UNIVERSITY OF CALIFORNIA, SAN DIEGO

# A Kodaira Vanishing Theorem for Formal Schemes

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

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2017

The dissertation of Daniel Smith is approved, and it is acceptable in quality and form for publication on microfilm:

Chair

University of California, San Diego

2017

DEDICATION

To Alyson.

# EPIGRAPH

*For in much wisdom is much vexation,*
*and he who increases knowledge increases sorrow.*
—Ecclesiastes 1:18 (ESV)

*Therefore I tell you, do not be anxious about your life, what you will eat or what you will drink, nor about your body, what you will put on. . . . Look at the birds of the air: they neither sow nor reap nor gather into barns, and yet your heavenly Father feeds them. Are you not of more value than they? . . . Consider the lilies of the field, how they grow: they neither toil nor spin, yet I tell you, even Solomon in all his glory was not arrayed like one of these. But if God so clothes the grass of the field, which today is alive and tomorrow is thrown into the oven, will he not much more clothe you, O you of little faith?*

—Matthew 6:25–29 (ESV)

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ACKNOWLEDGEMENTS

There are many people who deserve my thanks and acknowledgement, and I am certain this record will have some omissions.

First, it would not be possible for me to have even started this project had it not been for the many teachers and instructors I had up until this point. Far too often these individuals receive far too little credit. Similarly, my parents (and sister) deserve a lot of special thanks for their encouragement (albeit in the latter case motivated perhaps more by sibling rivalry, though I can take the blame for that).

I am also incredibly grateful to my peers, who stood by my during the whole process. A special thanks goes to members of my cohort as well as many members of the Graduate Christian Fellowship at UCSD.

I would particularly like to thank Calum Spicer, Perry Strahl, and Ryan Cooper for reinforcing my knowledge of algebraic geometry, both through answering questions of mine and asking questions of their own. My thanks also goes to Robbie Snellman for several helpful conversations about number-theoretic topics, as well as algebro-geometric ones. I must also extend my thanks to Susan Elle and Brian Longo, who, whether they know it or not, served as welcome mentors in navigating many of the oddities of this trek through graduate school.

The mathematics department at UCSD has also been very helpful in my time here. Special thanks should go to the front desk staff who, in spite of being overworked, seem always to have a quick and ready solution to any question or problem I might approach them with, even without warning. I'd also like to extend general thanks to the faculty of the department for their welcoming and helpful nature. I'm especially indebted to Elham Izadi, Kiran Kedlaya, Dragos Oprea, and Cristian Popescu, for conversations and advice both about my research and about various decisions to make during and after graduate school.

It almost goes without saying, but I could not have come anywhere close to completing (even starting!) this project without the aid of my advisor, James McKernan. Throughout my graduate career, he both gave me advice and motivation, even when the subjects I was researching were less than familiar to him. My debt of gratitude to him I'm not sure I can ever repay.

Although it may seem cliché to say it, my heartfelt thanks go to my wife, Alyson. Despite all the difficulties I've encountered, and all the help I've needed or demanded, and all of her own problems pursuing her own Ph. D., she has remained by my side, always willing to help and support. She is a gift beyond words.

Finally, thank God. To him be all glory, for ever and ever.

VITA

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ABSTRACT OF THE DISSERTATION

**A Kodaira Vanishing Theorem for Formal Schemes**

by

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Doctor of Philosophy in Mathematics

University of California San Diego, 2017

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*Formal schemes* are (topologically) ringed spaces that Grothendieck introduced in EGA. They are simultaneously analogues for admissible rings of schemes for general commutative rings and a “bridge” between analytic and algebraic geometry. More recently, formal schemes have been a subject of interest in studies of rigid analytic geometry, where, due to work by Raynaud, Bosch, and Lütkebohmert, they act as models for rigid analytic varieties. The recent interest in these objects has led to study of them in their own right. In this thesis, we investigate whether a minimal model program could exist for formal schemes.

# Introduction

Let us begin with a heuristic definition and a little bit of history.

*Formal schemes* are, in a very rough sense, the analogs for topological rings of schemes for rings. Thus, if one of the great achievements of scheme theory is to provide a geometric setting for the study of rings, one might hope that formal schemes accomplish the same for topological rings.

Of course, the immediate question is why topological rings should deserve such attention. There are two quick answers:

- many rings of number-theoretic interest actually are topological rings. For example, the $p$-adic integers $\mathbb{Z}_p$, beyond being a ring, are endowed with a topology (in fact, most constructions of $\mathbb{Z}_p$ proceed via completion of $\mathbb{Z}$ with respect to a certain topology). More generally, any nonarchimedean field is a topological ring, as is any (complete) subring thereof (such as its ring of integers).

- completions of rings with respect to powers of an ideal are, in a natural sense, topological rings, and naturally arise in algebraic geometry. For example, given a scheme $X$ and a point $x \in X$, the local ring $\mathscr{O}_{X,x}$ in some sense gives a "zoomed in" perspective of how $x$ sits in $X$. The completion of this local ring along its maximal ideal, morally, "zooms in" even more, to a finer degree than the Zariski topology "should be able to see".

Grothendieck actually defined formal schemes alongside ordinary schemes in [Gro60]. (We'll use the term *ordinary* scheme to indicate the usual notion of scheme, i.e., a locally ringed space which admits, for each point, a neighborhood isomorphic to Spec $R$ for some ring $R$. When Grothendieck needs to distinguish the two notions, he refers to ordinary schemes as *preschemás usuels*.) Despite the fact that their definition would have allowed them to arise in number theoretic contexts to which the first bullet above alludes, their use was mostly in the context of the second point. For example, Grothendieck used formal schemes to prove that the étale fundamental groups of a scheme and a subscheme that is a locally complete intersection of dimension at least 2 are isomorphic. Hartshorne [Har68] used them to study the cohomology of varieties over a field.

One of the first big applications of formal schemes along the lines of the first point above was Raynaud's theory of formal models. This theory was an attempt to provide an algebro-geometric framework for rigid analytic varieties, i.e., analytic varieties over a nonarchimedean field. Together with Bosch and Lütkebohmert, Raynaud systematically worked out this theory in the '90s. Heuristically, the way the theory works is that, given a rigid analytic variety $X$ over a nonarchimedean field $k$, one constructs a formal scheme $\mathfrak{X}$ over the ring of integers $R$ of $k$, such that the "generic fiber", in the category of locally ringed spaces, is isomorphic to $X$.

Formal schemes also started to see use in foliation theory. Roughly speaking, a foliation (on an algebraic variety, say,) is a collection of local differential equations. A solution to these local equations is called a leaf. Depending on the source, the exact definition of a foliation might differ slightly, but for purposes of exposition let's content ourselves with the following: a foliation is a subbundle of the tangent bundle that is closed under Lie bracket. Under these conditions, such a system of equations admits a $C^\infty$ solution, which may not be analytic. However, it does still have a formal power series expansion, which formal schemes can represent. Thus, given a formal scheme representing the leaf of a foliation, one might ask if the formal scheme represents an analytic object (which is essentially a question whose answer relies on analytic techniques), or if in fact it represents an *algebraic* object, that is, whether the leaf was (morally) a subvariety. The answer to this question usually boils down to whether the formal scheme in question is *algebraizable*, which is a question one can ask (and answer) independent of this context.

In light of these more recent developments, there has been a certain desire to acquire a better understanding of the fundamentals of formal schemes. It has also become natural to ask whether theorems known to hold for ordinary schemes hold for (suitable classes of) formal schemes. For example, Kollár writes in [Kol08] that "it is high time to work out the whole MMP over an arbitrary base scheme, especially over complete local rings". One of the immediate things to recognize about complete local rings, however, is that they are points in the category of formal schemes. Thus, it seems reasonable to hope that an MMP might hold in the category of formal schemes, at least over complete local rings, and that such a program would have desirable consequences.

To that end, this project attempts to make some headway into whether such a program could exist for (a suitable class of) formal schemes. In brief, we find that many of the ingredients required for an MMP are in place, but there are still some that need addressing.

# Chapter 1

# Algebraic Preliminaries

The purpose of this chapter is to record a large number of results and conventions that we'll use in our later discussions. What we record here holds very little intrinsic interest, apart from various foundations of the material.

When we construct the category of formal schemes, we will do so by creating a subcategory of the category of topologically ringed spaces. We are not interested in general topological rings, however; the (much smaller) class of our interest is the subject of this section.

Recall that a topological ring is a ring $R$ equipped with a topology so that the operations of addition and multiplication are continuous. Consequently, $R$ under addition with its topology forms a topological (abelian) group. Since (left) translation is a homeomorphism for any topological group, to specify the topology, it suffices to specify a neighborhood basis of any single element. In particular, specifying a neighborhood basis for 0 suffices.

Without further remark, we will insist that all rings we mention are commutative with unity, and that a morphism of rings $R \to S$ satisfies $1_R \mapsto 1_S$.

## 1.1 Topological abelian groups

Although our aim is to discuss rings, some of the statements we make will only make use of the underlying topological abelian group structure. The results we state and prove here are all straightforward, and surely they have record in plenty of other places; we include them here only for completeness.

**Definition 1.1.1.** A topological group is a group $G$ with operation $\bullet$ equipped with a topology on the underlying set of $G$ such that $\bullet : G \times G \to G$ is continuous.

A few quick observations regarding this definition:

- Let $\ell_a$ denote the map $G \to G$ given by $x \mapsto ax$, where $a \in G$. By definition, if $U \subseteq G$

is open, and $y = ax \in U$, there exist $U_a, U_x \subseteq G$ such that the image of $U_x \times U_a$ under multiplication is contained in $U$. In particular, $az \in U$ for $z \in U_x$. Thus $\ell_a$ is in fact continuous. The same analysis applies to $\ell_{a^{-1}}$, which, by elementary group theory, we know to be $(\ell_a)^{-1}$. So in fact $\ell_a$ is a *homeomorphism* of $G$ with itself.

- As a consequence of the last point, if $\{U_\lambda\}_{\lambda\in\Lambda}$ is a neighborhood basis for $x \in G$, a neighborhood basis for $a \in G$ is given by $\{\ell_{ax^{-1}}(U_\lambda)\}_{\lambda\in\Lambda}$. In particular, we may specify a topology on $G$ by specifying a neighborhood basis for a single point.

Topological groups are frequent, but not exactly of particular interest to us. The objects of our interest are *abelian* groups, and, in light of the last observation above, we will make the following definition:

**Definition 1.1.2.** An abelian topological group is a topological group that is abelian such that there exists a neighborhood basis of 0 consisting of subgroups.

Given an abelian topological group $G$, there are two natural questions we might ask:

- Is $G$ separated (i.e., Hausdorff)? (We might ask this of any topological space.)
- Is $G$ complete (i.e., does every "Cauchy sequence" converge)? (The Cauchy condition is particularly easy to state in this setting, though it has formulations in more general contexts.)

As a tool to answer the first question, we note the following:

**Proposition 1.1.3** ([AM69, Proposition 10.2])**.** *Let $G$ be an abelian topological group and $\{G_\lambda\}_{\lambda\in\Lambda}$ a neighborhood basis of* 0 *consisting of subgroups.*

1. *$H = \bigcap_{\lambda\in\Lambda} G_\lambda$ is the closure of $\{0\}$.*
2. *$G$ is separated (i.e., Hausdorff) if and only if $H = \{0\}$.*

*Proof.* It is easy to establish claim 2 given claim 1: a topological space is separated if and only if every singleton set is closed. Thus, if $G$ is separated, we must have $H = \{0\}$; if $H = \{0\}$, then because $\ell_a$ is a homeomorphism, $\{a\}$ is closed for each $a \in G$, hence $G$ is separated.

To establish 1, let us first observe that, if $H \leq G$ is an open subgroup, then $H$ is also closed: fix a fundamental set $F$ for $G/H$, which uses 0 for the coset $0 + H$. Then the complement of $H$ is

$$\bigcup_{0\neq x\in F} x + H = \bigcup_{0\neq x\in F} \ell_x(H),$$

which is open.

Therefore, the closure of $\{0\}$ is certainly contained in $H$; we will be done once we show that, if $V$ is a closed subset of $G$ containing $\{0\}$, then $V \supseteq H$. Suppose $x$ is not in $V$. Then, since $V$ is closed, there is some neighborhood of $x$ not meeting $V$. In particular, for some $\lambda$, $(x + G_\lambda) \cap V = \emptyset$. Since $0 \in V$, we know $0 \notin x + G_\lambda$, so $x \notin G_\lambda$. In particular, $x \notin H$. So indeed $H \subseteq V$, and our desired conclusion follows. □

Regarding completeness, first let us make a definition.

**Definition 1.1.4.** A *net* in an abelian topological group $G$ is a function $\Lambda \to G$, where $\Lambda$ is a directed set. We represent a net $f : \Lambda \to G$ by $\{x_\lambda\}_{\lambda \in \Lambda}$, where $x_\lambda = f(\lambda)$.

A net $(x_\lambda)$ is *Cauchy* if, for any neighborhood $U$ of 0, there exists some $\lambda_U \in \Lambda$ such that, for $\lambda, \lambda' \geq \lambda_U$, $x_\lambda - x_{\lambda'} \in U$.

An abelian topological group $G$ is *complete* if every Cauchy net in $G$ converges.

Given a topological abelian group, we can form a separated and complete topological group:

**Proposition 1.1.5.** *Let $G$ be an abelian topological group, and let $\{G_\lambda\}_{\lambda \in \Lambda}$ be a neighborhood basis of* 0 *consisting of subgroups. Consider*

$$\hat{G} = \varprojlim_{\lambda \in \Lambda} G/G_\lambda,$$

*that is, the usual subset of $(x_\lambda) \in \prod G/G_\lambda$, with $\pi_\lambda^{\lambda'}(\pi_{\lambda'}(x)) = \pi_\lambda(x)$, where $\pi_\lambda : \prod G/G_\lambda \to G/G_\lambda$ is the projection map, and $\pi_\lambda^{\lambda'} : G/G_{\lambda'} \to G/G_\lambda$ is the canonical map. Equip $\hat{G}$ with the topology induced by a fundamental system of neighborhoods of* 0 *consisting of* $\ker \pi_\lambda$. *Then $\hat{G}$ is separated and complete.*

*Proof.* The claim that $\hat{G}$ is separated is easy to justify: if $(x_\lambda) \in \ker \pi_\lambda$, then $x_\lambda = 0$. Consequently, if $x_\lambda \in \bigcap \ker \pi_\lambda$, $x_\lambda = 0$ for all $\lambda$, hence $(x_\lambda) = 0$. The claim follows by proposition 1.1.3.

Because the notation gets unreadable really quickly, for this proof, we will use functional notation for nets.

Let $f : M \to \hat{G}$ be a Cauchy net. Define $y \in \hat{G}$ by $\pi_\lambda(y) = \pi_\lambda(f(\mu))$ for some $\mu \geq \mu_{\ker(\pi_\lambda)}$. Note that $y$ is well-defined, since if $\mu' \geq \mu_{\ker(\pi_\lambda)}$, then $f(\mu) - f(\mu') \in \ker(\pi_\lambda)$, hence $\pi_\lambda(f(\mu) - f(\mu')) = 0$. Note also that indeed $y \in \hat{G}$: if $\lambda \geq \lambda'$,

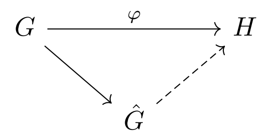
$$\begin{aligned} \pi_\lambda^{\lambda'}(\pi_{\lambda'}(y)) &= \pi_\lambda^{\lambda'}(\pi_{\lambda'}(f(\mu)) && \text{for some } \mu \geq \mu_{\ker(\pi_{\lambda'})} \\ &= \pi_\lambda(f(\mu)) \\ &= \pi_\lambda(y), \end{aligned}$$

where the last equality follows since $\ker(\pi_{\lambda'}) \subseteq \ker(\pi_\lambda)$, so $\mu_{\ker(\pi_{\lambda'})} \geq \mu_{\ker(\pi_\lambda)}$, hence $\mu \geq \mu_{\ker(\pi_\lambda)}$.

Now, $f$ converges to $y$ in $G$: given $\lambda \in \Lambda$, if $\mu \geq \mu_{\ker(\pi_\lambda)}$, we have $\pi_\lambda(y) = \pi_\lambda(f(\mu))$, thus $f(\mu) - y \in \ker(\pi_\lambda)$. So in fact $\hat{G}$ is complete. □

In fact, $\hat{G}$, as defined in the previous proposition, satisfies a universal property:

**Proposition 1.1.6.** *With notation as in the previous proposition, if $\varphi : G \to H$ is a continuous homomorphism of topological abelian groups, where $H$ is complete, then there exists a unique continuous homomorphism $\hat{G} \to H$ such that the following diagram commutes:*




*Proof.* The proof boils down to the following:

1. any element $x$ of $\hat{G}$ gives rise to a Cauchy net $\{x_\lambda\}$ in $G$;

2. since $H$ is complete, we can define a map $\phi : \hat{G} \to H$ by declaring $\phi(x) = \lim \phi(x_\lambda)$, and this gives us a continuous homomorphism;

3. any two such maps $\phi, \phi' : \hat{G} \to H$ must be equal because elements of $\hat{G}$ give rise to the Cauchy nets above.

Regarding the first claim: let $x \in \hat{G}$. Define a net $\{x_\lambda\}$ in $G$ by choosing elements $x_\lambda \in G$ so that $x_\lambda + G_\lambda = \pi_\lambda(x)$. This net is Cauchy: if $\lambda_1, \lambda_2 \geq \lambda$, then $x_{\lambda_1} + G_\lambda = \pi^\lambda_{\lambda_1}(\pi_{\lambda_1}(x)) = \pi_\lambda(x) = \pi^\lambda_{\lambda_2}(\pi_{\lambda_2}(x)) = x_{\lambda_2} + G_\lambda$. That is, given a neighborhood $G_\lambda$ of 0, $\lambda$ is the element of $\Lambda$ such that, if $\lambda_1, \lambda_2 \geq \lambda$, then $x_{\lambda_1} - x_{\lambda_2} \in G_\lambda$.

Now, if $\{x_\lambda\}$ is a Cauchy net obtained from $x$, $\{\varphi(x_\lambda)\}$ is a Cauchy net in $H$, hence converges. The proof is a simple matter of chasing definitions: if $U$ is a neighbhorhood of 0 in $H$, $\varphi^{-1}(U)$ is open in $G$, so there exists some $\lambda$ such that $\lambda_1, \lambda_2 \geq \lambda$ implies $x_{\lambda_1} - x_{\lambda_2} \in \varphi^{-1}(U)$, hence $\varphi(x_{\lambda_1} - x_{\lambda_2}) = \varphi(x_{\lambda_1}) - \varphi(x_{\lambda_2}) \in U$. So define $\phi(x) = \lim \varphi(x_\lambda)$. In fact $\phi$ is well-defined: if $\{x_\lambda\}$ and $\{x'_\lambda\}$ are two such nets described above, as above, if $\lambda_1 \geq \lambda$, then $x_{\lambda_1} + G_\lambda = \pi^\lambda_{\lambda_1}(\pi_{\lambda_1}(x)) = \pi^\lambda_{\lambda_1}(\pi_{\lambda_1}(x)) = x'_{\lambda_1} + G_\lambda$, so $\{x_\lambda - x'_\lambda\}$ converges to 0 in $G$. Thus $\lim \varphi(x_\lambda - x'_\lambda) = 0$. This observation also leads us to conclude $\phi$ is a homomorphism: if $x, y \in G$, $\phi(x+y) = \lim \varphi((x+y)_\lambda) = \lim \varphi((x_\lambda)+(y_\lambda)) = \lim \varphi(x_\lambda)+\varphi(y_\lambda) = \phi(x)+\phi(y)$. The continuity of $\phi$ is also straightforward: we need only check that $\phi^{-1}(U)$ is open for some open subgroup $U \subseteq H$. Since $\varphi$ was continuous, we know $\varphi^{-1}(U) \supseteq G_\lambda$ for some $\lambda$, hence $\phi(\ker \pi_\lambda) \subseteq U$, so that $\phi$ is continuous.

Suppose we had two maps $\phi, \phi' : \hat{G} \to H$ completing the diagram above. Let $(x_\lambda) \in \hat{G}$. Choose a Cauchy net $\{\tilde{x}_\lambda\} \subseteq G$ so that $(x_\lambda)$ is the limit in $\hat{G}$ of the net $\{\tilde{x}_\lambda\}$ under the natual map to $\hat{G}$. By continuity of the compositions, we must have $\phi(x_\lambda) = \lim \varphi(x_\lambda) = \phi'(x_\lambda)$, so the maps are equal as claimed. □

An immediate corollary is that we can use the above description for explicit calculations within complete topological abelian groups, and that such groups are in fact separated:

**Corollary 1.1.7.** *If $G$ is complete, then $\hat{G}$ is topologically isomorphic to $G$. In particular, $G$ is separated.*

*Proof.* The identity is a continuous morphism $G \to G$, so there is a unique map, say $\phi$, so that, if $i : G \to \hat{G}$ denotes the natural map, $\phi \circ i = \mathrm{id}_G$.

On the other hand, $i : G \to \hat{G}$ is a continuous map from $G$ to a complete topological abelain group, so there is some map $\hat{G} \to \hat{G}$ filling in the diagram above, and $\mathrm{id}_{\hat{G}}$ works, so we conclude that $i \circ \phi = \mathrm{id}_{\hat{G}}$. □

## 1.2 Pre-admissible, pre-adic, admissible, and adic

**Definition 1.2.1.** A topological ring $R$ is *linearly topologized* if there exists a neighborhood basis of 0 consisting of ideals.

Linearly topologized rings abound. Some elementary examples include:

- *Any* commutative ring under the discrete topology. A neighborhood basis for 0 is the collection $\{(0)\}$. Though trivial, this example will actually carry some weight later.
- The ring of formal power series $R[[x]]$ in a single variable over a ring $R$, where a system of neighborhoods of 0 is given by powers of the ideal $(x)$. More generally, $R[[x_1, \ldots, x_n]]$ is also an example, with the topology given by powers of $(x_1, \ldots, x_n)$.
- The ring of integers of a non-archimedean field.

Given the list above, it might seem hard to think of a topological ring that is *not* linearly topologized. However, there is also an easy-to-find class of such examples:

**Non-example 1.2.2.** Any topological field whose topology is not discrete is *not* linearly topologized. The simple observation to make is that the only ideal of a field is $\{0\}$; if this is a neighborhood basis of 0, then $\{0\}$ is open, so every singleton set is open, so the topology is discrete.

In particular, local fields such as $\mathbb{Q}_p$ and $\mathbb{F}_q((t))$ are *not* linearly topologized rings. Neither are the fields $\mathbb{R}$ and $\mathbb{C}$ with their usual topologies.

These linearly topologized rings are special cases of *non-archimedean* rings, which are rings with a topology given by additive subgroups. While non-archimedean fields are not linearly topologized rings, they are non-archimedean; very roughly speaking, the fact that linearly topologized rings are non-archimedean rings is what makes the category of adic spaces include the category of formal schemes.

Linearly topologized rings by themselves, however, are difficult to work with; formal schemes actually make use of rings with a bit more structure, as given in the following definition.

**Definition 1.2.3.** A linearly topologized ring is *pre-admissible* if there exists an open ideal $I$ such that, for any neighborhood $U$ of 0, there exists a positive integer $n$ so that $I^n \subseteq U$. Such an ideal is called an *ideal of definition*. If a pre-admissible ring is separated and complete (in the sense of a topological abelian group, i.e., every Cauchy sequence converges), we say it is *admissible*.

Ideals of definition provide useful tools. Consider, for example, the following (which will arise in our formulation of the formal spectrum of an admissible ring):

**Proposition 1.2.4.** *Let $\mathfrak{p}$ be a prime ideal of a pre-admissible ring $R$. The following are equivalent:*

1. *$\mathfrak{p}$ is open;*

2. *$\mathfrak{p}$ contains every ideal of definition.*

3. *$\mathfrak{p}$ contains an ideal of definition;*

*Proof.* $1 \implies 2$: given any ideal of definition $I$, by definition there exists some $n$ (depending on $I$) so that $I^n \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is prime, $I \subseteq \mathfrak{p}$.

$2 \implies 3$: trivial.

$3 \implies 1$: let $I$ be an ideal of definition, $I \subseteq \mathfrak{p}$. Recall that $I$ is open; given any $x \in \mathfrak{p}$, $x + I \subseteq \mathfrak{p}$ is a neighborhood of $x$ contained in $\mathfrak{p}$. So $\mathfrak{p}$ is open. □

Before we go on, another term we will frequently use is:

**Definition 1.2.5.** Let $R$ be a (pre)-admissible ring. An element $x \in R$ is *topologically nilpotent* if $\lim x^n = 0$.

Note that a nilpotent element is necessarily topologically nilpotent. Also, essentially by definition, every element of an ideal of definition is topologically nilpotent.

We make the following defintion, although we will not really need it beyond the present discussion (also, as far as I'm aware, this definition is nowhere else recorded).

**Definition 1.2.6.** The *topological nilradical* of a (pre-)admissible ring is the intersection of all *open* prime ideals.

**Proposition 1.2.7.** *The topological nilradical of a (pre-)admissible ring contains* all *topologically nilpotent elements.*

*Proof.* The proof is essentially the same as the one that the nilradical of a ring contains all nilpotent elements, but even easier: if $x$ is topologically nilpotent, then, since every open prime ideal is a neighborhood of 0, $x^n \in \mathfrak{p}$ for some $n$ for every such $\mathfrak{p}$. Hence $x \in \mathfrak{p}$ for every open $\mathfrak{p}$. If $x \notin \mathfrak{p}$ for some open prime $\mathfrak{p}$, then $x^n \notin \mathfrak{p}$ for all $n \geq 0$, hence $\lim x^n \neq 0$, as there is a neighborhood of 0 in which no power of $x$ lies. □

A natural question to ask is whether a given ring has a *largest* ideal of definition. First let us observe the following:

**Proposition 1.2.8** (Cf. [Gro60, O.7.1.6])**.** *Let $R$ be a (pre)-admissible ring. The following are equivalent, for an ideal of definition $I$:*

1. *$R/I$ is reduced;*

2. *$I$ is the largest ideal of definition;*

3. *$I$ is a maximal ideal of definition.*

*Proof.* $1 \implies 2$: suppose $I' \nsubseteq I$ is another ideal of definition. Then there exists some $n$ so that $(I')^n \subseteq I$. Given $x \in I' \setminus I$, we have $(x+I) \neq 0$, $(x+I)^n = 0$ in $R/I$, so that $R/I$ is not reduced.

$2 \implies 3$: trivial.

$3 \implies 1$: suppose $0 \neq x \in R/I$ is nilpotent, and let $\tilde{x} \in R$ be some lift. Then $I + (\tilde{x})$ is another ideal of definition: given an ideal of definition $I'$, take $n = n_x + n'$, where $I^{n'} \subseteq I'$ and $x^{n_x} \in I$. Then $(I + (\tilde{x}))^n \subseteq I'$. □

In light of the above, we immediately have:

**Corollary 1.2.9.** *If $R$ is Noetherian, $R$ has a largest ideal of definition.*

*Remark* 1.2.10*.* We will largely assume that the rings we deal with are Noetherian. However, this is yet one more place where formal schemes prove somewhat inadequate: many of the arguments for formal schemes (that we present) rely on the existence of such a largest ideal of definition, but, short of making Noetherian hypotheses, the assumption of such an ideal's existence seems *ad hoc*. For this reason adic spaces can be more appealing in the study of rings such as $\mathbb{Z}_p\{t^{p^{-\infty}}\}^\wedge$, where one has adjoined all $p$-th power roots of an indeterminate $t$ and then taken the completion. (This operation may be desirable, for example, to ensure that the Frobenius morphism remains surjective after modding out by $(p)$.) Note that this ring miserably fails to be Noetherian, since $(t) \subseteq (t^{p^{-1}}) \subseteq (t^{p^{-2}}) \subseteq \cdots$ is an infinitely strictly increasing sequence of ideals.

*Remark* 1.2.11*.* As we mentioned, the above treatment of (pre-)admissible rings follows the one in [Gro60]. The treatment in [McQ02] instead defines an ideal of definition as one in which all elements are topologically nilpotent. This definition automatically yields that there exists a largest ideal of definition, in light of proposition 1.2.7. Sometimes McQuillan's definition of admissible is called *weakly admissible*.

When we construct formal schemes, we will do so by defining the formal spectrum of an admissible ring. In practice, however, the formal schemes that arise are those whose open affines are *adic*:

**Definition 1.2.12.** A pre-admissible ring $R$ is *pre-adic* if there exists an ideal of definition $I$ such that the collection $\{I^n\}_{n>0}$ is a neighborhood basis for 0. A separated and complete pre-adic ring is *adic*.

Let us immediately make the following observation, commenting on it later:

**Lemma 1.2.13.** *Suppose $R$ is pre-adic for the ideal of definition $I$ (i.e., $\{I^n\}$ is a neighborhood basis for 0). Then, for $k > 0$, $I^k$ is also an ideal of definition, and $R$ is pre-adic for $I^k$ as well.*

*Proof.* By definition, since $\{I^n\}$ forms a neighborhood basis for 0, $I^n$ is open for $n > 0$. In particular, $I^k$ is open. Given any neighborhood $U$ of 0, there is some $\ell$ such that $I^\ell \subseteq U$; then $(I^k)^\ell \subseteq I^\ell \subseteq U$, so that $I^k$ is an ideal of definition.

To check that $\{(I^k)^n\}_{n>0}$ is a neighborhood basis of 0, we need only check that $I^{kn}$ is open. That it is open is an immediate consequence of the definition that $R$ is pre-adic for $I$. □

As we mentioned earlier, most (pre-)admissible rings that arise in practice are (pre-)adic.

**Example 1.2.14.** The following rings are pre-adic:

- The ring $R[x]$, where a basis of neighborhoods of 0 is given by powers of the ideal $(x)$. As before, more generally, $R[x_1, \ldots, x_m]$ with a neighborhood basis of 0 given by powers of the ideal $(x_1, \ldots, x_m)$. Remark that:
  - these rings are pre-adic essentially by our choice of the topology on them; and
  - by lemma 1.2.13, these rings are pre-adic for the ideal $(x)^2$, or $(x)^k$ (respectively, $(x_1, \ldots, x_n)^2$ or $(x_1, \ldots, x_m)^k$).
- $\mathbb{Z}$ equipped with the $m$-adic topology, for any $m \in \mathbb{Z}$. (Of course, the cases of primary interest are when $m$ is in fact a prime $p$.)

The following rings are adic:

- *Any* commutative ring under the discrete topology. As mentioned earlier, a neighborhood basis is $\{(0)\}$, which is also $\{(0)^n\}$.
- $R[[x]]$ (resp. $R[[x_1, \ldots, x_m]]$), with a neighborhood basis $\{(x)^n\}$ (resp. $\{(x_1, \ldots, x_m)^n\}$). This is just the completion of the first ring mentioned above.
- The ring of integers of a non-archimedean field.

In all of the examples 1.2.14 above, the ideal witnessing the adic nature of the ring in question was fairly easy to guess. However, there are cases in which there is some ambiguity.

**Example 1.2.15.** Consider the ring $R = \mathbb{Z}_p[x]$. $R$ is pre-adic for the following two topologies:

- the topology where a neighborhood basis for 0 is given by $\{(x)^n\}$; and

- the topology where a neighborhood basis for 0 is given by $\{(p)^n\}$.

These two topologies are *not* the same! Perhaps the easiest way to see this is to observe that the sequence $x^n$ tends toward 0 in the former topology, but not in the latter (the ideal $(p)$ is a neighborhood of 0 which contains no terms of the form $x^n$). A more sophisticated way to distinguish them would be to take their respective completions: the completion of the former is the ring $\mathbb{Z}_p[[x]]$, whereas the completion of the latter is $\mathbb{Z}_p\{x\}$ (the ring of restricted formal power series).

While it would appear that classifying a ring as adic depends on a choice of ideal, it turns out that any ideal of definition will do:

**Proposition 1.2.16.** *Suppose $R$ is a pre-adic ring for the ideal of definition $I$. If $J$ is another ideal of definition of $R$, then the collection $\{J^n\}_{n>0}$ is also a neighborhood basis for $0$.*

*Proof.* Since $J$ is an ideal of definition, we already know that, for any neighborhood $U$ of 0, there exists an $n$ such that $J^n \subset U$. The only thing to check is that $J^n$ is open. It will suffice to show that there is some $m$ so that $I^m \subseteq J^n$, since $I^m$ is open. Since $I$ is an ideal of definition and $J$ is open, there exists an $m'$ such that $I^{m'} \subseteq J$, so $(I^{m'})^n \subseteq J^n$; so, if we take $m = m'n$, we reach our desired conclusion. □

All of our discussion so far has been about rings. Many of the terms (and conclusions) above can be applied to modules.

**Definition 1.2.17.** Let $R$ be a linearly topologized ring. A topological $R$-module $M$ is linearly topologized if there exists a neighborhood basis of 0 of the form $\{I_\lambda M\}_{\lambda\in\Lambda}$, where $I_\lambda$ is an open ideal of $R$.

If $R$ is pre-admissible, $M$ is pre-admissible if there exists an ideal of definition $I$ of $R$ such that, for any neighborhood $U$ of 0 in $M$, there exists a positive integer $n$ such that $I^nM \subseteq U$.

If $R$ is pre-adic, $M$ is pre-adic if there exists an ideal of definition $I$ such that $\{I^nM\}_{n>0}$ is a neighborhood basis for 0 in $M$.

If $R$ is admissible (resp. adic), then $M$ is admissible (resp. adic) if $M$ is pre-admissible (resp. pre-adic), separated, and complete.

Of course, the terms above could be made more general: we could, for example, define $M$ to be pre-admissible if there exists an ideal I of $R$ (not necessarily an ideal of definition) so that, for any neighborhood of 0 in $M$, there exists an $n > 0$ with $I^nM \subseteq U$. However, we will not require this generality, and it seems not to yield any greater results to do so.

As we've been hinting, objects of interest for formal schemes will always be *admissible.* Similar to how we can always take the completion of a topological abelian group, we can take the completion of a pre-admissible ring to obtain an admissible ring:

**Proposition 1.2.18.** *Let $R$ be a pre-admissible ring with collection of ideals of definition $\{I_\lambda\}$. Define*

$$\hat{R} = \varprojlim_{\lambda} R/I_\lambda$$

*to be the usual subset of $\prod R/I_\lambda$ consisting of those $(x_\lambda)$ such that $\pi_\lambda^{\lambda'}(\pi_{\lambda'}(x)) = \pi_\lambda(x)$. Equip $\hat{R}$ with the topology induced by a fundamental system of neighborhoods of $0$ consisting of $\ker \pi_\lambda$. Then $\hat{R}$ is admissible, with ideals of definition $\ker \pi_\lambda$.*

*If $M$ is a pre-admissible $R$ module, then*

$$\hat{M} = \varprojlim_{\lambda} M/I_\lambda M$$

*is an admissible $\hat{R}$-module.*

*Proof.* Corollary 1.1.5 already informs us that $\hat{R}$ is separated and complete, and the inverse limit topology naturally makes $\hat{R}$ a topological ring. The only claim we need to justify is the one concerning the pre-admissibility of $\hat{R}$.

For convenience, let $\hat{I}_\lambda = \ker \pi_\lambda$. We need to show that, for any $\lambda'$, there exists some positive integer $n$ so that $\hat{I}_\lambda^n \subseteq \hat{I}_{\lambda'}$. Take $n$ so that $I_\lambda^n \subseteq I_{\lambda'}$ as ideals of $R$ (possible since $R$ is pre-admissible). Suppose $x \in \hat{I}_\lambda^n$; then write

$$x = \sum_{i=1}^{m} \prod_{j=1}^{n} x_{i,j},$$

where $x_{i,j} \in \hat{I}_\lambda$. As in the proof of proposition 1.1.6, for each $x_{i,j}$, we may form a Cauchy net $\{(x_{i,j})_\mu\} \subseteq R$ satisfying, for all $\mu$, $(x_{i,j})_\mu + I_\mu = \pi_\mu(x_{i,j})$. Then

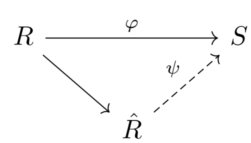
$$\pi_{\lambda'}(x) = \sum_{i=1}^{m} \prod_{j=1}^{n} x_{i,j} + I_{\lambda'} = 0 + I_{\lambda'},$$

since $I_\lambda^n \subseteq I_{\lambda'}$. So in fact $x \in \hat{I}_{\lambda'}$, as required.

The proof that $\hat{M}$ is admissible is the same as the above, modulo minor changes in notation. □

As in the case of groups, $\hat{R}$ satisfies a universal property:

**Corollary 1.2.19.** *If $\varphi : R \to S$ is a continuous homomorphism where $S$ is an admissible ring, then there exists a unique continuous $\psi : \hat{R} \to S$ such that the following diagram commutes:*




*Proof.* We already know from the proof of proposition 1.1.6 that there exists a unique $\psi : \hat{R} \to S$ that is a continuous morphism of abelian groups. It suffices to show that this $\psi$ is a bona-fide ring homomorphism.

Recalling the aforementioned proof, $\psi$ is defined by $\psi(x_\lambda) = \lim \varphi(\tilde{x}_\lambda)$, where $\tilde{x}_\lambda$ is a Cauchy net in $R$ whose image in $\hat{R}$ is $x_\lambda$. We need only show that $\psi(x_\lambda y_\lambda) = \psi(x_\lambda)\psi(y_\lambda)$, i.e.,

$$\lim \varphi(\tilde{x}_\lambda \tilde{y}_\lambda) = \lim(\varphi(\tilde{x}_\lambda)) \lim(\varphi(\tilde{y}_\lambda)).$$

Since $\varphi(x_\lambda y_\lambda) = \varphi(x_\lambda)\varphi(y_\lambda)$, the usual trick of writing

$$\begin{aligned} \lim(\varphi(x_\lambda)) \lim(\varphi(y_\lambda)) - \varphi(x_\lambda)\varphi(y_\lambda) &= (\lim(\varphi(x_\lambda)) - \varphi(x_\lambda))(\lim(\varphi(y_\lambda)) - \varphi(y_\lambda)) \\ &\quad - \lim(\varphi(x_\lambda))(\lim(\varphi(y_\lambda)) - \varphi(y_\lambda)) \\ &\quad - \lim(\varphi(y_\lambda))(\lim(\varphi(x_\lambda)) - \varphi(x_\lambda)) \end{aligned}$$

yields the equality we desire. (Note that the above makes use of the fact that the topologies in question are, in fact, linear; one can remove this assumption while still working with topologies of additive subgroups of rings, but then greater care is needed in this proof.) □

**Corollary 1.2.20.** *If $S$ is a pre-admissible $R$-algebra, then $\hat{S}$ is an admissible $\hat{R}$-algebra.*

*Proof.* $\hat{S}$ is an admissible $R$-algebra, so there is a unique morphism $\hat{R} \to \hat{S}$. □

Even better, however, is that, under some hypotheses, the *pre-adicity* of a module (in particular, of the ring itself) is preserved by completion. That is,

**Proposition 1.2.21.** *If $M$ is a pre-adic $R$-module for a finitely-generated ideal of definition $I$, then $\hat{M}$ is $\hat{I}$-adic.*

*Proof. (Cf. [Sta17, Tag 05GG], [Mat89, Theorem 8.11]).* We need to show that

- $\hat{I}^n \hat{M} \subseteq M$ is open, for any $n > 0$; and
- for any $m > 0$ there exists an $n > 0$ such that $\hat{I}^n \hat{M} \subseteq \widehat{I^n M}$ (since the latter generate the topology on $\hat{M}$).

What we will show, however, is that $\hat{I}^n \hat{M} = \widehat{I^n M}$, which simultaneously yields both the above.

First observe that $\hat{I}^n \hat{M} \subseteq \widehat{I^n M}$. Hence it suffices to show that $\hat{I}^n \hat{M} \supseteq \widehat{I^n M}$. In fact we will show that $I^n \hat{M} \supseteq \widehat{I^n M}$. To do so, first consider the map

$$\theta : M^{\oplus r} \to M, \quad (m_i) \mapsto \sum_{i=1}^{r} f_i m_i,$$

where $f_1, \ldots, f_r$ generate $I^n$. Since these elements generate $I^n$, $\theta$ surjects onto $I^n M$. So we have an exact sequence, for any $\ell$,

$$0 \to \ker \theta_\ell \to M^r / I^\ell M^r \xrightarrow{\theta_\ell} I^n M / I^\ell (I^n M) = I^n M / I^{\ell+n} M \to 0.$$

For the moment, let us admit that we can take inverse limits over $\ell$ in the above exact sequence to produce an exact sequence

$$0 \to \varprojlim \ker \theta_\ell \to \varprojlim M^r / I^\ell M^r \to \varprojlim I^n M / I^{\ell+n} M \to 0. \tag{1.2.1}$$

The middle term we may actually identify as $\hat{M}^r$, and the cokernel is just $\widehat{I^n M}$. In particular, we have a surjection $\hat{M}^r \to \widehat{I^n M}$ given by $(m_i) \mapsto \sum f_i m_i$, i.e., $I^n \hat{M} \supseteq \widehat{I^n M}$. Our desired conclusion follows.

There are only a few things left to justify. The first is that we actually do get the exact sequence (1.2.1). It is a standard result that taking limits produces an exact sequence if the images of the maps $\ker \theta_{\ell+m} \to \ker \theta_\ell$ are stationary. (This is referred to as the Mittag-Leffler condition; see, e.g., [Har77, II.9]). One way to ensure that the sequence is stationary is to show that $\ker \theta_{\ell+1} \to \ker \theta_\ell$ is surjective (cf. [AM69, Proposition 10.2]), which is true for our case of interest. Suppose we have $(m_i) \in M^r$ satisfying $\theta(m_i) \in I^{\ell+n} M$. Then

$$\theta(m_i) = \sum_{j=1}^{s} g_j m_j'$$

where $g_j$ are generators for $I^\ell$ and $m_j' \in I^n M$. Since $\theta$ is surjective, we may write

$$m_j' = \theta(m_{j,i}') = \sum_{i=1}^{r} f_i m_{j,i}'.$$

Therefore

$$\begin{aligned}
\theta\left((m_i) - \sum_{j=1}^{s} g_j (m_{i,j}')\right) &= \theta\left((m_i) - (\sum_{j=1}^{s} g_j m_{i,j}')\right) \\
&= \theta\left(m_i - \sum_{j=1}^{s} g_j m_{i,j}'\right) \\
&= \sum_{i=1}^{r} f_i m_i - f_i \sum_{j=1}^{s} g_j m_{i,j}' \\
&= \left(\sum_{i=1}^{r} f_i m_i\right) - \left(\sum_{i=1}^{r}\sum_{j=1}^{s} f_i g_j m_{i,j}'\right) \\
&= \left(\sum_{i=1}^{r} f_i m_i\right) - \left(\sum_{j=1}^{s} g_j \sum_{i=1}^{r} f_i m_{i,j}'\right) \\
&= \left(\sum_{i=1}^{r} f_i m_i\right) - \left(\sum_{j=1}^{s} g_j m_j'\right) \\
&= \theta(m_i) - \theta(m_i) = 0.
\end{aligned}$$

In particular, $((m_i) - \sum g_j(m_{i,j}') + I^{\ell+1} M^r) \in \ker \theta_{\ell+1}$, and since $g_j \in I^\ell$, the image of this element in $\ker \theta_\ell$ is $((m_i) + I^\ell M^r)$. Since any element of $\ker \theta_\ell$ can be written as $((m_i) + I^\ell M^r$ where $\theta(m_i) \in I^{\ell+n} M$, we have, indeed, that the map $\ker \theta_{\ell+1} \to \ker \theta_\ell$ is surjective, as we desired.

The other fact requiring justification is that $\hat{M}^r \simeq \varprojlim M^r / I^\ell M^r$. This fact follows from the observation that

$$M^r / I^\ell M^r \xrightarrow{\sim} \left(M / I^\ell M\right)^r,$$

and that

$$\left(\varprojlim_{\ell} M/I^{\ell}M\right)^{r} \xrightarrow{\sim} \varprojlim_{\ell}(M/I^{\ell}M)^{r},$$

both of which are simple unravelings of the definitions of the natural maps in question. □

*Remark* 1.2.22. Note that the above proof actually shows that $\hat{M}$ is $I$-adic, not just $\hat{I}$-adic. We will not make use of this observation in any way.

*Remark* 1.2.23. The hypothesis on the finitely-generated nature of $I$ is crucial in the above proposition. For the proof that we gave, this hypothesis came in, essentially, in verifying that (1.2.1) was an exact sequence. However, in the absense of this hypothesis, the conclusion of the proposition may be false. Consider, for example, $R = k[x_1, x_2, \dots]$ and $I = (x_1, x_2, \dots)$. Then

$$h = x_2x_3 + x_4x_5x_6 + x_8x_9x_{10}x_{11} + \cdots = \sum_{i=1}^{\infty} \prod_{j=2^i}^{2^i+i} x_j \in \widehat{I^2},$$

but $h \notin \hat{I}^2$. If, on the contrary, $h$ were an element of $\hat{I}^2$, we would be able to write

$$h = \sum_{i=1}^{n} f_{i,1}f_{i,2},$$

where $f_{i,j} \in \hat{I}$. Necessarily, then, we must have some $i$ such that the degree one part of $f_{i,1}$ contains $x_2$ and the degree one part of $f_{i,2}$ contains $x_3$. Then $f_{i,1}f_{i,2} = x_2x_3 + x_2g_{i,1} + x_3g_{i,2}$ for some $g_{i,1}, g_{i,2} \in \hat{I}$. Thus, after reindexing so that $i = 1$,

$$h - f_{1,1}f_{1,2} = (h - x_2x_3) - (x_2g_{1,1} + x_3g_{1,2}).$$

Repeating the above argument (with minor changes, since we don't know exactly which degree contains $x_{2^i}$, only that we can put a bound on this degree), we find that

$$0 = h - \sum_{i=1}^{n} f_{i,1}f_{i,2} = h - \left(\sum_{i=1}^{n} \prod_{j=2^i}^{2^i+i} x_j\right) - \sum_{i=1}^{n} x_{2^i}g_{i,1} + x_{2^i+1}g_{i,2}.$$

So

$$\sum_{i=n+1}^{\infty} \prod_{j=2^i}^{2^i+i} x_j = \sum_{i=1}^{n} x_{2^i}g_{i,1} + x_{2^i+i}g_{i,2}$$

But this is absurd: the right hand side contains no monomials composed exclusively of $x_j$ for $j > 2^{n+1}$, whereas the left hand side clearly does.

For our purposes, the finitely-generated nature of $I$ is almost never an issue, because we'll consider only Noetherian rings.

## 1.3 Fractions, tensor products, and power series rings

The entire theme of this subsection is the following: suppose we are given an admissible ring (or module). We'd like to carry out algebraic constructions, such as a ring (module) of fractions, and make sure that the objects these constructions yield preserve the admissibility property. As a general rule of thumb, we go about such constructions as follows:

1. Write our starting object (which may be a ring or module, but we'll use a module for sake of example) in the form $\varprojlim M/I_\lambda M$. Note that the objects of our interest will be *complete* topological abelian groups, so that, by corollary 1.1.7, we may actually do so.

2. Perform the desired operation to $M$, yielding a module $N$, and endow $N$ with the $I_\lambda N$ topology. This should make $N$ a pre-admissible $R$-module.

3. Take the result to be $\varprojlim_\lambda N/I_\lambda N$. By proposition 1.2.18, we should end up with an admissible $\hat{R}$-module.

Of course, the above procedure is somewhat vague. In particular, it's not clear that, after performing our desired opertion to $M$, we'll end up with a (meaninful) pre-admissible module as we had hoped. However, we will verify pre-admissiblity in every instance that we consider.

**Lemma 1.3.1.** *Let $M$ be a pre-admissible $R$-module, and let $S$ be a multiplicatively closed subset of $R$ with $1 \in S$. Then $S^{-1}M$, endowed with the $I_\lambda S^{-1}M$ topology, is pre-admissible.*

*Proof.* It suffices to show that, for any $\lambda, \lambda'$ there exists some $n$ so that $I_\lambda^n(S^{-1}M) \subseteq I_{\lambda'}(S^{-1}M)$. The same $n$ witnessing $I_\lambda^n \subseteq I_{\lambda'}$ suffices here. □

*Remark* 1.3.2. We will only consider sets $S$ such that $S \cap I_\lambda = \emptyset$ for every ideal of definition $I_\lambda$. This restriction might seem rather limiting, but it causes no trouble in practice, because we never want to invert topological nilpotents. More precisely, suppose we have $f \in I_\lambda \cap S$ for some $\lambda$. Then $m \in I_\lambda^n S^{-1}M$ for all $m \in M$, and in particular, the only non-empty open set of $S^{-1}M$ in this topology is $S^{-1}M$ itself. In particular, we would be considering $S^{-1}M$ with the trivial topology, and completing with respect to this topology gives 0. In short, if we want to restrict our attention to linearly topologized objects, we can't invert topological nilpotents because it destroys the linear topology in question.

**Definition 1.3.3.** Given an admissible $R$-module $M$ with collection of ideals of definition $\{I_\lambda\}$, and $S$ a multiplicatively closed subset of $R$ containing 1, the completed module of fractions with denominators in $S$ is

$$M_{\{S\}} = \varprojlim_\lambda (S^{-1}M)/(I_\lambda S^{-1}M).$$

If $S = \{1, f^n : n > 0\}$, we will write $M_{\{f\}}$ for $M_{\{S\}}$. If $S = R \setminus \mathfrak{p}$ for some prime ideal $\mathfrak{p}$, we will write $M_{\{\mathfrak{p}\}}$ for $M_{\{S\}}$.

*Remark* 1.3.4. In [Gro60], the definition of the completed module of fractions is not what we give above. Rather, it is defined as

$$\varprojlim_{\lambda} (\pi_\lambda(S))^{-1}(M/I_\lambda M).$$

The two definitions are equivalent, however. Observe that, in fact,

$$(\pi_\lambda(S))^{-1}(M/I_\lambda M) \simeq (S^{-1}M)/(I_\lambda S^{-1}M).$$

The isomorphism is easy to establish: there's a natural map

$$\varphi_\lambda : S^{-1}M \to (\pi_\lambda(S))^{-1}(M/I_\lambda M)$$

whose kernel is those $\frac{m}{s}$ such that there exists an $s' \in S$ so that $s'm \in I_\lambda M$. But

$$\begin{aligned} s'm \in I_\lambda M &\iff s'm = \alpha y \text{ in } M, \quad \alpha \in I_\lambda, y \in M \\ &\iff \frac{m}{1} = \alpha \frac{y}{s'} \text{ in } S^{-1}M \\ &\iff \frac{m}{s} = \alpha \frac{y}{s's} \text{ in } S^{-1}M \\ &\iff \frac{m}{s} \in I_\lambda S^{-1}M. \end{aligned}$$

So the two limits are isomorphic term-by-term, hence isomorphic.

The tensor product of two pre-admissible $R$ modules is straightforward:

**Lemma 1.3.5.** *Let $M$ and $N$ be two pre-admissible $R$-modules. Then $M \otimes N$, endowed with the $I_\lambda(M \otimes N)$ topology, is pre-admissible.*

*Proof.* The same as in lemma 1.3.1: any element of $I_\lambda(M \otimes N)$ can be written as

$$\sum_i r_i \left( \sum_j m_{i,j} \otimes n_{i,j} \right)$$

where $r_i \in I_\lambda$, $m_{i,j} \in M$, and $n_{i,j} \in N$. So if $n$ witnesses that $I_\lambda^n \subseteq I_{\lambda'}$, and $r_i \in I_\lambda^n$, then $r_i \in I_{\lambda'}$, hence $I_\lambda^n(M \otimes N) \subseteq I_{\lambda'}(M \otimes N)$. □

**Definition 1.3.6.** Suppose $M$ and $N$ are two pre-admissible $R$-modules, the completed tensor product of $M$ with $N$ is

$$M \hat{\otimes} N = \varprojlim_{\lambda} (M \otimes N)/I_\lambda(M \otimes N).$$

The tensor product of two pre-admissible $R$-algebras, however, requires a little bit more careful examination.

**Lemma 1.3.7.** *Suppose $A$ is an $\{I_\lambda\}$-pre-admissible $R$-algebra and $B$ is a $\{J_\mu\}$-pre-admissible $R$ algebra. Then $A \otimes_R B$ is an $\{(I_\lambda \otimes B) + (A \otimes J_\mu)\}$-pre-admissible $R$-algebra.*

*Proof.* Take $n$ so that $I_\lambda^n \subseteq I_{\lambda'}$ and $J_\mu^n \subseteq J_{\mu'}$. Then observe that

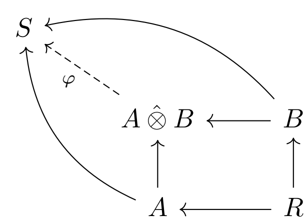
$$\left(\sum_{k=1}^{t}\left(\sum_{i=1}^{r}\alpha_{k,i}\otimes b_{k,i}\right)+\left(\sum_{j=1}^{s}a_{k,j}\otimes\beta_{k,j}\right)\right)^{2n}\in(I_{\lambda'}\otimes B)+(A\otimes J_{\mu'}),$$

if $\alpha_{k,i} \in I_\lambda$, $\beta_{k,j} \in J_\mu$, $a_{k,j} \in A$, and $b_{k,i} \in B$. The reason for this is that the above is a sum of tensors $a_\ell \otimes b_\ell$, where either $a_\ell \in I_\lambda^n$ or $b_\ell \in J_\mu^n$, hence $a_\ell \in I_{\lambda'}$ or $b_\ell \in J_{\mu'}$. $\square$

**Definition 1.3.8.** If $A$ is an $\{I_\lambda\}$-pre-admissible $R$-algebra and $B$ is a $\{J_\mu\}$-pre-admissible $R$ algebra, the completed tensor product of $A$ and $B$ is

$$A\hat{\otimes}B=\varprojlim_{\lambda,\mu}(A\otimes B)/(I_\lambda\otimes B+A\otimes J_\mu).$$

**Proposition 1.3.9.** *Let $A$ and $B$ be admissible $R$-algebras, where $R$ is an admissible ring. Then $A \hat{\otimes} B$ is the pushout in the category of admissible rings. That is, given an admissible ring $S$ and two continuous ring maps $A \to S$, $B \to S$ whose compositions with the morphisms $R \to A$, $R \to B$ are equal, there exists a unique continuous map $\varphi : A \hat{\otimes} B \to S$ making the following diagram commute:*




*Proof.* We already know that $A \otimes B$ is a pushout in the category of rings; so, given maps as above, we know there exists a unique map $\psi : A \otimes B \to S$. This map is continuous because $\psi^{-1}(U) = A \otimes \varphi_B^{-1}(U) + \varphi_A^{-1}(U) \otimes B$, which is open if $U$ is open. Hence by corollary 1.2.19 there exists a unique map of rings $A \hat{\otimes} B \to S$. $\square$

The next lemmae and definitions we have actually already observed in examples above, but we record them for posterity:

**Lemma 1.3.10.** *Let $R$ be an $\{I_\lambda\}$-pre-admissible ring. $R[x_1, \dots, x_n]$ can be considered pre-adic with respect to the $(x_1, \dots, x_n)^\ell$-topology and pre-admissible with respect to $\{I_\lambda R[x_1, \dots, x_n]\}$.*

*Proof.* The statement regarding the $(x_1, \dots, x_n)$-topology is clear. To demonstrate the pre-admissibility of the $\{I_\lambda R[x_1, \dots, x_n]\}$-topology, note that

$$\left(\sum_{i=1}^{m} r_i f_i(\bar{x})\right)^N = \sum_{j=1}^{m'} s_j g_j(\bar{x}),$$

where $r_i \in I_\lambda$, $N$ is such that $I_\lambda^N \subseteq I_{\lambda'}$, hence $s_j \in I_{\lambda'}$. That is, $(I_\lambda R[x_1, \dots, x_n])^N \subseteq I_{\lambda'} R[x_1, \dots, x_n]$. $\square$

**Definition 1.3.11.** The ring of formal power series over $R$ in indeterminates $x_1, \ldots, x_n$ is

$$R[[x_1, \ldots, x_n]] = \varprojlim_{\ell} R[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^{\ell}.$$

The ring of restricted power series over $R$ in indeterminates $x_1, \ldots, x_n$ is

$$R\{x_1, \ldots, x_n\} = \varprojlim_{\lambda} R[x_1, \ldots, x_n]/I_\lambda R[x_1, \ldots, x_n].$$

# Chapter 2

# Formal schemes

We will present formal schemes using the formulation in [Gro60]. Before delving into the material, though, let's take this opportunity to point out that some alternative formulations for formal schemes have been proposed, such as [McQ02] and [Yas09]. The purpose of these reformulations is to assist in work on formal schemes which are, *a priori*, not adic. Since the focus of our work is in the adic setting, we will mostly ignore the improvements this new perspective offers, except to make some occasional remarks.

## 2.1 The category of formal schemes

Analogous to the case of ordinary schemes, formal schemes are ringed spaces that are locally isomorphic to "affine" formal schemes. However, unlike the case of ordinary schemes, we will say that a formal scheme is a *topologically ringed space* locally isomorphic to an affine formal scheme. It turns out that formal schemes are, by means of this definition, in fact locally ringed spaces, but we will not pursue this idea beyond its mention.

There are three major objectives of this section:

1. We want to establish what the formal spectrum of an admissible ring $R$ is (hence what an affine formal scheme is). While definition 2.1.1 and the preceding observations settle this matter, they do so only in a minimal sense. Arguably, the rest of the subsection gives more flavor to what affine formal schemes are beyond their definition.

2. We want to establish that affine formal schemes are locally ringed spaces

3. We want to establish that the category of affine formal schemes is the dual category to that of admissible rings.

Throughout our discussion in this subsection, $R$ will be an admissible ring, and $\{I_\lambda\}$ will denote the collection of ideals of definition of $R$.

The definition of the formal spectrum of $R$ is surprisingly simple; what is more difficult is to show that it has all the properties we claim.

Before we make our definition of the formal spectrum, let us make a few observations:

1. $\operatorname{Spec} R/I = \operatorname{Spec} R/I'$ as topological spaces for any two ideals of definition $I$ and $I'$. Each is equipped with the subspace topology from the inclusions $\operatorname{Spec} R/I \to \operatorname{Spec} R$ and $\operatorname{Spec} R/I' \to \operatorname{Spec} R$, so it suffices to show the two sets are equal. Proposition 1.2.4 immediately yields equality of the two sets. In particular, the topological space $\operatorname{Spec} R/I_\lambda$ does not depend on the choice of ideal of definition $I_\lambda$.

2. If $\mathscr{I}_\lambda$ denotes the sheaf on $\operatorname{Spec} R$ associated to the ideal of definition $I_\lambda$, then $\{\mathscr{O}_R/\mathscr{I}_\lambda\}$ forms an inverse system of sheaves on the topological space $\operatorname{Spec} R/I$ (where $I$ is some ideal of definition). This follows from the previous point, that $\mathscr{O}_R/\mathscr{I}_\lambda$ is naturally a sheaf on $\operatorname{Spec} R/I_\lambda$, and that $\{R/I_\lambda\}$ is an inverse system of rings. As in [Har77, Ex. II.1.12], $\varprojlim_\lambda \mathscr{O}_R/\mathscr{I}_\lambda$ is a sheaf on $\operatorname{Spec} R/I$.

The above points make the following unambiguous:

**Definition 2.1.1.** Fix some ideal of definition $I_{\lambda_0}$. The *formal spectrum* of $R$ is the topological space $\operatorname{Spec} R/I_{\lambda_0}$ together with the sheaf of rings $\varprojlim_\lambda \mathscr{O}_R/\mathscr{I}_\lambda$, where $\mathscr{I}_\lambda$ is the sheaf of ideals on $\operatorname{Spec} R$ given by $I_\lambda$. We denote by $\operatorname{Spf} R$ the pair $(\operatorname{Spec} R/I_{\lambda_0}, \varprojlim \mathscr{O}_R/\mathscr{I}_\lambda)$.

**Definition 2.1.2.** An affine formal scheme is a topologically ringed space $(\mathfrak{X}, \mathscr{O}_\mathfrak{X})$ that is isomorphic to $\operatorname{Spf} R$ for some admissible ring $R$.

*Remark* 2.1.3. It is common practice to use non-Roman letters to denote formal schemes. Conventions vary from author to author; the convention established in [Gro60] is to use fraktur to refer to both formal schemes and sheaves (of modules) on them. (E.g., a formal scheme might be $\mathfrak{X}$, and a line bundle on it might be $\mathfrak{L}$.) Other practices include using calligraphic letters for both formal schemes and sheaves thereon (e.g., $\mathcal{X}$, $\mathcal{L}$) and using calligraphic letters for formal schemes, but script lettering for sheaves ($\mathscr{L}$). In all cases that I've encountered, the structure sheaf of a formal scheme is denoted by an 'O' in either a calligraphic or script glyph, subscripted by the letter denoting the formal scheme in question in the lettering used for formal schemes (e.g., $\mathscr{O}_\mathfrak{X}$).

I will adopt the convention of using fraktur to represent formal schemes ($\mathfrak{X}$) and script lettering for sheaves on them $\mathscr{L}$. This means that I will be using the same lettering for sheaves on both ordinary schemes and formal schemes, but it should result in no confusion. Of course, this convention adds nothing to the statements and arguments contained herein, but it can aid their comprehensibility.

The above definition is somewhat opaque; a better way to think of $\operatorname{Spf} R$ is as a space whose points correspond to the *open* prime ideals of $R$ equipped with a sheaf of rings whose global sections is $R$ itself. The latter requires proof, which we will give in a moment. Given $f \in R$, define $\mathfrak{V}(f)$ to be the set of open ideals which contain $f$, and say $\mathfrak{D}(f)$ is its complement. Naturally, $\mathfrak{V}(f) = V(f) \cap \operatorname{Spf} R$, and $\mathfrak{D}(f) = D(f) \cap \operatorname{Spf} R$, so that $\mathfrak{V}(f)$ is closed and $\mathfrak{D}(f)$ is open in $\operatorname{Spf} R$.

**Proposition 2.1.4.** *Suppose $R$ is an admissible ring, $f \in R$, and let $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ be its formal spectrum. Then $\Gamma(\mathfrak{D}(f), \mathscr{O}_{\mathfrak{X}}) \simeq R_{\{f\}}$ as topological rings. In particular, $\Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) = R$.*

*Proof.* This proof is essentially just chasing definitions. Since the presheaf $\varprojlim \mathscr{O}_R/\mathscr{I}_\lambda$ is in fact a sheaf, we know that

$$\mathscr{O}_{\mathfrak{X}}(\mathfrak{D}(f)) = \left(\varprojlim_{\lambda} \mathscr{O}_R/\mathscr{I}_\lambda\right)(\mathfrak{D}(f)) = \varprojlim_{\lambda}(\mathscr{O}_R/\mathscr{I}_\lambda)(\mathfrak{D}(f)).$$

$\mathfrak{D}(f)$ is, as we observed above, the intersection of $D(f)$ with $\operatorname{Spf} R$, i.e., $D(f) \cap \operatorname{Spec} R/I_\lambda$, for any $\lambda$. The latter set is the set of primes of $R$ which contain $I_\lambda$ but do not contain $f$; $\mathfrak{p}$ contains $f$ if and only if $\pi_\lambda(f) \in \pi_\lambda(\mathfrak{p})$, since $\mathfrak{p}$ contains $I_\lambda$. So $\mathfrak{D}(f) = D(\pi_\lambda(f))$ in $\operatorname{Spec} R/I_\lambda$, hence

$$(\mathscr{O}_R/\mathscr{I}_\lambda)(\mathfrak{D}(f)) = (\mathscr{O}_{R/I_\lambda})(D(\pi_\lambda(f))) = (R/I_\lambda)_{\pi_\lambda(f)}.$$

Substituting this last equality into the first equation gives the result, taking note of 1.3.4. The last statement of the proposition is just combining the formula given in the preposition with corollary 1.1.7. □

A word of caution: the above perspective might make tempting the thought that, analogous to the case of ordinary schemes, $\mathscr{O}_{\mathfrak{X},\mathfrak{p}} \simeq R_{\mathfrak{p}}$. Unfortunately, that is not quite the case. By definition,

$$\begin{aligned}
\mathscr{O}_{\mathfrak{X},\mathfrak{p}} &= \varinjlim_{U \ni \mathfrak{p}} \mathscr{O}_{\mathfrak{X}}(U) \\
&= \varinjlim_{f \notin \mathfrak{p}} \mathscr{O}_{\mathfrak{X}}(\mathfrak{D}(f)) \\
&= \varinjlim_{f \notin \mathfrak{p}} R_{\{f\}} \qquad \text{(Proposition 2.1.4)},
\end{aligned}$$

where the second equality holds because the sets $\mathfrak{D}(f)$ form a base of $\operatorname{Spf} R$ (because it is equipped with the subspace topology from $\operatorname{Spec} R$, where the sets $D(f)$ form a base). However, it is true that there is a map between $\mathscr{O}_{\mathfrak{X},\mathfrak{p}}$ and $R_{\{\mathfrak{p}\}}$.

**Proposition 2.1.5** (Cf. [Gro60, O.7.6.17])**.** *Let $R$ be an admissible ring and $\mathfrak{X}$ is formal spectrum. There is a natural map*

$$\mathscr{O}_{\mathfrak{X},\mathfrak{p}} \to R_{\{\mathfrak{p}\}}$$

*which is a local map of local rings (in particular, $\mathscr{O}_{\mathfrak{X},\mathfrak{p}}$ is local).*

*Proof.* First, let us establish the map in question. As the rings $R_{\{f\}}$ are initial among admissible rings $S$ to which there is a continuous map $R \to S$ in which the image of $f$ is invertible, we need only show that the image of $f$ in $R_{\{\mathfrak{p}\}}$ is invertible. The latter fact, however, is almost true by definition, since $R_{\{\mathfrak{p}\}}$ is initial among admissible rings to which there is a continuous map in which elements in the complement of $\mathfrak{p}$ (such as $f$) are invertible.

Let us call the map we obtain $\varphi$. To prove the remaining claims of the theorem, set

$$\mathfrak{m} = \varprojlim_{\lambda} \left((R \setminus \mathfrak{p})^{-1}\mathfrak{p}\right) / \left(I_\lambda (R \setminus \mathfrak{p})^{-1}\mathfrak{p}\right), \quad \mathfrak{n} = \varinjlim_{f \notin \mathfrak{p}} \mathfrak{p}_{\{f\}}.$$

Note that $\mathfrak{m}$ is the unique maximal ideal in $R_{\{\mathfrak{p}\}}$: if $f = (f_\lambda) \notin \mathfrak{m}$, then $f_\lambda$ is invertible in each term of the limit. Let $g_\lambda$ be an element of $R_\mathfrak{p}$ such that $f_\lambda g_\lambda - 1 \in I_\lambda$. Observe that, if $\lambda' \geq \lambda$, then

$$\begin{aligned}(f_\lambda + I_\lambda)(g_{\lambda'} - g_\lambda + I_\lambda) &= ((f_{\lambda'} + I'_\lambda)(g_{\lambda'} + I_{\lambda'}) + I_\lambda) - (f_\lambda g_\lambda + I_\lambda) \\ &= ((1 + I_{\lambda'}) + I_\lambda) - (1 + I_\lambda) \\ &= 1 - 1 + I_\lambda = 0 + I_\lambda.\end{aligned}$$

Since $f_\lambda + I_\lambda$ is a unit, we conclude that $g_{\lambda'} - g_\lambda \in I_\lambda$. In particular, $(g_\lambda)$ is an element of $R_{\{\mathfrak{p}\}}$ satisfying $(f_\lambda)(g_\lambda) = 1$.

Now we need only show that $\mathfrak{n}$ is the unique maximal idea in $\mathscr{O}_{\mathfrak{X},\mathfrak{p}}$. Elements of this ring may be represented as $(f, D(g))$, where $f \in R_{\{g\}}$. Suppose $f \notin \mathfrak{n}$, i.e., $\varphi(f) \notin \mathfrak{m}$. Write

$$f = \left(\frac{a_\lambda}{g^{n_\lambda}}\right),$$

where $a_\lambda \in R$. The definition of $\varphi$ makes it so that $\varphi(f)$ can be written in the same way, although we should think of $g$ as an element of $R \setminus \mathfrak{p}$. If $\varphi(f) \notin \mathfrak{m}$, then we can conclude that $a_\lambda \notin \mathfrak{p}$ for all $\lambda$. Fix some $\lambda_0$ and note that, in $R_{\{a_{\lambda_0} g\}} / \ker \pi_{\lambda_0}$, we have (denoting by $\bar{x}$ the image of an element $x$ of $R$ in $R_{\{a_{\lambda_0} g\}}$)

$$\left(\frac{\bar{g}^{n_{\lambda_0}}}{\bar{a}_{\lambda_0}}\right) \bar{f} = 1.$$

So we have some $h \in R$ such that $hf = 1 + x$ in $R_{\{a_{\lambda_0} g\}}$, where $x \in \ker \pi_{\lambda_0}$. But $\ker \pi_{\lambda_0}$ is an ideal of definition, so $x$ is topologically nilpotent. In particular, taking limits of each side of

$$(1 + x) \sum_{k=0}^{n} (-1)^k x^k = (-1)^n x^{n+1} + 1,$$

we find that $hf$ is a unit in $R_{\{a_{\lambda_0} g\}}$, so that $f$ is a unit in this ring, hence in the limit $\mathscr{O}_{\mathfrak{X},\mathfrak{p}}$.

Thus $\mathfrak{n}$ is maximal and the natural map $\mathscr{O}_{\mathfrak{X},\mathfrak{p}} \to R_{\{\mathfrak{p}\}}$ is a local map of local rings. □

**Proposition 2.1.6.** *A continuous ring homomorphism $A \to B$ of admissible rings induces a map on formal spectra* Spf $B \to$ Spf $A$.

*Proof.* Let $f : A \to B$ be a continuous morphism of admissible rings. Since it is a ring morphism, $f^{-1}(\mathfrak{p})$ is a prime ideal of $A$ for any prime ideal $\mathfrak{p}$ of $B$. Since $f$ is continuous, if $\mathfrak{p}$ is open, so is $f^{-1}(\mathfrak{p})$. Define $\varphi : \operatorname{Spf} B \to \operatorname{Spf} A$ by $\varphi(\mathfrak{p}) = f^{-1}(\mathfrak{p})$.

First, let us observe that $\varphi$ is continuous: let $a \in A$, and consider $\varphi^{-1}(\mathfrak{D}(a))$. By definition, this is the set of all open ideals $\mathfrak{p}$ in $B$ such that $a \notin f^{-1}(\mathfrak{p})$, which is true if and only if $f(a) \notin \mathfrak{p}$. That is, $\varphi^{-1}(\mathfrak{D}(\mathfrak{a})) = \mathfrak{D}(f(a))$. So $\varphi$ is continuous.

This observation also gives us a map on sheaves: for any $a \in A$, there is a natural map $A_{\{a\}} \to B_{\{f(a)\}}$, i.e., a natural map $\mathscr{O}_A(\mathfrak{D}(a)) \to \mathscr{O}_B(\mathfrak{D}(f(a))) = \mathscr{O}_B(\varphi^{-1}(\mathfrak{D}(a))) = \varphi_*\mathscr{O}_B(\mathfrak{D}(a))$. Since the map in question is natural, we get a unique map of sheaves $\mathscr{O}_A \to \varphi_*\mathscr{O}_B$. □

**Proposition 2.1.7.** *Let $R$ be an admissible ring, and $f \in R$. Set $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) = \operatorname{Spf} R$. We have*

$$\operatorname{Spf}\left(R_{\{f\}}\right) \simeq (\mathfrak{D}(f), \mathscr{O}_{\mathfrak{X}}|_{\mathfrak{D}(f)}).$$

*Proof.* This proof basically boils down to the following: we have a natural (continuous) map (of admissible rings)

$$i : R \to R_{\{f\}}$$

which induces a map on formal spectra

$$j : \operatorname{Spf} R_{\{f\}} \to \operatorname{Spf} R.$$

Then we go through the following motions:

1. Observe that $j$ takes image in $\mathfrak{D}(f)$, hence can actually be taken to be
$$j : \operatorname{Spf} R_{\{f\}} \to (\mathfrak{D}(f), \mathscr{O}_R|_{\mathfrak{D}(f)}).$$
2. Note that $j$ is open and bijective onto $\mathfrak{D}(f)$ as maps of topological spaces, which gives rise to an inverse (topological) map $h : \mathfrak{D}(f) \to \operatorname{Spf} R_{\{f\}}$.
3. Record that, in fact, $j^{\#} : \mathscr{O}_R|_{\mathfrak{D}(f)} \to j_*\mathscr{O}_{R_{\{f\}}}$ is an isomorphism on sets of the form $\mathfrak{D}(g)$, which yields a map of sheaves
$$h^{\#} : \mathscr{O}_{R_{\{f\}}} \to h_*\mathscr{O}_R|_{\mathfrak{D}(f)}.$$
4. Note that $j$ and $h$ are mutual inverses.

The first point is straightforward: by definition, $j(\mathfrak{p}) = i^{-1}(\mathfrak{p})$ for $\mathfrak{p} \in \operatorname{Spf} R_{\{f\}}$. Since $\mathfrak{p}$ is a prime of $R_{\{f\}}$, $1 \notin \mathfrak{p}$, so the image of $f$ must not be in $\mathfrak{p}$. That is, $i(f) \notin \mathfrak{p}$, so $f \notin i^{-1}(\mathfrak{p})$. Since $i^{-1}(\mathfrak{p})$ is an open prime of $R$, we have indeed that $j$ takes image in $\mathfrak{D}(f)$.

Now, let us show that $j$ is injective. Suppose $i^{-1}(\mathfrak{p}_1) = i^{-1}(\mathfrak{p}_2)$, and let $g \in \mathfrak{p}_1$. We may write

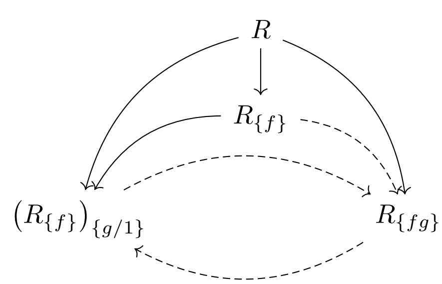
$$g = \left(\frac{a_\lambda}{f^{n_\lambda}}\right).$$

Fix some $\lambda_0$ and write $n_0 = n_{\lambda_0}$, so that $g = \frac{a_{\lambda_0}}{f^{n_0}} + x$, where $x \in \ker \pi_{\lambda_0}$. Since $\mathfrak{p}_1$ is open, $\ker \pi_{\lambda_0} \subseteq \mathfrak{p}_1$, hence $x \in \mathfrak{p}_1$, so $\frac{a_{\lambda_0}}{f^{n_0}} \in \mathfrak{p}_1$. So $\frac{a_{\lambda_0}}{1} \in \mathfrak{p}_1$, which means $a_{\lambda_0} \in i^{-1}(\mathfrak{p}_1)$. By hypothesis we then have $\frac{a_{\lambda_0}}{1} \in \mathfrak{p}_2$, so that $\frac{a_{\lambda_0}}{f^{n_0}} \in \mathfrak{p}_2$, and $\mathfrak{p}_2$ is open, so that $x \in \mathfrak{p}_2$. We conclude that $g \in \mathfrak{p}_2$. The opposite inclusion follows by symmetry.

Surjectivity follows from the following fact: if $\mathfrak{p} \subseteq R$ is an open prime, then $\mathfrak{p}R_{\{f\}}$ is an open prime of $R_{\{f\}}$, and $i^{-1}(\mathfrak{p}R_{\{f\}}) = \mathfrak{p}$. The ideal $\mathfrak{p}R_{\{f\}}$ clearly contains $I_\lambda R_{\{f\}} = \ker \pi_\lambda$, hence is open; we need only verify that it is prime. Suppose $g = (g_\lambda)$ and $h = (h_\lambda)$ satisfy $gh \in \mathfrak{p}R_{\{f\}}$. Again, fix some $\lambda_0$, and write $gh = g_{\lambda_0} h_{\lambda_0} + x$, where $x \in \ker \pi_{\lambda_0}$. Then we find that $g_{\lambda_0} h_{\lambda_0} \in \mathfrak{p}R_{\{f\}}$. Writing $g_{\lambda_0} = g'/f^n$, and similarly for $h_{\lambda_0}$, we have that $\frac{g'h'}{1} \in \mathfrak{p}R_f$ (note that we no longer are working in the completion!). From here it is the usual proof that $\frac{g'}{1} \in \mathfrak{p}R_f$ or $\frac{h'}{1} \in \mathfrak{p}R_f$: we know $f^m(g'h'f^n - r) = 0 \in \mathfrak{p} \subseteq R$, where $r \in \mathfrak{p}$, so, since $f \notin \mathfrak{p}$, we have $g'h'f^n \in \mathfrak{p}$, hence $g' \in \mathfrak{p}$ or $h' \in \mathfrak{p}$. Chasing back through our computations, we have $g \in \mathfrak{p}R_{\{f\}}$ or $h \in \mathfrak{p}R_{\{f\}}$.

That $j$ is open follows from the fact that $j(\mathfrak{D}(g)) = \mathfrak{D}(fg')$, where $g' \in R$ is some element such that $\frac{g'}{1} - g \in I_\lambda R_f$. We already know that in the case of ordinary schemes $D(fg') = D(f) \cap D(g')$, hence $\mathfrak{D}(fg') = \mathfrak{D}(f) \cap \mathfrak{D}(g')$, and we've already seen that $j$ takes image in $\mathfrak{D}(f)$, so we need only show that if $\mathfrak{p} \in \mathfrak{D}(g)$, then $j(\mathfrak{p}) \in \mathfrak{D}(g')$ and if $\mathfrak{q} \in \mathfrak{D}(g')$, then there exists some $\mathfrak{q}'$ with $j(\mathfrak{q}') = \mathfrak{q}$. The former implication is straightforward: if $g' \in i^{-1}(\mathfrak{p})$, then $\frac{g'}{1} \in \mathfrak{p}$, and by assumption, $\mathfrak{p}$ is open, so $\frac{g'}{1} - g \in \mathfrak{p}$, so $g \in \mathfrak{p}$, but this is absurd. The latter is also straightforward: take $q' = \mathfrak{q}R_{\{f\}}$. As in the above paragraph, we find that this is an open prime of $R_{\{f\}}$, and its image under $j$ is $\mathfrak{q}$.

Now we should settle the claim regarding $j^\#$. Let $g \in R$. We know that $j^\#(\mathfrak{D}(g)) : \mathscr{O}_R|_{\mathfrak{D}(f)}(\mathfrak{D}(g)) \to \mathscr{O}_{R_{\{f\}}}(i^{-1}(\mathfrak{D}(g)))$ is the natural map between these two rings. We just need to unravel what they are. From our previous work, we identify the domain with $R_{\{fg\}}$ and the codomain with $\left(R_{\{f\}}\right)_{\{g/1\}}$. But the natural map in question is an isomorphism, as the following diagram shows (dashed arrows indicate unique maps):




So we can assemble a morphism $(h, h^\#) : (\mathfrak{D}(f), \mathscr{O}_R|_{\mathfrak{D}(f)}) \to \operatorname{Spf} R_{\{f\}}$, as $h^{-1}(\mathfrak{D}(g)) =$

$j(\mathfrak{D}(g)) = \mathfrak{D}(fg')$, as described above. Checking that this is a mutual inverse with $(j, j^{\#})$ is routine. □

**Proposition 2.1.8.** *Let $\mathfrak{X} = \operatorname{Spf} R$ and $\mathfrak{Y} = \operatorname{Spf} S$ be two affine formal schemes. A morphism $(\varphi, \varphi^{\#}) : \mathfrak{X} \to \mathfrak{Y}$ is of the form of one given by proposition 2.1.6 if and only if, for any $x \in \mathfrak{X}$, the induced map $\varphi_x^{\#} : \mathscr{O}_{\mathfrak{Y},\varphi(x)} \to \mathscr{O}_{\mathfrak{X},x}$ is local.*

*Proof.* First suppose that $(\varphi, \varphi^{\#})$ is the map on formal spectra induced by some continuous $f : S \to R$. For convenience, let $\mathfrak{q} = f^{-1}(\mathfrak{p})$, i.e., $\mathfrak{q} = \varphi(\mathfrak{p})$. Then we have a commutative diagram

$$\begin{array}{ccc} \mathscr{O}_{\mathfrak{Y},\mathfrak{q}} & \longrightarrow & \mathscr{O}_{\mathfrak{X},\mathfrak{p}} \\ \downarrow & & \downarrow \\ S_{\{\mathfrak{q}\}} & \xrightarrow{f_{\mathfrak{q}}} & R_{\{\mathfrak{p}\}} \end{array}$$

where the vertical arrows are local maps by proposition 2.1.5. Thus we need only show that the bottom map is local. If $x \in \varprojlim_{\lambda} \mathfrak{q}/I_\lambda$, where $\{I_\lambda\}$ is a collection of ideals of definition for $S$, then we may write $x = \lim x_\lambda$, where $\{x_\lambda\}$ is a Cauchy net in $S_{\mathfrak{q}}$ converging to $x$ with $f_{\mathfrak{q}}(x_\lambda) \in \mathfrak{p}R_{\mathfrak{p}}$. So the image of $x$ is $\lim f_{\mathfrak{q}}(x_\lambda)$, hence $f_{\mathfrak{q}}(x) \in \varprojlim \mathfrak{p}/J_\mu$, i.e., $f_{\mathfrak{q}}$ is local.

Now suppose conversely that $\varphi_x^{\#}$ is local. Even without this assumption, we have a commutative diagram

$$\begin{array}{ccc} S = \Gamma(\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}) & \xrightarrow{\varphi^{\#}(\mathfrak{Y})} & \Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) = R \\ \downarrow & & \downarrow \\ \mathscr{O}_{\mathfrak{Y},f(x)} & \xrightarrow{\varphi_x^{\#}} & \mathscr{O}_{\mathfrak{X},x}. \end{array}$$

The commutative diagram informs us, by taking the inverse image of the maximal ideal of $\mathscr{O}_{\mathfrak{X},x}$, that if $\mathfrak{p}$ is the open prime of $R$ corresponding to $x$, then $\varphi(x) = (\varphi^{\#}(\mathfrak{Y}))^{-1}(\mathfrak{p})$. That is, as a map on topological spaces, $\varphi$ is the map induced by the map of rings $\varphi^{\#}(\mathfrak{Y})$.

Call the above map $f$. Then, for any $a \in S$, we have the following commutative diagram:

$$\begin{array}{ccc} S = \Gamma(\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}) & \longrightarrow & \Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) = R \\ \downarrow & & \downarrow \\ S_{\{a\}} = \Gamma(\mathfrak{D}(a), \mathscr{O}_{\mathfrak{Y}}) & \xrightarrow{\varphi^{\#}(\mathfrak{D}(a))} & \Gamma(\mathfrak{D}(f(a)), \mathscr{O}_{\mathfrak{X}}) = R_{\{f(a)\}}. \end{array}$$

The universal property of the ring $S_{\{a\}}$, however, tells us that $\varphi^{\#}(\mathfrak{D}(a))$ must be the map induced by $f = \varphi^{\#}(\mathfrak{Y})$. So in fact $\varphi^{\#}$ is the map on sheaves given by a map of rings, as we indicated. □

Keeping the above in mind, we define:

**Definition 2.1.9.** A morphism of formal schemes is a morphism $(\varphi, \varphi^{\#})$ of topologically ringed spaces that induces local morphisms on local rings.

As a consequence of proposition 2.1.8, we have:

**Proposition 2.1.10.** *The category of affine formal schemes is the dual category of admissible rings.*

## 2.2 First properties of formal schemes

Here we would like to record the analogues, for formal schemes, of several elementary properties of ordinary schemes (or morphisms thereof), such as proper and Noetherian.

It is not obvious from our exposition thus far, but, for reasons which will become clear in section 2.3, it is useful to define a large number of properties for formal schemes that actually refer to certain "subschemes of definition". For now, let us say that, not only do the first non-trivial examples of formal schemes arise in a way that makes these properties apply, but also that if we have "nice enough" formal schemes, we can use these properties to apply results from ordinary scheme theory and learn about the objects of our study. In order to make these definitions, we first need to discuss *ideals of definition* for formal schemes.

### Ideals of definition and subschemes of definition

The first thing we wish to discuss is an *ideal of definition* of a formal scheme. However, its construction follows that of a sheaf of modules on a formal scheme, so we will briefly detail that first.

Suppose we are given an admissible ring $R$ and an admissible module $M$. Then we may write $M = \varprojlim M/I_\lambda M$ for ideals of definition $I_\lambda$ of $R$. As in the definition of Spf, we note that $\operatorname{Spec} R/I_\lambda = \operatorname{Spec} R/I_{\lambda'}$ as topological spaces. Thus the sheaves of modules $(M/I_\lambda M)^\sim$, which are each defined on $\operatorname{Spec} R/I_\lambda$, are all defined on the same topological space. We then say:

**Definition 2.2.1.** Given $R$ and $M$ as above, the sheaf of modules associated to $M$ is the projective limit

$$M^\triangle = \varprojlim_\lambda (M/I_\lambda M)^\sim$$

of sheaves on the topological space $\operatorname{Spf} R$. The natural action of $R = \varprojlim R/I_\lambda$ on $M$ makes $M^\triangle$ a sheaf of $\mathscr{O}_R$-modules on $\operatorname{Spf} R$.

A sheaf of ideals is defined as for ordinary schemes. The following particular class of sheaves of ideals we will frequently use:

**Definition 2.2.2.** An ideal of definition for a formal scheme $\mathfrak{X}$ is a sheaf of ideals $\mathscr{I}$ such that there exists a covering of $\mathfrak{X}$ by open affine formal schemes $\mathfrak{U}_i = \operatorname{Spf} R_i$ where $\mathscr{I}|_{\mathfrak{U}_i} = J_i^\triangle$ for some ideal of definition $J_i$ of $R_i$.

These are particularly handy because they allow us to work with ordinary schemes closely related to our formal schemes:

**Lemma 2.2.3.** *If $\mathfrak{X}$ is a formal scheme and $\mathscr{I}_\lambda$ is an ideal of definition of $\mathfrak{X}$, the ringed space $(\mathfrak{X}, \mathscr{O}_\mathfrak{X}/\mathscr{I}_\lambda)$ is an ordinary scheme.*

*Proof.* Let $\mathfrak{U} = \operatorname{Spf} R$ be an open formal affine of $\mathfrak{X}$. Then $\mathscr{O}_{\mathfrak{X}}|_{\mathfrak{U}} \simeq \mathscr{O}_R$, and $\mathscr{I}_\lambda|_{\mathfrak{U}} \simeq I_\lambda^\triangle$. By definition $(\mathscr{O}_{\mathfrak{X}}/\mathscr{I}_\lambda)|_{\mathfrak{U}}$ is the sheafification of the presheaf that assigns, to any open $\mathfrak{V} \subseteq \mathfrak{U}$, $\mathscr{O}_{\mathfrak{X}}(\mathfrak{V})/\mathscr{I}_\lambda(\mathfrak{V})$. This is also the sheafification of the presheaf assigning $\mathfrak{V}$ to $\mathscr{O}_{\mathfrak{X}}|_{\mathfrak{U}}(\mathfrak{V})/\mathscr{I}_\lambda|_{\mathfrak{U}}(\mathfrak{V})$. Let $f \in R$, and observe that on $\mathfrak{D}(f)$, we have $\mathscr{O}_{\mathfrak{X}}|_{\mathfrak{U}}(\mathfrak{D}(f))/\mathscr{I}_\lambda|_{\mathfrak{U}}(\mathfrak{D}(f)) \simeq R_{\{f\}}/(I_\lambda)_{\{f\}} \simeq (R/I_\lambda)_f$. So this presheaf agrees, on a basis of $\operatorname{Spf} R = \operatorname{Spec} R/I_\lambda$, with the sheaf $\mathscr{O}_{R/I_\lambda}$. That is, $(\mathfrak{U}, \mathscr{O}_{\mathfrak{X}}|_{\mathfrak{U}}/\mathscr{I}_\lambda|_{\mathfrak{U}}) \simeq \operatorname{Spec} R/I_\lambda$. □

**Definition 2.2.4.** The subscheme of definition $X_\lambda$ associated to the ideal of definition $\mathscr{I}_\lambda$ of a formal scheme $\mathfrak{X}$ is the ringed space consisting of the topological space underlying $\mathfrak{X}$ equipped with the sheaf of rings $\mathscr{O}_{\mathfrak{X}}/\mathscr{I}_\lambda$.

*Remark* 2.2.5. We'll define a closed immersion for formal schemes as the exact analogue of a closed immersion for ordinary schemes. Under this definition, it is obvious that we have a closed immersion *of formal schemes* $X_\lambda \to \mathfrak{X}$. The point of the above lemma and definition is that we do get a *bona fide* ordinary scheme structure on this closed sub-formal scheme.

However, there's no guarantee that an ideal of definition exists for a formal scheme. For this reason, we'll restrict ourselves to locally noetherian formal schemes. The definition is parallel to the case of ordinary schemes:

**Definition 2.2.6.** A formal scheme $\mathfrak{X}$ is *adic* if there exists a cover of $\mathfrak{X}$ by open affine formal schemes $\mathfrak{U}_i = \operatorname{Spf} R_i$ where $R_i$ is adic.

A formal scheme $\mathfrak{X}$ is *locally Noetherian* if the $R_i$ are Noetherian *and adic*. $\mathfrak{X}$ is *Noetherian* if it is locally Noetherian and quasi-compact.

**Warning 2.2.7.** It's easy to fall into the mindset that a locally Noetherian formal scheme is merely a formal scheme admitting a covering by open affines $U_i = \operatorname{Spf} R_i$ where $R_i$ is just Noetherian. While this seems a sensible definition, we prefer the one above because most formal schemes that arise in practice are adic and conclusions are rather difficult to draw without the adic hypothesis.

Yasuda's paper [Yas09] is an attempt to construct formal schemes in a way that allows one to forego the adic hypotheses. In summary, Yasuda constructs formal schemes as *proringed spaces*, i.e., topological spaces equipped with sheaves taking values in the category of pro-rings. In order to avoid conflict with the terminology in the literature, Yasuda keeps the term *locally Noetherian* for describing formal schemes which are (in his paradigm) locally Noetherian and adic, and then introduces the term *locally pre-Noetherian* to describe the situation that retains Noetherian hypotheses, but drops adic ones.

Note that the definition of adic *does not* imply that there exists an ideal of definition. We'll shortly prove that locally Noetherian schemes have ideals of definitions, and note in remark 2.2.9 what issue could prevent the existence of an ideal of definition; the same issue stands for general adic formal schemes. However, it is easy to see that, if $\mathscr{I}$ is an ideal of definition for an adic formal scheme $\mathfrak{X}$, then so is $\mathscr{I}^n$ for any $n > 0$.

**Proposition 2.2.8** (Cf. [Gro60, I.10.5.4])**.** *A locally Noetherian formal scheme admits a* largest *ideal of definition $\mathscr{I}_0$. In particular, ideals of definition exist for locally Noetherian formal schemes.*

*Proof.* If $\mathfrak{X} = \operatorname{Spf} R$ is affine, where $R$ is Noetherian, take $\mathscr{I}_0 = (I_0)^\triangle$, where $I_0$ is the largest ideal of definition of $R$.

To deal with the general case, we need only deal with the situation where $\mathfrak{U} = \operatorname{Spf} S \subseteq \mathfrak{V} = \operatorname{Spf} T$ are noetherian open affines of $\mathfrak{X}$ and $\mathscr{I}_\mathfrak{V}$ is the largest ideal of definition of $\mathfrak{V}$. The question is whether $\mathscr{I}_\mathfrak{V}|_\mathfrak{U}$ is the largest (indeed, whether it is any) ideal of definition. The key observation is that $(\mathfrak{U}, \mathscr{O}_\mathfrak{X}|_\mathfrak{U}/\mathscr{I}_\mathfrak{V}|_\mathfrak{U})$ is a *reduced* ordinary scheme since $(\mathfrak{V}, \mathscr{O}_\mathfrak{X}|_\mathfrak{V}/\mathscr{I}_\mathfrak{V})$ is. There is a question of whether we can cover $\mathfrak{U}$ by open affines $\mathfrak{U}_i = \operatorname{Spf} S_i$ where $\mathscr{I}_\mathfrak{V}|_\mathfrak{U} = J_i^\triangle$ for an ideal of definition $J_i$, but this is true because $\mathscr{I}_\mathfrak{V}|_{\mathfrak{D}(f)} = (I_T)^\triangle_{\{f\}}$ for any $f \in T$ (by the same analysis that we applied to the structure sheaf in the previous section), so we can cover $\mathfrak{U}$ by sets $\mathfrak{D}(f)$ to establish the result. □

*Remark* 2.2.9. The grease that makes the wheels turn in this proof is that, when we restrict an ideal of definition, we know *which* ideal of definition we get. That is, the brief argument at the end indicating that we end up with an ideal of definition on $\mathfrak{U}$ shows that, if we restrict an ideal of definition to an affine open subset, we will get another ideal of definition. The problem, however, is that if we have ideals of definition $\mathscr{I}_1$ on $\mathfrak{V}_1$ and $\mathscr{I}_2$ on $\mathfrak{V}_2$, we don't know that we can glue them, because we have no guarantee that $\mathscr{I}_1|_{\mathfrak{V}_1 \cap \mathfrak{V}_2} \simeq \mathscr{I}_2|_{\mathfrak{V}_1 \cap \mathfrak{V}_2}$. But the fact that we end up with the largest ideal of definition in the locally Noetherian case saves the day.

*Remark* 2.2.10. Because ideals of definition may not exist for a general formal scheme, sometimes the term *admissible* appears in the literature to refer to formal schemes for which an ideal of definition exists. We will shortly assume that every formal scheme of our interest is locally Noetherian, however, so we will not need this term.

We'll frequently make use of the following:

**Definition 2.2.11.** If $\mathfrak{X}$ is a locally Noetherian formal scheme, the *reduced subscheme of definition $X_0$* is the ordinary scheme $(\mathfrak{X}, \mathscr{O}_\mathfrak{X}/\mathscr{I}_0)$.

## Formal schemes as limits and adic formal schemes

Now that we've defined subschemes of definition, there arises a natural question. We already know that, if $R$ is an admissible ring, $\operatorname{Spf} R$ is (essentially by definition) the limit (in the category of formal schemes) $\varinjlim_\lambda \operatorname{Spec} R/I_\lambda$. So, suppose we have a *fundamental system of ideals of definition* for a formal scheme $\mathfrak{X}$. That is, beyond assuming that we have an ideal of definition

$\mathscr{I}$ for $\mathfrak{X}$, suppose we have a system $\{\mathscr{I}_\lambda\}$ such that, for any open affine $\mathfrak{U} = \operatorname{Spf} S \subseteq \mathfrak{X}$, we have $\mathscr{I}_\lambda = I_\lambda^\triangle$, where $\{I_\lambda\}$ form a fundamental system of ideals of definition of $R$. Can we say that $\mathfrak{X} = \varinjlim X_\lambda$?

In short, the answer is *yes.* The proof essentially boils down to checking definitions. In light of this answer, one might ask, more generally, whether arbitrary limits of ordinary schemes exist in the category of formal schemes. Asking for that might be too much, but there is a nice sufficient condition:

**Proposition 2.2.12** (Cf. [Gro60, Proposition I.10.6.3])**.** *Let $\mathfrak{X}$ be a topological space and $(\mathscr{O}_i, f_{j,i})$ a projective system of sheaves of rings on $\mathfrak{X}$ indexed by $\mathbb{N}$. Denote by $\mathscr{I}_i$ the (sheaf) kernel of the morphism $f_{0,i} : \mathscr{O}_i \to \mathscr{O}_0$. Suppose further that:*

1. *$(\mathfrak{X}, \mathscr{O}_i)$ is an ordinary scheme; call it $X_i$;*
2. *for any $x \in X$ and any $i$, there exists an open neighborhood $U_i$ of $x$ in $\mathfrak{X}$ such that $\mathscr{I}_i|_{U_i}$ is nilpotent; and*
3. *the morphisms $f_{j,i}$ are surjective.*

*Then, taking $\mathscr{O}_{\mathfrak{X}} = \varprojlim_\ell \mathscr{O}_\ell$ to be the sheaf of topological rings on $\mathfrak{X}$, $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ is a formal scheme. Moreover:*

- *the natural morphisms $\mathscr{O}_{\mathfrak{X}} \to \mathscr{O}_\ell$ are surjective;*
- *the kernels $\mathscr{I}^{(\ell)}$ of the natural maps above form a fundamental system of ideals of definition of $\mathfrak{X}$; and*
- *$\mathscr{I}^{(0)} \simeq \varprojlim_\ell \mathscr{I}_\ell$.*

*Proof.* We must show that any $x \in \mathfrak{X}$ has a neighborhood which is isomorphic to $\operatorname{Spf} R$ for some admissible ring $R$. Given $x$, choose some open $\mathfrak{U} \subseteq \mathfrak{X}$ such that $(\mathfrak{U}, \mathscr{O}_0|_{\mathfrak{U}})$ is affine (possible by hypothesis 1). Note, then, that $(\mathfrak{U}, \mathscr{O}_\ell)$ is affine for any $\ell > 0$ since, by hypothesis 3, $\mathscr{O}_0 \simeq \mathscr{O}_\ell / \mathscr{I}_\ell$. (This fact is not obvious, but admit it for the moment.) So we only need show that the conclusion holds in the case when $X_i$ is affine and quasi-compact.

To that end, assume $X_i = \operatorname{Spec} A_i$. Then $f_{j,i}$ is actually the map of affine schemes arising from a map $\varphi_{j,i} : A_j \to A_i$. This claim is not immediately obvious, but neither is it very difficult: all we know is that $f_{j,i}$ is a map of sheaves $\mathscr{O}_j \to \mathscr{O}_i$. So we have a map $\varphi_{j,i} : A_j = \Gamma(\mathfrak{X}, \mathscr{O}_j) \to \Gamma(\mathfrak{X}, \mathscr{I}_i) = A_i$. This leads to a map on spectra $\tilde{\varphi}_{j,i} : \operatorname{Spec} A_i = X_i \to \operatorname{Spec} A_j = X_j$. We should check that this map agrees with $f_{j,i}$. It suffices to check that these agree on some confinal family of open subsets of $\mathfrak{X}$. Take $a \in A_i$, and let $a' = \varphi_{j,i}(a) = f_{j,i}(X)(a)$. Then $D(a) = D(a')$, and we also have that $f_{j,i}(D(a)) = (\varphi_{j,i})_a$, so that $f_{j,i} = \tilde{\varphi}_{j,i}$, as claimed. Then we also have that $\mathscr{I}_\ell = \ker \tilde{\varphi}_{0,\ell} = (\ker \varphi_{0,\ell})^\sim$ as $\mathscr{O}_\ell$-modules. Moreover, since $X_\ell$ is quasi-compact, we know there exists some $n$ such that $\mathscr{I}_\ell^n = 0$, so that $\ker \varphi_{0,\ell}$ is nilpotent.

So let $A = \varprojlim_\ell A_\ell$. Immediately, we know that $A$ is admissible and that a fundamental system of ideals of definition for $A$ is given by $\ker \pi_i$, where $\pi_i : A \to A_i$ is the natural projection. (This result is essentially 1.2.18, though we do need to make use of the fact that the kernels are nilpotent in order to ensure that $\ker \pi_0$ is a *bona fide* ideal of definition.) In particular, $A/\ker \pi_i \simeq A_i$, so $\operatorname{Spf} A$ consists of the topological space $\operatorname{Spec} A_i = \mathfrak{X}$ with the sheaf of rings $\varprojlim_\ell \mathscr{O}_{A_i} = \varprojlim_\ell \mathscr{O}_i$, and, essentially by construction, $\mathscr{I}^{(0)} = \varprojlim_\ell \mathscr{I}_\ell$. □

The last proposition insisted that the sheaves in question be indexed by $\mathbb{N}$, which seems rather inconsistent with our practice so far. Momentarily, however, we will see that it causes no issues for the applications we have in mind.

**Corollary 2.2.13** (Cf. [Gro60, Corollary I.10.6.4])**.** *Keeping the same hypotheses as in proposition 2.2.12, suppose further that, for $i \geq j$, $\ker f_{j,i} = \mathscr{I}_i^{j+1}$ and that $\mathscr{I}_1/\mathscr{I}_1^2$ is of finite type over $\mathscr{O}_0 = \mathscr{O}_1/\mathscr{I}_1$. Then $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ is* adic. *Moreover, $\mathscr{I}^{(n)} = \mathscr{I}^{n+1}$ and $\mathscr{I}/\mathscr{I}^2 \simeq \mathscr{I}_1$. If $X_0$ is (locally) Noetherian, then so is $\mathfrak{X}$.*

*Proof.* We may assume that $X_i$ is affine, in which case, as in the previous proposition, we have $\mathfrak{X} = \operatorname{Spf} \varprojlim A_\ell$. The result, then is simply one of algebra; again, it is essentially proposition 1.2.18, with a bit of added attention.

Let $A = \varprojlim_\ell A_\ell$ and $I = \ker \pi_0$. It suffices to show that $I^{n+1} = \ker \pi_n$.

One containment is straightfoward: if $(x_\ell) \in I^{n+1}$, we may write

$$(x_\ell) = \sum_i \prod_{j=1}^{n+1} ((x_{i,j})_\ell)$$

where $((x_{i,j})_\ell) \in I$. Then we need only show that $\prod_{j=1}^{n+1} \pi_n((x_{i,j})_\ell) = 0$. But $\pi_n((x_{i,j})_\ell) = (x_{i,j})_n$, and since $((x_{i,j})_\ell) \in I$, we must have $f_{0,n}((x_{i,j})_n) = \pi_0((x_{i,j})_\ell) = 0$, i.e., $(x_{i,j})_n \in I_n$. By hypothesis, $I_n^{n+1} = \ker f_{n,n}$, and $f_{n,n} = \mathrm{id}_{A_n}$, so we must have that $\prod_{j=1}^{n+1} \pi_n((x_{i,j})_\ell) = 0$, as desired.

For the other, let $(a_\ell)_j$ be $r$ elements of $I$ such that $(a_1)_1, \ldots, (a_1)_r$ generate $I_1/I_1^2 = I_1$ as an $A_0$-module. It will suffice to show that $S_{n+1}((a_\ell)_j)$ generates $\ker \pi_n$, where $S_n((a_\ell)_j)$ is the set of monomials of degree $n$ in the $(a_\ell)_j$. In turn, it will suffice to show for every $n$ that the residue classes of $S_{n+1}((a_\ell)_j)$ generate $\ker \pi_n / \ker \pi_{n+1}$: given $x \in \ker \pi_n$, we may then write

$$x = \left( \sum_{s \in S_{n+1}} a_s s \right) + y,$$

where $y \in \ker \pi_{n+1}$. Thus, for any $m \geq n$, we have

$$\pi_m(x) = \sum_{k=n+1}^{m} \left( \sum_{s \in S_k} \pi_m(a_{s,k} s) \right) = \sum_{s \in S_{n+1}} \pi_m(a'_{s,m} s).$$

So $\sum a'_{s,m} s$ is a sequence of elements in $A$ that converge to $x$, so it does suffice to show that the residues of $S_{n+1}((a_\ell)_j)$ generate $\ker \pi_n / \ker \pi_{n+1}$. Note that the above reasoning also shows that we need only show that the residues generate $\ker \pi_n / \ker \pi_{n+1}$ *as an $A_{n+1}$-module.* Moreover, $\ker \pi_n / \ker \pi_{n+1} \simeq I_{n+1}^{n+1}$ as an $A_{n+1}$-module, and the residues of $\{(a_\ell)_j\}$ are precisely $\{(a_n)_j\}$. In turn, it suffices to show for any $m$ that $I_m$ is generated by monomials of degree at most $m$ in the $(a_m)_j$. By our choice of the $(a_\ell)_j$, this holds for $m = 1$. Inductively, let $I_n'$ be the $A_n$-submodule of $I_n$ generated by monomials of degree at most $n$ in the $(a_n)_j$, so that we have $I_{n-1} = I_{n-1}'$. Note that $I_{n-1} \simeq I_n / I_n^n$, which allows us to write $I_n' + I_n^n = I_n$. Taking $n$-th powers of both sides of this last relation, we find $I_n'^n = I_n^n$ (note that $I_n' \subseteq I_n$, and $I_n^{n+1} = 0$). So $I_n = I_n' + I_n'^n \subseteq I_n'$, and our desired conclusion follows. □

Finally we arrive at one of the most important characterizations of locally Noetherian formal schemes:

**Corollary 2.2.14.** *Any locally Noetherian formal scheme is the limit of ordinary (locally Noetherian) schemes. Namely, if $\mathfrak{X}$ is a locally Noetherian formal scheme and $\mathscr{I}_0$ denotes its largest ideal of definition, write $X_\ell = (\mathfrak{X}, \mathscr{O}_\mathfrak{X} / \mathscr{I}^{\ell+1})$; then*

$$\mathfrak{X} \simeq \varinjlim_{\ell} X_\ell.$$

*Proof.* Since $\mathfrak{X}$ is locally Noetherian, we know that, for any open affine $\mathfrak{U}$, $\mathscr{O}_\mathfrak{X} / \mathscr{I}_0^2(\mathfrak{U})$ is Noetherian, hence the kernel is finitely generated, so that the hypotheses of corollary 2.2.13 are satisfied. The only thing left to show is that $\mathscr{O}_\mathfrak{X} \simeq \varprojlim_\ell \mathscr{O}_\mathfrak{X} / \mathscr{I}_0^{\ell+1}$. However, since $\mathfrak{X}$ is locally Noetherian, and in particular adic, this follows. □

**Proposition 2.2.15** (Cf. [Gro60, Corollary 10.6.5])**.** *Let $R$ be an admissible ring.* Spf $R$ *is Noetherian if and only if $R$ is Noetherian and adic.*

*Proof.* One direction is trivial: if $R$ is Noetherian (and adic), then Spf $R$ is Noetherian.

Convsersely, let $\mathfrak{X} = \operatorname{Spf} R$, where $R$ is admissible, and suppose $\mathfrak{X}$ is Noetherian. Let $I$ be an ideal of definition of $R$ (note that we cannot, at this point, take the *largest* ideal of definition of $R$, because we do not know $R$ is Noetherian; we only know that open ideals satisfy the ACC), and let $\mathscr{I}$ be the corresponding ideal of definition of $\mathfrak{X}$. Then $X_n = (\mathfrak{X}, \mathscr{O}_\mathfrak{X} / \mathscr{I}^{n+1})$ are affine Noetherian schemes. We are now in a position to apply proposition 2.2.12 and, in light of Noetherianness, corollary 2.2.13 to conclude that $(\mathfrak{X}, \varprojlim \mathscr{O}_\mathfrak{X} / \mathscr{I}^{n+1})$ is an adic formal scheme. So we are reduced to showing that $\mathscr{O}_\mathfrak{X} \simeq \varprojlim \mathscr{O}_\mathfrak{X} / \mathscr{I}^{n+1}$.

Since $\mathfrak{X}$ is Noetherian, it is in particular locally Noetherian, hence adic. So on some cover $\mathfrak{U}_i$ we have $\mathscr{O}_\mathfrak{X}|_{\mathfrak{U}_i} \simeq \varprojlim_\ell \mathscr{O}_\mathfrak{X} / \mathscr{I}^{\ell+1}|_{\mathfrak{U}_i}$, but these isomorphisms glue to give us, indeed, $\mathscr{O}_\mathfrak{X} = \varprojlim_\ell \mathscr{O}_\mathfrak{X} / \mathscr{I}^{\ell+1}$. Taking global sections of the sheaf, we find $R \simeq \varprojlim_\ell R / I^{\ell+1}$ (and this is a topological isomorphism), i.e., $R$ is adic. □

*Remark* 2.2.16. The interesting takeaway from this proof is *not that an admissible Noetherian ring is adic.* Rather, the interesting takeaway is that, if we're given an adic ring, we can check if it is Noetherian by checking if it satisfies the ACC on *open* ideals.

In light of the above proposition, most treatments of formal schemes actually only treat the adic case. The reason for this is that, frequently, it is useful to have a largest ideal of definition, because then some statements (e.g., those concerning the topology of $\mathfrak{X}$) can be studied by looking at the *reduced* subscheme of definition of $\mathfrak{X}$. But since the only real (not *ad-hoc*) way of ensuring that $\mathfrak{X}$ has an ideal of definition is to assume that $\mathfrak{X}$ is locally Noetherian, and in that case, we have that $\mathfrak{X}$ is adic, it greatly simplifies exposition to consider only adic formal schemes.

We will also assume, from here on, that our formal schemes are locally Noetherian (hence adic), unless otherwise stated. However, for convenience, we will continue to make note of the locally Noetherian hypotheses in proposition and theorem statements.

When working with ordinary schemes, we frequently consider the construction of affine and projective space over some scheme. The latter bears special importance because it is part of the definition of a projective morphism. One can carry out the same constructions for formal schemes, which we will do here. One of the interesting new aspects of formal schemes, however, is that we can construct "formal disc space" over a base formal scheme. We will remark on this more when we arrive at its definition.

First, we remark the following:

**Proposition 2.2.17.** *Fiber products exist in the category of formal schemes (not just locally Noetherian formal schemes).*

*Proof sketch.* The essential observation to make is that fiber products exist for *affine* formal schemes. In light of proposition 2.1.10, we need only observe that the category of admissible rings admits pushouts. This is exactly proposition 1.3.9.

Once we know that fiber products exist for affine formal schemes, the proof that fiber products exist in general follows the same route as that for ordinary schemes. □

Now we dispense with the definitions of formal affine, projective, and disk space; the first two are exact analogs of their definitions for ordinary schemes.

**Definition 2.2.18.** Let $\mathfrak{X}$ be a formal scheme, and consider $\mathbb{Z}[x_1, \ldots, x_n]$ with the discrete topology. Formal affine $n$-space over $\mathfrak{X}$ is

$$\mathbb{A}^n_{\mathfrak{X}} = \mathfrak{X} \times \operatorname{Spf} \mathbb{Z}[x_1, \ldots, x_n] \quad (= \mathfrak{X} \times \mathbb{A}^n_{\mathbb{Z}}).$$

**Definition 2.2.19.** Let $\mathfrak{X}$ be a formal scheme and consider $\mathbb{P}^n_{\mathbb{Z}}$ as a formal scheme. Formal projective $n$-space over $\mathfrak{X}$ is

$$\mathbb{P}^n_{\mathfrak{X}} = \mathfrak{X} \times \mathbb{P}^n_{\mathbb{Z}}$$

**Definition 2.2.20.** Let $\mathfrak{X}$ be a formal scheme. Formal $n$-disc space over $\mathfrak{X}$ is

$$\mathbb{D}^n_{\mathfrak{X}} = \mathfrak{X} \times \operatorname{Spf} \mathbb{Z}[[x_1, \ldots, x_n]].$$

Formal disc space earns its name because, in Raynaud's theory of formal models, $\mathbb{D}^n_{\operatorname{Spf} R}$ is a model for the *open* unit $n$-disc over $K = \operatorname{Frac}(R)$. By contrast, $\mathbb{A}^n_{\operatorname{Spf} R}$ is a model for the closed unit disc. It is not hard to see how these claims might be reasonable: the global sections of $\mathbb{D}^n_{\operatorname{Spf} R}$ should be $R[[x_1, \ldots, x_n]]$, while the global sections of $\mathbb{A}^n_{\operatorname{Spf} R}$ should be $R\{x_1, \ldots, x_n\}$. Elements of the former "should" converge on the open unit disc because, in a nonarchimedean setting, a power series converges if and only if the limit of its terms is 0, which is true if $|x_i| < 1$ (recall that, in a nonarchimedean field, the ring of integers consists of those elements $r$ such that $|r| \leq 1$). Elements of the latter, likewise, "should" converge on the closed unit disc because coefficients of such power series tend to zero, hence terms of the series converge to zero even when $|x_i| = 1$.

Note that, if $R$ is a ring with the discrete topology, $\mathbb{A}^n_{\operatorname{Spf}(R)} = \mathbb{A}^n_R$, and the same goes for $\mathbb{P}^n_{\operatorname{Spf} R}$. In particular, if $R = k$ a field, formal affine and projective space are the usual affine and projective space.

We might hope to define properties of morphisms such as separated, finite type, projective, etc. in the category of formal schemes. We can do so, but essentially because we are dealing with ring maps that are not guaranteed to be finite, we have to modify our definitions a little bit. First, let us say that

**Definition 2.2.21.** A morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of adic formal schemes is *adic* if there exists an ideal of definition $\mathscr{J}$ of $\mathfrak{Y}$ such that $f^*(\mathscr{I}_{\mathfrak{Y}})\mathscr{O}_{\mathfrak{X}}$ is an ideal of definition of $\mathfrak{X}$.

(Note that, since the induced map of sheaves $f^\# : f^*\mathscr{O}_{\mathfrak{Y}} \to \mathscr{O}_{\mathfrak{X}}$ is a map of sheaves of topological rings, we must have that $f^\#(U)$ is continuous for every open $U$. In particular, $(f^\#(U))^{-1}(I) \supseteq J$, where $I$ is any ideal of definition of $\mathscr{O}_{\mathfrak{X}}(U)$ and $J$ is some ideal of definition of $\mathscr{O}_{\mathfrak{Y}}(U)$. So $f(J) \subseteq I$, i.e., we always have $f^*(\mathscr{I}_{\mathfrak{Y}})\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{I}_{\mathfrak{X}}$ for some ideal of definition $\mathscr{I}_{\mathfrak{X}}$. Note also that, under the convention of regarding ordinary schemes as formal schemes with sheaves of rings with 0-adic topologies, every morphism of ordinary schemes is adic.)

With the above in mind, let's return to the locally Noetherian setting and make the following definition:

**Definition 2.2.22.** A morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of locally Noetherian formal schemes is of *pseudo-finite type* if the induced map on reduced subschemes of definiton $f_0 : X_0 \to Y_0$ is of finite type. We say $f$ is of *finite type* if $f$ is pseudo-finite type and adic.

(Note that every morphism of ordinary schemes of finite type is of finite type according to this definition.)

Analogously, many properties of morphisms of locally Noetherian formal schemes are termed pseudo- if the property is satisfied on the reduced subschemes of definition. (For example, a morphism is pseudo-proper if $f_0 : X_0 \to Y_0$ is proper.)

One interesting exception is that a *pseudo-separated* morphism is in fact separated:

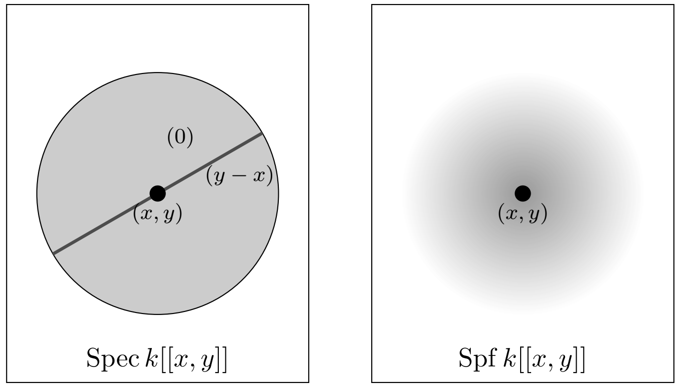



Figure 2.1: $\operatorname{Spec} k[[x,y]]$ vs. $\operatorname{Spf} k[[x,y]]$.

**Proposition 2.2.23.** *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of locally Noetherian formal schemes such that $f_0 : X_0 \to Y_0$ is separated. Then $f$ is separated.*

*Proof.* This result follows because separatedness is a purely topological condition. That is, we first note that the underlying topological space of $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is the underlying topological space of $X_0 \times_{Y_0} X_0$. Since $f_0$ is separated, we know that the (topological) image of $\Delta_0$ is closed in $X_0 \times_{Y_0} X_0$ is closed. So the image of $\Delta$ is closed in $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$, i.e., $\Delta$ is a closed immersion. □

## 2.3 First examples: completions

Now that we've introduced formal schemes, we should spend some time looking for actual examples thereof. As noted earlier, since any ring can be considered admissible (in fact, adic) with respect to the $\{0\}$-adic topology, we can realize any ordinary scheme as a formal scheme. And, of course, if $R$ is an admissible ring with respect to a topology that is not the $\{0\}$-adic one, $\operatorname{Spf} R$ and $\operatorname{Spec} R$ are two different spaces.

**Example 2.3.1.** Let $R = k[[x,y]] = \varprojlim_{\ell} k[x,y]/(x,y)^{\ell}$. $R$ is manifestly adic with respect to a topology that is not the 0-adic topology. $R$ has only one open prime ideal, namely, $(x,y)$, so $\operatorname{Spf} R$ is a one-point space. However, $\operatorname{Spec} R$ has many points: it is a two-dimensional topological space, and contains, for example, points corresponding to ideals $(y - ax)$ for $a \in k$. (More generally, it contains points corresponding to ideals $(y - g(x))$ where $g(x) \in k[[x]]$.) See figure 2.1.

The most elementary "exotic" example of a formal scheme is the *completion* of a (formal) scheme along a closed (formal) subscheme:

**Definition 2.3.2.** Let $X$ be a locally Noetherian scheme, and let $Y$ be a closed subscheme given by a sheaf of ideals $\mathscr{I}_Y$. The *completion of $X$ along $Y$* is the ringed space consisting of the underlying topological space $Y$ with the sheaf of rings given by $\varprojlim_\ell \mathscr{O}_X/\mathscr{I}_Y^\ell$. We denote the completion by $X_{/Y}$.

**Proposition 2.3.3.** *$X_{/Y}$ is a formal scheme.*

*Proof.* We need to show that every point admits an open neighborhood isomorphic to Spf $R$ for some admissible ring $R$. Let $\operatorname{Spec} S = U \subseteq X$ be an open affine; it will suffice to show that $U_{/Y\cap U}$ is a formal affine. Let $I = \Gamma(U, \mathscr{I}_Y)$. Since $U$ is affine, we have

$$\begin{aligned}\left(\varprojlim_\ell \mathscr{O}_X/\mathscr{I}_Y^\ell\right)|_U &\simeq \varprojlim_\ell \mathscr{O}_X|_U/\mathscr{I}_Y^\ell|_U \\ &\simeq \varprojlim_\ell \mathscr{O}_R/\tilde{I}^\ell\end{aligned}$$

Then we just observe that $U \cap Y$ is the underlying topological space of $\operatorname{Spec} R/I$, so that $U_{/Y\cap U}$ has the same definition as $\operatorname{Spf} \varprojlim R/I^\ell$. (Note that this does make use of the Noetherian hypothesis, as otherwise funny things may happen when considering the $I$-adic completion of $R$.) □

*Remark* 2.3.4. One could, more generally, discuss the completion of a (locally Noetherian) formal scheme along a closed formal subscheme. That is, given a closed immersion of formal schemes $\mathfrak{Y} \to \mathfrak{X}$ with sheaf of ideals $\mathscr{I}_\mathfrak{Y}$, we might consider the space $\mathfrak{Y}$ equipped with the sheaf of rings $\varprojlim_\ell \mathscr{O}_\mathfrak{X}/\mathscr{I}_\mathfrak{Y}^{\ell+1}$. The only issue here is whether the resulting (topologically) ringed space is a genuine formal scheme: what we'd like to do is, as above, start with an formal affine open $\mathfrak{U} = \operatorname{Spf} R \subseteq \mathfrak{X}$ and conclude that $(\mathfrak{U}, \varprojlim \mathscr{O}_\mathfrak{X}|_\mathfrak{U}/\mathscr{I}_\mathfrak{Y}^{\ell+1}|_\mathfrak{U}) \simeq \operatorname{Spf} S$ where $S = \varprojlim \mathscr{O}_\mathfrak{X}(\mathfrak{U})/\mathscr{I}_\mathfrak{Y}^{\ell+1}(\mathfrak{U})$. As usual, the limit presents no problem: the real question is whether

$$\left(\mathscr{O}_\mathfrak{X}/\mathscr{I}_\mathfrak{Y}^{\ell+1}\right)(\mathfrak{U}) \simeq \mathscr{O}_\mathfrak{X}(\mathfrak{U})/\mathscr{I}_\mathfrak{Y}^{\ell+1}(\mathfrak{U}).$$

The functor $M \rightsquigarrow M^\triangle$ is an equivalence of categories between *coherent* $\mathscr{O}_\mathfrak{X}$-modules and $\mathscr{O}_\mathfrak{X}(U)$ modules of finite type. Thus, if $\mathscr{I}_\mathfrak{Y}$ is known to be coherent, there are no troubles. (One might worry about taking the completion above, but this completion is well behaved, at least in the locally Noetherian case.)

**Example 2.3.5.** In fact, example 2.3.1 is the completion of $\mathbb{A}_k^2$ along the origin with embedding given by ideal $(x, y)$ (or, in fact, $(x, y)^n$ for any $n > 0$).

But for a non-affine example, consider the completion of $\mathbb{P}_k^2$ along a line given by, say, $\mathscr{I}_L \simeq \mathscr{O}(-1)$. By definition, the completion is the topological space $L$ equipped with the sheaf of rings $\varprojlim_\ell \mathscr{O}_{\mathbb{P}^2}/\mathscr{I}_L^\ell$. The topological space $L$ is isomorphic to $\mathbb{P}^1$; choose some point $P \in L$, and let $L'$ be a line in $\mathbb{P}^2$ meeting $L$ at $P$. If $U = \mathbb{P}^2 \setminus L'$, then $(U, \mathscr{O}_{\mathbb{P}^2}|_U) \simeq \mathbb{A}_k^2$. Since $\mathscr{I}_L \simeq \mathscr{O}(-1)$, and $U$ is affine, we have

$$\begin{aligned}(\mathscr{O}_{\mathbb{P}^2}|_U/\mathscr{I}_L|_U^n)(U) &\simeq \mathscr{O}_{\mathbb{P}^2}|_U(U)/\mathscr{I}_L|_U(U)^n \\ &\simeq k[x,y]/(f)^n, \quad \text{for some } f = \alpha x + \beta y + \gamma \\ &\simeq k[x,y]/(x)^n, \quad \text{after a change of coordinates.}\end{aligned}$$

So if $V = L - P$, $\left(\varprojlim_\ell \mathscr{O}_{\mathbb{P}^2}/\mathscr{I}_L^\ell\right)(V) \simeq (k[y])[[x]]$.

A fairly natural question is whether *every* (say locally Noetherian) formal scheme arises as a completion of some (locally Noetherian) scheme along a closed subscheme. The answer is a decisive *no*. In fact, criteria for determining when a formal scheme is a completion are known as *algebraicity criteria*, and arise in a few different contexts. (See, for example, [Băd04, Chapter 10], or [Bos01] for an example with formal schemes over number fields.)

## 2.4 Differentials on formal schemes

Recall that, for a morphism of ordinary schemes $f : X \to Y$, we can consider the sheaf of 1-forms $\Omega^1_f$, which frequently receives the notation $\Omega^1_{X/Y}$. This sheaf can either be constructed locally on affines (where it is just the sheafification of the module $\Omega_{A/B}$), or as the pullback of $\mathscr{I}/\mathscr{I}^2$, where $\mathscr{I}$ is the sheaf of ideals of $\Delta(X)$ in some open subset of $X \times_Y X$.

Commutative algebra informs us that, if $f : X \to Y$ is a morphism of finite type between Noetherian schemes, $\Omega_{X/Y}$ should be coherent. Unfortunately, the analogous statement fails spectacularly for pseudo-finite type morphisms:

**Example 2.4.1.** Let $\mathfrak{X} = \operatorname{Spf} k[[x]]$. Then $\Omega^1_{\mathfrak{X}/k}$ is not coherent.

Intuitively, the reason for this is that, for a general power series

$$g = \sum_{m=0}^{\infty} b_m x^m$$

we "should have"

$$dg = \lim_{\ell\to\infty}\left(\sum_{m=0}^{\ell}(m+1)b_{m+1}x^m dx\right).$$

However, the module $\Omega^1_{k[[x]]/k}$ is not complete, so we cannot pass the limit inside the parentheses. Consequently, we should have distinct elements $dg$ for some (necessarily infinite) class of power series that span $k[[x]]$ as a $k[x]$-module.

In more detail: suppose, to the contrary, that it were, say, by $df_1, \ldots, df_n$. Then, for a general $f \in k[[x]]$, we should be able to write $df = \sum_{i=1}^n g_i df_i$. We know, however, that if we endow $\Omega_{k[[x]]/k}$ with the $x$-adic topology, then

$$df = \lim_{\ell\to\infty}\left(\sum_{m=0}^{\ell}(m+1)a_{m+1}x^m dx\right),$$

where

$$f = \sum_{m=0}^{\infty} a_m x^m.$$

In particular, we should have that

$$\left(\sum_{m=0}^{\ell}(m+1)a_{m+1}x^m\right) dx = \sum_{i=1}^{\ell}\left(\sum_{j=1}^{n} b_{i,j} f_i\right) x^i,$$

where $b_{i,j} \in k$, for arbitrarily large $\ell$. But this is possible only if $b_{i,j} = 0$ for arbitrarily large $i$, which is true if and only if $f$ was a $k[x]$-linear combination of the $f_i$. In particular, we may find an $f$ not in the $k[[x]]$-span of the $df_i$.

However, all is not lost. In fact, the studies [TL07] and [TLR09] show that the sheaf $\hat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}$ (which is locally of the form $\hat{\Omega}^{1\triangle}_{A/B}$) retains many of the good properties, for locally Noetherian schemes, that $\Omega^1_{X/Y}$ holds for ordinary schemes. For example:

**Proposition 2.4.2** (Cf. [TL07, Proposition 4.6])**.** *A morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of pseudo-finite type between two locally Noetherian formal schemes is unramified if and only if $\hat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}} = 0$.*

**Proposition 2.4.3** (Cf. [TL07, Proposition 4.8])**.** *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a smooth morphism of locally Noetherian formal schemes. Then $f$ is flat and $\hat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}$ is a locally free $\mathscr{O}_{\mathfrak{X}}$-module of finite rank.*

**Proposition 2.4.4** (Cf. [TL07, Corollary 4.10])**.** *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be an étale morphism of locally Noetherian formal schemes. Given a pseudo-finite type morphism $\mathfrak{Y} \to \mathfrak{S}$ of locally Noetherian formal schemes, we have*

$$f^*\hat{\Omega}^1_{\mathfrak{Y}/\mathfrak{S}} \simeq \hat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}}.$$

This last proposition is especially interesting in light of the fact that, if $\mathfrak{X} = X_{/Y}$ is the completion of a variety over $k$ along a closed subscheme, then the natural map $\kappa : \mathfrak{X} \to X$ is etale. In particular, we have $f^*\Omega^1_{X/k} \simeq \hat{\Omega}^1_{\mathfrak{X}/k}$. We'll later find the desire to have a suitable analogue, for formal schemes, of a "canonical divisor" (though we'll always be working with Cartier divisors, hence line bundles). With this last fact in mind, a potential analogue (which actually works well) might be $\det \hat{\Omega}^1_{\mathfrak{X}/k}$.

# Chapter 3

# Cohomology and duality

As with ordinary schemes, cohomology is an indispensible tool for formal schemes. To that end, this chapter is dedicated to collecting results that we will need for our later proofs, and has little motivation other than their assembly.

Unlike the previous chapter, whose purpose was to introduce our setting and a flavor of elementary proofs, some of the results in this chapter are far beyond the scope of our writing. Where appropriate, we'll offer some proofs, or sketches, but, especially with the results on duality, we'll prefer to take the results as given and point to their sources.

## 3.1 Sheaf cohomology for formal schemes

Here we would like to accomplish the following:

- record the definition of sheaf cohomology for formal schemes (which is the same as sheaf cohomology for ordinary schemes, but there are a couple different behaviors);
- establish definitions of and notations for the derived (and bounded variants thereof) category of sheaves of (quasi-coherent and/or quasi-coherent torsion) $\mathscr{O}_{\mathfrak{X}}$-modules for a formal scheme $\mathfrak{X}$, and interpret the definition of cohomology in this framework;
- establish a result for computing sheaf cohomology for a particular class of formal schemes (which includes completions of smooth varieties along closed subvarieties).

The definition of sheaf cohomology we will use is the derived functor one. This definition is due to Grothendieck,though it has seen treatment in several other places, such as [Har77].

In brief: given any ringed space $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$, the category of sheaves of $\mathscr{O}_{\mathfrak{X}}$-modules has enough injectives, which implies that every sheaf of $\mathscr{O}_{\mathfrak{X}}$-modules admits an injective resolution. That is, given an $\mathscr{O}_{\mathfrak{X}}$-module $\mathscr{F}$, we may find sheaves $\mathscr{I}^n$ for $n \geq 0$ such that there exists an exact sequence

$$0 \to \mathscr{F} \to \mathscr{I}^0 \to \mathscr{I}^1 \to \mathscr{I}^2 \to \cdots,$$

where $\mathscr{I}^n$ is *injective*, i.e., $\mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\cdot, \mathscr{I}^n)$ is an exact functor. We define $H^i(\mathfrak{X}, \mathscr{F})$ to be the $i$-th right derived functor of the global sections functor: the functor $\Gamma(\mathfrak{X}, \cdot)$ applied to the exact sequence above produces a (not necessarily exact) sequence

$$\Gamma(\mathfrak{X}, \mathscr{I}^0) \xrightarrow{\delta^0} \Gamma(\mathfrak{X}, \mathscr{I}^1) \xrightarrow{\delta^1} \Gamma(\mathfrak{X}, \mathscr{I}^2) \to \cdots,$$

and we say $H^i(\mathfrak{X}, \mathscr{F}) = \ker \delta^i / \operatorname{im} \delta^{i-1}$. We should that the above definition is independent of the choice of injective resolution $\{\mathscr{I}^n\}$. This is essentially a fact from homological algebra; for our discussion, we'll note that, in fact, any two injective resolutions are homotopy equivalent, and (though this does not follow from that observation, per se) that we get isomorphic cohomology groups regardless of the resolution chosen.

Note that we did not stipulate anything about the structure sheaf $\mathscr{O}_{\mathfrak{X}}$. In particular, we could take $\mathscr{O}_{\mathfrak{X}}$ to be the constant sheaf $\mathbb{Z}$, which means that the above cohomology groups are well-defined for sheaves of abelian groups. We could also, in the case of formal schemes, take $\mathscr{O}_{\mathfrak{X}}$ to be the structure sheaf, and we arrive at cohomology for $\mathscr{O}_{\mathfrak{X}}$-modules. Whether we choose an injective resolution of $\mathscr{O}_{\mathfrak{X}}$-modules or sheaves of abelian groups for a sheaf of $\mathscr{O}_{\mathfrak{X}}$-modules does not impact the cohomology groups $H^i(\mathfrak{X}, \mathscr{F})$: injective $\mathscr{O}_{\mathfrak{X}}$-modules are flasque, hence $\Gamma(\mathfrak{X}, \cdot)$-acyclic, hence can be used for the computations of cohomology (of $\mathscr{O}_{\mathfrak{X}}$-modules as sheaves of abelian groups).

The above definition is already abstract, but concrete in the sense that there is a "procedure" for computing cohomology: given a sheaf $\mathscr{F}$, take an injective resolution, apply the global sections functor, and then compute kernel mod image. The principal difficulty is, of course, finding an injective resolution. A nice way to "streamline" the definition is to do the following: the category of $\mathscr{O}_{\mathfrak{X}}$-modules is an abelian category, so we can construct its derived category $\mathbf{D}(\mathfrak{X})$ as follows. First we consider the chain category consisting of chains $\cdots \to \mathscr{F}^i \to \mathscr{F}^{i+1} \to \cdots$ with morphisms $\mathscr{F}^\bullet \to \mathscr{G}^\bullet$ that are maps $\varphi^i : \mathscr{F}^i \to \mathscr{G}^i$ compatible with the morphisms $\mathscr{F}^i \to \mathscr{F}^{i+1}$. We then form the homotopy category by identifing morphisms that are chain homotopic. Finally, we form the derived category by localizing at quasi-isomorphisms, i.e., those maps of complexes that induce isomorphisms on cohomology. That is, a morphism $X \to Y$ in the derived category is a "roof"

$$X \xleftarrow{\ p\ } X' \longrightarrow Y$$

where $p$ is a quasi-isomorphism.

There is a natural map from the category of $\mathscr{O}_{\mathfrak{X}}$-modules to $\mathbf{D}(\mathfrak{X})$ that places $\mathscr{F}$ in degree 0. In this parlance, the cohomology groups of $\mathscr{F}$ are just the cohomology groups of the complex obtained by applying $\Gamma(\mathfrak{X}, \cdot)$ to the injective resolution of $\mathscr{F}$ in $\mathbf{D}(\mathfrak{X})$ (note that the terminology is well-founded, as any two injective resolutions are homotopy equivalent, hence isomorphic in the homotopy category and derived category).

With the general cohomology definitions out of the way, we'd like to know if there are tools for computing cohomology for formal schemes in general. The most basic is the following:

**Proposition 3.1.1** (Cf. [Har68, Proposition 4.1]). *Let $\mathfrak{X}$ be a locally Noetherian formal scheme pseudo-proper over a field $k$ of characteristic $0$, and let $\mathscr{F}$ be a quasi-coherent sheaf on $\mathfrak{X}$. Then, for each $i \geq 0$,*

$$H^i(\mathfrak{X}, \mathscr{F}) = \varprojlim_{\ell} H^i(X_\ell, \mathscr{F}_\ell),$$

*where $X_\ell$ and $\mathscr{F}_\ell$ denote the $\ell$-th infinitesimal neighborhood of $X_0$ in $\mathfrak{X}$ and the restriction of $\mathscr{F}$ to that subscheme, respectively.*

## 3.2 From Grothendieck duality to Serre duality for ordinary schemes

We'll see in the next chapter that one of the obstacles we face in establishing vanishing theorems is a lack of an exact analogue for the following theorem:

**Theorem 3.2.1** (Serre Duality). *Let $X$ be a nonsingular projective variety of dimension $n$ over an algebraically closed field $k$, and let $\mathscr{F}$ be a locally free sheaf on $X$. Then*

$$H^i(X, \mathscr{F}) \simeq H^{n-i}(X, \mathscr{F}^\vee \otimes \omega_X)^\vee. \tag{3.2.1}$$

The form above, however, can be frustrating even in the study of ordinary schemes. In particular:

- one might hope to apply the above theorem in a situation where $X$ is not nonsingular (but not too singular);
- one might want to work over a more general ring than an algebraically closed field.

Addressing the first point leads to the notions of *Gorenstein* and *Cohen-Macaulay* schemes: nonsingular schemes are those whose local rings are all regular local rings, while Gorenstein (resp. Cohen-Macaulay) schemes are those whose local rings are all Gorenstein (resp. Cohen-Macaulay). We will not recall the definitions of Gorenstein or Cohen-Macaulay here, and instead content ourselves with the knowledge that one can recover, more or less, the above formula. In brief, the adjustments that need to be made are: $X$ must be assumed to be equidimensional (if we remove the nonsingular hypothesis, we might have multiple components), and $\det \Omega^1_X$ must be replaced by some other sheaf, called the dualizing sheaf, of $X$ (in the Gorenstein case, the two coincide).

Addressing the second point, roughly speaking, requires study of the injective hulls of rings. In part this is what leads to the definition of Gorenstein and Cohen-Macaulay rings.

In any case, one reason for introducing the classes of Gorenstein and Cohen-Macaulay rings is that, in these cases, the more general *Grothendieck* duality reduces to a statement like equation (3.2.1). As we remarked previously, if a formal scheme $\mathfrak{X}$ is projective over a field, it is in fact an ordinary projective scheme over a field. In particular, Serre duality holds for formal schemes over fields with hypotheses as stated. The situation of our interest, however, is not that of projective formal schemes over a field, but that of pseudo-projective formal schemes over a field. We might hope to recover something like equation (3.2.1) by working backward from a more general Grothendieck duality, if such a theory exists for formal schemes.

For reasons that we discuss in the next section, there is motivation beyond ours to establish such a theory of Grothendieck duality for formal schemes, and some progress has been made toward this goal. The purpose of this section is to show how one can move from a statement of Grothendieck duality to one of Serre duality under appropriate hypotheses.

Without further ado, we state Grothendieck duality.

**Theorem 3.2.2** (Grothendieck Duality)**.** *Let $f : X \to Y$ be a proper morphism of (ordinary) Noetherian schemes. If $\mathscr{F} \in \overline{\mathbf{D}}_c^-(X)$, there is a natural isomorphism, for $\mathscr{G} \in \overline{\mathbf{D}}_{qc}^+(Y)$,*

$$\mathbf{R}f_* \, \mathbf{R}\mathscr{H}om^\bullet_X(\mathscr{F}, f^!\mathscr{G}) \xrightarrow{\sim} \mathbf{R}\mathscr{H}om^\bullet_Y(\mathbf{R}f_*\mathscr{F}, \mathscr{G}). \tag{3.2.2}$$

To work from this statement to Serre duality, first let's take $f : X \to \operatorname{Spec} k$ to be the structure morphism, and take $\mathscr{G}$ to be the structure sheaf $\mathscr{O}_k$ in degree 0. Let's also take $\mathscr{F}$ to be a line bundle on $X$ in degree 0. Then the right hand side of (3.2.2) becomes

$$\mathbf{R}\mathscr{H}om^\bullet_k(\mathbf{R}^\bullet f_*\mathscr{F}, \mathscr{G}) \simeq \mathbf{R}\mathscr{H}om^\bullet_k(H^\bullet(X, \mathscr{F}), \mathscr{G}).$$

Moreover, since $\mathscr{G} = k$ has an injective resolution given by $k$ itself (i.e., the sequence $0 \to k \to k \to 0$ is exact, and $k$ is injective) in degree 0, we can explicitly compute

$$\mathbf{R}\mathscr{H}om^\bullet_k(H^\bullet(X, \mathscr{F}), k) \simeq \mathscr{H}om^\bullet_k(H^\bullet(X, \mathscr{F}), k) \simeq \operatorname{Hom}^\bullet_k(H^\bullet(X, \mathscr{F}), k).$$

By definition,

$$\begin{aligned} \operatorname{Hom}^m_k(H^\bullet(X, \mathscr{F}), k) &= \prod_{i \in \mathbb{Z}} \operatorname{Hom}_k(H^i(X, \mathscr{F}), k^{i+m}) \\ &\simeq \operatorname{Hom}_k(H^{-m}(X, \mathscr{F}), k) \\ &\simeq H^{-m}(X, \mathscr{F})^\vee. \end{aligned}$$

Now, to compute the left hand side of (3.2.2), we first need the highly nontrivial result that $f^!(\mathscr{O}_k) \simeq \omega_X[n]$, i.e., the sheaf $\omega_X$ shifted left by $n$. If $X$ is merely Cohen-Macaulay, we know that $f^!(\mathscr{O}_k)$ is supported in a single degree, and we might call this sheaf $\omega_X^\circ$. If, moreover,

$X$ is Gorenstein, then we know that $\omega_X^\circ$ is locally free of rank 1. When $X$ is smooth projective over $k$, we actually do obtain $\omega_X^\circ \simeq \det \Omega_X^1$. Supposing $\mathscr{J}^\bullet$ is an injective resolution for $\omega_X$, shifting this resolution left by $n$ (so that, for example, the term in degree 0 is actually $\mathscr{J}^n$), we compute

$$\begin{aligned}\mathscr{H}om_X^m(\mathscr{F},\omega_X[n]) &= \prod_{i\in\mathbb{Z}} \mathscr{H}om_X(\mathscr{F},\mathscr{J}^{i+m+n}) \\ &\simeq \mathscr{H}om_X(\mathscr{F},\mathscr{J}^{m+n}) \\ &\simeq \mathscr{E}xt_X^{m+n}(\mathscr{F},\omega_X).\end{aligned}$$

Naturally, applying $\mathbf{R}f_*$ gives global sections, i.e., $\operatorname{Ext}^{m+n}(\mathscr{F},\omega_X)$.

Thus, comparing left and right hand sides, we have

$$\operatorname{Ext}^m(\mathscr{F},\omega_X) \simeq H^{n-m}(X,\mathscr{F})^\vee.$$

Now, if $\mathscr{F}$ is locally free of finite rank, take $\mathscr{E} = \mathscr{H}om_X(\mathscr{F},\omega_X) = \mathscr{F}^{-1}\otimes\omega_X$, and apply the above formula for $\mathscr{E}$. The right-hand side is simply $H^{n-m}(X,\mathscr{F}^{-1}\otimes\omega_X)^\vee$. The left-hand side, using the formulas we know for $\operatorname{Hom}_X$, becomes (since $\omega_X$ is locally free)

$$\begin{aligned}\operatorname{Ext}_X^m(\mathscr{F}^{-1}\otimes\omega_X,\omega_X) &\simeq \operatorname{Ext}^m(\omega_X,\mathscr{H}om_X(\mathscr{F}^{-1},\omega_X)) \\ &\simeq \operatorname{Ext}^m(\omega_X,\mathscr{F}\otimes\omega_X) \\ &\simeq \operatorname{Ext}^m(\omega_X,\mathscr{H}om_X(\omega_X^{-1},\mathscr{F})) \\ &\simeq \operatorname{Ext}^m(\omega_X^{-1}\otimes\omega_X,\mathscr{F}) \\ &\simeq \operatorname{Ext}^m(\mathscr{O}_X,\mathscr{F}) \simeq H^m(X,\mathscr{F}).\end{aligned}$$

That is, we have

$$H^m(X,\mathscr{F}) \simeq H^{n-m}(X,\mathscr{F}^{-1}\otimes\omega_X),$$

which is equation (3.2.1).

## 3.3 Grothendieck duality for formal schemes

As we mentioned in the previous section, Serre duality fails for formal schemes in general. Of course, it fails for ordinary schemes in general, as well, but we mean to indicate that nonsingular pseudo-projective formal schemes over an algebraically closed field $k$ do not obviously satisfy the conclusion. First, the techniques used to prove Serre duality "more directly", as in, e.g., [Har77, III.7] do not apply, essentially because they rely on properties of $\mathbb{P}_k^n$ and embedding a scheme into $\mathbb{P}_k^n$ (recall that a pseudo-projective formal scheme $\mathfrak{X}$ over a field $k$ embeds into projective space if and only if $\mathfrak{X}$ is in fact an ordinary scheme). Second, the formulae of the first section give us no aid, essentially because we would require

$$\omega_{\mathfrak{X}}\otimes\mathscr{O}_X \simeq \omega_X,$$

whereas what actually holds is adjunction (this requires $X_0$ to be smooth over $k$):

$$\omega_{\mathfrak{X}} \otimes \det \mathscr{N}_{X/\mathfrak{X}} \simeq \omega_X.$$

Since we cannot hope to acquire a Serre duality in the ordinary way, we might instead hope to find a Grothendieck duality statement, then deduce a statement resembling Serre duality, as we did in the previous section. In this section, we will give a brief overview of what is known regarding Grothendieck duality for formal schemes, then explain what statements akin to Serre duality we can produce.

The major strides toward establishing a Grothendieck duality theory for formal schemes are due to Alonso Tarrío, Jeremías Lopez, Lipman, Nayak, and Sastry in the two volumes [TLL99b] and [LNS05]. Aside from merely trying to generalize Grothendieck duality to a larger class of objects, one of the main motivations for undergoing these studies is the following (cf. [LNS05, Preface]): among other things, one of the main results of Grothendieck duality is the existence of a functor $f^! : \mathbf{D}^+_{qc}(Y) \to \mathbf{D}^+_{qc}(X)$ for a given finite-type map of separated schemes $f : X \to Y$. This functor behaves nicely with other already-known functors: for an étale morphism $f : X \to Y$, $f^!$ should be (naturally isomorphic to) $f^*$, and if $f : X \to Y$ is proper, $f^!$ should be a right-adjoint for $\mathbf{R}f_*$. (Note that this latter statement is, in some sense, the content of theorem 3.2.2. There are other compatibilities that we do not mention, which collectively bear the name "the six functors".) One "trouble" that one encounters is that, if $f : X \to Y$ is proper, we know that $f$ induces a map on stalks $\mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$, hence a map of affine schemes $f_x : \operatorname{Spec} \mathscr{O}_{X,x} \to \mathscr{O}_{Y,f(y)}$. The map $f_x$, however, may not be proper, so we can't hope to treat $f_x^!$ as we do $f^!$. An interesting observation, however, is that, we get a map of formal schemes $\hat{f} : \mathfrak{X} \to \mathfrak{Y}$, where both $X$ and $Y$ are completions of $X$ and $Y$ along compatible closed subschemes, and similarly an induced map of formal spectra $\hat{f}_x : \operatorname{Spf} \hat{\mathscr{O}}_{X,x} \to \operatorname{Spf} \hat{\mathscr{O}}_{Y,f(x)}$. Both $\hat{f}$ and $\hat{f}_x$ are *pseudo-proper*; so establishing analogous results for Grothendieck duality for pseudo-proper maps might allow one to think of global and local duality as different realizations of a larger theory.

One of the first questions in attempting to generalize Grothendieck duality to formal schemes is what categories to consider. In essence, the derived category of quasi-coherent sheaves on a formal scheme (even a Noetherian one) doesn't behave well enough to work out such statements. [TLL99a] establishes two "Grothendieck duality statements"; one of them considers a functor $\mathbf{D}(\mathfrak{Y}) \to \mathbf{D}(\mathfrak{X})$, but objects of the essential image are all (equivalence classes of chains of) *direct limits* of coherent $\mathscr{O}_{\mathfrak{X}}$-modules. The other is "Grothendieck duality for quasi-coherent torsion sheaves", i.e., sheaves $\mathscr{F}$ satisfying $\mathscr{F} \simeq \varinjlim_n \mathscr{H}om_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{O}_X/\mathscr{I}^n_{\mathfrak{X}}, \mathscr{F})$, where $\mathscr{I}_{\mathfrak{X}}$ is an ideal of definition for $\mathfrak{X}$. (All statements are made in the case where $\mathfrak{X}$ is assumed to be at least locally Noetherian, if not Noetherian.) In some sense, this is the more "correct" generalization of Grothendieck duality for ordinary schemes, since for an ordinary scheme $X$, $\mathscr{I}_X = 0$, so all sheaves of $\mathscr{O}_X$-modules are torsion by this definition.

The main result of interest to us from [TLL99a] is the following:

**Theorem 3.3.1** (Cf. [TLL99a, Theorem 5]). *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Noetherian formal schemes and let $f : \mathfrak{X} \to \mathfrak{Y}$ be a pseudo-proper map. Then there is a natural isomorphism*

$$\mathbf{R}f_* \, \mathbf{R}\mathscr{H}om^\bullet_{\mathfrak{X}}(\mathscr{G}, f^! \mathscr{F}) \xrightarrow{\sim} \mathbf{R}\mathscr{H}om^\bullet_{\mathfrak{Y}}(\mathbf{R}f_* \mathscr{G}, \mathscr{F})$$

*for $\mathscr{G} \in \mathbf{D}_{qct}(\mathfrak{X})$, $\mathscr{F} \in \widetilde{\mathbf{D}}^+_{qc}(\mathfrak{Y})$.*

In order to work back to a Serre duality-like statment, since the functor $f^!$ in the above theorem must inherently account for torsion, the authors introduce cohomology groups $H'^i_{\mathfrak{X}}(\mathscr{E})$, for any $\mathscr{E} \in \mathbf{D}(\mathfrak{X})$. In the case of our interest, these turn out to be akin to cohomology with supports:

**Proposition 3.3.2** (Cf. [TLL99a, 2.3.4]). *If $\mathfrak{X}$ is the completion of a Noetherian scheme $X$ along a closed subscheme $Z$, then for $\mathscr{F} \in \mathbf{D}(X)$,*

$$H'^i_{\mathfrak{X}}(\kappa^* \mathscr{F}) \simeq H^i_Z(X, \mathscr{F}).$$

Finally, essentially by the same analysis that we carried out in the previous section, although with more care to detail about torsion properties, the authors arrive at:

**Theorem 3.3.3** (Cf. [TLL99a, Remark 2.3.8]). *If $\kappa : \mathfrak{X} \to X$ is the completion of a Gorenstein of pure dimension $s$ separated scheme $X \to Y$ along a closed subset $Z$, proper over $Y$, then there is an isomorphism*

$$H^{s-i}(\mathfrak{X}, \mathscr{F}) \xrightarrow{\sim} (H'^i_{\mathfrak{X}}(\mathscr{F}^{-1} \otimes \omega_{\mathfrak{X}}))^\vee.$$

A few final remarks: in this case, $\omega_{\mathfrak{X}} = \kappa^* \omega_X$, as we generally expect from the previous chapter.

# Chapter 4

# Vanishing theorems

Let's start the discussion of this chapter with the following "definition": a *vanishing theorem for sheaf cohomology* is a set of conditions on a topological space $X$, a sheaf $\mathscr{F}$ (of abelian groups) on $X$, and $i$ (an integer) such that $H^i(X, \mathscr{F}) = 0$. (Note the abuse of notation: strictly speaking, $H^i(X, \mathscr{F})$ should be the group with a single element.) We could more generally speak of vanishing theorems for other cohomology theories, but we will focus our attention on the aforementioned case, so we'll content ourselves with the given definition.

Vanishing theorems typically hold little intrinsic interest:

> Why should one care about this problem? Actually one doesn't, but in various situations these cohomology groups come up as intermediate objects, and understanding them helps to solve the original problem. [Kol87]

As already mentioned in the introduction, our interest in such theorems stems from the desire to establish a minimal model program for (some subcollection of) formal schemes, where such theorems permit for "lifting of sections" and inductive arguments. We will comment on this more in the next chapter. However, deformation theory is another example where these cohomology groups arise as intermediary objects:

Heuristically, in deformation theory, the question of focus is whether there are "continuous" formations of a particular geometric object, i.e., whether one can find a "continuous path" between a point representing the object in question and some other object within a "parameter space". For example, the "parameter space" might be the "space of subvarieties" of a particular algebraic variety. This subject quickly becomes far more technical than our scope allows, so let it suffice to say that tangent spaces to these parameter spaces frequently correspond to cohomology groups $H^0$ or $H^1$ and local equations frequently correspond to $H^1$ or $H^2$.

The "classical" vanishing theorem is Kodaira vanishing:

**Theorem 4.0.1** (Kodaira vanishing)**.** *Let $X$ be a smooth projective variety of dimension $n$ over* *a field $k$ of characteristic $0$, and let $\mathscr{L}$ be an ample line bundle on $X$. Then*

$$H^i(X, \mathscr{L} \otimes \omega_X) = 0 \quad \text{for} \quad i > 0.$$

*Equivalently, by Serre duality,*

$$H^i(X, \mathscr{L}^{-1}) = 0 \quad \text{for} \quad i < n.$$

Several variants of this theorem have been established:

**Kodaira–Akizuki–Nakano** vanishing, which is sometimes called just Kodaira–Nakano vanishing, makes use of the same hypotheses, but instead makes conclusions on the groups $H^i(X, \Omega_X^j \otimes \mathscr{L})$ depending on the values $i$, $j$, and $n$. As with classical Kodaira, there is also a form for $\mathscr{L}^{-1}$.

**Kawamata–Viehweg** vanishing instead focuses on divisors $D$ which are big and nef, instead of ample, and draws conclusions for $H^i(X, \mathscr{O}_X(K_X + D))$. Kawamata-Viehweg vanishing often refers to an analogous statment where $D$ is a $\mathbb{Q}$-divisor.

**Kollár's** vanishing theorem considers a surjective morphism $f : X \to Y$ of projective varieties, where $X$ is smooth, and draws conclusions on $R^i f_* \mathscr{O}_X(K_X)$ depending on $i$ and $\dim X - \dim Y$.

**Grauert–Riemenschneider** vanishing is similar to Kollár's vanishing theorem, except that $f$ is required to be finite and generically surjective, and the conclusion is for all $i > 0$.

An interesting thing to note is that most proofs of vanishing theorems are analytic in nature. Deligne and Illusie manage to provide a proof [DI87] of Akizuki–Nakano–Kodaira vanishing that used only algebraic methods, although this proof is not known to extend to any of the others. See also [EV92]. This might seem to present a problem for formal schemes (because a formal scheme need not have a realization as an analytic variety), but our proofs will actually make use of the case of ordinary schemes, and we will sidestep the issue.

The general idea is that the basic hypothesis for a vanishing theorem is some sort of positivity of the line bundle in question. As a general rule of thumb, this extends to vanishing theorems for vector bundles. In our investigation of vanishing theorems for formal schemes, we will make use of vanishing theorems for vector bundles; the first section of this chapter outlines some of the results that we will use. The second section is where we establish our main result. The third is a discussion of a rather general class of examples where undesirable behavior occurs.

## 4.1 Positivity

The basic notion of positivity for vector bundles is ampleness, which is defined as follows:

**Definition 4.1.1.** Let $\mathscr{E}$ be a vector bundle on $X$, and let (as usual) $\pi : \mathbf{P}(\mathscr{E}) = \mathbf{Proj}(S(\mathscr{E})) \to X$ be the projective bundle of $\mathscr{E}$. $\mathscr{E}$ is ample if $\mathscr{O}_{\mathbf{P}(\mathscr{E})}(1)$ is ample.

As an easy consequence of this definition, we have that quotients of ample vector bundles are ample: a surjection $\mathscr{E} \to \mathscr{F}$ induces an embedding $\mathbf{P}(\mathscr{F}) \to \mathbf{P}(\mathscr{E})$, where $\mathscr{O}_{\mathbf{P}(\mathscr{F})}(1) = \mathscr{O}_{\mathbf{P}(\mathscr{E})}(1)|_{\mathbf{P}(\mathscr{F})}$. But restrictions of ample line bundles are ample, so we have our claim.

As consequences, we have the following facts:

**Proposition 4.1.2** (Cf. [Har68, Lemma 5.6])**.** *If $Y$ is a non-singular subscheme of $\mathbb{P}^n$, then its normal bundle is ample.*

**Proposition 4.1.3.** *Suppose $\mathscr{E}$ is an ample vector bundle of rank $r$ on a nonsingular scheme $X$. Then so is $S^{r+\ell}\mathscr{E} \otimes \det \mathscr{E}^{-1}$ for any $\ell \geq 1$.*

This actually follows from the following statement of commutative algebra:

**Lemma 4.1.4.** *Let $M$ and $N$ be $R$-modules with submodules $M'$ and $N'$, respectively. Then the natural map*

$$M \otimes N \to (M/M') \otimes (N/N')$$

*is surjective.*

*Proof.* First tensor the exact sequence

$$0 \to M' \to M \to M/M' \to 0$$

with $N$ to conclude that the natural map $M \otimes N \to (M/M') \otimes N$ is surjective (since tensoring is right-exact). Then tensor the exact sequence

$$0 \to N' \to N \to N/N' \to 0$$

with $(M/M')$ to conclude that $(M/M') \otimes N \to (M/M') \otimes (N/N')$ is surjective. Then the composition $M \otimes N \to (M/M') \otimes (N/N')$ is surjective, as desired. □

**Corollary 4.1.5.** *Suppose $M_1, \ldots, M_n$ are free $R$-modules, and that $N_1, \ldots, N_n$ are submodules of $M_1^{\otimes r_1}, \ldots, M_n^{\otimes r_n}$, respectively. Then the natural map*

$$M_1^{\otimes r_1} \otimes \cdots \otimes M_n^{\otimes r_n} \to \left(M_1^{\otimes r_1}/N_1\right) \otimes \cdots \otimes \left(M_n^{\otimes r_n}/N_n\right)$$

*is surjective. In particular, the codomain is isomorphic to a quotient of the domain.*

*Proof.* Immediate. However, here is an explicit computation: by induction on $n$. The case $n = 1$ is trivial.

Suppose the result holds for $n = m$. Let

$$M = M_1^{\otimes r_1} \otimes \cdots \otimes M_m^{\otimes r_m}$$
$$\overline{M} = \left(M_1^{\otimes r_1}/N_1\right) \otimes \cdots \otimes \left(M_m^{\otimes r_m}/N_m\right).$$

We aim to show that the map

$$M \otimes M_{m+1}^{\otimes r_{m+1}} \to \overline{M} \otimes \left(M_{m+1}^{\otimes r_{m+1}}/N_{m+1}\right), \quad m \otimes a \mapsto \bar{m} \otimes \bar{a}$$

is surjective. Say that $M_{m+1}$ has rank $\ell$, $r_{m+1} = s$, and choose a basis $e_1, \ldots, e_\ell$ for $M_{m+1}$. Suppose we are given $\bar{m} \otimes \bar{a} \in \overline{M} \otimes \left(M_{m+1}^{\otimes s}/N_{m+1}\right)$, where $m \in M$ and $a = a_1 \otimes \cdots \otimes a_s \in M_{m+1}^s$. Write

$$a_i = \sum_{j=1}^{\ell} a_{i,j} e_j$$

and observe that

$$\begin{aligned}
\sum_{j_1,\ldots,j_s} \left(\prod_{i=1}^{s} a_{i,j_i}\right) m \otimes e_{j_1} \otimes \cdots \otimes e_{j_s} \\
&\mapsto \sum_{j_1,\ldots,j_s} \left(\prod_{i=1}^{s} a_{i,j_i}\right) \bar{m} \otimes \overline{(e_{j_1} \otimes \cdots \otimes e_{j_s})} \\
&= \sum_{j_1,\ldots,j_s} \bar{m} \otimes \overline{(a_{1,j_1} e_{j_1} \otimes \cdots \otimes a_{s,j_s} e_{j_s})} \\
&= \sum_{j_1,\ldots,j_{s-1}} \bar{m} \otimes \overline{\left(a_{1,j_1} e_{j_1} \otimes \cdots \otimes a_{s-1,j_{s-1}} e_{j_{s-1}} \otimes \left(\sum_{j=1}^{\ell} a_{s,j} e_j\right)\right)} \\
&= \sum_{j_1,\ldots,j_{s-1}} \bar{m} \otimes \overline{(a_{1,j_1} e_{j_1} \otimes \cdots \otimes a_{s-1,j_{s-1}} e_{j_{s-1}} \otimes a_s)} \\
&= \bar{m} \otimes \overline{a_1 \otimes \cdots \otimes a_s} = \bar{m} \otimes \bar{a}.
\end{aligned}$$

So indeed the map is surjective, as we desire.

□

*Proof of proposition 4.1.3.* Simply observe that:

- $S^{r+\ell}\mathscr{E}$ is a quotient of $\mathscr{E}^{\otimes r+\ell}$; and
- $\det \mathscr{E}^{-1} = \bigwedge^r \mathscr{E}^{-1}$ is a quotient of $(\mathscr{E}^{-1})^{\otimes r}$.

By the lemma, $(\mathscr{E}^{\otimes r+\ell}) \otimes (\mathscr{E}^{-1})^{\otimes r} \to S^{r+\ell}\mathscr{E} \otimes \det \mathscr{E}^{-1}$ is surjective on stalks, so $S^{r+\ell}\mathscr{E} \otimes \det \mathscr{E}^{-1}$ is isomorphic to a quotient of $\mathscr{E}^{\otimes r+\ell} \otimes (\mathscr{E}^{-1})^{\otimes r} \simeq \mathscr{E}^{\otimes \ell}$. Since $\mathscr{E}$ is ample, so is $\mathscr{E}^{\ell}$, and any quotient thereof. So $S^{r+\ell}\mathscr{E} \otimes \det \mathscr{E}^{-1}$ is ample. □

Unfortunately, vanishing theorems for vector bundles are trickier than they are for line bundles. Amplitude of a given vector bundle $\mathscr{E}$ is usually not enough to conclude vanishing of cohomology groups of $\mathscr{E}$ itself. The Griffiths and Le Potier vanishing theorems operate under the hypothesis that $\mathscr{E}$ is ample, but their conclusions concern cohomology groups of $\omega_X \otimes \bigwedge^m \mathscr{E}$ and $\omega_X \otimes S^m(\mathscr{E}) \otimes \det \mathscr{E}$. There is a rather strong form of positivity, however, which produces familiar-looking vanishing theorems:

**Theorem 4.1.6** (Cf. [Laz04, 7.3.18–7.3.19], [SS85, Chapter VI]). *Let $X$ be a compact Kähler manifold of dimension $n$, and let $\mathscr{E}$ be a vector bundle equipped with a Hermitian metric that is Nakano semipositive on $X$ and which is Nakano positive in a neighborhood of a point $x \in X$. Then*

$$H^i(X, \mathscr{E} \otimes \omega_X) = 0 \quad \textit{for} \quad i > 0.$$

Nakano positivity is the same as ampleness for line bundles, but for general vector bundles, it is a much stronger notion of positivity than ampleness (see [Laz04, 6.1.D, Theorem 6.1.25]). Interestingly, one can check for Nakano positivity by checking for ampleness of a particular vector bundle:

**Proposition 4.1.7** (Cf. [LSY13, Theorem 1.1], [Ber09]). *Let $X$ be a compact Kähler manifold, $\mathscr{E}$ a holomorphic vector bundle on $X$, and $\mathscr{F}$ a line bundle on $X$. Let $r$ be the rank of $\mathscr{E}$ and $\ell \geq 0$ an arbitrary nonnegative integer. If $S^{r+\ell}(\mathscr{E}) \otimes \det \mathscr{E}^{-1} \otimes \mathscr{F}$ is ample over $X$, then there exists a smooth Hermitian metric $f$ on $S^\ell(\mathscr{E}) \otimes \mathscr{F}$ such that $(S^\ell(\mathscr{E}) \otimes \mathscr{F}, f)$ is Nakano positive.*

## 4.2 A Kodaira vanishing theorem for formal schemes

**Theorem 4.2.1** (Kodaira vanishing for pseudo-projective formal schemes). *Let $\mathfrak{X}$ be a locally Noetherian formal scheme that is pseudo-projective over a field $k$ of characteristic 0, and let $\mathscr{L}$ be a pseudo-ample line bundle on $\mathfrak{X}$. Assume further that:*

1. *the reduced subscheme of definition $X_0$ of $\mathfrak{X}$ is a locally complete intersection in $\mathfrak{X}$;*
2. *the normal bundle $\mathscr{N}_{X_0/\mathfrak{X}}$ of $X_0$ in $\mathfrak{X}$ is ample; and*
3. *$X_0$ is nonsingular.*

*Then*

$$H^i(\mathfrak{X}, \mathscr{L}^{-1}) = 0 \quad \textit{for} \quad i < d = \operatorname{dimtop} \mathfrak{X}.$$

*Proof.* The pseudo-projectivity of $\mathfrak{X}$ in particular makes $\mathfrak{X}$ pseudo-proper, and $\mathscr{L}^{-1}$ is certainly quasi-coherent, so proposition 3.1.1 allows us to write

$$H^i(\mathfrak{X}, \mathscr{L}^{-1}) \simeq \varprojlim_{\ell} H^i(X_\ell, \mathscr{L}_\ell).$$

Thus we are reduced to showing that $H^i(X_\ell, \mathscr{L}_\ell) = 0$ for each $i < d$. The basic idea is to use classical Kodaira vanishing for $H^i(X_0, \mathscr{L}_0)$, and then use the ampleness of the normal bundle to conclude inductively that $H^i(X_\ell, \mathscr{L}_\ell) = 0$.

To that end, let $\ell \geq 1$, and consider the short exact sequence of sheaves

$$0 \to \mathscr{I}_{\mathfrak{X}}^{\ell} / \mathscr{I}_{\mathfrak{X}}^{\ell+1} \to \mathscr{O}_{\mathfrak{X}} / \mathscr{I}_{\mathfrak{X}}^{\ell+1} \to \mathscr{O}_{\mathfrak{X}} / \mathscr{I}_{\mathfrak{X}}^{\ell} \to 0.$$

$\mathscr{L}^{-1}$ is locally free of rank 1, so we may tensor by $\mathscr{L}^{-1}$ to produce another short exact sequence

$$0 \to \mathscr{I}_{\mathfrak{X}}^{\ell}/\mathscr{I}_{\mathfrak{X}}^{\ell+1} \otimes \mathscr{L}^{-1} \to \mathscr{L}_{\ell}^{-1} \to \mathscr{L}_{\ell-1}^{-1} \to 0.$$

In turn, we get a long exact sequence on cohomology

$$\cdots \longrightarrow H^i(X_0, \mathscr{I}_{\mathfrak{X}}^{\ell}/\mathscr{I}_{\mathfrak{X}}^{\ell+1} \otimes \mathscr{L}_0^{-1}) \longrightarrow H^i(X_\ell, \mathscr{L}_\ell^{-1}) \longrightarrow H^i(X_{\ell-1}, \mathscr{L}_{\ell-1}^{-1}) \longrightarrow H^{i+1}(X_0, \mathscr{I}_{\mathfrak{X}}^{\ell}/\mathscr{I}_{\mathfrak{X}}^{\ell+1} \otimes \mathscr{L}_0^{-1}) \longrightarrow \cdots.$$

We will show that the terms $H^i(X_0, \mathscr{I}_{\mathfrak{X}}^{\ell}/\mathscr{I}_{\mathfrak{X}}^{\ell+1} \otimes \mathscr{L}_0^{-1})$ are 0 for each $i < d$. This will inform us that, for $i < d-1$,

$$H^i(X_\ell, \mathscr{L}_\ell^{-1}) \simeq H^i(X_{\ell-1}, \mathscr{L}_{\ell-1}^{-1}),$$

and, for $i = d-1$, the sequence

$$0 \to H^i(X_\ell, \mathscr{L}_\ell^{-1}) \to H^i(X_{\ell-1}, \mathscr{L}_{\ell-1}^{-1})$$

is exact. By classical Kodaira vanishing, we know that $H^i(X_0, \mathscr{L}_0^{-1}) = 0$ for $i < d$, so induction on $\ell$ gives us our desired result.

We need only show, then that, those terms are 0. Now, $\mathscr{I}_{\mathfrak{X}}^{\ell}/\mathscr{I}_{\mathfrak{X}}^{\ell+1} \simeq S^\ell(\mathscr{I}_{\mathfrak{X}}/\mathscr{I}_{\mathfrak{X}}^2)$. Also, $X_0$ is smooth projective of dimension $d$, so we may write

$$\begin{aligned} H^i(X_0, \mathscr{I}_{\mathfrak{X}}^{\ell}/\mathscr{I}_{\mathfrak{X}}^{\ell+1} \otimes \mathscr{L}_0^{-1}) &\simeq H^i(X_0, S^\ell(\mathscr{I}_{\mathfrak{X}}/\mathscr{I}_{\mathfrak{X}}^2) \otimes \mathscr{L}_0^{-1}) \\ &\simeq H^{d-i}(X_0, (S^\ell(\mathscr{I}_{\mathfrak{X}}/\mathscr{I}_{\mathfrak{X}}^2) \otimes \mathscr{L}_0^{-1})^{-1} \otimes \omega_{X_0}). \end{aligned}$$

Since we are working in characteristic 0, taking symmetric powers of vector bundles commutes with taking duals, and we may write $S^\ell(\mathscr{I}_{\mathfrak{X}}/\mathscr{I}_{\mathfrak{X}}^2)^{-1} \simeq S^\ell((\mathscr{I}_{\mathfrak{X}}/\mathscr{I}_{\mathfrak{X}}^2)^{-1}) \simeq S^\ell(\mathscr{N}_{X_0/\mathfrak{X}})$. Thus

$$\begin{aligned} H^{d-i}(X_0, (S^\ell(\mathscr{I}_{\mathfrak{X}}/\mathscr{I}_{\mathfrak{X}}^2) \otimes \mathscr{L}_0^{-1})^{-1} \otimes \omega_{X_0}) &\simeq H^{d-i}(X_0, S^\ell((\mathscr{I}_{\mathfrak{X}}/\mathscr{I}_{\mathfrak{X}}^2)^{-1}) \otimes \mathscr{L}_0 \otimes \omega_{X_0}) \\ &\simeq H^{d-i}(X_0, S^\ell(\mathscr{N}_{X_0/\mathfrak{X}}) \otimes \mathscr{L}_0 \otimes \omega_{X_0}). \end{aligned}$$

So now we find ourselves wishing to show

$$H^{d-i}(X_0, S^\ell(\mathscr{N}_{X_0/\mathfrak{X}}) \otimes \mathscr{L}_0 \otimes \omega_{X_0}) = 0.$$

In consideration of theorem 4.1.6, it will suffice to show that $S^\ell(\mathscr{N}_{X_0/\mathfrak{X}}) \otimes \mathscr{L}_0$ is Nakano positive. In turn, proposition 4.1.7 informs us that we need only show $S^{r+\ell}(\mathscr{N}) \otimes \det(\mathscr{I}_{\mathfrak{X}}/\mathscr{I}_{\mathfrak{X}}^2) \otimes \mathscr{L}_0$ is ample. By hypothesis 2, $\mathscr{N}_{X_0/\mathfrak{X}}$ is ample, so by proposition 4.1.3, since $\ell \geq 1$, $S^{r+\ell}(\mathscr{N}_{X_0/\mathfrak{X}}) \otimes \det(\mathscr{I}_{\mathfrak{X}}/\mathscr{I}_{\mathfrak{X}}^2)$ is as well. Since the tensor product of two ample vector bundles is again ample, we have $S^{r+\ell}(\mathscr{N}_{X_0/\mathfrak{X}}) \otimes \det(\mathscr{I}_{\mathfrak{X}}/\mathscr{I}_{\mathfrak{X}}^2) \otimes \mathscr{L}_0$ is as ample, and our desired conclusion follows. □

## 4.3 Nonvanishing

As we remarked in the last section, some of the hypotheses in theorem 4.2.1 seem to be necessary only for the method of proof, as opposed to the truth of the theorem. Recall also that the Kodaira vanishing theorem for 'ordinary' schemes has a form for the sheaf $\omega_X \otimes \mathscr{L}$. If, indeed, we wish to bootstrap some form of the minimal model program, we would like to have a vanishing theorem of that form, since most statements of the minimal model program involve understanding $\omega_X$ as a divisor (class).

The purpose of this section is to introduce and prove the following result:

**Proposition 4.3.1.** *Let $\mathfrak{X}$ be the completion of a smooth projective $d$-dimensional variety $X$ over a field $k$ of characteristic $0$ along a hyperplane $Y$ (we assume $d > 0$, and that $Y$ is a hyperplane that does not contain $X$, i.e., $Y \cap X$ is a divisor on $X$), and let $\mathscr{L} = \kappa^* \mathscr{F}$, where $\kappa : \mathfrak{X} \to X$ is the completion morphism, and $\mathscr{F}$ is ample on $X$. Then*

$$H^i(\mathfrak{X}, \omega_{\mathfrak{X}} \otimes \mathscr{L}) = 0 \quad \textit{if} \quad i \neq 0,\, d-1$$

*and, perhaps more importantly,*

$$H^{d-1}(\mathfrak{X}, \omega_{\mathfrak{X}} \otimes \mathscr{L}) \neq 0,$$
$$H^0(\mathfrak{X}, \omega_{\mathfrak{X}} \otimes \mathscr{L}) \simeq H^d(X, \mathscr{F}^{-1}).$$

Before we go into the proof, let us discuss the relevance of this result. $\mathfrak{X}$, as described in the theorem, can be taken to have nonsingular reduced subscheme of definition by Bertini's theorem (if we assume $k$ is algebraically closed). Moreover, in this case, $H \cap \mathfrak{X}$ will be a smooth subvariety of $\mathfrak{X}$ of codimension 1, hence will be a smooth prime divisor. In particular, the ideal sheaf of $Y$ in $X$ is locally free, thus $Y \subseteq \mathfrak{X}$ is a locally complete intersection. However, the normal bundle of $Y$ need not be ample in $X$. So this proposition forms a counterpoint of sorts to theorem 4.2.1 and suggests that the ampleness of the normal bundle may be more than just a technical assumption in vanishing theorems for formal schemes. That is, while theorem 4.2.1 seems a suitable analogue of one of the statements of Kodaira vanishing, what we would like to be true is

$$H^i(\mathfrak{X}, \omega_{\mathfrak{X}} \otimes \mathscr{L}) = 0 \quad \text{for} \quad i > 0.$$

In particular, we really would like $H^{d-1}(\mathfrak{X}, \omega_{\mathfrak{X}} \otimes \mathscr{L})$ to be 0. In going through the proof, however, we will find that this cohomology group is actually isomorphic to the global sections of a (nonzero) sheaf on an affine variety. Metaphorically, there's "no hope" that we could end up with this group being zero in general.

This result also has some rather negative implications for our hopes to apply it to lifting sections. That is, in very rough terms, we'd like to consider the following situation: we have a (say smooth) projective variety $X$ and a smooth prime divisor $S \subset X$. We'd like to gain information about divisors on $X$ from divisors on $S$. A starting point is that we already have the relation of adjunction between their canonical divisors: $\omega_S \simeq \omega_X \otimes \det \mathscr{N}_{S/X}$. So we might hope to compare divisors of the form $K_X + S + \Delta$. Again, very roughly speaking, we can do so, but instead of arriving at an equality of divisors, we end up with an equality of linear systems

$$|m(K_S + \Delta'|_S)| = |m(K_X + \Delta)|_S.$$

The key point in proving this fact is that, for a divisor $D$, $|D| = H^0(X, \mathscr{O}_X(D))$; so the technique is to set up some short exact sequence of sheaves on $X$

$$0 \to \mathscr{O}_X(K_X + \Delta - S) \to \mathscr{O}_X(K_X + \Delta) \to \mathscr{O}_S(K_X + \Delta) \to 0,$$

which leads to an induced long exact sequence in cohomology

$$\cdots \to H^0(X, K_X + \Delta) \to H^0(S, (K_X + \Delta)|_S) \to H^1(X, \mathscr{O}_X(K_X + \Delta - S)) \to \cdots.$$

What we'd really like to say is that the $H^1$ term above is 0, and we can do so, in the case of ordinary schemes with suitable hypotheses on $\Delta$. However, with our theorem above, we know there will be times when we cannot conclude $H^1 = 0$, namely, in cases when $d = 2$. The problem is that we use the situation above in inductive proofs, so, unless there is a way to deal with the case $d = 2$ differently, we find ourselves "stuck".

*Proof of proposition 4.3.1.* By theorem 3.3.3, we have

$$H^{d-i}(\mathfrak{X}, \omega_{\mathfrak{X}} \otimes \mathscr{L}) \simeq (H'^{i}_{\mathfrak{X}}((\omega_{\mathfrak{X}} \otimes \mathscr{L})^{-1} \otimes \omega_{\mathfrak{X}}))^{\vee} \simeq (H'^{i}_{\mathfrak{X}}(\mathscr{L}^{-1}))^{\vee}.$$

By proposition 3.3.2, since $\mathscr{L} = \kappa^* \mathscr{F}$,

$$H'^{i}_{\mathfrak{X}}(\mathscr{L}^{-1}) \simeq H^i_Y(X, \mathscr{F}^{-1}).$$

Ordinarily, cohomology with supports is difficult to compute, but we are in a fairly nice situation. We know there is a long exact sequence

$$\cdots \to H^{i-1}(X - Y, \mathscr{F}^{-1}) \to H^i_Y(X, \mathscr{F}^{-1}) \to H^i(X, \mathscr{F}^{-1}) \to H^i(X - Y, \mathscr{F}^{-1}) \to \cdots.$$

The key observation now is that $X - Y$ is in fact affine, so we know that the terms $H^i(X - Y, \mathscr{F}^{-1})$ are zero for $i > 0$. Thus, we have

$$H^i_Y(X, \mathscr{F}^{-1}) \simeq H^i(X, \mathscr{F}^{-1}) \quad \text{for} \quad i > 1$$

and a long exact sequence

$$0 \longrightarrow H^0_Y(X, \mathscr{F}^{-1}) \longrightarrow H^0(X, \mathscr{F}^{-1}) \longrightarrow H^0(X - Y, \mathscr{F}^{-1}) \longrightarrow H^1_Y(X, \mathscr{F}^{-1}) \longrightarrow H^1(X, \mathscr{F}^{-1}) \longrightarrow 0 \longrightarrow \cdots.$$

By classical Kodaira vanishing, we know $H^i(X, \mathscr{F}^{-1}) = 0$ for $i < d$. So, for $1 < i < d$, we have $H^i_Y(X, \mathscr{F}^{-1}) = 0$, $H^0_Y(X, \mathscr{F}^{-1}) = 0$, and $H^0(X - Y, \mathscr{F}^{-1})$ injects into $H^1_Y(X, \mathscr{F}^{-1})$. So $H^i(\mathfrak{X}, \omega_{\mathfrak{X}} \otimes \mathscr{L}) \simeq H^{d-i}_Y(X, \mathscr{F}^{-1})$, and the latter is zero if $d - i = 0$, i.e., $i = d$, or $1 < d - i < d$, i.e., $0 < i < d - 1$. In the case $d - i = 1$, i.e., $i = d - 1$, we know $H^0(X - Y, \mathscr{F}^{-1})$ injects into $H^i(\mathfrak{X}, \omega_X \otimes \mathscr{L})$, and in the case $d - i = d$, i.e., $i = 0$, we know that $H^0(\mathfrak{X}, \omega_{\mathfrak{X}} \otimes \mathscr{L}) \simeq H^d(X, \mathscr{F}^{-1})$. □

# Chapter 5

# Conclusions and further directions

Let us return to the question we asked so long ago: could there exist a minimal model program for some class of formal schemes?

It might seem premature to ask this question now, as we've only discussed vanishing theorems. Certainly, such theorems are useful (even necessary) tools in birational geometry, but they alone do not yield something as vast as a minimal model program. Recent work of Cascini, Corti, and Lazić, however, suggets that a "good enough" vanishing theorem, along with a sufficiently sophisticated theory of divisors, allows one to set up a minimal model program. We'll briefly summarize their work (and its contrast with the "conventional" way to establish a minimal model program) here, but for more details, see [CL12] and [CL13].

## 5.1 The conventional MMP and a new outlook

### History and traditions

We've talked an awful lot about the MMP without giving any notion of what it is. Here we'll give a rough sketch of the ideas behind it. We caution the reader that this is *not* a rigorous treatment, and that several such have been written on the subject. See, for example, [KM98] and [Deb01, Chapters 6–7]. Our sketch here will follow the latter.

The MMP, also known as Mori theory, grew out of a desire to generalize, for higher-dimensional varieties, a collection of results known to be true for (smooth) algebraic surfaces. To review these, let $S$ be a smooth projective algebraic surface over $\mathbb{C}$.

Denote by $K_S$ the canonical divisor of $S$. Thus far in our writing, we've largely avoided the term *divisor* and instead preferred to consider line bundles; we will comment on this more in the next section. For now, since $S$ is a smooth variety over a field, we have a correspondence between Weil and Cartier divisors, and we can freely move back and forth between the two. The motivating results behind Mori theory for surfaces come down to the following idea: understanding the (birational) geometry of $S$ is closely tied to understanding numerical properties of $K_S$. That is, the geometry of $S$ can more or less be determined by understanding how $K_S$ intersects divisors on $S$. The key result is the following:

**Theorem 5.1.1** (Termination of MMP for surfaces)**.** *Given a smooth projective surface $S$ as above, there exists a smooth surface $T$, birational to $S$, such that either*

1. *$K_T$ is* nef, *i.e., $K_T \cdot C \geq 0$ for any curve $C \subseteq T$; or*
2. *$T$ is birational to $\mathbb{P}^2$ or a $\mathbb{P}^1$-bundle over a curve.*

Actually, the MMP itself is more than just the above statement: it is a procedure to produce the birational morphism between $S$ and $T$ mentioned above. The procedure roughly works as follows:

1. Start with $S_0 = S$ as above.
2. If $K_{S_i}$ is nef, set $T = S_i$, and the birational morphism of interest is the composition of morphisms $S_0 \to S_1 \to \cdots \to T$.
3. If not, there exists a curve $C \subseteq S_i$ such that $K_{S_i} \cdot C < 0$. By the cone theorem (see theorem 5.1.2 below), there exists a birational morphism $S_i \to S_{i+1}$ where either:
   - $\dim S_{i+1} \leq 1$. In this case, set $T = S_{i+1}$, and the birational morphism of interest is the composition of morphisms $S_0 \to S_1 \to \cdots \to T$; or
   - $\dim S_{i+1} = 2$; continue with step 2, replacing $i = i + 1$

The result that pushes the above through is the following:

**Theorem 5.1.2** (Cone theorem)**.** *If $S$ is a smooth projective surface, there are countably many extremal rays $\{R_i\}$ of the closed cone of curves on $S$ which $K_S$ intersects negatively. Moreover,*

$$\overline{\mathrm{NE}}(S) = \overline{\mathrm{NE}}(S)_{K_X \geq 0} + \sum R_i.$$

*Furthermore, for such an extremal ray, there is a birational morphism $\pi : S \to Z$ that contracts a curve $C$ if and only if $C$ spans the ray. Either $\dim Z \leq 1$, which leads to the two cases above, or $Z$ is a smooth surface and $\pi$ contracts a $-1$-curve.*

There are quite a few terms in this theorem which we have not defined. Loosely speaking, however, the cone of curves is an $\mathbb{R}$-span of cohomology classes of curves in $S$. The theorem above essentially says that this cone breaks into two pieces: one which consists of divisor classes that

$K_X$ intersects nonnegatively, and one whose members $K_X$ intersects negatively. Moreover, the latter class is "polyhedral", with possible accumulation points at the $K_X = 0$ locus. Finally, the extreme rays of the polyhedral part of this cone may be contracted to produce a minimal model.

The MMP for higher-dimensional varieties hopes to follow the procedure above, but there are quite a few changes that need to be made:

- Instead of working with smooth varieties, one has to allow some mild singularities. Essentially this requirement comes from the fact that the contraction theorem for surfaces actually produces a smooth surface, while in the higher-dimensional setting, the variety it yields may have singularities itself.

- Rather than working with just $K_{X_i}$, one works with *log pairs* $(X_i, \Delta_i)$ where $\Delta_i$ is a *boundary divisor*, i.e., an effective ($\mathbb{R}$- or) $\mathbb{Q}$-divisor that has simple normal crossings after some birational transformation $Y_i \to X_i$. Roughly speaking, one thinks of $\Delta_i$ as a "boundary" because the birational tranformations in question produce isomorphisms of the open set $X_i - \Delta_i$. Consequently, rather than considering numerical properties of $K_{X_i}$, one considers numerical properties of $K_{X_i} + \Delta_i$. Also, the singularities mentioned above actually refer to singularities of the pair $(X_i, \Delta_i)$.

- Sometimes contractions aren't enough; one has to consider other birational morphisms called *flips* and *flops*. The introduction of this class of morphisms gives rise to a big question, namely, whether performing one of these operations meaningfully brings one closer to a termination point of the above procedure.

In fact, one actually considers "different sorts" of MMPs for higher-dimensional varieties (e.g., an MMP *with scaling*, a *relative* MMP, etc.). The celebrated paper [BCHM10] proves that the MMP exists for smooth projective varieties *of general type*. A lot of research now focuses on generalizing the methods in these proofs to other settings and applications.

## A new perspective

Ordinarily, one uses the MMP to establish a result like the following:

**Theorem 5.1.3** (Cf. [CL12, Theorem 1.1])**.** *Let $X$ be a smooth projective variety and $\Delta$ a $\mathbb{Q}$-divisor with simple normal crossings such that $\lfloor \Delta \rfloor = 0$. Then the log canonical ring*

$$R(X, K_X + \Delta) := \bigoplus_m H^0\left(X, \mathscr{O}_X(\lfloor m(K_X + \Delta) \rfloor)\right)$$

*is finitely generated.*

The surprising result of [CL13] is that, knowing the above result, one can actually establish an MMP. Roughly speaking, finite generation of the log canonical ring characterises the cone of curves sufficiently well enough to detect extremal rays, which, as before, we'd like to contract. The finite generation also (after a bit of work) demonstrates that the procedure terminates.

In turn, and even more surprising, is that [CL12] demonstrates that theorem 5.1.3 follows simply from Kawamata–Viehweg vanishing. In a bit more detail, the idea is the following: Kawamata–Viehweg vanishing allows one to "lift sections" for suitable divisors. That is, for a suitable divisor $D$, one can show

$$H^0(X, \mathscr{O}_X(K_X + D)) \twoheadrightarrow H^0(S, \mathscr{O}_S(K_X + D)),$$

where $S \subseteq X$ is a smooth divisor. (Again, this is a sketch; some of these statements are quite imprecise.) Then careful bookkeeping in the study of adjoint rings and diophantine approximation allows one to characterise the finite generation of $R(X, K_X + D)$ in terms of the finite generation of the "restricted canonical ring"

$$\operatorname{res}_S R(X, K_X + D) := \bigoplus \operatorname{res}_S H^0(X, \mathscr{O}_S(\lfloor m(K_X + D) \rfloor)).$$

Then induction on the dimension of $X$ establishes the result.

With the above outline of establishing of the MMP, it seems reasonable to hope that sufficiently nice vanishing theorems could lead to minimal model programs in other contexts, such as formal schemes.

## 5.2 Where do formal schemes stand?

### Resolution of singularities and birational geometry

Of course, for any of the above to make sense for formal schemes, we need to have an appropriate analogues, for formal schemes, of the following notions:

**Birational morphisms** of formal schemes don't appear, at least not in a standard way with any frequency, in the literature. However, *admissible blow-ups* of formal schemes receive regular attention, and if any morphisms should be considered birational, blow-ups should be among them. One problem, however, is that it is not necessarily clear what the birational inverse of this admissible blow-up is. In Raynaud's theory of formal models, one formally inverts such morphisms by working in a category of formal schemes that has been localized at such morphisms.

**Smooth (or mildly singular) formal schemes** are notions with somewhat precise meaning. The papers [TL07] and [TLR09] propose notions for smooth morphisms of formal schemes. The term *nonsingular* has traditionally been applied to formal schemes in the same sense that it applies to ordinary schemes (i.e., all local rings are regular local rings); however, see the next point. Classes of “mild” singularities as are present in the MMP for varieties have not yet been defined.

**Resolution of singularities** in characteristic zero now exists for formal schemes, due to the work of Temkin. See [Tem11] for a heuristic overview of the known results, and [Tem12] and [Tem17] for rigorous treatments. Of particular note is that these treatments make the singular locus of a formal scheme a *bona fide* formal subscheme of the formal scheme of interest, rather than a subset of the topological space (as is done for varieties).

**Intersection theory** on formal schemes hasn't received much attention, though it seems likely it would, in the case of locally Noetherian formal schemes, be the same as intersection theory on the reduced subschemes of definition.

**Divisors on formal schemes** at this point only manifest as Cartier divisors; see the next section for further commentary on this subject.

The takeaway from the above points, except perhaps the last one, is that it seems plausible that the underlying machinery for a minimal model program might exist, especially upon restriction to a “nice enough” class of formal schemes. It is clear, however, that quite a few definitions would need to be made.

## Divisors on formal schemes

One of the obstructions to carrying out the development of a minimal model program is what a good notion of a divisor on a formal scheme would be. In the case of smooth varieties over a field, we have a correspondence between Weil divisors and Cartier divisors, and we can freely move back and forth between the two. For formal schemes, however, this is not the case:

**Example 5.2.1.** Consider the formal scheme $\mathfrak{X} = \operatorname{Spf}\left(\varprojlim k[x,y]/(xy)^n\right)$. This is a regular formal scheme, isomorphic to the completion of $\mathbb{A}^2_k$ along the two axes (see figure 5.1). The fact that this is regular boils down to the fact from commutative algebra that the completion of a regular local ring along its maximal ideal is again regular. (So that, in particular, the completion of any smooth variety along a closed subscheme is regular.) Moreover, $\mathfrak{X}$ is smooth over $k$, since its structure morphism is the composition of the completion morphism (which is étale) $\mathfrak{X} \to \mathbb{A}^2_k$ with the projection $\mathbb{A}^2_k \to k$ (which is smooth).

However, the line bundle generated by $\left(\frac{1}{y}\right)$ is *not* a Weil divisor, as its support is the entire $x$-axis, i.e., a codimension 0 subset!

What exactly goes wrong in the example above is a little tricky to place. On the one hand, as the example indicates, smooth, in the case of formal schemes over a field, does not imply irreducible. On the other, one might say that the traditional formulation of Cartier divisors

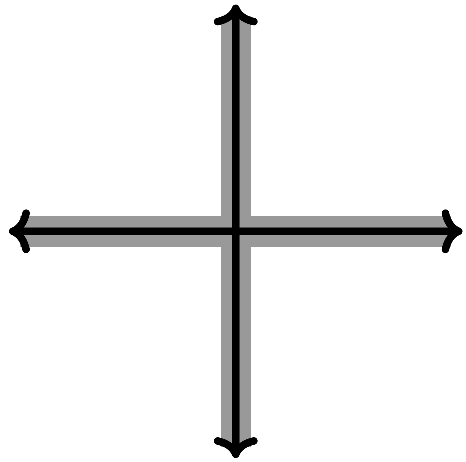


Figure 5.1: Spf $\left(\operatorname{proj\,lim}_n k[x,y]/(xy)^{n+1}\right)$

somehow allows for "too many sections": the section $y$ in the example above is topologically nilpotent, so it should be considered "almost a zero divisor".

In any case, it is clear that some additional thought needs to go into how to discuss divisors on formal schemes. We can, of course make use of Cartier (even, formally, $\mathbb{Q}$-Cartier) divisors, but it seems like they are not yet quite the tools we need.

So, could there be a minimal model program for formal schemes? At this point, the answer seems to depend on the optimism of the one answering. Some of the machinery that goes into building an MMP does exist for formal schemes. And, where it exists, it seems to work. The problem thus seems to be a lack of fully-developed machinery, which, hopefully, time will solve.

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