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Jerome A. Rosenthal
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Contents

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ABSTRACT

A pulse modulator that offers several useful features is introduced. Some of its desirable features are that it can respond without delay to a change in input signal and that it is of very simple construction. The modulator can be used as an amplifier, a multiplier, or a divider. It can also be used to control either proportional or on-off systems.

I. INTRODUCTION

This pulse modulator resulted from a desire to build a simple and efficient power amplifier. It has been known for some time that an amplifier with a saturated pulse output can satisfy both of those requirements, since such an amplifier is easy to construct and dissipates very little power in its output stages. Most commonly, pulse-width modulation amplifiers have been built to satisfy these needs. Operationally, a pulse-width modulator periodically samples the amplitude of an input signal and generates a pulse whose width is proportional to this amplitude (Fig. 1c). In terms of physical components, a pulse-width modulator usually consists of a ramp generator and a comparator.

The pulse modulator discussed in this paper can be of simple construction, requiring only a relay, a resistor, and a capacitor, although more complicated designs will give better performance. The modulator operates by continuously sensing (not sampling) the input signal. The output of the modulator is a pulse train whose frequency and pulse duration both vary with the input signal. By continuously sensing the input signal, the modulator can respond instantaneously and with maximum effort to a change in the input signal. Unlike pulse-width modulation, the output pulses of this modulator have a minimum duration, thereby placing less stringent requirements on the frequency response of the elements in the system. As with other pulse modulators, this modulator is capable of both proportional and on-off control.

While analyzing the modulator, we discovered that it could easily be converted into an analogue multiplier or divider. All three uses of the modulator (amplification, multiplication, and division) can be performed separately or in combination.

Before we discuss the modulator, it might be of interest to review some of the other forms of modulation.

A. Continuous-Wave Modulation

Methods of modulation can be divided into two convenient categories — — continuous-wave modulation and pulse modulation. In continuous-wave modulation one of the parameters of a sinusoid are

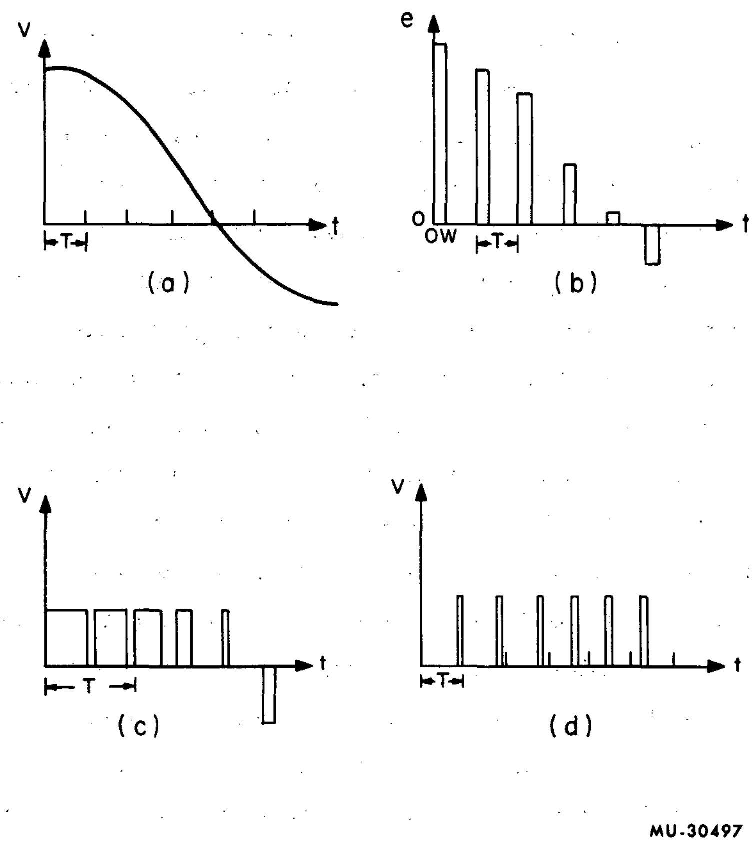

Fig. 1. Various kinds of pulse modulation

(a) The signal to be modulated
(b) Pulse amplitude modulation
(c) Lead type pulse width modulation
(d) Pulse position modulation.

varied as a function of frequency. The parameters that are usually varied are amplitude (amplitude modulation), instantaneous frequency (frequency modulation), or phase (phase modulation). The principle of superposition applies to amplitude modulation, but does not apply, in general, to the other forms of continuous-wave modulation. In all forms of continuous-wave modulation, the modulated wave form can respond instantaneously to a change in the modulating signal.

B. Pulse Modulation

Pulse modulation consists of varying some of the parameters of a pulse train as a function of the modulating signal. The parameters that are usually varied, whether singly or in combination, are pulse amplitude, pulse width, pulse position, or pulse frequency.

C. Pulse-Amplitude Modulation

Operationally, pulse-amplitude modulation (PAM) consists of multiplying a carrier (pulse train) by the modulating signal. An analytic expression for a PAM signal, with constant period and pulse width is

$$m(t) = e(t) \sum_{k=-\infty}^{\infty} u(t - kT) - u(t - kT - w) , \qquad (1)$$

where u is the unit step, T is the period of the pulse train, and w is the pulse width. Figure 1b shows a PAM signal.

At times the pulse width or pulse rate of a PAM signal is not constant, but may vary either with or independently of the modulating signal. Often that is the case when a time-shared digital computer is part of the system. The pulse rate might vary independently of the signal if each sampling pulse were generated only after the computer had finished processing other information. On the other hand, a signal dependent sampling rate might arise if one wished to make economical use of the computer by sampling frequently when the signal were changing rapidly, and sampling infrequently when the signal were relatively constant.

If the pulse rate is much greater than the highest significant frequency in the modulating signal, then, as a mathematical artifice, the pulse train may be replaced by an impulse train. An impulse train has a mathematically tractable Laplace transform and lends itself to z-transform techniques; z-transforms are widely used to analyze PAM systems.[1]

Pulse-amplitude modulation is used in time multiplex systems and sampled-data control systems. A PAM system, unlike continuous-wave systems, cannot always respond instantaneously to a change in the modulating signal, since pulses are present only a fraction of the time. This property introduces an inherent delay in the PAM process.

D. Pulse-Width Modulation

Pulse-width modulation (PWM) consists of operating on a pulse train by varying the width of its member pulses as a function of the modulating signal. The pulse width can be modulated by varying the position of the leading edge of the pulse (the position of the trailing edge being held constant), by varying the position of the trailing edge (keeping the position of the leading edge constant), or by varying the position of both edges. These three types of PWM are called, respectively, lead type, lag type and lead-lag type.[1, 2, 3]

An analytic expression for lead type PWM is

$$m(t) = \sum_{k=-\infty}^{+\infty} \left\{ u(t - kT) - u\Big(t - kT - w(k)\Big) \right\} \operatorname{sgn} e(kT) , \qquad (2)$$

where $e(kT)$ means the value of e at time kT, sgn is the signum function,

$$\operatorname{sgn} x = \begin{cases} 1 & \text{if} \quad x > 0 \\ -1 & \text{if} \quad x < 0 \end{cases} ,$$

$w(k)$ is the width of the $\underline{k}$th pulse, which is expressed by

$$w(k) = T \text{ sat } \frac{d}{t} \left|e(kT)\right| ,$$

where d is a positive constant, and the saturation function is defined as

$$\text{sat } x = \begin{cases} 1 & \text{if} \quad x > 1 \\ x & \text{if} \quad |x| \leq 1 \\ -1 & \text{if} \quad x < -1 \end{cases} .$$

An analytic expression for lag type PWM is

$$m(t) = \sum_{k=-\infty}^{+\infty} \left\{ u[t - (k+1)T + w(k)] - u[t - (k+1)T] \right\} \text{sgn } e(kT) . \quad (3)$$

An analytic expression for lead-lag PWM is

$$m(t) = \sum_{k=-\infty}^{+\infty} \left\{ u\left(t - \frac{kT + (k+1)T}{2} + \frac{w(kT)}{2}\right) - u\left(t - \frac{kT + (k+1)T}{2} - \frac{w(kT)}{2}\right)\right\} \text{sgn } e(kT) . \quad (4)$$

Figure 1c shows lead-type PWM.

Pulse-width modulation is used in sampled-data systems and in power amplifiers. This form of modulation is well suited, from a device point of view, to power amplifiers because it results in minimum power dissipation in the stages driving the load. Minimizing the dissipation in the driving stages is especially important when power is scarce (as in space ships) or where increased dissipation increases the complexity of the hardware (as in transistorized power stages).

From an analytical point of view, PWM is more difficult than PAM. This difficulty results from the fact that the principle of superposition does not apply to PWM. Therefore, linear theory cannot be freely used to analyze PWM systems. Some of the analytical tools that have been used to study this form of modulation[4-8] are z-transforms, difference equations, the second theorem of Lyapunov, describing functions, and phase-plane analysis.

As with PAM, a pulse-width modulator is insensitive to changes in the modulating signal at times other than the sampling instants. Therefore, PWM exhibits an inherent delay.

This discussion has been limited to a pulse-width modulator that acted on the magnitude of the modulating signal. Of course, the modulator could act on any of a number of functions of the modulating signal.[2]

E. Pulse-Position Modulation

In pulse-position modulation[3] (PPM) one operates on a pulse train by varying the position of its member pulses as a function of the modulating signal. An analytic expression for PPM is

$$m(t) = \sum_{k=-\infty}^{+\infty} \left\{ u[t - T(k + \tfrac{1}{2}) - p(kT)] - u[t - T(k + \tfrac{1}{2}) - p(kT) + w] \right\}, \tag{5}$$

where

$$p(kT) = \frac{T}{2} \text{ sat } \frac{d}{T} e(kT).$$

Figure 1d shows a PPM signal.

Pulse-position modulation has not been widely applied, although it has found some uses in telephony.[3] As with PWM, the principle of superposition does not apply to PPM. In general, a pulse-position modulator is incapable of responding immediately to a change in the modulating signal. Thus, PPM exhibits an inherent delay.

F. Pulse-Ratio Modulation

Pulse-ratio modulation (PRM), which is most like the type of modulation described in this paper, consists of varying both the pulse frequency and pulse width as a function of the input signal.[9] This has been done by turning two current sources on and off. The current in one source is proportional to the input voltage and the current in the other source is proportional to 1 minus the input voltage: we have

$$I_1 = X I_c ,$$

and

$$I_2 = (1 - X) I_c ,$$

where X is the normalized input voltage, I_c is a constant, and I_1 and I_2 are the currents in the two sources.

The time that the pulse is on is the time it takes to change the voltage across a capacitor by ΔV when the first current source is connected to the capacitor. The time that the pulse is off is the time that it takes to change the voltage across the capacitor by ΔV when the second current source is connected across the capacitor. The on and off times, then, are determined by the time taken to move a fixed distance along a voltage trajectory, the slope of the trajectory being a function of the input signal. Schaefer describes a circuit in which a negative resistance is used to switch the two current sources at the appropriate times.[9] His expressions for t_{on} and t_{off} are

$$t_{on} = \frac{CR_n}{1 - X} ,$$

and

$$t_{off} = \frac{CR_n}{X} ,$$

where C is the capacitor being charged, and R_n is the value of the negative resistance.

These equations are somewhat similar to the equations derived for a tristable variety of our modulator [Eqs. (44) and (45)]. This similarity occurs because both the Schaefer modulator[9] and our modulator operate on timing trajectories whose slopes are functions of the input signals (the modulator described in this paper uses an exponential trajectory).

A pulse-ratio modulator is capable of responding instantaneously to a change in the input signal and, therefore, it does not exhibit delay. The principle of superposition does not apply to PRM systems.

G. Pulse-Code Modulation

In pulse-code modulation (PCM),[3] the amplitude of the modulating signal is transformed into a coded train of pulses (usually a binary code) and this train of pulses constitutes the modulated signal. An easy way of performing PCM is to sample the modulating signal and process the samples in an analogue-to-digital converter. Thus, PCM is basically PAM with the amplitude information appearing digitally, rather than as an analogue signal. The principle of superposition does not apply to PCM, which exhibits the same type of inherent delay as do the other pulse modulators discussed thus far.

H. Some Other Types of Pulse Modulation

In pulse-frequency modulation (PFM) the repetition rate of a constant width and amplitude pulse is varied as a function of the modulating signal. As with other forms of modulation, the principle of superposition does not apply and the modulator exhibits an inherent delay.

In integral pulse-frequency modulation (IPFM), the modulator produces a pulse of fixed width and amplitude whenever the time-integrated value of the modulating signal reaches a fixed value. The integrator is reset to zero after each output pulse. This form of modulation is used in electrical analogues of nerve action. The principle of superposition does not apply and the modulator exhibits an inherent delay.

II. DESCRIPTION OF THE MODULATOR

As can be seen from Fig. 2, the main element of the modulator is a relay (or regenerative switch). There is a feedback path going from the output to the input of the relay; the circuit elements of the feedback path are a low pass filter and an attenuator. The contact voltages of the relay are $+A$ volts and $-A$ volts, the hysteresis of the relay is $2\Delta V$ and the transfer function of the attenuator is B.

In a very general way, certain things can be said of the waveforms at different parts of the circuit. Assume that there is a dc voltage impressed on the input. The output voltage, V_0, oscillates between two values, $+A$ and $-A$. The feedback path passes only the low frequency components of the output and attenuates them by a factor of B. The feedback voltage, V_f, is subtracted from the input voltage to form the error voltage, V_e. Now, the action of the circuit is such as to keep the magnitude of the error voltage small. If the magnitude of the error voltage exceeds ΔV, the output changes polarity, thereby decreasing the magnitude of the error voltage. Therefore, if the input is a dc voltage, the amplitude of the error voltage will oscillate between $+\Delta V$ and $-\Delta V$. Since the error voltage is kept small, the feedback voltage, V_f, must be approximately equal to the input voltage, V_1. This means that the output of the low pass filter must be approximately equal to $1/B$ times the input voltage. Therefore, the dc value of the output must be approximately equal to $1/B$ times the input voltage. To summarize this paragraph, we have seen that the modulator has converted the dc input into a signal that oscillates between $+A$ volts and $-A$ volts and that the dc value of the output is approximately equal to $1/B$ times the input (the gain of the modulator is $1/B$). We may also note that the low pass filter in the feedback loop acts as a demodulator of the output voltage.

Figure 3 shows some waveforms that were obtained from an experimental model of the modulator. The results of the experiment are discussed in detail in Sec. V. It might be helpful now to examine the waveforms in order to better understand the operation of the modulator. The upper trace in Fig. 3 is the input waveform, V_1. The middle trace

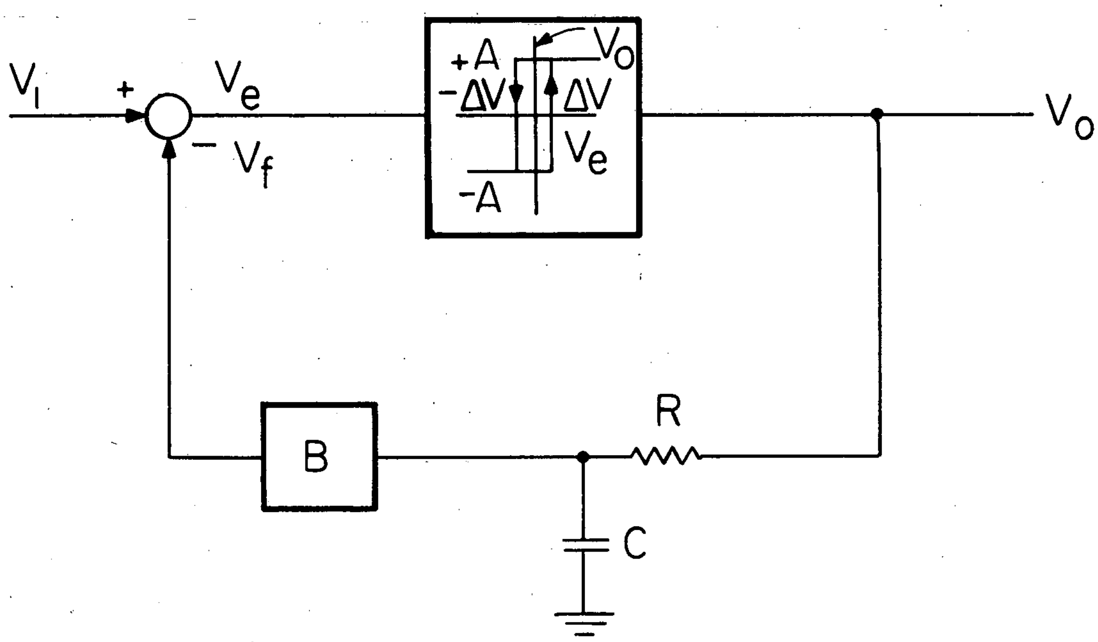

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Fig. 2. The basic modulator (bistable version).

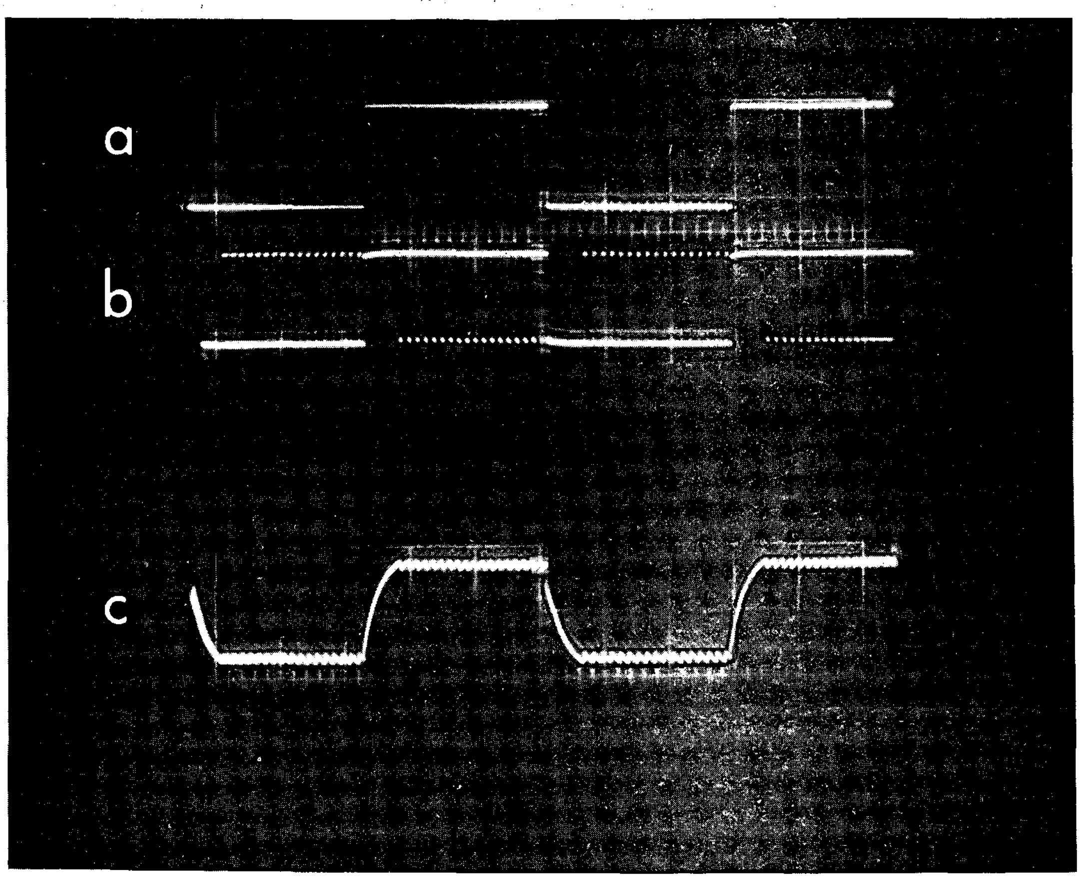

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Fig. 3. Wave forms exhibited in experimental modulator

(a) Input voltage Time scale: 2 msec/division
(b) Output voltage Voltage scale: (a) and (c) 1 V/division
(c) Feedback voltage (b) 10 V/division.

is the output voltage, V_0. It can be seen that, for a positive input, the swinger of the relay spends most of its time on the contact that has +A volts on it. For a negative input voltage the swinger spends most of its time on the -A volt contact. The lowest trace is a picture of the feedback voltage, V_f. It looks very much like the input voltage, except that it has a ripple that is approximately equal to $2\Delta V$.

A more precise relationship between the input and the output voltage can be derived with the aid of Fig. 4. As indicated in that figure, the feedback voltage, V_f, will follow an exponential trajectory in time. The trajectory may be either positive going or negative going. Two representative trajectories are shown in Fig. 4.

Consider the steady state response to a dc input, V_1. The dwell time on the positive contact equals the time taken to travel from $V_1 - \Delta V$ to $V_1 + \Delta V$ on the positive going trajectory. The dwell time on the negative contact equals the time taken to travel from $V_1 + \Delta V$ to $V_1 - \Delta V$ on the negative going trajectory. These times are functions of V_1.

The positive and negative dwell times may be derived as follows* (Fig. 4):

$$V_1 + \Delta V = -AB(1 - 2e^{-t_1/T}) = AB(1 - 2e^{-t_4/T}) , \tag{6}$$

and

$$V_1 - \Delta V = -AB(1 - 2e^{-t_2/T}) = AB(1 - 2e^{-t_3/T}) , \tag{7}$$

where $T = RC$. From Eqs. (6) and (7), we obtain

$$\frac{V_1 + \Delta V + AB}{V_1 - \Delta V + AB} = \frac{e^{-t_1/T}}{e^{-t_2/T}} . \tag{8}$$

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* The modulator has a relatively linear gain function for $-AB + \Delta V \leq V_1 \leq AB - \Delta V$. Outside of this range the modulator saturates. Unless otherwise noted, all equations apply only to inputs within the linear range of the modulator.

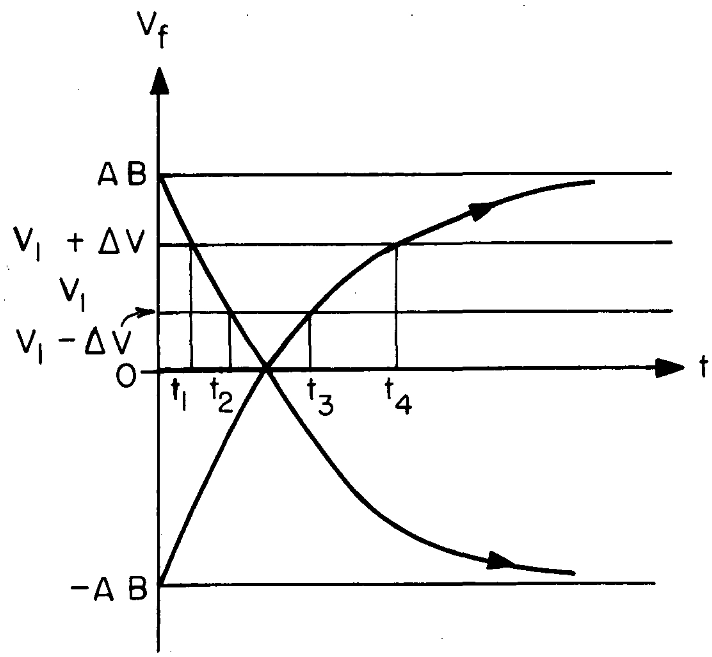

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Fig. 4. Typical operating trajectories for the bistable modulator.

If the negative dwell time is called t_{off}, then we have

$$t_{off} = t_2 - t_1 = T \ln\left(\frac{V_1 + \Delta V + AB}{V_1 - \Delta V + AB}\right) = T \ln\left(\frac{1 + \frac{\Delta V}{AB + V_1}}{1 - \frac{\Delta V}{AB + V_1}}\right). \quad (9)$$

To find the expression for the positive dwell time, t_{on}, we again use Eqs. (6) and (7):

$$\frac{AB - V_1 + \Delta V}{AB - V_1 - \Delta V} = \frac{e^{-t_3/T}}{e^{-t_4/T}},$$

$$t_{on} = t_4 - t_3 = T \ln\left(\frac{AB - V_1 + \Delta V}{AB - V_1 - \Delta V}\right),$$

and

$$t_{on} = T \ln\left(\frac{1 + \frac{\Delta V}{AB - V_1}}{1 - \frac{\Delta V}{AB - V_1}}\right). \quad (10)$$

Plots of t_{on}, t_{off} and the period, $P = t_{on} + t_{off}$, for $\Delta V = 0.01$ V, $AB = 1$ V, and $T = 10^{-3}$ sec, appear in Fig. 5.

An alternative, but only approximate derivation of the positive and negative dwell times (Fig. 6) is based on the assumption that the slope of the exponential is constant during the dwell time. Thus, on the positive going trajectory (Fig. 6),

$$V_1 = AB(1 - 2e^{-t_1/T}). \quad (11)$$

The slope of the positive going trajectory at V_1 is

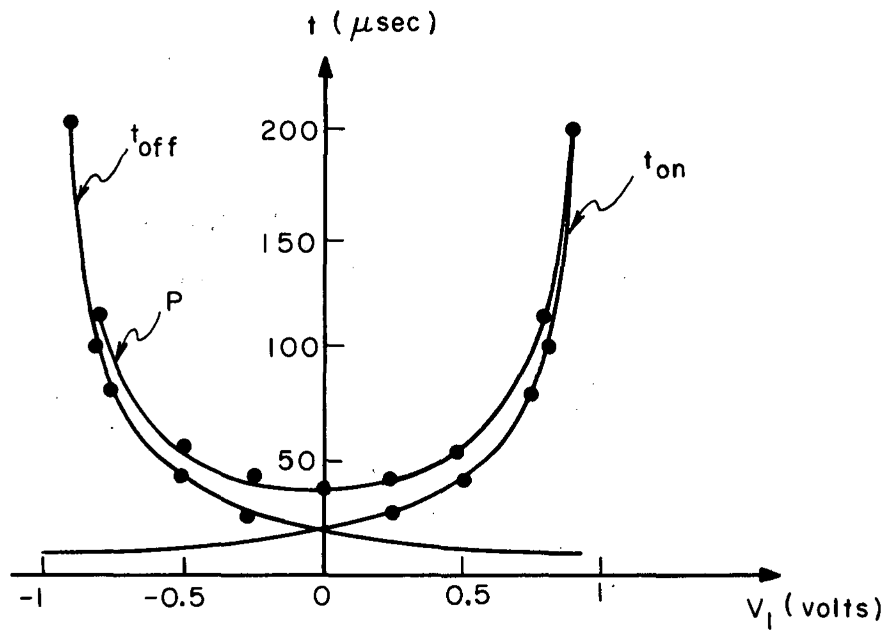

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Fig. 5. t_{on}, t_{off} and $P = t_{on} + t_{off}$ when $\Delta V = 0.01$ volts, AB = 1 volt, and $T = 10^{-3}$ sec (bistable modulator).

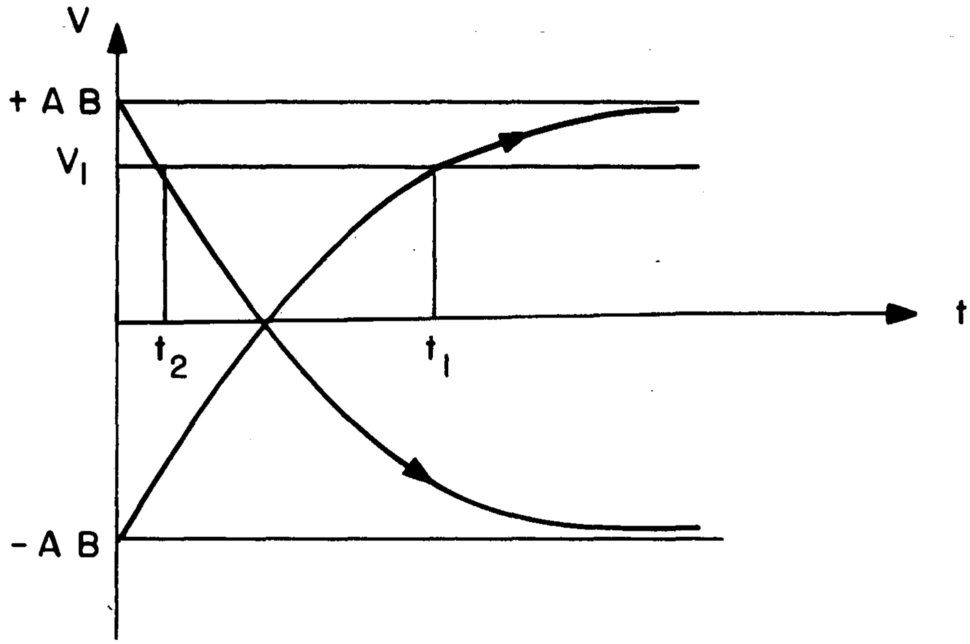

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Fig. 6. Typical trajectories for the bistable modulator (approximate method).

$$m_1 = \frac{2AB}{T} e^{-t_1/T} . \tag{12}$$

Combining Eqs. (11) and (12), we obtain

$$2AB\, e^{-t_1/T} = AB - V_1 \quad \text{and} \quad m_1 = \frac{AB - V_1}{T} . \tag{13}$$

If we call the approximate value of the positive dwell time that is derived via these equations t^*_{on}, then

$$t^*_{on} = \frac{2\, \Delta V}{m_1} . \tag{14}$$

Therefore we have

$$t^*_{on} = \frac{2\, \Delta V T}{AB - V_1} . \tag{15}$$

To find the approximate expression for t_{off}, t^*_{off}, it is necessary to find the slope of the negative going trajectory at V_1 (Fig. 6). We have

$$V_1 = -AB(1 - 2e^{-t_2/T}) . \tag{16}$$

The slope of the negative going trajectory at V_1 is

$$m_2 = - \frac{2\, AB}{T} e^{-t_2/T} . \tag{17}$$

Combining Eqs. (16) and (17),

$$m_2 = - \frac{V_1 + AB}{T} . \tag{18}$$

But $t_{off}^* = -2\ \Delta V/m_2$. So, we see that

$$t_{off}^* = \frac{2\ \Delta V T}{AB + V_1}\ . \tag{19}$$

Plots of t_{on}^*, t_{off}^* and $P^* = t_{on}^* + t_{off}^*$ for $AB = 1\ V$, $\Delta V = 0.01\ V$ and $T = 10^{-3}$ sec, appear in Fig. 7.

Since the approximate expressions for t_{on} and t_{off} [Eqs. (15) and (19)] are more tractable than the exact expressions [Eqs. (9) and (10)], it is convenient to use the approximate expressions. We can justify doing this as follows:

From Eqs. (10) and (15) we have

$$\frac{t_{on}}{T} = \ln\left(\frac{1 + t_{on}^*/2T}{1 - t_{on}^*/2T}\right), \tag{20}$$

and by writing the series expansion for t_{on}/T we obtain

$$\frac{t_{on}}{T} = \frac{t_{on}^*}{T} + \frac{2}{3}\left(\frac{t_{on}^*}{2T}\right)^3 + \frac{2}{5}\left(\frac{t_{on}^*}{2T}\right)^5 + \cdots\ . \tag{21}$$

So if we have $t_{on}^*/2T << 1$, then $t_{on} \cong t_{on}^*$.

The relative error is

$$\frac{t_{on} - t_{on}^*}{t_{on}} = \frac{\frac{2}{3}\left(\frac{t_{on}^*}{2T}\right)^3 + \frac{2}{5}\left(\frac{t_{on}^*}{2T}\right)^5 + \frac{2}{7}\left(\frac{t_{on}^*}{2T}\right)^7 + \cdots}{\frac{t_{on}^*}{T} + \frac{2}{3}\left(\frac{t_{on}^*}{2T}\right)^3 + \frac{2}{5}\left(\frac{t_{on}^*}{2T}\right)^5 + \cdots}\ . \tag{22}$$

When the modulator is in its linear range, V_1 ranges from $-AB + \Delta V$ to $AB - \Delta V$. This range for V_1 corresponds to a range for $t_{on}^*/2T$ [Eq. (15)] that goes from $\frac{\Delta V}{2\ AB - \Delta V}$ to 1 ($\frac{\Delta V}{2\ AB - \Delta V}$ is much less than 1 if $AB \gg \Delta V$). Since we find $t_{on}^*/2T \leqslant 1$ for all values of

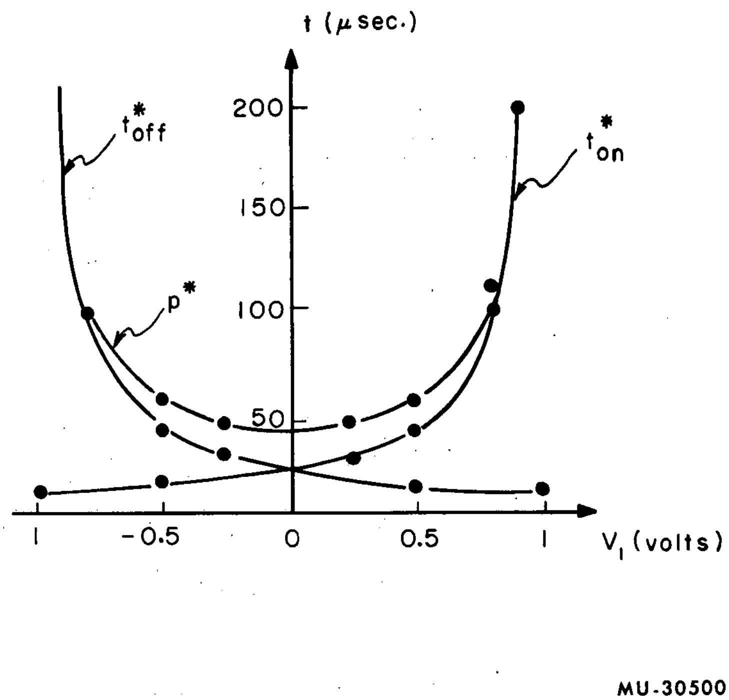

Fig. 7. t^*_{on}, t^*_{off}, and $P^* = t^*_{on} + t^*_{off}$ when $\Delta V = 0.01$ volts, AB = 1 volt, and $T = 10^{-3}$ sec (bistable modulator).

V_1 within the linear range of the modulator and since the infinite series $2/3 + 2/5 + 2/7 + \cdots$ diverges, the maximum relative error is 1.

To determine the relationship between t_{off} and t^*_{off}, examine Eqs. (9) and (19):

$$\frac{t_{off}}{T} = \ln\left(\frac{1 + t^*_{off}/2T}{1 - t^*_{off}/2T}\right), \tag{23}$$

and

$$\frac{t_{off}}{T} = \frac{t^*_{off}}{T} + \frac{2}{3}\left(\frac{t^*_{off}}{2T}\right)^3 + \frac{2}{5}\left(\frac{t^*_{off}}{2T}\right)^5 + \cdots . \tag{24}$$

So if we have $\frac{t^*_{off}}{2T} << 1$, then $t_{off} \cong t^*_{off}$.

The relative error is

$$\frac{t_{off} - t^*_{off}}{t_{off}} = \frac{\frac{2}{3}\left(\frac{t^*_{off}}{2T}\right)^3 + \frac{2}{5}\left(\frac{t^*_{off}}{2T}\right)^5 + \frac{2}{7}\left(\frac{t^*_{off}}{2T}\right)^7 + \cdots}{\frac{t^*_{off}}{T} + \frac{2}{3}\left(\frac{t^*_{off}}{2T}\right)^3 + \frac{2}{5}\left(\frac{t^*_{off}}{2T}\right)^5 + \cdots} . \tag{25}$$

It can be seen that the expression for the relative error in t_{off} is of the same form as that for t_{on}. Furthermore, $t^*_{off}/2T$ also ranges between 1 and $\frac{\Delta V}{2\,AB - \Delta V}$. Therefore, the relative error in t_{off} never exceeds 1. A representation of the per cent error in t_{on} as a function of V_1 and as a function of t^*_{on}/T (when $\Delta V = 0.01$ V, $AB = 1$ V, and $T = 10^{-3}$ sec) is shown in Fig. 8. The representation for the per cent error in t_{off} would be of the same shape. As can be seen from Fig. 8, the maximum value of V_1 (with the modulator still being in its linear range) is 0.99 V. When V_1 is as high as 0.9 V, the error is only 0.6%. We may then conclude that the approximation is a fairly valid one over 90% of the range of the modulator.

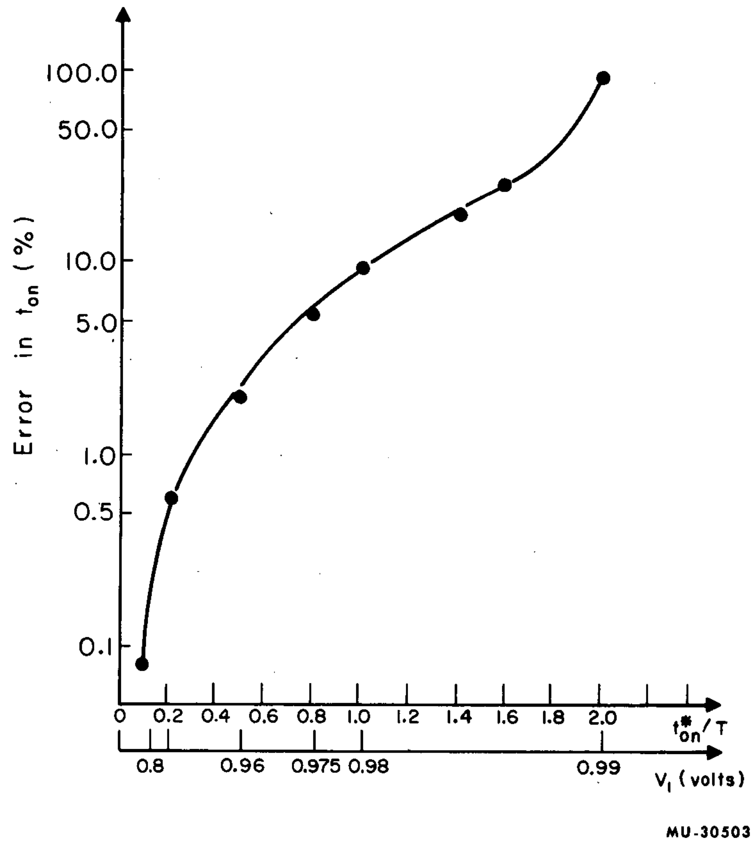

Fig. 8. Percent error for t^*_{on} relative to t_{on} when $\Delta V = 0.01$ volts, AB = 1 volt, and $T = 10^{-3}$ sec.

In the rest of this analysis we will use the simple expressions for t_{on} and t_{off} and not use the starred notation.

The period of the pulse train is (Fig. 9)

$$P = t_{on} + t_{off} = 4\,\Delta VT\left(\frac{AB}{(AB)^2 - V_1^2}\right). \tag{26}$$

The duty cycle of the pulse train is (Fig. 10)

$$D_0 = \frac{t_{on}}{t_{on} + t_{off}} = \frac{AB + V_1}{2\,AB}. \tag{27}$$

The dc value of the output voltage is

$$V_0 = AD_0 - A(1 - D_0) = A(2D_0 - 1). \tag{28}$$

Combining Eqs. (27) and (28)

$$V_0 = \frac{V_1}{B}. \tag{29}$$

The dc gain of the circuit is

$$K = \frac{V_0}{V_1} = \frac{1}{B}. \tag{30}$$

III. THE TIME RESPONSE OF THE MODULATOR TO OTHER THAN STEADY STATE DC INPUTS

It is difficult to find the time response of the modulator to a generalized input signal. However, it is possible, for certain types of signals, to find useful analytic expressions for the time response.

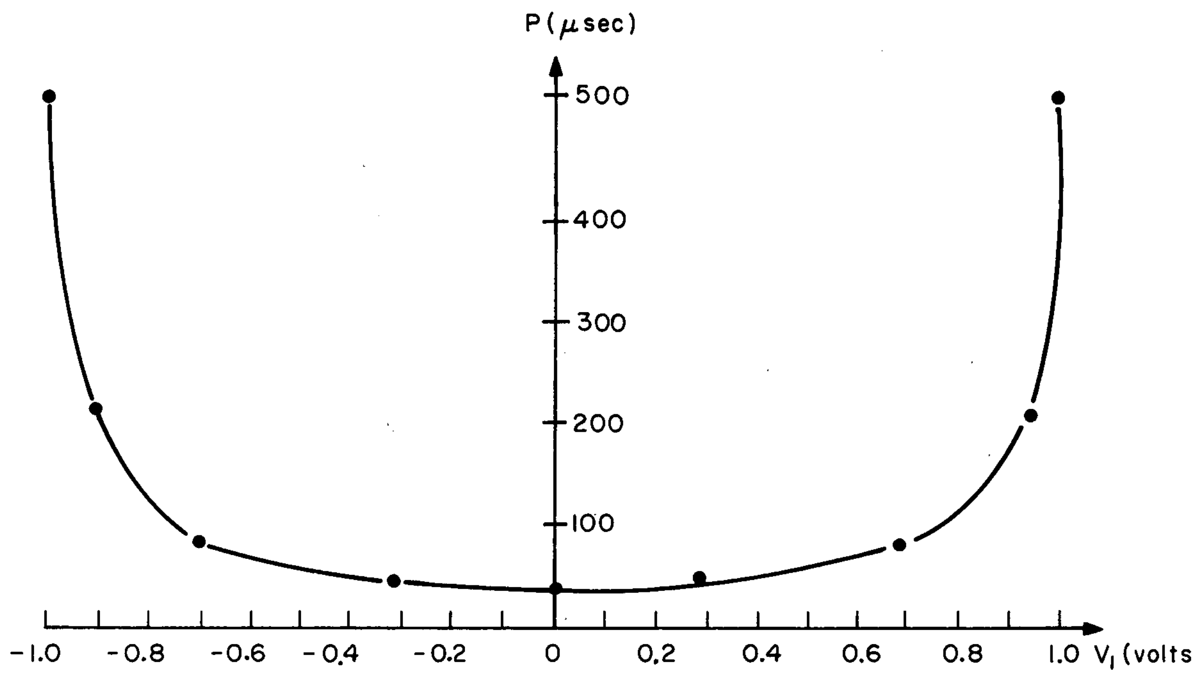

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Fig. 9. The period of the pulse train when $\Delta V = 0.01$ volts, AB = 1 volt, and $T = 10^{-3}$ sec (bistable modulator).

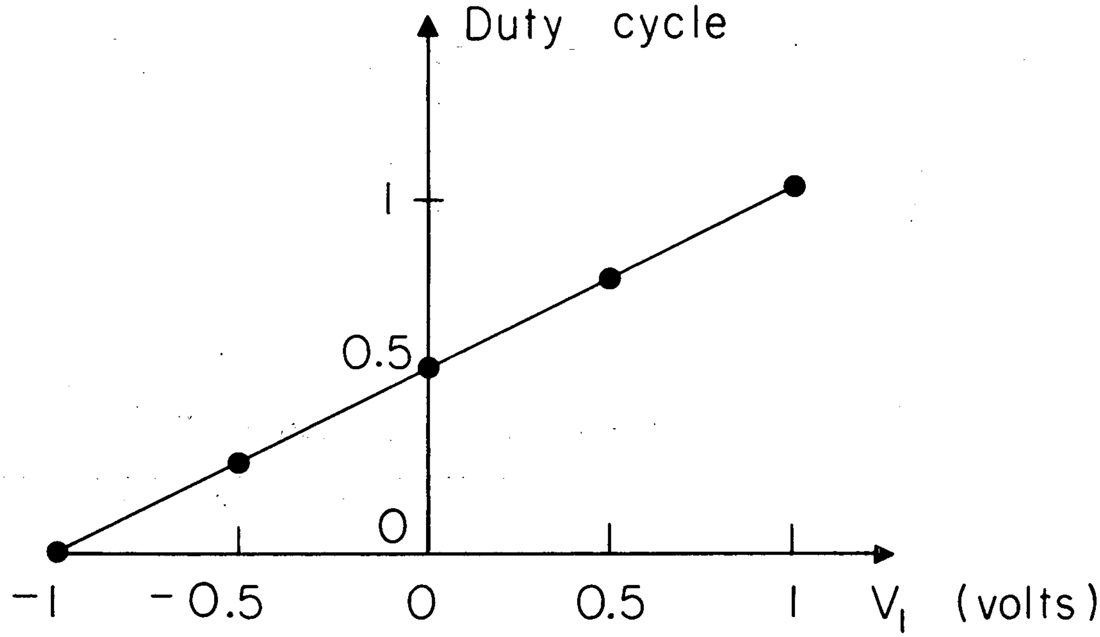

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Fig. 10. Duty cycle of pulse train when AB = 1 volt (bistable modulator).

A. Step Inputs

If the modulator receives a step input of amplitude C, it responds by saturating. That is, its initial response will be a step of amplitude A. The output will continue to be A volts until the RC network has integrated to $(C + \Delta V)/B$. At that time, the transient response will have been completed and the modulator will have a steady state output corresponding to an input voltage of C volts.

If we assume that the input step occurs at that instant when $V_f = D$ volts ($|D| \leq \Delta V$), then the transient response of the modulator will be a pulse of width

$$W = T \ln\left(\frac{AB - D}{AB - C - \Delta V}\right). \tag{31}$$

It is interesting that this modulator, unlike other pulse modulators, offers an instantaneous maximum effort transient response. In that regard it is similar to a maximum effort (bang-bang) control system.

B. Response to Slowly Varying Signals

If we assume that the slope of the input signal and the slope of V_f are both constant during the dwell time (a kind of piecewise linear approximation), we can arrive at useful expressions for the response.

Assume that the modulator has just switched positive at the instant (t_1) that the input signal first appears. Label the dwell times T_1, T_2, $\cdots T_n$ (Fig. 11), the positive dwell times having odd subscripts and the negative dwell times having even subscripts. Label the switching instants t_1, t_2, $\cdots t_n$. Now, we have

$$T_n = \frac{\Delta V_n}{|m_n|}, \tag{32}$$

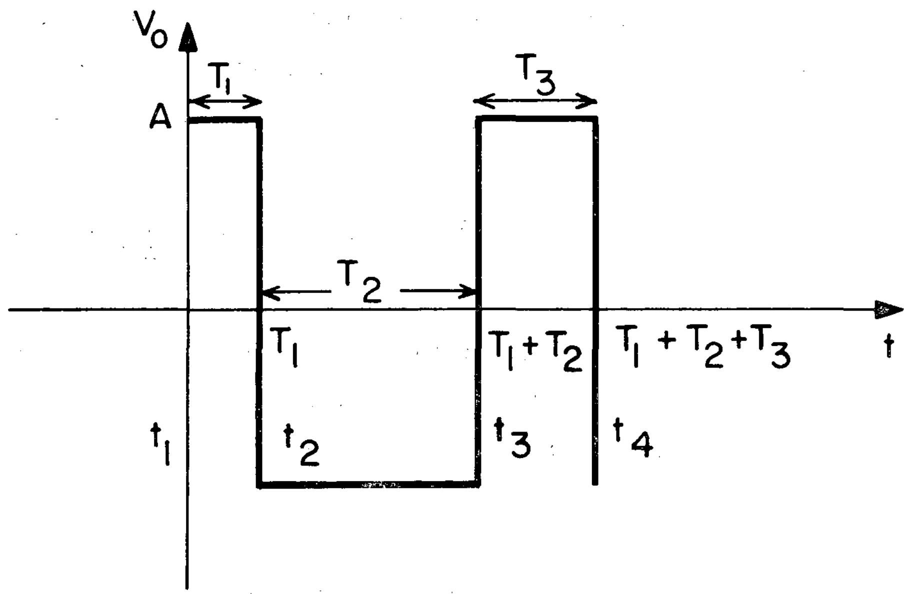

Fig. 11. Response of the modulator to slowly varying signals.

where ΔV_n is the change in V_f during the nth dwell time and m_n is the slope of V_f during the nth dwell time. For odd n (positive dwell times)

$$\Delta V_n = 2\Delta V + V_1'(t_n)(t_{n+1} - t_n), \tag{33}$$

and

$$m_n = \frac{2AB}{T} e^{-t_n/T} = \frac{AB - V_1(t_n)}{T}. \tag{34}$$

Therefore, we get

$$T_n = t_{n+1} - t_n = \frac{2\Delta V T}{AB - V_1(t_n) - V_1'(t_n)T}. \tag{35}$$

For negative dwell times (n even), we have

$$\Delta V_n = 2\Delta V - V_1'(t_n)(t_{n+1} - t_n), \tag{36}$$

and

$$m_n = - \frac{V_1(t_n) + AB}{T}. \tag{37}$$

Therefore, we obtain

$$T_n = t_{n+1} - t_n = \frac{2\Delta V T}{AB + V_1(t_n) + V_1'(t_n)T}. \tag{38}$$

If $V_1'(t_n) = 0$, Eqs. (35) and (38) reduce to Eqs. (15) and (19), which describe the steady state dc response of the modulator.

As an example of the application of Eqs. (35) and (38), let V_1 be a ramp with slope a. Then, we get

$$T_n = \frac{2\Delta V T}{AB - V_1(t_n) - aT} \qquad (n \text{ odd}),$$

and

$$T_n = \frac{2\Delta V T}{AB + V_1(t_n) + aT} \qquad (n \text{ even}),$$

where $t_n = \sum_1^n T_k$.

The response to a sine input can be found if we let $V_1 = C \sin 2\pi ft$:

$$T_n = \frac{2\Delta V T}{AB - C\sin 2\pi ft_n - TC\cos 2\pi ft_n} \qquad (n \text{ odd}),$$

and

$$T_n = \frac{2\Delta V T}{AB + C\sin 2\pi ft_n + TC\cos 2\pi ft_n} \qquad (n \text{ even}).$$

C. Response to Rapidly Varying Signals

If a signal has a slope much greater than $1/T$, then we may consider the feedback loop of the modulator to be open circuited. That is, the signal changes faster than the RC network can respond. For such signals the modulator degenerates to a relay (or regenerative switch).

IV. APPLICATIONS FOR THE MODULATOR

The modulator (or slightly modified forms of it) has at least four applications. It can be used as a modulator, an amplifier, a multiplier, and a divider.

A. As a Modulator

The operation of the circuit as a modulator has been described in the last two sections, where the response to various kinds of signals has been derived. In comparison to other pulse modulators, it has two desirable features (both of which are also found in PRM,[9] but not in other pulse-modulation schemes).

The first feature is that it responds instantaneously and with maximum effort to changes in the input. The modulator does not have to wait for sampling instants, and therefore does not exhibit a delay.

The second feature is that t_{on} and t_{off} have minimum values greater than zero. This means that the transmitting medium has to have a bandwidth no greater than that required to transmit the minimum pulse width. The minimum t_{on} occurs when $V_1 = AB - \Delta V$ and the minimum t_{off} occurs when $V_1 = -AB + \Delta V$. From Eqs. (15) and (19), we have

$$\min t_{on} = \min t_{off} = \frac{2T\Delta V}{2AB - \Delta V}. \tag{39}$$

B. As an Amplifier

An undesirable feature of the modulator is that it has a square-wave output (dc value equal to zero) for an input of zero volts. This results in a large amount of dissipation in the load when the input is zero volts. It would be better, from the viewpoint of efficiency, for the output to be identically zero when the input is zero.

There are at least two ways of making the amplifier output be zero volts for an input of zero volts and still maintain a bipolar output. One method is to detect the absolute value of V_1 and gate the output of the amplifier when $|V_1|$ is greater than some threshold voltage. The previous analysis would apply to such an amplifier except that the amplifier would have a dead zone for input signals whose absolute values were below the threshold level.

A second way of achieving the same end (Fig. 12) is to have a tristable relay in the forward loop, rather than the bistable relay of

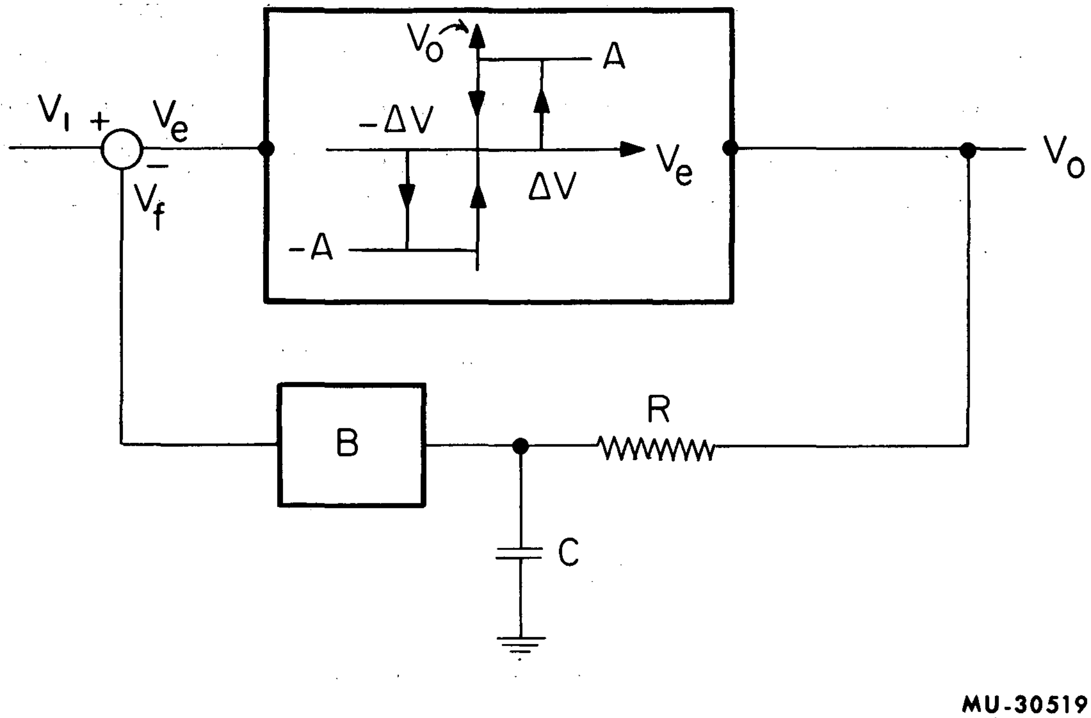

Fig. 12. The tristable modulator.

Fig. 2. Tristable relays are manufactured by several companies and they are usually called bipolar relays. The swinger of a bipolar relay is connected to one of two contacts according to whether the coil voltage is positive or negative. If the coil voltage is insufficient to energize the relay, the swinger comes in contact with neither contact.

A bipolar relay can be constructed from two bistable relays, by making each relay sensitive to a different polarity input. This can be accomplished by placing a diode in series with each coil. The output of the two relays must then be added (Fig. 13).

An electronic bipolar relay can be constructed with two regenerative switches in parallel, the output of the switches being added (Fig. 13).

The sensitivity of the bipolar relay can be modified by inserting an amplifier between the summing junction and the relay. This has the effect of dividing the dead zone and hysteresis of the relay by the gain of the amplifier. The dead zone can be made zero by properly biasing the relay (or regenerative amplifier).

The analytic expressions for the amplifier of Fig. 12 are different than those for the amplifier of Fig. 2. The operation of the tristable amplifier can be explained with the aid of Fig. 14.† From Fig. 14, we see that

$$V_1 + \Delta V = ABe^{-t_1/T} = AB(1 - e^{-t_4/T}), \tag{40}$$

$$V_1 = ABe^{-t_2/T} = AB(1 - e^{-t_3/T}), \tag{41}$$

$$t_2 - t_1 = t_{off} = T \ln\left(1 + \frac{\Delta V}{|V_1|}\right), \tag{42}$$

† The amplifier is in its linear range for $\Delta V \leqslant |V_1| \leqslant AB - \Delta V$. These equations apply only when the input signal is within the above limits.

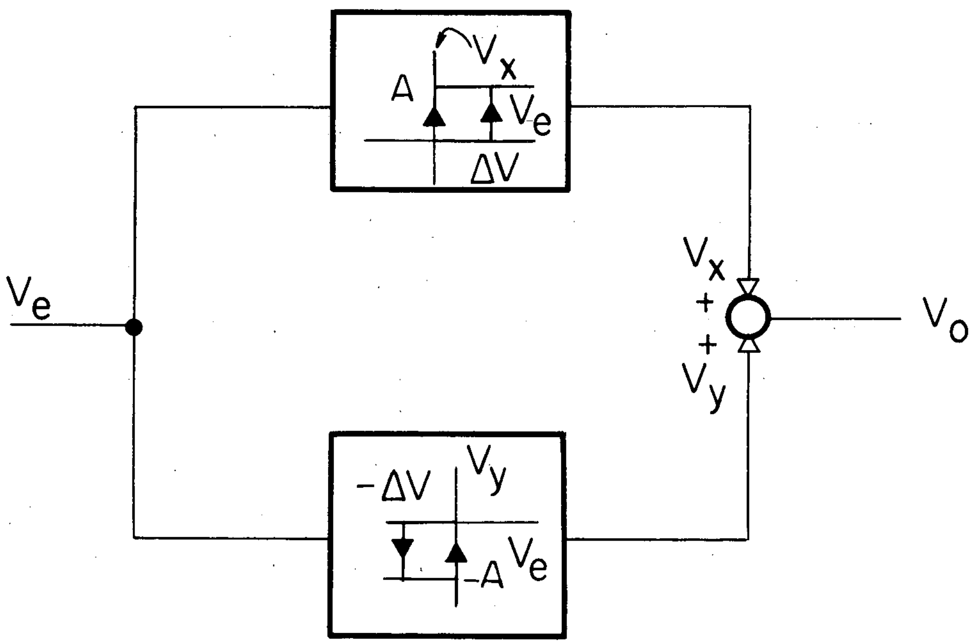

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Fig. 13. The synthesis of a bipolar relay using two bistable relays.

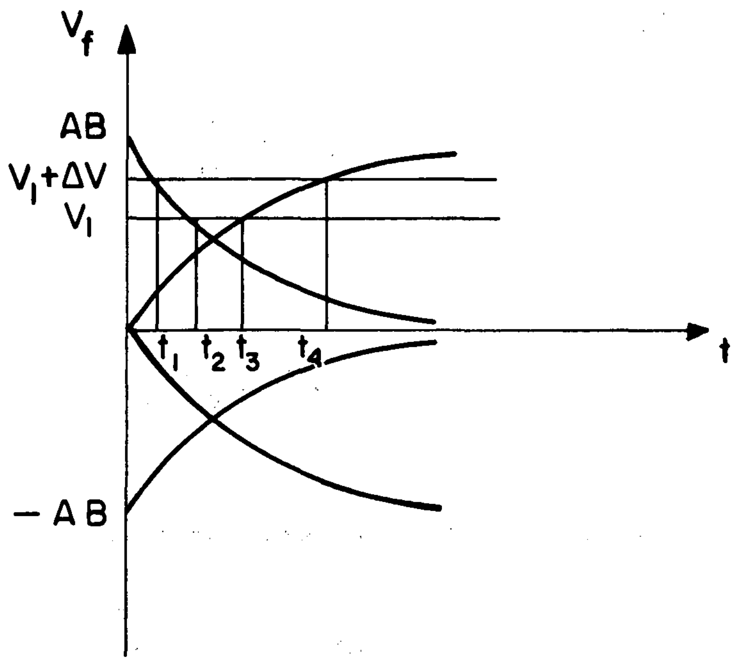

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Fig. 14. Typical trajectories for the tristable modulator.

and

$$t_4 - t_3 = t_{on} = T \ln\left(\frac{1}{1 - \dfrac{\Delta V}{|AB| - |V_1|}} \right). \tag{43}$$

Where the polarity of the output voltage is the same as the sign of V_1, and t_{off} is the time that $V_0 = 0$.

The approximate equations, analogous to Eqs. (15) and (19) of the bistable case are

$$t_{on}^{*} = \frac{\Delta V T}{|AB| - |V_1|}, \tag{44}$$

and

$$t_{off}^{*} = \frac{\Delta V T}{|V_1|}. \tag{45}$$

The approximate and exact formulas are related by

$$\frac{t_{off}}{T} = \ln\left(1 + \frac{t_{off}^{*}}{T}\right) \tag{46}$$

and

$$\frac{t_{on}}{T} = \ln\left(\frac{1}{1 - \dfrac{t_{on}^{*}}{T}}\right). \tag{47}$$

Therefore, we have

$$\frac{t_{off}}{T} = \frac{t_{off}^{*}}{T} - \frac{1}{2}\left(\frac{t_{off}^{*}}{T}\right)^2 + \frac{1}{3}\left(\frac{t_{off}^{*}}{T}\right)^3 + \cdots,$$

which indicates that t^*_{off}/T is close to t_{off}/T for small values of t^*_{off}/T.

We can also observe that

$$\frac{t_{on}}{T} = \ln\left(1 + \frac{t^*_{on}}{T} + \left(\frac{t^*_{on}}{T}\right)^2 + \left(\frac{t^*_{on}}{T}\right)^3 + \cdots\right)$$

$$= \frac{t^*_{on}}{T} + \frac{1}{2}\left(\frac{t^*_{on}}{T}\right)^2 + \frac{1}{3}\left(\frac{t^*_{on}}{T}\right)^3 + \cdots ,$$

which indicates that t^*_{on}/T is close to t_{on}/T for small values of t^*_{on}/T.

Plots of t_{off} and t^*_{off} are shown in Fig. 15, t_{on} and t^*_{on} in Fig. 16, and $P = t_{on} + t_{off}$ and $P^* = t^*_{on} + t^*_{off}$ in Fig. 17. The gain of the tristable amplifier is (using the approximate expressions for t_{on} and t_{off}),

$$K = \frac{A}{V_1} \frac{t_{on}}{t_{on} + t_{off}} = \frac{1}{B} . \tag{48}$$

This amplifier is especially well suited for use with transistorized power stages. The advantage results from the fact that the power transistors are always, except during switching, in cutoff or in saturation. While in cutoff the current through the power transistors is small and while in saturation the voltage drop across the transistors is small. Therefore, compared to linear dc amplifiers, relatively little power is dissipated in the transistors for a given power delivered to the load. Since transistors are severely limited in the amount of power that they can dissipate (that is, heat), this type of amplifier results in simpler driving stages.

C. As a Multiplier and as a Divider

If we modify the modulator by placing a saturating amplifier (a regenerative switch or relay would do as well) in the feedback loop, we

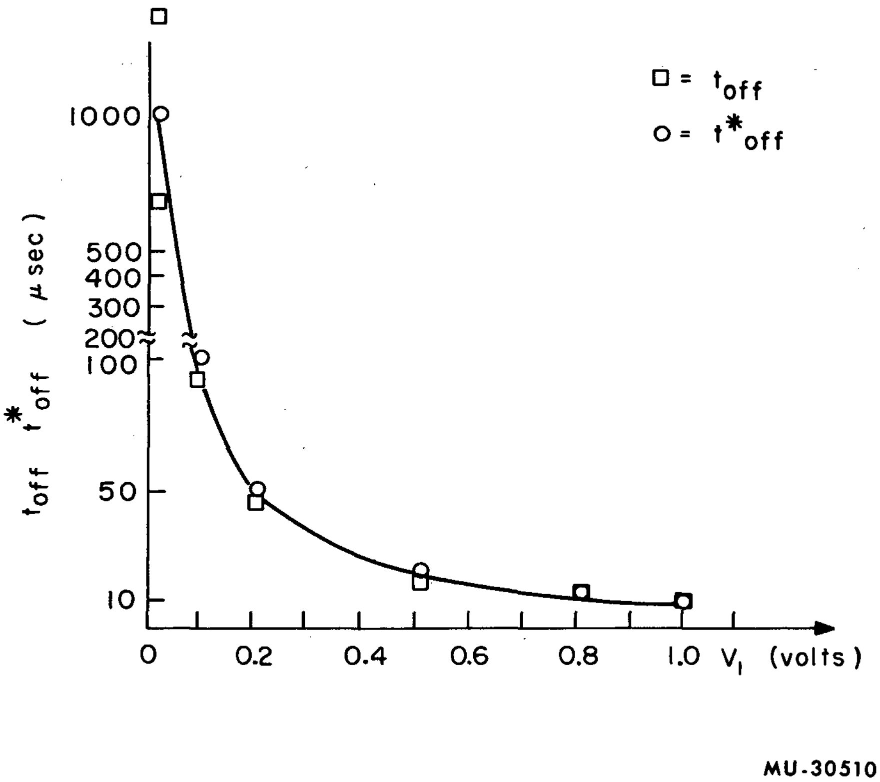

Fig. 15. t_{off} and t_{off}^{*} when $\Delta V = 0.01$ volts, AB = 1 volt, and $T = 10^{-3}$ sec (tristable modulator).

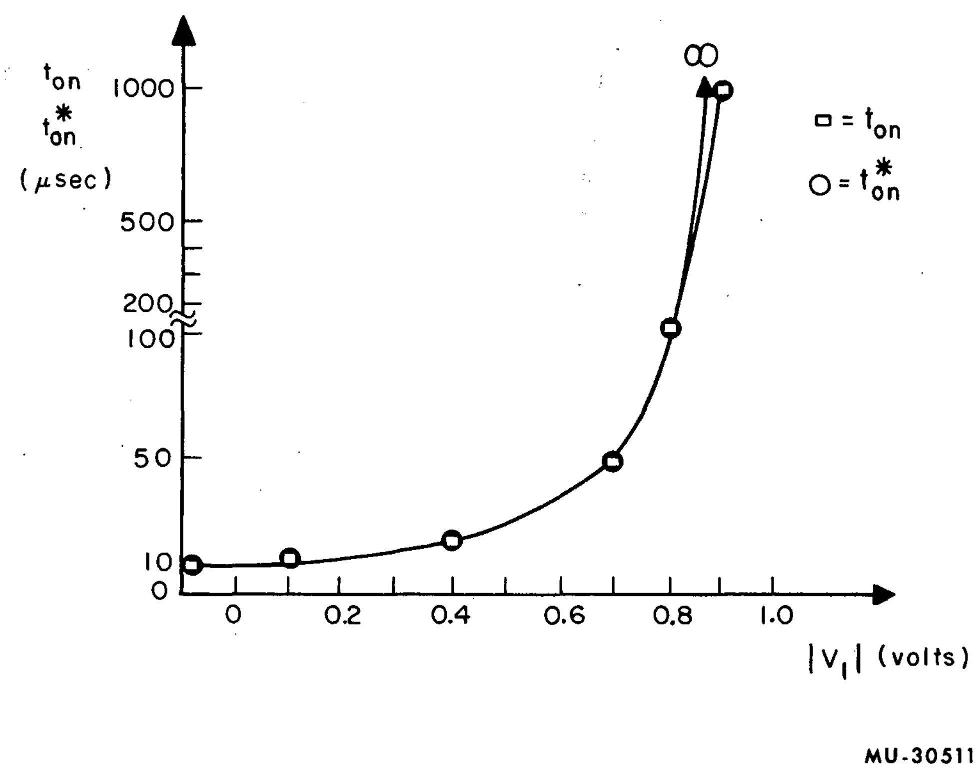

Fig. 16. t_{on} and t^*_{on} when $\Delta V = 0.01$ volts, AB = 1 volt, and $T = 10^{-3}$ sec (tristable modulator).

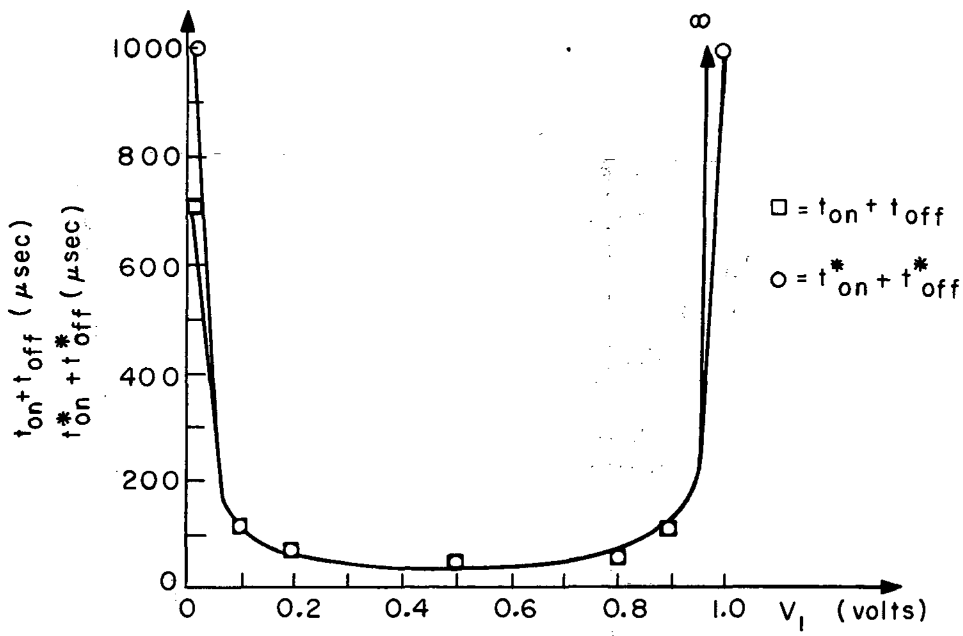

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Fig. 17. The pulse period and the approximate pulse period when $\Delta V = 0.01$ volts, AB = 1 volt, and $T = 10^{-3}$ sec (tristable modulator).

can use the modulator as a divider, or a multiplier, or both (Fig. 18). The bistable relay in the forward loop switches between +E and -E volts, while the saturating amplifier switches between +A and -A volts. From Eqs. (27) and (28) we have

$$V_0 = \frac{EV_1}{AB}, \tag{49}$$

where Eq. (27) applies directly, while in Eq. (28) the "A" must be changed to "E."

If we allow A to be time varying, the modulator can function as a divider. The modulation will still be of the same type discussed previously. The equations of Secs. II and III still apply [with the exception of Eqs. (29) and (30)]. In equation (28) one must change "A" to "E."

If we cause E to be time varying, the modulator functions as a multiplier. In this case, the type of modulation has changed, since the frequency, width, and amplitude of the pulse train all vary as a function of EV_1. All the expressions of Secs. II and III apply. In Eq. (28) one must substitute "E" for "A."

It is obvious that the modulator can function simultaneously as a multiplier and as a divider.

V. EXPERIMENTAL RESULTS

Two electronic versions of the modulator were constructed — a bistable model (Fig. 20) and a tristable model (Figs. 22 and 23). The regenerative switch of the forward loop of the bistable model consists of a dc amplifier with positive feedback (via the 75K resistor) from the output to the base of the input transistor. Important parameters that were determined are $C = 0.5\ \mu F$, $R = 930\ \Omega$, $T = 462\ \mu sec$, $AB = 1.1\ V$ and $2\Delta V = 0.150\ V$. The gain of the amplifier without regenerative feedback is 300. The potentiometer in the emitters of the differential stage was set for zero dc output with zero input. The

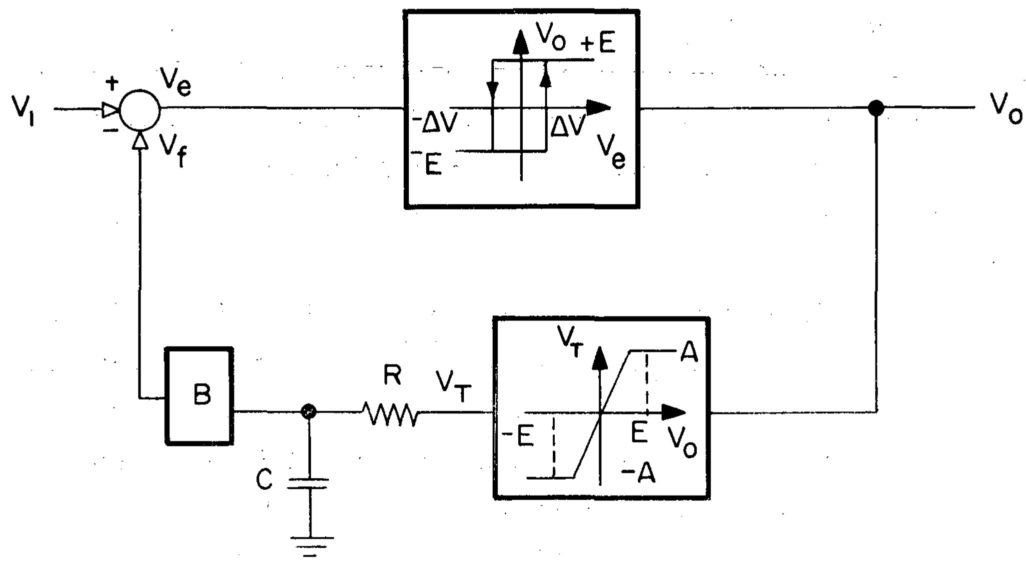

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Fig. 18. The multiplier or divider: $V_0 = \left(\frac{1}{B}\right)\left(\frac{EV_1}{A}\right)$.

positive and negative dwell times for different input voltages were measured with an Eput meter (Fig. 21). The results of these measurements are shown in Fig. 19. Also shown in Fig. 19 are some typical values for the dwell times that were calculated by means of Eqs. (9) and (10). The minimum t_{on} and the minimum t_{off} were both 32 μsec. Minimum t_{on} occurred when $V_1 = 1\,V$ and minimum t_{off} occurred when $V_1 = -1\,V$. Using Eq (39), one calculates a minimum dwell time of 31.6 μsec. Some waveforms exhibited by this modulator are shown in Fig. 3.

The results of this experiment were in close agreement with theory. However, the largest discrepancy occurs when V_1 is close in value to AB and when $\frac{\Delta V}{V_1 + AB}$ is close in value to 1, because it is for those values that Eqs. (9) and (10) become most sensitive to experimental error in determining ΔV and AB.

The tristable modulator (Figs. 22 and 23) was used to drive a 100 W (output) dc printed circuit motor. The amplifier delivered as much as 15A at 12 V to the motor. The gain of the modulator section of the amplifier was about 50 and the voltage gain of the power stage was 2. Tachometer feedback was placed around the amplifier and motor and a dc gain of 30, flat to 30 cycles, was achieved (the output velocity being underdamped).

VI. CONCLUSION

It has been shown that a pulse modulator, which can be constructed with simple components, can be used as an amplifier, a divider, or as a multiplier. As an amplifier, the modulator has fairly linear gain over most of its range.

The accuracy of the modulator is mainly determined by how well one can control ΔV. All of the other parameters (T, A, and B) can easily be realized such that they are both precise and drift free. If a relay is used in the forward loop of the modulator, ΔV will not be very stably or accurately determined (although this might not be a limitation for many applications). If a regenerative switch (whose basic

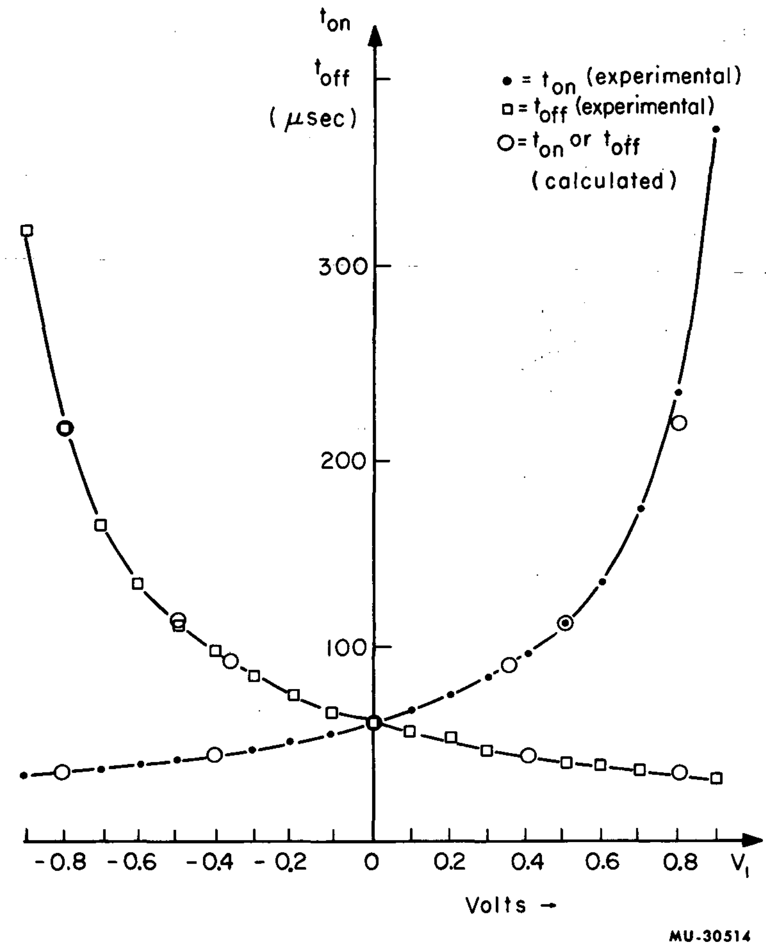

Fig. 19. The results of the experiment using the bistable modulator.

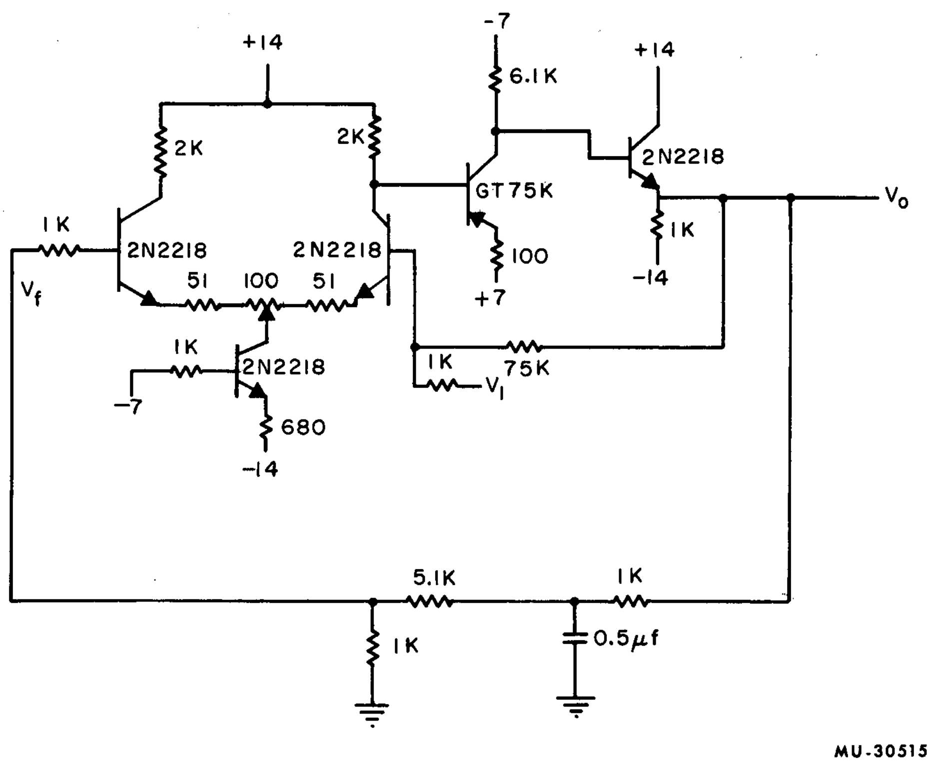

Fig. 20. Circuit diagram of bistable modulator used in experimental setup.

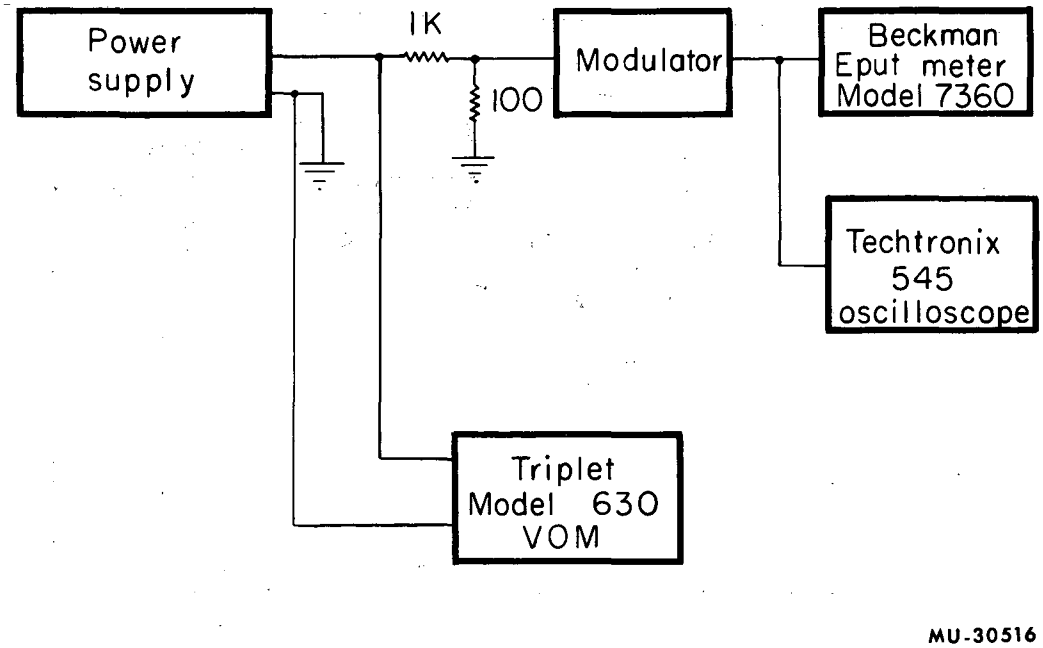

Fig. 21. Experimental setup used to test the modulator.

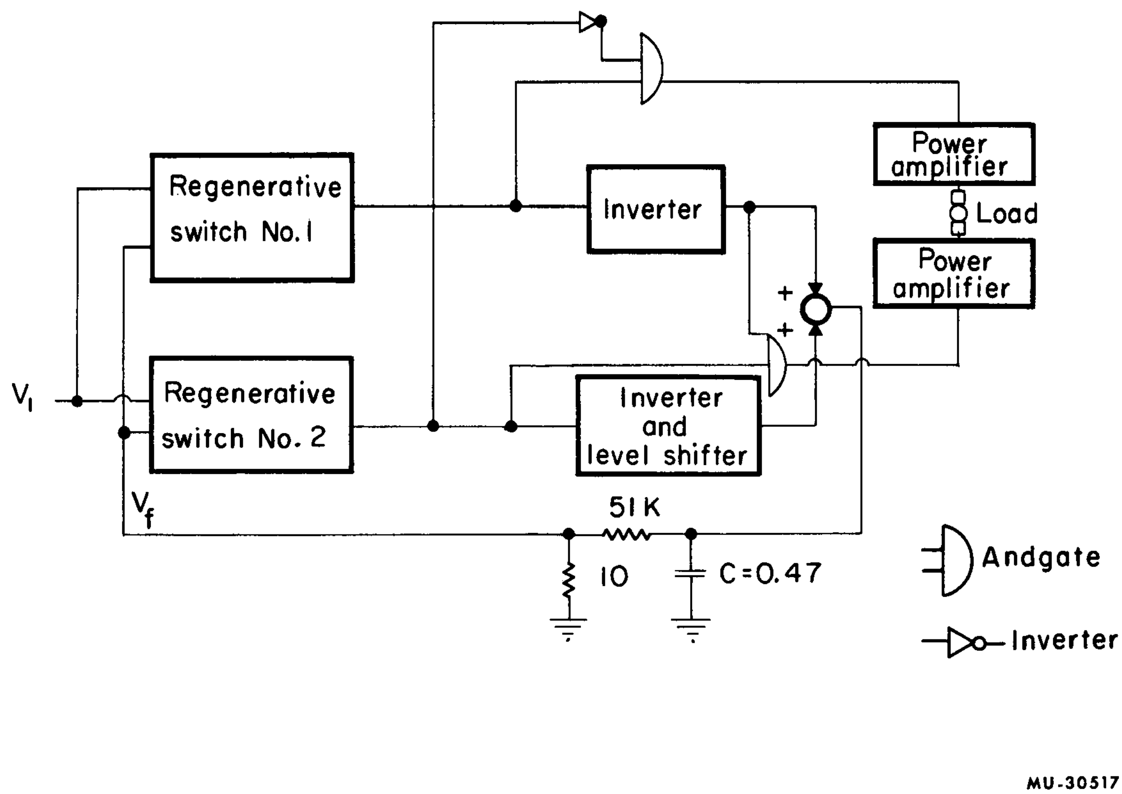

Fig. 22. Block diagram of experimental tristable amplifier.

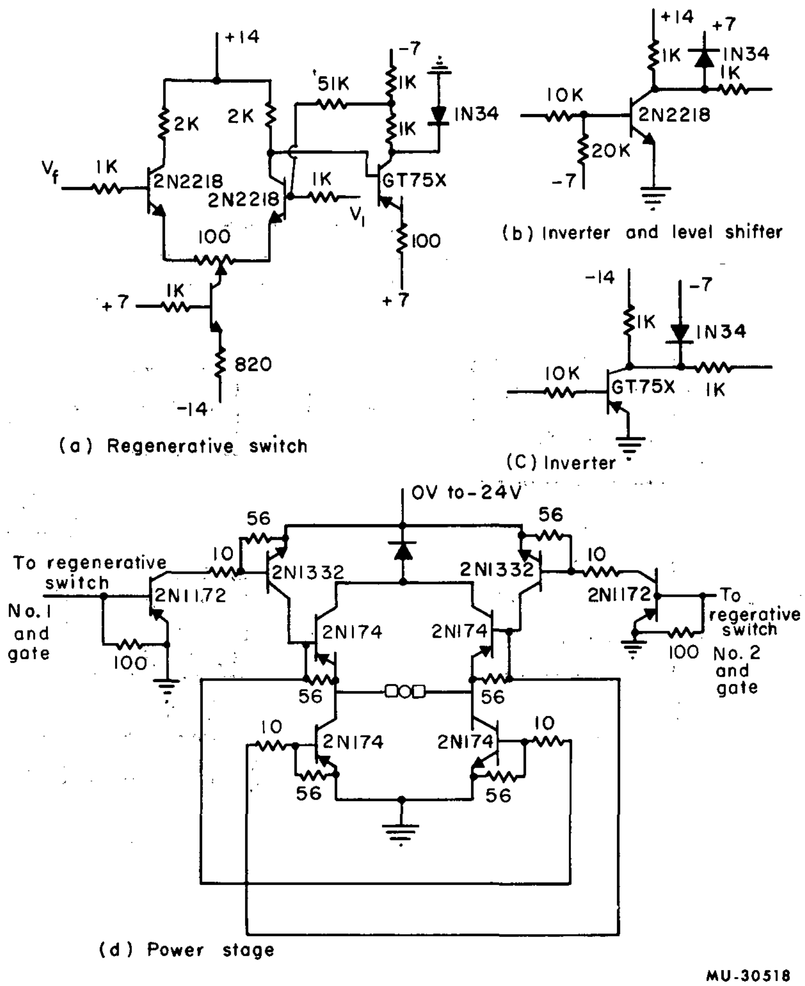

Fig. 23. The experimental tristable amplifier.

element is a differential amplifier) is used in the forward loop, then ΔV can be much more precisely and stably determined.

Our modulator and Schaefer's[9] share some advantages over other forms of pulse modulation. They both respond instantaneously with maximum effort to a change in input signal. They both have minimum on and off times (which conserves bandwidth).

The high frequency gain of our modulator is excessive, although this fault can be mitigated by proper filtering. In terms of realizing a variety of configurations our modulator seems to offer many advantages. In his article,[9] Schaefer offers only a unipolar amplifier and it is not obvious how a bipolar model could be simply constructed. Besides being used in a variety of bipolar and unipolar configurations, the modulator described in this paper can also be used as a multiplier or divider.

There are still some variations of this circuit that might be worth investigating. One such variation would be to place something other than a low pass filter in the feedback path (e.g., a delay line). It might also be worthwhile to place in the feedback path an integrating network whose time constant is a function of input voltage. This could be accomplished by replacing the R of the RC network with a resistor whose resistance was a function of the voltage across it.

Still another variation would be to place the low pass filter in the forward loop, between the summing junction and the relay (Fig. 24). The results of such a change would be a circuit very similar to the one discussed, except that the circuit would not respond instantaneously to a change in input voltage (there would be a small delay). However, since the high frequency response of the circuit would be less, the circuit would be less sensitive to high frequency noise. If a literal relay were used in the circuit, the coil of the relay could serve as the low pass filter.

It is perhaps interesting that such a simple circuit lends itself to such varied and useful applications. This is especially striking when one realizes that the modulator is directly analogous to so simple a system as a thermostatically controlled heater. In that analogy, the relay in the forward loop of the modulator is analogous to the bimetallic

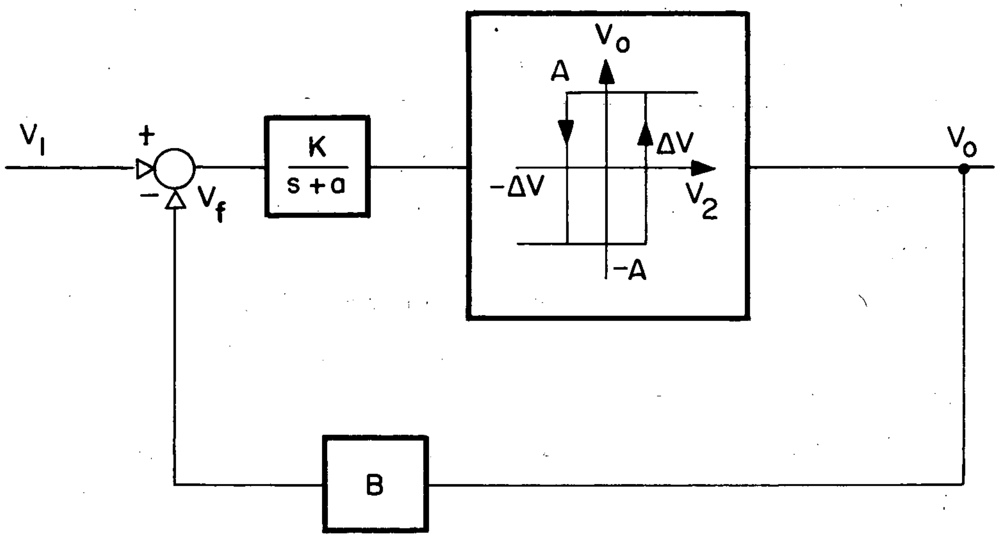

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Fig. 24. Modulator with low pass filter in forward loop.

strip in the thermostat, V_1 is analogous to the setting of the thermostat, the RC element is analogous to the thermal medium coupling the heater to the thermostat, and V_0 is analogous to the pulses of heat that are emitted by the heating element.

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