Bond Pricing with Default Risk*

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March 23, 1999

We offer a new model for pricing bonds subject to default risk. The event of default is modelled as the first time that a state variable that captures the solvency of the issuer goes below a certain level. The payoff to the bond in case of default is a constant fraction of the value of a security with the same promised payoffs but without the risk of default. We show that our model is very tractable under different models of interest rate risk and of the interaction between default risk and interest rate risk, with closed-form solutions for corporate bond prices in special cases. The model is seen to produce term structures of default yield spreads and forward spreads with more reasonable properties than other models that have recently been proposed. We illustrate the use of our model by estimating its parameters and backing up both the default writedown and the state variable that governs default risk from a panel data set of bond prices issued by RJR Nabisco.

*Thanks to Frank de Jong, Olivier Ledoit, Eduardo Schwartz, René Stulz and an anonymous referee, for useful comments. This research was partially supported by the Center for International Business Education and Research at the Anderson School.

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1 Introduction

We can write the time t value of a (corporate) default risky zero coupon bond with maturity T as

$$C(t,T) = P(t,T) - \mathrm{E}_t[W(T)\mathbf{1}_{\{D<T\}}e^{-\int_t^T r(u)du}] \tag{1}$$

where $P(t,T)$ is the value of a (Treasury) default riskless zero coupon bond with the same maturity; $W(T)$ is the writedown in case of default (assumed to be incurred at the maturity of the bond); $\mathbf{1}_{\{D<T\}}$ is the indicator function of default happening before the maturity of the bond; r is the default riskless instantaneous interest rate; and the expectation is taken with respect to a risk adjusted probability measure.[1]

There are thus three elements to specify when modeling risky bonds: The maturity writedown in case of default, $W(T)$; the event of default, $\{D < T\}$; and the dynamics of the default riskless spot rate, r, and the corresponding bond prices, $P(t,T)$.

Two broad approaches have been taken to modeling $W(T)$. The traditional approach, that started with Black and Scholes (1973) and Merton (1974), assumes strict priority rules are enforced, and distributes the value of the firm's assets among its claimants accordingly. In these models, $W(T)$ is the difference between the value of the assets remaining after paying claimants with higher priority and the value of the bond. Although conceptually appealing, there are two problems with this approach: it is only tractable to price simple securities issued by firms that have simple capital structures; and, it does not encompass violations of strict priority that are often observed in practice.

Instead, we take $W(T)$ to be an exogenously given constant, that needs to be estimated.[2] This approach has the advantage of allowing the valuation of a security independently of the other securities issued by the firm, and therefore, makes it possible to use study complex liabilities issued by firms with complicated capital structures. In particular, a corporate coupon bond can be valued as a weighted sum of the values of its coupons. The drawback of this approach is, of course, that the bankruptcy game is left unspecified.

As regards the event that triggers default, we can again identify two types of models.[3] The first type of model is stylised, defining the occurrence of default as the first jump of some Poisson process. An example of such "statistical" models is Duffie and Singleton (1995), who show that the pricing formulas developed for certain dynamics of default riskless interest rates carry over to suitably defined default risky interest rates.[4] This makes this approach obviously attractive for pricing derivatives and doing empirical work.

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[1] The existence of which in the frictionless markets we assume (with no taxes, transaction costs, informational asymmetries or agency problems) is roughly equivalent to the inexistence of arbitrage opportunities. We do not assume this probability measure to be unique.

[2] This approach has also recently been taken by Jarrow, Lando and Turnbull (1997) and Longstaff and Schwartz (1995) and has been extended by Duffie and Singleton (1995) to make $W(T)$ exogenously dependent on any state variables, including the value of the claim to be priced just prior to default.

[3] A third class of models, more deeply rooted in corporate finance, endogeneizes the default decision by making it a choice variable of the shareholders. This category includes the papers of Anderson and Sundaresan (1992), Leland and Toft (1996), and Mella-Barral and Perraudin (1996). Although interesting conceptually, this approach places strong restrictions on the capital structure of the firm issuing the securities to be valued.

[4] Jarrow and Turnbull (1995), Madan and Unal (1993) and Lando(1994) have also produced models in this spirit.

There are however some drawbacks to this class of models. Since there is no clear connection between the pricing formulas and the state of the firm, the interpretation of default is rather abstract. More importantly, the simple parametrizations that have been used in empirical work imply that the probability of default in the future, conditional on no prior default, does not go to zero with the length of the time horizon but remains in general positive. This means that there is no resolution of the uncertainty about the event of default, which is contrary to the intuition that if a firm survives for a long time, it has a lower probability of defaulting after that. We will show that our model does not have this feature.

Our model is "structural", in that we model the assets and the liabilities of the firm. We start from a process for the value of the firm's assets, V, and define the occurence of default by the first time this value crosses some threshold, K, representing insolvency. Examples of this approach start with Black and Scholes (1973) and Merton (1974) and continue with Black and Cox (1976) and, more recently, Longstaff and Schwartz (1995).

Black and Scholes (1973) and Merton (1974) posit a single point default boundary, such that default can only happen at maturity, with K being the face value of the debt. Black and Cox (1976) and Longstaff and Schwartz (1995) allow for a continuous, deterministic boundary, so that default may occur at any point in time rather than only at maturity. This captures the idea that, in reality, default occurs whenever the firm is unable to meet any payment on any of its debt issues outstanding, or when it fails to meet some criterion stipulated in the covenants of these debt contracts, which can happen at any time.

There is however a conceptual problem with the way the continuous default boundary has been defined in the latter models. The big advantage of "structural" models as compared to "statistical" models lies in giving some economic meaning to the event of default. Now, in order to have economic meaning, we want default to be triggered by the value of assets falling below the level of liabilities, since, in a frictionless world, default will only happen when the value of liquidating the assets of the firm is not sufficient to cover the value of the liablities. The firm will only be unable to face a due payment by raising new funds if its net worth is not positive.

From a technical point of view, V and K play no direct role in the analysis, what matters is the risk adjusted probability of the ratio of V by K, call it the solvency ratio, hitting one. Now, if V and K are assets that some agent is willing to hold, their risk adjusted drift will be equal to the instantaneous interest rate.[5] Therefore, from Ito's lemma, the drift of the log of the solvency ratio, X, will not depend on the instantaneous interest rate.[6]

Unfortunately, Black and Cox (1976) and Longstaff and Schwartz (1995) make the drift of X dependent on r, which does not allow the event of default in these models to be interpreted as being caused by insolvency.[7] This is not the case of our model.

We must be careful here. We do not have a full model of all the securities issued by the firm, including the covenants that trigger default, nor do we necessarily presume to observe the value

[5] Minus some payout rate.

[6] Unless, of course, r shows up in the diffusion coefficients of V or K, which is not the premise of any of these models.

[7] Unless we assume that the liabilities have a payout rate equal to the instantaneous interest rate. See comments below.

of the assets and the value of all the liabilities. We just argue that, in the assumed world with no frictions, the only economically meaningful definition of default is the event of the value of the assets falling bellow the value of the liabilities.[8] It is the fact that these assets are held by investors that puts the (economic) constraint on the risk adjusted drift of X.

The final element in a model of risky bonds is the dynamics of default riskless interest rates. Our model is tractable for a wide variety of models of default riskless bond prices. In the special case of independence between interest rate and default risks, we obtain closed form solutions. In other cases, we provide tractable numerical solutions of bond prices.

Our model can be used for pricing and hedging default risky bonds and their derivatives. It may also have an application in rating, since we can extract both the implied (risk adjusted) probability of default and the implied writedown in case of default from observed bond prices.

2 Bond Pricing

We assume trading occurs continuously, in perfect and frictionless financial markets with no taxes, transaction costs or informational asymmetries. We posit the existence of a risk adjusted probability measure that prices all assets in this economy as an expectation of their payoffs discounted at the instantaneous interest rate. All processes bellow are defined under this risk adjusted probability measure.

2.1 Default Risk

We consider a firm whose capital structure may include a variety of securities with different maturities, coupons, or covenants in general. The process followed by the value of the assets of this firm is assumed to be independent of the financing decisions taken by the firm. We further assume that the value of the assets of this firm, denoted V, follows

$$\frac{dV(t)}{V(t)} = [r(t) - \delta_v]dt + \sigma_v dZ_v(t) \tag{2}$$

where σ_v is a positive constant and Z_v a standard Brownian motion under the risk adjusted probability measure Q. We allow for a constant payout rate to the investors of the firm, δ_v.

The instantaneous interest rate, r, follows a process with uncertainty driven by a Brownian motion $Z_r(t)$.[9] We leave the parametrization of this diffusion for now unspecified. We just assume Z_v to be correlated with $Z_r(t)$ with correlation coefficient ρ_{rv}.

We assume that, because of contractual provisions, default is triggered simultaneously for all the debt issues[10] the first time that the value of the assets of the firm reaches a critical level, defined by a process K. We assume that this default barrier corresponds to the value of the liabilities of

[8] V could be the value of only part of the assets of the firm, namely those that can be readily liquidated.

[9] We will assume $Z_r(t)$ to be one-dimensional, althought the extension to higher dimensions is straightforward.

[10] This is in accordance with cross default provisions that are widely used in practice.

the firm. The risk adjusted dynamics of K is then modelled as a joint diffusion with V and r

$$\frac{dK(t)}{K(t)} = [r(t) - \delta_k]dt + \sigma_{kr} dZ_r(t) + \sigma_{kv} dZ_v(t)$$

where σ_{kr} and σ_{kv} are positive constants and δ_k is a constant payout rate to the debtholders of the firm. Being related to the debt of the firm, K has uncertainty related to the interest rates and to the value of the assets of the firm. The default boundary that we propose is therefore stochastic.

It is beyond the scope of this paper to derive a corporate finance model which endogenizes the contractual provisions defining default. Both the existence of default and the process followed by the default boundary are taken as given. However, we can provide an interpretation of the modeling approach to default adopted here.

In practice, default is triggered either by the value of the assets of the firm falling below the value of the liabilities (stock-based insolvency) or by the firm's failure to make a cash payment (flow-based insolvency). However, flow-based default only reflects the firm's incapability to obtain financing for its due payments. Facing a due payment, the firm can use its own available cash flow or funds raised by issuing new securities. Therefore, only the incapability of the firm to raise new funds will ultimately trigger flow-based default. Certainly, in a frictionless world, flow-based insolvency can only be due to stock-based insolvency, since the firm will only be unable to raise new funds when the total market value of its assets in place and investment opportunities is smaller than the current market value of the outstanding contractual obligations[11].

To be consistent with stock-based default, we must be able to define the event that triggers default as the first time the value of some assets, V, hits the value of some liabilities, K. Now, depending on the legal framework and the covenants negotiated in the debt issues, V can be the value of the total assets of the firm or their liquidation value, and K can be the (market) value of the total debt, or only of the senior debt, or can even be the present value, discounting at the default riskless rate, of the face value of the liabilities. In any of these cases, V and K are values of assets. Hence, under the risk adjusted probability measure, their drift should be equal to r, minus the payout rate.

We assume that, in the event of default at or prior to maturity, the bond pays a fixed value $1 - W$ per unit of face value at maturity. In reality, the payoff to a particular security in case of default depends on its degree of subordination, its collateral, and, more generally, on the nature of the bargaining game among the different corporate claimants to be played when default occurs. Positing a constant writedown has the advantage of being general in the sense that the outcome of financial distress does not have to be specified, that is, the firm can be liquidated or some form of restructuring can occur. Also, no explicit assumption has to be made about the priority of the security under study, and the security can be valued independently of the other securities issued by the firm. Thus, for example, the value a coupon bond will be equal to the value of a portfolio of zero-coupon bonds with the same promised payments. Additionally, it should be noted that, prior to default, there is very little information about the magnitude of the writedown.

[11] However, due to a number of reasons like asymmetric information problems, agency and legal conflicts, bankruptcy costs and others, this comparison of market values may not be the only criterion for investors to decide whether or not to invest new funds in the firm. Therefore, in the presence of market frictions stock-based insolvency and flow-based insolvency may be different.

The risky payoff of a corporate zero-coupon bond with maturity T is equivalent to the certain maturity payoff

$$C(T,T) = 1 - W\mathbf{1}_{\{\tau \leq T\}} \tag{3}$$

where τ is the first passage time of the value-of-the-firm process through the default boundary, so that $\mathbf{1}_{\{\tau \leq T\}}$ is an indicator function which takes the value one if the process of the value of the assets hits the default boundary during the life of the bond, and zero otherwise.

The first passage time is defined formally by

$$\tau = \inf\{u \geq t, V(u) = K(u)\} = \inf\{u \geq t, X(u) \equiv \log V(u) - \log K(u) = 0\} \tag{4}$$

We thus see that V and K do not matter directly to the valuation of default risky bonds but only through their ratio, which is a measure of the solvency of the firm. From Ito's lemma, the risk adjusted dynamics of X are given by

$$dX(t) = \mu dt + \sigma dZ_x(t) \tag{5}$$

where the constant drift and diffusion coefficients are given by

$$\mu = \delta_k - \delta_v - \frac{1}{2}\left(\sigma_v^2 - (\sigma_{kv}^2 + \sigma_{kr}^2 + 2\rho_{rv}\sigma_{kv}\sigma_{kr})\right) \tag{6}$$

and

$$\sigma^2 = (\sigma_v - \sigma_{kv})^2 + \sigma_{kr}^2 - 2\rho_{rv}(\sigma_v - \sigma_{kv})\sigma_{kr} \tag{7}$$

We define Z_x as a new Brownian motion

$$\sigma Z_x(s) = (\sigma_v - \sigma_{kv})Z_v(s) - \sigma_{kr}Z_r(s) \tag{8}$$

which is correlated with Z_r, with correlation coefficient

$$\rho = \frac{\rho_{rv}(\sigma_v - \sigma_{kv}) - \sigma_{kr}}{\sigma} \tag{9}$$

Note that, for zero payout rates,[12] when the volatility of the assets is higher than the volatility of the debt, the drift is negative. In this case, the (risk adjusted) probability of default goes to 1 as T goes to infinity.

In Longstaff and Schwartz (1995), the drift of the process that corresponds to our X is equal to the spot rate, r, minus a constant. We then see that their model can be made compatible with ours if the difference in the payout rates of the assets and the liabilities is equal to the spot rate. This assumption seems unduly restrictive and with no clear economic meaning.

[12] Which mean that any payments that are made to the debtholder of the company are financed by issuing new debt and payments to the equity holders are finance by new equity issues.

2.2 Pricing Bonds and Derivatives

We are now ready to price default risky bonds. Given our definition of the event of default and a constant writedown, we can rewrite (1) as

$$\begin{aligned} C(t,T) &= P(t,T) - W\mathrm{E}_t\left[\mathbf{1}_{\{\tau<T\}}e^{-\int_t^T r(u)du}\right] \\ &= P(t,T) - WP(t,T)\mathrm{E}_t^T[\mathbf{1}_{\{\tau<T\}}] \\ &= P(t,T) - WP(t,T)Q_t^T(\{\tau<T\}) \end{aligned} \quad (10)$$

where $Q_t^T(\{\tau<T\})$ is the forward, risk adjusted, probability of default before time T. The forward risk adjusted probability is the measure under which asset prices normalized by the price of the T-maturity default riskless bond are martingales.[13] Note that the writedown suffered at maturity, $W(T)$, can be random in our model, as long as it is uncorrelated with both the solvency ratio of the firm and interest rates. In this case, we would replace W in (10) with $\mathrm{E}_t[W(T)]$.

Under this forward risk adjusted probability measure, the dynamics of X are given by

$$dX(t) = [\mu - \rho\sigma s(t,T)]\,dt + \sigma dW_x(t) \quad (11)$$

where $s(t,T)$ is the (percent) volatility of the T-maturity Treasury bond price and W_x is a standard Brownian motion.[14]

The added term in the drift $\rho\sigma s(t,T)$ serves to correct for interest rate risk and its correlation with default risk, ρ. If this correlation between the two sources of risk is positive, which corresponds to $\rho<0$, there is an increase in the drift of X, making default less likely under the forward risk adjusted probability measure.[15] The reverse happens when ρ is positive.

To obtain more intuition, we can yet again rewrite (1) as

$$C(t,T) = P(t,T) - WP(t,T)Q_t(\{\tau<T\}) - W\mathrm{Cov}_t[\mathbf{1}_{\{\tau<T\}}, e^{-\int_t^T r(u)du}] \quad (12)$$

[13] See Duffie (1996). We will now refer to the risk adjusted probability measure with discounting at the instantaneous interest rate as the "spot risk adjusted probability measure" to distinguish it from the forward measure.

[14] Under the spot risk adjusted probbility measure, we can decompose Z_x into ρZ_r and $\sqrt{1-\rho^2}Z_o$, with Z_o a standard Brownian motion orthogonal to Z_r. Now define a new two-dimensional process W with dynamics

$$d\begin{pmatrix} W_1(t) \\ W_2(t) \end{pmatrix} = d\begin{pmatrix} Z_r(t) \\ Z_o(t) \end{pmatrix} - \begin{pmatrix} s(t,T) \\ 0 \end{pmatrix} dt$$

The two Brownian motions W_1 and W_2 are still orthogonal and therefore constitute a two-dimensional standard Brownian motion under the forward risk adjusted probability measure. The dynamics of X can be written with respect to W as

$$dX(t) = (\mu + \rho\sigma s(t,T))\,dt + \rho\sigma dW_1(t) + \sqrt{1-\rho^2}\sigma dW_2(t)$$

To simplify, we can define a standard Brownian motion W_x

$$W_x(t) = \rho W_1(t) + \sqrt{1-\rho^2}W_2(t)$$

and (11) follows.

[15] In this case, even if $\mu<0$, the probabiliy of default does not have to go to 1 as T goes to infinity.

where we use the definition of covariance. So, we see that, in general, the hitting time probabilities are different under the forward and spot risk adjusted measures. Only when the risk of default is independent of interest rates are the two equal. If the covariance is positive, the writedown will be incurred when interest rates are high, so that its expected value is low and corporate bond prices are higher.

The forward risk adjusted probability of default $Q_t^T(\{\tau < T\})$ can in general be computed by simulation. It is only necessary to generate a large number of sample paths of X,[16] from date t to date T, and compute the proportion of these that hit zero. We will see in the next section that in some special cases it is possible to obtain closed form solutions for the above probability and, in other cases, make use of efficient numeric approximations.

Finally, we note that coupon bonds can be valued simply as a portfolio of zero coupon bonds, with weights equal to the promised cashflows. Pricing risky bond derivatives is straightforward. The value at time t of a future random payoff $\Phi(T)$ at date T is given by

$$\Phi(t) = P(t,T)\mathrm{E}_t^T[\Phi(T)] \tag{13}$$

3 Examples

3.1 Independent Risks

When the event of default is independent of the instantaneous default riskless interest rate[17], $\rho = 0$, we have, from (12),

$$C(t,T) = P(t,T) - WP(t,T)Q_t(\{\tau < T\}) \tag{14}$$

and we can obtain a closed-form solution for corporate bond prices since $Q_t(\{\tau < T\})$ is just the probability that an arithmetic Brownian motion, X, will hit zero between times t and T, starting from an initial value $X(t) > 0$.

From Karatzas and Shreve (1991), the first passage time density of X evaluated at $\tau > t$ is

$$\phi(\tau) = \frac{X(t)}{\sigma(2\pi)^{1/2}(\tau - t)^{3/2}} \exp\left\{-\frac{[X(t) - \mu(\tau - t)]^2}{2\sigma^2(\tau - t)}\right\} \tag{15}$$

so that

$$Q_t(\{\tau < T\}) = 1 - \mathcal{N}\left(\frac{X(t) - \mu(T-t)}{\sigma\sqrt{T-t}}\right) + e^{\frac{2\mu}{\sigma^2}X(t)}\mathcal{N}\left(\frac{-X(t) - \mu(T-t)}{\sigma\sqrt{T-t}}\right) \tag{16}$$

where $\mathcal{N}$ denotes the standard normal cumulative distribution function.

We note that this model gives closed form corporate bond prices when coupled with any dynamic model of the term structure that produces closed form Treasury bond prices, such as Vasicek (1977)

[16] The process X may be a diffusion jointly with other state variables, depending on the form of the bond price volatility $s(t,T)$. In this case, it is necessary to generate the joint sample paths.

[17] That is, default risk is idiosyncratic or it is only related to systematic risks that are independent of interest rates.

or Cox, Ingersoll and Ross (1985), in single factor or multifactor versions. The solvency ratio X acts as a single additional state variable that, if assumed not to be directly observable, can be extracted from bond prices. Note that, given bond prices and the current level of X, only one parameter needs to be estimated, σ. This model is therefore as tractable as the usual implementations of Duffie and Singleton (1995),[18] who also impose independence between the factors to obtain closed form solutions for corporate bond prices.

The difference between our formulation and Longstaff and Schwartz (1995) is readily apparent in the case of independent risks. In their model the drift of X under the forward risk adjusted probability measure is $r + \mu$, instead of simply μ in our model. Thus, under their formulation, when interest rates go up, so does the drift of X, making the (forward) risk adjusted probability of default go down. This effect can be so important as to make the price of the risky bond increasing in r, implying negative duration. Note that this effect has nothing to do with a price adjustment for the covariation between the two sources of risk. It is only due to their modeling of the default boundary as a deterministic process rather than as an asset value.

3.2 Deterministic Bond Volatilities

Also of interest is the general case of deterministic (default riskless) bond volatilities.[19] Prominent examples in this category are given by Vasicek (1977) and several models in the Heath, Jarrow and Morton (1992) framework.

We assume that the percent volatility of bond prices, $s(t,T)$, is a deterministic function of time. In this case, we only need to compute the first hitting time probability of X (that starts at $X(t) > 0$) through zero in the interval $[t,T]$, where the dynamics of X are give by (11). This problem can be restated as the computation of the first hitting time probability of a standard Brownian motion, through a boundary that is a deterministic function of time. This boundary is given, for y in the interval $[t,T]$, by

$$B(y) = -\frac{X(t) + \mu(y-t)}{\sigma} - \rho \int_t^y s(u,T)du \tag{17}$$

This probability can be computed analytically only for the case of constant bond volatility, which does not make much sense. In general, we can use the results from Durbin (1992), who shows that the above first hitting time probability can be well approximated by

$$\begin{aligned} Q_t^T(\{\tau < T\}) \;\approx\; & \int_t^T \left(\frac{B(u)}{u} - B'(u)\right)\varphi(u)du \\ & - \int_t^T \int_t^u \left(\frac{B(v)}{v} - B'(v)\right)\left(\frac{B(u)-B(v)}{u-v} - B'(u)\right)\varphi(u,v)dvdu \end{aligned} \tag{18}$$

where $B'(u)$ denotes the slope of the boundary at u; $\varphi(u)$ is the density of ot the Brownian motion at time u, evaluated at $B(u)$; and $\varphi(u)$ is the joint density of the Brownian motion at times u and

[18]As multifactor term structure models, of the Cox, Ingersoll and Ross (1985) type, with one factor to model the term structure of default yield spreads.

[19]Coupled with any model for the default boundary, rather than the very special case of the previous subsection.

v, evaluated at $B(u)$ and $B(v)$. It is well known that

$$\varphi(u) = (2\pi(u-t))^{-1/2} \exp\left\{\frac{-(B(u)-B(t))^2}{2(u-t)}\right\} \tag{19}$$

and

$$\varphi(u,v) = \varphi(u)(2\pi(v-u))^{-1/2} \exp\left\{\frac{-(B(v)-B(u))^2}{2(v-u)}\right\} \tag{20}$$

The first hitting time probability can thus be easily computed numerically for most cases where the volatility of bond prices is tractable.

The numerical approximation for the first passage time probability from Durbin (1992) can be used with any model in the Heath, Jarrow and Morton (1992) framework, as long as the volatility function is a deterministic function of time. The simplest example in this class is the well known Ho and Lee (1986) model, for which $s(t,T) = s(T-t)$ with constant s.

3.3 Other Term Structure Models

In the more general case where Treasury bond prices depend on a set of state variables, corporate bond prices can still be valued by simulation. Only now the trajectories of X have to be simulated jointly with these state variables.

As a simple example, consider the one-factor Cox, Ingersoll and Ross (1995) model.[20] In this model, the state variable is the instantaneous interest rate, which is assumed to follow a square root process

$$dr(t) = a\,(b - r(t))\,dt + s\sqrt{r(t)}dZ_r(t) \tag{21}$$

Bonds can be priced in closed form and have volatilities equal to $s(t,T) = sB(t,T)\sqrt{r(t)}$ where

$$B(t,T) = \frac{2\left(e^{\gamma(T-t)}-1\right)}{(\gamma+a)\left(e^{\gamma(T-t)}-1\right)+2\gamma} \tag{22}$$

and $\gamma = \sqrt{a^2+2s^2}$.

In order to price corporate bonds, or credit risky derivatives, it is necessary to simulate jointly the following two processes

$$\begin{aligned} dr(t) &= \left[a\,(b-r(t)) + s^2 B(t,T) r(t)\right] dt + s\sqrt{r(t)}dW_1(t) \\ dX(t) &= \left[\mu + \rho\sigma s B(t,T)\sqrt{r(t)}\right] dt + \rho\sigma dW_1(t) + \sqrt{1-\rho^2}\sigma dW_2(t) \end{aligned} \tag{23}$$

where the two Brownian motions are orthogonal.

[20] The extension to the multi-factor case is straightforward.

4 Yield Spreads and Forward Spreads

We can obtain a better understanding of our model by looking at default yield spreads and forward spreads. We can write corporate yields as

$$\begin{aligned} y_c(t,T) &= -\frac{\log C(t,T)}{T-t} \\ &= -\frac{\log P(t,T)}{T-t} - \frac{\log\left(1 - WQ_t^T(\{\tau < T\})\right)}{T-t} \\ &\approx y_p(t,T) + \frac{WQ_t^T(\{\tau < T\})}{T-t} \end{aligned} \tag{24}$$

where the subscripts c and p refer to corporate and Treasury bonds.

We see that, since W is a fixed parameter and $Q_t^T(\{\tau < T\})$ (being a probability) is bounded above by 1, the term structure of default yield spreads must eventually go to zero.

For intuition, imagine that the first passage time density, of which $Q_t^T(\{\tau < T\})$ is the time integral, is constant forever.[21] Then, $Q_t^T(\{\tau < T\})$ would be the (constant) probability of default per year, conditional on no previous default, times $(T-t)$ years. So that the spread would be proportional to the yearly (conditional) probability of default.

Define now default risky instantaneous forward rates[22] by

$$C(t,T) = \exp\left\{-\int_t^T f_c(t,u)du\right\} \tag{25}$$

Then

$$\begin{aligned} f_c(t,T) &= -\frac{\partial \log C(t,T)}{\partial T} \\ &= -\frac{\partial \log P(t,T)}{\partial T} - \frac{\partial \log\left(1 - WQ_t^T(\{\tau < T\})\right)}{\partial T} \\ &= f_p(t,T) + \frac{W}{1 - WQ_t^T(\{\tau < T\})}\frac{\partial Q_t^T(\{\tau < T\})}{\partial T} \\ &= f_p(t,T) + \frac{W\left(1 - Q_t^T(\{\tau < T\})\right)}{1 - WQ_t^T(\{\tau < T\})}\frac{\partial Q_t^T(\{\tau < T\})/\partial T}{1 - Q_t^T(\{\tau < T\})} \\ &= f_p(t,T) + l(t,T)h(t,T) \end{aligned} \tag{26}$$

where l is the percent writedown on the value of the corporate bond in the event of current default, termed the "default loss rate"; and h is the conditional density of default at time T,[23] given survival

[21] Which obviously it cannot be.

[22] These are not literally forward rates in the sense that they would be interest rates contracted today for instantaneous borrowing or lending in the future by the firm, but we still define them by analogy to default riskless forward rates.

[23] The first passage time density of X through 0 evaluated at time T

until time T, which is the forward hazard rate. The difference in forward rates is therefore the expected "default loss rate".

Since the first passage time density goes to zero as T goes to infinity, the expected loss rate goes to zero, and the corporate forward rate converges to the Treasury forward rate as T increases without bound. This is in contrast to usual implementations of Duffie and Singleton (1995), where the expected loss rate is modeled as a state variable following a square root process. In this case, the difference in forward rates does not converge to zero but to a positive constant. There is no asymptotic resolution of uncertainty: the fact that the firm survives for a very long time does not affect its expected loss rate.

5 An Empirical Illustration

This section offers an illustration of the empirical uses of our model. We consider two parametric models examined in section 3: the model with independent risks (I) and the model with deterministic term structure of volatilities (II). In both models, we treat the solvency ratio X as an abstract state variable to be extracted from the data.

Note that our econometric method allows the joint estimation of the writedown parameter and the filtering of the solvency ratio. This means that we can recover both the (risk adjusted) term structure of probabilities of default and the expected payoff in case of default.

5.1 The Data

All data is obtained from Datastream. We obtain 42 monthly prices, from November 1993 to February 1997, of 22 bonds issued by RJR Nabisco.[24] All are simple coupon bonds, with no embedded options or sinking fund provisions, with ratings below investment grade.

We select a sample of 5 bonds from the above 22.[25] The selection tries to: obtain a variety of maturities;[26] select bonds that have existed for more than 4 years, to construct a sufficiently long sample size; and collect prices that seem to be reasonably liquid. This leaves us with a sample of 5 bonds, which are described in Table 5.1. Finally, we adjust prices for accrued interest.

The S&P Bond Guide lists bonds for RJR Nabisco and its subsidiary, Nabisco Inc. In November 1993, there were 20 bonds listed for RJR Nabisco, with a total outstanding amount of $9 billion. The rating for the notes and senior notes, including 17 of the 20 issues, was "BBB-" and the rating of the other 3 issues, all subordinated debentures, was "BB+". Of the 20 bonds outstanding, 13 had no call or sinking fund provisions.

By December 1996, RJR Nabisco had 12 bonds tracked in the S&P Bond Guide for a total outstanding amount of $3.7 billion. There were no subordinated issues left at this time and all remaining

[24] All prices refer to the 24th of the month, or the previous work day when the 24th was a holiday.

[25] The data is obtained from Datastream and checked against the S&P Bond Guides for the years 1993 through 1996.

[26] Of no more than 10 years, which is the longest risk free rate in our data.

Bond	1999 8.3%	2002 8 5/8%	2003 7 5/8%	2004 8 3/4%	2005 8 3/4%
Type Debt	Senior Notes	Notes	Notes	Senior Notes	Notes
Redemption Date	04/15/99	11/22/02	09/15/03	04/15/04	08/15/05
Life Remaining	2.13	5.76	6.55	7.13	8.46
Coupon Rate	8.300%	8.625%	7.625%	8.750%	8.750%
Coupon Dates	04/15 10/15	06/01 12/01	03/15 09/15	04/15 10/15	02/15 08/15
Issue Year	1992	1992	1992	1992	1993
Options Embedded	None	None	None	None	None
Sinking Fund	None	None	None	None	None
Bond Form	R	BE	BE	R	BE
Amount Issued	$600 million	$875 million	$750 million	$600 million	$500 million
Amount Outstanding	$61 million	$875 million	$750 million	$600 million	$500 million
S&P Rating 1993-97	BBB-	BBB-	BBB-	BBB-	BBB-

Table 1: This table reports the characteristics of the corporate bonds used in the empirical application. All bonds are plain vanilla coupon bonds, with no sinking fund provisions or option features. All amounts in millions of Dollars. The amount of the 1999 8.3% notes was reduced to $61.9 million in July of 1995. All data in the table as of December 1996.

issues were rated "BBB-". This rating did not change since the beginning of our sample in November 1993. Of the 12 bonds listed in December 1996, 11 were non-callble and had no sinking fund provisions.

The Nabisco subsidiary had 2 issues outstanding in November 1993. Both issues had sinking fund provisions and accounted for only $40 million. The rating was "BBB-". By December 1996, there were 7 issues outstanding for a total of $2.5 billion. All were non-callable, with no sinking fund provisions, and were rated "BBB".

The five bonds used in this study are all denoted as non-callable, with no sinking fund provisons. The amount of the issue outstanding was constant over our time period for four of the five bonds. The first bonds' amount outstanding decreased from $600 million in June of 1995 to $61.9 million in July of 1995. This amount remained constant thereafter. This decrease was the result of an exchange offer where the remaining $539.9 million was exchanged into a Nabisco bond with the exact same features.

Three of the five bonds are denoted as "notes" while the other two are denoted as "senior notes". The senior notes are registered while the notes are listed as book entry. There is no rating differentiation between notes and senior notes. The senior notes were formerly of RJR Nabisco Capital.

Finally, these bonds have sufficient liquidity. The S&P Bond Guide shows the month-end prices to be sale prices for over 4 of the 5 bonds every month. These issues also have a substantial amount issued, with $500 million being the minimum amount issued. Note that this is not the case of all RJR Nabisco bonds. For the bonds not in the sample, we often found exactly the same prices in the database for more than three months in a row. This suggests that the prices reported are not for the date give but for the last date in which they traded, which was possibly in a previous month. We selected our sample such that the bonds used have prices that change every month. Nevertheless, it is still possible that these prices are not all from the same day.

For the term structure of default riskless interest rates, we use daily data on Libor and swap rates,[27] between August 1993 and February 1997.[28] We obtain zero coupon rates by fitting the Libor and swap data with piecewise constant forward rates,[29] using least squares, as proposed by Coleman, Fisher and Ibbotson (1992). The prices of default riskless bonds to use in our pricing formulas, at each sample date, are obtained by taking the exponential of the relevant time integral of the forward rates.

For the model I, the data above is enough for the estimation method. Model II however requires estimates of the term structure of volatilities at each sample date.[30]

To obtain the term structure of volatilities, we first estimate the standard deviations of the changes in the forward rates. At each sample date, we use the previous 40 daily changes in the corresponding forward rate to compute its standard deviation. Then, we construct the volatilities of bond prices by integrating the piecewise constant forward rate standard deviations until the relevant maturity. We make use of the fact that the diffusion term of the T maturity bond, $P(t,T)s(t,T)$, can be written as $P(t,T)\int_t^T \sigma(t,y)dy$, where $\sigma(t,y)$ is the diffusion of the instantaneous forward rate with maturity y.

5.2 The Method

According to our model, the time t price of corporate bond j with coupon rate κ and M_j remaining coupon payment dates is

$$\hat{V}_{jt} = \sum_{m=1}^{M_j} C(t, t_m; \theta, X(t))\kappa + C(t, t_{M_j}; \theta, X(t)) \tag{27}$$

where $C(t,T;\theta)$ is computed using model I or II, and we make explicit the dependence on the parameter vector $\theta = (\mu, \sigma, \rho, W)$ and the state variable X.

We assume that at the true parameters and realization of the state variable, the observed bond

[27]Although, strictly speaking, these rates have some default risk, Grinblatt (1995) argues that they are closer to the true default riskless interest rates than Treasury bond rates, which have important liquidity premia incorporated.

[28]The Libor rates use have maturities of 7 days, 1, 3 and 6 months and 1 year. The swap rates have maturities of 2, 3, 4, 5, 7 and 10 years.

[29]With nodes at 3 and 6 months, 1, 2, 3, 4, 5, 7 and 10 years.

[30]Being deterministic, the term structure of volatilities should not change throughout the sample. However, following the usual practice in applying Heath, Jarrow and Morton (1992) models, we estimate a different term structure of volatilities at each date in the sample.

price, V_{jt}, is equal to the model price, $\hat{V}_{jt}$, plus a mean zero error term

$$V_{jt} = \hat{V}_{jt} + \varepsilon_{jt} \tag{28}$$

We further assume that the norm of the error term is minimized at the true parameter values and realized value of the state variable. The econometric method thus estimates the parameters and extracts the state variable X by minimizing the norm of the vector ε, with dimensions $j = 1, ..., 5$ and $t = t_1, ..., t_{42}$.

For model I, assuming independent risks, zero coupon default risky bond prices are computed with the formula (14), and, for model II, with deterministic term structure of bond volatilities, bonds are priced by simulation. In this case, at each sample date, t, and for each promised cash flow (of each bond in the sample), we produce 1000 simulated paths of a standard Brownian motion W_x and obtain the corresponding (forward risk adjusted) discretized process $\hat{X}$, starting at $X(t)$, from

$$\begin{aligned} \hat{X}(t+(m+1)h) &= \hat{X}(t+mh) + [\mu - \rho\sigma s(t+mh,T)]\,h \\ &+ \sigma\,[W_x(t+(m+1)h) - W_x(t+mh)] \end{aligned} \tag{29}$$

for $m = 1, ..., M$, where h is a daily interval (expressed in years) and M is the number of days between the sample date and the cash flow date.[31] We approximate $Q_t^T(\{\tau < T\})$ by the proportion of simulated paths of $\hat{X}$ that fall below zero and plug this estimated probability in our general formula for pricing corporate bonds (10).

The econometric method starts with an initial guess of the values of the parameter vector θ and of the state variable X at each sample date and then numerically minimizes the norm of the differences between the observed bond prices and the prices computed with our model. We state the estimation problem, in the context of GMM, as the minimization of a distance function of the form

$$J(\theta, X) = \sum_{t=1}^{42} (\hat{V}_t(\theta, X(t)) - V_t)' S^{-1} (\hat{V}_t(\theta, X(t)) - V_t) \tag{30}$$

where V_t and $\hat{V}_t(\theta, X(t))$ are the vectors of observed and computed model prices for the five bonds at date t, and S is a five-by-five, symmetric, positive definite weighting matrix.

We treat X as a vector of additional parameters. In doing this, we do not make full use of the implications of the model. In particular, we do not make use of the transition distribution of X, and thus avoid having to specify the drift of X under the true probability measure. Although not efficient, the method remains consistent.

The only remaining problem is determining S. We follow a two-step procedure. In the first step, we take S equal to the identity matrix. In the second step, we make S equal to the covariance matrix of the errors estimated in the first step. This is a consistent estimator of Hansen's (1982) optimal weighting matrix, under the assumption of no serial correlation in the errors.

[31] We use antithetic variables. In order to decrease the variance of the simulation, we obtain the increments of 500 simulated paths of the Brownian motion from a (pseudo) random number generator and the other 500 paths by taking the negatives of the increments generated in the first place. Prices vary less than 2 basis points when we take different sets of simulated paths. This variation is presumably smaller than the errors in the data. Finally, pricing of the 5 bonds, with 85 promised cash flows, at the 42 sample dates, takes approximately 5 minutes on a personal computer.

5.3 The Results

Table 2 shows parameter estimates, their respective standard deviations, and the value of the objective function at the optimum, for both models I and II.

Model	μ	σ	ρ	W	J
Model I	0.0466 (0.0035)	0.4501 (0.0087)	0	0.6816 (0.2765)	25.0530
Model II	-0.0324 (0.0014)	0.1492 (0.0007)	-0.0705 (0.0233)	0.4551 (0.1965)	15.4145

Table 2: This table reports parameter estimates and their standard errors for model I and II. The value of the criterium function, J, at the optimum is also shown for both models.

We can now test the constraint that $\rho = 0$. For this, we use a pseudo likelihood ratio test

$$l = J(\hat{\theta}_I, \hat{X}_I) - J(\hat{\theta}_{II}, \hat{X}_{II}) \tag{31}$$

Under the null, l is distributed as a χ_1^2. The p-value of the test is 0.0019, which leads us to reject the hypothesis. Interestingly thus, the dependence between interest rate and default risks appears to be important in explaining corporate bond prices.

The parameter estimates under model II are of the right magnitudes and signs. In particular, we estimate ρ to be negative, which corresponds to a positive correlation between the interest rate risk and default risk. The coefficient multiplied by the volatility of the Treasury bond prices increases the forward risk adjusted drift of X, amaking default less likely. Another parameter of interest is the writedown, which we estimate to be of the order of 45%. Therefore, in case of default, the market is expecting that only 55% of the face value of these securities may be recovered.

Figure 1 shows the estimated sample path of the state variable X under model II. For ease of interpretation we show the exponential of X, which can be interpreted as the solvency ratio of the firm.

Finally, Figure 2 provides a graphical illustration of the pricing errors under our parameter estimates and filtered state variable path. Although economically significant, they do not seem to be of a higher order of magnitude than the noise in the data. We thus conclude that the model thus a good job at pricing this sample of corporate bonds.

Note that there are other ways of fitting the model. The parameter W can be estimated from historical data rather than the implicit approach taken here. The forward risk adjusted hazard rate can as well be estimated from historical data on probabilities of default and the corresponding forward risk premium.

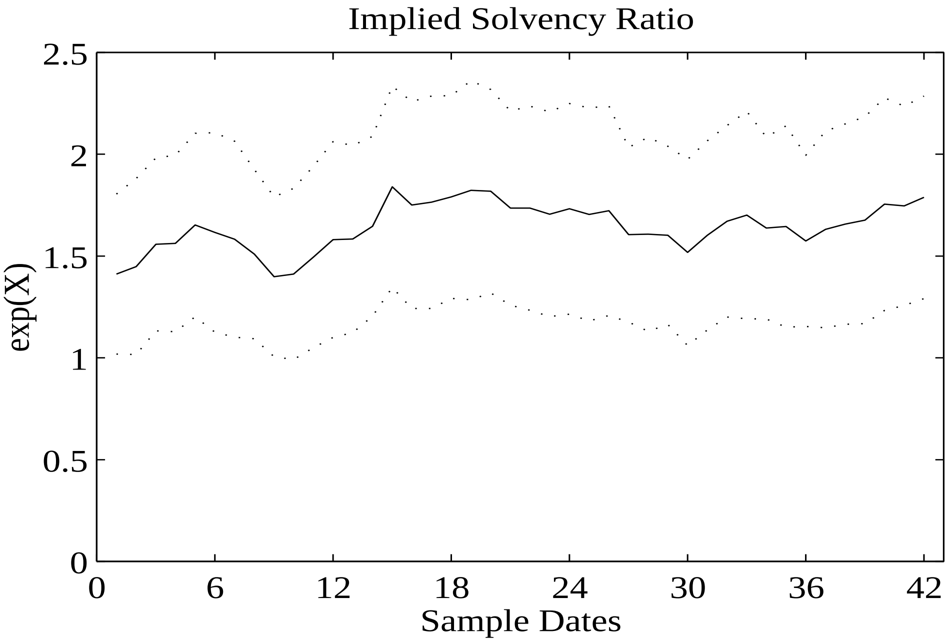

Figure 1: Model II implied sample path of the sovency ratio with two-standard-deviation interval. The related parameter estimates are presented in Table 2.

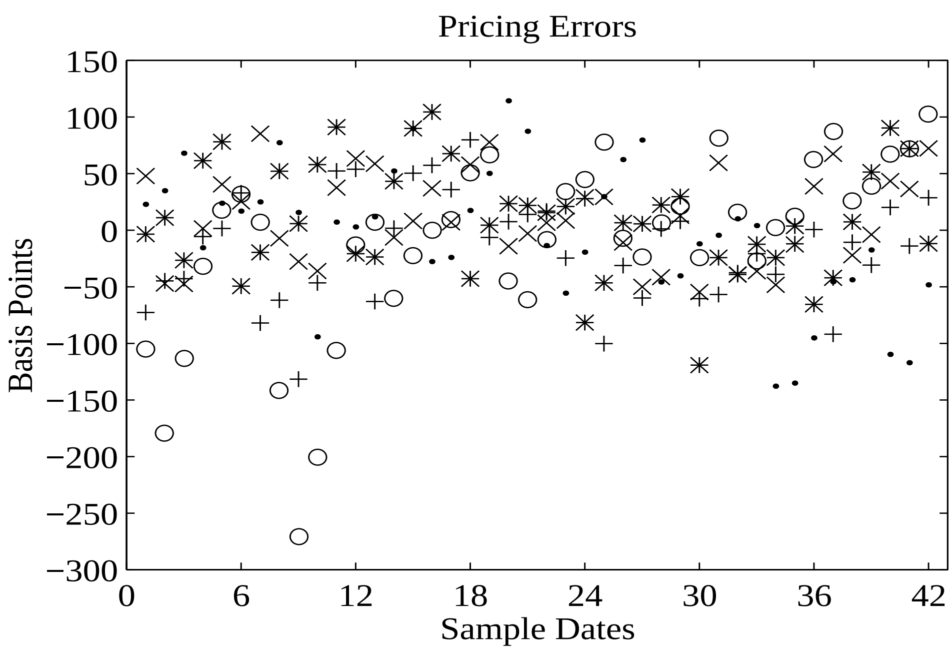

Figure 2: For the five bonds in the sample, at each sample date, the figure shows the deviation between the observed price and the price computed from the model, with the parameter estimates for model II presented in Table 2.

6 Conclusion

This article presents a model of default risk that allows for tractable pricing of default risky claims under a variety of models for the Treasury term structure of interest rates, and dependence between interest rate and default risks. We show that our model has better theoretical properties than others that have been proposed in the recent literature. We estimate the model from a panel data set of bond prices from RJR Nabisco, and show its good empirical performance.

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