of the function $f(s,x,t,y)$ consisted of using the Kolmogorov forward equation where the parameters were t and y, but similarly the Kolmogorov backward equation that holds for the parameters s and x, we see that the solution enjoys the same smoothness and integrability properties away from the line $\{s = t\}$ (it is formally just the adjoint problem). Hence we can differentiate inside the integral sign to get

$$\lim_{x\uparrow 0}\frac{\partial}{\partial x}\hat{m}(s,x,s+\epsilon,\phi'(s+\epsilon)) = \int_0^{\infty} e^{\phi'(s+\epsilon)y}\lim_{x\uparrow 0}\frac{\partial}{\partial x}m(s,x,s+\epsilon,y)dy$$

since we have that

$$\begin{aligned} \lim_{x\uparrow 0}\frac{\partial}{\partial x}m(s,x,s+\epsilon,y) = -\frac{2y}{\sqrt{2\pi\epsilon^3}}e^{-\frac{y^2}{2\epsilon}} \times \\ \mathbb{E}\left[\exp\left(-\int_s^{s+\epsilon}\phi''(u)B(u)du\right)|B(s)=0, B(s+\epsilon)=y\right] \end{aligned}$$

Thus

$$\lim_{x\uparrow 0}\frac{\partial}{\partial x}\hat{m}(s,x,s+\epsilon,\phi'(s+\epsilon)) = -\int_0^\infty e^{\phi'(s+\epsilon)y-\frac{y^2}{2\epsilon}}\frac{2y}{\sqrt{2\pi\epsilon^3}}\times$$
$$\mathbb{E}\left[\exp\left(-\int_s^{s+\epsilon}\phi''(u)B(u)du\right)|B(s)=0,B(s+\epsilon)=y\right]dy$$

However, the density of a three-dimensional Bessel process is given by

$$\mathbb{P}[B(s+\epsilon)\in dy|B(s)=0] = \frac{2y^2}{\sqrt{2\pi\epsilon^3}}e^{-\frac{y^2}{2\epsilon}}dy \tag{3.27}$$

Hence

$$\lim_{x\uparrow 0}\frac{\partial}{\partial x}\hat{m}(s,x,s+\epsilon,\phi'(s+\epsilon)) =$$
$$-\mathbb{E}\left[\frac{1}{B(s+\epsilon)}\exp\left(\phi'(s+\epsilon)B(s+\epsilon)-\int_s^{s+\epsilon}\phi''(u)B(u)du\right)|B(s)=0\right]$$
$$= -\mathbb{E}\left[\frac{1}{B(\epsilon)}\exp\left(\phi'(s+\epsilon)B(\epsilon)-\int_0^{\epsilon}\phi''(u+s)B(u)du\right)|B(0)=0\right]$$

However by Brownian scaling, we know that

$$(B(u),u\geq 0)\stackrel{d}{=}(\sqrt{\epsilon}B\left(\frac{u}{\epsilon}\right),u\geq 0)$$

Hence

$$\lim_{x\uparrow 0}\frac{\partial}{\partial x}\hat{m}(s,x,s+\epsilon,\phi'(s+\epsilon)) = -\mathbb{E}\left[\frac{1}{\sqrt{\epsilon}B(1)}\exp\left(\phi'(s+\epsilon)\sqrt{\epsilon}B(1)-\right.\right.$$
$$\left.\left.\sqrt{\epsilon^3}\int_0^1\phi''(\epsilon u+s)B(u)du\right)|B(0)=0\right]$$
$$= -\frac{1}{\sqrt{\epsilon}}\mathbb{E}\left[\frac{1}{B(1)}\right]+\phi'(s)+O(\sqrt{\epsilon})$$

It follows then that

$$\lim_{x\uparrow 0}\frac{\partial}{\partial x}\hat{m}(s,x,s+\epsilon,\phi'(s+\epsilon)) = -\frac{2}{\sqrt{2\pi\epsilon}}+\phi'(s)+O(\sqrt{\epsilon}) \tag{3.28}$$

for ϵ small. The expectation of the inverse of $B(1)$ is computed using the density given in (3.27). Now, on the other hand for the second term in (3.26), we have

$$\lim_{x\uparrow 0}\frac{\partial}{\partial x}\Upsilon(s,x,u) = -\partial_x\Phi(s,0,u)\exp\left(\frac{1}{2}\int_s^u\phi'(\omega)^2d\omega\right)$$
$$= \frac{1}{\sqrt{2\pi(u-s)^3}}\mathbb{E}\left[\exp\left(-\int_s^u\phi''(z)\mathbf{e}(z)dz\right)\right]$$

Hence

$$\int_{s+\epsilon}^{\infty} e^{-\frac{1}{2}\int_s^u \phi'(\omega)^2 d\omega} \lim_{x\uparrow 0} \frac{\partial}{\partial x}\Upsilon(s,x,u)du = \int_{\epsilon}^{\infty} \frac{l^s(u)}{\sqrt{2\pi u^3}}du \tag{3.29}$$

and thus, from combining (3.26), (3.28) and (3.29) we get

$$j(s) = \int_{\epsilon}^{\infty} \frac{l^s(u)}{\sqrt{2\pi u^3}}du - \frac{2}{\sqrt{2\pi\epsilon}} - \phi'(s) + O(\sqrt{\epsilon})$$

Finally, see that

$$\int_{\epsilon}^{\infty} \frac{du}{\sqrt{2\pi u^3}} = \frac{2}{\sqrt{2\pi\epsilon}}$$

and then send ϵ to zero to finish the proof. □

Remark 3.4.9 *When ϕ is parabolic ($\phi(y) = y^2$), the term ϕ'' in the PDE (3.24) of m becomes a constant and thus it takes the simple form*

$$\partial_t m = \frac{1}{2}\partial^2_{yy} m - 2ym$$

By taking the Fourier transform in time we get

$$\frac{1}{2}(\hat{m}(\tau,y))'' = (i\tau + 2y)\hat{m}(\tau,y)$$

This is a Sturm-Liouville equation. Its solution can be expressed in terms of Airy functions, from which follows all the analytical descriptions that Groeneboom found in [25]. It is clear that when ϕ'' is not constant, this method fails which makes the study more delicate as one doesn't have any asymptotic or regularity properties of the function m, which was a crucial part in the analysis of Groeneboom. For those reasons, we had to take advantage of the space Laplace transform.

As a consequence of the explicit formula of j and Φ, we are able to provide the joint distribution of the maximum of the process $(W(\omega) - \phi(\omega))_{\omega \geq s}$ and its location. This is given by the expression of Φ and j and using Lemma 3.3.5. However, the formula is involving many terms, in particular the Bessel bridge area. On the other hand, the density of the location of the maximum takes a simpler formula. This is a generalization of Chernoff distribution, where the parabolic drift is replaced by any strictly convex drift ϕ.

Theorem 3.4.10 *Let ω_M be the location of the unique maximum of the process $(S(\omega) = W(\omega) - \phi(\omega))_{\omega\in\mathbb{R}}$, its density is equal to*

$$\frac{\mathbb{P}[\omega_M \in dt]}{dt} = \frac{1}{2} j(t)\tilde{j}(-t)$$

where $\tilde{j}$ is the analogue of j for the process $\tilde{S}(\omega) := W(\omega) - \phi(-\omega)$.

Proof. We will prove the equality for $t \geq 0$, the case $t \leq 0$ is completely identical. From Lemma 3.3.5 with $s = 0$ and any $x > z$

$$\frac{\mathbb{P}[\text{argmax}_{\omega\geq 0} S(\omega) \in dt, \max_{\omega\geq 0} S(\omega) \in dz | S(0) = x]}{dtdz} = \frac{1}{2} j(t)\partial_y f(0, x - z, t, 0)$$

Hence

$$\mathbb{P}[\omega_M \in dt|S(0) = x] = \int_x^{+\infty} \mathbb{P}[\text{argmax}_{\omega\geq 0} S(\omega) \in dt, \max_{\omega\geq 0} S(\omega) \in dz, \max_{\omega\leq 0} S(\omega) < z|S(0) = x]$$

$$= \int_x^{+\infty} \frac{1}{2} j(t)\partial_y f(0, x - z, t, 0)\mathbb{P}[S(\omega) < z \text{ for all } \omega \leq 0|S(0) = x]dzdt$$

by independence of the paths $(S(\omega), \omega \leq 0)$ and $(S(\omega), \omega \geq 0)$. However by time reversal of the Brownian motion we have

$$\begin{aligned}
\mathbb{P}[S(\omega) < z \text{ for all } \omega \leq 0|S(0) = x] &= \mathbb{P}[\tilde{S}(\omega) < z \text{ for all } \omega \geq 0|\tilde{S}(0) = x] \\
&= \mathbb{P}[\tilde{S}(\omega) < 0 \text{ for all } \omega \geq 0|\tilde{S}(0) = x - z] \\
&= \tilde{J}(0, x - z)
\end{aligned}$$

Thus

$$\frac{\mathbb{P}[\omega_M \in dt|S(0) = x]}{dt} = \int_{-\infty}^0 \frac{1}{2} j(t)\partial_y f(0, z, t, 0)\tilde{J}(0, z)dz$$

Notice that the right hand-side is independent of x, so we can drop the conditional probability in the left hand-side. Moreover by (3.18), we have

$$\partial_y f(0, z, t, 0) = \partial_x \tilde{f}(-t, 0, 0, z) \tag{3.30}$$

Using the expression of the entrance law of the process $\tilde{S}^{\downarrow}$ in (3.14), we have

$$\mathbb{P}[\tilde{S}^{\downarrow}(0) \in dz|\tilde{S}^{\downarrow}(-t) = 0] = \frac{\tilde{J}(0, z)}{\tilde{j}(-t)}\partial_x \tilde{f}(-t, 0, 0, z)dz \tag{3.31}$$

Hence combining (3.30) and (3.31) we get

$$\int_0^{\infty} \partial_y f(0, z, t, 0)\tilde{J}(0, z)dz = \tilde{j}(-t) \int_{-\infty}^0 \mathbb{P}[\tilde{S}^{\downarrow}(0) \in dz|\tilde{S}^{\downarrow}(-t) = 0] = \tilde{j}(-t)$$

which completes the proof. □

Remark 3.4.11 *This last theorem is exactly Theorem 3.1.10 by noticing that* $f^{\phi}(t) = -j(t)$ *and* $f^{\phi(-\cdot)}(-t) = -\tilde{j}(-t)$.

Remark 3.4.12 *From [25] results in the parabolic drift case, the Chernoff distribution can be expressed as*

$$\frac{\mathbb{P}[argmax_{z\in\mathbb{R}}(W(z)-z^2)\in dt]}{dt}=\frac{1}{2}k(t)k(-t)$$

where $k(t)=e^{\frac{2}{3}t^3}g(t)$ *and* g *has the Fourier transform given by*

$$\hat{g}(\tau):=\int_{-\infty}^{\infty}e^{it\tau}g(t)dt=\frac{2^{\frac{1}{3}}}{\mathrm{Ai}(i2^{-\frac{1}{3}}\tau)}$$

This expression is not clear from the formula we provided in Theorem 3.1.10. We will prove thus in the following proposition that those two indeed coincide.

Proposition 3.4.13 *For any* $t\in\mathbb{R}$ *we have*

$$2t+\int_0^{\infty}\frac{1}{\sqrt{2\pi u^3}}\left(1-e^{-\frac{2}{3}((u+t)^3-t^3)}\mathbb{E}\left[exp\left(-2\int_0^u \boldsymbol{e}(z)dz\right)\right]\right)du=\frac{e^{\frac{2}{3}t^3}}{2\pi}\int_{-\infty}^{\infty}e^{-itv}\hat{g}(v)dv$$

Proof. From equation (1.6) in [27] [3] , we have that

$$\begin{aligned}\frac{1}{2\pi}\int_{v=-\infty}^{\infty}\frac{\mathrm{Ai}(i\xi-4^{\frac{1}{3}}x)}{\mathrm{Ai}(i\xi)}\int_{u=0}^{\infty}e^{iuv-\frac{2}{3}((u+t)^3-t^3)}dudv=\\ e^{-2tx}-\frac{e^{\frac{2}{3}t^3}}{4^{\frac{2}{3}}}\int_{v=-\infty}^{\infty}e^{-itv}\frac{\mathrm{Ai}(i\xi)\mathrm{Bi}(i\xi-4^{\frac{1}{3}}x)-\mathrm{Ai}(i\xi-4^{\frac{1}{3}}x)\mathrm{Bi}(i\xi)}{\mathrm{Ai}(i\xi)}dv\end{aligned}\tag{3.32}$$

where $\xi=2^{-\frac{1}{3}}v$, and Bi is the second Airy function. By differentiating both sides with respect to x and sending x to zero, we get

$$\frac{e^{\frac{2}{3}t^3}}{4^{\frac{1}{3}}\pi}\int_{v=-\infty}^{\infty}\frac{e^{-itv}}{\mathrm{Ai}(i\xi)}dv=2t+\lim_{x\uparrow 0}\frac{\partial}{\partial x}\frac{1}{2\pi}\int_{v=-\infty}^{\infty}\frac{\mathrm{Ai}(i\xi-4^{\frac{1}{3}}x)}{\mathrm{Ai}(i\xi)}\int_{u=0}^{\infty}e^{iuv-\frac{2}{3}((u+t)^3-t^3)}dudv\tag{3.33}$$

as the Wronskian of the Airy functions Ai and Bi is constant and equal to $\frac{1}{\pi}$. In the right-hand side of (3.32), we cannot differentiate inside the integral sign because it becomes divergent. However for fixed $x<0$, the integrand is absolutely integrable and thus we can use Fubini theorem. Now from [28][Equation 384, Page 141] we have that

$$-\int_0^{\infty}e^{-\lambda s}\mathbb{E}\left[\exp\left(-2\int_0^s B(u)du\right)|B(s)=-x\right]\frac{x}{\sqrt{2\pi s^3}}e^{-\frac{x^2}{2s}}ds=\frac{\mathrm{Ai}(2^{-\frac{1}{3}}\lambda-4^{\frac{1}{3}}x)}{\mathrm{Ai}(2^{-\frac{1}{3}}\lambda)}$$

where B is as usual a three-dimensional Bessel process. Thus, by inverse Laplace transform we have

$$-\mathbb{E}\left[\exp\left(-2\int_0^u B(z)dz\right)|B(u)=-x\right]\frac{x}{\sqrt{2\pi u^3}}e^{-\frac{x^2}{2u}}=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{iuv}\frac{\mathrm{Ai}(i\xi-4^{\frac{1}{3}}x)}{\mathrm{Ai}(i\xi)}dv$$

[3] There is a typo in the published paper, the term $4^{\frac{2}{3}}$ in the denominator should be there instead of $4^{\frac{1}{3}}$.

Hence the integral in the RHS of (3.33) is equal to

$$-\int_0^\infty e^{-\frac{2}{3}((u+t)^3-t^3)}\frac{x}{\sqrt{2\pi u^3}}e^{-\frac{x^2}{2u}}\mathbb{E}\left[\exp\left(-2\int_0^u B(z)dz\right)|B(u)=-x\right]du \tag{3.34}$$

By splitting this integral on $(0,\epsilon)$ and (ϵ,∞), we can interchange the integral and the differentiation for the integral on (ϵ,∞), and so we get after sending x to zero

$$-\int_\epsilon^\infty e^{-\frac{2}{3}((u+t)^3-t^3)}\frac{1}{\sqrt{2\pi u^3}}\mathbb{E}\left[\exp\left(-2\int_0^u \mathbf{e}(z)dz\right)\right]du \tag{3.35}$$

where $\mathbf{e}$ is as usual a Brownian excursion on the corresponding interval. As for the first term (the integral on $(0,\epsilon)$), by the change of variable $y=\frac{x}{\sqrt{u}}$ ($dy=-\frac{x}{2\sqrt{u^3}}du$), it is equal to

$$\begin{aligned}-\int_0^\epsilon e^{-\frac{2}{3}((u+t)^3-t^3)}\frac{x}{\sqrt{2\pi u^3}}e^{-\frac{x^2}{2u}}\mathbb{E}\left[\exp\left(-2\int_0^u B(z)dz\right)|B(u)=-x\right]du\\
=\int_{-\infty}^{\frac{x}{\sqrt{\epsilon}}} e^{-\frac{2}{3}((\frac{x^2}{y^2}+t)^3-t^3)}\frac{2}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\mathbb{E}\left[\exp\left(-2\frac{x^3}{y^3}\int_0^1 B(z)dz\right)|B(1)=-y\right]dy\end{aligned}$$

by Brownian scaling on the Bessel process B. Differentiating with respect to x, we get by Leibniz rule

$$\frac{2}{\sqrt{2\pi\epsilon}}e^{-\frac{2}{3}((\epsilon+t)^3-t^3)}e^{-\frac{x^2}{2\epsilon}}\mathbb{E}\left[\exp\left(-2\sqrt{\epsilon^3}\int_0^1 B(z)dz\right)|B(1)=-\frac{x}{\sqrt{\epsilon}}\right]+F^\epsilon(x) \tag{3.36}$$

where F^ϵ is equal to

$$\begin{aligned}F^\epsilon(x)=\frac{2}{\sqrt{2\pi}}\int_{-\infty}^{\frac{x}{\sqrt{\epsilon}}}\left(-4\frac{x^5}{y^6}-8t\frac{x^3}{y^4}-4t^2\frac{x}{y^2}-6\frac{x^2}{y^3}\right)e^{-\frac{y^2}{2}}e^{-\frac{2}{3}((\frac{x^2}{y^2}+t)^3-t^3)}\times\\
\mathbb{E}\left[\exp\left(-2\frac{x^3}{y^3}\int_0^1 B(z)dz\right)|B(1)=-y\right]dy\end{aligned}$$

However we have that for x small enough (such that $|\frac{x}{\sqrt{\epsilon}}|=-\frac{x}{\sqrt{\epsilon}}\leq 1$)

$$\begin{aligned}\left|\int_{-\infty}^{\frac{x}{\sqrt{\epsilon}}}\frac{x}{y^2}e^{-\frac{y^2}{2}}e^{-\frac{2}{3}((\frac{x^2}{y^2}+t)^3-t^3)}\mathbb{E}\left[\exp\left(-2\frac{x^3}{y^3}\int_0^1 B(z)dz\right)|B(1)=-y\right]dy\right|\leq\\
|x|\int_{-\frac{x}{\sqrt{\epsilon}}}^\infty\frac{e^{-\frac{y^2}{2}}}{y^2}dy\leq|x|(1-\frac{\sqrt{\epsilon}}{x}+\int_1^\infty e^{-\frac{y^2}{2}}dy)\end{aligned}$$

so

$$\limsup_{x\uparrow 0}\left|\int_{-\infty}^{\frac{x}{\sqrt{\epsilon}}}\frac{x}{y^2}e^{-\frac{y^2}{2}}e^{-\frac{2}{3}((\frac{x^2}{y^2}+t)^3-t^3)}\mathbb{E}\left[\exp\left(-2\frac{x^3}{y^3}\int_0^1 B(z)dz\right)|B(1)=-y\right]dy\right|\leq\sqrt{\epsilon}$$

Similarly with the other terms we find that there is a constant $C>0$ (that depends on t) such that

$$\limsup_{x\uparrow 0}|F^\epsilon(x)|\leq C\sqrt{\epsilon}$$

Hence, by combining (3.35) and (3.36), the limit of the derivative of the expression in (3.34) when x goes to zero is equal to

$$-\int_{\epsilon}^{\infty} e^{-\frac{2}{3}((u+t)^3-t^3)}\frac{1}{\sqrt{2\pi u^3}}\mathbb{E}\left[\exp\left(-2\int_0^u \mathbf{e}(z)dz\right)\right]du$$
$$+\frac{2}{\sqrt{2\pi\epsilon}}e^{-\frac{2}{3}((\epsilon+t)^3-t^3)}\mathbb{E}\left[\exp\left(-2\sqrt{\epsilon^3}\int_0^1 \mathbf{e}(z)dz\right)\right]+\limsup_{x\uparrow 0} F^{\epsilon}(x)$$

Now it suffices to see that

$$\begin{aligned}\frac{2}{\sqrt{2\pi\epsilon}}e^{-\frac{2}{3}((\epsilon+t)^3-t^3)}\mathbb{E}\left[\exp\left(-2\sqrt{\epsilon^3}\int_0^1 \mathbf{e}(z)dz\right)\right] &= \frac{2}{\sqrt{2\pi\epsilon}}+O(\sqrt{\epsilon})\\ &= \int_{\epsilon}^{\infty}\frac{1}{\sqrt{2\pi u^3}}du+O(\sqrt{\epsilon})\end{aligned}$$

□

We are now ready to prove the Theorem 3.1.8.

Proof of Theorem 3.1.8. Recall that our solution is expressed as

$$\rho(x,t)=L'\left(\frac{y(x,t)-x}{t}\right)=L'\left(\frac{\Psi^{tL(\frac{\cdot}{t})}(x)-x}{t}\right)$$

Hence, ρ is stationary by Theorem 3.2.2, and so it is a time-homogenous Markov process, its generator is determined by

$$\begin{aligned}\mathcal{A}^t\varphi(y) &= \lim_{h\to 0}\frac{\mathbb{E}[\varphi(\rho(h,t))-\varphi(\rho_-)|\rho(0,t)=\rho_-]}{h}\\ &= \lim_{h\to 0}\frac{\mathbb{E}[\varphi(L'(\frac{\Psi^{tL(\frac{\cdot}{t})}(h)-h}{t}))-\varphi(\rho_-)|\Psi^{tL(\frac{\cdot}{t})}(0)=tH'(\rho_-)]}{h}\\ &= -\frac{1}{t}L''(H'(\rho_-))\varphi'(\rho_-)+\mathcal{A}^{tL(\frac{\cdot}{t})}\varphi(L'(\frac{\cdot}{t}))(tH'(\rho_-))\\ &= -\frac{\varphi'(\rho_-))}{tH''(\rho_-)}+\mathcal{A}^{tL(\frac{\cdot}{t})}\varphi(L'(\frac{\cdot}{t}))(tH'(\rho_-))\\ &= -\frac{\varphi'(\rho_-)}{tH''(\rho_-)}+\int_{\rho_-}^{\infty}(\varphi(\rho_+)-\varphi(\rho_-))n(\rho_-,\rho_+,t)d\rho_+\end{aligned}$$

where

$$n(\rho_-,\rho_+,t)=tH''(\rho_+)\frac{j^{tL(\frac{\cdot}{t})}(tH'(\rho_+))}{j^{tL(\frac{\cdot}{t})}(tH'(\rho_-))}K^{tL(\frac{\cdot}{t})}(tH'(\rho_-),tH'(\rho_+))$$

By a change of variables we have

$$K^{tL(\frac{\cdot}{t})}(tH'(\rho_-),tH'(\rho_+))=\frac{\rho_+-\rho_-}{\sqrt{2\pi t^3(H'(\rho_+)-H'(\rho_-))^3}}\times$$
$$\exp\left(-\frac{t}{2}\int_{\rho_-}^{\rho_+}(\rho_*)^2H''(\rho_*)d\rho_*\right)\mathbb{E}\left[\exp\left(-\int_{\rho_-}^{\rho_+}\mathbf{e}(tH'(\rho_*))d\rho_*\right)\right]$$

Similarly

$$-j^{tL(\frac{\cdot}{t})}(tH'(\rho_-)) = \rho_- + \int_{\rho_-}^{\infty} \frac{1 - p(\rho_-, \rho, t)}{\sqrt{2\pi t(H'(\rho) - H'(\rho_-))^3}} H''(\rho) d\rho$$

where

$$p(\rho_-, \rho, t) = \exp\left(-\frac{t}{2}\int_{\rho_-}^{\rho} (\rho_*)^2 H''(\rho_*) d\rho_*\right) \mathbb{E}\left[\exp\left(-\int_{\rho_-}^{\rho} \mathbf{e}(tH'(\rho_*)) d\rho_*\right)\right]$$

The theorem then follows by appropriately defining the kernel K. □

Remark 3.4.14 *While our main study focused on the case where the initial potential is a two-sided Brownian motion. It is not hard to see that we can extend the result about the profile of the entropy solution when the potential is a spectrally positive Lévy process with non-zero Brownian exponent. The main ingredients that were used were respectively the path decomposition of Markov processes at their ultimate maximum and the regularity properties of the transition function f. Both these facts hold true in the Lévy case when the initial potential U_0 has a non-zero Brownian exponent, as the only difference in the Kolmogorov forward equation is an added integral operator accounting for the jumps of the Lévy process. A similar approach will lead to the same smoothness property away from the singularity line $\{t = s\}$ (the presence of the heat operator $\partial_t - \frac{1}{2}\partial_y^2$ is key to have parabolic smoothing), which will allow all the operations in the second section to be valid. Moreover, one should be able to extract similar expression for the jump kernel n by using the Girsanov theorem version for Lévy processes. We chose in this chapter to only discuss the Brownian motion case because it gives a general idea on how things work and also because it simplifies greatly the computations. One would expect to have similar formulas where the equivalent of the Brownian excursion will be the Lévy bridge informally defined as a Lévy process conditioned to stay positive and to start and end at zero. Those bridges are discussed in [62].*

3.5 Structure of shocks of the entropy solution

A priori, from the involved expression of the generator in Theorem 3.1.8, one cannot easily claim whether if the structure of shocks of the solution ρ is discrete or not. Indeed, this amounts to checking if the following integrability condition on the jump kernel n holds

$$\lambda(\rho_-) = \int_{\rho_-}^{\infty} n(\rho_-, \rho_+, t) d\rho_+ < \infty \text{ for all } \rho_- \in \mathbb{R}$$

However, using the recent theory of Lipschitz minorants of Lévy processes developed in [2] and [21], and following some of the arguments from the study of the structure of shocks in Burgers equation of [1], it turns out that when the initial potential is an *abrupt* spectrally positive Lévy process, one can prove that the set of jump times of the solution ρ is discrete. As we did with Theorem 3.1.8, we will prove a general statement for the process Ψ^ϕ from which Theorem 3.1.14 will follow. We state thus the following theorem

Theorem 3.5.1 *Assume that U_0 is an abrupt spectrally positive Lévy process and ϕ is a strictly convex function such that $\lim_{|y|\to\infty}|\phi'(y)|= +\infty$ and $\displaystyle\lim_{|y|\to+\infty}\frac{U_0(y)}{\phi(y)} = 0$ almost surely, then the range of Ψ^ϕ is a.s discrete.*

Proof. From Theorem 3.2.2, we know that for every $n \in \mathbb{Z}$

$$(\Psi^\phi(x+n)-n)_{x\in\mathbb{R}} \stackrel{d}{=} (\Psi^\phi(x))_{x\in\mathbb{R}}$$

hence it suffices to prove that the set range$(\Psi^\phi) \cap [0,1]$ is a.s discrete. Moreover, we can restrict the process Ψ^ϕ on $[-M,M]$. Indeed, we claim that the probability of the event

$$A_M := \{\text{there exists } a \text{ such that } |a|\geq M \text{ and } \Psi^\phi(a) \in [0,1]\}$$

goes to zero as M goes to infinity. To show this claim, assume that there exists a sequence $(a_n)_{n\in\mathbb{N}}$ such that $\lambda_n := \Psi^\phi(a_n) \in [0,1]$ and $|a_n|\to\infty$. By definition we have that

$$U_0(\lambda_n) - \phi(\lambda_n - a_n) \geq U_0(y) - \phi(y-a_n) \text{ for all } y \tag{3.37}$$

Up to taking subsequences, we have either that $a_n \to \infty$ or $a_n \to -\infty$. If $a_n \to \infty$, take $y = a_n - 1$ in (3.37), then

$$U_0(\lambda_n) - \phi(\lambda_n - a_n) \geq U_0(a_n - 1) - \phi(-1) \tag{3.38}$$

As ϕ' is strictly increasing, we must have $\lim_{y\to-\infty}\phi'(y) = -\infty$, and thus ϕ is decreasing for $y \to -\infty$. Hence from (4.37) and the fact that $\lambda_n \leq 1$, we get

$$U_0(\lambda_n) - U_0(a_n-1) \geq \phi(\lambda_n - a_n) - \phi(-1) \geq \phi(1-a_n) - \phi(-1) \tag{3.39}$$

for n large enough. However, because $(U_0(y))_{y\in\mathbb{R}}$ has the same distribution as $(-U_0((-y)-))_{y\in\mathbb{R}}$, then almost surely $\lim_{n\to\infty}\frac{U_0(a_n-1)}{\phi(1-a_n)} = 0$, which is a contradiction with (4.42). The case $a_n \to -\infty$ is similar by taking $y = a_n$ in (3.37), proving thus our claim.
Define now the event B_M as

$$B_M = \left\{\text{Card}\left(\text{range}(\Psi^\phi_{|[-M,M]|}) \cap [0,1]\right) = \infty\right\}$$

It suffices to prove that $\displaystyle\lim_{M\to\infty}\mathbb{P}[B_M] = 0$.
Suppose initially that $\mathbb{E}[|U_0(1)|] < \infty$ and let $C_M := \displaystyle\sup_{t\in[-2M,2M]}|\phi'(t)|$. Because of our assumption on ϕ, then for M large enough we have that $\mathbb{E}[|U_0(1)|] < C_M$. For any $a \in [-M,M]$ such that $\lambda_a := \Psi^\phi(a) \in [0,1]$, we have for all $t \in [-M,M]$

$$U_0(t) - U_0(\lambda_a) \leq \phi(t-a) - \phi(\lambda_a - a) \leq C_M|t-\lambda_a| \tag{3.40}$$

For $\alpha > 0$ such that $\mathbb{E}[|U_0(1)|] < \alpha$, let us consider now the process L_0^α that is the α-Lipschitz majorant of U_0, defined formally as

$$L_0^\alpha(y) = \sup_{z\in\mathbb{R}}\{U_0(z) - \alpha|z-y|\}$$

We refer the reader to the two papers [2] and [21] for a detailed study of the Lipschitz minorant of a Lévy process. Consider G_t^α (resp. D_t^α) to be the last contact point before t (resp. the first contact point after t) of L_0^α with U_0, i.e.

$$G_t^\alpha = \sup\{y < t : L_0^\alpha(y) = U_0(y)\} \text{ and } D_t^\alpha = \inf\{y > t : L_0^\alpha(y) = U_0(y)\}$$

for any $t \in \mathbb{R}$. Moreover, let $\mathcal{Z}_\alpha$ be the contact set of L_0^α and U_0 defined as

$$\mathcal{Z}_\alpha := \{y \in \mathbb{R} : L_0^\alpha(y) = U_0(y)\}$$

Then on the event $\{G_0^{C_M}, D_1^{C_M} \in [-M, M]\}$, from the inequality (3.40), we have

$$U_0(G_0^{C_M}) - U_0(\lambda_a) \le C_M(\lambda_a - G_0^{C_M}) \text{ and } U_0(D_1^{C_M}) - U_0(\lambda_a) \le C_M(D_1^{C_M} - \lambda_a)$$

Hence for $t \ge M$, we have

$$\begin{aligned} U_0(t) - U_0(\lambda_a) &\le U_0(D_1^{C_M}) + C_M(t - D_1^{C_M}) - U_0(\lambda_a) \\ &\le C_M(D_1^{C_M} - \lambda_a) + C_M(t - D_1^{C_M}) = C_M|t - \lambda_a| \end{aligned}$$

Similarly for $t \le -M$ we get the same result. Together with (3.40), we deduce that for any $a \in [-M, M]$ such that $\lambda_a := \Psi^\phi(a) \in [0, 1]$, λ_a is in the contact set $\mathcal{Z}_{C_M}$. However when U_0 is abrupt, we know from [2][See proof of Proposition 6.1] that this set is discrete, and hence $\mathcal{Z}_{C_M} \cap [0, 1]$ is finite. Thus

$$\mathbb{P}[B_M] \le \mathbb{P}[G_0^{C_M} \le -M] + \mathbb{P}[D_1^{C_M} \ge M] \tag{3.41}$$

Now it is not hard to see that for $\alpha < \alpha'$, we have that $\mathcal{Z}_\alpha \subset \mathcal{Z}_{\alpha'}$. Hence, for M large enough we have

$$D_1^{C_M} \le D_1^\beta, \quad G_0^{C_M} \ge G_0^\beta \tag{3.42}$$

where $\beta = \mathbb{E}[|U_0(1)|] + 1$ is independent of M. However, from [2][Theorem 2.6] we know that the set $\mathcal{Z}_\beta$ is stationary and regenerative (see [22] for the precise definition of stationary regenerative sets), thus the random variables $D_1^\beta - 1$ and $-G_0^\beta$ have the same distribution as D_0^β. Moreover from [2][Equation (4.7)], we have that

$$\mathbb{P}[D_0^\beta - G_0^\beta \in dx] = \frac{x\Lambda^\beta(dx)}{\int_{\mathbb{R}+} x\Lambda^\beta(dx)}$$

where Λ^β is the Lévy measure of the subordinator associated with the contact set $\mathcal{Z}_\beta$ (the stationarity of $\mathcal{Z}_\beta$ ensuring that $\int_{\mathbb{R}+} x\Lambda^\beta(dx) < \infty$). It follows thus from (3.42) that the right-hand side of (3.41) goes to zero when $M \to \infty$, from which we get the desired result that the range of Ψ^ϕ is discrete when $\mathbb{E}[|U_0(1)|] < \infty$.
Now, if $\mathbb{E}[|U_0(1)|] = \infty$, consider for any $N \in \mathbb{N}$ the truncated process U_0^N, that is the process U_0 started at zero and with its jumps of size greater than N removed. It is formally defined as :

$$U_0^N(y) = \begin{cases} U_0(y) - \sum_{0 \le z \le y}(U_0(z) - U_0(z-))\mathbb{1}_{\{U_0(z) - U_0(z-) \ge N\}} & \text{if} \quad y \ge 0 \\ U_0(y) + \sum_{y \le z \le 0}(U_0(z) - U_0(z-))\mathbb{1}_{\{U_0(z) - U_0(z-) \ge N\}} & \text{if} \quad y \le 0 \end{cases} \tag{3.43}$$

We have that $\mathbb{E}[|U_0^N(1)|] < \infty$ as any Lévy process with uniformly bounded jumps has finite moments of any order (see [57][Lemma 8.2]). Hence, if we denote by Ψ_N^ϕ the process Ψ^ϕ where we replace U_0 by U_0^N. By what we proved previously, we have that almost surely, the set $\text{range}(\Psi_N^\phi) \cap [0,1]$ is finite for every $N \in \mathbb{N}$ (as the finiteness of the moment of order 1 of $U_0^N(1)$ ensures by the law of large numbers that $U_0^N(y) = o(\phi(y))$). By the arguments provided before, it suffices to prove that $\text{range}(\Psi_{|[-M,M]}^\phi) \cap [0,1]$ is finite for every $M \geq 0$. Now, for $N \geq 1$ and $y \geq 0$ we have that

$$\begin{aligned} |U_0^N(y)| &\leq |U_0(y)| + \sum_{0 \leq z \leq y} (U_0(z) - U_0(z-)) \mathbb{1}_{\{U_0(z) - U_0(z-) \geq N\}} \\ &\leq |U_0(y)| + |U_0(y) - U_0^1(y)| \leq 2|U_0(y)| + |U_0^1(y)| \end{aligned}$$

and similarly for $y \leq 0$. Thus almost surely

$$\lim_{|y| \to \infty} \sup_{N \geq 1} \frac{|U_0^N(y)|}{\phi(y \pm M)} = 0$$

as by the law of large numbers $U_0^1(y) = O(|y|)$. Let $K_1 > 0$ such that for all $|y| \leq K_1$, we have almost surely

$$\sup_{N \geq 1} \frac{|U_0^N(y)|}{\phi(y \pm M)} \leq \frac{1}{2}$$

Let $K_2 > 0$ such that $y \mapsto \phi(y)$ is increasing on $[K_2, +\infty)$ and decreasing on $(-\infty, -K_2)$, then for $|y| \geq \max(K_1, K_2) + M$ and $a \in [-M, M]$, we have

$$\begin{aligned} U_0^N(y) - \phi(y-a) &\leq U_0^N(y) - \phi(y \pm M) \\ &\leq -\frac{1}{2}\phi(y \pm M) \underset{y \to +\infty}{\to} -\infty \end{aligned}$$

Hence there exists $K > 1$ large enough such that

$$\sup_{|y| \geq K} \sup_{a \in [-M,M]} \sup_{N \geq 1} \left(U_0^N(y) - \phi(y-a)\right) \leq B := \inf_{\lambda \in [0,1], a \in [-M,M]} (U_0(\lambda) - \phi(\lambda - a)) \tag{3.44}$$

Now, the largest jump size of the process U_0 on any compact interval $[-R, R]$ is almost surely finite, because

$$\mathbb{P}[\exists y \in [-R,R], U_0(y) - U_0(y-) \geq N] = 1 - e^{-2R\Pi([N,+\infty))} \underset{N \to \infty}{\to} 0$$

where Π is the Lévy measure of U_0. Hence there exists a random $\tilde{N}$ such that $U_0^{\tilde{N}}(y) = U_0(y)$ on $[-K, K]$, and thus if $\text{range}(\Psi_{|[-M,M]}^\phi) \cap [0,1]$ is infinite, then there exists infinitely many $\lambda_a \in [0,1]$ such that

$$U_0(\lambda_a) - \phi(\lambda_a - a) \geq U_0(y) - \phi(y-a) \text{ for all } y$$

which in light of (3.44) implies that

$$U_0^{\tilde{N}}(\lambda_a) - \phi(\lambda_a - a) \geq U_0^{\tilde{N}}(y) - \phi(y-a) \text{ for all } y$$

and this is a contradiction with the fact that range$(\Psi^{\phi}_{\tilde{N}}) \cap [0,1]$ is finite, thus completing the proof. $\square$

Finally, we are left to prove Theorem 3.1.14

Proof of Theorem 3.1.14 In light of Theorem 3.5.1, it suffices to check that for any $t > 0$ we have

$$\lim_{|x|\to\infty} \left| L'\left(\frac{x}{t}\right) \right| = +\infty$$

However, due to the convexity of L, the function L' is increasing and thus the limits,

$$l^+ := \lim_{x\to\infty} L'(x) \text{ and } l^- := \lim_{x\to-\infty} L'(x)$$

exist. However, due to the superlinear growth of H (and thus of L), it must be that $l^+ = \infty$ and $l^- = -\infty$, which gives the desired result. $\square$

Remark 3.5.2 *The class of abrupt Lévy processes mentioned in Theorem 3.1.14 is quite large. Indeed, it contains any linear combination of Brownian motion with linear drift and stable Lévy processes with index $\alpha \in (1,2)$ with its negative jumps removed.*

Chapter 4

Random tessellations and Gibbsian solutions to Hamilton-Jacobi equations

This chapter is based on the article [43] written in collaboration with *Fraydoun Rezakhanlou* that is published in *Communications in Mathematical Physics*.

4.1 Introduction

In numerous models of nonequilibrium statistical mechanics we encounter an interface that separates different phases and is evolving with time. It is often the case that the evolution of such an interface depends on the location x, the time t, and the inclination ρ of the interface at x. If the interface is represented by a graph of a *height* function $u : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$, then a natural model for its evolution is a Hamilton-Jacobi PDE:

$$u_t = H(x, t, u_x). \tag{4.1}$$

Since in practice the exact form of the Hamiltonian function H is not known to us, it is a common practice to assume that H is random. A natural question is whether or not we can describe the stochastic law ν_t of the height function $u(\cdot, t)$ as t varies. Ideally we would like to derive a tractable/explicit evolution equation for ν_t. Alternatively, we may keep track of the inclination $\rho = u_x$, and wonder whether or not the law of $\rho(\cdot, t)$ follows an explicitly describable evolution equation. This is indeed the case for a small number of exactly solvable one dimensional discrete models. In this article however, we pursue a very different strategy: we search for a natural class of stochastic laws that is invariant with respect to the evolution of the Hamilton-Jacobi PDE (4.1). This strategy has already been tested in dimension one: If initially the process $x \mapsto \rho(x, 0)$ evolves as an ODE that is interrupted by Markovian jumps, then the same is true at later times, and the evolution of the jump rates can be described by a kinetic equation. In the present article we examine this strategy in higher dimensions. It turns out that the evolution of the height function is significantly more complex when $d > 1$. Fortunately, when H is independent of (x, t), and convex in the *momentum* variable ρ, the dynamics simplify and the classical formulas of Hopf, Lax and Oleinik lead to variational representations of solutions. A particularly tractable case is when the height function is piecewise linear and convex. As our first step, we offer a recipe for a Gibbsian measure on the set of piecewise linear and convex functions. For our second step, we study the evolution of this measure with respect to the Hamilton-Jacobi dynamics.

4.1.1 Hamilton-Jacobi semigroup

As a preparation for the statement of our main results, we first recall two classical variational formulas for solutions of (4.1) when H is convex and independent of (x,t):

(1) *(Hopf Formula)* If g is convex, then

$$u(x,t) = (g^* - tH)^*(x), \tag{4.2}$$

where $g(x) = u(x,0)$ is the initial condition, and g^* denotes the Legendre transform of g. More explicitly,

$$u(x,t) = \sup_{\rho} \left(x \cdot \rho - g^*(\rho) + tH(\rho)\right). \tag{4.3}$$

(2) *(Hopf-Lax-Oleinik Formula)* If H is convex, then

$$u(x,t) = \sup_{y} \left(g(y) - tL\left(\frac{y-x}{t}\right)\right), \tag{4.4}$$

where $L = H^*$ is the Legendre transform of H (see for example [20]).

In this article we assume that both the Hamiltonian function and the initial data are convex. An immediate consequence of the convexity of H is that the flow of (4.1) is *strongly monotone*. More precisely, if we write Φ_t for the flow of our PDE:

$$\Phi_t(g)(x) = u(x,t) \quad \Leftrightarrow \quad u_t = H(u_x), \text{ and } u(x,0) = g(x),$$

Then $\Phi_t(\sup_\rho h^\rho) = \sup_\rho \Phi_t(h^\rho)$, which would follow from (4.4). In particular, if we choose h^ρ to be a linear function of the form $h^\rho(x) = x \cdot \rho - g^*(\rho)$, then

$$\Phi_t(h^\rho)(x) = x \cdot \rho - g^*(\rho) + tH(\rho),$$

which in turn implies (4.2)-(4.3). For our purposes, it is more convenient to keep track of the slope $\rho(x,t) = u_x(x,t)$. Its evolution with respect to time can be represented by a semigroup $\widehat{\Phi}_t$;

$$\widehat{\Phi}_t(\nabla g)(x) = \rho(x,t).$$

Definition 1.1(i) Given a convex set Λ, we write $\mathcal{C}(\Lambda)$ for the set of convex functions $g : \Lambda \to \mathbb{R}$. The set of piecewise linear functions $g \in \mathcal{C}(\Lambda)$ is denoted by $\mathcal{C}_0(\Lambda)$. **(ii)** We write $\widehat{\mathcal{C}}_0(\Lambda)$ for the set of functions $\rho : \Lambda \to \mathbb{R}^d$ such that $\rho = \nabla g$, for some $g \in \mathcal{C}_0(\Lambda)$. □

Observe that $\Phi_t(\mathcal{C}(\mathbb{R}^d)) \subset \mathcal{C}(\mathbb{R}^d)$ by (4.2). Moreover, the set of piecewise linear convex functions $\mathcal{C}_0(\mathbb{R}^d)$ is also invariant with respect to Φ_t. As one of our main contribution, we construct a family $\mathcal{M}_G$ of Gibbsian measures on $\widehat{\mathcal{C}}_0$ that is expected to be invariant under the flow $\widehat{\Phi}_t$.

4.1.2 Tessellations

It turns out that there is a one-to-one correspondence between the members of $\widehat{\mathcal{C}}_0(\mathbb{R}^d)$, and the *Laguerre tessellations* of $\mathbb{R}^d$. Henceforth our aforementioned set $\mathcal{M}_G$ offers a natural family of Gibbsian measures on the set of tessellations. To explain this further, let us remark that $g \in \mathcal{C}_0$ means that there exists a *discrete set* $\mathcal{S} \subset \mathbb{R}^d$ such that

$$g(x) = \sup_{\rho \in \mathcal{S}} (x \cdot \rho - g^*(\rho)).$$

If we set

$$X(\rho) = \{x \in \mathbb{R}^d : \ g(x) = x \cdot \rho - g^*(\rho)\},$$

then the *cell* $X(\rho)$ is a *convex polytope* and the collection

$$\{(\rho, X(\rho)) : \ \rho \in \mathcal{S}\},$$

is a *Laguerre tessellation* of $\mathbb{R}^d$. Moreover, the function ∇g is piecewise constant, and has a representation of the form

$$\rho(x) := \nabla g(x) = \sum_{\rho \in \mathcal{S}} \rho \ \mathbb{1}(x \in X(\rho)). \tag{4.5}$$

It is this geometric interpretation that is at the heart of our strategy for constructing our Gibbsian measures.

Before embarking on our construction, we first need to come up with criteria that would guarantee that a polytope tessellation does come from a function $g \in \mathcal{C}_0$. Since we will be mostly studying planar tessellations in this article, let us assume that $d = 2$. Indeed if

$$X(\rho^-, \rho^+) := X(\rho^-) \cap X(\rho^+) \neq \emptyset,$$

for a planar tessellation, then generically the set $X(\rho^-, \rho^+)$ is a line segment, and if $\tau(\rho^-, \rho^+)$ is a vector that is parallel to this line segment, then we must have

$$\tau(\rho^-, \rho^+) \cdot (\rho^+ - \rho^-) = 0. \tag{4.6}$$

We can readily verify this using the fact that the linear functions $h^{\pm}(x) = x \cdot \rho^{\pm} - g^*(\rho^{\pm})$ must agree on the set $X(\rho^-, \rho^+)$. It is worth mentioning that if a function ρ is given by (4.5), then its *weak derivative* $D\rho$ is a matrix measure that is concentrated on the union of edges $X(\rho^-, \rho^+)$, $\rho^{\pm} \in \mathcal{S}$. Indeed,

$$D\rho(dx) = \sum_{\rho^{\pm} \in \mathcal{S}} \mathbb{1}(x \in X(\rho^-, \rho^+)) \ [(\rho^+ - \rho^-) \otimes n(\rho^-, \rho^+)] \ m(dx),$$

where dm denotes the one-dimensional Lebesgue measure (on the union of the edges), and $n(\rho^-, \rho^+)$ is a unit normal that is orthogonal to $\tau(\rho^-, \rho^+)$, and is pointing from the $X(\rho^-)$ side to the $X(\rho^+)$ side of $X(\rho^-, \rho^+)$. For ρ to be a gradient ∇g, the matrix $D\rho$ must be symmetric. The matrix $D\rho$ is symmetric if and only if (4.6) holds. For the convexity of g, we need $D\rho \geq 0$, which is equivalent to saying that $\rho^+ - \rho^-$ is pointing from the $X(\rho^-)$ side to the $X(\rho^+)$ side of $X(\rho^-, \rho^+)$. In other words,

$$n(\rho^-, \rho^+) = \frac{\rho^+ - \rho^-}{|\rho^+ - \rho^-|}.$$

We summarize our discussion in the next definition.

Definition 1.2(i) Let Λ be a convex polytope in $\mathbb{R}^2$. By a *(generic Laguerre) tessellation* of Λ we mean a countable collection $\mathbf{X} = \{(\rho, X(\rho)) : \ \rho \in \mathcal{S}\}$ such that

- Each $X(\rho)$ is a convex polytope, and

$$\bigcup_{\rho \in \mathcal{S}} X(\rho) = \Lambda.$$

  We refer to each $X(\rho)$ as a *cell* of $\mathbf{X}$.

- If $\rho^\pm \in \mathcal{S}$ are distinct, and $X(\rho^-, \rho^+) \neq \emptyset$, then $X(\rho^-, \rho^+)$ is a line segment orthogonal to $\rho^+ - \rho^-$. We refer to such $X(\rho^-, \rho^+)$ as an *edge* of $\mathbf{X}$.

- If $\rho^\pm, \rho^* \in \mathcal{S}$ are distinct, and

$$X(\rho^-, \rho^*, \rho^+) := X(\rho^-) \cap X(\rho^*) \cap X(\rho^+) \neq \emptyset,$$

  then $X(\rho^-, \rho^*, \rho^+)$ consists of a single point. We refer to this point as a *vertex* of the tessellation $\mathbf{X}$.

- For each edge $X(\rho^-, \rho^+)$, the vector $\rho^+ - \rho^-$ is pointing from the $X(\rho^-)$ side to the $X(\rho^+)$ side of $X(\rho^-, \rho^+)$.

We write $\mathcal{X}(\Lambda)$ for the set of all generic tessellations of Λ. **(ii)** We set

$$\Gamma = \{(\rho^-, \rho^+) \in \mathbb{R}^2 \times \mathbb{R}^2 : \ \rho^- \neq \rho^+\}.$$

By an *orientation* τ, we mean a continuous function $\tau : \Gamma \to \mathbb{R}^2$, such that (4.6) holds.

4.1.3 Gibbsian measures on $\mathcal{X}$ or $\mathcal{C}_0$

Our definition of generic tessellations can be readily extended to any dimension. Observe that when $d = 1$, each *cell* $X(\rho)$ is an interval on which the nondecreasing function $\rho(\cdot)$ is constant. As a natural candidate for a measure on $\widehat{\mathcal{C}}_0$, we may pick a continuous *kernel* $f(x, \rho^-, d\rho^+)$ (a measure in ρ^+ for every $(x, \rho^-) \in \mathbb{R}^2$) with

$$(x, \rho^-) := \int_{\rho^-}^{\infty} f(x, \rho^-, d\rho^+) < \infty,$$

and set ν^f to be the law of an *inhomogeneous Markov process* $x \mapsto \rho(x)$ with a jump rate given by f. In other words, as x increases, the Markov process $\rho(x)$ has an infinitesimal generator

$$\mathcal{L}_x F(\rho^-) = \int (F(\rho^+) - F(\rho^-))\ f(x, \rho^-, d\rho^+).$$

Given an initial law $\ell^0(d\rho)$, and $a^- \in \mathbb{R}$, we may construct a measure on piecewise constant functions $\rho : [a^-, \infty) \to \mathbb{R}$, so that $\rho(a^-)$ is selected according to ℓ^0, and evolves in a Markovian fashion with the generator $\mathcal{L}_x$. Note that if the law of $\rho(x)$ is given by $\ell(x, d\rho)$, then ℓ satisfies the forward equation $\ell_x = \mathcal{L}_x^* \ell$, subject to the initial condition $\ell(a^-, d\rho) = \ell^0(d\rho)$. Given $a^+ > a^-$, we may *also* interpret $(\rho(x) :\ x \in [a^-, a^+])$, as a Markov process that starts at time a^+ with law $\ell(a^+, d\rho)$, and evolves backward in a Markovian fashion as we *decrease* x. In order to have the same law on $(\rho(x) :\ x \in [a^-, a^+])$, the jump kernel of this (backward) Markov process must be selected appropriately (see (4.98) below).

In the same manner, we wish to construct a Gibbsian measure ν^f on $\widehat{\mathcal{C}}_0$ for a bounded continuous kernel $f(x, \rho^-, d\rho^+)$, which is a measure in ρ^+ for any given (x, ρ^-), and depends continuously on x. We carry out this construction when $d = 2$ in the present article. Though, as our method of construction suggests, it seems plausible that one can carry out similar constructions in higher dimensions in an inductive manner. Indeed our method of construction takes advantage of the fact that we already have a natural candidate for such measures in dimension one, namely Markov jump processes. Once our measures are constructed for $d = 2$, we may use them to construct Gibbsian measures in dimension 3 in a similar manner. We should mention that if $\nabla g \in \widehat{\mathcal{C}}_0(\mathbb{R}^2)$, and $\hat{\rho}(x,t) = (u_x, u_t)(x,t)$, then $\hat{\rho} \in \widehat{\mathcal{C}}_0(\mathbb{R}^2 \times [0,\infty))$. A Gibbsian choice of ∇g leads to a probability measure on $\widehat{\mathcal{C}}_0(\mathbb{R}^d \times [0,\infty))$, which has a similar flavor as our construction when $d = 2$. We expand on this in Section 1.6.

Given a kernel $f(x, \rho^-, d\rho^+)$ in $\mathbb{R}^2$, we wish to construct a measure on $\widehat{\mathcal{C}}_0$ so that $f(x, \rho^-, d\rho^+)$ represents the rate at which ρ^- changes to ρ^+ as we cross an edge of $X(\rho^-)$ at a point x. To achieve this we adopt the following strategy:

(i) We take a convex planar set Λ (for example a box), and construct a measure $\nu^{f,\Lambda}$ on the set of tessellations of Λ. This is carried out by first constructing the one dimensional tessellation

$$\{(\rho, X(\rho) \cap \partial\Lambda) :\ \rho \in \mathcal{S}\},$$

in a Markovian fashion with a jump rate that is expressed in terms of f. We then use the information coming from the boundary to build the tessellation inside. More edges will be added inside Λ in a Markovian fashion.

(ii) We then show that when f satisfies a suitable kinetic PDE, the measures $\nu^{f,\Lambda}$ are consistent as we enlarge Λ. As we attempt to carry out the above strategy, we encounter two problems. Our treatment of these problems are responsible for our final recipe of our Gibbsian measures.

Problem 1. According to our strategy, we would like to construct $\rho(x)$ on the boundary as a Markov process. Imagine that we start from a point $a \in \partial\Lambda$ and select a slope ρ^- for $\rho(a)$. We then move counterclockwise on the boundary and change $\rho(x)$ as a jump process. If a jump from ρ^- to ρ^+ occurs at a point $b \in \partial\Lambda$ to the right of a, then there will be an edge of our tessellation separating $X(\rho^-)$ from $X(\rho^+)$. If no other jump occurs as we traverse the whole boundary, the restriction of the desired tessellation to the set Λ consists of exactly two cells. However the edge $X(\rho^-, \rho^+)$ intersects the boundary at a second point b' that was not a jump point of our Markov process. In other words, the restriction of the desired tessellation to $\partial\Lambda$ cannot be realized as a Markov process.

Our Remedy: We resolve this problem by giving an orientation to the edges. This orientation will be used to decide what points of the boundary tessellation will be created as a Markov jump process. In other words, given an orientation τ, we consider a Markov process on the boundary with the jump rate

$$(\tau(\rho^-, \rho^+) \cdot n(x))^+ \ f(x, \rho^-, \rho^+),$$

where $n(x)$ denotes the inward unit normal to $\partial\Lambda$ at x. This Markov process would allow us to determine the *entering edges* only, not the *exiting edges*.

Problem 2. After determining all the entering edges, we need to use them to build our tessellation inside Λ. These edges may intersect to produce vertices. How can this be done in an orderly manner?

Our Remedy: Given a fixed direction v, we expect/insist

$$\rho^\pm \in R, \quad \rho^+ \neq \rho^- \quad \implies \quad (\rho^+ - \rho^-) \cdot v \neq 0,$$

in the support of our measure. This is equivalent to saying that $\tau(\rho^-, \rho^+)$ is not parallel to v. Without loss of generality, we may choose $v = e_1 = (1, 0)$, so that τ is never horizontal. A continuous choice of τ forces a fixed sign for $\tau(\rho^-, \rho^+) \cdot e_2$. Without loss of generality, we require that $\tau(\rho^-, \rho^+) \cdot e_2 > 0$. For the sake of definiteness, we choose

$$\tau(\rho^-, \rho^+) = (- [\rho^-, \rho^+], 1), \quad \text{where} \quad [\rho^-, \rho^+] := \frac{\rho_2^+ - \rho_2^-}{\rho_1^+ - \rho_1^-}. \tag{4.7}$$

We may treat τ as a velocity for a point/particle that is created at a point b on the boundary, and its trajectory determines the edge emanating from b. Here, we are treating x_2 as a time parameter. We will use this time parameter to order the creation of particles, and the occurrence of particle collisions as we increase x_2. □

We now have all the ingredients to describe our Gibbsian measure in a convex set Λ. For this construction, we use the orientation (4.7). Note that because of our choice of τ, we may talk about a cell $X(\rho^-)$ (respectively $X(\rho^+)$) that lies on the left (respectively right) of the edge $\tau(\rho^-, \rho^+)$. With this convention, the fourth condition in Definition 1.2**(i)** is equivalent to saying that for each (x, ρ^-), the support of the measure $f(x, \rho^-, d\rho^+)$ is contained in the set

$$R(\rho^-) := \{\rho^+ = (\rho_1^+, \rho_2^+) : \ \rho_1^+ > \rho_1^-\}. \tag{4.8}$$

To simplify our presentation, we give a precise recipe for our measure when Λ is a box:

$$\Lambda = \Lambda(a^-, a^+, t_0, t_1) := [a^-, a^+] \times [t_0, t_1].$$

Because of our choice of τ, there will be no entering edge from the top side of the box. To ease the notation, we simply write t for the second coordinate. Pictorially, we may decorate each edge with an arrow that always points upward (see Figure 1 below). Our Gibbsian measure will be supported on generic tessellations of which each vertex is of degree 3. With an orientation at our disposal, we may interpret each vertex as either a *coagulation point* (when two intersecting edges are replaced with one edge as time increases), or a *fragmentation point* (when an edge splits into two edges as time increases). With these interpretations, we may fully determine the tessellation inside Λ in terms a collection of particles that travel according to their velocities, and may experience coagulation and fragmentation. More precisely, the function $x_1 \mapsto \rho(x_1, t)$ can be expressed as

$$\rho(x_1, t) = \sum_{i \in J(t)} \rho^i(t)\ \mathbb{1}(x_1 \in (z_i(t), z_{i+1}(t))),$$

with the interpretation that $z_i(t)$ represents the position of the i-th particle. Writing $q_i = (z_i, \rho^i)$, and $\mathbf{q}(t) = (q_i(t) :\ i \in J(t))$, the dynamics of $\mathbf{q}$ can be conveniently described as a Markov process.

Definition 1.3(i) Given a pair $\rho^\pm \in \mathbb{R}^2$, we write $\rho^- \prec \rho^+$ if $\rho^+ \in R(\rho^-)$, where $R(\rho^-)$ was defined in (4.8). Similarly, we define the set $L(\rho^+)$ as

$$L(\rho^+) := \{\rho^- \in \mathbb{R}^2 : \rho^- \prec \rho^+\},$$

and the set $D(\rho^-, \rho^+)$ for $\rho^- \prec \rho^+$ to be

$$D(\rho^-, \rho^+) := \{\rho^* : \rho^- \prec \rho^* \prec \rho^+\}.$$

(ii) We write $\Delta = \bigcup_{n=0}^{\infty} \Delta_n$, where Δ_n denotes the set of $\mathbf{q} = (q_0, \dots, q_n)$ such that $q_i = (z_i, \rho^i) \in \mathbb{R}^3$, and

$$z_0 = a^- < z_1 < \dots < z_n < z_{n+1} := a^+, \qquad \rho^0 \prec \rho^1 \prec \dots \prec \rho^n.$$

(iii) We write $\mathcal{M}$ for the set of measures on $\mathbb{R}^2$, and equip $\mathcal{M}$ with the topology of weak convergence. The set of probability measures is denoted by $\mathcal{M}_1$. We write $\mathcal{F}(\Lambda)$ for the set of kernels

$$f : \Lambda \times \mathbb{R}^2 \to \mathcal{M},$$

with the following properties:

- The map $(x, \rho^-) \mapsto f(x, \rho^-, d\rho^+)$ is measurable, and

$$\sup_{(x,\rho^-)} \int |\tau(\rho^-, \rho^+)|\ f(x, \rho^-, d\rho^+) < \infty.$$

- For every (x, ρ^-),

$$f(x, \rho^-, \mathbb{R}^2 \setminus R(\rho^-)) = 0.$$

(iv) Given $\beta \in \mathcal{M}$, we write $\mathcal{F}(\beta,\Lambda)$ for the set of $f \in \mathcal{F}(\Lambda)$ such that $f(x,\rho^-,d\rho^+) \ll \beta(d\rho^+)$. With a slight abuse of notation, we write $f(x,\rho^-,\rho^+)$ for the Radon-Nikodym derivative of $f(x,\rho^-,d\rho^+)$ with respect to β.

(v) Given constants $P^\pm$, with $P^- < P^+$, and positive constants V_∞ and δ_0, we put

$$\Gamma^{V_\infty} = \left\{(\rho^-,\rho^+) \in [P^-,P^+]^2 \times [P^-,P^+]^2 :\ \rho^- \prec \rho^+,\ |[\rho^-,\rho^+]| \le V_\infty\right\},$$

and write $\mathcal{F}(\beta,\Lambda,V_\infty,\delta_0)$ for the set of $f \in \mathcal{F}(\beta,\Lambda)$ such that

$$\begin{aligned} x \in \mathbb{R}^2,\ (\rho^-,\rho^+) \in \Gamma^{V_\infty} \quad &\Longrightarrow \quad f(x,\rho^-,\rho^+) \ge \delta_0, \\ x \in \mathbb{R}^2,\ (\rho^-,\rho^+) \notin \Gamma^{V_\infty} \quad &\Longrightarrow \quad f(x,\rho^-,\rho^+) = 0. \end{aligned}$$

We now introduce some definitions that will allow us to construct a Markov process taking values in Δ.

Definition 1.4 **(i)** Given $\ell^0 \in \mathcal{M}_1$, $f \in \mathcal{F}(\beta,\Lambda)$, and $t_0 \in \mathbb{R}$, we write $\gamma(d\mathbf{q};t_0,\ell^0,f)$ for a probability measure on Δ that represents a Markov jump process with the jump rate $f((x_1,t_0),\rho^-,d\rho^+)$, and the initial law ℓ^0. More precisely, if $\mathbf{q}$ is selected according to $\gamma(d\mathbf{q};t_0,\ell^0,f)$, and the function $\rho(\cdot;\mathbf{q})$ is defined by

$$\rho(x_1;\mathbf{q}) = \sum_{i=0}^{n} \rho^i\ \mathbb{1}(z_i \le x_1 < z_{i+1}),$$

then $\rho(a^-;\mathbf{q}) = \rho^0$ is distributed according to ℓ^0, and the Markov jump process $x_1 \mapsto \rho(x_1;\mathbf{q})$ makes its i-th jump from ρ^{i-1} to ρ^i at *time* z_i, with the rate $f((z_i,t_0),\rho^{i-1},d\rho^i)$.

(ii) Given $\ell^0 \in \mathcal{M}_1$ and $f \in \mathcal{F}(\Lambda,\beta,V_\infty,\delta_0)$ as in the Definition 1.3**(v)**, we define a Markov process $(\mathbf{q}(t):\ t \ge t_0)$ that takes value in the set Δ. This Markov process induces a function

$$\rho : \Lambda \to \mathbb{R}, \qquad \rho(x_1,x_2) = \rho(x_1,t) := \rho(x_1;\mathbf{q}(t)),$$

that belongs to $\widehat{\mathcal{C}}_0(\Lambda)$. The (initial) law of $\mathbf{q}(t)$ at $t=t_0$ is given by $\gamma(d\mathbf{q};t_0,\ell^0,f)$. This process induces a probability measure on $\widehat{\mathcal{C}}_0(\Lambda)$, that is denoted by $\nu^{f,\Lambda} = \nu^{\ell^0,f,\Lambda}$. The Markovian dynamics of $\mathbf{q}(t)$ is as follows:

1. The particle z_i travels with velocity $-[\rho^{i-1},\rho^i]$. When z_1 reaches a^-, or z_n reaches respectively a^+, the number of particles reduces by one. In the former case, we relabel (z_i,ρ^i) as (z_{i-1},ρ^{i-1}) for $i \ge 1$.

2. If at some time t, we have $z_i(t) = z_{i+1}(t)$, then we remove the i-th particle from the system, and relabel (z_j,ρ^j) as (z_{j-1},ρ^{j-1}) for $j > i$.

3. At the boundary point a^-, the function $t \mapsto \rho(a^-,t)$ can change from ρ^0 to ρ^* with the rate

$$[\rho^0,\rho^*]^- \frac{\ell(a^-,d\rho^*) f((a^-,t),\rho^*,d\rho^0)}{\ell(a^-,d\rho^0)},$$

When this happens, we relabel (z_i,ρ^i), as (z_{i+1},ρ^{i+1}), for $i \ge 0$, and declare ρ^* to be our new ρ^0.

4. At the boundary point a^+, the function $t \mapsto \rho(a^+, t)$ can change from ρ^n to ρ^* with the rate

$$[\rho^n, \rho^*]^+ f((a^+, t), \rho^n, d\rho^*),$$

When this happens, a new particle has been born at a^+. That is, we now have $n+1$ many particles with $z_{n+1} = a^+$, and $\rho^{n+1} = \rho^*$.

5. The i-th particle can *fragment* into two particles. This occurs at time t with the rate density

$$\sigma(\rho^{i-1}, \rho^*, \rho^i)^- \ \frac{f((z_i, t), \rho^{i-1}, \rho^*) f((z_i, t), \rho^*, \rho^i)}{f((z_i, t), \rho^{i-1}, \rho^i)}, \tag{4.9}$$

where

$$\sigma(\rho^-, \rho^*, \rho^+) = [\rho^*, \rho^+] - [\rho^-, \rho^*]. \tag{4.10}$$

By fragmentation we mean that the particle (z_j, ρ^j) is relabeled as (z_{j+1}, ρ^{j+1}) for $j \geq i$, and the i-th particle at the location z_i is associated with a new label $\rho^i = \rho^*$.

(iii) We write $\mathcal{M}_G(\Lambda)$ for the set of measures of the form $\nu^{\ell^0, f, \Lambda}$, as we vary $\ell^0 \in \mathcal{M}_1$ and $f \in \mathcal{F}(\beta, \Lambda, V_\infty, \delta_0)$.

Because of our choice (4.9), the fragmentation mechanism in **5** is the time reversal of the coagulation mechanism in **2**. The choice (4.9) plays an essential role in the validity of our first main result, namely, the consistency of our measures $\nu^{f,\Lambda}$ as we vary Λ. We remark that the fragmentation occurs only when $\sigma < 0$, so that the resulting particles move away from each other, with the $i+1$-th particle to the right of the newly born particle.

4.1.4 Consistency

We now turn our attention to the question of the consistency of our measures $\nu^{f,\Lambda}$, as we vary the sides of the box $\Lambda = \Lambda(a^-, a^+, t_0, t_1)$. We first vary the horizontal sides. If

$$\Lambda = \Lambda(a^-, a^+, t_0, t_1), \qquad \Lambda' = \Lambda(a^-, a^+, t, t_1),$$

with $t \in (t_0, t_1)$, the consistency of $\nu^{f,\Lambda}$ and $\nu^{f,\Lambda'}$, requires that the process $x \mapsto \rho(x, t)$ under $\nu^{f,\Lambda}$ to be a Markov process with the jump rate

$$f((x_1, t), \rho^-, d\rho^+). \tag{4.11}$$

This turns out to be equivalent to the requirement that the kernel f satisfies a kinetic type PDE of the form

$$\tau(\rho^-, \rho^+) \cdot f_x(x, \rho^-, d\rho^+) = Q(f)(x, \rho^-, d\rho^+), \tag{4.12}$$

for a suitable quadratic function Q. To ease the notation, we suppress the dependence on x in our notations, and write

$$\tau f = (-\alpha f, f) := (-f^2, f^1), \tag{4.13}$$

where $\alpha(\rho^-, \rho^+) = [\rho^-, \rho^+]$. With these conventions, the operator Q equals

$$Q(f) = Q(f^1, f^2) = Q^+(f^1, f^2) - Q^-(f^1, f^2),$$

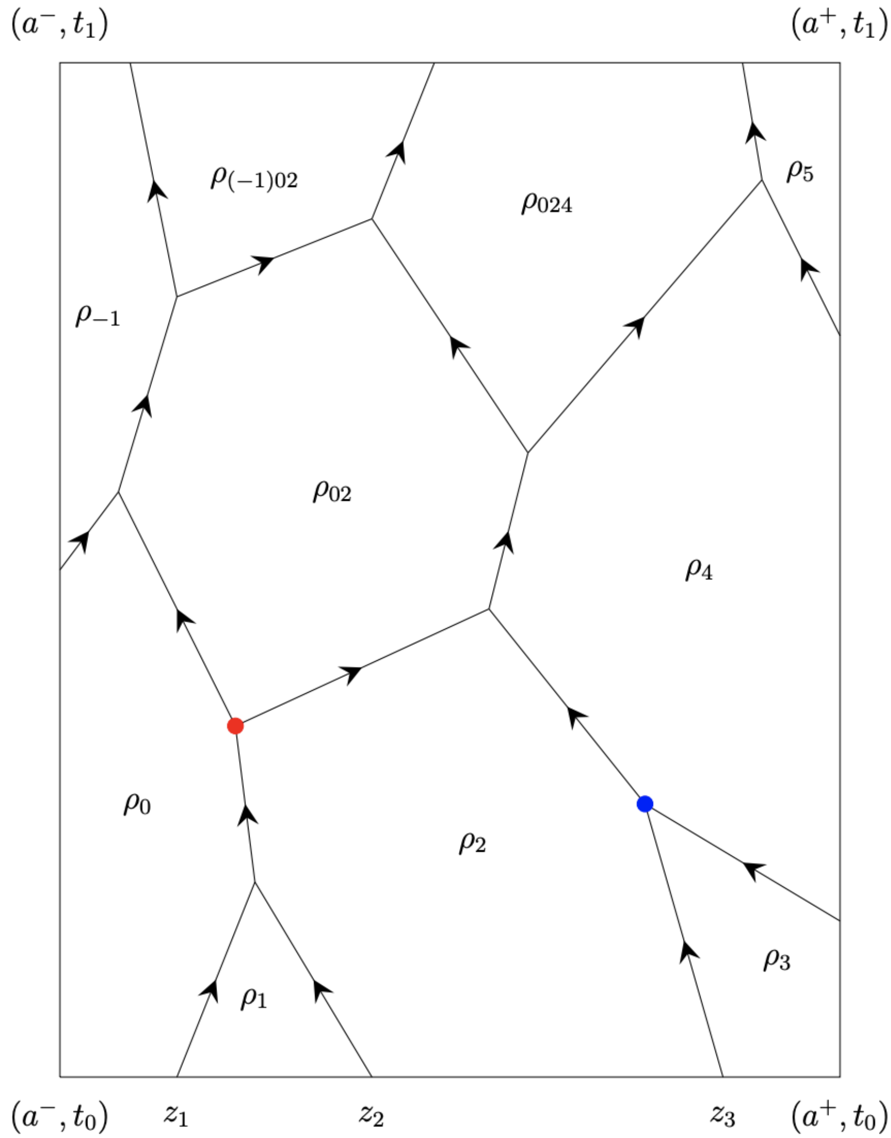

Figure 4.1: The blue dot represents the coagulation of the particles with labels (ρ_2, ρ_3) and (ρ_3, ρ_4) into the particle with label (ρ_2, ρ_4). The red dot represents the fragmentation of the particle with label (ρ_0, ρ_2) into two particles of respective labels (ρ_0, ρ_{02}) and (ρ_{02}, ρ_2).

with

$$\begin{aligned} Q^+(f^1,f^2)(\rho^-,d\rho^+) &= (f^1 * f^2)(\rho^-,d\rho^+) - (f^2 * f^1)(\rho^-,d\rho^+), \\ Q^-(f^1,f^2)(\rho^-,d\rho^+) &= \big(A(f^2)(\rho^+) - A(f^2)(\rho^-)\big)\, f^1(\rho^-,d\rho^+) \\ &\quad - \big(A(f^1)(\rho^+) - A(f^1)(\rho^-)\big)\, f^2(\rho^-,d\rho^+). \end{aligned} \tag{4.14}$$

Here,

$$(h * k)(\rho^-,d\rho^+) := \int h(\rho^-,d\rho^*)\; k(\rho^*,d\rho^+), \qquad A(h)(\rho) := \int h(\rho,d\rho^*).$$

Remark 4.1.1 *Given a pair of kernels* (f^1,f^2)*, define the quadratic operator*

$$\mathcal{Q}(f^1,f^2) = f^1 * f^2 - A(f^1) \otimes f^2 - f^1 \otimes A(f^2),$$

where

$$(h \otimes k)(\rho^-,d\rho^+) = h(\rho^-)k(\rho^-,d\rho^+), \qquad (k \otimes h)(\rho^-,d\rho^+) = h(\rho^+)k(\rho^-,d\rho^+).$$

Then (4.12) *can be written as*

$$div\ (\tau f) = f^1_{x_2} - f^2_{x_1} = \mathcal{Q}(f_1,f_2) - \mathcal{Q}(f_2,f_1).$$

As our first main result, we verify the consistency of the measures $\nu^{f,\Lambda}$ as we vary the horizontal sides of Λ.

Theorem 4.1.2 *Assume that the kernel* $f \in \mathcal{F}(\beta,\Lambda,\Lambda_\infty,\delta_0)$ *is a* C^1 *function, and satisfies* (4.12)*, for* $\Lambda = \Lambda(a^-,a^+,t_0,t_1)$*. Assume that* $\ell^0 \geq \delta_0$*. Then for every* $x_2 \in [t_0,t_1]$*, the law of the function* $x_1 \mapsto \rho(x_1,x_2)$ *with respect to* $\nu^{\ell^0,f,\Lambda}$ *coincides with the law of a Markov jump process with the jump rate* $f((x_1,x_2),\rho^-,d\rho^+)$*.*

Example 1.1 Given a continuous function $K : \mathbb{R} \to \mathbb{R}$, we may consider

$$\beta(d\rho) = \beta(d\rho_1,d\rho_2) = \delta_{K(\rho_1)}(d\rho_2)\; d\rho_1,$$

which is a measure that is supported on the graph of the function K. In this case, the corresponding convex function g, with $\nabla g = \rho$ satisfies the PDE

$$g_{x_2} = K(g_{x_1}), \tag{4.15}$$

inside the cells of the corresponding tessellation. We put

$$\tilde{f}^i(x_1,x_2,\rho_1^-,\rho_1^+) = f^i(x_1,x_2,\rho_1^-,K(\rho_1^-),\rho_1^+,K(\rho_2^+)),$$

write $\tilde{f}$ for $\tilde{f}^1$, and write $\mu^{\ell^0,\Lambda,\tilde{f},K}$ for the corresponding measure $\nu^{\ell^0,\Lambda,f}$. We write $\widehat{\mathcal{M}}_G$ for the set of such measures as we vary f and K.

(i) When K is a convex function, then g is a viscosity solution of (4.15). Moreover, there would be no fragmentation because $\sigma \geq 0$. If we also assume that f is independent of x_2 and that K is increasing, then Theorem 1.1 was established in [32], confirming affirmatively a conjecture of Menon and Srinivasan [36]. We may rephrase Theorem 1.1 as follows: If g solves the Hamilton-Jacobi PDE (4.15) in $d = 1$, and initially $g_{x_1}(x_1, t_0)$ is a Markov jump process with the rate density $\tilde{f}^1(x_1, t_0, \rho_1^-, \rho_1^+)$, then for every $t > t_0$, the process $g_{x_1}(x_1, t)$ is also a Markov jump process with the rate density $\tilde{f}^1(x_1, t, \rho_1^-, \rho_1^+)$.

(ii) If we assume that K is concave, then there would be no collision as we increase $t = x_2$ because $\sigma \leq 0$. In fact g is a viscosity solution for a final value problem i.e., as we reverse (decrease) time x_2. We may rephrase Theorem 1.1 in this case as a statement for the reversed dynamics: If we reverse x_2 in part **(i)**, then the dynamics can be described as a particle system with stochastic fragmentation; the fragmentation rate is given by (4.9).

(iii) If K is neither convex, nor concave, then g is not a viscosity solution of the PDE (4.15) no matter what direction for the coordinate x_2 is adopted.

Remark 4.1.3 *Observe that a choice of an orientation for edges allowed us to have a natural time direction for the Markov processes on the boundary sides of the box Λ, and the dynamics inside the box. For the undirected tessellation, this choice is irrelevant, and there should be a formulation of the consistency criteria that is independent of the orientation. As an illustration, we will demonstrate in Proposition 4.2 below how reversing a direction, or interchanging coordinates can be performed on the solutions of* (4.12)*.*

As we mentioned before, the Markov process $\mathbf{q}(t) = ((z_i(t), \rho_i(t)) : \ i \in J(t))$, yields a random tessellation

$$\mathbf{X}_\Lambda = \{(\rho, X(\rho)) : \ \rho \in \mathcal{S}_\Lambda\},$$

of the box Λ. The set $\mathcal{S}_\Lambda$ is simply defined by

$$\mathcal{S}_\Lambda = \left\{\rho^i(t) : \ t \in [t_0, t_1], \ i \in J(t)\right\},$$

and the cells of $\mathbf{X}_\Lambda$ are the connected components of the set

$$\Lambda \setminus \{(z_i(t), t) : \ t \in [t_0, t_1], \ i \in J(t)\}.$$

The law of $\mathbf{X}_\Lambda$ is denoted by $\eta^{\ell^0, f, \Lambda}$.

Proposition 4.1.4 *Under the assumptions of Theorem 1.1, we have that $\eta^{\ell^0, f, \Lambda}(\chi_\Lambda) = 1$. In words, the tessellation $\mathbf{X}_\Lambda$ is generic in the sense of Definition* **1.2(i)**, *with probability one with respect to $\eta^{\ell^0, f, \Lambda}$.*

The proof of this Proposition is rather straightforward and follows from our Proposition 3.1 in Section 3.4.

We next examine the question of the consistency as we vary the vertical sides of Λ. Note however that although the boundary dynamics on the lower side is Markovian, the dynamics on the lateral sides may depend on the configuration inside Λ. This can be avoided if we assume that τ always points to the left.

Theorem 4.1.5 *Let f and Λ be as in Theorem 4.1.2. Assume that the measure $f(x,\rho^-,d\rho^+)$ is supported in the set*

$$R_0(\rho^-)=\{\rho^+=(\rho_1^+,\rho_2^+):\ \ \rho_2^+>\rho_2^-,\ \rho_1^+>\rho_1^-\}. \tag{4.16}$$

for every (x,ρ^-). Then the law of the function $x_2\mapsto\rho(x_1,x_2)$ with respect to $\nu^{\ell^0,f,\Lambda}$ coincides with the law of a Markov jump process with the jump rate given by $[\rho^-,\rho^+]f((x_1,x_2),\rho^-,d\rho^+)$.

Our assumptions on f allow us to reduce Theorem 4.1.5 from Theorem 4.1.2. The details can be found in Section 5.

Remark 4.1.6 *More generally, define*

$$R_c(\rho^-)=\{\rho^+=(\rho_1^+,\rho_2^+):\ \ \rho_2^+-\rho_2^-+c(\rho_1^+-\rho_1^-)>0,\ \rho_1^+>\rho_1^-\}, \tag{4.17}$$

and assume that there exists $c\geq 0$ such that the measure $f(x,\rho^-,d\rho^+)$ is supported in the set $R_c(\rho^-)$, for every (x,ρ^-). Define $S_c(x_1,x_2)=(x_1+cx_2,x_2)$, and

$$S_cg(x_1,x_2)=g(x_1+cx_2,x_2),\qquad T_c(\rho_1,\rho_2)=(\rho_1,\rho_2+c\rho_1),$$

so that if $\rho=\nabla g$, then

$$\nabla(S_cg)(x)=(T_c\rho)(x_1+cx_2,x_2)=:\rho'(x).$$

We also define $f':=T_c^{\sharp}f$, i.e., for every bounded continuous function $\varphi:\mathbb{R}\to\mathbb{R}$, we have

$$\int\varphi(\rho^+)\ f'(x,\rho^-,d\rho^+)=\int(\varphi\circ T_c)(\rho^+)f(x_1+cx_2,x_2,T_c\rho^-,d\rho^+).$$

Then the kernel $f'(x,\rho^-,d\rho^+)$ is supported in the set $R_0(\rho^-)$, and Theorem 1.2 is applicable to f'. As a result, under $\nu^{\ell^0,f',\Lambda}$, the law of ρ, restricted to a line of slope c is a Markov jump process. This in turn implies the consistency for the measures $\nu^{f,\Lambda'}$, provided that $\Lambda'=S_c\Lambda$, for a box Λ.

Example 1.1(iv) *(continued)* When K is increasing and convex, we claim that for each x_1, the process $x_2\mapsto\rho(x_1,x_2)$ is a Markov jump process with the rate $\tilde{f}^2(x_1,t,\rho_1^-,\rho_1^+)$. To see this observe that we may write $g_{x_1}=K^{-1}(g_{x_2})$, which suggests that we should regard x_1 as the time variable now. With this choice of time, we now have a scenario that resembles Example 1.1(i), except for few non-essential differences: We are initially at $x=a^+$, and go backward by *decreasing* x_1. The function K^{-1} is now concave, which implies that the convex function g is a viscosity solution for the final-value Hamilton-Jacobi PDE $g_{x_1}=K^{-1}(g_{x_2})$. The initial jump process with the jump rate density $\tilde{f}^2(a^+,t,\rho_1^-,\rho_1^+)$, evolves to a jump process with the jump rate density $\tilde{f}^2(a,t,\rho_1^-,\rho_1^+)$ as we decrease x_1 from a^+ to a.

4.1.5 The invariance of $\mathcal{M}_G$ and $\widehat{\mathcal{M}}_G$

We now examine the question of the invariance of the set $\mathcal{M}_G$ under the flow $\widehat{\Phi}$ of the Hamilton-Jacobi PDE

$$u_t = H(u_x), \quad u(x,0) = g(x), \tag{4.18}$$

with $H : \mathbb{R}^2 \to \mathbb{R}$ convex, and $g \in \mathcal{C}_0(\Lambda)$.

From Example 1.1**(i)**, we already know that $\mathcal{M}_G(\Lambda)$ is invariant under $\widehat{\Phi}$, when $d = 1$. In details, we choose β to be the one-dimensional Lebesgue measure, and pick $f^0 \in \mathcal{F}(\Lambda, \beta)$, with $\Lambda = [a^-, a^+]$ or $[a^-, \infty)$. The measure $\nu^{\ell^0, f^0, \Lambda}$ is the law of a Markov jump process $x \mapsto \rho^0(x) = g'(x)$, $x \in \Lambda$, with the rate density f^0, and the initial $\rho(a^0, 0)$ distributed according to ℓ^0. Then $\widehat{\Phi}_t(\rho^0)$ is a Markov jump process associated with a rate $\Theta_t(f)$, where $f(x,t,\rho^-,\rho^+) = \Theta_t(f)(x,\rho^-,\rho^+)$ solves the kinetic equation

$$f_t - v^H f_x = Q^H(f), \qquad f(x,0,\rho^-,\rho^+) = f^0(x,\rho^-,\rho^+), \tag{4.19}$$

where $v^H = v^H(\rho^-,\rho^+) = (H(\rho^-) - H(\rho^+))/(\rho^- - \rho^+)$, and Q^H is as Q of (4.14), except that α is replaced with v^H. We write θ_t^H for the flow of the kinetic equation (4.19):

$$f(x,t,\rho^-,\rho^+) =: \theta_t^H(f^0)(x,\rho^-,\rho^+).$$

Before giving precise statements for our results in dimension 2, we need to address a technical issue concerning the domain of the definition of the function u. We remark that because of the quadratic nature of the right-hand of our kinetic equation (4.12), generically non-negative solutions are defined only locally in spatial variables (in the Appendix, we provide a local well-posedness for (4.12) under some natural assumptions). Because of this, we will consider the PDE (4.12) in a convex domain Λ on which our Markovian kernels can be defined. In order to solve (4.18) in Λ, we need to assign suitable boundary conditions. These boundary conditions are selected so that the law of the corresponding solutions are consistent as we vary Λ.

As we mentioned earlier, since the function $u : \Lambda \times [0,T] \to \mathbb{R}$ is a piecewise linear convex function, it induces a tessellations of $\hat{\Lambda} = \Lambda \times [0,T]$. The vector $\hat{\rho} = (u_x, u_t) = (\rho, H(\rho))$ lies on the graph of H, and if a 2-dimensional face F separates $\hat{\rho}^+ = (\rho^+, H(\rho^+))$ from $\hat{\rho}^- = (\rho^-, H(\rho^-))$, then any vector $\hat{v} = (v, v_3) = (v_1, v_2, v_3) \in \mathbb{R}^2 \times \mathbb{R}$ parallel to F must satisfy

$$v \cdot (\rho^+ - \rho^-) + v_3 \big(H(\rho^+) - H(\rho^-)\big) = 0. \tag{4.20}$$

We address the question of invariance of $\mathcal{M}_G$ in two settings.

Setting 1.1(i) *(Hamiltonian Function)* We assume that the convex function $H(\rho_1, \rho_2) = H(\rho_1)$, depends on ρ_1 only. To simplify our presentation, we also assume that H is an increasing function (see Remark 4.1.6 for general H).

(ii) *(Initial Condition)* Assume that $\Lambda = [a^-, a^+] \times [t_0, t_1]$, and $\nabla g = \rho(x, 0)$ is distributed according to $\nu^{\ell^0, f, \Lambda}$, for a function f that satisfies the kinetic equation (4.12) in the set Λ. We additionally assume that the measure $f(x, \rho^-, d\rho^+)$ is supported in the set $R_0(\rho^-)$, so that Theorem 1.2 is applicable.

(iii) *(Kernel)* We define a kernel $\hat{f}(x, t, \rho^-, \rho^+)$ by

$$\hat{f}(x_1, x_2, t, \rho^-, \rho^+) = \Theta_t^H(f)(x_1, x_2, \rho^-, \rho^+) := \theta_t^H(h^{x_2})(x_1, \rho^-, \rho^+),$$

where $h^{x_2}(x_1, \rho^-, \rho^+) := f(x_1, x_2, \rho^-, \rho^+)$. We put

$$\left(\hat{f}^1, \hat{f}^2, \hat{f}^3\right) := \left(\hat{f}, \alpha \hat{f}, v^H \hat{f}\right),$$

where

$$\alpha(\rho^-, \rho^+) = [\rho^-, \rho^+], \qquad v^H(\rho^-, \rho^+) = \left(H(\rho_1^-) - H(\rho_1^+)\right) / \left(\rho_1^- - \rho_1^+\right).$$

(iii) *(Boundary Condition)* We assign a boundary condition at $x = a^+$. The law of $(x_2, t) \mapsto u_x(a^+, x_2, t) = \rho(a^+, x_2, t)$ is denoted by μ. We assume that under μ, the process $t \mapsto \rho(a^+, x_2, t)$ is a Markov jump process with the jump rate density $\hat{f}^3(a^+, x_2, t)$ for every $x_2 \in [t_0, t_1]$.

Note that for fixed x_2, we may regard the equation (4.18) as a HJ equation in dimension one in x_1 variable. The process

$$(x_1, t) \in [a^-, a^+] \times [0, T] \to m(x_1, t) = m(x_1, t; x_2) := \rho(x_1, x_2, t),$$

is piecewise constant and induces a tessellation in $[a^-, a^+] \times [0, T]$. The process $x_1 \mapsto m(x_1, 0)$ is a Markov jump process. Its discontinuity points $z_1 < \cdots < z_n$ travel with time t with velocity $-v^H(\rho_1^-, \rho_1^+)$. We are tempted to use either Theorem 1.1 or Theorem 1.2 (or [32]) to determine the Markovian law of the process $(x_1, t) \mapsto \rho(x_1, x_2, t)$. However these theorems cannot be applied directly because the relation between the particle velocities $-v^H$ and the slopes $\rho^\pm$ is not exactly what we had in these theorems (namely $-[\rho^-, \rho^+]$). We note that what we have is slightly different from the setting of [32], as the jump rate depends on a vector ρ, not just its first coordinate $\rho_1 = u_{x_1}$. Nonetheless a verbatim proof would allow us to have a similar result. In other words, given a *velocity function* $v(\rho^-, \rho^+)$ satisfying some natural conditions, we may consider a particle system as in the Definition 1.3**(vii)** such that $[\rho^-, \rho^+]$ is replaced with $v(\rho^-, \rho^+)$, and the analogs of Theorems 1.1 and 1.2 are still true. In summary we have the following result:

Theorem 4.1.7 *Under the Setting 1.1, the following statements are true:*

(i) *For every $(x_2, t) \in [t_0, t_1] \times [0, T]$, the process $x_1 \mapsto \rho(x_1, x_2, t)$ is a Markov jump process with the jump rate density $\hat{f}(x_1, x_2, t, \rho^-, \rho^+)$.*

(ii) *For every $(x_1, x_2) \in \Lambda$, the process $t \mapsto \rho(x_1, x_2, t)$ is a Markov jump process with the jump rate density $\hat{f}^3(x_1, x_2, t, \rho^-, \rho^+)$.*

Naturally, we may wonder whether or not the law of the process $(x_1, x_2) \mapsto \rho(x_1, x_2, t)$ is given by $\nu^{\ell^t, \Theta_t^H(f), \Lambda}$, where ℓ^t represents the law of $\rho(a^-, t_0, t)$. To examine this possibility, let us switch to a more symmetric notation and write x_3 for t, $\hat{x}$ for (x_1, x_2, x_3), and ρ_3 for $H(\rho_1)$. In this way,

$$\hat{f}^i(\hat{x}, \rho^-, \rho^+) = [\rho^-, \rho^+]_i \hat{f}(\hat{x}, \rho^-, \rho^+),$$

where

$$[\rho^-,\rho^+]_i = \frac{\rho_i^- - \rho_i^+}{\rho_1^- - \rho_1^+}.$$

Now if the law of the process $(x_1,x_2)\mapsto \rho(x_1,x_2,t)$ is given by $\nu^{\ell^t,\Theta_t^H(f),\Lambda}$, then we know that $x_i\mapsto\rho(\hat{x})$ is a Markov jump process with the jump rate density $\hat{f}^i$. Let us write $\mathcal{L}^i$ for the infinitesimal generator of this jump process. If $\ell(\hat{x},\rho)$ is the law of $\rho(\hat{x})$, then it must satisfy the forward equations

$$\ell_{x_i} = \left(\mathcal{L}^i\right)^* \ell, \quad i\in\{1,2,3\},$$

where $(\mathcal{L}^i)^*$ denotes the adjoint of the operator $\mathcal{L}^i$. From the compatibility of these equations, namely $\ell_{x_ix_j}=\ell_{x_jx_i}$, we derive the kinetic equation

$$f^i_{x_j} - f^j_{x_i} = Q(f^i,f^j). \tag{4.21}$$

In summary, the function f must satisfy the kinetic equation (4.21) for all $i,j\in\{1,2,3\}$. When these equations hold, we may choose a Gibbsian measure for the boundary condition at $x_1=a^+$, and we expect that the law of the marginals $(x_1,x_2)\to\rho(x_1,x_2,t)$, and $(x_2,t)\to\rho(x_1,x_2,t)$ to be Gibbsian of the type we have constructed. We leave further investigation of the system (4.21) for future.

Remark 4.1.8 *Assume that $H(\rho_1,\rho_2)=H_1(\rho_1)+H_2(\rho_2)$ with H_1 and H_2 convex and increasing. To display the dependence on the Hamiltonian function, we write $\Phi_t^{H_i}$ for the Hamilton-Jacobi flow Φ_t (as was defined in Section 1.1) associated with H_i. Writing $g^{x_2}(x_1):=g(x_1,x_2)$, and $\hat{g}(x_1,x_2)=\hat{g}^{t,x_1}(x_2)=\Phi_t^{H_1}(g^{x_2})(x_1)$, then using (4.4), it is not hard to show that solution u of (4.18) can be expressed as*

$$u(x_1,x_2,t)=\Phi_t^{H_2}(\hat{g}^{t,x_1})(x_2).$$

Now if g is distributed according to $\nu^{\ell^0,f^0,\Lambda}$, then we may apply Theorem 1.3 to assert the marginal $x_1\mapsto\nabla\hat{g}(x_1,x_2)$ is a jump process with the jump rate that are expressed in terms of $\Theta_t^{H_1}f$. If the law of $\hat{g}$ is also a Gibbsian measure associated with $\Theta_t^{H_1}f$, then another application of Theorem 1.3 would allow us to assert that the marginals of $\rho=u_x$ are jump processes with the jump rates that are expressed in terms of $\Theta_t^{H_2}\Theta_t^{H_1}f$.

So far we have described some of the challenges we encounter as we try to examine the invariance of the set $\mathcal{M}_G$ under the Setting 1.1. Fortunately these challenges can be avoided when we examine the invariance of $\widehat{\mathcal{M}}_G$.

Setting 1.2(i) *(Hamiltonian Function)* We assume that the convex function $H(\rho_1,\rho_2)$ is increasing with respect to both ρ_1 and ρ_2.

(ii) *(Initial Condition)* Let $\Lambda=[a^-,a^+]\times[t_0,t_1]$, and let $K:\mathbb{R}\to\mathbb{R}$ be an increasing continuous function. We assume that $\nabla g=\rho(x,0)$ is distributed according to $\nu^{\ell^0,\tilde{f},\Lambda,K}$ as in Example 1.1. Recall

$$\tilde{f}(x_1,x_2,\rho_1^-,\rho_1^+) = f(x_1,x_2,\rho_1^-,K(\rho_1^-),\rho_1^+,K(\rho_1^-)),$$

with f a kernel that satisfies the kinetic equation (4.12) in the set Λ.

(iii) *(Kernel)* We define a kernel $\hat{f}(x, t, \rho^-, \rho^+)$ as in the Setting 1.1(iii). We also continue to use our notations $\hat{f}^i$ and $\hat{x} = (x_1, x_2, x_3)$ as in Setting 1.1. With a slight abuse of notation, we define $\Theta_t(\tilde{f})$ as

$$\Theta_t(\tilde{f})(x_1, x_2, \rho_1^-, \rho_1^+) = \hat{f}(x_1, x_2, t, \rho_1^-, K(\rho_1^-), \rho_1^+, K(\rho_1^-)).$$

(iv) *(Boundary Condition)* We assign a boundary condition at $x = a^+$ so that the law of $(x_2, t) \mapsto u_x(a^+, x_2, t) = \rho(a^+, x_2, t)$ is again a Gibbsian measure μ of the type we defined in Example 1.1, but now $(\rho_2, \rho_3) = (u_{x_2}, u_t)$ lies on the graph of $\hat{K}(m) = H(K^{-1}(m), m)$. In particular, $u_t = \hat{K}(u_{x_2})$ in the support of μ.

Conjecture 1.1 Under the Setting 1.2, the law of $(x_1, x_2) \mapsto \rho(x_1, x_2, t)$ is given by the measure $\nu^{\ell^0, \Theta_t(\tilde{f}), \Lambda, K}$.

We have been able to partially verify this conjecture:

Theorem 4.1.9 *Under the Setting 1.2, the process $x_i \mapsto \rho(\hat{x})$, $i \in \{1, 2, 3\}$, is a Markov jump process with the jump rate density $\hat{f}^i$.*

In fact we can readily establish Theorem 1.4 with the aid of Theorem 1.3. We explain this in three short steps:

(Step 1) We first argue that the relationship $u_{x_2} = K(u_{x_1})$ that is assumed at $t = 0$, also holds at later times. To see this, observe that if initially

$$g(x_1, x_2) = \sup_{\rho_1 \in R} (x_1 \rho + x_2 K(\rho_1) - \alpha(\rho_1)),$$

for a (discrete) set R and a function $\alpha(m) = g^*(m, K(m))$, then on account of (4.3), a similar formula is true at a later time t, where $\alpha(\rho_1)$ is replaced with $\alpha(\rho_1) - t\tilde{H}(\rho_1)$, for $\tilde{H}(\rho_1) = H(\rho_1, K(\rho_1))$. We can take advantage of this property to reduce the question of invariance to Case 1. After all if $u_{x_2} = K(u_{x_1})$ holds for a solution u of (4.18), then such a solution also solves the equation $u_t = \tilde{H}(u_{x_1})$.

(Step 2) We note that $\tilde{H}$ is convex if H is convex. Let us present a short proof of this when H is C^2 and K is differentiable:

$$\begin{aligned}\tilde{H}''(m) =& H_{\rho_1\rho_1}(m, K(m)) + 2H_{\rho_1\rho_2}(m, K(m))K'(m) + H_{\rho_2\rho_2}(m, K(m))K'(m)^2 \\ =& (D^2 H)(m, K(m)) \begin{bmatrix} 1 \\ K'(m) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ K'(m) \end{bmatrix} \geq 0.\end{aligned}$$

Furthermore, if K is increasing, and H is increasing in both arguments, then $\tilde{H}$ is also increasing. This allows us to apply Theorem 1.3 to assert that the process $x_i \mapsto \rho(\hat{x})$ is a Markov jump process with the jump rate density $\hat{f}^i$, for $i = 1$ and $i = 3$.

(Step 3) Since K is increasing we can interchange the role of x_1 with x_2. That is, from $u_{x_1} = K^{-1}(u_{x_2})$, we learn that $u_t = \bar{H}(u_{x_2})$, where $\bar{H}(\rho_2) = H(K^{-1}(\rho_2), \rho_2)$. Since $\bar{H}$ is

convex and increasing, and the boundary dynamics at $x_2 = t_1$ is Markovian with the jump rate density $\hat{f}^3$, we are at a position to apply Theorem 1.3 once more to assert that the process $x_2 \mapsto \rho(\hat{x})$, is Markov jump process with the jump rate density $\hat{f}^2$.

Remark 4.1.10 *When K is also concave or convex, then Conjecture 1.1 would follow from Theorem 1.4. For example, if K is concave, then we can fully determine the function $(x_1, x_2) \to \rho(x_1, x_2, t)$ from its boundary values on $\partial\Lambda$. Simply because as we decrease x_2, there would be no (stochastic) fragmentation and the corresponding particle system involves free motion and collisions.*

4.1.6 Generalization to higher dimensions $d > 2$.

As we mentioned earlier, a Gibbsian measure $\nu^{f,\Lambda}$ for the distribution of ∇g in (4.18) would lead to a Gibbsian measure $\hat{\nu}^{\hat{f},\hat{\Lambda}}$ for the distribution of $\hat{\rho}(x,t) = (u_x, u_t)(x,t)$, where $\hat{\Lambda} = \Lambda \times [0,T]$, and

$$\hat{f}(x,t,\rho^-,\rho^+) =: g\left(x,t,\rho^-,H(\rho^-),\rho^+,H(\rho^+)\right),$$

represents a density rate at which $\hat{\rho}^- = (\rho^-, H(\rho^-))$ changes to $\hat{\rho}^+ = (\rho^+, H(\rho^+))$ at the point $(x,t) \in \mathbb{R}^3$. Indeed the function $u(\hat{x}) = u(x_1,x_2,x_3) = u(x_1,x_2,t)$ is a piecewise linear convex function such that its gradient $\hat{\rho}$ lies on the graph of H. The piecewise constant function $\hat{\rho}$ yields a Laguerre tessellation

$$\widehat{\mathbf{X}} = \left\{(\hat{\rho}, X(\hat{\rho})) : \ \hat{\rho} \in \widehat{\mathcal{S}}\right\},$$

of $\mathbb{R}^3$. We may wonder whether or not we can apply our approach to build more general Gibbsian measure on the set of Laguerre tessellations of $\mathbb{R}^3$. More specifically we wish to relax the restriction $\hat{\rho} \in \{(a, H(a)) : \ a \in \mathbb{R}^2\}$. To describe a strategy for achieving this, let us first discuss some of the features of Laguerre tessellations in $\mathbb{R}^3$. Generically the following statements are true:

- If $\hat{\rho}^+ \neq \hat{\rho}^-$, and $\widehat{X}(\hat{\rho}^-, \hat{\rho}^-) := \widehat{X}(\hat{\rho}^-) \cap \widehat{X}(\hat{\rho}^+) \neq \emptyset$, then $\widehat{X}(\hat{\rho}^-, \hat{\rho}^+)$ is a (2-dimensional) convex polygon. The vector $\hat{\rho}^+ - \hat{\rho}^-$ points from the $\hat{\rho}^-$ side to the $\hat{\rho}^-$ side of $\widehat{X}(\hat{\rho}^-, \hat{\rho}^+)$.

- If $\hat{\rho}^-$, $\hat{\rho}^+$, and $\hat{\rho}^*$ are distinct, and $\widehat{X}(\hat{\rho}^-, \hat{\rho}^+, \hat{\rho}^*) := \widehat{X}(\hat{\rho}^-) \cap \widehat{X}(\hat{\rho}^+) \cap \widehat{X}(\hat{\rho}^*) \neq \emptyset$, then $\widehat{X}(\hat{\rho}^-, \hat{\rho}^+, \hat{\rho}^*)$ is an edge (a line segment). We can uniquely determine a vector direction

  $$\tau = \tau(\hat{\rho}^-, \hat{\rho}^+, \hat{\rho}^*) = (v(\hat{\rho}^-, \hat{\rho}^+, \hat{\rho}^*), 1), \qquad v \in \mathbb{R}^2,$$

  of this edge by solving the system of linear equations $\tau \cdot (\hat{\rho}^\pm - \rho^*) = 0$.

- If $\hat{\rho}^i$, $i = 1, \dots, 4$ are distinct, and $\widehat{X}(\hat{\rho}^1, \dots, \hat{\rho}^4) := \widehat{X}(\hat{\rho}^1) \cap \dots \cap \widehat{X}(\hat{\rho}^4) \neq \emptyset$, then $\widehat{X}(\hat{\rho}^1, \dots, \hat{\rho}^4)$ consists of a single point which is a vertex of our tessellation.

We wish to build a random tessellation $\widehat{\mathbf{X}}$ so that its intersection with the plane $\{\hat{x} : x_3 = t\}$ is a random planar tessellation of the type we have constructed in the Section 1.3. Let us ignore the lateral boundary dynamics and focus on our strategy for building such a tessellation inside the box $\hat{\Lambda}$. In other words, we start from a planar tessellation that represents the restriction of $\widehat{\mathbf{X}}$ to the plan $\{x_3 = 0\}$, and evolve it in a Markovian fashion as we increase x_3. The law of this planar tessellation is a suitable $\nu^{f^0,\Lambda}$, except that the kernel $f^0(x_1, x_2, \hat{\rho}^-, \hat{\rho}^+)$ is defined for $\hat{\rho}^\pm \in \mathbb{R}^3$. In other words, the cells of the initial tessellations labeled/decorated by vectors in $\mathbb{R}^3$. To build our random tessellation initially, we need to assume that the kernel f^0 satisfies the kinetic equation (4.12), where only the first two coordinates of $\hat{\rho}$ are used for determining the speed α. The evolution of our tessellation consists of the deterministic and the stochastic parts. As for the deterministic part, a vertex associated with vector densities $\hat{\rho}^-, \hat{\rho}^+$, and $\hat{\rho}^*$ travels with the velocity $v(\hat{\rho}^-, \hat{\rho}^+, \hat{\rho}^*)$. As x_3 increases, it is possible that an edge of vertices a^- and a^+ collapses, or equivalently a^- and a^+ collide. There would be two possibilities for the type of a collision that can occur. To explain this, let us write C^- and C^+ for the cells which are sharing the edge a^-a^+.

- One of the cells $C^\pm$ is a triangle (generically not both C^- and C^+ could be triangle). When this is the case, the whole triangular cell collapses and becomes a vertex. The tessellation has lost a cell at time of such a collision.

- Neither C^- nor C^+ are triangle. If a^- is a vertex associated with cells of labels $(\hat{m}^-, \hat{\rho}^-, \hat{\rho}^+)$, and a^+ is a vertex associated with cells of labels $(\hat{m}^+, \hat{\rho}^-, \hat{\rho}^+)$, then after the collision a new edge is created with vertices b^- and b^+. The new vertices b^- and b^+ are now associated with cells of marks $(\hat{\rho}^-, \hat{m}^-, \hat{m}^+)$, and $(\hat{\rho}^+, \hat{m}^-, \hat{m}^+)$, respectively (the role of $\hat{m}$ and $\hat{\rho}$ are swapped).

Our dynamics also involves a stochastic fragmentation; at a random time t, a vertex a, associated with $(\hat{\rho}^1, \hat{\rho}^2, \hat{\rho}^3)$, can give birth to a triangle with vertices b_1, b_2 and b_3, and a random label $\hat{\rho}^*$. These vertices start their journey at the location a at time t, and move away from each others with velocities that are determined in terms of $(\hat{\rho}^1, \hat{\rho}^2, \hat{\rho}^3)$, and $\hat{\rho}^*$.

In order to carry out our program in dimension 3, we need to work out the form of the fragmentation rate. We emphasis that when $\hat{\rho}$ lies on a graph of a convex function H, there would be no fragmentation. We conjecture that when the rate $\hat{f}$ satisfies a system of kinetic equations analogous to (4.21), our outlined strategy would yield a consistent family of Gibbsian measures on the set of tessellations of $\mathbb{R}^3$.

4.1.7 Bibliography and the outline of the chapter

Most of the earlier works on stochastic solutions of Hamilton-Jacobi PDEs have been carried out in the Burgers context. For example, Groeneboom [25] determined the statistics of solutions to Burgers equation ($H(p) = p^2/2$, $d = 1$) with white noise initial data. Burgers equation is not explicitly mentioned—the paper discusses convex minorants of Brownian motion with parabolic drift—but these problems are connected by the Hopf-Lax-Oleinik solution formula (4.4). Recently the author in [41] has extended this result to arbitrary convex Hamiltonian function H. The special cases of $H(p) = \infty \mathbb{1}(p \notin [-1, 1])$, and $H(p) =$

p^+ were already studied in references Abramson-Evans [2], Evans-Ouaki [21] and Pitman-Tang [48].

Carraro and Duchon ([15],[14]) considered *statistical* solutions, which need not coincide with genuine (entropy) solutions, but realized in this context that Lévy process initial data should interact nicely with Burgers equation. Bertoin [12] showed this intuition was correct on the level of entropy solutions, arguing in a Lagrangian style (by analyzing the inverse Lagrangian process).

Developing an alternative treatment to that given by Bertoin, which relies less on particulars of Burgers equation and happens to be more Eulerian, was among the goals of Menon and Srinivasan [36]. Most notably, [36] formulates an interesting conjecture for the evolution of the infinitesimal generator of the solution $\rho(\cdot,t)$ which is equivalent to our kinetic equation (4.14) when there is no fragmentation, f is independent of x, and ρ lies on a graph of a convex function (see Example 1.1**(i)**). When the initial data $\rho(x,0)$ is allowed to assume values only in a fixed, finite set of states, the infinitesimal generators of the processes $x \mapsto \rho(x,t)$ and $t \mapsto \rho(x,t)$ can be represented by triangular matrices. The integrability of this matrix evolution has been investigated by Menon [34] and Li [33]. For generic matrices—where the genericity assumptions unfortunately exclude the triangular case—this evolution is completely integrable in the Liouville sense. The full treatment of Menon and Srinivasan's conjecture was achieved in papers [32] and [31] (we also refer to [52] for an overview). The works of ([32],[31]) have been recently extended in [51] to allow inhomogeneous HJ equation of the type (4.1) in dimension one.

The organization of the chapter is as follows: In Section 2 we give a precise construction of the particle system $\mathbf{q}(t)$ that we described in Definition 1.3**(vii)**. In Section 3, we derive a *forward equation* for the law of $\mathbf{q}(t)$ and establish Theorem 1.1. Section 4 is devoted to the proof of Theorem 1.2. In the Appendix we address the question of well-posedness and the regularity of solutions of the kinetic equation.

4.2 Construction of the Particle System

As discussed in the introduction section, we use z for the variable x_1 and time t for the variable x_2. We wish to build a probability measure on the space of *Laguerre tessellations* with orientation $\tau : \{(\rho^-,\rho^+) \in (\mathbb{R}^2)^2 : \rho^- \prec \rho^+\} \to \mathbb{R}^2$ that was given by (4.7). We have already given a rough description for this probability measure in Section 1.3. In this section we give more thorough details and make some rudimentary preparations for the proof of our main results. Our strategy will be to build a consistent family of probability measures on the space of labeled interacting particle systems on fixed boxes.

Fix $t_0, t_1, a^-, a^+ \in \mathbb{R}$, with $a^- < a^+$ and $t_0 < t_1$. We will build a probability measure on the space of stochastic processes $(\mathbf{q}(t))_{t\in[t_0,t_1]}$ that take values in the space of particle systems of the form

$$\mathbf{q}(t) := \big((z_0,\rho^0(t)),(z_1(t),\rho^1(t)),\cdots,(z_{n(t)}(t),\rho^{n(t)}(t))\big)\,,$$

with $z_0 = a^-$. To be more precise, let us introduce some notation.

Notation 2.1(i) Let $P^- < P^+$ be two fixed constants and define the state space $\Omega = \Omega_{a^-,a^+}$ of particle systems as the following disjoint union

$$\Omega = \bigsqcup_{n=0}^{\infty} \Omega^n$$

where $\Omega^n := \overline{\Delta^n} \times R^n$, with $\Delta^n := \{(z_1, \cdots, z_n) : a^- < z_1 < \cdots < z_n < a^+\}$, $\overline{\Delta^n}$ is the topological closure of Δ^n in $\mathbb{R}^n$, and

$$R^n := \left\{(\rho^0, \rho^1, \cdots, \rho^n) \in ([P^-, P^+]^2)^{n+1} : \rho^0 \prec \rho^1 \prec \cdots \prec \rho^n\right\}.$$

(ii) For any $\mathbf{q} \in \Omega$, let $\mathbf{n}(\mathbf{q})$ be the unique integer n such that $\mathbf{q} \in \Omega^n$.

(iii) For any three labels $\rho, \rho', \rho'' \in [P^-, P^+]^2$ such that $\rho \prec \rho' \prec \rho''$, set

$$[\rho, \rho', \rho''] = [\rho', \rho''] - [\rho, \rho']$$

(iv) For any real number r, let $r^+ = \max(r, 0)$ and $r^- = \max(-r, 0)$. For any two integers $m \le n$, we denote by m, n the set $\{m, m+1, \cdots, n\}$.

(v) Let Γ be the space of vector-valued right-continuous piecewise-constant functions on $[a^-, a^+]$, an define the map $\mathcal{V} : \Omega \to \Gamma$ as follows. For any $\mathbf{q} \in \Omega$ of the form

$$\mathbf{q} = \big((a^-, \rho^0), (z_1, \rho^1), \cdots, (z_n, \rho^n)\big) \in \Omega^n$$

Define

$$\mathcal{V}(\mathbf{q})(z) := \sum_{i=0}^{n} \rho^i \, \mathbb{1}(z_i \le z < z_{i+1}) \quad \text{for all } z \in [a^-, a^+], \tag{4.22}$$

with the convention that $z_0 = a^-$ and $z_{n+1} = a^+$.

(vi) Let Ω' to be the subset of Ω such that for any $\mathbf{q} = ((a^-, \rho^0), (z_1, \rho^1), \cdots, (z_n, \rho^n)) \in \Omega' \cap \Omega^n$, and if $z_i = z_{i+1}$ for some $i \in 1, n$, then we have

$$[\rho^{i-1}, \rho^i, \rho^{i+1}] \ge 0.$$

To construct our measure $\nu^{\ell,f,\Lambda}$, we take a kernel $f \in \mathcal{F}(\beta, \Lambda)$,

$$f(z, t, \rho^-, d\rho^+) = f(z, t, \rho^-, \rho^+) \; \beta(d\rho^+),$$

that satisfies the kinetic equation (4.12) in Λ. We then use f to define a Markov process $(\mathbf{q}(t) : t \in [t_0, t_1])$ in Ω. The law of this Markov process for a suitable initial distribution ℓ is the desired measure $\nu^{\ell,f,\Lambda}$. For the reader's convenience, we recall the kinetic equation,

$$\tau(\rho^-, \rho^+) \cdot \nabla f = Q(f)(z, t, \rho^-, \rho^+), \tag{4.23}$$

where $Q(f) = Q^+(f) - f \; Lf$, with

$$Q^+(f)(z, t, \rho^-, \rho^+) = \int_{D(\rho^-, \rho^+)} [\rho^-, \rho^*, \rho^+] f(z, t, \rho^-, \rho^*) f(z, t, \rho^*, \rho^+) \; \beta(d\rho^*),$$
$$(Lf)(z, t, \rho^-, \rho^+) = A(z, t, \rho^+) - A(z, t, \rho^-) - [\rho^-, \rho^+] \left(\lambda(z, t, \rho^+) - \lambda(z, t, \rho^-)\right).$$

Here,

$$\lambda(z,t,\rho) = \int_{R(\rho)} f(z,t,\rho,\rho^*)\ \beta(d\rho^*),$$
$$A(z,t,\rho) = \int_{R(\rho)} [\rho,\rho^*] f(z,t,\rho,\rho^*)\ \beta(d\rho^*).$$

Now, consider a nonnegative function $(z,t,\rho) \mapsto \ell(z,t,\rho)$, verifying the following equations

$$\ell_z(z,t,\rho) = \int_{L(\rho)} f(z,t,\rho^*,\rho)\ell(z,t,\rho^*)\ \beta(d\rho^*) - \lambda(z,t,\rho)\ell(z,t,\rho), \tag{4.24}$$
$$\ell_t(z,t,\rho) = \int_{L(\rho)} [\rho^*,\rho] f(z,t,\rho^*,\rho)\ell(z,t,\rho^*)\ \beta(d\rho^*) - A(z,t,\rho)\ell(z,t,\rho). \tag{4.25}$$

Moreover for any fixed z and t we assume $\int \ell(z,t,\rho)\ \beta(d\rho) = 1$. As we will see in Proposition 4.1 below, the two equations (4.24) and (4.25) are compatible because of the kinetic equation verified by f. Mainly, the two flows generated in both direction z and t commute, as the corresponding vector fields commute in the sense of having a zero Lie Bracket. This last fact is exactly a reformulation of the equation (4.23).

Without loss of generality let us assume that $t_0 = 0$ to alleviate the notation. The purpose of this section is to construct the law of a stochastic process $(\mathbf{q}(t))_{t\geq 0}$ that takes values in Ω. Let us introduce first the following probability measure on Ω. For any $t \geq 0$, let

$$\mu(d\mathbf{q},t) = \sum_{n=0}^{\infty} \mathbb{1}(\mathbf{n}(\mathbf{q}) = n)\ \mu^n(d\mathbf{q},t), \tag{4.26}$$

where $\mu^n(d\mathbf{q},t) = g^n(\mathbf{z},\boldsymbol{\rho})\ d\mathbf{z}\ \beta(d\boldsymbol{\rho})$ is a measure defined on Ω^n, with

$$\beta(d\boldsymbol{\rho}) := \prod_{i=0}^{n} \beta(d\rho^i), \qquad d\mathbf{z} = \prod_{i=0}^{n} dz_i,$$
$$g^n(\mathbf{z},\boldsymbol{\rho}) := \ell(a^-,t,\rho^0)\ \prod_{i=1}^{n} f(z_i,t,\rho^{i-1},\rho^i)\ \exp\left(-\sum_{i=0}^{n}\int_{z_i}^{z_{i+1}} \lambda(z,t,\rho_i)\ dz\right).$$

Here $\mathbf{q} = ((z_0,\rho^0),(z_1,\rho^1),\cdots,(z_n,\rho^n))$, and by convention $z_0 = a^-$ and $z_{n+1} = a^+$.

4.2.1 The deterministic flow

We first start by defining a deterministic flow on Ω. Let $t \geq 0$, and $\mathbf{q} \in \Omega^n$. We distinguish two cases:

(1) If $\mathbf{q} \in \mathrm{Int}(\Omega^n)$, i.e., $\mathbf{q} = ((a^-,\rho^0),(z_1,\rho^1),\cdots,(z_n,\rho^n))$ where $a^- < z_1 < \cdots < z_n < a^+$. Let $(z_i(t))_{i=1}^n$ be defined as

$$\dot{z}_i(t) = -[\rho^{i-1},\rho^i], \qquad z_i(0) = z_i,$$

for $t \in [0, T^*)$, where

$$T^* = \inf \{t > 0 : z_i(t) = z_{i+1}(t) \text{ for some } i \in \{0, 1, \cdots, n\}\},$$

the time of the first collision. By convention, we always take $z_0(t) = a^-$ and $z_{n+1}(t) = a^+$ when we consider a particle system of n particles. This defines a flow $\psi^t \mathbf{q}$:

$$\psi^t \mathbf{q} = \big((a^-, \rho^0), (z_1(t), \rho^1), \cdots, (z_n(t), \rho^n)\big) \in \Omega^n \quad \text{for} \quad t \in [0, T^*).$$

At time $t = T^*$, for any i such that $z_i(T^*-) = z_{i+1}(T^*-)$, we remove the $(i+1)$-th particle $(z_{i+1}(T^*-), \rho^{i+1})$ from the particle system and relabel the particle $(z_j(T^*-), \rho^j)$ as $(z_{j-1}(T^*-), \rho^{j-1})$, for $j > i+1$. We keep doing this procedure until we end up with an element in $\text{Int}(\Omega^m)$ for some $m \leq n-1$ (it might be possible that multiple collisions happen at the same time T^*), we then define $\psi^{T^*}\mathbf{q}$ to be this element. Repeating the same process again starting from this new configuration, we get a sequence of collision times $0 < T_1^* < T_2^* < \cdots$, where $t \mapsto \psi^t \mathbf{q}$ evolves with free motion in each interval $[T_k^*, T_{k+1}^*)$.

(2) If $\mathbf{q} \in \partial\Omega^n$, then for any i such that $z_i = z_{i+1}$, there are two cases to consider:

- If $[\rho^{i+1}, \rho^i] \geq [\rho^i, \rho^{i-1}]$, then we delete the particle (z_{i+1}, ρ^{i+1}) and relabel the particle (z_j, ρ^j) as (z_{j-1}, ρ^{j-1}), for $j > i+1$.
- If $[\rho^{i+1}, \rho^i] < [\rho^i, \rho^{i-1}]$, we keep both (sticky) particles at $z = z_i = z_{i+1}$.

Denote

$$\bar{\mathbf{q}} := \big((a^-, \bar{\rho}^0), (y_1, \bar{\rho}^1), \cdots, (y_m, \bar{\rho}^m)\big) \in \Omega^m$$

to be the resulting configuration after doing this modification. Notice that $\bar{\mathbf{q}}$ is not necessarily in $\text{Int}(\Omega^m)$, as we may still have an index j such that $y_j = y_{j+1}$ and $[\bar{\rho}^{j-1}, \bar{\rho}^j, \bar{\rho}^{j+1}] < 0$. The flow $\psi^t \mathbf{q} := \psi^t \bar{\mathbf{q}}$ for any $t > 0$ is now defined in the same fashion as before, by letting each particle $y_j(t)$ have a free motion with the corresponding velocity $v_j := -[\bar{\rho}^{j-1}, \bar{\rho}^j]$. Instantaneously, for any small $t > 0$, the configuration $\psi^t \mathbf{q}$ belongs to $\text{Int}(\Omega^m)$. To see this, observe that for any j such that $y_j = y_{j+1}$ we have

$$\frac{d}{dt}(y_{j+1}(t) - y_j(t)) = -[\bar{\rho}_{j-1}, \bar{\rho}_j, \bar{\rho}_{j+1}] > 0.$$

The motion of the particle system then encounters collisions and behaves similarly to the first case when we start from a particle system in the interior of the state space.

More generally for any $s \leq t$, we define the deterministic flow between time s and time t to be $\psi_s^t \mathbf{q} := \psi^{t-s} \mathbf{q}$ for any $\mathbf{q} \in \Omega$. By construction, this flow verifies the semi-group property

$$\psi_{t_1}^{t_3} \mathbf{q} = \psi_{t_2}^{t_3} \psi_{t_1}^{t_2} \mathbf{q} \quad \text{for any} \quad t_1 \leq t_2 \leq t_3, \quad \text{and} \quad \mathbf{q} \in \Omega.$$

4.2.2 The stochastic flow and Markov process

We will define a stochastic process $(\mathbf{q}(t))_{t \geq 0}$, that takes values in Ω, and such that $(t, \mathbf{q}(t))_{t \geq 0}$ is strong Markov. Equivalently, this amounts to constructing a probability measure $\mathbb{P}_t^{\mathbf{q}}$ for any $\mathbf{q} \in \Omega$ and $t \geq 0$. This probability measure should be understood as the law of $(\mathbf{q}(\theta))_{\theta \geq t}$ conditionally on $\mathbf{q}(t) = \mathbf{q}$. Naturally, the measure $\mathbb{P}_t^{\mathbf{q}}$ is concentrated on the set measurable maps $\mathbf{q} : [t, +\infty) \to \Omega$ such that $\mathbf{q}(t) = \mathbf{q}$. Let us start with some notation.

Notation 2.2(i) For any $(\rho^-, \rho^+) \in [P^-, P^+]^2$ with $\rho^- \prec \rho^+$ and $t \geq 0$, we define the two quantities

$$\begin{aligned}\mathfrak{c}_-(t, \rho^-, \rho^+) &:= [\rho^-, \rho^+]^- \frac{\ell(a^-, t, \rho^-) f(a^-, t, \rho^-, \rho^+)}{\ell(a^-, t, \rho^+)}, \\ \mathfrak{c}_+(t, \rho^-, \rho^+) &:= [\rho^-, \rho^+]^+ f(a^+, t, \rho^-, \rho^+).\end{aligned}$$

They correspond to the rates of creation of particles respectively at $z = a^-$ and $z = a^+$. For $\rho, \rho', \rho'' \in [P^-, P^+]^2$ with $\rho \prec \rho' \prec \rho''$, $z \in [a^-, a^+]$ and $t \geq 0$, define the fragmentation rate at position z and time t as

$$\mathfrak{f}(z, t, \rho, \rho', \rho'') := [\rho, \rho', \rho'']^- \frac{f(z, t, \rho, \rho') f(z, t, \rho', \rho'')}{f(z, t, \rho, \rho'')}.$$

(ii) For any $\rho \in [P^-, P^+]^2$ and $t \geq 0$, let

$$\mathfrak{C}_-(t, \rho) := \int_{L(\rho)} \mathfrak{c}_-(t, \rho^-, \rho)\ \beta(d\rho^-) \quad \text{and} \quad \mathfrak{C}_+(t, \rho) := \int_{R(\rho)} \mathfrak{c}_+(t, \rho, \rho^+)\ \beta(d\rho^+),$$

and for any $\rho^-, \rho^+ \in [P^-, P^+]^2$ such that $\rho^- \prec \rho^+$, and $t \geq 0$, let

$$\mathfrak{F}(z, t, \rho^-, \rho^+) := \int_{D(\rho^-, \rho^+)} \mathfrak{f}(z, t, \rho^-, \rho^*, \rho^+)\ \beta(d\rho^*).$$

(See Definition 1.3**(i)** for the definition of $R(\rho^-)$, $L(\rho^+)$, and $D(\rho^-, \rho^+)$.) For any particle system $\mathbf{q} \in \Omega^n$ of the form $\mathbf{q} = ((a^-, \rho^0), (z_1, \rho^1), \cdots, (z_n, \rho^n))$, we define its *particle rate* at time t by

$$\mathfrak{r}(t, \mathbf{q}) := \mathfrak{C}_-(t, \rho^0) + \sum_{i=1}^{n} \mathfrak{F}(z_i, t, \rho^{i-1}, \rho^i) + \mathfrak{C}_+(t, \rho^n).$$

(iii) We introduce now a notation that corresponds to the state of the particle system after a creation or fragmentation. For any $\rho^* \prec \rho^0$, we define the new particle configuration $E_-^{\rho^*}\mathbf{q}$ as

$$E_-^{\rho^*}\mathbf{q} := \left((a^-, \rho^*), (a^-, \rho^0), (z_1, \rho^1), \cdots, (z_n, \rho^n)\right),$$

where a new particle is added at $z = a^-$. Similarly for the barrier $z = a^+$, for any ρ^* such that $\rho^n \prec \rho^*$, define the new particle configuration $E_+^{\rho^*}\mathbf{q}$ by

$$E_+^{\rho^*}\mathbf{q} := \left((a^-, \rho^0), (z_1, \rho^1), \cdots, (z_n, \rho^n), (a^+, \rho^*)\right).$$

Finally for ρ^* such that $\rho^{i-1} \prec \rho^* \prec \rho^i$ for some $i \in \{1, \cdots, n\}$, let

$$E_i^{\rho^*}\mathbf{q} := \left((a^-, \rho^0), (z_1, \rho^1), \cdots, (z_{i-1}, \rho^{i-1}), (z_i, \rho^*), (z_i, \rho^i), (z_{i+1}, \rho^{i+1}), \cdots, (z_n, \rho^n)\right),$$

denote the particle configuration we obtain after the fragmentation of the i-th particle.

Construction of the Markov process

Consider now a probability space $(\Omega_0, \mathcal{F}, \mathbb{P})$ on which is already defined an infinite i.i.d sequence $(\tau_i)_{i\geq 1}$ of standard exponential random variables. We will define a process $t \mapsto \mathbf{q}(t) = \mathbf{q}(t,\omega) \in \Omega$ on this probability space with $\mathbf{q}(0) = \mathbf{q} \in \Omega$ and $\omega \in \Omega_0$. The reader should keep in mind that our construction only works up to a time T^* before the solution of the kinetic equation cease to be positive (see Appendix A), but to ease the notation we will assume that this state to be infinite. Thus, any temporal quantity t in the future should be thought of implicitly as $t \wedge T^*$. Define now the stopping time T_1 as

$$T_1 = \inf \left\{ t \geq 0 : \int_0^t \mathfrak{r}(\theta, \psi_0^\theta \mathbf{q}) \, d\theta \geq \tau_1 \right\}.$$

For any $t \in [0, T_1)$, put $\mathbf{q}(t) = \psi_0^t \mathbf{q}$. Conditionally on T_1, write

$$\psi_0^{T_1}\mathbf{q} = \big((a^-, \rho^0), (z_1, \rho^1), \cdots, (z_n, \rho^n)\big).$$

- With probability $\dfrac{\mathfrak{C}_-(T_1, \rho^0)}{\mathfrak{r}(T_1, \psi_0^{T_1}\mathbf{q})}$, we sample ρ^* with density $\dfrac{\mathfrak{c}_-(T_1, \rho^*, \rho^0)}{\mathfrak{C}_-(T_1, \rho^0)}$ (with respect to the measure $\beta(d\rho^*)$) and put $\mathbf{q}(T_1) = E_-^{\rho^*}\psi_0^{T_1}\mathbf{q}$.
- With probability $\dfrac{\mathfrak{F}(z_i, T_1, \rho^{i-1}, \rho^i)}{\mathfrak{r}(T_1, \psi_0^{T_1}\mathbf{q})}$ for $i \in \{1, 2, \cdots, n\}$, sample ρ^* with density $\dfrac{\mathfrak{f}(z_i, T_1, \rho^{i-1}, \rho^*, \rho^i)}{\mathfrak{F}(T_1, \rho^{i-1}, \rho^i)}$ and put $\mathbf{q}(T_1) = E_i^{\rho^*}\psi_0^{T_1}\mathbf{q}$.
- With probability $\dfrac{\mathfrak{C}_+(T_1, \rho^n)}{\mathfrak{r}(T_1, \psi_0^{T_1}\mathbf{q})}$, sample ρ^* with density $\dfrac{\mathfrak{c}_+(T_1, \rho^n, \rho^*)}{\mathfrak{C}_+(T_1, \rho^n)}$ and put $\mathbf{q}(T_1) = E_+^{\rho^*}\psi_0^{T_1}\mathbf{q}$.

We repeat this process again by defining

$$T_2 := \inf \left\{ t \geq T_1 : \int_{T_1}^t \mathfrak{r}(\theta, \psi_{T_1}^\theta \mathbf{q}(T_1)) \, d\theta \geq \tau_2 \right\},$$

putting $\mathbf{q}(t) = \psi_{T_1}^t \mathbf{q}(T_1)$ for $t \in [T_1, T_2)$ and resampling again a ρ^* with the analogous above probabilities to define $\mathbf{q}(T_2)$. This constructs a sequence of random times $T_1, T_2, \cdots$. To ensure that our process $\mathbf{q}(t)$ is defined for any time $t \in [0, T]$ and that only finitely many jumps happens, we would assume

$$M := \sup_{t\in[0,T]} \sup_{z\in[a^-,a^+]} \sup_{\rho^-\prec\rho^+} \max\big(\mathfrak{C}_-(t,\rho^-), \mathfrak{F}(z,t,\rho^-,\rho^+), \mathfrak{C}_+(t,\rho^+)\big) < \infty. \tag{4.27}$$

To this end, let us define $N^T(\mathbf{q})$ to be the number of stochastic jumps up to time $T > 0$ when we starts from $\mathbf{q}$, i.e.,

$$N^T(\mathbf{q}) = \sup\{n \geq 0 : T_n < T\}.$$

The following lemma will be used in several occasions:

Lemma 4.2.1 *Assume that* (4.27) *holds. Then there exists a constant* $C_0 = C_0(T, M) > 0$, *independent of* $\mathbf{q}$ *such that*

$$[N^T(\mathbf{q})] \leq C_0(\mathbf{n}(\mathbf{q}) + 2)^2, \qquad \mathbb{P}(T_n < T) \leq C_0(\mathbf{n}(\mathbf{q}) + 2)^2 n^{-2}. \tag{4.28}$$

In particular $N^T(\mathbf{q})$ *is almost surely finite.*

Proof. For any $k \geq 0$ we have that

$$\int_{T_k}^{T_{k+1}} \mathfrak{r}(\theta, \psi^\theta_{T_k}\mathbf{q}(T_k))\, d\theta \geq \tau_{k+1}, \tag{4.29}$$

with the convention that $T_0 = 0$, and where $(\tau_i)_{i\geq 1}$ is an i.i.d sequence of standard exponential random variables. The number of particles of $\mathbf{q}(T_k)$ is at most $\mathbf{n}(\mathbf{q}) + k$, therefore for $\theta \in [T_k, T_{k+1})$, we have that $\mathfrak{r}(\theta, \psi^\theta_{T_k}\mathbf{q}(T_k)) \leq M(\mathbf{n}(\mathbf{q}) + k + 2)$. From this and (4.29), we deduce that

$$T_{k+1} - T_k \geq \frac{\tau_{k+1}}{M(\mathbf{n}(\mathbf{q}) + k + 2)}.$$

From this we learn,

$$\mathbb{P}(N^T(\mathbf{q}) > n) = \mathbb{P}(T_{n+1} < T) \leq \mathbb{P}\left(\sum_{k=1}^{n} \frac{\tau_{k+1}}{\mathbf{n}(\mathbf{q}) + k + 2} \leq MT\right).$$

Hence for any $\lambda > 0$, by Markov inequality

$$\begin{aligned}
\mathbb{P}\left(\sum_{k=1}^{n} \frac{\tau_{k+1}}{\mathbf{n}(\mathbf{q}) + k + 2} \leq MT\right) &= \mathbb{P}\left(\exp\left(-\lambda\sum_{k=1}^{n} \frac{\tau_{k+1}}{\mathbf{n}(\mathbf{q}) + k + 2}\right) \geq \exp(-\lambda MT)\right) \\
&\leq e^{\lambda MT} \prod_{k=1}^{n} \mathbb{E}\left[\exp\left(-\frac{\lambda\tau_{k+1}}{\mathbf{n}(\mathbf{q}) + k + 2}\right)\right] \\
&= e^{\lambda MT} \prod_{k=1}^{n} \left(1 + \frac{\lambda}{\mathbf{n}(\mathbf{q}) + k + 2}\right)^{-1}.
\end{aligned}$$

Using the fact that $\log(1 + x) \geq x - \frac{x^2}{2}$ for any positive x, we have

$$\log\left(\prod_{k=1}^{n}\left(1 + \frac{\lambda}{\mathbf{n}(\mathbf{q}) + k + 2}\right)\right) \geq \lambda(H_{\mathbf{n}(\mathbf{q})+n+2} - H_{\mathbf{n}(\mathbf{q})+2}) - \frac{\lambda^2}{2}\left(\sum_{k=\mathbf{n}(\mathbf{q})+3}^{\mathbf{n}(\mathbf{q})+n+2} \frac{1}{k^2}\right),$$

where $H_n := \sum_{k=1}^n \frac{1}{k}$ is the harmonic series. It is well-known however that

$$\log(n + 1) \leq H_n \leq \log n + 1.$$

Hence,

$$\log\left(\prod_{k=1}^{n}\left(1 + \frac{\lambda}{\mathbf{n}(\mathbf{q}) + k + 2}\right)\right) \geq \lambda \log\left(\frac{\mathbf{n}(\mathbf{q}) + n + 3}{\mathbf{n}(\mathbf{q}) + 2}\right) - \lambda - \frac{\lambda^2\pi^2}{12},$$

which in turn implies

$$\mathbb{P}(N^T(\mathbf{q}) > n) \leq e^{\lambda MT - \lambda \log\left(\frac{\mathbf{n}(\mathbf{q})+n+3}{\mathbf{n}(\mathbf{q})+2}\right) + \lambda + \frac{\lambda^2\pi^2}{12}}.$$

In particular for $\lambda = 2$, we get the bound

$$\mathbb{P}(T_{n+1} < T) = \mathbb{P}(N^T(\mathbf{q}) > n) \leq e^{2MT+2+\frac{\pi^2}{3}} \left(\frac{\mathbf{n}(\mathbf{q}) + 2}{\mathbf{n}(\mathbf{q}) + n + 3}\right)^2.$$

This certainly implies the second inequality in (4.28). Finally

$$[N^T(\mathbf{q})] \leq \frac{\pi^2}{6} e^{2MT+2+\frac{\pi^2}{3}} (\mathbf{n}(\mathbf{q}) + 2)^2 < \infty, \tag{4.30}$$

which implies the first inequality in (4.28). □

By construction, the process $(t, \mathbf{q}(t))_{t\geq 0}$ is a piecewise-deterministic process in the sense of Davis [19], and thus we have the following proposition

Proposition 4.2.2 *The process $(t, \mathbf{q}(t))_{t\geq 0}$ has the strong Markov property.*

It is not homogeneous because of the time dependence of the rates. Due to this Markovian property, we can talk about the stochastic flow $\Psi_s^t \mathbf{q}$ for any $s \leq t$ being defined as the realization of the particle system $\mathbf{q}(t)$ at time t conditioned to start at time s at $\mathbf{q}(s) = \mathbf{q}$. The Markov property, ensures that this stochastic flow enjoys the semigroup property in distribution

$$\Psi_{t_1}^{t_3} \mathbf{q} \stackrel{d}{=} \Psi_{t_2}^{t_3} \Psi_{t_1}^{t_2} \mathbf{q}, \tag{4.31}$$

for any $t_1 \leq t_2 \leq t_3$, where on the right-hand side of (4.31) the stochastic flow $\Psi_{t_2}^{t_3}$ is independent of $\Psi_{t_1}^{t_2}$. We can also replace the times t_i's by appropriate stopping times due the strong Markov property. The Markovian nature of our construction comes essentially from the memoryless property of exponential random variables.

4.3 Forward Equation

Recall the measure μ that was defined by (4.26). The goal of this section is to prove the following theorem.

Theorem 4.3.1 *For any measurable function G and $t \geq 0$, we have*

$$\int_\Omega \mathbb{E}[G(\Psi_0^t \mathbf{q})]\ \mu(d\mathbf{q}, 0) = \int_\Omega G(\mathbf{q})\ \mu(d\mathbf{q}, t). \tag{4.32}$$

Theorem 4.3.1 proves that if $\mathbf{q}(0)$ is distributed according to the measure $\mu(d\mathbf{q}, 0)$, then $\mathbf{q}(t)$ has the law $\mu(d\mathbf{q}, t)$. By a density argument, it suffices to prove the equality (4.32) for a suitable class of functions G on Ω, of the form

$$G(\mathbf{q}) = \exp\left(\int_{a^-}^{a^+} J(z)\mathcal{V}(\mathbf{q})(z)\ dz\right), \tag{4.33}$$

where $\mathcal{V}$ was defined in (4.22), and J is a continuous function on $[a^-, a^+]$. For (4.32), it is enough to show

$$\frac{d}{ds}\int_\Omega \mathbb{E}[G(\Psi_s^t \mathbf{q})]\ \mu(d\mathbf{q}, s) = 0, \quad \text{for all} \quad s \in [0,t]. \tag{4.34}$$

From now on, we fix $t \geq 0$, and define the function

$$G(\mathbf{q}, s) = \mathbb{E}[G(\Psi_s^t \mathbf{q})] \ \text{ for all } 0 \leq s \leq t.$$

4.3.1 Lipschitzness of G

We will start by proving the following crucial theorem.

Theorem 4.3.2 *There exists a constant $C_1 = C_1(P^-, P^+, V_\infty, J, a^-, a^+, t) > 0$ such that*

$$|G(\mathbf{q}, s) - G(\mathbf{q}, s')| \leq C_1(\mathbf{n}(\mathbf{q}) + 2)^2\ |s - s'|,$$

for all $s, s' \in [0,t]$ and for any $\mathbf{q} \in \Omega$.

The proof of this Lipschitz property is carried out in two steps. These steps are formulated as Lemmas 4.3.3 and 4.3.4.

Lemma 4.3.3 *Let $0 \leq s' \leq s \leq t$, and put $\theta := s - s'$. There exists a constant $C_2 = C_2(P^-, P^+, V_\infty, J, a^-, a^+, t)$ such that*

$$\left|G(\mathbf{q}, s') - \mathbb{E}\left[G(\Psi_{s'}^{t-\theta}\mathbf{q})\right]\right| \leq C_2(\mathbf{n}(\mathbf{q}) + 2)^2\theta. \tag{4.35}$$

Lemma 4.3.4 *There exists a constant $C_2' = C_2'(P^-, P^+, V_\infty, J, a^-, a^+, t)$ such that*

$$\left|G(\mathbf{q}, s) - \mathbb{E}\left[G(\Psi_{s'}^{t-\theta}\mathbf{q})\right]\right| \leq C_2'(\mathbf{n}(\mathbf{q}) + 2)^2\theta. \tag{4.36}$$

Proof of Lemma 4.3.3. *(Step 1)* In this step, we show that there exists a constant $C_3 = C_3(P^-, P^+, \|J\|_\infty, a^-, a^+)$ such that

$$\left|G(\mathbf{q}, s') - \mathbb{E}[G(\psi_{t-\theta}^t \Psi_{s'}^{t-\theta}\mathbf{q})\right| \leq C_3(\mathbf{n}(\mathbf{q}) + 2)^2\theta. \tag{4.37}$$

We first use the Markov property to write

$$G(\mathbf{q}, s') = \mathbb{E}[G(\Psi_{t-\theta}^t \Psi_{s'}^{t-\theta}\mathbf{q})].$$

Let $\mathcal{E}$ be the event

$$\mathcal{E} := \Big\{\text{there exists a stochastic jump in } (t-\theta, t)\Big\}.$$

Here by stochastic jump, we mean the creation of a new particle either at $z = a^-$ or at $z = a^+$, or the fragmentation of one of the particles. We claim that there exists a constant $C_4 = C_4(t)$ such that

$$\mathbb{P}(\mathcal{E}) \leq C_4\left(\mathbf{n}(\mathbf{q}) + 2\right)^2\theta. \tag{4.38}$$

To see this, observe

$$\mathbb{P}(\mathcal{E}) = \mathbb{P}\left(\int_{t-\theta}^{t} \mathfrak{r}(u, \psi_{t-\theta}^{u}\Psi_{s'}^{t-\theta}\mathbf{q})\ du \geq \tau\right), \tag{4.39}$$

where τ is a standard exponential random variable that is independent of $\{\Psi_{s'}^{v}\mathbf{q}, s' \leq v \leq t-\theta\}$. Let $N_{s'}^{t-\theta}(\mathbf{q})$ be the number of stochastic jumps of the particle system started at time s' at $\mathbf{q}$ up to time $t-\theta$. By Lemma 4.2.1 we have that

$$\mathbb{E}[N_{s'}^{t-\theta}(\mathbf{q})] \leq C_0(\mathbf{n}(\mathbf{q})+2)^2, \tag{4.40}$$

where $C_0 = C_0(t)$ is a constant that depends only on t. As the number of particles of $\psi_{t-\theta}^{u}\Psi_{s'}^{t-\theta}\mathbf{q}$ is at most $\mathbf{n}(\mathbf{q}) + N_{s'}^{t-\theta}(\mathbf{q})$ for all $u \in [t-\theta, t]$, then from (4.39), it follows that

$$\begin{aligned}
\mathbb{P}(\mathcal{E}) &\leq \mathbb{P}\Big(M\theta\left(\mathbf{n}(\mathbf{q}) + N_{s'}^{t-\theta}(\mathbf{q}) + 2\right) \geq \tau\Big) \\
&= \mathbb{E}\left[1 - e^{-M\theta(\mathbf{n}(\mathbf{q})+N_{s'}^{t-\theta}(\mathbf{q})+2)}\right] \\
&\leq M\theta\mathbb{E}\left[\mathbf{n}(\mathbf{q}) + N_{s'}^{t-\theta}(\mathbf{q}) + 2\right] \\
&\leq C_4\left(\mathbf{n}(\mathbf{q})+2\right)^2\theta,
\end{aligned}$$

where the constant M is the uniform bound on the rates defined in (4.27), and $C_4 = M(1+C_0)$. This completes the proof of (4.38).
Evidently,

$$G(\mathbf{q}, s') = \mathbb{E}\left[\mathbb{1}_{\mathcal{E}^c}\ G(\psi_{t-\theta}^{t}\Psi_{s'}^{t-\theta}\mathbf{q})\right] + \mathbb{E}\left[\mathbb{1}_{\mathcal{E}}\ G(\Psi_{s'}^{t}\mathbf{q})\right]. \tag{4.41}$$

However, from the expression of G in (4.33), we can find a constant $C_5 = C(P^-, P^+, \|J\|_\infty, a^-, a^+)$ such that

$$|G(\mathbf{q})| \leq C_5 \quad \text{for all} \quad \mathbf{q} \in \Omega.$$

From this and (4.41), we learn

$$\left|G(\mathbf{q}, s') - \mathbb{E}[G(\psi_{t-\theta}^{t}\Psi_{s'}^{t-\theta}\mathbf{q})\right| \leq 2C_5\mathbb{P}(\mathcal{E}) \leq C_3(\mathbf{n}(\mathbf{q})+2)^2\theta,$$

for a constant $C_3 = 2C_5C_4$. This completes the proof of (4.37).

(Step 2) On account of (4.37), it remains to show

$$\left|\mathbb{E}[G(\psi_{t-\theta}^{t}\Psi_{s'}^{t-\theta}\mathbf{q})] - \mathbb{E}[G(\Psi_{s'}^{t-\theta}\mathbf{q})]\right| \leq C_6(\mathbf{n}(\mathbf{q})+2)^2\theta, \tag{4.42}$$

for some constant $C_6 = C_6(P^-, P^+, V_\infty, J, t)$. As a preparation, we first show that there exists a constant $C_7 = C_7(P^-, P^+, V_\infty, J, a^-, a^+, t)$ such that

$$|G(\psi_0^r\mathbf{q}) - G(\mathbf{q})| \leq C_7\mathbf{n}(\mathbf{q})r, \quad \text{for all} \quad r \in [0,t] \quad \text{and} \quad \mathbf{q} \in \Omega', \tag{4.43}$$

where the set Ω' is a set of full measure that was defined in Notation 2.1**(vi)**. To prove (4.43), we fix $r > 0$, and let $\rho := \mathcal{V}(\mathbf{q})$ and $\rho' := \mathcal{V}(\psi_0^r\mathbf{q})$. As the exponential function is locally Lipschitz, for (4.43) it suffices to show that there exists a constant $C_8 = C_8(P^-, P^+, V_\infty, a^-, a^+, t)$, such that

$$\int_{a^-}^{a^+} |\rho'(z) - \rho(z)|\ dz \leq C_8\mathbf{n}(\mathbf{q})r. \tag{4.44}$$

Note that $\rho = \mathcal{V}(\bar{\mathbf{q}})$ where $\bar{\mathbf{q}}$ is the particle configuration obtained from $\mathbf{q}$ after deleting the redundant particles, so without loss of generality we can assume that $\mathbf{q} \in \mathrm{Int}(\Omega^n)$ for some $n \leq \mathbf{n}(\mathbf{q})$. Let

$$a^- < z_1 < z_2 < \cdots < z_n < a^+,$$

be the discontinuity points of ρ. Let $\delta := V_\infty r$. If I is the interval

$$I := \bigcup_{i=1}^{n} [z_i - \delta, z_i + \delta],$$

then for any $z \notin I$, we have $\rho'(z) = \rho(z)$, as the discontinuity points of the function $(z, v) \in [a^-, a^+] \times [0, r] \mapsto \mathcal{V}(\psi_0^v \mathbf{q})(z)$ travel with speed at most V_∞. Therefore

$$\int_{a^-}^{a^+} |\rho'(z) - \rho(z)| \; dz \leq (\|\rho\|_\infty + \|\rho'\|_\infty) |I| \leq 4 \max(|P^-|, |P^+|) V_\infty n r,$$

which proves (4.44). This in turn implies (4.43).

We are now ready to establish (4.42). We have that almost surely $\Psi_{s'}^{t-\theta} \mathbf{q} \in \Omega'$, as this is equivalent to not having a stochastic jump at time $t - \theta$, an event that happens with probability one. This allows us to apply (4.43) to assert

$$\mathbb{E}\left[\left| G(\psi_{t-\theta}^t \Psi_{s'}^{t-\theta} \mathbf{q}) - G(\Psi_{s'}^{t-\theta} \mathbf{q}) \right| \right] \leq C_7 \mathbb{E}\left[\left(\mathbf{n}(\Psi_{s'}^{t-\theta} \mathbf{q}) + 2 \right) \right] \theta.$$

This, the bound $\mathbf{n}(\Psi_{s'}^{t-\theta} \mathbf{q}) \leq \mathbf{n}(\mathbf{q}) + N_{s'}^{t-\theta}(\mathbf{q})$, and (4.40) imply (4.42). From (4.37) and (4.42), we get (4.3.3). $\square$

To finish the proof of Theorem 4.3.2, it remains to establish Lemma 3.2. To achieve this, we will define a coupling of $(\Psi_{s'}^{t-\theta} \mathbf{q}, \Psi_s^t \mathbf{q})$ for $t \geq s$. Or equivalently, we define a coupling of $(\Psi_{s'}^t \mathbf{q}, \Psi_{s'+\theta}^{t+\theta} \mathbf{q})$ for $t \geq s'$.

Construction of the coupling

We fix $0 \leq s' \leq s$ and as before put $\theta := s - s'$. We wish to construct two processes $(\mathbf{q}(t))_{t \geq s'}$ and $(\mathbf{q}'(t))_{t \geq s'}$ on the same probability space such that $(\mathbf{q}(t))_{t \geq s'}$ has the law $(\Psi_{s'}^t \mathbf{q})_{t \geq s'}$ and $(\mathbf{q}'(t))_{t \geq s'}$ has the law $(\Psi_{s'+\theta}^{t+\theta} \mathbf{q})_{t \geq s'}$.

We start from a sequence of i.i.d exponential random variables of the form $(\tau_i^j, i \geq 1, 1 \leq j \leq 3)$. We define first the *coupling* rates as follows.

Let $t \geq s'$ and for any $\rho \in [P^-, P^+]^2$, let

$$\mathfrak{C}_-^{\text{coupling}}(t, \rho) := \int_{L(\rho)} \mathfrak{c}_-(t, \rho^-, \rho) \wedge \mathfrak{c}_-(t + \theta, \rho^-, \rho) \; \beta(d\rho^-),$$

$$\mathfrak{C}_+^{\text{coupling}}(t, \rho) := \int_{R(\rho)} \mathfrak{c}_+(t, \rho, \rho^+) \wedge \mathfrak{c}_+(t + \theta, \rho, \rho^+) \; \beta(d\rho^+),$$

and for $\rho^-, \rho^+ \in [P^-, P^+]^2$ such that $\rho^- \prec \rho^+$, let

$$\mathfrak{F}^{\text{coupling}}(z, t, \rho^-, \rho^+) := \int_{D(\rho^-,\rho^+)} \mathfrak{f}(z, t, \rho^-, \rho, \rho^+) \wedge \mathfrak{f}(z, t+\theta, \rho^-, \rho, \rho^+)\ \beta(d\rho).$$

The *particle coupling rate* is defined as

$$\mathfrak{r}^{\text{coupling}}(t, \mathbf{q}) := \mathfrak{C}_-^{\text{coupling}}(t, \rho^0) + \sum_{i=1}^{n} \mathfrak{F}^{\text{coupling}}(z_i, t, \rho^{i-1}, \rho^i) + \mathfrak{C}_+^{\text{coupling}}(t, \rho^n),$$

for any

$$\mathbf{q} = ((a^-, \rho^0), (z_1, \rho^1), \cdots, (z_n, \rho^n)) \in \Omega^n.$$

Now, define

$$T_1^1 := \inf\left\{ t \geq s' : \int_{s'}^{t} \mathfrak{r}^{\text{coupling}}(u, \psi_{s'}^u \mathbf{q})\ du \geq \tau_1^1 \right\},$$
$$T_1^2 := \inf\left\{ t \geq s' : \int_{s'}^{t} (\mathfrak{r}(u, \psi_{s'}^u \mathbf{q}) - \mathfrak{r}^{\text{coupling}}(u, \psi_{s'}^u \mathbf{q}))\ du \geq \tau_1^2 \right\},$$
$$T_1^3 := \inf\left\{ t \geq s' : \int_{s'}^{t} (\mathfrak{r}(u+\theta, \psi_{s'}^u \mathbf{q}) - \mathfrak{r}^{\text{coupling}}(u, \psi_{s'}^u \mathbf{q}))\ du \geq \tau_1^3 \right\}.$$

Put $T_1 = \min(T_1^1, T_1^2, T_1^3)$, and for $t \in [s', T_1)$, set $\mathbf{q}(t) = \mathbf{q}'(t) = \psi_{s'}^t \mathbf{q} = \psi_{s'+\theta}^{t+\theta} \mathbf{q}$. Now, write

$$\psi_{s'}^{T_1} \mathbf{q} = ((a^-, \rho^0), (z_1, \rho^1), \cdots, (z_n, \rho^n)) \in \Omega^n.$$

Conditionally on T_1, we consider the following cases:

- If $T_1 = T_1^2$, we set $\mathbf{q}'(T_1) = \psi_{s'+\theta}^{T_1+\theta} \mathbf{q}$, and define $\mathbf{q}(T_1)$ by making a stochastic jump (either a creation of a particle at $z = a^{\pm}$ or a fragmentation of a particle) using the rate $\mathfrak{r}(T_1, \psi_{s'}^{T_1} \mathbf{q}) - \mathfrak{r}^{\text{coupling}}(T_1, \psi_{s'}^{T_1} \mathbf{q})$. For $t \geq T_1$, conditionally on $(\mathbf{q}(T_1), \mathbf{q}'(T_1))$, we let the two processes $(\mathbf{q}(t))_{t \geq T_1}$ and $(\mathbf{q}'(t))_{t \geq T_1}$ evolve independently with respectively the law of $(\Psi_{T_1}^t \mathbf{q}(T_1))_{t \geq T_1}$ and $(\Psi_{T_1+\theta}^{t+\theta} \mathbf{q}'(T_1))_{t \geq T_1}$.
- If $T_1 = T_1^3$, we do the same as previously by switching the roles of $\mathbf{q}$ and $\mathbf{q}'$ and using the rates at $T_1 + \theta$ instead of T_1, in a way that only $\mathbf{q}'$ makes a jump at time T_1 and not $\mathbf{q}$.
- If $T_1 = T_1^1$, both $\mathbf{q}$ and $\mathbf{q}'$ make a stochastic jump at T_1 using the rate $\mathfrak{r}^{\text{coupling}}(T_1, \psi_{s'}^{T_1} \mathbf{q})$. As $\mathbf{q}(T_1) = \mathbf{q}'(T_1)$, we redo the same process again by using now random variables $(\tau_2^1, \tau_2^2, \tau_2^3)$ and defining (T_2^1, T_2^2, T_2^3), etc ...

We claim that this defines a coupling. Indeed, each one of the processes $(\mathbf{q}(t))_{t \geq s'}$ and $(\mathbf{q}'(t))_{t \geq s}$ is a piecewise-deterministic Markov process by construction, and the law of the first jump time and the value it takes at that time is easily verified to be the same as that of the two processes $(\Psi_{s'}^t \mathbf{q})_{t \geq s'}$ and $(\Psi_{s'+\theta}^{t+\theta} \mathbf{q})_{t \geq s'}$. Using this coupling, we can now prove Lemma 3.2, which in turn finishes the proof of Theorem 4.3.2.

Proof of Lemma 4.3.4 Keeping the notation from the construction of the coupling, we will exhibit an event $\mathcal{E}$ on which $\mathbf{q}(u) = \mathbf{q}'(u)$ for all $u \in [s', t-\theta]$. Writing T_i for the time at which the coupled process experiences a jump for the i-th time, we see that as long as $T_i^1 = T_i$ for all i such that $T_i < t-\theta$, then we must have $\mathbf{q}(t-\theta) = \mathbf{q}'(t-\theta)$, as we jump using the same value ρ^* at every step. Let

$$\mathcal{E} = \{T_i^1 = T_i \text{ for all } i \leq N_t\},$$

where $N_t = \sup\{n \geq 0 : T_n < t - \theta\}$. Evidently,

$$\left|\mathbb{E}\left[G(\Psi_s^t \mathbf{q})\right] - \mathbb{E}[G(\Psi_{s'}^{t-\theta}\mathbf{q})]\right| = \left|\mathbb{E}\left[G(\mathbf{q}'(t-\theta))\right] - \mathbb{E}[G(\mathbf{q}(t-\theta))]\right| \leq 2\|G\|_\infty \mathbb{P}(\mathcal{E}^c).$$

Hence for (4.3.4), it suffices to show that there is a constant $C_9 = C_9(P^-, P^+, V_\infty, t) > 0$, such that

$$\mathbb{P}(\mathcal{E}^c) \leq C_9 \theta (\mathbf{n}(\mathbf{q}) + 2)^2. \tag{4.45}$$

To achieve this, first observe,

$$\mathcal{E}^c \subset \bigcup_{n=1}^{\infty} \left\{T_i^1 = T_i \text{ for all } i \leq n-1,\ T_{n-1} < t-\theta,\ T_n^1 > T_n\right\}.$$

As a result,

$$\mathbb{P}(\mathcal{E}^c) \leq \sum_{n=1}^{\infty} \mathbb{P}(T_n^1 - T_{n-1} > T_n - T_{n-1},\ T_{n-1} < t - \theta). \tag{4.46}$$

Recall that we can write

$$T_n^1 - T_{n-1} = \inf\left\{v \geq 0 : \int_{T_{n-1}}^{T_{n-1}+v} \mathfrak{r}^{\text{coupling}}(u, \psi_{s'}^u \mathbf{q})\ du \geq \tau_n^1\right\},$$

and similarly for $T_n^2 - T_{n-1}$ and $T_n^3 - T_{n-1}$, with appropriately replacing the rates $\mathfrak{r}$ used inside the integral. Note that the conditional probability

$$P_n := \mathbb{P}\Big(T_n^1 - T_{n-1} > T_n - T_{n-1} \;\Big|\; T_{n-1}\Big),$$

satisfies

$$\begin{aligned} P_n &= \mathbb{E}\left[1 - \exp\left(-\int_{T_{n-1}}^{T_n^1} \left|\mathfrak{r}(u, \psi_{s'}^u \mathbf{q}) - \mathfrak{r}(u+\theta, \psi_s^{u+\theta}\mathbf{q})\right|\ du\right) \;\middle|\; T_{n-1}\right] \\ &\leq \mathbb{E}\left[\int_{T_{n-1}}^{T_n^1} \left|\mathfrak{r}(u, \psi_{s'}^u \mathbf{q}) - \mathfrak{r}(u+\theta, \psi_{s'}^u \mathbf{q})\right|\ du \;\middle|\; T_{n-1}\right]. \end{aligned} \tag{4.47}$$

Now, for any fixed $\mathbf{q} \in \Omega$, we have that

$$\partial_t \mathfrak{r}(t, \mathbf{q}) = \mathcal{R}^-(t, \mathbf{q}) + \sum_{i=1}^{\mathbf{n}(\mathbf{q})} \mathcal{R}_i(t, \mathbf{q}) + \mathcal{R}^+(t, \mathbf{q})$$

where

$$\mathcal{R}^-(t,\mathbf{q}) := \int_{L(\rho^0)} [\rho^-,\rho_0]^- \ \partial_t \left(\frac{\ell(a^-,t,\rho^-) f(a^-,t,\rho^-,\rho^0)}{\ell(a^-,t,\rho^0)} \right) \ \beta(d\rho^-),$$
$$\mathcal{R}_i(t,\mathbf{q}) := \int_{D(\rho^{i-1},\rho^i)} [\rho^{i-1},\rho,\rho^i]^- \ \partial_t \left(\frac{f(z_i,t,\rho^{i-1},\rho) f(z_i,t,\rho,\rho^i)}{f(z_i,t,\rho^{i-1},\rho^i)} \right) \ \beta(d\rho),$$
$$\mathcal{R}^+(t,\mathbf{q}) := \int_{R(\rho^{\mathbf{n}(\mathbf{q})})} [\rho^{\mathbf{n}(\mathbf{q})},\rho^+]^+ \ \partial_t \left(f(a^+,t,\rho^{\mathbf{n}(\mathbf{q})},\rho^+) \right) \ \beta(d\rho^+).$$

Hence by using the uniform upper and lower bound of the kernel f and ℓ, and the uniform upper bound of their first-order derivatives, there exists a uniform constant $M' > 0$ such that

$$\partial_t \mathfrak{r}(t,\mathbf{q}) \le M' \mathcal{A}(\mathbf{q}),$$

where $\mathcal{A}(\mathbf{q})$ is given by

$$\int_{L(\rho^0)} [\rho^-,\rho^0]^- \ \beta(d\rho^-) + \sum_{i=1}^{\mathbf{n}(\mathbf{q})} \int_{D(\rho^{i-1},\rho^i)} [\rho^{i-1},\rho,\rho^i]^- \ \beta(d\rho) + \int_{R(\rho^{\mathbf{n}(\mathbf{q})})} [\rho^{\mathbf{n}(\mathbf{q})},\rho^+]^+ \ \beta(d\rho^+).$$

From this and (4.47) we deduce,

$$P_n \le M'\theta \ \mathbb{E}\left[\int_{T_{n-1}}^{T_n^1} \mathcal{A}(\psi_{s'}^u \mathbf{q}) \ du \ \Big| \ T_{n-1} \right]. \tag{4.48}$$

Moreover, using the uniform upper and lower bound of the kernel f and ℓ, we can find $\delta_1 > 0$ such that

$$\mathfrak{r}^{\text{coupling}}(u,\mathbf{q}) \ge \delta_1 \mathcal{A}(\mathbf{q}). \tag{4.49}$$

On the other hand, since by definition,

$$\tau_n^1 = \int_{T_{n-1}}^{T_n^1} \mathfrak{r}^{\text{coupling}}(u,\psi_{s'}^u \mathbf{q}) \ du,$$

we use (4.49) to assert

$$\tau_n^1 \ge \delta_1 \int_{T_{n-1}}^{T_n^1} \mathcal{A}(\psi_{s'}^u \mathbf{q}) \ du.$$

This and (4.48) yield,

$$P_n \le \delta_1^{-1} M'\theta \ \mathbb{E}[\tau_n^1 | T_{n-1}] = \delta_1^{-1} M'\theta,$$

as τ_n^1 is independent of T_{n-1}. From this and (4.46) we learn

$$\mathbb{P}\left(\mathcal{E}^c\right) \le \delta_1^{-1} M'\theta \sum_{n=1}^{\infty} \mathbb{P}(T_{n-1} < t).$$

From this and (4.28) we deduce (4.45). This completes the proof of (4.3.4). □

4.3.2 Differentiation of G

Recall that for Theorem 2.1, it suffices to verify (4.34). To carry out the differentiation in (4.34), we first learn how to differentiate the integrand G with respect to s. Since $G(\mathbf{q}, s) = [G(\Psi_s^t \mathbf{q})]$, we expect a Kolmogorov type equation of the form

$$G_s(\mathbf{q}, s) = -\mathbb{E}[(\mathcal{L}G)(\Psi_s^t \mathbf{q})], \tag{4.51}$$

where $\mathcal{L}$ denotes the generator of the process $\mathbf{q}(\cdot)$. Since the deterministic part of the dynamics is discontinuous, the verification of (4.51) poses some challenges that are handled in this subsection. The integrated version of (4.51) is our next result.

Theorem 4.3.5 *For any $s \geq 0$ we have that the limit*

$$\lim_{s' \uparrow s} \frac{1}{s - s'} \int (G(\mathbf{q}, s) - G(\mathbf{q}, s'))\ \mu(d\mathbf{q}, s), \tag{4.52}$$

equals to

$$-\sum_{n=0}^{\infty} (\Gamma_n^-(s) + \Gamma_n^+(s)) + \sum_{n=1}^{\infty} (\Gamma_n^{\mathfrak{f}}(s) + \Gamma_n^t(s) + \Gamma_n^d(s)),$$

where for $n \geq 1$, we have

$$\Gamma_n^-(s) = \int_{\Omega^n} \int_{L(\rho^0)} \mathfrak{c}_-(s, \rho^*, \rho^0)(G(E_-^{\rho^*} \mathbf{q}, s) - G(\mathbf{q}, s))\ \beta(d\rho^*)\ \mu^n(d\mathbf{q}, s),$$

$$\Gamma_n^+(s) = \int_{\Omega^n} \int_{R(\rho^n)} \mathfrak{c}_+(s, \rho^n, \rho^*)(G(E_+^{\rho^*} \mathbf{q}, s) - G(\mathbf{q}, s))\ \beta(d\rho^*) \mu^n(d\mathbf{q}, s),$$

$$\Gamma_n^{\mathfrak{f}}(s) = \sum_{i=1}^{n} \int_{\Omega^n} \int_{D(\rho^{i-1}, \rho^i)} \mathfrak{f}(z_i, s, \rho^{i-1}, \rho^*, \rho^i)(G(E_i^{\rho^*} \mathbf{q}, s) - G(\mathbf{q}, s))\ \beta(d\rho^*) \mu^n(d\mathbf{q}, s),$$

$$\Gamma_n^t(s) = \sum_{i=1}^{n} \int_{\Omega^n} [\rho^{i-1}, \rho^i] \left(\frac{f_z(z_i, s, \rho^{i-1}, \rho^i)}{f(z_i, s, \rho^{i-1}, \rho^i)} + (z_i, s, \rho^i) - (z_i, s, \rho^{i-1}) \right) G(\mathbf{q}, s) \beta(d\rho^*) \mu^n(d\mathbf{q}, s),$$

and we can write $\Gamma_n^d(s) = \Gamma_n^{d,-}(s) + \Gamma_n^{d,\mathfrak{f}}(s) + \Gamma_n^{d,+}(s)$, with

$$\Gamma_n^{d,+}(s) = -\int_{\Omega^{n-1}} \int_{R(\rho^{n-1})} [\rho^{n-1}, \rho^*] f(a^+, s, \rho^{n-1}, \rho^*) G(E_+^{\rho^*} \mathbf{q}, s)\ \beta(d\rho^*) \mu^{n-1}(d\mathbf{q}, s),$$

$$\Gamma_n^{d,-}(s) = \int_{\Omega^{n-1}} \int_{L(\rho^0)} [\rho^*, \rho^0] \frac{\ell(a^-, s, \rho^*) f(a^-, s, \rho^*, \rho^0)}{\ell(a^-, s, \rho^0)} G(E_-^{\rho^*} \mathbf{q}, s)\ \beta(d\rho^*) \mu^{n-1}(d\mathbf{q}, s),$$

$$\Gamma_n^{d,\mathfrak{f}}(s) = \sum_{i=1}^{n-1} \int_{\Omega^{n-1}} \int_{D(\rho^{i-1}, \rho^i)} [\rho^{i-1}, \rho^*, \rho^i] \frac{f(z_i, s, \rho^{i-1}, \rho^*) f(z_i, s, \rho^*, \rho^i)}{f(x_i, s, \rho^{i-1}, \rho^i)}$$
$$G(E_i^{\rho^*} \mathbf{q}, s)\ \beta(d\rho^*) \mu^{n-1}(d\mathbf{q}, s).$$

For $n = 1$, $\Gamma_n^{d,\mathfrak{f}}(s) = 0$, and without ambiguity the other terms have the same expression as for $n > 1$.

Proof. (Step 1) In this step, we use the Markov property to derive a formula for $G(\mathbf{q}, s')$ (see (4.58) at the end of this step). To begin, observe that by the Markov property,

$$G(\mathbf{q}, s') = \mathbb{E}[G(\Psi^t_{s'}\mathbf{q})] = \mathbb{E}[G(\Psi^t_s \Psi^s_{s'}\mathbf{q})] = \mathbb{E}[G(\Psi^s_{s'}\mathbf{q}, s)] \tag{4.53}$$

Using the notation that we used previously in our construction of the Markov process, let

$$T_1 = \inf\left\{t \geq s' : \int_{s'}^{t} \mathfrak{r}(u, \psi^u_{s'}\mathbf{q})\ du \geq \tau_1\right\},$$

where τ_1 is an independent standard exponential random variable. Consider again $\mathcal{E}$ to be the event $\{T_1 \in [s', s]\}$, then

$$\mathbb{E}[G(\Psi^s_{s'}\mathbf{q}, s)] = \mathbb{E}[G(\Psi^s_{s'}\mathbf{q}, s)\mathbb{1}_{\mathcal{E}}] + \mathbb{P}(\mathcal{E}^c)G(\psi^s_{s'}\mathbf{q}, s). \tag{4.54}$$

For the first term, we can use the strong Markov property at the stopping time T_1 to get

$$\mathbb{E}\left[G(\Psi^s_{s'}\mathbf{q}, s)\mathbb{1}_{\mathcal{E}}\right] = \mathbb{E}\left[G(\Psi^{T_1}_{s'}\mathbf{q}, T_1)\mathbb{1}_{\mathcal{E}}\right]. \tag{4.55}$$

Using Theorem 4.3.2, we have

$$\begin{aligned}\left|\mathbb{E}\left[(G(\Psi^{T_1}_{s'}\mathbf{q}, T_1) - G(\Psi^{T_1}_{s'}\mathbf{q}, s))\mathbb{1}_{\mathcal{E}}\right]\right| &\leq C_1\mathbb{E}\left[\left(\mathbf{n}\left(\Psi^{T_1}_{s'}\mathbf{q}\right) + 2\right)^2 |T_1 - s'|\mathbb{1}_{\mathcal{E}}\right] \\ &\leq C_1(\mathbf{n}(\mathbf{q}) + 3)^2\ \mathbb{P}(\mathcal{E})\ (s - s').\end{aligned}$$

We certainly have

$$\begin{aligned}\mathbb{P}(\mathcal{E}) =& \mathbb{P}\left(\tau_1 \leq \int_{s'}^{s} \mathfrak{r}(u, \psi^u_{s'}\mathbf{q})du\right) = 1 - \exp\left(-\int_{s'}^{s} \mathfrak{r}(u, \psi^u_{s'}\mathbf{q})\ du\right) \\ &\leq \int_{s'}^{s} \mathfrak{r}(u, \psi^u_{s'}\mathbf{q})\ du \leq M(\mathbf{n}(\mathbf{q}) + 2)(s - s'),\end{aligned} \tag{4.56}$$

where M is a uniform bound on the rates. From this and the previous display we learn

$$\left|\left[\left(G\left(\Psi^{T_1}_{s'}\mathbf{q}, T_1\right) - G\left(\Psi^{T_1}_{s'}\mathbf{q}, s\right)\right)\mathbb{1}_{\mathcal{E}}\right]\right| \leq C_1 M(\mathbf{n}(\mathbf{q}) + 3)^3(s - s')^2. \tag{4.57}$$

From this, (4.53), (4.54), and (4.55) we deduce

$$G(\mathbf{q}, s') = [G(\Psi^{T_1}_{s'}\mathbf{q}, s)\mathbb{1}_{\mathcal{E}}] + \mathbb{P}(\mathcal{E}^c)G(\psi^s_{s'}\mathbf{q}, s) + R_1(\mathbf{q})(s - s')^2, \tag{4.58}$$

where $|R_1(\mathbf{q})| \leq C_1 M(\mathbf{n}(\mathbf{q}) + 3)^3$.

(Step 2) The main goal of this step is to use (4.58) to establish the following decomposition:

$$\begin{aligned}G(\mathbf{q}, s) - G(\mathbf{q}, s') =& \mathcal{S}'(\mathbf{q}) + \mathcal{S}^-(\mathbf{q}) + \mathcal{S}^+(\mathbf{q}) + \sum_{i=1}^{\mathbf{n}(\mathbf{q})} \mathcal{S}_i(\mathbf{q}) \\ &+ R_2(\mathbf{q})\left[(s - s')^2 + (s - s')\mathbb{1}(\mathbf{q} \in \hat{\Omega})\right],\end{aligned} \tag{4.59}$$

where

$$\mathcal{S}'(\mathbf{q}) = \exp\left(-\int_{s'}^{s} \mathfrak{r}(u, \psi_{s'}^{u}\mathbf{q})\ du\right)(G(\mathbf{q}, s) - G(\psi_{s'}^{s}\mathbf{q}, s)),$$

$$\mathcal{S}^{-}(\mathbf{q}) = \int_{s'}^{s}\int_{L(\rho^0)} \mathfrak{c}_{-}(\theta, \rho^*, \rho^0)(G(\mathbf{q}, s) - G(E_{-}^{\rho^*}\psi_{s'}^{\theta}\mathbf{q}, s))\ \beta(d\rho^*)d\theta,$$

$$\mathcal{S}^{+}(\mathbf{q}) = \int_{s'}^{s}\int_{R(\rho^{\mathbf{n}(\mathbf{q})})} \mathfrak{c}_{+}(\theta, \rho^{\mathbf{n}(\mathbf{q})}, \rho^*)(G(\mathbf{q}, s) - G(E_{+}^{\rho^*}\psi_{s'}^{\theta}\mathbf{q}, s))\ \beta(d\rho^*)d\theta,$$

$$\mathcal{S}_i(\mathbf{q}) = \int_{s'}^{s}\int_{D(\rho^{i-1}, \rho^i)} \mathfrak{f}(z_i, \theta, \rho^{i-1}, \rho^*, \rho^i)(G(\mathbf{q}, s) - G(E_{i}^{\rho^*}\psi_{s'}^{t}\mathbf{q}, s))\ \beta(d\rho^*)d\theta,$$

and the term R_2 satisfies the bound

$$|R_2(\mathbf{q})| \le c_1(\mathbf{n}(\mathbf{q}) + 3)^3,$$

for a constant $c_1 = c_1(P^-, P^+, J, a^-, a^+, t) > 0$ that does not depend on $\mathbf{q}$, with the set $\hat{\Omega}$ is defined as

$$\hat{\Omega} = \hat{\Omega}_{s'}^{s} := \left\{\mathbf{q} \in \Omega :\ u \mapsto \psi_{s'}^{u}\mathbf{q} \text{ experiences a collision in } [s', s]\right\}.$$

We can readily show that there exists a universal constant c_2 such that

$$\mu(\hat{\Omega}_{s'}^{s}, s) \le c_2(s - s'). \tag{4.60}$$

and so upon integrating with respect to $\mu(d\mathbf{q}, s)$, the error terms are all of order $O((s-s')^2)$.

To achieve (4.59), we first examine the first term on the right-hand side of (4.58). From the boundedness of G and (4.56) we deduce

$$\mathbb{E}\left[G\left(\Psi_{s'}^{T_1}\mathbf{q}, s\right)\mathbb{1}_{\mathcal{E}}\right] = \mathbb{E}\left[G\left(\Psi_{s'}^{T_1}\mathbf{q}, s\right)\mathbb{1}_{\mathcal{E}}\right]\mathbb{1}(\mathbf{q} \notin \hat{\Omega}) + R_3(\mathbf{q})\mathbb{1}(\mathbf{q} \in \hat{\Omega})(s - s'), \tag{4.61}$$

with $|R_3(\mathbf{q})| \le c_3(\mathbf{n}(\mathbf{q}) + 2)$, for a constant $c_3 = c_3(t)$. Moreover for $\mathbf{q} \notin \hat{\Omega}$,

$$\begin{aligned}
\mathbb{E}\left[G\left(\Psi_{s'}^{T_1}\mathbf{q}, s\right)\mathbb{1}_{\mathcal{E}}\right] =& \mathbb{E}\left[\frac{\mathfrak{C}_{-}(T_1, \rho^0)}{\mathfrak{r}(T_1, \psi_{s'}^{T_1}\mathbf{q})} G\left(E_{-}^{\rho^*}\psi_{s'}^{T_1}\mathbf{q}, s\right)\mathbb{1}_{\mathcal{E}}\right] \\
&+ \mathbb{E}\left[\frac{\mathfrak{C}_{+}(T_1, \rho^{\mathbf{n}(\mathbf{q})})}{\mathfrak{r}(T_1, \psi_{s'}^{T_1}\mathbf{q})} G\left(E_{+}^{\rho^*}\psi_{s'}^{T_1}\mathbf{q}, s\right)\mathbb{1}_{\mathcal{E}}\right] \\
&+ \sum_{i=1}^{\mathbf{n}(\mathbf{q})}\mathbb{E}\left[\frac{\mathfrak{F}(z_i - [\rho^{i-1}, \rho^i](T_1 - s'), T_1, \rho^{i-1}, \rho^i)}{\mathfrak{r}(T_1, \psi_{s'}^{T_1}\mathbf{q})} G\left(E_{i}^{\rho^*}\psi_{s'}^{T_1}\mathbf{q}, s\right)\mathbb{1}_{\mathcal{E}}\right] \\
&=: \mathcal{T}_{-} + \mathcal{T}_{+} + \sum_{i=1}^{\mathbf{n}(\mathbf{q})}\mathcal{T}_i,
\end{aligned}$$

where each ρ^* is distributed according to the density previously described in the construction of the stochastic flow. Note that the distribution function of T_1 is given by

$$\mathbb{P}(T_1 \ge \theta) = \exp\left(-\int_{s'}^{\theta} \mathfrak{r}(u, \psi_{s'}^{u}\mathbf{q})\ du\right),$$

or equivalently,

$$\mathbb{P}(T_1 \in d\theta) = \mathfrak{r}(\theta, \psi_{s'}^{\theta}\mathbf{q}) \exp\left(-\int_{s'}^{\theta} \mathfrak{r}(u, \psi_{s'}^{u}\mathbf{q})\ du\right) d\theta \tag{4.62}$$

We can certainly write

$$\begin{aligned}\mathcal{T}_- =&\mathbb{E}\left[\mathbb{1}_{\mathcal{E}} \int_{L(\rho^0)} \frac{\mathfrak{c}_-(T_1, \rho^*, \rho^0)}{\mathfrak{r}(T_1, \psi_{s'}^{T_1}\mathbf{q})} G\left(E_-^{\rho^*}\psi_{s'}^{T_1}\mathbf{q}, s\right)\ \beta(d\rho^*)\right]\\ =&\int_{s'}^{s}\int_{L(\rho^0)} \mathfrak{c}_-(\theta, \rho^*, \rho^0) \exp\left(-\int_{s'}^{\theta} \mathfrak{r}(u, \psi_{s'}^{u}\mathbf{q})\ du\right) G\left(E_-^{\rho^*}\psi_{s'}^{\theta}\mathbf{q}, s\right)\ \beta(d\rho^*)d\theta,\end{aligned}$$

where we have used (4.62) for the second equality. On the other hand, from the boundedness of G, we learn that the expression

$$\left|\int_{s'}^{s}\int_{L(\rho^0)} \mathfrak{c}_-(\theta, \rho^*, \rho^0) \left(\exp\left(-\int_{s'}^{\theta} \mathfrak{r}(u, \psi_{s'}^{u}\mathbf{q})du\right) - 1\right) G\left(E_-^{\rho^*}\psi_{s'}^{\theta}\mathbf{q}, s\right)\ \beta(d\rho^*)d\theta\right|,$$

is bounded above by $c_4(s-s')^2$ for some constant $c_4 = c_4(P^-, P^+, J, a^-, a^+)$. This in turn implies

$$\begin{aligned}\mathcal{T}_-\ \mathbb{1}(\mathbf{q} \notin \hat{\Omega}) =& \int_{s'}^{s}\int_{L(\rho^0)} \mathfrak{c}_-(\theta, \rho^*, \rho^0) G\left(E_-^{\rho^*}\psi_{s'}^{\theta}\mathbf{q}, s\right)\ \beta(d\rho^*)d\theta\ \ \mathbb{1}(\mathbf{q} \notin \hat{\Omega}) + R_4(\mathbf{q})(s-s')^2\\ =& \int_{s'}^{s}\int_{L(\rho^0)} \mathfrak{c}_-(\theta, \rho^*, \rho^0) G\left(E_-^{\rho^*}\psi_{s'}^{\theta}\mathbf{q}, s\right)\ \beta(d\rho^*)d\theta\\ &+ R_5(\mathbf{q})\left[(s-s')^2 + (s-s')\mathbb{1}(\mathbf{q} \in \hat{\Omega})\right],\end{aligned}$$

where $R_4(\mathbf{q}), R_5(\mathbf{q}) \le c_5$, for a constant c_5. We treat the terms $\mathcal{T}_+$ and $\mathcal{T}_i$ in the same fashion. For example,

$$\begin{aligned}\mathcal{T}_i =& \int_{s'}^{s}\int_{\rho^{i-1}}^{\rho^i} \mathfrak{f}(z_i - [\rho^{i-1}, \rho^i](\theta - s'), \theta, \rho^{i-1}, \rho^*, \rho^i) G(E_i^{\rho^*}\psi_{s'}^{\theta}\mathbf{q}, s)\ d\theta d\beta(\rho^*) + R_5(\mathbf{q})(s-s')^2\\ =& \int_{s'}^{s}\int_{\rho^{i-1}}^{\rho^i} \mathfrak{f}(z_i, \theta, \rho^{i-1}, \rho^*, \rho^i) G(E_i^{\rho^*}\psi_{s'}^{\theta}\mathbf{q}, s)\ d\theta\beta(d\rho^*) + R_6(\mathbf{q})(s-s')\end{aligned}$$

where $R_5(\mathbf{q}), R_6(\mathbf{q}) \le c_6$, for a constant c_6. Here for the last equality, we have used the Lipschitzness of the rate $\mathfrak{f}$. This in turn implies

$$\begin{aligned}\mathcal{T}_i\ \mathbb{1}(\mathbf{q} \notin \hat{\Omega}) =& \int_{s'}^{s}\int_{\rho^{i-1}}^{\rho^i} \mathfrak{f}(z_i, \theta, \rho^{i-1}, \rho^*, \rho^i) G(E_i^{\rho^*}\psi_{s'}^{\theta}\mathbf{q}, s)\ d\theta\beta(d\rho^*)\\ &+ R_7(\mathbf{q})\left[(s-s')^2 + (s-s')\mathbb{1}(\mathbf{q} \notin \hat{\Omega})\right],\end{aligned}$$

where $R_7(\mathbf{q}) \leq c_7$, for a constant c_7. From these representations of $\mathcal{T}_\pm\ \mathbb{1}(\mathbf{q} \notin \hat{\Omega})$ and $\mathcal{T}_i\ \mathbb{1}(\mathbf{q} \notin \hat{\Omega})$, (4.56), (4.58) and (4.61) we deduce

$$\begin{aligned} G(\mathbf{q}, s') = &\int_{s'}^{s} \int_{L(\rho^0)} \mathfrak{c}_-(\theta, \rho^*, \rho^0) G\left(E_-^{\rho^*} \psi_{s'}^t \mathbf{q}, s\right)\ d\theta d\rho^* \\ &+ \sum_{i=1}^{\mathbf{n}(\mathbf{q})} \int_{s'}^{s} \int_{\rho^{i-1}}^{\rho^i} \mathfrak{f}(z_i, \theta, \rho^{i-1}, \rho^*, \rho_i) G(E_i^{\rho^*} \psi_{s'}^\theta \mathbf{q}, s)\ d\theta d\rho^* \\ &+ \int_{s'}^{s} \int_{\rho^{\mathbf{n}(\mathbf{q})}}^{P^+} \mathfrak{c}_+(\rho^{\mathbf{n}(\mathbf{q})}, \rho^*, \theta) G(E_+^{\rho^*} \psi_{s'}^\theta \mathbf{q}, s)\ d\theta d\rho^* \\ &+ e^{-\int_{s'}^{s} \mathfrak{r}(\theta, \psi_{s'}^t \mathbf{q})\ d\theta}\ G\left(\psi_{s'}^s \mathbf{q}, s\right) + R_1(\mathbf{q})(s - s')^2 \\ &+ R_8(\mathbf{q}) \left[(s - s')^2 + (s - s')\mathbb{1}(\mathbf{q} \notin \hat{\Omega})\right], \end{aligned} \tag{4.63}$$

where $|R_8(\mathbf{q})| \leq c_8$, for a constant c_8. On the other-hand,

$$\begin{aligned} 1 =& e^{-\int_{s'}^{s} \mathfrak{r}(\theta, \psi_{s'}^t \mathbf{q})\ d\theta} + \int_{s'}^{s} \mathfrak{r}(\theta, \psi_{s'}^\theta \mathbf{q})\ d\theta + R_9(\mathbf{q})(s - s')^2 \\ =& e^{-\int_{s'}^{s} \mathfrak{r}(\theta, \psi_{s'}^\theta \mathbf{q})\ d\theta} + \mathbb{1}(\mathbf{q} \notin \hat{\Omega}) \int_{s'}^{s} \mathfrak{r}(\theta, \psi_{s'}^\theta \mathbf{q})\ d\theta \\ &+ R_{10}(\mathbf{q}) \left[(s - s')^2 + (s - s')\mathbb{1}(\mathbf{q} \in \hat{\Omega})\right], \end{aligned}$$

with $|R_9(\mathbf{q})|, |R_{10}(\mathbf{q})| \leq c_9(\mathbf{n}(\mathbf{q}) + 2)^2$, for a constant c_9 that is independent of $\mathbf{q}$. We now use the Lipschitzness of our rate $\mathfrak{f}$ to replace $z_i - [\rho^{i-1}, \rho^i](t - s')$ with z_i. As a result,

$$\begin{aligned} 1 - e^{-\int_{s'}^{s} \mathfrak{r}(\theta, \psi_{s'}^\theta \mathbf{q})\ d\theta} = &\int_{s'}^{s} \int_{L(\rho^0)} \mathfrak{c}_-(\theta, \rho^*, \rho^0)\ \beta(d\rho^*) d\theta \\ &+ \sum_{i=1}^{\mathbf{n}(\mathbf{q})} \int_{s'}^{s} \int_{D(\rho^{i-1}, \rho^i)} \mathfrak{f}(z_i, \theta, \rho^{i-1}, \rho^*, \rho^i)\ \beta(d\rho^*) d\theta \\ &+ \int_{s'}^{s} \int_{R(\rho^{\mathbf{n}(\mathbf{q})})} \mathfrak{c}_+(z_i, \theta, \rho^{i-1}, \rho^*, \rho^i)\ \beta(d\rho^*) d\theta \\ &+ R_{11}(\mathbf{q}) \left[(s - s')^2 + (s - s')\mathbb{1}(\mathbf{q} \in \hat{\Omega})\right], \end{aligned} \tag{4.64}$$

with again $|R_{11}(\mathbf{q})|$ bounded by a constant multiple of $(\mathbf{n}(\mathbf{q}) + 2)^2$. Here again we have used the Lipschitzness of our rate $\mathfrak{f}$ to replace $z_i - [\rho^{i-1}, \rho^i](t - s')$ with z_i. We now multiply both sides of (4.64) by $G(\mathbf{q}, s)$ and subtract the outcome from (4.63) to arrive at (4.59).

(Step 3) Fix now $n \geq 1$, and let us analyze each term of the sum in (4.59), integrated against the probability measure $\mu^n(d\mathbf{q}, s)$. We start from $\mathcal{S}'$, and focus on the spatial integration. To prepare for this, we need some definitions. Let us write $u_i,\ i \in 0, n$ for the relative velocities of the particles:

$$u_0 := v_1, \qquad u_n := -v_n, \qquad u_i := v_{i+1} - v_i.$$

We also write U for the set of particle configurations $\mathbf{z} \in \Delta^n$ that do not experience any collision in the interval $[s', s]$, and define the sets A_i and B_i, $i \in 0, n$ by:

$$A_0 = \Big\{\mathbf{z} \in \Delta^n : z_1 < a^- - u_0(s - s')\Big\}, \qquad B_0 = \Big\{\mathbf{z} \in \Delta^n : \ z_1 < a^- + |u_0|(s - s')\Big\},$$
$$A_n = \Big\{\mathbf{z} \in \Delta^n : \ a^+ - u_n(s - s') < z_n\Big\}, \qquad B_n = \Big\{\mathbf{z} \in \Delta^n : \ a^+ - |u_n|(s - s') < z_n\Big\},$$
$$A_i = \Big\{\mathbf{z} \in \Delta^n : \ z_{i+1} - z_i < -u_i(s - s')\Big\}, \qquad B_i = \Big\{\mathbf{z} \in \Delta^n : \ z_{i+1} - z_i < |u_i|(s - s')\Big\}.$$

for $i \in 1, n-1$. Note that the action of the flow $\psi^s_{s'}$ on the set U is simply a translation in the $\mathbf{z}$-space:

$$\psi^s_{s'}(\mathbf{z}, \boldsymbol{\rho}) =: (\phi^s_{s'}(\mathbf{z}), \boldsymbol{\rho}), \qquad \phi^s_{s'}(\mathbf{z}) = (z_i + v_i(s - s'))_{i=1}^n,$$

where $v_i = -[\rho^{i-1}, \rho^i]$. Then the set U can be expressed as $U = \Delta^n \setminus \bigcup_{i=0}^n A_i$. Moreover, writing $|A|$ for the Lebesgue measure of the set A, it is not hard to show that there exists a constant c_{10} such that

$$|A_i| \leq |B_i| \leq c_{10}(s - s') \ \frac{(a^+ - a^-)^n}{n!}, \tag{4.65}$$

$$|A_i \cap A_j| \leq |B_i \cap B_j| \leq c_{10}(s - s')^2 \ \frac{(a^+ - a^-)^n}{n!}. \tag{4.66}$$

In the present step, we fix $\boldsymbol{\rho} := (\rho^0, \rho^1, \cdots, \rho^n)$, satisfying $\rho^0 \prec \rho^1 \prec \cdots \prec \rho^n$, and focus on the integration with respect to the space variable $\mathbf{z} := (z_1, \cdots, z_n) \in \Delta^n$. More specifically we will show

$$\int_{\Delta^n} \mathcal{S}'(\mathbf{q}) g^n(\mathbf{z}, \boldsymbol{\rho}) \ d\mathbf{z} = \sum_{i=0}^n \mathbb{1}_{\{u_i > 0\}} \int_{B_i} G(\mathbf{q}, s) g^n(\mathbf{z}, \boldsymbol{\rho}) \ d\mathbf{z} - \sum_{i=0}^n \mathbb{1}_{\{u_i < 0\}} \int_{B_i} G(\psi^s_{s'}\mathbf{q}, s) g^n(\mathbf{z}, \boldsymbol{\rho}) \ d\mathbf{z}$$
$$+ \int_U G(\mathbf{q}, s) \left(g^n(\mathbf{z}, \boldsymbol{\rho}) - g^n(\phi^{s'}_s(\mathbf{z}), \boldsymbol{\rho})\right) \ d\mathbf{z} + \gamma(s - s') R(\boldsymbol{\rho}), \tag{4.67}$$

where R satisfies the bound $|R(\boldsymbol{\rho})| \leq c_{12} c_{11}^n / n!$, for positive constants c_{11} and c_{12} and γ is an increasing non-negative function such that $\gamma(\theta)/\theta \to 0$ as $\theta \to 0$.

To prove (4.67), we use (4.66), and the boundedness of f and ℓ, to assert

$$\int_{\Delta^n} \mathcal{S}'(\mathbf{q}) g^n(\mathbf{z}, \boldsymbol{\rho}) \ d\mathbf{z} = \mathcal{S}''(\boldsymbol{\rho}) + \sum_{i=0}^n \mathcal{S}'_i(\boldsymbol{\rho}) + (s - s')^2 R_0(\boldsymbol{\rho}), \tag{4.68}$$

with the term R_0 satisfying $|R_0| \leq c_{13} c_{11}^n / n!$, for constants c_{11} and c_{13}, and the other terms given by

$$\mathcal{S}'_i(\boldsymbol{\rho}) = \int_{A_i} \exp\left(- \int_{s'}^s \mathfrak{r}(u, \psi^u_{s'}\mathbf{q}) \ du\right) \left(G(\mathbf{q}, s) - G(\psi^s_{s'}\mathbf{q}, s)\right) g^n(\mathbf{z}, \boldsymbol{\rho}) \ d\mathbf{z},$$
$$\mathcal{S}''(\boldsymbol{\rho}) = \int_U \exp\left(- \int_{s'}^s \mathfrak{r}(u, \psi^u_{s'}\mathbf{q}) \ du\right) \left(G(\mathbf{q}, s) - G(\psi^s_{s'}\mathbf{q}, s)\right) g^n(\mathbf{z}, \boldsymbol{\rho}) \ d\mathbf{z}.$$

Observe that the replacement of the exponential

$$\exp\left(- \int_{s'}^s \mathfrak{r}(u, \psi^u_{s'}\mathbf{q}) \ du\right),$$

with

$$\text{either} \quad \exp\left(-\int_{s'}^{s} \mathfrak{r}(u,\mathbf{q})\ du\right) \quad \text{or} \quad \exp\left(-\int_{s'}^{s} \mathfrak{r}(u,\psi_{s'}^{s}\mathbf{q})\ du\right)$$

results in an error of size $O((n+2)(s-s')^2)$ by the Lipschitzness of the rates with respect to the space variable. Because of this, we can write,

$$\begin{aligned}\mathcal{S}''(\boldsymbol{\rho}) = &\int_U \exp\left(-\int_{s'}^{s} \mathfrak{r}(u,\mathbf{q})\ du\right) G(\mathbf{q},s) g^n(\mathbf{z},\boldsymbol{\rho})\ d\mathbf{z}\\ &- \int_U \exp\left(-\int_{s'}^{s} \mathfrak{r}(u,\psi_{s'}^{s}\mathbf{q})\ du\right) G(\psi_{s'}^{s}\mathbf{q},s) g^n(\mathbf{z},\boldsymbol{\rho})\ d\mathbf{z} + (s-s')^2 R_1(\boldsymbol{\rho}),\end{aligned}$$

with the term R_1 satisfying $|R_1(\boldsymbol{\rho})|\leq c_{14}(n+2)c_{11}^n/n!$, for a constant c_{14}. By a change of variables,

$$\begin{aligned}\mathcal{S}''(\boldsymbol{\rho}) = &\int_U \exp\left(-\int_{s'}^{s} \mathfrak{r}(u,\mathbf{q})\ du\right) G(\mathbf{q},s) g^n(\mathbf{z},\boldsymbol{\rho})\ d\mathbf{z}\\ &- \int_{\phi_{s'}^{s}(U)} \exp\left(-\int_{s'}^{s} \mathfrak{r}(u,\mathbf{q}) du\right) G(\mathbf{q},s) g^n(\phi_{s}^{s'}(\mathbf{z}),\boldsymbol{\rho})\ d\mathbf{z} + (s-s')^2 R_1(\boldsymbol{\rho}),\end{aligned}$$

where

$$\phi_{s}^{s'}(\mathbf{z}) = \left(\phi_{s'}^{s}\right)^{-1}(\mathbf{z}) = \left(z_i - v_i(s-s')\right)_{i=1}^{n}, \qquad \psi_{s}^{s'}(\mathbf{z}) = \left(\phi_{s}^{s'}(\mathbf{z}),\boldsymbol{\rho}\right), \qquad \psi_{s}^{u} := \psi_{s'}^{u}\circ\psi_{s}^{s'},$$

are the reverse flows. This allows us to assert

$$\mathcal{S}''(\boldsymbol{\rho}) = \mathcal{S}_1''(\boldsymbol{\rho}) + \mathcal{S}_2''(\boldsymbol{\rho}) + (s-s')^2 R_1(\boldsymbol{\rho}), \tag{4.69}$$

where

$$\begin{aligned}\mathcal{S}_1''(\boldsymbol{\rho}) &:= \int_{\phi_{s'}^{s}(U)} \exp\left(-\int_{s'}^{s} \mathfrak{r}(u,\mathbf{q})\ du\right) G(\mathbf{q},s)\left(g^n(\mathbf{z},\boldsymbol{\rho}) - g^n(\phi_{s}^{s'}(\mathbf{z}),\boldsymbol{\rho})\right) d\mathbf{z},\\ \mathcal{S}_2''(\boldsymbol{\rho}) &:= \left(\int_U - \int_{\phi_{s'}^{s}(U)}\right) \exp\left(-\int_{s'}^{s} \mathfrak{r}(u,\mathbf{q})\ du\right) G(\mathbf{q},s) g^n(\mathbf{z},\boldsymbol{\rho})\ d\mathbf{z}.\end{aligned}$$

We now compare the set U with its translate $\phi_{s}^{s'}(U)$. Observe,

$$\begin{aligned}U =&\Big\{\mathbf{z}\in\overline{\Delta^n}:\ z_1 > \max(a^-, a^- - u_0(s-s')),\ z_n < \min(a^+, a^+ + u_n(s-s')),\\ &\qquad z_{i+1} - z_i > \max(0, -u_i(s-s')) \ \text{ for } \ i\in 1,n-1\Big\},\\ \phi_{s'}^{s}(U) =&\Big\{\mathbf{z}\in\overline{\Delta^n}:\ z_1 > \max(a^-, a^- + u_0(s-s')),\ z_n < \min(a^+, a^+ - u_n(s-s'))\\ &\qquad z_{i+1} - z_i > \max(0, u_i(s-s')) \ \text{ for } \ i\in 1,n-1\Big\}.\end{aligned}$$

Hence the symmetric difference of the sets U and $\phi_{s'}^{s}(U)$ can be represented as

$$U\Delta\ \phi_{s'}^{s}(U) = \cup_{i=0}^{n} B_i. \tag{4.70}$$

Note that we can replace the exponential with 1 at a cost of $O((n+1)(s-s'))$. From this and (4.65) we deduce,

$$\mathcal{S}_2''(\boldsymbol{\rho}) = \left(\int_U - \int_{\phi^s_{s'}(U)}\right) G(\mathbf{q}, s) g^n(\mathbf{z}, \boldsymbol{\rho})\ d\mathbf{z} + (s-s')^2 R_2(\boldsymbol{\rho}),$$

with the term R_2 satisfying $|R_2(\boldsymbol{\rho})| \leq c_{15}(n+1)c_{11}^n/n!$, for a constant c_{15}. This, (4.70) and (4.66) allow us to ignore the overlaps of the sets B_i, $i \in 1, n$, so that we can write

$$\mathcal{S}_2''(\boldsymbol{\rho}) = \sum_{i=0}^{n} \text{sign}(u_i) \int_{B_i} G(\mathbf{q}, s) g^n(\mathbf{z}, \boldsymbol{\rho})\ d\mathbf{z} + (s-s')^2 R_3(\boldsymbol{\rho}), \tag{4.71}$$

with the term R_3 satisfying $|R_3(\boldsymbol{\rho})| \leq c_{16}(n+1)c_{11}^n/n!$, for a constant c_{16}.

We now turn our attention to $\mathcal{S}_1''$. By the Taylor expansion we have that

$$\frac{g^n(\phi^{s'}_s(\mathbf{z}), \boldsymbol{\rho})}{g^n(\mathbf{z}, \boldsymbol{\rho})} = 1 + (s'-s) \sum_{i=1}^{n} v_i \frac{g^n_{z_i}(\mathbf{z}, \boldsymbol{\rho})}{g^n(\mathbf{z}, \boldsymbol{\rho})} + R_4(\mathbf{z}, \boldsymbol{\rho}, s-s')\gamma(s-s'), \tag{4.72}$$

where $|R_4| \leq c_{17}(n+2)$, for a constant c_{12}, and γ is a function satisfying $\gamma(\theta)/\theta \to 0$ as $\theta \to 0$. In particular, there exists a constant c_{18} such that

$$\frac{|g^n(\phi^{s'}_s(\mathbf{z}), \boldsymbol{\rho}) - g^n(\mathbf{z}, \boldsymbol{\rho})|}{g^n(\mathbf{z}, \boldsymbol{\rho})} \leq c_{18}(n+2)(s-s'). \tag{4.73}$$

This allows us to make two changes in $\mathcal{S}_1''$ at a cost of a constant multiple of $\gamma(s-s')$, namely replacing the exponential with 1, and replacing the set $\phi^s_{s'}(U)$ with U. As a result,

$$\mathcal{S}_1''(\boldsymbol{\rho}) = \int_U G(\mathbf{q}, s) \left(g^n(\mathbf{z}, \boldsymbol{\rho}) - g^n(\phi^{s'}_s(\mathbf{z}), \boldsymbol{\rho})\right)\ d\mathbf{z} + R_5(\boldsymbol{\rho})\gamma(s-s'), \tag{4.74}$$

with again the term R_5 satisfying $|R_5(\boldsymbol{\rho})| \leq c_{20}(n+1)c_{11}^n/n!$, for a constant c_{20}. Coming back to the second term $\mathcal{S}_i'$ in (4.68), we see use (4.65) to replace the exponential term of the integrand with 1:

$$\mathcal{S}_i'(\boldsymbol{\rho}) = \int_{A_i} (G(\mathbf{q}, s) - G(\psi^s_{s'}\mathbf{q}, s)) g^n(\mathbf{z}, \boldsymbol{\rho})\ d\mathbf{z} + R_6(\boldsymbol{\rho})\gamma(s-s'), \tag{4.75}$$

with the term R_6 satisfying $|R_6(\boldsymbol{\rho})| \leq c_{21}(n+1)c_{11}^n/n!$, for a constant c_{21}. Now, notice that for $i \in 0, n$ if $u_i > 0$, then $A_i = \emptyset$, otherwise if $u_i < 0$, then $A_i = B_i$. From this, (4.68), (4.69), (4.71), (4.74), and (4.75), we obtain (4.67)

(Step 4) In this step, we use (4.67) to show

$$\int_{\Omega^n} \mathcal{S}'(\mathbf{q})\ \mu^n(d\mathbf{q}, s) = -\ (s-s') \left(\Gamma^d_n(s) + \Gamma^t_n(s)\right) + \hat{R}^n(s, s')\gamma(s-s'), \tag{4.76}$$

with $\hat{R}^n$ satisfying $|\hat{R}^n| \leq c_{22}(n+1)^2 c_{11}^n/n!$, for a constant c_{22}. Put

$$\hat{B}_i = B_i \setminus \cup_{j \neq i} B_j,$$

so that the sets $(\hat{B}_i : i \in 0, n)$ are mutually disjoint. The bound (4.66) allows us to replace B_i with $\hat{B}_i$ in (4.67) at a small cost:

$$\int_{\overline{\Delta^n}} \mathcal{S}'(\mathbf{q}) g^n(\mathbf{z}, \boldsymbol{\rho})\, d\mathbf{z} = \sum_{i=0}^{n} \left(\widehat{\mathcal{S}}_i^+(\boldsymbol{\rho}) - \widehat{\mathcal{S}}_i^-(\boldsymbol{\rho}) \right) + \mathcal{S}'''(\boldsymbol{\rho}) + R_8(\boldsymbol{\rho})\gamma(s - s'), \tag{4.77}$$

where R_7 satisfies $|R_7(\boldsymbol{\rho})| \leq c_{23}(n+1)c_{11}^n/n!$, for a constant c_{23}, and

$$\begin{aligned}
\widehat{\mathcal{S}}_i^+(\boldsymbol{\rho}) &= \mathbb{1}_{\{u_i > 0\}} \int_{\hat{B}_i} G(\mathbf{q}, s) g^n(\mathbf{z}, \boldsymbol{\rho})\, d\mathbf{z}, \\
\widehat{\mathcal{S}}_i^-(\boldsymbol{\rho}) &= \mathbb{1}_{\{u_i < 0\}} \int_{\hat{B}_i} G(\psi_{s'}^s \mathbf{q}, s) g^n(\mathbf{z}, \boldsymbol{\rho})\, d\mathbf{z}, \\
\mathcal{S}'''(\boldsymbol{\rho}) &= \int_U G(\mathbf{q}, s) \left(g^n(\mathbf{z}, \boldsymbol{\rho}) - g^n(\phi_s^{s'}(\mathbf{z}), \boldsymbol{\rho}) \right)\, d\mathbf{z}.
\end{aligned}$$

We establish (4.77) by proving

$$\int \sum_{i=0}^{n} \left(\widehat{\mathcal{S}}_i^+(\boldsymbol{\rho}) - \widehat{\mathcal{S}}_i^-(\boldsymbol{\rho}) \right)\, \beta(d\boldsymbol{\rho}) = -(s - s')\Gamma_n^d(s) + \hat{R}_0(s, s')(s - s')^2, \tag{4.78}$$

$$\int \mathcal{S}'''(\boldsymbol{\rho})\, \beta(d\boldsymbol{\rho}) = -(s - s')\Gamma_n^t(s) + \hat{R}_1(s, s')(s - s')^2, \tag{4.79}$$

with $|\hat{R}_0|, |\hat{R}_1| \leq c_{24}(n+1)c_{11}^n/n!$, for a constant c_{24}.

To prove (4.78), first observe that if $\sigma := \sigma(\mathbf{q}, s')$ denotes the first collision time of the deterministic flow starting at time s' at $\mathbf{q} = (\mathbf{z}, \boldsymbol{\rho})$, and $\sigma' = \sigma \wedge T_1$, then for sure $\sigma = \sigma' < s$ provided that $u_i < 0$, $\mathbf{z} \in \hat{B}_i$, and no stochastic jump occurs in the interval $[s', s]$ (equivalently, $T_1 > s$). From this, the strong Markov property, (4.56), and Lemma 4.3.2 we deduce

$$\begin{aligned}
G(\psi_{s'}^s \mathbf{q}, s) &= \mathbb{E}[G(\Psi_s^t \psi_{s'}^s \mathbf{q})] = \mathbb{E}[G(\Psi_s^t \psi_{s'}^s \mathbf{q})\ \mathbb{1}_{\mathcal{E}^c}] + O(s - s') \\
&= \mathbb{E}[G(\Psi_{\sigma'}^t \psi_{s'}^{\sigma'} \mathbf{q})\ \mathbb{1}_{\mathcal{E}^c}] + O(s - s') = \mathbb{E}[G(\Psi_{\sigma'}^t \psi_{s'}^{\sigma'} \mathbf{q})] + O(s - s') \\
&= \mathbb{E}[G(\psi_{s'}^{\sigma'} \mathbf{q}, \sigma')] + O(s - s') = G(\psi_{s'}^{\sigma'} \mathbf{q}, s) + O(s - s'),
\end{aligned}$$

provided that $u_i < 0$, and $\mathbf{z} \in \hat{B}_i$. On $\hat{B}_i$, and when $u_i < 0$, we have that

$$G(\psi_{s'}^{\sigma} \mathbf{q}, s) = G(E_i^{\rho^i} \mathbf{q}_i, s),$$

for $i \in 0, n$, with $\mathbf{q}_i = (\mathbf{z}_i, \boldsymbol{\rho}_i)$ the configuration $\mathbf{q}$ with the particle i removed, and where $E_0^{\rho^0}$ and $E_n^{\rho^n}$ should be understood as $E_-^{\rho^0}$ and $E_+^{\rho^n}$. After replacing $G(\psi_{s'}^s \mathbf{q}, s)$ with $G(E_i^{\rho^i} \mathbf{q}_i, s)$ at a cost of $O(s - s')$, we replace back the set $\hat{B}_i$ with B_i. This allows us to write

$$\widehat{\mathcal{S}}_i^-(\boldsymbol{\rho}) = (s - s')\mathcal{T}_i^-(\boldsymbol{\rho}) + R_9(\rho)(s - s')^2, \tag{4.80}$$

where R_9 satisfies $|R_9|\leq c_{25}(n+1)c_{11}^n/n!$, for a constant c_{25}, and

$$\begin{aligned}\mathcal{T}_0^-(\boldsymbol{\rho}) =& u_0^- \frac{\ell(a^-,s,\rho^0)f(a^-,s,\rho^0,\rho^1)}{\ell(a^-,s,\rho^1)}\int_{\Delta^{n-1}} G(E_-^{\rho^0}\mathbf{q}_0,s)g^{n-1}(\mathbf{z}_0,\boldsymbol{\rho}_0)\ d\mathbf{z}_0,\\ \mathcal{T}_i^-(\boldsymbol{\rho}) =& u_i^- \frac{f(z_i,s,\rho^{i-1},\rho^i)f(z_i,s,\rho^i,\rho^{i+1})}{f(z_i,s,\rho^{i-1},\rho^{i+1})}\int_{\Delta^{n-1}} G(E_i^{\rho^i}\mathbf{q}_i,s)g^{n-1}(\mathbf{z}_i,\boldsymbol{\rho}_i)\ d\mathbf{z}_i,\\ \mathcal{T}_n^-(\boldsymbol{\rho}) =& u_n^- f(a^+,s,\rho^{n-1},\rho^n)\int_{\Delta^{n-1}} G(E_i^{\rho^n}\mathbf{q}_n,s)g^{n-1}(\mathbf{z}_n,\boldsymbol{\rho}_n)\ d\mathbf{z}_n,\end{aligned}$$

for $i\in 1,n-1$, where $u^- = \mathbb{1}_{\{u<0\}}|u|$.

The terms $\widehat{\mathcal{S}}_i^+$ can be treated likewise: Fix $i\in 0,n$, and define the time $\sigma' := s-\frac{z_{i+1}-z_i}{u_i}$, with the convention that $z_0=a^-$ and $z_{n+1}=a^+$. When $\mathbf{z}\in\hat{B}_i$ and $u_i>0$, there would be no collision in the interval $[\sigma',s]$, and the reverse flow $\psi_s^{\sigma'}\mathbf{q}$ is well-defined for $\mathbf{q}=(\mathbf{z},\boldsymbol{\rho})$. We then define $\tilde{\mathbf{q}}$ to be $\tilde{\mathbf{q}}=\psi_s^{\sigma'}\mathbf{q}$. By similar arguments we have

$$G(\mathbf{q},s)=G(\tilde{\mathbf{q}},\sigma')+O(s-s')=G(\tilde{\mathbf{q}},s)+O(s-s').$$

Again, we see that $G(\tilde{\mathbf{q}},s)=G(E_i^{\rho^i}\mathbf{q}_i,s)$, and hence we get the analog of (4.80), namely

$$\widehat{\mathcal{S}}_i^+(\boldsymbol{\rho})=(s-s')\mathcal{T}_i^+(\boldsymbol{\rho})+R_{10}(\rho)(s-s')^2, \tag{4.81}$$

where R_{10} satisfies $|R_{10}|\leq c_{25}(n+1)c_{11}^n/n!$, and the expression for $\mathcal{T}_i^+$ is the same as $\mathcal{T}_i^-$, except that u_i^- is replaced with u_i^+. We integrate both sides of (4.80) and (4.81) against β, and take the difference to arrive at (4.78). After integrating out (4.78) and (4.82) with respect to $\boldsymbol{\rho}$ and by relabeling ρ^i as ρ^* and ρ^j for $j>i+1$ by ρ^{j-1}, we arrive at (4.76).

We now focus on (4.79). From (4.72) and the straightforward computation

$$\frac{g^n_{z_i}(\mathbf{z},\boldsymbol{\rho})}{g^n(\mathbf{z},\boldsymbol{\rho})}=\frac{f_z(z_i,s,\rho^{i-1},\rho^i)}{f(z_i,s,\rho^{i-1},\rho^i)}+(z_i,s,\rho^i)-(z_i,s,\rho^{i-1}),$$

we deduce

$$\mathcal{S}'''(\boldsymbol{\rho})=-(s-s')\sum_{i=1}^n\mathcal{S}_i'''(\boldsymbol{\rho})+R_{10}(\boldsymbol{\rho})\gamma(s-s'), \tag{4.82}$$

where R_{10} satisfies $|R_{10}(\boldsymbol{\rho})|\leq c_{26}(n+1)c_{11}^n/n!$, for a constant c_{26}, and

$$\mathcal{S}_i'''(\boldsymbol{\rho})=\int_U G(\mathbf{q},s)[\rho^{i-1},\rho^i]\left(\frac{f_z(z_i,s,\rho^{i-1},\rho^i)}{f(z_i,s,\rho^{i-1},\rho^i)}+(z_i,s,\rho^i)-(z_i,s,\rho^{i-1})\right)g^n(\mathbf{z},\boldsymbol{\rho})\ d\mathbf{z}.$$

Note that (4.65) provides us a bound on the Lebesgue measure of the set U^c. This bound and (4.73) allow us to replace the domain of integration from U to the whole simplex Δ^n at a cost of replacing R_{10} with R_{11}, that satisfies a similar bound. After such a replacement, we integrate both sides with respect to β to deduce (4.79).

(Final Step) Note that our error terms are bounded by constant multiples of $\bar{c}_n\gamma(s-s')$, with $\bar{c}_n=(n+1)^3c_{11}^n/n!$. Since $\sum_n\bar{c}_n<\infty$, and $\gamma(\theta)/\theta\to 0$ as $\theta\to 0$, these error terms

can be ignored as we calculate the limit in (4.52). On account of this, (4.59), and (4.76), it remains to verify

$$\int_{\Omega^n} \mathcal{S}^{\pm}(\mathbf{q})\ \mu^n(d\mathbf{q},s) = -\,(s-s')\Gamma_n^{\pm}(s) + (s-s')^2 R^{\pm}(s), \tag{4.83}$$
$$\sum_{i=1}^{n}\int_{\Omega^n} \mathcal{S}_i(\mathbf{q})\ \mu^n(d\mathbf{q},s) = -\,(s-s')\Gamma_n^{\mathfrak{f}}(s) + (s-s')^2 R^{\mathfrak{f}}(s),$$

with $R^{\pm}, R^{f}$ satisfying $|R^{\pm}|, |R^{\mathfrak{f}}| \leq c_{27}(n+1)c_{11}^n/n!$, for a constant c_{27}.

We only verify (4.83) in the case of $\mathcal{S}^{-}$, as the other cases can be treated in the same fashion. For this, first observe that because of (4.60), we may write

$$\int_{\Omega^n} \mathcal{S}^{-}(\mathbf{q})\ \mu^n(d\mathbf{q},s) = \mathcal{S}_1^{-} - \mathcal{S}_2^{-} + R_n^{-}(s)(s-s')^2, \tag{4.84}$$

where R_0^{-} satisfies $|R_0^{-}| \leq c_{28}(n+1)c_{11}^n/n!$, for a constant c_{28}, and

$$\mathcal{S}_1^{-} = \int_{\Omega^n}\int_{s'}^{s}\int_{L(\rho^0)} \mathfrak{c}_{-}(\theta,\rho^*,\rho^0)G(\mathbf{q},s)\ \beta(d\rho^*)d\theta\ \mu^n(d\mathbf{q},s),$$
$$\mathcal{S}_2^{-} = \int_{\Omega^n}\int_{s'}^{s}\int_{L(\rho^0)} \mathfrak{c}_{-}(\theta,\rho^*,\rho^0)G(E_0^{\rho^*}\psi_{s'}^{\theta}\mathbf{q},s)\ 1\!\!1(\mathbf{q}\notin\hat{\Omega})\ \beta(d\rho^*)d\theta\ \mu^n(d\mathbf{q},s).$$

On the other-hand, for $\mathbf{q}\notin\hat{\Omega}$, the the flow $\psi_{s'}^{t}$ experiences no collision, and is just a translation that preserves the volume. Again focusing on the spatial integration first, we certainly have

$$\int_{\Delta^n} G(E_0^{\rho^*}\psi_{s'}^{\theta}\mathbf{q},s)1\!\!1(\mathbf{q}\notin\hat{\Omega})g^n(\mathbf{z},\boldsymbol{\rho})\ d\mathbf{z} = \int_{\Delta^n} G(E_0^{\rho^*}\mathbf{q},s)1\!\!1(\psi_{s'}^{\theta}\mathbf{q}\notin\hat{\Omega})g^n(\phi_{\theta}^{s'}\mathbf{z},\boldsymbol{\rho})\ d\mathbf{z}.$$

We can then use (4.73) (with s replaced with θ) to replace $g^n(\phi_{\theta}^{s'}\mathbf{z},\boldsymbol{\rho})$ with $g^n(\mathbf{z},\boldsymbol{\rho})$ at a cost that is bounded by a constant multiple of $\bar{c}_n(s-s')^2$. As in (4.60), we can readily show that at a cost of $O((s-s')^2)$, we can now drop $1\!\!1(\psi_{s'}^{\theta}\mathbf{q}\notin\hat{\Omega})$. From all this we conclude

$$\mathcal{S}_2^{-} = \int_{\Omega^n}\int_{s'}^{s}\int_{L(\rho^0)} \mathfrak{c}_{-}(\theta,\rho^*,\rho^0)G(E_0^{\rho^*}\mathbf{q},s)\ 1\!\!1(\mathbf{q}\notin\hat{\Omega})\ \beta(d\rho^*)d\theta\ \mu^n(d\mathbf{q},s) + \hat{R}_n^{-}(s)(s-s')^2,$$

with $|\hat{R}_n^{-}| \leq c_{29}\bar{c}_n$, for a constant c_{29}. From this and (4.84) we deduce (4.83), completing the proof of our theorem. □

4.3.3 Proof of Theorem 4.3.1

(Step 1) We need to check that

$$\lim_{s'\to s}\frac{1}{s-s'}\left(\int G(\mathbf{q},s)\mu(d\mathbf{q},s) - \int G(\mathbf{q},s')\ \mu(d\mathbf{q},s')\right) = 0.$$

We can certainly write

$$\int G(\mathbf{q},s)\ \mu(d\mathbf{q},s) - \int G(\mathbf{q},s')\mu(d\mathbf{q},s') = \mathcal{X}_1(s',s) + \mathcal{X}_2(s',s) - \mathcal{X}_3(s',s), \tag{4.85}$$

where

$$\begin{aligned}
\mathcal{X}_1(s',s) &= \int (G(\mathbf{q},s) - G(\mathbf{q},s'))\ \mu(d\mathbf{q},s),\\
\mathcal{X}_2(s',s) &= \int G(\mathbf{q},s)\ (\mu(d\mathbf{q},s) - \mu(d\mathbf{q},s')),\\
\mathcal{X}_3(s',s) &= \int ((G(\mathbf{q},s) - G(\mathbf{q},s'))\ (\mu(d\mathbf{q},s) - \mu(d\mathbf{q},s')).
\end{aligned}$$

We work out $\mathcal{X}_2$ by differentiating μ with respect to the time s. Evidently

$$\lim_{s'\uparrow s}(s-s')^{-1}\mathcal{X}_2(s',s) = \int G(\mathbf{q},s)\ \dot{\mu}(d\mathbf{q},s), \tag{4.86}$$

where $\dot{\mu}$ represents the s-derivative of μ. We may write $\dot{\mu}^n = X^n\mu^n$, where

$$X^n(\mathbf{q},s) = \frac{\dot{\ell}(a^-,s,\rho^0)}{\ell(a^-,s,\rho^0)} + \sum_{i=1}^{n}\frac{f_s(z_i,s,\rho^{i-1},\rho^i)}{f(z_i,s,\rho^{i-1},\rho^i)} - \sum_{i=0}^{n}\int_{z_i}^{z_{i+1}}\lambda_s(z,s,\rho^i)dz.$$

Now using the kinetic equations verified by both ℓ and f, let us find an explicit expression of the Radon-Nidokym derivative X^n. For the first term we have

$$\frac{\dot{\ell}(a^-,s,\rho^0)}{\ell(a^-,s,\rho^0)} = \int_{L(\rho^0)}[\rho^*,\rho^0]\ \frac{\ell(a^-,s,\rho^*)f(a^-,s,\rho^*,\rho^0)}{\ell(a^-,s,\rho^0)}\ \beta(d\rho^*) - A(a^-,s,\rho^0).$$

For the second term we get for $i\in 1,n$,

$$\begin{aligned}
\frac{f_s(z_i,s,\rho^{i-1},\rho^i)}{f(z_i,s,\rho^{i-1},\rho^i)} =& [\rho^{i-1},\rho^i]\ \frac{f_z(z_i,s,\rho^{i-1},\rho^i)}{f(z_i,s,\rho^{i-1},\rho^i)}\\
&+ \int_{D(\rho^{i-1},\rho^i)}[\rho^{i-1},\rho^*,\rho^i]\ \frac{f(z_i,s,\rho^{i-1},\rho^*)f(z_i,s,\rho^*,\rho^i)}{f(z_i,s,\rho^{i-1},\rho^i)}\ \beta(d\rho^*)\\
&+ [\rho^{i-1},\rho^i](\lambda(z_i,s,\rho^i) - \lambda(z_i,s,\rho^{i-1})) - (A(z_i,s,\rho^i) - A(z_i,s,\rho^{i-1})).
\end{aligned}$$

For the last term, let us first show $\lambda_s = A_z$. To see this, observe

$$\begin{aligned}
\lambda_s(z,s,\rho) =& \int_{R(\rho)} f_s(z,s,\rho,\rho^+)\ \beta(d\rho^+) = \int_{R(\rho)}[\rho,\rho^+]f_z(z,s,\rho,\rho^+)\ \beta(d\rho^+)\\
&+ \iint_{\{\rho\prec\rho^*\prec\rho\}}[\rho,\rho^*,\rho^+]f(z,s,\rho,\rho^*)f(z,s,\rho^*,\rho^+)\ \beta(d\rho^*)\ \beta(d\rho^+)\\
&+ \int_{R(\rho)}[\rho,\rho^+](\lambda(z,s,\rho^+) - \lambda(z,s,\rho)) + (A(z,s,\rho) - A(z,s,\rho^+))f(z,s,\rho,\rho^+)\ \beta(d\rho^+).
\end{aligned}$$

By integrating out ρ^+, we get that the double integral equals to

$$\int_{R(\rho)} (f(z,s,\rho,\rho^*)A(z,s,\rho^*) - [\rho,\rho^*]f(z,s,\rho,\rho^*)\lambda(z,s,\rho^*))\ \beta(d\rho^*).$$

However ρ^* and ρ^+ are both just dummy variables in our integrals, so summing over the terms, we get

$$\begin{aligned}\lambda_s(z,s,\rho) = &\int_{R(\rho)} [\rho,\rho^+]f_z(z,s,\rho,\rho^+)\ \beta(d\rho^+) - \lambda(z,s,\rho)\int_{R(\rho)} [\rho,\rho^+]f(z,s,\rho,\rho^+)\ \beta(d\rho^+)\\ &+ A(z,s,\rho)\int_{R(\rho)} f(z,s,\rho,\rho^+)\ \beta(d\rho^+) = \int_{R(\rho)} [\rho,\rho^+]f_z(z,s,\rho,\rho^+)\ \beta(d\rho^+),\end{aligned}$$

confirming our claim $\lambda_s = A_z$. As a result,

$$\begin{aligned}\int_{z_i}^{z_{i+1}} \lambda_s(s,z,\rho^i)\ dz &= \int_{R(\rho)} [\rho^i,\rho^+]\ \beta(d\rho^+)\int_{z_i}^{z_{i+1}} f_z(z,s,\rho^i,\rho^+)\ dz\\ &= A(z_{i+1},s,\rho^i) - A(z_i,s,\rho^i).\end{aligned}$$

Summing over everything, we get that

$$\begin{aligned}X^n(\mathbf{q}) = &- A(a^+,s,\rho^n) + \int_{L(\rho^0)} [\rho^*,\rho^0]\frac{\ell(a^-,s,\rho^*)f(a^-,s,\rho^*,\rho^0)}{\ell(a^-,s,\rho^0)}\ \beta(d\rho^*)\\ &+ \sum_{i=1}^n [\rho^{i-1},\rho^i]\left(\frac{f_z(z_i,s,\rho^{i-1},\rho^i)}{f(z_i,s,\rho^{i-1},\rho^i)} + \lambda(z_i,s,\rho^i) - \lambda(z_i,s,\rho^{i-1})\right)\\ &+ \sum_{i=1}^n \int_{D(\rho^{i-1},\rho^i)} [\rho^{i-1},\rho^*,\rho^i]\frac{f(z_i,s,\rho^{i-1},\rho^*)f(z_i,s,\rho^*,\rho^i)}{f(z_i,s,\rho^{i-1},\rho^i)}\ \beta(d\rho^*).\end{aligned} \tag{4.87}$$

(Step 2) From (4.87), we learn that there is a uniform constant $C > 0$ such that $|X^n(\mathbf{q})| \leq C(n+1)$. Hence using this observation and Theorem 4.3.2 we get

$$\left|\int_\Omega (G(\mathbf{q},s) - G(\mathbf{q},s'))(\mu(d\mathbf{q},s) - \mu(d\mathbf{q},s'))\right| \leq C(s-s')^2\int_\Omega (\mathbf{n}(\mathbf{q})+2)^3\ \mu(d\mathbf{q},s).$$

As a result,

$$\lim_{s'\uparrow s}(s-s')^{-1}\mathcal{X}_3(s',s) = 0.$$

Because of this, (4.85), and (4.86) we are done if we can show

$$\lim_{s'\to s}(s-s')^{-1}\mathcal{X}_1(s',s) = -\sum_{n=0}^\infty \int_{\Omega^n} X^n(\mathbf{q})G(\mathbf{q},s)\ \mu^n(d\mathbf{q},s). \tag{4.88}$$

By (4.87),

$$\int_{\Omega^n} X^n(\mathbf{q})G(\mathbf{q},s)\ \mu^n(d\mathbf{q},s) = \lambda_n^-(s) + \Gamma_n^t(s) + \lambda_n^{\mathfrak{f}}(s) + \lambda_n^+(s),$$

where

$$\lambda_n^-(s) = \int_{\Omega^n}\int_{L(\rho^0)}[\rho^*,\rho^0]\frac{f(a^-,s,\rho^*,\rho^0)\ell(a^-,s,\rho^*)}{\ell(a^-,s,\rho^0)}G(\mathbf{q},s)\ \beta(d\rho^*)\ \mu^n(d\mathbf{q},s),$$
$$\lambda_n^+(s) = -\int_{\Omega^n}\int_{R(\rho^n)}[\rho^n,\rho^*]f(a^+,s,\rho^n,\rho^*)G(\mathbf{q},s)\ \beta(d\rho^*)\ \mu^n(d\mathbf{q},s),$$
$$\lambda_n^{\mathfrak{f}}(s) = \int_{\Omega_L^n}\sum_{i=1}^n\int_{D(\rho^{i-1},\rho^i)}[\rho^{i-1},\rho^*,\rho^i]\frac{f(z_i,s,\rho^{i-1},\rho^*)f(z_i,s,\rho^*,\rho^i)}{f(z_i,s,\rho^{i-1},\rho^i)}G(\mathbf{q},s)\ \beta(d\rho^*)\mu^n(d\mathbf{q},s),$$

where of course $\lambda_0^{\mathfrak{f}}(s) = \Gamma_0^t(s) = 0$. On account of Theorem 3.3, we will be done if we show that for any $n \geq 0$, the following equalities hold:

$$\lambda_n^-(s) = \Gamma_n^-(s) + \Gamma_{n+1}^{d,-}(s), \tag{4.89}$$
$$\lambda_n^+(s) = \Gamma_n^+(s) + \Gamma_{n+1}^{d,+}(s),$$
$$\lambda_n^{\mathfrak{f}}(s) = \Gamma_n^{\mathfrak{f}}(s) + \Gamma_{n+1}^{d,\mathfrak{f}}(s).$$

We will verify this only for the first equality, as the others are done in a similar fashion. By the construction of our Markov process, we know that when $[\rho^*,\rho^0] > 0$, then $G(E_-^{\rho^*}\mathbf{q},s) = G(\mathbf{q},s)$ (as in this case the particle at $z = a^-$ corresponds to an exit of the interval $[a^-,a^+]$ and becomes irrelevant instantaneously), thus by splitting the two cases whether $[\rho^*,\rho^0]$ is negative or nonnegative we get

$$\Gamma_{n+1}^{d,-}(s) = \int_{\Omega^n}\int_{L(\rho^0)}[\rho^*,\rho^0]^+\frac{\ell(a^-,s,\rho^*)f(a^-,s,\rho^*,\rho^0)}{\ell(a^-,s,\rho^0)}G(\mathbf{q},s)\ \beta(d\rho^*)\mu^n(d\mathbf{q},s)$$
$$-\int_{\Omega^n}\int_{L(\rho^0)}\mathfrak{c}_-(s,\rho^*,\rho^0)G(E_-^{\rho^*}\mathbf{q},s)\ \beta(d\rho^*)\mu^n(d\mathbf{q},s).$$

This immediately implies the first identity in (4.89).

For the terms corresponding to the fragmentation i.e., $\lambda_n^{\mathfrak{f}}$ and $\Gamma_n^{\mathfrak{f}}$, we use now the observation that when $[\rho^{i-1},\rho^*,\rho^i] > 0$, then $G(E_i^{\rho^*}\mathbf{q},s) = G(\mathbf{q},s)$ to get similarly the desired equality. □

4.3.4 Proof of the genericity of the tessellation $\mathbf{X}_\Lambda$

We now state and prove a Proposition that guarantees the genericity of the tessellation $\mathbf{X}_\Lambda$, which is induced by the process $\mathbf{q}(t) = (q_i(t):\ 1 \leq i \leq \mathbf{n}(t))$, where $\mathbf{n}(t) := \mathbf{n}(\mathbf{q}(t))$.

Proposition 4.3.6 *Under the assumptions of Theorem 1.1, the probability of the occurrence of the event $z_{i-1}(t) = z_i(t) = z_{i+1}(t)$ is zero. In words, no three particles arrive at the same location almost surely.*

Proof. (Step 1) The main idea is that since particles have uniformly bounded velocities, the probability of the occurrence of two collisions in a time interval of size δ is of order $O(\delta^2)$. The reason for this is that if we trace back the colliding particles to the boundary of the box, we will have a configuration in which two pairs of particles have distances of order $O(\delta)$.

Let us write $\sigma(t)$ for the smallest time $\sigma > t$ such that $z_{i-1}(\sigma) = z_i(\sigma) = z_{i+1}(\sigma)$ for some index $i \in \{2, \dots, \mathbf{n}(t) - 1\}$. We first claim that there exists a constant c_1 such that

$$\mathbb{P}\left(\sigma(t) \in (t, t+\delta)\right) \le c_1 \delta^2, \tag{4.90}$$

for every $\delta > 0$. Once this is established, we can then choose $\delta = T/n$, and argue

$$\mathbb{P}\left(\sigma(0) \in [0, T]\right) \le n \sup_i \mathbb{P}\left(\sigma(t_i) \in [t_i, t_{i+1}]\right) \le c_1 T^2 n^{-1},$$

for $t_i = iT/n, \; i = 0, \dots, n-1$. We then send $n \to \infty$ to deduce that there is no triple collision almost surely.

It remains to prove (4.90). Let us write $\mathcal{E}_r(t_1, t_2)$ for the event that at least r stochastic jumps occur in the interval $[t_1, t_2]$. We claim that there exists a constant $c_2 = c_2(T)$ such that

$$\mathbb{P}\left(\mathcal{E}_2(t, t+\delta)\right) \le c_2 \delta^2, \tag{4.91}$$

for every $t \in [0, T]$, and $\delta \in (0, 1)$. This is an immediate consequence of the strong Markov property of the process $\mathbf{q}(t)$, and the bound (4.38): Indeed if σ_1 and σ_2 denote the times of the first and second stochastic jumps after time t, then

$$\begin{aligned}
\mathbb{P}\left(\mathcal{E}_2(t, t+\delta)\right) &\le \mathbb{E} 1_{\mathcal{E}_1(0, t+\delta)} \; \mathbb{P}^{\mathbf{q}(\sigma_1)}\left(\mathcal{E}_1(\sigma_1, t+\delta)\right) \\
&\le c_3 \delta \; \mathbb{E}\left(\mathbf{n}(\mathbf{q}(\sigma_1)) + 2\right)^2 \; \mathbb{1}_{\mathcal{E}_1(t, t+\delta)} \\
&\le c_3 \delta \mathbb{E}\left(\mathbf{n}(\mathbf{q}(t)) + 3\right)^2 \; \mathbb{1}_{\mathcal{E}_1(t, t+\delta)} \\
&\le c_4 \delta^2 \mathbb{E}\left(\mathbf{n}(\mathbf{q}(t)) + 3\right)^4 = c_5 \delta^2,
\end{aligned}$$

for some constants c_3, c_4, and c_5. Because of (4.91), the bound (4.90) would follow if we can show

$$\mathbb{P}\left(\sigma(t) \in (t, t+\delta), \; \sigma_2 > t+\delta\right) \le c_6 \delta^2, \tag{4.92}$$

for some constant c_6.

(Step 2) It remains to establish (4.92). Let us write θ_1 and θ_2 for the first and the second times of particle collisions after t. By convention, $\theta_1 = \theta_2 = \theta$ when at time θ there is a double collisions (which includes the case $z_{i-1}(t) = z_i(t) = z_{i+1}(t)$). We claim

$$\mathbb{P}\left(\theta_1, \theta_2 \in (t, t+\delta), \; \theta_2 \le \sigma_1\right) \le c_7 \delta^2. \tag{4.93}$$

To see this, observe that the particle z_i travels with the velocity $v_i = -[\rho^{i-1}, \rho^i] \in [-V_\infty, V_\infty]$. Hence, if there is a collision between z_i and z_{i+1}, and between z_j and z_{j+1} at time θ_1, then $|z_i(t) - z_{i+1}(t)| = O(\delta)$ and $|z_j(t) - z_{j+1}(t)| = O(\delta)$. Since $i \ne j$, the probability of such event is of order $O(\delta^2)$ because the law of $\mathbf{q}(t)$ has a bounded density with respect to the Lebesgue measure. In summary,

$$\mathbb{P}\left(\theta_1, \theta_2 \in (t, t+\delta), \; \theta_2 \le \sigma_1\right) \le c_8 \delta^2 \mathbb{E}(\mathbf{n}(\mathbf{q}(t)))^2 \le c_7 \delta^2,$$

for constants c_7 and c_8, proving (4.93).

(Final Step) On account of (4.93), the bound (4.92) would follow if we can show

$$\mathbb{P}\left(\sigma_1 \le \theta_1 = \theta_2 = \sigma(t) \le t + \delta\right) \le c_9 \delta^2, \tag{4.94}$$

for a constant c_9. At σ_1 a new particle is created either at boundary point, or as a result of a fragmentation. We only treat the latter because the former can be treated likewise. Let us assume that the particle (z_i, ρ^i) is replaced with two particles $(z_{i,1}, \rho^*)$ and $(z_{i,2}, \rho^i)$. At the time $\sigma(t)$ a double collision occurs. If none of $z_{i,1}$ or $z_{i,2}$ are involved in this collision, then can be treated as in *Step 2*. If, for example the particle $z_{i,2}$ is to the right of $z_{i,1}$, and is involved in the double collision at $\theta := \theta_1 = \theta_2 = \sigma(t)$, then $z_{i,2}(\theta) = z_{i+1}(\theta) = z_{i+2}(\theta)$. In spite of a change of velocity of the i-th particle, we still must have that $|z_i(t) - z_{i+1}(t)|, |z_{i+1}(t) - z_{i+2}(t)| = O(\delta)$, which results in a bound of order $O(\delta^2)$ for the probability of such event. This completes the proof of (4.94). □

4.4 Proof of Theorem 4.1.5

As we stated in the Introduction, our assumption on the support of f allows us to deduce Theorem 4.1.5 from Theorem 4.1.2. To explain our strategy, observe that by Theorem 4.1.2 we already know that the process $x_1 \mapsto \rho(x_1, x_2)$ is a Markov process for every $x_2 \in [t_0, t_1]$. Under the assumptions of Theorem 1.2, no new particle is created on the left side of Λ, and the process $x_2 \mapsto \rho(a^+, x_2)$ is an independent Markov jump process with the jump rate density $[\rho^-, \rho^+] f((a^+, x_2), \rho^-, \rho^+)$. Though its initial state $\rho(a^+, t_0)$ depends on the dynamics of the lower side of Λ. We may deduce Theorem 1.2 from Theorem 1.1 by interchanging x_1 with x_2, and reversing direction on both axes. We explain the consequences of such operations on our variables in steps **(1)** and **(2)** below. In **(1)** we verify the compatibility of the forward equations (4.24) and (4.25) that are satisfied by the marginal ℓ. We use ℓ in **(2)** to give a recipe for the jump rates of the reversed processes, and see how the time reversal and variables swap operations are compatible with the kinetic equation (4.12).

(1) We first address the effect of a time reversal on our Markov jump processes on the lower and the right sides of Λ. Given a kernel $h(x, \rho^-, d\rho^+)$, define the linear operator $\mathcal{L}(x, h)$ by

$$(\mathcal{L}(x, h)F)(\rho^-) := \int \left(F(\rho^+) - F(\rho^-)\right)\, h(x, \rho^-, d\rho^+).$$

If $\ell(x, d\rho)$ is the law of $\rho(x)$ with respect to the measure $\nu^{f,\Lambda}$, then it satisfies the forward equation associated with the operator $\mathcal{L}(x, f) = \mathcal{L}(x, f^1)$:

$$\ell_{x_1} = \mathcal{L}(x, f)^*\, \ell = \ell * f - A(f)\, \ell. \tag{4.95}$$

where $\mathcal{L}(x, f)^*$ denotes the adjoint of the operator $\mathcal{L}(x, f)$, and

$$(\ell * h)(x, d\rho^+) := \int \ell(x, d\rho^-)\, h(x, \rho^-, d\rho^+).$$

The equation (4.95) is an immediate consequence of Theorem 4.1.2. This and the kinetic equation (4.12) imply a similar equation for the second partial derivative, namely

Proposition 4.4.1 *Assume that f and ℓ are bounded functions, f is C^1 in x-variable and that ℓ is C^2 in x-variable. Also assume that f satisfies (4.12), ℓ satisfies (4.95), and that the equation*

$$\ell_{x_2} = \mathcal{L}(x, f^2)^* \ \ell = \ell * f^2 - A(f^2) \ \ell, \tag{4.96}$$

holds when $x_1 = a^+$. Then (4.96) holds for $x_1 \in [a^-, a^+]$.

Proof. It is not hard to show that the right-hand side of (4.12), integrated with respect to ρ^+ is 0. As a result,

$$A(f^1)_{x_2} - A(f^2)_{x_1} = 0.$$

Let us set

$$\xi = \ell_{x_2} - \ell * f^2 + A(f^2) \ \ell.$$

From differentiating both sides of (4.95) with respect to x_2 we learn

$$\begin{aligned}
\ell_{x_1 x_2} =& \ell_{x_2} * f^1 + \ell * f^1_{x_2} - A(f^1)_{x_2} \ \ell - A(f^1) \ \ell_{x_2} \\
=& \xi * f^1 + \ell * f^2 * f^1 - (A(f^2)\ell) * f^1 + \ell * f^1_{x_2} \\
& - A(f^1)_{x_2} \ \ell - A(f^1)\xi - A(f^1)(\ell * f^2) + A(f^1)A(f^2)\ell \\
=& \ell * [f^2 * f^1 - A(f^2) \otimes f^1 - f^2 \otimes A(f^1) + f^1_{x_2}] \\
& - \ell[A(f^1)_{x_2} - A(f^1)A(f^2)] + \xi * f^1 - A(f^1)\xi,
\end{aligned}$$

where the operation $h \otimes k$ was defined in Remark 4.1.1. Similarly,

$$\begin{aligned}
\ell_{x_2 x_1} =& \xi_{x_1} + \ell_{x_1} * f^2 + \ell * f^2_{x_1} - A(f^2)_{x_1} \ \ell - A(f^2) \ \ell_{x_1} \\
=& \xi_{x_1} + \ell * f^1 * f^2 - (A(f^1)\ell) * f^2 + \ell * f^2_{x_1} \\
& - A(f^2)_{x_1} \ \ell - A(f^2)(\ell * f^1) + A(f^2)A(f^1)\ell \\
=& \ell * [f^1 * f^2 - A(f^1) \otimes f^2 - f^1 \otimes A(f^2) + f^2_{x_1}] \\
& - \ell[A(f^2)_{x_1} - A(f^1)A(f^2)] + \xi_{x_1}.
\end{aligned}$$

From $\ell_{x_2 x_1} = \ell_{x_1 x_2}$, $A(f^2)_{x_1} = A(f^1)_{x_2}$, and (4.12) we deduce

$$\xi_{x_1} = \xi * f - A(f)\xi. \tag{4.97}$$

This means that $\xi(\cdot, x_2, \rho)$ satisfies the forward equation for the Markov jump process $x_1 \mapsto \rho(x_1, x_2)$ associated with the kernel f. We wish to use the condition $\xi(a^+, x_2, \rho) = 0$, to deduce that $\xi(x_1, x_2, \rho) = 0$ for $x_1 \in [a^-, a^2]$. This being true for every $x_2 \in [t_0, t_1]$ yields the desired result. Indeed if $\varphi : \mathbb{R} \to [0, \infty)$ is a C^1 Lipschitz function such that $\varphi(0) = 0$, and $\varphi(r) \geq |r| - c_0$, for some constant c_0, then

$$\begin{aligned}
\frac{d}{dx_1} \int \varphi(\xi(x, \rho)) \ \beta(d\rho) =& \int \varphi'(\xi(x, \rho)) \ (\xi * f - A(f)\xi)(x, \rho) \ \beta(d\rho) \\
\leq& c_1 \int |\xi(x, \rho)| \ \beta(d\rho) \leq c_1 \int \varphi(\xi(x, \rho)) \ \beta(d\rho) + c_2,
\end{aligned}$$

for constants c_1 and c_2. From this, Gronwall's inequality, and the condition $\xi(a^+, x_2, \rho) = 0$, we deduce that $\varphi(\xi) = 0$. This completes the proof because we can approximate $|\xi|$ by functions of the form $\varphi(\xi)$, with φ as above. □

(2) As it is well-known, a time reversal of a Markov process can be realized as a Markov process with a generator that can be described in terms of original process and its marginals. Indeed if we decrease $x_1 \in [a^-, a^+]$, the process $x_1 \mapsto \rho(x_1, x_2)$ is a Markov process with the jump rate

$$\hat{f}(x, \rho^+, d\rho^-) := \eta(x, \rho^-, \rho^+)\ f(x, \rho^-, d\rho^+), \quad \text{where} \quad \eta(x, \rho^-, \rho^+) = \frac{\ell(x, d\rho^-)}{\ell(x, d\rho^+)}.$$

Similarly, as we decrease $x_2 \in [t_0, t_1]$, the process $x_2 \mapsto \rho(a^+, x_2)$ is a Markov process with the jump rate $\hat{f}^2(a^+, x_2, \rho^+, d\rho^-)$, where

$$\hat{f}^2(x, \rho^+, d\rho^-) := \eta(x, \rho^-, \rho^+)\ f^2(x, \rho^-, d\rho^+).$$

We also define

$$\tilde{f}(x, \rho^+, d\rho^-) := \hat{f}(-x, \rho^+, d\rho^-), \tag{4.98}$$

to represent the jump rate density of the process $x \mapsto \rho(-x)$.

If g is a convex function such that $\rho = (\rho_1, \rho_2) = \nabla g$ is distributed according to $\nu^{f,\Lambda}$, and $\varphi(x_1, x_2) = (-x_2, -x_1)$, then $\hat{g} = g \circ \varphi$ is a convex function that is defined on

$$\hat{\Lambda} := \varphi(\Lambda) = [-t_1, -t_0] \times [-a^+, -a^-],$$

and $\hat{\rho} := \nabla \hat{g} = (-\rho_2, -\rho_1) \circ \varphi$ is distributed according to a probability measure that is denoted by $\hat{\nu}$. Note $[\hat{\rho}^-, \hat{\rho}^+] = [\rho^-, \rho^+]^{-1}$. According to $\hat{\nu}$, the process $x_1 \mapsto \hat{\rho}(x_1, -a^+)$ is a jump process with the jump rate density

$$\tilde{f}^2(-a^+, x_1, -\rho^-, -\rho^+),$$

with respect to the measure $\hat{\beta}$ which is the push-forward of β under the map $\theta(\rho) := -\rho$. Similarly, the process $x_2 \mapsto \hat{\rho}(-t_0, x_2)$ is a jump process with the jump rate density

$$\tilde{f}^1(x_2, -t_0, -\rho^-, -\rho^+).$$

We are now in a position to apply Theorem 1.1 to assert that the process $x_1 \mapsto \hat{\rho}(x_1, x_2)$ is a Markov jump process for every $x_2 \in [-a^+, -a^-]$. This in turn implies that as we decrease x_2, the process $x_2 \mapsto \rho(x_1, x_2)$ is a (reversed) Markov jump process for every $x_1 \in [a^-, a^+]$, completing the proof of Theorem 1.2. To apply Theorem 1.1 though, we need to make sure that our candidate the jump rate density of the jump process $x_1 \mapsto \hat{\rho}(x_1, x_2)$, namely

$$\bar{f}\left(x_1, x_2, \rho_1^-, \rho_2^-, \rho_1^+, \rho_2^+\right) := \tilde{f}^2\left(x_2, x_1, -\rho_2^-, -\rho_1^-, -\rho_2^+, -\rho_1^+\right), \tag{4.99}$$

satisfies the kinetic equation. This will be carried out in Proposition 4.2.

Proposition 4.4.2 *Let $f = f^1$ be a solution of (4.12). Then the following statements are true:*

(i) *The reversed kernel $\tilde{f}$, given by (4.98) satisfies (4.12).*

(ii) *The kernel $\bar{f}$ given by (4.99) satisfies (4.12) where α is replaced with $\hat{\alpha} := \alpha^{-1}$.*

Proof. **(i)** Observe that (4.95) and (4.96) can be rewritten as

$$\frac{\ell_{x_1}}{\ell} = \frac{\ell * f^1}{\ell} - A(f^1) = A(\hat{f}^1) - A(f^1),$$
$$\frac{\ell_{x_2}}{\ell} = \frac{\ell * f^2}{\ell} - A(f^2) = A(\hat{f}^2) - A(f^2),$$

As a consequence,

$$\left(\tau \cdot \frac{\nabla \eta}{\eta}\right)(x, \rho^-, \rho^+) = \frac{Q^-(f)}{f}(x, \rho^-, \rho^+) + \frac{Q^-(\hat{f})}{\hat{f}}(x, \rho^+, \rho^-), \tag{4.100}$$

because

$$\frac{Q^-(f)}{f}(x, \rho^-, \rho^+) = A(f^2)(x, \rho^+) - A(f^2)(x, \rho^-) - [\rho^-, \rho^+](A(f^1)(x, \rho^+) - A(f^1)(x, \rho^-)).$$

On the other hand, we can readily show

$$\frac{\hat{f}^1 * \hat{f}^2}{\hat{f}}(x, \rho^+, \rho^-) = \frac{f^2 * f^1}{f}(x, \rho^-, \rho^+), \qquad \frac{\hat{f}^2 * \hat{f}^1}{\hat{f}}(x, \rho^+, \rho^-) = \frac{f^1 * f^2}{f}(x, \rho^-, \rho^+),$$

which is an immediate consequence of $\eta(x, \rho^-, \rho^+) = \eta(x, \rho^-, \rho^*)\eta(x, \rho^*, \rho^+)$. From this, (4.100) and our assumption on f we deduce

$$\begin{aligned}\left(\tau \cdot \frac{\nabla \hat{f}}{\hat{f}}\right)(x, \rho^+, \rho^-) &= \left(\tau \cdot \frac{\nabla f}{f}\right)(x, \rho^-, \rho^+) + \left(\tau \cdot \frac{\nabla \eta}{\eta}\right)(x, \rho^-, \rho^+) \\ &= \frac{Q^+(f)}{f}(x, \rho^-, \rho^+) - \frac{Q^-(f)}{f}(x, \rho^-, \rho^+) + \tau \cdot \frac{\nabla \eta}{\eta}(x, \rho^-, \rho^+) \\ &= -\frac{Q^+(\hat{f})}{\hat{f}}(x, \rho^+, \rho^-) + \frac{Q^-(\hat{f})}{\hat{f}}(x, \rho^+, \rho^-) = -\frac{Q(\hat{f})}{\hat{f}}(x, \rho^+, \rho^-),\end{aligned}$$

which is the reversed kinetic equation. This implies that $\tilde{f}$ satisfies (4.12) because

$$\nabla \tilde{f}(x, \rho^-, \rho^+) = -\nabla \hat{f}(-x, \rho^-, \rho^+).$$

(ii) Observe that if $\bar{f}^2 = \hat{\alpha} \bar{f}$, then

$$\bar{f}^2(x_1, x_2, \rho_1^-, \rho_2^-, \rho_1^+, \rho_2^+) == \tilde{f}(x_2, x_1, -\rho_2^-, -\rho_1^-, -\rho_2^+, -\rho_1^+).$$

By **(i)**, we know that $\alpha \tilde{f}_{x_1} - \tilde{f}_{x_2} = -Q(\tilde{f})$. After swapping x_1 with x_2 we deduce

$$\bar{f}_{x_2} - \hat{\alpha} \bar{f}_{x_1} = -Q(\tilde{f}).$$

Finally observe that $-Q(\tilde{f}) = Q(\bar{f})$ because when $\tilde{f}^1$ is swapped with $\tilde{f}^2$, the sign of Q changes. □

Remark 4.4.3 *An alternative strategy for completing the proof of Theorem 4.1.5 is to use Proposition 1.1. We already know that ℓ satisfies (4.96). On the other hand, since we also know that the process $x_2 \mapsto \rho(x_1, x_2)$ is a Markov jump process, the measure ℓ also satisfies*

$$\ell_{x_2} = \ell * h - A(h)\ell,$$

where h is its jump rate. One should be able to deduce from this that $h = f^2$.

4.5 The Kinetic Equation

The purpose of this section is to prove the existence of a solution of the kinetic equation. To have a more conventional notation, we write (x, t) for (x_1, x_2) throughout this section. We start first with the following notation:

Notation A.1 (i) We fix $P^- < P^+$ two real numbers, such that the range of our piecewise constant function ρ is in the box $[P^-, P^+]^2$.

(ii) For any measure space $\mathcal{E}$, let $\mathcal{F}_b(\mathcal{E})$ be the space of real-valued bounded measurable functions defined on $\mathcal{E}$.

(iii) We introduce the function space $\mathcal{X}$ to be the set kernels $h \in \mathcal{F}_b(\mathbb{R} \times ([P^-, P^+]^2)^2)$ such that $x \mapsto h(x, \rho^-, \rho^+)$ is C^1 and Lipschitz for all ρ^- and ρ^+.

(iv) We equip $\mathcal{X}$ with the following norm

$$\|h\|_{\mathcal{X}} := \sup_{x\in\mathbb{R}} \sup_{\rho^-,\rho^+} \left[|h(x, \rho^-, \rho^+)| + |\partial_x h(x, \rho^-, \rho^+)|\right].$$

It is standard that $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is a Banach space.

(v) For any $v \geq 0$, let Γ^v and Γ^v_+ be the sets

$$\begin{aligned}
\Gamma^v :=& \Big\{(\rho^-, \rho^+) \in ([P^-, P^+]^2)^2 : \ \rho^- \prec \rho^+, \ |[\rho^-, \rho^+]| \leq v\Big\} \\
=& \Big\{(\rho^-, \rho^+) \in ([P^-, P^+]^2)^2 : \ \rho^+ - \rho^- \in C^v \setminus \{0\}\Big\}, \\
\Gamma^v_+ :=& \Big\{(\rho^-, \rho^+) \in ([P^-, P^+]^2)^2 : \ \rho^+ - \rho^- \in C^v_+ \setminus \{0\}\Big\},
\end{aligned}$$

where C^v and C^v_+ are the cones

$$\begin{aligned}
C^v &= \Big\{m = (m_1, m_2) \in \mathbb{R}^2 : \ m_1 \geq 0, \ |m_2| \leq v m_1\Big\}, \qquad (4.101) \\
C^v_+ &= \Big\{m = (m_1, m_2) \in \mathbb{R}^2 : \ m_1, m_2 \geq 0, \ m_2 \leq v m_1\Big\}.
\end{aligned}$$

(vi) Let $V_\infty \geq 0$, and $\delta_0 > 0$. We write $\mathcal{X}(V_\infty, \delta_0)$ for the set of $h \in \mathcal{X}$ with the following properties:

- **(1)** The function $h(\cdot, \rho^-, \rho^+)$ is zero for $(\rho^-, \rho^+) \notin \Gamma^{V_\infty}$.
- **(2)** There exists a constant $\delta_0 > 0$ such that $\inf_{x\in\mathbb{R}} \inf_{(\rho^-,\rho^+)\in\Gamma^{V_\infty}} h(x, \rho^-, \rho^+) \geq \delta_0$.
- **(3)** For all ρ^-, ρ^+, the function $x \mapsto h(x, \rho^-, \rho^+)$ is C^2 such that

$$\sup_{x\in\mathbb{R}} \sup_{\rho^-,\rho^+} |\partial_x^2 h(x, \rho^-, \rho^+)| < \infty.$$

Likewise, we write $\mathcal{X}_+(V_\infty, \delta_0)$ for the set of $h \in \mathcal{X}_+(V_\infty, \delta_0)$ with the similar properties, except that the set Γ^{V_∞} in **(i)** and **(iii)** is replaced with $\Gamma^{V_\infty}_+$.

The following theorem proves the existence of a local solution of the kinetic equation.

Theorem 4.5.1 *Given $h \in \mathcal{X}(V_\infty, \delta_0)$, denote by $M_0 := \sup_{x\in\mathbb{R}} \sup_{\rho^-,\rho^+} h(x,\rho^-,\rho^+)$, and define the time*

$$T^* := \min\left(\frac{1}{12V_\infty M_0}\ ,\ \frac{\delta_0}{48V_\infty M_0^2}\right).$$

Then, there exists a unique solution

$$f : \mathbb{R} \times [0,T^*] \times ([P^-,P^+]^2)^2 \to \mathbb{R},$$

of the kinetic equation

$$f_t - [\rho^-,\rho^+]f_x = Q(f) =: Q^+(f) - Q^-(f),$$

where

$$Q^+(f)(x,t,\rho^-,\rho^+) = \int \left([\rho^+,\rho^*] - [\rho^*,\rho^-]\right) f(x,t,\rho^-,\rho^*)f(x,t,\rho^*,\rho^+)\ \beta(d\rho^*),$$

$$Q^-(f)(x,t,\rho^-,\rho^+) = \left(\int ([\rho^+,\rho^*] - [\rho^-,\rho^+])f(x,t,\rho^+,\rho^*)\ \beta(d\rho^*) - \int ([\rho^-,\rho^*] - [\rho^-,\rho^+])f(x,t,\rho^-,\rho^*)\ \beta(d\rho^*)\right) f(x,t,\rho^-,\rho^+),$$

with $f(\cdot,0,\cdot,\cdot) = h$. The function f is C^1 in the variables (x,t) for all fixed ρ^-,ρ^+ and $f(\cdot,\cdot,\rho^-,\rho^+) \equiv 0$ for all $(\rho^-,\rho^+) \notin \Gamma^{V_\infty}$. Furthermore, we have that

$$\sup_{t\in[0,T^*]} \|f(\cdot,t,\cdot,\cdot)\|_{\mathcal{X}} < \infty, \qquad \inf_{t\in[0,T^*]} \inf_{x\in\mathbb{R}} \inf_{(\rho^-,\rho^+)\in\Gamma^{V_\infty}} f(x,t,\rho^-,\rho^+) \geq \frac{\delta_0}{2}.$$

Moreover if $h \in \mathcal{X}_+(V_\infty,\delta_0)$, then there exists a unique solution f with similar properties except that the set Γ^{V_∞} must be replaced with $\Gamma_+^{V_\infty}$.

Proof. *(Step 1)* We assume here without loss of generality that the measure β has total mass 1 on the box $[P^-,P^+]^2$. By the following standard change of variables, we transform the previous PDE to an ODE. For instance, define the function g as

$$g(x,t,\rho^-,\rho^+) = f(x - [\rho^-,\rho^+]t, t, \rho^-,\rho^+). \tag{4.102}$$

Then by the chain rule, g must verify the following ODE

$$g_t = \tilde{Q}^+(g) - \tilde{Q}^-(g) = \tilde{Q}^+(g) - \tilde{L}(g)g,$$

where

$$\tilde{Q}^+(g)(x,t,\rho^-,\rho^+) = \int [\rho^-,\rho^*,\rho^+]\ g\left(x - ([\rho^-,\rho^+] - [\rho^-,\rho^*])t, t, \rho^-,\rho^*\right) g(x - ([\rho^-,\rho^+] - [\rho^+,\rho^*])t, t, \rho^*,\rho^+)\ \ \beta(d\rho^*),$$

$$\tilde{L}(g)(x,t,\rho^-,\rho^+) = \int ([\rho^+,\rho^*] - [\rho^-,\rho^+])\ g(x - ([\rho^-,\rho^+] - [\rho^+,\rho^*])t, t, \rho^+,\rho^*)\ \beta(d\rho^*) - \int ([\rho^-,\rho^*] - [\rho^-,\rho^+])\ g(x - ([\rho^-,\rho^+] - [\rho^-,\rho^*])t, t, \rho^-,\rho^*)\ \beta(d\rho^*).$$

We will prove the existence of a solution g by an approximation scheme and then recover the desired f via the equation (4.102). Define the functional $\mathcal{H} : \mathcal{X} \times \mathbb{R} \to \mathcal{X}$, $\mathcal{H}(g,t) := \mathcal{H}^+(g,t) - \mathcal{K}(g,t)g$, by

$$\begin{aligned}
\mathcal{H}^+(h,t)(x,\rho^-,\rho^+) := \int [\rho^+,\rho^*,\rho^-]\; & h(x - ([\rho^-,\rho^+] - [\rho^-,\rho^*])t, \rho^-, \rho^*) \\
& h(x - ([\rho^-,\rho^+] - [\rho^+,\rho^*])t, \rho^*, \rho^+)\; \beta(d\rho^*), \\
\mathcal{K}^+(h,t)(x,\rho^-,\rho^+) := \int ([\rho^+,\rho^*] - [\rho_-,\rho^+])\; & h(x - ([\rho^-,\rho^+] - [\rho^+,\rho^*])t, \rho^+, \rho^*)\; \beta(d\rho^*) \\
- \int ([\rho^-,\rho^*] - [\rho^-,\rho^+])\; & h(x - ([\rho^-,\rho^+] - [\rho^-,\rho^*])t, \rho^-, \rho^*)\; \beta(d\rho^*).
\end{aligned}$$

Our goal is to prove the existence of a *local* solution $g : [0,T^*] \mapsto \mathcal{X}$ to the inhomogeneous ODE

$$\dot{g}(t) = \mathcal{H}(g(t),t) \tag{4.103}$$

under the initial condition $g(0) = h$. As the function space $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is clearly Banach, we will construct a Cauchy sequence $(g_n)_{n\in\mathbb{N}}$ of elements in $C([0,T^*],\mathcal{X})$ that will converge to our desired solution g.

(Step 2) For any fixed $n \in \mathbb{N}$, we define the polygonal function g_n such that $g_n(0) = h$ and

$$\dot{g}_n(t) = \mathcal{H}\left(g_n\left(\frac{j}{n}\right), \frac{j}{n}\right) \text{ for all } t \in \left(\frac{j}{n}, \frac{j+1}{n}\right)$$

for all $j \geq 0$. Let us denote $g_n^j = g_n\left(\frac{j}{n}\right)$, then it is clear that all g_n^j are C^2 in the variable x. We have that

$$n(g_n^{j+1} - g_n^j) = \mathcal{H}\left(g_n^j, \frac{j}{n}\right). \tag{4.104}$$

Let us prove first that $g_n^j(\cdot,\rho^-,\rho^+) \equiv 0$ for all $(\rho^-,\rho^+) \notin \Gamma^{V_\infty}$ by induction on j. Suppose this is true for j and we wish to prove it for $j+1$. Take $x \in \mathbb{R}$, $(\rho^-,\rho^+) \notin \Gamma^{V_\infty}$, and take any ρ^* such that $\rho^- \prec \rho^* \prec \rho^+$. Since C^v of (4.101) is a cone, we have that either $(\rho^-,\rho^*) \notin \Gamma^{V_\infty}$ or $(\rho^*,\rho^+) \notin \Gamma^{V_\infty}$. In either cases

$$g_n^j\left(x - \left([\rho^-,\rho^+] - [\rho^-,\rho^*]\right)\frac{j}{n}, \rho^-, \rho^*\right) g_n^j\left(x - ([\rho^-,\rho^+] - [\rho^+,\rho^*])\frac{j}{n}, \rho^*, \rho^+\right) = 0,$$

by the induction hypothesis. As a result, $g_n^{j+1}(x,\rho^-,\rho^+) = 0$, as desired.

Next, let us define

$$\begin{aligned}
m_j &:= \inf_{x\in\mathbb{R}} \inf_{(\rho^-,\rho^+)\in\Gamma^{V_\infty}} g_n^j(x,\rho^-,\rho^+), & M_j &:= \sup_{x\in\mathbb{R}} \sup_{(\rho^-,\rho^+)\in\Gamma^{V_\infty}} |g_n^j(x,\rho^-,\rho^+)|, \\
M_j' &:= \sup_{x\in\mathbb{R}} \sup_{(\rho^-,\rho^+)\in\Gamma^{V_\infty}} |\partial_x g_n^j(x,\rho^-,\rho^+)|, & M_j'' &:= \sup_{x\in\mathbb{R}} \sup_{(\rho^-,\rho^+)\in\Gamma^{V_\infty}} |\partial_x^2 g_n^j(x,\rho^-,\rho^+)|,
\end{aligned}$$

It is clear from the expression of $\mathcal{H}$ that we have for all $j \geq 0$,

$$n(m_{j+1} - m_j) \geq -6V_\infty M_j^2, \quad n(M_{j+1} - M_j) \leq 6V_\infty M_j^2. \tag{4.105}$$

Let us prove first by induction on j the following inequality,

$$rj < 1 \quad \Longrightarrow \quad M_j \leq M_0(1-rj)^{-1}. \tag{4.106}$$

where $r = \frac{6V_\infty M_0}{n}$. The verification for $j=0$ is trivial. Assume it is true for j, then from the second inequality in (4.105), it suffices to prove that

$$(1-rj)^{-1}\left(1+r(1-rj)^{-1}\right) \leq (1-r(j+1))^{-1}.$$

This inequality is equivalent to

$$(1-(j-1)r)(1-(j+1)r) \leq (1-jr)^2,$$

which is clearly true. As an immediate consequence of (4.106) we have that

$$\sup_{t\in[0,T^*]} \|g_n(t)\|_{L^\infty} \leq M_0 \sup_{t\in[0,T^*]} \left(1 - \frac{6\lfloor nt\rfloor V_\infty M_0}{n}\right)^{-1} \leq 2M_0.$$

By differentiating the identity (4.104), we also have that

$$n(M'_{j+1} - M'_j) \leq 12V_\infty M'_j M_j,$$

So for all j such that $\frac{j+1}{n} \leq T^*$, we have that

$$M'_{j+1} \leq M'_j\left(1 + \frac{24V_\infty M_0}{n}\right),$$

from which it follows that

$$M'_j \leq M'_0\left(1+\frac{24V_\infty M_0}{n}\right)^j \leq M'_0 e^{\frac{2j}{nT^*}} \leq M'_0 e^2,$$

and hence

$$\sup_{t\in[0,T^*]} \|\partial_x g_n(t)\|_{L^\infty} \leq M'_0 e^2.$$

Likewise, by differentiating twice the identity (4.104), we get that

$$n(M''_{j+1} - M''_j) \leq 12V_\infty M_j M''_j + 12V_\infty(M'_j)^2 \leq 24V_\infty M_0 M''_j + 12V_\infty(M'_0)^2 e^4.$$

From this, it follows by similar arguments as before that

$$M''_j \leq \left(M''_0 + \frac{(M'_0)^2 e^4}{2M_0}\right) e^{\frac{24V_\infty M_0 j}{n}},$$

and hence

$$\sup_{t\in[0,T^*]} \|\partial_x^2 g_n(t)\|_{L^\infty} \leq \left(M''_0 + \frac{(M'_0)^2 e^4}{2M_0}\right) e^2.$$

Now since we have for every j such that $\frac{j+1}{n} \le T_*$,

$$m_{j+1} \ge m_j - \frac{24V_\infty M_0^2}{n},$$

it follows easily that

$$\inf_{t\in[0,T_*]} \inf_{x\in\mathbb{R}} \inf_{(\rho^-,\rho^+)\in\Gamma^{V_\infty}} g_n(t)(x,\rho^-,\rho^+) \ge \frac{\delta_0}{2} > 0.$$

We have hence proved that all the approximating functions $(g_n)_{n\in\mathbb{N}} \in \mathcal{C}([0,T^*],\mathcal{X})$ are supported on Γ^{V_∞} in the (ρ^-,ρ^+) variables, and are uniformly bounded from above and below by positive constants in their supports.

(Step 3) To finish the proof, we shall show that the sequence $\{g_n\}$ is Cauchy. This is achieved by obtaining Lipschitz estimates on g_n. Observe that for any $s<t$, and $k_1,k_2\in\mathcal{X}$ that are C^2 in the x-variable and supported on Γ^{V_∞} such that

$$\max(\|\partial_x^2 k_1\|_{L^\infty}, \|\partial_x^2 k_2\|_{L^\infty}) < \infty,$$

it is straightforward to show

$$\begin{aligned}
\|\mathcal{H}(k_1,t)-\mathcal{H}(k_2,t)\|_{\mathcal{X}} &\le 6V_\infty\left(\|k_1\|_{\mathcal{X}}+\|k_2\|_{\mathcal{X}}\right)\|k_1-k_2\|_{\mathcal{X}},\\
\|\mathcal{H}(k_1,t)-\mathcal{H}(k_1,s)\|_{\mathcal{X}} &\le 72V_\infty^2\|k_1\|_{\mathcal{X}}\left(\|k_1\|_{\mathcal{X}}+\|\partial_x^2 k_1\|_{L^\infty}\right)(t-s).
\end{aligned}$$

Let us denote

$$M := \max\left(2M_0, M_0'e^2, \left(M_0''+\frac{(M_0')^2e^4}{2M_0}\right)e^2\right)$$

The constant M is a uniform upper bound on the supremum norm of $g_n(t), \partial_x g_n(t), \partial_x^2 g_n(t)$ for all $t\in[0,T^*]$ and $n\in\mathbb{N}$. We have that

$$\begin{aligned}
\|\dot{g}_n(t)-\dot{g}_m(t)\|_{\mathcal{X}} =& \left\|\mathcal{H}\left(g_n\left(\frac{\lfloor nt\rfloor}{n}\right),\frac{\lfloor nt\rfloor}{n}\right)-\mathcal{H}\left(g_m\left(\frac{\lfloor mt\rfloor}{m}\right),\frac{\lfloor mt\rfloor}{m}\right)\right\|_{\mathcal{X}}\\
\le& \left\|\mathcal{H}\left(g_n\left(\frac{\lfloor nt\rfloor}{n}\right),\frac{\lfloor nt\rfloor}{n}\right)-\mathcal{H}\left(g_n\left(\frac{\lfloor nt\rfloor}{n}\right),t\right)\right\|_{\mathcal{X}}\\
&+\left\|\mathcal{H}\left(g_n\left(\frac{\lfloor nt\rfloor}{n}\right),t\right)-\mathcal{H}\left(g_n(t),t\right)\right\|_{\mathcal{X}}\\
&+\|\mathcal{H}(g_n(t),t)-\mathcal{H}(g_m(t),t)\|_{\mathcal{X}}\\
&+\left\|\mathcal{H}(g_m(t),t)-\mathcal{H}\left(g_m\left(\frac{\lfloor mt\rfloor}{m}\right),t\right)\right\|_{\mathcal{X}}\\
&+\left\|\mathcal{H}\left(g_m\left(\frac{\lfloor mt\rfloor}{m}\right),t\right)-\mathcal{H}\left(g_m\left(\frac{\lfloor mt\rfloor}{m}\right),\frac{\lfloor mt\rfloor}{m}\right)|\right\|_{\mathcal{X}}\\
\le& 144V_\infty^2M^2\left(\frac{1}{n}+\frac{1}{m}\right)+12V_\infty M\left(\left\|g_n\left(\frac{\lfloor nt\rfloor}{n}\right)-g_n(t)\right\|_{\mathcal{X}}\right.\\
&\left.+\left\|g_m\left(\frac{\lfloor mt\rfloor}{m}\right)-g_m(t)\right\|_{\mathcal{X}}\right)+12V_\infty M\|g_n(t)-g_m(t)\|_{\mathcal{X}}.
\end{aligned}$$

On the other hand,

$$\left\|g_n\left(\frac{\lfloor nt\rfloor}{n}\right)-g_n(t)\right\|_{\mathcal{X}}\leq\frac{1}{n}\left\|\mathcal{H}\left(g_n\left(\frac{\lfloor nt\rfloor}{n}\right)\right)\right\|_{\mathcal{X}}\leq\frac{6V_\infty M^2}{n}$$

and similarly for the term concerning m. Hence there exist two positive constants C_1, C_2 that only depend on V_∞ and M, such that for all $t\in[0,T^*]$,

$$\|\dot{g}_n(t)-\dot{g}_m(t)\|_{\mathcal{X}}\leq C_1\left(\frac{1}{n}+\frac{1}{m}\right)+C_2\|g_n(t)-g_m(t)\|_{\mathcal{X}},$$

which implies

$$||g_n(t)-g_m(t)||_{\mathcal{X}}\leq C_1\left(\frac{1}{n}+\frac{1}{m}\right)t+C_2\int_0^t\|g_n(s)-g_m(s)\|_{\mathcal{X}}\;ds.$$

This, and the Gronwall's inequality give

$$\sup_{t\in[0,T^*]}\|g_n(t)-g_m(t)\|_{\mathcal{X}}\leq C_1\left(\frac{1}{n}+\frac{1}{m}\right)T^*\left(1+c_2T^*e^{C_2T^*}\right)$$

which implies that $(g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and therefore admits a limit $g_\infty\in\mathcal{C}([0,T^*],\mathcal{X})$. The function g_∞ (that we now regard as a function of the four variables (x,t,ρ^-,ρ^+)) is C^1 in the variables x and t, and verify the inhomogeneous ODE (4.103) and is bounded uniformly from below by $\frac{\delta_0}{2}$ and is such that

$$\sup_{t\in[0,T^*]}\sup_{x\in\mathbb{R}}\sup_{\rho^-,\rho^+}g_\infty(x,t,\rho^-,\rho^+)\leq M.$$

Moreover, for any fixed x and t in its domain of definition, the function $(\rho^-,\rho^+)\mapsto g_\infty(x,t,\rho^-,\rho^+)$ is supported on Γ^{V_∞}. Now defining

$$f(x,t,\rho^-,\rho^+)=g_\infty(x+[\rho^-,\rho^+]t,t,\rho^-,\rho^+)\tag{4.107}$$

f is again C^1 in x and t, verify the same properties as g_∞ and verifies the desired kinetic equation.

Finally we remark that in the above proof, we may replace the set Γ^{V_∞} with $\Gamma^{V_\infty}_+$. □

For the second part of this section, we will prove the existence of the solution to the Kolmogorov forward equation both in space x and time t. More precisely, we wish to address the existence of a unique uniformly positive solution ℓ of the equations (4.24) and (4.25), provided that the kernel f is uniformly positive. We remark that these equations are consistant by Proposition 4.1. Because of this, we only need to solve (4.24) in $[a^-,a^+]$ for an initial condition $\ell(a^-,t,\cdot)$ that solves (4.25). The existence of a solution to (4.24) can be carried out by standard arguments. However, we need to ensure the constructed solution is uniformly positive in Λ, if the initial $\ell^0(\rho)=\ell(a^-,t_0,\rho)$ is uniformly positive. Observe that if ℓ solves (4.24), then

$$\frac{d}{dx}\int\ell(x,t,\rho)\;\beta(d\rho)=0,$$

because the β-integral of the right-hand side of (4.24) is 0. This means

$$\int \ell(x,t)\ \beta(d\rho) = 1, \tag{4.108}$$

if this is the case for $x = a^-$. On the other hand, if the total integral of β is one, $f \geq \delta_1$ for some positive constant δ_1, and ℓ is a solution of (4.24) satisfying (4.108), then

$$\ell_x(x,t,\rho) \geq \delta_1 - \lambda(x,t,\rho)\ell(x,t,\rho),$$

which leads to the lower bound

$$\ell(x,t,\rho) \geq \ell(a^-,t,\rho)e^{-\int_{a^-}^x \lambda(\theta,t,\rho)\ d\theta} + \delta_1 \int_{a^-}^x e^{-\int_y^x \lambda(\theta,t,\rho)\ d\theta}\ dy.$$

From this we learn that ℓ is uniformly positive in Λ if this is the case on the left boundary side of Λ. By assumption, ℓ is uniformly positive at (a^-, t_0), and as t varies, the function $t \mapsto \ell(a^-,t,\rho)$ satisfies (4.25). If the kernel f is supported in $\Gamma_+^{V_\infty}$, then $[\rho^-,\rho^+]f \geq 0$, and a repetition of the above reasoning guarantees

$$\ell(a^-,t,\rho) \geq \ell(a^-,t_0,\rho)e^{-\int_{t_0}^x A(a^-,\theta,\rho)\ d\theta}.$$

In summary, when the kernel f is supported in $\Gamma_+^{V_\infty}$, and is uniformly positive on its support, we can construct a unique uniformly positive solution ℓ to forward equations (4.24) and (4.25) by standard arguments. However some care is needed if $[\rho^-,\rho^+]$ can change sign in the support of our kernel f. In this case, we can guarantee the existence of a uniformly positive solution to (4.24) and (4.25) if we either replace the time interval $[0,T^*]$ with a shorter interval, or assume that the initial $\ell(a^-,t_0,\rho)$ is sufficiently positive. As an example, we demonstrate how a lower bound of 1/6 on the initial ℓ can guarantee the positivity of the solution.

Theorem 4.5.2 *Fix $a^- < a^+$. Let $\ell^0 : [P^-,P^+]^2 \to [0,+\infty)$ be a measurable function such that there exists two constants $c, C > 0$ with*

$$c \leq \ell^0(\rho) \leq C \text{ for all } \rho$$

and $\int \ell^0(\rho)\ \beta(d\rho) = 1$. Moreover, assume that $c \geq \frac{1}{6}$. Then there exists a C^1 solution $\ell : [a^-,a^+] \times [0,T^*] \times [P^-,P^+]^2 \to [0,+\infty)$ to the equations (4.24) and (4.25) such that $\ell(a^-,0,\cdot) = \ell^0$, and such that ℓ is uniformly bounded below by a positive constant and

$$\int \ell(x,t,\rho)\ \beta(d\rho) = 1,$$

for all $(x,t) \in [a^-,a^+] \times [0,T^*]$.

Proof. Without loss of generality let us assume that $a^- = 0$ and denote $a^+ = a$. We will construct a two-parameter function $\ell : [-V_\infty T^*, a] \times [0,T^*] \to \mathcal{F}_b([P^-,P^+]^2)$. The reason why we extend the space domain to $[-V_\infty T^*, a]$ instead of $[0,a]$ will be made clear later.

Let us define first $\ell(\cdot,0)$ on $[-V_\infty T^*, a]$ using the first ODE in the x-direction. The utility of the condition $c \geq \frac{1}{6}$ is to ensure the non-negativity of ℓ as we run the ODE backwards from $0 \to -V_\infty T^*$. Our strategy for proving the existence of the solution of the ODE at $t = 0$ is done in a similar fashion as the kinetic equation via an approximation scheme. In other words, we construct a polygonal approximating $\ell_n : [-V_\infty T^*, a] \to \mathcal{F}_b([P^-, P^+]^2)$ by putting $\ell_n(0) = \ell^0$, and for any $k \in \mathbb{Z}$ by the inductive relation. More precisely, we put $f^k(\rho, \rho_*) := f(k/n, 0, \rho, \rho_*)$, and require that functions $\ell_n^k := \ell_n\left(\frac{k}{n}\right) \in \mathcal{F}_b([P^-, P^+]^2)$ to satisfy

$$n\left(\ell_n^{k+1}(\rho) - \ell_n^k(\rho)\right) = \int f^k\left(\rho^*, \rho\right) \ell_n^k(\rho^*)\ \beta(d\rho^*) - \left(\int f^k\left(\rho, \rho^+\right)\ \beta(d\rho^+)\right) \ell_n^k(\rho),$$

for $k \geq 0$, and

$$-n\left(\ell_n^{k-1}(\rho) - \ell_n^k(\rho)\right) = \int f^k\left(\rho^*, \rho\right) \ell_n^k(\rho^*)\ \beta(d\rho^*) - \left(\int f^k\left(\rho, \rho^+\right)\ \beta(d\rho^+)\right) \ell_n^k(\rho),$$

for $k \leq 0$. The intermediate values $\ell_n(x)$ for $x \in (\frac{k}{n}, \frac{k+1}{n})$ are obtained by linear interpolation. As an initial observation, remark that

$$\int \ell_n^k(\rho)\ \beta(d\rho) = \int \ell_n^{k\pm1}(\rho)\ \beta(d\rho),$$

and hence

$$\int \ell_n^k(\rho)\ \beta(d\rho) = 1, \qquad \text{for all} \quad k \in \mathbb{Z}.$$

Now, if we take $n \geq M_0$ where $M_0 = \|f(\cdot,0,\cdot,\cdot)\|_{L^\infty}$, then by induction it follows that $\ell_n^k \geq 0$ for all $k \geq 0$, as we have that

$$\begin{aligned}\ell_n^{k+1}(\rho) &= \ell_n^k(\rho) + \frac{1}{n}\left(\int f^k\left(\rho^*, \rho\right) \ell_n^k(\rho^*)\ \beta(d\rho^*) - \left(\int f^k\left(\rho, \rho^+\right)\ \beta(d\rho^+)\right) \ell_n^k(\rho)\right) \\ &\geq \ell_n^k(\rho) - \frac{M_0}{n} \ell_n^k(\rho),\end{aligned}$$

which in turn implies the following lower bound

$$\ell_n^k(\rho) \geq \ell^0(\rho)\left(1 - \frac{M_0}{n}\right)^k \geq \ell^0(\rho) e^{-\frac{M_0 k}{n}} \quad \text{for all} \quad k \geq 0.$$

On the other hand, for $k \leq 0$ we have

$$\ell_n^{k-1}(\rho) = \ell_n^k(\rho) - \frac{1}{n}\int f^k(\rho^*, \rho) \ell_n^k(\rho^*)\ \beta(d\rho^*) + \frac{1}{n}\left(\int f^k\left(\rho, \rho^+\right)\ \beta(d\rho^+)\right) \ell_n^k(\rho),$$

which leads to

$$\ell_n^k(\rho) \geq \ell^0(\rho) - \frac{M_0 k}{n},$$

because

$$\int f^k\left(\rho^*, \rho\right) \ell_n^k(\rho^*)\ \beta(d\rho^*) \leq M_0 \int \ell_n^k(\rho^*)\ \beta(d\rho_*) = M_0.$$

In particular, if $\frac{k}{n} \geq -V_\infty T^*$, then $\frac{M_0 k}{n} \geq -\frac{1}{2}$, and

$$\inf_\rho \ell_n^k(\rho) \geq c - \frac{1}{12} \geq \frac{1}{12}.$$

We have therefore constructed the polygonal approximating function $\ell_n : [-V_\infty T^*, a] \to \mathcal{F}_b([P^-, P^+]^2)$ such that it is uniformly bounded from below by $\min(1/12, ce^{-M_0 a})$. The sequence $(\ell_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the space $C([-V_\infty T^*, a], \mathcal{F}_b([P^-, P^+]^2)$ where $\mathcal{F}_b([P^-, P^+]^2)$ is viewed as a Banach space equipped with the uniform norm. We obtain that the limit $\ell_\infty := \lim_{n\to\infty} \ell_n$ is a solution to the ODE

$$(\ell_\infty)_x(x,\rho) = \int f(x,0,\rho^*,\rho)\ell_\infty(x,\rho^*)\ \beta(d\rho^*) - \left(\int f(x,0,\rho,\rho^+)\ \beta(d\rho^+)\right)\ell_\infty(x,\rho)$$

We define $\ell(\cdot, 0) = \ell_\infty$. We will move on now to prove the existence of the solution ℓ as an ODE in the time variable t. In order to preserve the non-negativity of ℓ, we have taken advantage in the ODE in the x-direction of the positivity of the kernel $f(x,t,\rho^-,\rho^+)$, however in the t-direction the kernel is equal to $[\rho^-,\rho^+]f(x,t,\rho^-,\rho^+)$. To circumvent this difficulty, we take advantage of the finite speed propagation (this also explains why we have constructed $\ell(0,\cdot)$ on $[-V_\infty T^*, a]$ instead of just $[0,a]$). For any $x \in \mathbb{R}$ and $t \in [0,T^*]$ we define

$$\tilde{f}(x,t,\rho^-,\rho^+) = f(x + V_\infty t, t, \rho^-, \rho^+) \text{ for all } \rho^-, \rho^+.$$

We define a function $\tilde{\ell} : [-V_\infty T^*, a] \times [0,T^*] \to \mathcal{F}_b([P^-, P^+]^2)$ that satisfies the initial condition $\tilde{\ell}(x,0) = \ell_\infty(x) = \ell(x,0)$. Now, for $x = -V_\infty T^*$ we define $\tilde{\ell}(-V_\infty T^*, \cdot) : [0,T^*] \to \mathcal{F}_b([P^-, P^+]^2)$ by solving the ODE

$$\begin{aligned}\tilde{\ell}_t(-V_\infty T^*, t, \rho) = &\int ([\rho^*,\rho] + V_\infty)\,\tilde{f}(-V_\infty T^*, t, \rho^*, \rho)\tilde{\ell}(-V_\infty T^*, t, \rho^*)\ \beta(d\rho^*) \\ &- \left(\int ([\rho,\rho^+] + V_\infty)\,\tilde{f}(-V_\infty T^*, t, \rho, \rho^+)\ \beta(d\rho^+)\right)\tilde{\ell}(-V_\infty T^*, t, \rho),\end{aligned}$$

with initial condition $\tilde{\ell}(-V_\infty T^*, 0) = \ell_\infty(-V_\infty T^*)$. Now, for any fixed $t \in (0,T^*]$ we define $\tilde{\ell}(\cdot, t) : [-V_\infty T^*, a] \to \mathcal{F}_b([P^-, P^+]^2)$ by solving the ODE on $[-V_\infty T^*, a]$

$$\tilde{\ell}_x(x,t,\rho) = \int \tilde{f}(x,t,\rho^*,\rho)\tilde{\ell}(x,t,\rho^*)\ \beta(d\rho^*) - \left(\int \tilde{f}(x,t,\rho,\rho^+)\ \beta(d\rho^+)\right)\tilde{\ell}(x,t,\rho)$$

with initial condition determined by $\tilde{\ell}(-V_\infty T^*, t)$. The existence of these solutions is done by exactly the same approximation scheme than before, and the function $\tilde{\ell}$ is bounded uniformly from below on the box $[-V_\infty T^*, a] \times [0,T^*]$ due to the non-negativity of the kernels $([\rho^-,\rho^+] + V_\infty)f(x,t,\rho^-,\rho^+)$ and $f(x,t,\rho^-,\rho^+)$. Moreover, if we assume that initially $f(0,\cdot)$ is C^3 then we get that f is C^2 in the variables (x,t), it follows that ℓ is also C^2 and thus from Proposition 5.1, the ODE in t is verified for all $x \in [-V_\infty T^*, a]$, i.e

$$\begin{aligned}\tilde{\ell}_t(x,t,\rho) = &\int ([\rho^*,\rho] + V_\infty)\,\tilde{f}(x,t,\rho^*,\rho)\tilde{\ell}(x,t,\rho^*)\ \beta(d\rho^*) \\ &- \left(\int ([\rho,\rho^+] + V_\infty)\,\tilde{f}(x,t,\rho,\rho^+)\ \beta(d\rho^+)\right)\tilde{\ell}(x,t,\rho)\end{aligned}$$

Now, it suffices to define

$$\ell(x,t,\rho) = \tilde{\ell}(x - V_\infty t, t, \rho) \text{ for all } (x,t) \in [0,a] \times [0,T^*] \text{ and } \rho \in [P^-, P^+]^2$$

then ℓ is C^1 in (x,t) and verify the desired ODEs. Moreover, the total of mass of ℓ is conserved through space and time. $\square$

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