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UNIVERSITY OF CALIFORNIA, SAN DIEGO

**Ihara zeta functions of irregular graphs**

A dissertation submitted in partial satisfaction of the  
requirements for the degree  
Doctor of Philosophy

in

Mathematics

by

Matthew D. Horton

Committee in charge:

Professor Audrey Terras, Chair  
Professor Mihir Bellare  
Professor Ron Evans  
Professor Herbert Levine  
Professor Harold Stark

2006

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The dissertation of Matthew D. Horton is approved, and it is acceptable in quality and form for publication on microfilm:

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Chair

University of California, San Diego

2006

To my wife and family

*Never hold discussions with the monkey  
when the organ grinder is in the room.*

—Sir Winston Churchill

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## ABSTRACT OF THE DISSERTATION

### **Ihara zeta functions of irregular graphs**

by

Matthew D. Horton

Doctor of Philosophy in Mathematics

University of California San Diego, 2006

Professor Audrey Terras, Chair

We explore three seemingly disparate but related avenues of inquiry: expanding what is known about the properties of the poles of the Ihara zeta function, determining what information about a graph is recoverable from its Ihara zeta function, and strengthening the ties between the Ihara zeta functions of graphs which are related to each other through common operations on graphs.

Using the singular value decomposition of directed edge matrices, we give an alternate proof of the bounds on the poles of Ihara zeta functions. We then give an explicit formula for the inverse of directed edge matrices and use the inverse to demonstrate that the sum of the poles of an Ihara zeta function is zero.

Next we discuss the information about a graph recoverable from its Ihara zeta function and prove that the girth of a graph as well as the number of cycles whose length is the girth can be read directly off of the reciprocal of the Ihara zeta function. We demonstrate that a graph's chromatic polynomial cannot in general be recovered from its Ihara zeta function and describe a method for constructing families of graphs which have the same chromatic polynomial but different Ihara zeta functions. We also show that a graph's Ihara zeta function cannot in general be recovered from its chromatic polynomial.

Then we make the deletion of an edge from a graph less jarring (from the

perspective of Ihara zeta functions) by viewing it as the limit as  $k$  goes to infinity of the operation of replacing the edge in the original graph we wish to delete with a walk of length  $k$ . We are able to prove that the limit of the Ihara zeta functions of the resulting graphs is in fact the Ihara zeta function of the original with the edge deleted.

We also improve upon the bounds on the poles of the Ihara zeta function by considering digraphs whose adjacency matrices are directed edge matrices.

# 1

## Introduction

### 1.1 Preliminaries

We pause here before defining the Ihara zeta function to fix some basic (and thus, crucial) graph theory terminology:

A *graph*  $G$  is an ordered pair  $(V, E)$  where  $V$  is a set and  $E$  is a set of unordered pairs whose elements are taken from  $V$ . An element of the set  $V$  is called a *vertex* and is represented by a dot. An element of the set  $E$  is called an *edge* and is represented by a line (not necessarily straight) connecting the two vertices of the pair. An edge in which connects a vertex to itself is called a *loop*. Two edges are called *parallel* if they are drawn as distinct edges but actually correspond to the same unordered pair. A graph is *simple* if it contains no loops and no parallel edges. A simple graph in which there is an edge between every pair of its  $n$  vertices is called the *complete graph on  $n$  vertices* and denoted  $K_n$ . A *bipartite graph* is a graph for which there is a partition  $\{V_1, V_2\}$  of  $V$  such that for any  $u_1, v_1$  in  $V_1$  and any  $u_2, v_2$  in  $V_2$ ,  $\{u_1, v_1\}, \{u_2, v_2\} \notin E$ . Where necessary to avoid confusion, we will use  $V(G), E(G)$  to refer to the vertices and edges respectively of  $G$ . We will use  $|E|, |V|$  to represent the number of edges and vertices respectively. A *finite graph* is a graph in which  $|E|, |V| < \infty$ .

An edge  $e$  is *incident* to a vertex  $v$  if  $v$  is one of the elements of the pair  $e$ . The *degree* of a vertex  $v$ , written  $\deg(v)$ , is the number of edges (loops being counted twice) incident to  $v$ . A graph in which every vertex has the same degree is called *regular*. An  $n$ -regular graph is a regular graph whose vertices all have degree  $n$ . An *irregular graph*, rather unsurprisingly, is a graph which is not regular. A *bi-regular bipartite graph* is a graph  $G$  for which there is a partition  $\{V_1, V_2\}$  of  $V(G)$  and integers  $d_1, d_2$  such that for any  $v_1, v_2$  in  $V_1, V_2$  respectively,  $\{v_1, v_2\} \notin E(G)$ ,  $\deg(v_1) = d_1$ ,  $\deg(v_2) = d_2$ .

A *deletion* of an edge  $e$  from a graph  $G$ , denoted  $G - e$ , is the graph such that  $V(G - e) = V(G)$  and  $E(G - e) = E(G) - \{e\}$ . A *contraction* of an edge  $e = \{u, v\}$  in a graph  $G$ , denoted  $G/e$ , is the graph formed by treating the vertices  $u$  and  $v$  as being the same vertex in the graph  $G - e$ .

A *walk* is an alternating sequence of vertices and edges starting and ending with a vertex such if vertex  $v$  is appears next to edge  $e$  in the sequence then  $e$  is incident to  $v$ . If  $\mathcal{W}_1 = \{v_0, e_1, v_1, \dots, e_n, v_n\}$  and  $\mathcal{W}_2 = \{v_n, e_{n+1}, v_{n+1}, \dots, e_{n+m}, v_{n+m}\}$  are walks, then we define the product  $\mathcal{W}_1\mathcal{W}_2 = \{v_0, e_1, v_1, e_2, v_2, \dots, e_{n+m}, v_{n+m}\}$ . Also, we define  $\mathcal{W}_1^{-1} = \{v_n, e_n, v_{n-1}, \dots, v_1, e_1, v_0\}$ . It is common to leave out the vertices in describing a walk in a graph without loops since the intended vertices can be inferred from the listed edges. The *length* of a walk is the number of edges it contains. The distance  $d(u, v)$  between two vertices  $u, v$  is the length of a shortest walk from  $u$  to  $v$  if such a walk exists (otherwise,  $d(u, v) = \infty$ ). A graph is *connected* if there is a walk between any two vertices of the graph. A walk  $\{v_0, e_1, v_1, e_2, v_2, \dots, e_n, v_n\}$  is called *closed* if  $v_0 = v_n$ . A *cycle* is a closed walk  $\{v_0, e_1, v_1, e_2, v_2, \dots, e_n, v_n\}$  such that  $\{v_1, v_2, \dots, v_n\}$  are distinct. The *girth* of a graph is the length of the shortest cycle contained within the graph. A *tree* is a connected graph which contains no cycles. A *spanning tree* of a (connected) graph  $G$  is a tree  $S$  such that  $V(S) = V(G)$  and  $E(S) \subseteq E(G)$ . A *cycle graph* is a graph whose vertices and edges are contained within a single cycle.

A *subgraph*  $H$  of a graph  $G$  is a graph such that  $V(H) \subseteq V(G)$  and  $E(H) \subseteq$

$E(G)$ . A *connected component* of a (not necessarily connected) graph  $G$  is a connected subgraph of  $G$  which is not contained in any other connected subgraph of  $G$ . An edge is a *bridge* in  $G$  if  $G$  with the edge removed has more connected components than  $G$ . Another characterization of a bridge is an edge  $e$  in  $G$  such that any walk from one vertex of  $e$  to the other contains  $e$  itself.

If  $G$  is a graph with vertex set  $V = v_1, v_2, \dots, v_{|V|}$ , then the matrix  $A = (a_{ij})$  defined by

$$a_{ij} = \begin{cases} \text{the number of edges of the form } v_i, v_j & \text{if } i \neq j \\ \text{twice the number of loops at } v_i & \text{if } i = j. \end{cases}$$

is an *adjacency matrix* of  $G$ .

A *directed graph* (or *digraph* for short) is a graph as defined above except that the edge set  $E_d$  is a set of ordered pairs called *directed edges*. The first vertex in an ordered pair  $e$  in  $E_d$  is the *start vertex* and will be denoted  $s(e)$ . The second vertex is the *terminal vertex* and will be denoted  $t(e)$ . The *indegree*  $id(v)$  of a vertex  $v$  is the number of edges whose terminal vertex is  $v$ . The *outdegree*  $od(v)$  of a vertex  $v$  is the number of edges whose start vertex is  $v$ . A directed edge is represented by an arrow drawn from the start vertex to the terminal vertex. A directed edge whose start vertex is the same as its terminal vertex is called a (directed) *loop*. Two directed edges are called *parallel* if they are drawn as distinct edges but actually correspond to the same ordered pair. A directed graph is *simple* if it contains no loops and no parallel edges. If  $e = (u, v)$  is a directed edge, then we define  $e^{-1} = (v, u)$ . A *bidirected edge* of a directed graph  $G$  is an edge  $e \in E_d$  such that  $e^{-1} \in E_d$ .

A (*directed*) *walk* in a directed graph is as defined above except that we insist that  $e_i$  feeds into  $e_{i+1}$  (and for a walk to be closed, we also require  $e_n$  to feed into  $e_0$ ). It is common to leave the vertices out when describing a directed walk since the vertices are implied by the directed edges. Also, for a directed walk  $\mathcal{W}_0 = \{e_1, e_2, \dots, e_n\}$ , we define  $\mathcal{W}_0^{-1} = \{e_n^{-1}, e_{n-1}^{-1}, \dots, e_2^{-1}, e_1^{-1}\}$ .

If  $G$  is a directed graph with vertex set  $V = v_1, v_2, \dots, v_{|V|}$ , then the matrix



$A = (a_{ij})$  defined by

$$a_{ij} = \text{the number of edges of the form } (v_i, v_j)$$

is an *adjacency matrix* of  $G$ .

Occasionally, it is useful to think of an undirected graph as a directed graph by replacing each of its undirected edges with a bidirected edge. (We will use  $E$  to denote the set of undirected edges and  $E_d$  to denote the set of directed edges.) Note that if we think of an undirected graph in this way, then our two definitions of an adjacency matrix are equivalent. We may also *orient* an undirected edge by forcing an order on its unordered pair (or we can orient a bidirected edge by throwing the unwanted ordering of the ordered pairs out of the directed edge set). Conversely, we may occasionally want to make a directed graph undirected by making its ordered pairs unordered (and throwing out duplicates due to bidirected edges). When we do this, we call the resulting undirected graph the *underlying graph* of the original directed graph.

We say that two graphs  $G, H$  are *isomorphic* if there exists a one-to-one function  $f$  from  $V(G)$  onto  $V(H)$  such that  $g$  defined by  $g((x, y)) = (f(x), f(y))$  for all  $(x, y) \in E_d(G)$  is a one-to-one function from  $E_d(G)$  onto  $E_d(H)$ . Such a function  $f$  is a *graph isomorphism*.

## 1.2 Ihara zeta function of a graph

We begin by defining what we mean by a *prime*  $[C]$  of  $G$ . Let  $C$  be a closed walk  $\{v_0, e_1, v_1, e_2, v_2, \dots, e_n, v_n = v_0\}$  in  $G$  such that  $C^2$  (that is, the product  $CC$ ) contains no backtracks and  $C$  is not  $C_0^j$  for any closed walk  $C_0$  and integer  $j \geq 2$ . (A *backtrack* is a subsequence  $e_i, v_i, e_{i+1}, v_{i+1}$  contained in the walk  $v_0, e_1, v_1, e_2, v_2, \dots, e_n, v_n$  such that  $e_i = e_{i+1}$ .) Then the prime  $[C]$  is the equivalence class  $\{\{v_n, e_1, v_1, e_2, v_2, \dots, e_n, v_n\}, \{v_1, e_2, v_2, e_3, v_3, \dots, e_n, v_n, e_1, v_1\}, \{v_2, e_3, v_3, \dots, e_n, v_n, e_1, v_1, e_2, v_2\}, \dots, \{v_{n-1}, e_n, v_n, e_1, v_1, \dots, e_{n-2}, v_{n-2}, e_{n-1}, v_{n-1}\}\}$ .

We will now give a preliminary definition of the Ihara zeta function:

**Definition 1.1.** The *Ihara zeta function* of a graph  $G$  is defined (for sufficiently small values of the complex number  $u$ ) to be

$$\zeta_G(u) = \prod_{[C]} (1 - u^{\nu(C)})^{-1}$$

where the product is over the primes  $[C]$  of  $G$  and  $\nu(C)$  is the length of  $C$ .

Note that the product in the definition of the Ihara zeta function is a finite one if and only if the graph  $G$  is a cycle graph. (Note also that, for example, there are two equivalence classes of primes in a cycle graph, one for each direction the cycle may be traversed.) Since the product is infinite except for the set of cycle graphs, we must of course be concerned about convergence issues, which is the reason for the requirement of a sufficiently small  $u$  in the definition. However, since “sufficiently small” for very large graphs may be very small indeed, we extend our definition of the Ihara zeta function of a graph to its analytic continuation with the following theorem (whose author is debatable due to the Ihara zeta function’s origins in  $p$ -adic groups):

**Theorem 1.2.** (*Bass [1]*) Let  $G$  be an undirected graph with vertex set  $V = \{v_1, v_2, \dots, v_{|V|}\}$  and adjacency matrix  $A = (a_{ij})$ . Let  $Q$  be the diagonal matrix whose  $i$ th diagonal entry is one less than the degree of the vertex  $v_i$ . Let  $r$  be the rank of the fundamental group of  $G$ . Then

$$\zeta_G(u)^{-1} = (1 - u^2)^{r-1} \det(I - Au + Qu^2).$$

The *fundamental group*  $\pi_1(G, v)$  of a connected graph  $G$  is the free group consisting of all closed walks starting and ending at the vertex  $v$  together with the operation which concatenates walks. The *rank*  $r$  of the fundamental group  $\pi_1(G, v)$  of a connected graph  $G$  is the number of elements in a minimal generating set of  $\pi_1(G, v)$  which is also the number of edges left out of a spanning tree of  $G$ . Thus,

$r = |E| - |V| + 1$ . The theorem still holds for graphs which are not connected if we simply take this as our definition of  $r$ .

Note that, by Theorem 1.2, the Ihara zeta function of a graph is the reciprocal of a degree  $2|E|$  polynomial with integer coefficients. Since the reciprocal of this polynomial agrees with our original definition of the Ihara zeta function within a small circle about zero in the complex plane and is analytic everywhere but at the isolated zeros of the polynomial, we take this analytic continuation as our new definition of the Ihara zeta function of a graph.

If the graph  $G$  is  $(q + 1)$ -regular, then we have the simplification  $\zeta_G(u)^{-1} = (1 - u^2)^{r-1} \det(I - Au + qIu^2) = (1 - u^2)^{r-1} \prod_{\lambda \in \text{spec}(A)} (1 - \lambda u + qu^2)$  where  $\text{spec}(A)$  is the spectrum of  $A$ . A great deal of the difficulty in working with the Ihara zeta functions of irregular graphs is due to the lack of this simplification. Also, extending results from the regular to the irregular case is complicated by the many ways of characterizing the value  $q$  in the regular case which are not equivalent in the irregular case.

Now we will present an alternate formulation of the Ihara zeta function which requires us to define a directed edge matrix of a graph  $G$ :

**Definition 1.3.** Arbitrarily orient the edges  $e_1, e_2, \dots, e_{|E|}$  of a graph  $G$  and let  $e_{|E|+i} = e_i^{-1}$  for all  $i$ ,  $1 \leq i \leq |E|$ . The  $2|E| \times 2|E|$  matrix  $M$  given by

$$(M)_{ij} = \begin{cases} 1 & \text{if } t(e_i) = s(e_j) \text{ and } s(e_i) \neq t(e_j), \\ 0 & \text{otherwise.} \end{cases}$$

is defined to be a *directed edge matrix* of  $G$ .

A directed edge matrix  $M$  of a graph  $G$  is related to the Ihara zeta function of  $G$  by the following theorem:

**Theorem 1.4.** (*Bass [1]*) *If  $M$  is a directed edge matrix of the graph  $G$ , then*

$$\zeta_G(u)^{-1} = \det(I - Mu).$$

Which formulation of the Ihara zeta function to use depends on the context. The Theorem 1.2 formulation offers the benefits of symmetric matrices (which allow for the simplification described above for the regular case) while the Theorem 1.4 formulation allows us to identify the poles of the Ihara zeta function with the reciprocals of the eigenvalues of the matrix  $M$ .

*Remark 1.5.* Note  $M$  has the structure  $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$  where  $A, B, C$  are  $|E| \times |E|$  matrices with the following properties:

- (i)  $B = B^T$ , and  $C = C^T$ ,
- (ii)  $D = A^T$ ,
- (iii) the diagonal entries of  $B, C$  are zeros,
- (iv) the diagonal entries of  $A, D$  are zeros if the graph contains no loops,
- (v)  $M^T = JMJ$  where  $J = \begin{pmatrix} 0 & I_{|E|} \\ I_{|E|} & 0 \end{pmatrix}$ ,
- (vi)  $A + B + C + A^T$  is an adjacency matrix of the line graph of  $G$ . (The *line graph* of  $G$  is the graph whose vertices are the edges of  $G$  and whose edges are such that two vertices are adjacent if and only if they are incident to each other as edges in  $G$ .)

Properties (i)-(iii) are contained in Lemma 4 of Stark and Terras [12]. Property (v) is a consequence of (i) and (ii). Properties (iv) and (vi) follow easily from the definition of  $M$ . The special properties of directed edge matrices are explored further in Chapter 2.

Armed with an expanded and reformulated definition of the Ihara zeta function, we will be prepared to tackle some results after a short pause (which supplants many short pauses later on) to discuss our simplifying assumptions.

### 1.3 Simplifying assumptions

In all that follows, except where noted, we will assume that our graphs are finite, simple, connected, and without vertices of degree one. Since these assumptions may seem a bit presumptuous, we will briefly defend each.

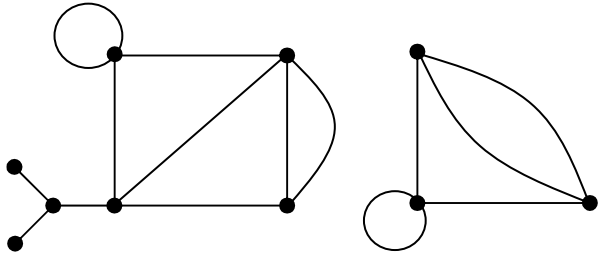
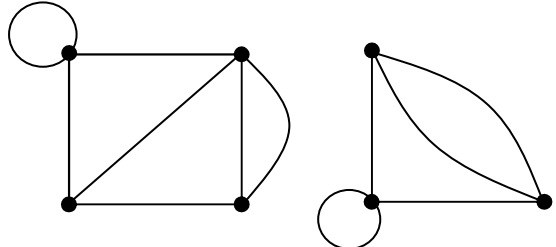
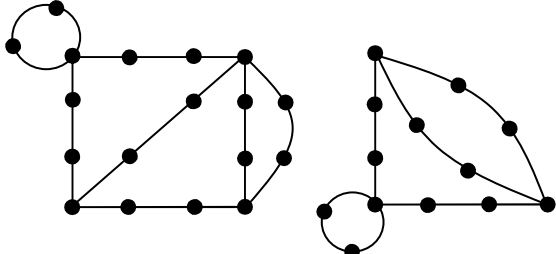
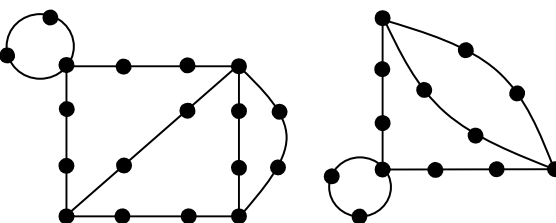
The Ihara zeta function of a graph  $G$  is the same as the Ihara zeta function of  $G$  with a tree attached by a single edge, so it is reasonable to expect our graphs to come pre-pruned by insisting that  $G$  contains no vertices of degree one. In most cases, results for pre-pruned graphs will be easily extended to their leafier counterparts.

Since the Ihara zeta function of a disconnected graph is just the product of the Ihara zeta functions of each of its components, we also lose little that we cannot reconstitute later by insisting on our graphs being connected.

We can create a simple graph  $H$  from any non-simple graph  $G$  by dividing each edge  $\{u, v\}$  in  $G$  into three edges  $\{u, w\}, \{w, x\}, \{x, v\}$  where  $w, x$  are new vertices of degree two. Then it can be shown that the Ihara zeta function of  $H$  will be  $\zeta_G(u^3)$  where  $\zeta_G(u)$  is the Ihara zeta function of  $G$ . So, any intelligent statement we can make about the Ihara zeta function of a simple graph can be transformed into at least a semi-intelligent statement about the Ihara zeta function of a non-simple graph. Figure 1.1 demonstrates the process of overcoming our simplifying assumptions given a graph which is not simple, not connected, and has vertices of degree one.

We justify our assumption that our graphs are finite only by saying that such an assumption is fairly standard and that the subject is meaty enough without dropping finiteness.

Table 1.1 Overcoming our simplifying assumptions given a graph which fails all but finiteness

Graph	Description
<p style="text-align: center;"><math>G_1</math></p> 	<p>Original graph which does not satisfy our simplifying assumptions</p>
<p style="text-align: center;"><math>G_2</math></p> 	<p>Vertices of degree one removed;  <math>\zeta_{G_2}(u) = \zeta_{G_1}(u)</math></p>
<p style="text-align: center;"><math>G_3</math></p> 	<p>Two vertices added to each edge;  <math>\zeta_{G_3}(u) = \zeta_{G_2}(u^3)</math></p>
<p style="text-align: center;"><math>G_4</math>                      <math>G_5</math></p> 	<p>Connected components considered as separate graphs;  <math>\zeta_{G_4}(u)\zeta_{G_5}(u) = \zeta_{G_3}(u)</math></p>

## 2

# Poles of the Ihara zeta function

Much of the interest in the Ihara zeta function is its similarity in feel to the Riemann zeta function,

$$\zeta(s) = \sum_{n \geq 1} n^{-s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}$$

where primes here are integers greater than one which are divisible only by themselves and one. However, whereas the zeros of the Riemann zeta function have been of primary interest, it is the poles of the Ihara zeta function (which are the just the zeros of the reciprocal of the Ihara zeta function, assuming of course that we patch up singularities in a reasonable way) which call out for attention.

In this chapter, we investigate some of the properties of the poles of the Ihara zeta function by taking advantage of the special properties of the singular value decomposition and the inverse of the directed edge matrix.

## 2.1 Bounds on the poles

We will use the singular value decomposition of the directed edge matrix of a graph to find bounds on the poles of the Ihara zeta function of the graph where the singular value decomposition of a square matrix with real entries is defined as follows:

**Definition 2.1.** The *singular values* of a real  $n \times n$  matrix  $A$  are  $\sigma_1, \sigma_2, \dots, \sigma_n$  where  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0$  and  $U^T A V = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_n)$  for some orthogonal  $U, V$  (which exist by the Singular Value Decomposition Theorem – see Golub and Van Loan [3]).  $U \cdot \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_n) \cdot V^T$  then is a *singular value decomposition* (SVD) of  $A$ .

Non-square matrices also have singular value decompositions, but the definition above suffices for our purposes.

The singular values of the directed edge matrix of a graph turn out to be surprisingly nice. Not only are they integers, but they are integers which carry familiar information about the graph itself as revealed by the following theorem:

**Theorem 2.2.** *Suppose  $G$  is a connected graph with no loops, multi-edges, or vertices of degree one. Then the singular values of a directed edge matrix  $M$  of  $G$  are*

$$\{q_1, q_2, \dots, q_n, \underbrace{1, \dots, 1}_{2m-n \text{ times}}\}$$

where  $q_1 + 1, q_2 + 1, \dots, q_n + 1$  are the degrees of the  $n$  vertices of  $G$  (counting multiplicities) and  $m$  is the number of edges in  $G$ .

*Proof.* Let  $G$  be a connected graph on  $n$  vertices with no loops, multi-edges, or vertices of degree one. Let  $\{e_1, e_2, \dots, e_{2m}\}$  where  $m$  is the number of edges in  $G$  be the set of directed edges such that each  $e_i$  (without its direction) is an edge of  $G$ . Choose the indexing of the  $e_i$  such that edges ending at the same vertex are listed together. That is, if  $t(e_i) = t(e_j)$  for some  $i < j$  then  $t(e_k) = t(e_i)$  for all  $k, i < k < j$ . Let  $\tilde{M}$  be the  $2m \times 2m$  matrix defined by

$$(\tilde{M})_{ij} = \begin{cases} 1 & \text{if } t(e_i) = s(e_j) \text{ and } s(e_i) \neq t(e_j), \\ 0 & \text{otherwise.} \end{cases}$$

Then for each  $i$  and  $j$ ,  $(\tilde{M}\tilde{M}^T)_{ij}$  is a count of the number of edges in  $\{e_1, e_2, \dots, e_{2m}\}$ —



$\{e_i^{-1}, e_j^{-1}\}$  whose start vertex is the end vertex of both  $e_i$  and  $e_j$ . So,

$$\left(\tilde{M}\tilde{M}^T\right)_{ij} = \begin{cases} 0 & \text{if } t(e_i) \neq t(e_j), \\ \deg(t(e_i)) - 2 & \text{if } t(e_i) = t(e_j) \text{ and } i \neq j, \\ \deg(t(e_i)) - 1 & \text{if } i = j. \end{cases}$$

Now choose  $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$  such that  $1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_n \leq 2m$  and  $\{t_{\alpha_1}, t_{\alpha_2}, \dots, t_{\alpha_n}\}$  is the vertex set of  $G$ . Define  $q_i = \deg(v_{\alpha_i}) - 1$  and note  $\tilde{M}\tilde{M}^T = \text{diag}(A_1, A_2, \dots, A_n)$  where  $A_i$  is a  $(q_i + 1) \times (q_i + 1)$  matrix with  $q_i$  on the diagonal and  $q_i - 1$  everywhere else. For each  $i$ , define  $(q_i + 1) \times (q_i + 1)$

matrixes  $B_i = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & -q_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & -q_i \end{pmatrix}$ , which is nonsingular (by row reduction),

and  $C_i = \text{diag}(q_i^2, 1, \dots, 1)$ . Then

$$\begin{aligned} A_i B_i &= \begin{pmatrix} q_i & q_i - 1 & q_i - 1 & \cdots & q_i - 1 \\ q_i - 1 & q_i & q_i - 1 & \cdots & q_i - 1 \\ q_i - 1 & q_i - 1 & q_i & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & q_i - 1 \\ q_i - 1 & q_i - 1 & \cdots & q_i - 1 & q_i \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & -q_i & 1 & \cdots & 1 \\ 1 & 1 & -q_i & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 1 & 1 & \cdots & 1 & -q_i \end{pmatrix} \\ &= \begin{pmatrix} q_i^2 & 1 & 1 & \cdots & 1 \\ q_i^2 & -q_i & 1 & \cdots & 1 \\ q_i^2 & 1 & -q_i & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ q_i^2 & 1 & \cdots & 1 & -q_i \end{pmatrix} = B_i C_i. \end{aligned}$$

Now define  $X = \text{diag}(B_1, B_2, \dots, B_n)$ . Note  $X$  is nonsingular and  $(\tilde{M}\tilde{M}^T)X = \text{diag}(A_1, A_2, \dots, A_n) \cdot \text{diag}(B_1, B_2, \dots, B_n) = \text{diag}(B_1, B_2, \dots, B_n) \cdot \text{diag}(C_1, C_2, \dots, C_n) = X \cdot \text{diag}(C_1, C_2, \dots, C_n)$ . Thus, the columns of  $X$  are eigenvectors of  $\tilde{M}\tilde{M}^T$  with corresponding eigenvalues appearing on the diagonal of the matrix

$\text{diag}(C_1, C_2, \dots, C_n)$ . Let  $U \cdot \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_{2m}) \cdot V^T$  be an SVD of  $\tilde{M}$ . Then  $(\tilde{M}\tilde{M}^T)U = U \cdot \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_{2m})^2$  which implies the eigenvectors of  $\tilde{M}\tilde{M}^T$  are the columns of  $U$  and the eigenvalues of  $\tilde{M}\tilde{M}^T$  are the squares of the singular values of  $\tilde{M}$ . So, the singular values of  $\tilde{M}$  are  $\{q_1, q_2, \dots, q_n, \underbrace{1, \dots, 1}_{2m-n \text{ times}}\}$ . Note that for any directed edge matrix  $M$  of  $G$ ,  $\tilde{M} = P^T M P$  for some permutation matrix  $P$ . Therefore, the singular values for any directed edge matrix of  $G$  are also  $\{q_1, q_2, \dots, q_n, \underbrace{1, \dots, 1}_{2m-n \text{ times}}\}$  since  $M = (P U) \cdot \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_{2m}) \cdot (P V)^T$  is an SVD of  $M$ .  $\square$

This theorem, besides being an interesting curiosity in and of itself, allows us to bound the poles of the Ihara zeta function of a graph as illustrated by the following corollary:

**Corollary 2.3.** *Suppose  $G$  is a connected graph with no loops, multi-edges, or vertices of degree one. Then the poles of the Ihara zeta function of  $G$  are contained within  $\{u : \frac{1}{q} \leq |u| \leq 1\}$  where  $q + 1$  is the largest vertex degree. (See Theorem 1.3 in Kotani and Sunada [6] for an alternate proof of this result.)*

*Proof.* Let  $M$  be a directed edge matrix of the graph  $G$ . By the theorem, the smallest singular value of  $M$ ,  $\sigma_{\min}(M)$ , is 1 and the largest singular value of  $M$ ,  $\sigma_{\max}(M)$ , is  $q$  where  $q + 1$  is the largest vertex degree. Since

$$\sigma_{\min}(M) \leq \min\{|\lambda| : \lambda \in \lambda(M)\} \leq \max\{|\lambda| : \lambda \in \lambda(M)\} \leq \sigma_{\max}(M),$$

where  $\lambda(M)$  is the spectrum of  $M$  (see Golub and Van Loan [3]), the poles of the Ihara zeta function of  $G$  (which are the reciprocals of the eigenvalues of  $M$ ) are contained within  $\{u : \frac{1}{q} \leq |u| \leq 1\}$ .  $\square$

## 2.2 Relations among the poles

The poles of the Ihara zeta function have many other interesting and informative properties due to the structure forced upon them by their corresponding

graphs as illustrated by the following theorem:

**Theorem 2.4.** (Hashimoto [4]) *Let  $G$  be a graph and let  $U$  be the set of poles of  $\zeta_G(u)$  including multiplicities. Define  $N_k$  to be the number of backtrackless, tailless closed walks in  $G$  of length  $k$ . Then  $\prod_{u \in U} u^{-1} = (-1)^{|V|} \prod_{v \in V} (\deg(v) - 1)$  and  $\prod_{u \in U} u^{-k} = N_k$  for any positive integer  $k$ .*

In particular, this theorem implies that in simple graphs (which by definition have no closed walks of length less than three), both the sum of the reciprocals of the poles of the Ihara zeta function and the sum of the squares of the reciprocals of the poles are zero. Oddly enough, it turns out also to be true that the sum of the poles (without taking reciprocals) is zero by a corollary to the following theorem:

**Theorem 2.5.** *Suppose  $G$  is a graph (which satisfies our simplifying assumptions) and let  $M$  be the directed edge matrix of  $G$ . Define  $J = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$  and  $Q = \text{diag}(M[1 \cdots 1]^T)$ . Then*

$$M^{-1} = J(Q^{-1}MJ + Q^{-1} - I).$$

*Proof.* Note that

$$(MM^T)_{ij} = \begin{cases} \deg(t_i) - 1 & \text{if } t_i = t_j \text{ and } i = j, \\ \deg(t_i) - 2 & \text{if } t_i = t_j \text{ and } i \neq j, \\ 0 & \text{if } t_i \neq t_j. \end{cases}$$

since the  $ij$  entry of  $MM^T$  is just the number of edges into which both  $e_i$  and  $e_j$  feed. So,

$$(MM^T + MJ)_{ij} = \begin{cases} \deg(t_i) - 1 & \text{if } t_i = t_j. \\ 0 & \text{if } t_i \neq t_j, \end{cases}$$

since

$$(MJ)_{ij} = \begin{cases} 1 & \text{if } t_i = t_j \text{ and } i \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

Thus,  $(Q^{-1}(MM^T + MJ))_{ij} = \begin{cases} 1 & \text{if } t_i = t_j. \\ 0 & \text{if } t_i \neq t_j, \end{cases}$

So,  $(Q^{-1}(MM^T + MJ) - MJ)_{ij} = I_{2m}$ .

Now we will use this fact to show that  $M^{-1} = J(Q^{-1}MJ + Q^{-1} - I)$ . Recall from Remark 1.5 that  $M^T = JMJ$  and note  $J(Q^{-1}MJ + Q^{-1} - I)M = J(Q^{-1}MJM + Q^{-1}M - M) = J(Q^{-1}MJMJ + Q^{-1}MJ - MJ)J = J(Q^{-1}MM^T + Q^{-1}MJ - MJ)J = J(Q^{-1}(MM^T + MJ) - MJ)J = JI_{2m}J = I$ . Therefore,  $M^{-1} = J(Q^{-1}MJ + Q^{-1} - I)$  as desired.  $\square$

For regular graphs, we have the following simplification:

**Corollary 2.6.** *Suppose  $G$  is a  $q+1$ -regular graph (which satisfies our simplifying assumptions). Let  $M, J$  be as in the theorem. Then*

$$M^{-1} = q^{-1}M^T + (q^{-1} - 1)J.$$

*Proof.* By the theorem,  $M^{-1} = J(Q^{-1}MJ + Q^{-1} - I) = J(q^{-1}MJ + (q^{-1} - 1)I) = q^{-1}JMJ + (q^{-1} - 1)J = q^{-1}M^T + (q^{-1} - 1)J$ .  $\square$

For both regular and irregular graphs, the explicit formula given in Theorem 2.5 for the inverse of a directed edge matrix implies the following relation among the poles of the Ihara zeta function:

**Corollary 2.7.** *Let  $u_0, u_1, \dots, u_{2m-1}$  be the poles (including multiplicities) of the Ihara zeta function of a graph. Then  $\sum_{k=0}^{2m-1} u_k = 0$ .*

*Proof.* The poles of the Ihara zeta function of a graph  $G$  are the reciprocals of the eigenvalues of a directed edge matrix  $M$  of  $G$ . Also, the reciprocals of the eigenvalues of  $M$  are the eigenvalues of  $M^{-1}$ . Using the structure of  $M$  from Remark 1.5 and the preceding theorem, we find that the diagonal elements of  $M^{-1}$  are all zeros. So, the trace of  $M^{-1}$  is zero. Therefore,  $\sum_{k=0}^{2m-1} u_k = 0$ .  $\square$

The previous corollary also holds if we exclude the poles on the unit circle:

**Corollary 2.8.** *Let  $u_0, u_1, \dots, u_{2m-1}$  be the poles (including multiplicities) of the Ihara zeta function of a graph. Then  $\sum u_k = 0$  where the sum is just over those poles whose magnitude is not one.*

*Proof.* Define  $U_1 = \{k : |u_k| = 1\}$  where the  $u_k$  are as defined above. Note that since the complex poles occur in conjugate pairs and  $u_k^{-1} = \overline{u_k}$  for all  $k \in U_1$ ,  $\sum_{k \in U_1} u_k = \sum_{k \in U_1} u_k^{-1}$ . Since  $\sum_{k=0}^{2m-1} u_k = 0 = \sum_{k=0}^{2m-1} u_k^{-1}$ , this implies  $\sum_{k \notin U_1} u_k = \sum_{k \notin U_1} u_k^{-1}$ .  $\square$

# 3

## Recovering information

### 3.1 The hope

For information about a graph to be recoverable, the Ihara zeta function must at the very least be able to distinguish between graphs for which the information differs. However, we generally also would like some systematic way of recovering the information. For instance, fingerprints may be able to distinguish between any two people, but there is no way (besides using some sort of database) of attaching a name to a print. In a fit of optimism, we might expect that a great deal of information about a graph is recoverable from its Ihara zeta function. Our optimism should be tempered somewhat however by the fact that the Ihara zeta function is not always able to distinguish between two non-isomorphic graphs, as illustrated in Figure 3.2.

Since the information about the graph used to create the Ihara zeta function concerns the lengths of closed walks in the graph, it is perhaps unsurprising that the greatest successes in extracting information from the Ihara zeta function also concerns information about the lengths of closed walks in the graph. What may be surprising is that the counts of closed walks we are able to extract using Theorem 2.4 have dropped the condition that the walks be prime. That is, a closed walk

$C$  is still counted even if it is  $C_0^j$  for some closed walk  $C_0$ . This, however, only prompts us to ask how much other information we have unintentionally collected while forming the Ihara zeta function. Table 3.1 reveals some notable successes as well as a couple sad truths concerning recoverable information. (The Ihara zeta functions of coverings hinted at in the table will be explored further in Section 4.3.)

## 3.2 Recovering Girth

By Theorem 2.4, we can calculate the girth of a graph given only its Ihara zeta function by finding the smallest integer  $k$  such that  $\sum_{u \in U} u^{-k} > 0$ . However, as the next theorem demonstrates, there is a much easier method of extracting this information from the Ihara zeta function.

**Theorem 3.1.** *Let  $G$  be a simple connected graph. Define  $c_i \in \mathbb{Z}$ ,  $1 \leq i \leq 2|E|$ , such that  $\zeta_G(u)^{-1} = 1 + c_1u + c_2u^2 + \cdots + c_{2|E|}u^{2|E|}$ . Then the girth of  $G$  is  $\min\{i \mid c_i \neq 0, 1 \leq i \leq 2|E|\}$ .*

*Proof.* For the sake of the readability of what follows, define  $\zeta = \zeta_G(u)$ .

Let  $N_m$  be the number of closed walks of length  $m$  without backtracking or tails where closed walks of the same equivalence class which have different start vertices are counted as different.

Let  $k = \min\{i \mid c_i \neq 0, 1 \leq i \leq 2|E|\}$  and  $g$  be the girth of  $G$ . Then  $u(\log \zeta)' = \sum_{m=1}^{\infty} N_m u^m = \sum_{m=g}^{\infty} N_m u^m$  (see page 137 of [12] for a derivation of  $u(\log \zeta)' = \sum_{m=1}^{\infty} N_m u^m$ ).

Note also that  $\lim_{u \rightarrow 0} u(\log \zeta)' = 0$ ,  $\lim_{u \rightarrow 0} \zeta = 1$ , and  $\lim_{u \rightarrow 0} u\zeta' = 0$ . Thus,

$$\begin{aligned} g &= \frac{gN_g}{N_g} = \frac{\lim_{u \rightarrow 0} \sum_{m=g}^{\infty} mN_m u^{m-g}}{\lim_{u \rightarrow 0} \sum_{m=g}^{\infty} N_m u^{m-g}} = \lim_{u \rightarrow 0} \frac{\sum_{m=g}^{\infty} mN_m u^{m-g}}{\sum_{m=g}^{\infty} N_m u^{m-g}} \\ &= \left( \lim_{u \rightarrow 0} \frac{u^g}{u^g} \right) \left( \lim_{u \rightarrow 0} \frac{\sum_{m=g}^{\infty} mN_m u^{m-g}}{\sum_{m=g}^{\infty} N_m u^{m-g}} \right) = \lim_{u \rightarrow 0} \frac{u^g \sum_{m=g}^{\infty} mN_m u^{m-g}}{u^g \sum_{m=g}^{\infty} N_m u^{m-g}} \end{aligned}$$

$$\begin{aligned}
&= \lim_{u \rightarrow 0} \frac{\sum_{m=g}^{\infty} m N_m u^m}{\sum_{m=g}^{\infty} N_m u^m} = \lim_{u \rightarrow 0} \frac{u \left( \sum_{m=g}^{\infty} N_m u^m \right)'}{\sum_{m=g}^{\infty} N_m u^m} \\
&= \lim_{u \rightarrow 0} \frac{u(u(\log \zeta)')'}{u(\log \zeta)'} = \lim_{u \rightarrow 0} \frac{(u(\log \zeta)')'}{(\log \zeta)'} = \lim_{u \rightarrow 0} \frac{(\log \zeta)' + u(\log \zeta)''}{(\log \zeta)'} \\
&= 1 + \lim_{u \rightarrow 0} \frac{u(\log \zeta)''}{(\log \zeta)'} = 1 + \lim_{u \rightarrow 0} \frac{u(\zeta' \zeta^{-1})'}{(\log \zeta)'} = 1 + \lim_{u \rightarrow 0} \frac{u(\zeta'' \zeta^{-1} - (\zeta' \zeta^{-1})^2)}{(\log \zeta)'} \\
&= 1 + \lim_{u \rightarrow 0} \frac{u \zeta'' \zeta^{-1} - u(\zeta' \zeta^{-1})^2}{(\log \zeta)'} = 1 + \lim_{u \rightarrow 0} \frac{u \zeta'' \zeta^{-1} - u(\zeta' \zeta^{-1})^2}{\zeta' \zeta^{-1}} \\
&= 1 + \lim_{u \rightarrow 0} \left( \frac{u \zeta''}{\zeta'} - \frac{u \zeta'}{\zeta} \right) = 1 + \lim_{u \rightarrow 0} \frac{u \zeta''}{\zeta'} - \lim_{u \rightarrow 0} \frac{u \zeta'}{\zeta} \\
&= 1 + \lim_{u \rightarrow 0} \frac{u \zeta''}{\zeta'} - \lim_{u \rightarrow 0} u(\log \zeta)' = 1 + \lim_{u \rightarrow 0} \frac{u \zeta''}{\zeta'} \stackrel{L'H}{=} 1 + \lim_{u \rightarrow 0} \frac{u \zeta' - \zeta + 1}{\zeta - 1} \\
&= 1 + \lim_{u \rightarrow 0} \left( \frac{u \zeta'}{\zeta - 1} - \frac{\zeta - 1}{\zeta - 1} \right) = 1 + \lim_{u \rightarrow 0} \frac{u \zeta'}{\zeta - 1} - \lim_{u \rightarrow 0} \frac{\zeta - 1}{\zeta - 1} \\
&= 1 + \lim_{u \rightarrow 0} \frac{u \zeta'}{\zeta - 1} - 1 = \lim_{u \rightarrow 0} \frac{u \zeta'}{\zeta - 1} = \left( \lim_{u \rightarrow 0} \frac{u \zeta'}{\zeta - 1} \right) \left( \lim_{u \rightarrow 0} \frac{1}{\zeta} \right) \\
&= \lim_{u \rightarrow 0} \frac{u \zeta'}{(\zeta - 1)\zeta} = \lim_{u \rightarrow 0} \frac{u \zeta'(-\zeta^{-2})}{\zeta^{-1} - 1} = \lim_{u \rightarrow 0} \frac{u(\zeta^{-1})'}{\zeta^{-1} - 1} \\
&= \lim_{u \rightarrow 0} \frac{u(1 + c_1 u + c_2 u^2 + \dots + c_{2|E|} u^{2|E|})'}{c_1 u + c_2 u^2 + \dots + c_{2|E|} u^{2|E|}} \\
&= \lim_{u \rightarrow 0} \frac{u(c_1 + 2c_2 u + \dots + 2|E|c_{2|E|} u^{2|E|-1})}{c_1 u + c_2 u^2 + \dots + c_{2|E|} u^{2|E|}} \\
&= \lim_{u \rightarrow 0} \frac{c_1 u + 2c_2 u^2 + \dots + 2|E|c_{2|E|} u^{2|E|}}{c_1 u + c_2 u^2 + \dots + c_{2|E|} u^{2|E|}} \\
&= \lim_{u \rightarrow 0} \frac{k c_k u + (k+1)c_{k+1} u^2 + \dots + 2|E|c_{2|E|} u^{2|E|}}{c_k u + c_{k+1} u^2 + \dots + c_{2|E|} u^{2|E|}} = k
\end{aligned}$$

and so  $g = k$  as desired.  $\square$

The coefficient of the  $u^g$  term is also informative as revealed by the following corollary:

**Corollary 3.2.** *Let  $g$  be the girth of a simple connected graph  $G$ . Define  $c_i \in \mathbb{Z}$ ,  $g \leq i \leq 2|E|$ , such that  $\zeta_G(u)^{-1} = 1 + c_g u^g + c_{g+1} u^{g+1} + \dots + c_{2|E|} u^{2|E|}$ . For each*



positive integer  $m$ , let  $N_m$  be as defined in the proof above and  $\pi(m)$  be the number of primes  $[P]$  of length  $m$  in  $G$ . Then  $c_g = -N_g/g = -\pi(g)$ .

*Proof.* Note that

$$\frac{N_g}{c_g} = \lim_{u \rightarrow 0} \frac{\sum_{m=g}^{\infty} N_m u^m}{\zeta^{-1} - 1} = \lim_{u \rightarrow 0} \frac{u \zeta'}{\zeta(\zeta^{-1} - 1)} = -\lim_{u \rightarrow 0} \frac{u \zeta'}{\zeta - 1} = -g$$

implies  $c_g = -N_g/g$ . Since  $g$  is the length of the smallest cycle in  $G$ ,  $N_g = g\pi(g)$ .

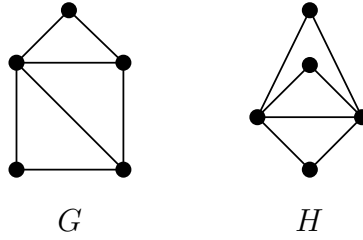
Therefore,  $c_g = -N_g/g = -\pi(g)$ .  $\square$

### 3.3 Chromatic polynomials and Ihara zeta functions

Besides demonstrating that the Ihara zeta function is not always able to distinguish between two non-isomorphic graphs, the graphs in Figure 3.2 also happen to have different chromatic polynomials (as can be demonstrated by calculating the number of three-colorings for each) where the chromatic polynomial of a graph is defined as follows:

**Definition 3.3.** The *chromatic polynomial*  $\chi_G(u)$  of a graph  $G$  is the degree  $|V|$  polynomial such that for each integer  $n$ ,  $1 \leq n \leq |V|$ ,  $\chi_G(n)$  is the number of different ways the vertices of  $G$  can be colored using exactly  $n$  colors including permutations of the  $n$  colors such that no two adjacent vertices are the same color. (A bipartite graph then is a graph  $G$  such that  $\chi_G(2) > 0$ .)

Since the Ihara zeta function of a graph does not in general contain enough information to recover the graph's chromatic polynomial, it seems reasonable to ask whether the chromatic polynomial contains enough information to recover a graph's Ihara zeta function. As demonstrated by the graphs in Figure 3.1, the answer in general is no. In fact, it is relatively easy to construct a family of graphs having the same chromatic polynomials but different Ihara zeta functions as we will now illustrate.



$$\begin{aligned}\zeta_G(u)^{-1} &= (-1 + u)^3(1 + u)^2(1 + u^2)^2(1 + u + 3u^2)(-1 + 2u^2 + 3u^3) \\ \zeta_H(u)^{-1} &= (-1 + u)^3(1 + u)^2(1 + u + 2u^2)(-1 + u + 3u^3)(1 + u + 2u^2 + u^3 + 2u^4) \\ \chi_G(u) &= \chi_H(u) = (-2 + u)^3(-1 + u)u\end{aligned}$$

Figure 3.1 Two graphs with the same chromatic polynomial but different Ihara zeta functions

The deletion-contraction method (see Biggs [2]) makes repeated use of the relation

$$\chi_G(u) = \chi_{G-e}(u) - \chi_{G/e}(u) \quad (3.1)$$

to reduce the problem of calculating the chromatic polynomial of a graph to calculating the chromatic polynomials of many graphs which are less complicated (that is, have fewer edges) than the original. We will use (3.1) in reverse to construct a family  $F_G$  from a graph  $G$ . For each edge  $e \in E(G)$ , define  $G_e$  to be the graph such that  $V(G_e) = V(G) \cup \{x\}$  and  $E(G_e) = E(G) \cup \{\{v_1, x\}, \{x, v_2\}\}$  where  $\{v_1, v_2\} = e$  and  $x$  is a new vertex not in  $V(G)$ . Now define  $F = \{G_e : e \in E(G)\}$ . Note that for each  $G_e \in F$ , if we define  $f_1, f_2$  to be the edges added to  $G$  to form  $G_e$ , then

$$\begin{aligned}\chi_{G_e}(u) &= \chi_{G_e-f_1}(u) - \chi_{G_e/f_1}(u) \\ &= (\chi_{(G_e-f_1)-f_2}(u) - \chi_{(G_e-f_1)/f_2}(u)) - \chi_{G_e/f_1}(u) \\ &= (u\chi_G(u) - \chi_G(u)) - \chi_G(u) \\ &= u\chi_G(u) - 2\chi_G(u)\end{aligned}$$

which is independent of our choice of  $e$ . Thus, every graph in  $F$  has the same chromatic polynomial. So, we need only judiciously choose our base graph  $G$  such

that  $\zeta_{G_e}(u)$  is not the same function as  $\zeta_{G_f}(u)$  for  $e, f \in E(G)$ ,  $e \neq f$ . This means of course that we want to choose  $G$  such that the only isomorphism from  $G$  to itself is the identity.

Table 3.1 Information about  $G$  recoverable from  $\zeta_G$ 

Information about $G$	How the information can be recovered from $\zeta_G$ :	Justification
# of edges, $ E $	$= \frac{\deg(\zeta_G(u)^{-1})}{2}$	Theorem 1.2
# of vertices, $ V $	$= \frac{\deg(\zeta_G(u)^{-1})}{2} - \text{order}_{u=1}(\zeta_G(u)) + 1$ unless $\zeta_G(u)^{-1} = (1 - u^n)^2$ for some $n$ in which case $ V  = n$	follows from theorem in [4]
rank $r$ of the fundamental group of $G$	$= \text{order}_{u=1}(\zeta_G(u))$	see [4]
$G$ is bipartite	$\Leftrightarrow \text{order}_{u=1}(\zeta_G(u)) = \text{order}_{u=-1}(\zeta_G(u))$	follows from theorem in [4]
complexity, $\kappa$	$= \frac{(-1)^{r+1}}{2^r(r-1)r! (\zeta_G(u)^{-1})^{(r)} \Big _{u=1}}$ where $r = \text{order}_{u=1}(\zeta_G(u))$	see [4]
girth, $g$	$= \lim_{u \rightarrow 0} \frac{u \zeta_G'(u)}{\zeta_G(u)^{-1}} = \lim_{u \rightarrow 0} \frac{u(\zeta_G(u)^{-1})'(u)}{\zeta_G(u)^{-1} - 1}$	Theorem 3.1
$G$ is a covering of $H$	$\Rightarrow \zeta_H(u)^{-1} \mid \zeta_G(u)^{-1}$	see [12]
chromatic polynomial	cannot, in general, be determined from $\zeta_G(u)$ alone	Figure 3.2
$G$ is isomorphic to $H$	cannot, in general, be determined from $\zeta_G(u), \zeta_H(u)$ alone	Figure 3.2

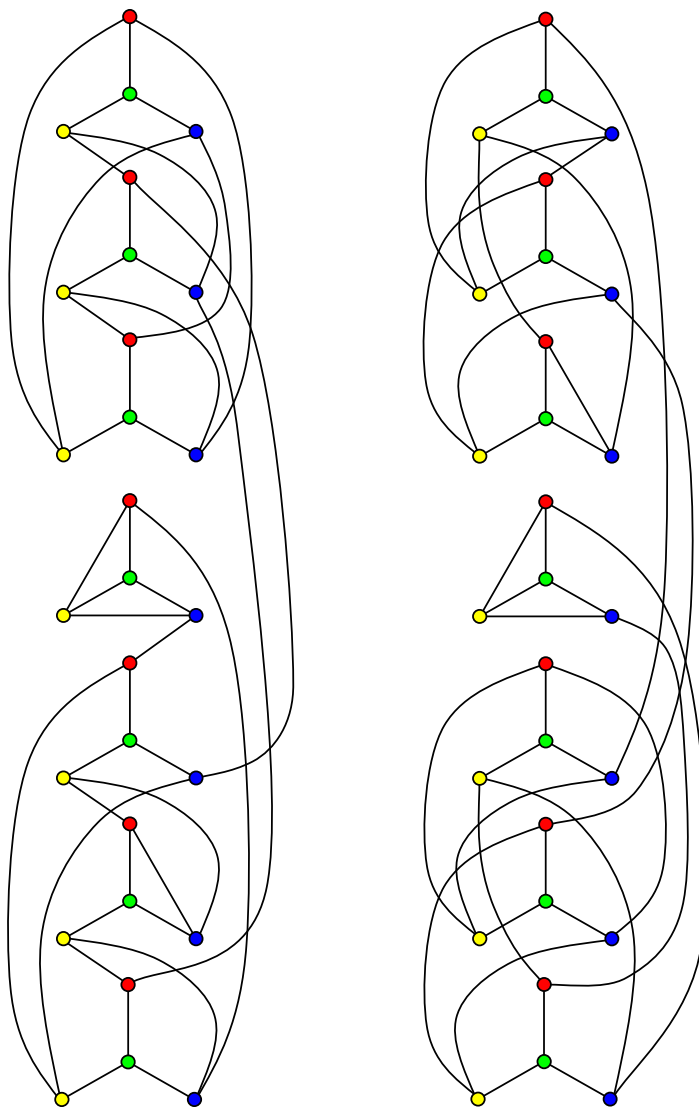


Figure 3.2 Non-isomorphic graphs with the same Ihara zeta function (redrawn from Stark and Terras [13]) which also happen to have different chromatic polynomials

# 4

## Relations among Ihara zeta functions

### 4.1 The problem

One of the more challenging aspects of attempting to investigate the relationship between a graph and its Ihara zeta function is the fact that small changes in the graph generally render the Ihara zeta function of the resulting graph unrecognizable as having any relation to the Ihara zeta function of the original. Deleting a single edge from a graph, for example, removes an infinite number of factors from its Ihara zeta function product. In the next section, we make this deletion less jarring by viewing it as the limit of an operation on the original graph.

### 4.2 Limits of Ihara zeta functions

For small values of  $u$ , it is evident from Definition 1.1 that, loosely speaking, one very long closed walk in a graph has less of an effect on the graph's Ihara zeta function overall than one very short closed walk since the factor corresponding to the long walk is much closer to being one. So, if we replaced a single edge  $e$  with

a walk of length  $k$ , we would expect that those closed walks through  $e$  (which are now some multiple of  $k - 1$  longer) would have less effect on the Ihara zeta function than they originally had. Even though there are an infinite number of closed walks through  $e$ , we may still hope that if  $k$  runs off to infinity, the Ihara zeta function of the graph would run toward the Ihara zeta function of the graph with the edge  $e$  deleted. In fact, that is essentially the content of the following theorem:

**Theorem 4.1.** *Let  $G$  be a graph with edge set  $E$ . Let  $F \subseteq E$  and for each non-negative integer  $k$  define  $G(F, k)$  to be the graph  $G$  but with every edge contained in  $F$  divided into  $k + 1$  edges. Then for each  $u$  such that  $|u| < \frac{1}{q_{max}}$  where  $q_{max} + 1$  is the largest vertex degree in  $G$ ,*

$$\lim_{k \rightarrow \infty} \zeta_{G(F, k)}(u) = \zeta_{(G-F)}(u).$$

*Proof.* Let  $q_{max} + 1$  be the largest vertex degree in  $G$  and note that the largest vertex degree in  $G(F, k)$  for each  $k \geq 0$  is exactly  $q_{max} + 1$ . For any graph  $X$ , define  $P(X)$  to be the set of all primes  $[C]$  in  $X$ . Then for each  $k \geq 0$  and any  $u$  with magnitude less than  $\frac{1}{q_{max}}$ , the product  $\prod_{[C] \in P(G(F, k))} (1 - u^{\nu(C)})^{-1}$  converges to  $\zeta_{G(F, k)}(u)$  by Kotani and Sunada. Moreover, since the largest vertex degree in  $(G - F)$  is at most  $q_{max} + 1$ , the product  $\prod_{[C] \in P(G-F)} (1 - u^{\nu(C)})^{-1}$  converges to  $\zeta_{(G-F)}(u)$  for any  $u$  such that  $|u| < \frac{1}{q_{max}}$ .

For any walk  $C$  in  $G$ , define  $\eta(C)$  to be the number of edges of  $C$  (counting multiplicities) which are also in  $F$ . Define the set  $S = P(G) - P(G - F)$ . Now fix a  $u$  such that  $|u| < \frac{1}{q_{max}}$ . Then for each  $k \geq 0$ ,

$$\begin{aligned} \zeta_{G(F, k)}(u) &= \prod_{[C] \in P(G(F, k))} (1 - u^{\nu(C)})^{-1} = \prod_{[C] \in P(G)} (1 - u^{\nu(C) + k\eta(C)})^{-1} \\ &= \left( \prod_{[C] \in P(G-F)} (1 - u^{\nu(C)})^{-1} \right) \left( \prod_{[C] \in S} (1 - u^{\nu(C) + k\eta(C)})^{-1} \right) \\ &= \zeta_{(G-F)}(u) \prod_{[C] \in S} (1 - u^{\nu(C) + k\eta(C)})^{-1}. \end{aligned}$$

So, we want to show that  $\lim_{k \rightarrow \infty} \prod_{[C] \in \mathcal{S}} (1 - u^{\nu(C) + k\eta(C)})^{-1} = 1$  or equivalently that  $\lim_{k \rightarrow \infty} \sum_{[C] \in \mathcal{S}} \log(1 - u^{\nu(C) + k\eta(C)}) = 0$ . Note that

$$\begin{aligned}
\left| \sum_{[C] \in \mathcal{S}} \log(1 - u^{\nu(C) + k\eta(C)}) \right| &= \left| \sum_{[C] \in \mathcal{S}} \sum_{n \geq 1} \frac{u^{n(\nu(C) + k\eta(C))}}{n} \right| \\
&\leq \sum_{[C] \in \mathcal{S}} \sum_{n \geq 1} \frac{|u|^{n\nu(C)} |u|^{nk\eta(C)}}{n} \\
&\leq \sum_{[C] \in \mathcal{S}} \sum_{n \geq 1} \frac{|u|^{n\nu(C)} |u|^k}{n} \\
&\leq |u|^k \sum_{[C] \in \mathcal{S}} \sum_{n \geq 1} \frac{|u|^{n\nu(C)}}{n} \\
&= |u|^k \sum_{[C] \in \mathcal{S}} \log(1 - |u|^{\nu(C)}) \\
&= |u|^k \log \left( \prod_{[C] \in \mathcal{S}} (1 - |u|^{\nu(C)}) \right) \\
&= |u|^k \log \left( \frac{\zeta_{(G-F)}(|u|)}{\zeta_G(|u|)} \right) \rightarrow 0 \text{ as } k \rightarrow \infty.
\end{aligned}$$

Thus,  $\lim_{k \rightarrow \infty} \sum_{[C] \in \mathcal{S}} \log(1 - u^{\nu(C) + k\eta(C)}) = 0$ . Therefore,  $\lim_{k \rightarrow \infty} \zeta_{G(F,k)}(u) = \zeta_{(G-F)}(u)$  as desired.  $\square$

While this theorem does lend credence to our intuition, the region on which the limit seems to hold is too restrictive. In fact, it excludes the very region on which the result would be most interesting, namely the region in which the poles of the Ihara zeta function are located. As we shall see in Section 6.3, this result extends to everything within distance one of the origin, which is a gloriously large region for the study of the poles of the Ihara zeta function.



### 4.3 Ihara zeta functions of coverings

As previously mentioned, the Ihara zeta function of a disconnected graph is just product of the Ihara zeta functions of its components. Much more interestingly, if a graph  $H$  is a covering of a graph  $G$ , then  $\zeta_G(u)^{-1}$  divides  $\zeta_H(u)^{-1}$  (see Stark and Terras [12]). A *covering* of a graph  $G$  is a graph which is an  $n$ -covering of  $G$  for some  $n$ . An  $n$ -*covering*  $G_n$  of a graph  $G$  is a graph (which we will also refer to as  $G_n$ ) together with a function  $f$  from the vertices of  $G_n$  onto the vertices of  $G$  such that if  $u, v$  are adjacent vertices in  $G_n$  then  $f(u), f(v)$  are adjacent vertices in  $G$  and for every vertex  $v$  in  $G$ ,  $|f^{-1}(v)| = n$ . Also, for every vertex  $v$  in  $G_n$ ,  $\deg(v) = \deg(f(v))$ .

The following theorem gives us a method of constructing a bipartite two-covering of a graph so that the quotient of the Ihara zeta function of the covering and the Ihara zeta function of the original graph is easily recognizable.

**Theorem 4.2.** *Let  $G$  be a simple connected graph with adjacency matrix  $A$ . Define  $G_2$  to be the graph with adjacency matrix  $A_2 = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$ . Then  $G_2$  is a bipartite two-covering of  $G$  such that  $\zeta_G(u)\zeta_G(-u) = \zeta_{G_2}(u)$  where  $\zeta_G(u), \zeta_{G_2}(u)$  are the Ihara zeta functions of  $G, G_2$  respectively.*

*Proof.* Let  $M$  be the directed edge matrix of  $G$ . Let  $e_1, \dots, e_m$  and  $v_1, \dots, v_n$  be the edges and vertices of  $G$  respectively such that the  $e_i, v_j$  are indexed as in the definitions of  $M, A$ . Define a graph  $H$  with edges  $f_i, 1 \leq i \leq 2m$ , and vertices  $u_j, 1 \leq j \leq 2n$ , such that  $e_i = (v_j, v_k)$  implies  $f_i = (u_j, u_{n+k})$  and  $f_{n+i} = (u_{n+j}, u_k)$ .

Then  $H$  has adjacency matrix  $\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$  and directed edge matrix  $\begin{pmatrix} 0 & MJ \\ JM & 0 \end{pmatrix}$

where  $J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ . So, we can take the directed edge matrix of  $G_2$  (from the

theorem) to be  $M_2 = \begin{pmatrix} 0 & MJ \\ JM & 0 \end{pmatrix}$ .

Now define  $\tilde{M}_2 = \begin{pmatrix} M & M \\ 0 & -M \end{pmatrix}$  and  $S = \begin{pmatrix} I & 0 \\ J & J \end{pmatrix}$ . Then

$$\begin{aligned} M_2 S &= \begin{pmatrix} 0 & MJ \\ JM & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ J & J \end{pmatrix} = \begin{pmatrix} M & M \\ JM & 0 \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ J & J \end{pmatrix} \begin{pmatrix} M & M \\ 0 & -M \end{pmatrix} = S \tilde{M}_2. \end{aligned}$$

So,  $M_2$  is similar to  $\tilde{M}_2$  which implies  $\zeta_{G_2}(u)^{-1} = \det(I - M_2 u) = \det(I - \tilde{M}_2 u) = \det(I - Mu) \det(I + Mu) = \zeta_G(u)^{-1} \zeta_G(-u)^{-1}$ .  $\square$

*Remark 4.3.* Note that the theorem is trivially true if  $G$  itself is bipartite since  $G_2$  would then just be two copies of  $G$  and so  $\zeta_{G_2}(u) = (\zeta_G(u))^2$ .

In the next theorem, we will show how the radius of convergence of Ihara zeta functions relates to coverings. First however, we will define a directed edge operator which will free us from some painful indices:

**Definition 4.4.** For a graph  $G$ , we define the *directed edge operator*  $M$  on all functions  $f : E_d(G) \rightarrow \mathbb{C}$  by

$$Mf(e_0) = \sum_{\substack{e \in E_d(G) \\ e_0 \text{ feeds into } e \neq e_0}} f(e) \text{ for each } e_0 \in E_d(G). \quad (4.1)$$

An *eigenfunction* of  $M$  (with corresponding to eigenvalue  $\alpha$ ) is a function  $f : E_d(G) \rightarrow \mathbb{C}$  which is not identically zero such that  $Mf(e) = \alpha f(e)$  for all  $e \in E_d(G)$ .

Note that the directed edge operator is just a coordinate-free interpretation of a directed edge matrix. So, the directed edge operator  $M$  of a graph  $G$  has the following properties:

1. There exists an eigenfunction  $f$  of  $M$  with corresponding eigenvalue  $u_0^{-1}$  if and only if  $u_0$  is a pole of  $\zeta_G(u)$ .

2. Assuming  $G$  satisfies our simplifying assumptions and is not a cycle, there exists a positive eigenfunction  $f$  of  $M$  with corresponding eigenvalue  $u_0^{-1}$  if and only if  $u_0$  is the pole of smallest magnitude of  $\zeta_G(u)$  (that is, if  $u_0$  is the radius of convergence of  $\zeta_G(u)$ ).

The first property follows from Theorem 1.4. The second follows from Theorem 1.4 and Perron-Frobenius (see for instance Horn and Johnson [5]).

**Theorem 4.5.** *If  $G$  is a covering of a graph  $H$ , then  $\zeta_G(u), \zeta_H(u)$  have the same radius of convergence.*

*Proof.* Let a graph  $G$  with covering function  $f$  be a covering of a graph  $H$ . Let  $M_G, M_H$  be the directed edge operators corresponding to  $G, H$  respectively. Define  $u_0^{-1}$  to be the radius of convergence of  $H$ . Then by Property 2 above, there exists an eigenfunction  $g_H > 0$  such that  $M_H g_H(e) = u_0^{-1} g_H(e)$  for all  $e \in E_d(H)$ . Now define function  $g_G : E_d(G) \rightarrow \mathbb{C}$  such that  $g_G(e) = g_H(f(e))$  for all  $e \in E_d(G)$ . Note that  $g_G > 0$  and for all  $e_0 \in E_d(G)$ ,

$$\begin{aligned} M_G g_G(e_0) &= M_G g_H(f(e_0)) = \sum_{\substack{e \in E_d(G) \\ e_0 \text{ feeds into } e \neq e_0}} g_H(f(e)) \\ &= \sum_{\substack{e \in E_d(H) \\ f(e_0) \text{ feeds into } e \neq e_0}} g_H(e) = M_H g_H(f(e_0)) \\ &= u_0^{-1} g_H(f(e_0)) = u_0^{-1} g_G(e_0). \end{aligned}$$

So, by Property 2,  $u_0^{-1}$  is the radius of convergence of  $G$  □

**Corollary 4.6.** *If  $G, H$  are coverings of a graph  $K$ , then  $\zeta_G(u), \zeta_H(u)$  have the same radius of convergence.*

*Proof.* The corollary follows immediately from the theorem. □

It should be noted that the pole of smallest magnitude greater than the radius of convergence of a covering of a graph  $G$  may be smaller in magnitude than the

pole of smallest magnitude greater than the radius of convergence of  $G$  itself as illustrated by Figure 4.1. This is unfortunate since this pole, loosely speaking, informs us of how easy it is to get lost on a backtrackless walk in the graph.

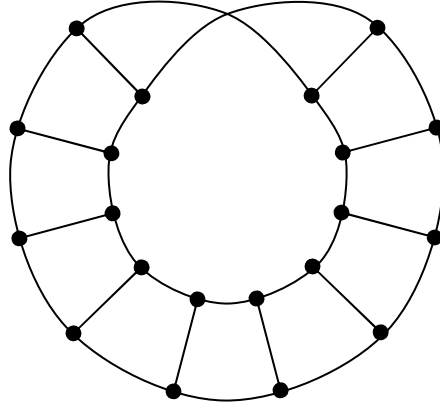


Figure 4.1 A 5-covering of  $K_4$  with poles in  $\{u : u_0 < |u| < |u_1|\}$  where  $u_0 = \frac{1}{2}$  is the radius of convergence of  $\zeta_{K_4}(u)$  and  $u_1 = \frac{1}{\sqrt{2}}$  is a pole of  $\zeta_{K_4}(u)$  of smallest magnitude greater than  $u_0$

## 4.4 Covering trees

A *covering tree* of a graph  $G$  is an infinite tree which satisfies the conditions for being a covering of  $G$  except that now  $|f^{-1}(v)| = \infty$ . So, for instance, Figure 4.2 shows part of the covering tree for  $K_4 - e$ .

By Corollary 4.6, every graph covered by the covering tree in Figure 4.2 has the same radius of convergence  $R$  since each of these graphs is a covering for  $K_4 - e$ . This of course begs the question of what significance the value  $R$  has to the covering tree itself. Theorem 4.14 below answers this question. First, however, we need some additional terminology, a couple borrowed theorems, and a lemma:

**Definition 4.7.** An irreducible matrix  $M$  is *primitive* if  $M^k > 0$  for some positive integer  $k$ .

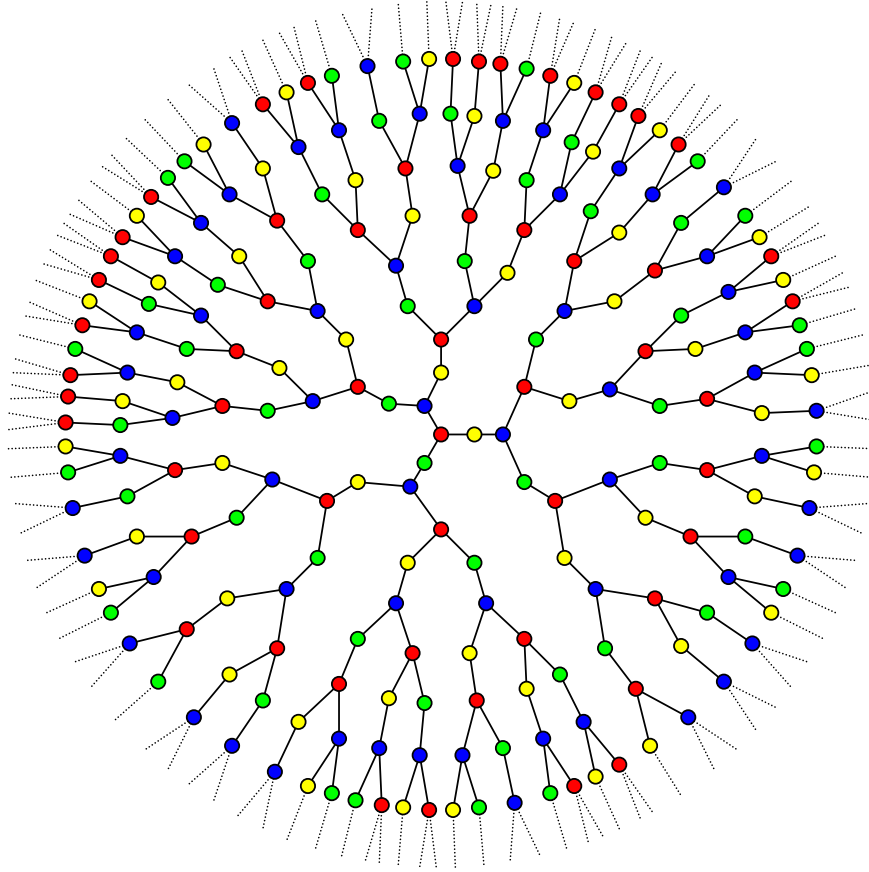


Figure 4.2 The covering tree of  $K_4 - e$

**Definition 4.8.** A digraph  $G$  is *strong* if there is a directed walk between every pair of vertices in  $G$ .

**Definition 4.9.** The *index of imprimitivity*  $d(G)$  of a digraph  $G$  is  $\gcd(S)$  where  $S = \{l : G \text{ contains a directed cycle of length } l\}$ .

**Theorem 4.10.** Let  $G$  be a digraph with adjacency matrix  $A$ . Then  $A$  is primitive if and only if  $G$  is strong and  $d(G) = 1$ .

*Proof.* This is a specialization of Theorem 3.2.3 in [7]. □

**Theorem 4.11.** Let  $T$  be a primitive matrix. Define  $\lambda_0$  to be the largest positive eigenvalue of  $T$  with positive right and left eigenvectors  $x, y^T$  chosen so that  $y^T x = 1$ .

Define  $\lambda_1$  to be an eigenvalue of  $T$  of largest magnitude less than  $\lambda_0$  such that no eigenvalue of equal magnitude has greater multiplicity. Then as  $k \rightarrow \infty$ ,  $T^k = \lambda_0^k xy^T + O(k^{m_2-1}|\lambda_1|^k)$  elementwise where  $m_2$  is the multiplicity of  $\lambda_1$ .

*Remark 4.12.* The existence of  $\lambda_0$  and its corresponding positive eigenvectors in the theorem above is guaranteed by Perron-Frobenius.

*Proof.* This is essentially Theorem 1.2 in [11]. □

**Lemma 4.13.** *Let  $G$  be a graph (which satisfies our simplifying assumptions). Further, assume  $\gcd(\{\nu(C) : [C] \text{ is a prime of } G\}) = 1$ . Let  $M$  be an edge matrix of  $G$ . Define  $u_0$  be the radius of convergence of  $\zeta_G(u)$  and  $u_1$  to be pole of  $\zeta_G(u)$  of smallest magnitude greater than  $u_0$  such that no pole of equal magnitude has greater multiplicity. Then as  $k \rightarrow \infty$ ,  $M^k = u_0^{-k} gh^T + O(k^{m_2-1}|u_1|^{-k})$  elementwise where  $m_2$  is the multiplicity of  $u_1$  and  $g, h^T$  are positive right and left eigenvectors corresponding to the eigenvalue  $u_0^{-1}$  of  $M$  chosen such that  $h^T g = 1$ .*

*Proof.* By Theorem 4.11, we need only show that  $T$  is primitive. Define  $H$  to be the digraph with adjacency matrix  $M$ . By Theorem 4.10,  $W$  is primitive if and only if  $H$  is strong and  $d(H) = 1$ .

First we will show that  $H$  is strong. This is equivalent to showing that for any two directed edges  $e, f$  in  $G$ , there is a directed walk which begins with  $e$  and ends with  $f$ . Note that since  $\gcd(\{\text{length}(C) : [C] \text{ is a prime of } G\}) = 1$ ,  $G$  is not a cycle. Since  $G$  is a connected graph which is not a cycle and  $G$  has no vertices of degree one, there is a directed walk beginning with  $e$  and ending with  $f$  for any two directed edges  $e, f$  in  $G$ . (This is due to the fact that these properties guarantee that we have room in the graph to turn around and come back to an edge going in the opposite direction without backtracking.) Thus,  $H$  is strong.

Now we will show that  $d(H) = 1$ . This is equivalent to showing that  $g = 1$  where  $g$  is defined to be  $\gcd(\{\nu(C) : C \text{ is a closed backtrackless, tailless directed walk in } G \text{ with no repeated directed edges}\})$ . We shall prove this by contradiction,

so suppose  $g > 1$ . Since  $\gcd(\{\nu(C) : [C] \text{ is a prime of } G\}) = 1$ , there exists some prime  $[C_0]$  in  $G$  whose length is not divisible by  $g$ . Note  $[C_0]$  must have some repeated directed edge  $e_0$ . Let  $[C_2], [C_3]$  be the closed backtrackless, tailless directed walks such that  $[C_2C_3] = [C_0]$  and  $C_3$  is the walk from the first appearance of  $e_0$  in  $C_0$  to the second appearance of  $e_0$  in  $C_0$ . Both  $[C_2]$  and  $[C_3]$  contain fewer repeated directed edges than  $[C_0]$ . Repeat this process until we have a set of closed backtrackless, tailless directed walks  $B_0, \dots, B_t$  such that  $[B_0 \cdots B_t] = [C_0]$  and none of the  $B_i$  contain repeated directed edges. Each  $\nu(B_i)$  is divisible by  $g$  which implies  $\nu(C_0)$  is divisible by  $g$ , a contradiction. Thus,  $d(H) = 1$ .

Since  $H$  is strong and  $d(H) = 1$ , the result holds.  $\square$

**Theorem 4.14.** *Let  $T$  be a covering tree for some finite graph  $G$  (which satisfies our simplifying assumptions). Further, assume  $\gcd(\{\nu(C) : [C] \text{ is a prime of } G\}) = 1$ . Define  $R$  to be the radius of convergence of  $\zeta_G(u)$ . Fix a vertex  $x \in V(T)$ . Then*

$$R^{-1} = \lim_{d \rightarrow \infty} \sqrt[d-1]{|\{y \in V(T) : d(x, y) = d\}|}$$

where  $d(x, y)$  is the distance in  $T$  between the vertices  $x$  and  $y$ .

*Proof.* Let  $T, G, R$  be as in the theorem. Define  $f$  to be the covering function. Also, define  $M$  to be the directed edge matrix of  $G$  respectively. Since  $G$  satisfies our simplifying assumptions, there are vectors  $g, h > 0$  such that  $Mg = R^{-1}g$  and  $h^T M = R^{-1}h^T$ . Assume  $h^T g = 1$ .

Fix a vertex  $x \in V(T)$ . For any vertex  $y$  adjacent to  $x$  in  $T$ , define

$$V(x, y, d) = \{v \in V(T) : d = d(v, x) = d(v, y) + 1\}.$$

So,  $V(x, y, d)$  is the set of all vertices distance  $d$  from  $x$  in the direction of  $y$  (that is, every backtrackless walk from  $x$  to a vertex of  $V(x, y, d)$  starts with the directed edge  $(x, y)$  and is of length  $d$ ).

For any each directed edge  $(x, y)$ , define  $i_{xy}$  to be the index of the row corresponding to  $(x, y)$  according to the labeling used to form  $M$ . Define  $f_{xy}$  to be

a vector of zeros which has a 1 in the  $i_{xy}$ th position. Note then that for any  $y$  adjacent to  $x$ ,

$$|V(x, y, d)| = f_{xy}^T M^{d-1} \vec{1}$$

where  $\vec{1}$  is the constant vector of ones. By the previous lemma,  $M^{d-1} \rightarrow R^{-d} g h^T$  elementwise as  $d \rightarrow \infty$ . So, as  $d \rightarrow \infty$ ,

$$|V(x, y, d)| = f_{xy}^T M^{d-1} \vec{1} \sim f_{xy}^T R^{1-d} g h^T \vec{1} = R^{1-d} g_{i_{xy}} |h| = c_{xy} R^{1-d}$$

where  $c_{xy}$  is defined to be the positive constant  $g_{i_{xy}} |h|$ . Thus,

$$\begin{aligned} |\{y \in V(T) : d(x, y) = d\}| &= \\ \sum_{y \text{ adjacent to } x} |V(x, y, d)| &\sim \sum_{y \text{ adjacent to } x} c_{xy} R^{1-d} = c R^{1-d} \end{aligned}$$

where  $c$  is defined to be the positive constant  $\sum_{y \text{ adjacent to } x} c_{xy}$ . So,

$$\lim_{d \rightarrow \infty} \frac{|\{y \in V(T) : d(x, y) = d + 1\}|}{|\{y \in V(T) : d(x, y) = d\}|} = R^{-1}$$

which (by Theorem 3.37 in Rudin [10]) implies

$$R^{-1} = \lim_{d \rightarrow \infty} \sqrt[d-1]{|\{y \in V(T) : d(x, y) = d\}|}$$

since  $\sqrt[d-1]{c^{-1}} \rightarrow 1$  as  $d \rightarrow \infty$ . □

*Remark 4.15.* Another interpretation of the value

$$|\{y \in V(T) : d(x, y) = d\}|$$

is the number of backtrackless walks of length  $d$  in  $G$  (the graph covered by  $T$ ) starting at the vertex  $f(x)$  where  $f$  is the covering function.

This result can be easily generalized further as demonstrated by the following corollary:



**Corollary 4.16.** *Let  $T$  be a covering tree for some graph  $G$  (which satisfies our simplifying assumptions). Further, assume  $\gcd(\{\nu(C) : [C] \text{ is a prime of } G\}) = 1$ . Define  $R$  to be the radius of convergence of  $\zeta_G(u)$ . Let  $X \subseteq E_d(T)$  such that  $|X| < \infty$ . Then*

$$R^{-1} = \lim_{d \rightarrow \infty} \sqrt[d-1]{|\{v : d = d(v, x) = d(v, y) + 1 \text{ and } (x, y) \in X\}|}.$$

*Proof.* Take  $g, h$  to be right and left eigenvectors as in the proof of the theorem. Also, define  $V(x, y, d)$  and  $c_{xy}$  as before except that here we are not fixing  $x$ . Note that

$$\{v : d = d(v, x) = d(v, y) + 1 \text{ and } (x, y) \in X\} = \bigsqcup_{(x,y) \in X} V(x, y, d).$$

So,

$$\begin{aligned} |\{v : d = d(v, x) = d(v, y) + 1 \text{ and } (x, y) \in X\}| &= \sum_{(x,y) \in X} |V(x, y, d)| \\ &= \sum_{(x,y) \in X} f_{xy}^T M^{d-1} \vec{1} \sim \sum_{(x,y) \in X} f_{xy}^T R^{1-d} g h^T \vec{1} \\ &= \sum_{(x,y) \in X} R^{1-d} g_{i_{xy}} |h| = \sum_{(x,y) \in X} c_{xy} R^{1-d} = c R^{1-d} \end{aligned}$$

where of course we define  $c$  to be the positive constant  $\sum_{(x,y) \in X} c_{xy}$ . Therefore,

$$R^{-1} = \lim_{d \rightarrow \infty} \sqrt[d-1]{|\{v : d = d(v, x) = d(v, y) + 1 \text{ and } (x, y) \in X\}|}.$$

□

# 5

## Eigenfunctions of directed edge operators

We saw in Chapter 2 that information about a graph is encoded in the poles of its Ihara zeta function, which are just the reciprocals of the eigenvalues of the directed edge operator. In this chapter, we consider how a graph's structure is reflected in the corresponding eigenfunctions.

### 5.1 Eigenfunctions corresponding to $\lambda \neq 1$

For any function  $f : E_d(G) \rightarrow \mathbb{C}$  and any walk  $\mathcal{W}$  in  $G$ , we will define

$$f(\mathcal{W}) = \sum_{\substack{e \in E_d(G) \\ e \text{ in } \mathcal{W}}} f(e)$$

where the sum includes any repetition of directed edges in  $\mathcal{W}$ .

**Theorem 5.1.** *Let  $G$  be a graph with directed edge operator  $M$ . Let  $(\lambda, f)$ ,  $\lambda \neq 1$ , be an eigenvalue and eigenfunction pair of  $M$ . Then for any closed backtrackless walk  $C$  in  $G$ ,  $f(C) = f(C^{-1})$ .*

*Proof.* Fix an eigen-pair  $(\lambda, f)$  of  $M$ . Then

$$\lambda f(e_0) = \sum_{\substack{e \neq e_0^{-1} \\ t(e_0)=s(e)}} f(e)$$

for each  $e_0 \in E_d(G)$ . So,

$$\begin{aligned} \lambda f(e_0) + f(e_0^{-1}) &= f(e_0^{-1}) + \sum_{\substack{e \neq e_0^{-1} \\ t(e_0)=s(e)}} f(e) \\ &= \sum_{t(e_0)=s(e)} f(e). \end{aligned}$$

Thus, for any two edges  $a, b$  such that  $t(a) = s(b)$  and  $a \neq b^{-1}$ ,

$$\lambda f(a) + f(a^{-1}) = \lambda f(b^{-1}) + f(b).$$

Let  $C = \{e_1, e_2, \dots, e_n\}$  be a closed backtrackless walk in  $G$  where the  $e_i$  are oriented edges. Then

$$\sum_{i=1}^n (\lambda f(e_i) + f(e_i^{-1})) = \sum_{i=1}^n (\lambda f(e_i^{-1}) + f(e_i))$$

which implies

$$(\lambda - 1)f(C) = (\lambda - 1)f(C^{-1}).$$

Since  $\lambda \neq 1$ ,  $f(C) = f(C^{-1})$  as desired.  $\square$

*Remark 5.2.* Note that it is not generally the case that  $f(\mathcal{W}) = f(\mathcal{W}^{-1})$  for an arbitrary walk  $\mathcal{W}$  in  $G$ . In particular, it is not necessarily true that  $f(e) = f(e^{-1})$  for  $e \in E_d(G)$ .

## 5.2 Eigenfunctions corresponding to $\lambda = 1$

Next we consider those eigenfunctions of the directed edge operator which were excluded from Theorem 5.1. After a series of lemmas, we will make explicit the

relationship between  $\pi(G, v)$  (the fundamental group of the graph  $G$ ) and the eigenfunctions of the directed edge operator corresponding to  $\lambda = 1$ .

The following lemma defines a family of functions (a subset of which, we will show in Lemma 5.4, are eigenfunctions of the directed edge operator) and reveals some of their useful properties.

**Lemma 5.3.** *For any walk  $\mathcal{W}$  in a graph  $G$ , define  $f_{\mathcal{W}} : E_d(G) \rightarrow \mathbb{C}$  by  $f_{\mathcal{W}}(e) = \eta_{\mathcal{W}}(e) - \eta_{\mathcal{W}}(e^{-1})$  where  $\eta_{\mathcal{W}}(e)$  is the number of times  $e$  appears as an edge of  $\mathcal{W}$ . Then*

$$(i) \quad f_{\mathcal{W}}(e) = -f_{\mathcal{W}}(e^{-1}),$$

$$(ii) \quad f_{\mathcal{W}} = -f_{\mathcal{W}^{-1}},$$

$$(iii) \quad f_{\mathcal{W}_1\mathcal{W}_2} = f_{\mathcal{W}_1} + f_{\mathcal{W}_2} \text{ for any walks } \mathcal{W}_1, \mathcal{W}_2 \text{ for which the product } \mathcal{W}_1\mathcal{W}_2 \text{ is defined,}$$

$$(iv) \quad f_{\mathcal{W}^k} = kf_{\mathcal{W}} \text{ for every integer } k,$$

$$(v) \quad f_{\mathcal{W}_1^{k_1}\mathcal{W}_2^{k_2}\dots\mathcal{W}_j^{k_j}} = k_1f_{\mathcal{W}_1} + k_2f_{\mathcal{W}_2} + \dots + k_jf_{\mathcal{W}_j} \text{ for integers } k_1, k_2, \dots, k_j \text{ and any walks } \mathcal{W}_1, \mathcal{W}_2, \dots, \mathcal{W}_j \text{ for which the product } \mathcal{W}_1\mathcal{W}_2 \dots \mathcal{W}_j \text{ is defined.}$$

*Proof.* By definition,

$$f_{\mathcal{W}}(e) = \eta_{\mathcal{W}}(e) - \eta_{\mathcal{W}}(e^{-1}) = -(\eta_{\mathcal{W}}(e^{-1}) - \eta_{\mathcal{W}}(e)) = -f_{\mathcal{W}}(e^{-1}).$$

So, (i) holds.

Note also that

$$f_{\mathcal{W}}(e) = \eta_{\mathcal{W}}(e) - \eta_{\mathcal{W}}(e^{-1}) = \eta_{\mathcal{W}^{-1}}(e^{-1}) - \eta_{\mathcal{W}^{-1}}(e) = f_{\mathcal{W}^{-1}}(e^{-1}).$$

Combined with (i), this implies (ii).

Part (iii) follows from the fact that

$$\eta_{f_{\mathcal{W}_1\mathcal{W}_2}}(e) = \eta_{f_{\mathcal{W}_1}}(e) + \eta_{f_{\mathcal{W}_2}}(e).$$

Together, (i) and (iii) imply (iv). Part (v) follows from repeated application of (iii) and (iv).  $\square$

**Lemma 5.4.** *For any prime  $[C]$  in  $G$ ,  $(1, f_C)$  where  $f_C$  is as in Lemma 5.3 is an eigen-pair of the directed edge operator of  $G$  if  $f_C(e) \neq f_C(e^{-1})$  for some  $e \in E_d(G)$ .*

*Proof.* Let  $M$  be the directed edge operator of  $G$ . Let  $[C]$  be a prime in  $G$ . Let  $e_0 \in E_d(G)$ . Then

$$\begin{aligned}
-f_C(e_0) + Mf_C(e_0) &= -f_C(e_0) + \sum_{\substack{e \neq e_0^{-1} \\ t(e_0)=s(e)}} f_C(e) \\
&= f_C(e_0^{-1}) + \sum_{\substack{e \neq e_0^{-1} \\ t(e_0)=s(e)}} f_C(e) \\
&= \sum_{t(e_0)=s(e)} f_C(e) \\
&= \sum_{t(e_0)=s(e)} (\eta_C(e) - \eta_C(e^{-1})) \\
&= \left( \sum_{t(e_0)=s(e)} \eta_C(e) \right) - \left( \sum_{t(e_0)=s(e)} \eta_C(e^{-1}) \right) \\
&= \left( \sum_{t(e_0)=s(e)} \eta_C(e) \right) - \left( \sum_{t(e_0)=t(e)} \eta_C(e) \right) \\
&= 0
\end{aligned}$$

since  $\sum_{t(e_0)=s(e)} \eta_C(e)$  is the number of times  $C$  enters the vertex  $t(e_0)$  and  $\sum_{t(e_0)=s(e)} \eta_C(e)$  is the number of times  $C$  leaves the vertex  $t(e_0)$  (and these two values must be equal since  $C$  is closed). So, the result holds.  $\square$

In the following lemma, we show that we can choose a linearly independent basis for the space of eigenfunctions of the directed edge operator of a graph from amongst the eigenfunctions which satisfy the conditions of Lemma 5.4. Moreover, we relate our chosen basis to a particular generating set for the fundamental group of the graph.

**Lemma 5.5.** *Let  $S$  be a spanning tree of a graph  $G$ . Let  $r = |E(G)| - |V(G)| + 1$  and  $\{e_1, \dots, e_r\} = E(G) - E(S)$ . Arbitrarily orient the undirected edges  $e_1, \dots, e_r$ . Fix a vertex  $v$  of  $G$ . For each  $i$ ,  $1 \leq i \leq r$ , define  $C_i$  to be the unique closed back-trackless (but not necessarily tailless) walk which starts and ends with  $v$  and whose directed edges are taken from  $E_d(S) \cup \{e_i\}$ . Then  $\{C_1, \dots, C_r\}$  is a minimal generating set for  $\pi(G, v)$ , the fundamental group of  $G$ , and  $\{f_{C_1}, \dots, f_{C_r}\}$  (where  $f_w$  is as in Lemma 5.4) is a linearly independent basis for the space of eigenfunctions of the directed edge operator of  $G$  corresponding to the eigenvalue 1.*

*Proof.* Let  $M$  be the directed edge operator of  $G$ . Since  $f_{C_i}(e_i) = 1$  for each  $i$ ,  $\{f_{C_1}, \dots, f_{C_r}\}$  is a set of eigenfunctions of  $M$  by Lemma 5.4. Also, since  $f_{C_i}(e_j) = 0$  if  $i \neq j$ , the eigenfunctions are linearly independent. By Hashimoto [4], the dimension of the space of eigenfunctions of the directed edge matrix of  $G$  corresponding to the eigenvalue 1 is  $r$ . By the argument in the discussion of multipath zeta functions in Stark and Terras [13],  $\{C_1, \dots, C_r\}$  is a minimal generating set for  $\pi(G, v)$ . Thus, the result holds.  $\square$

We will now extend Lemma by relaxing the restrictions on the minimal generating set for the fundamental group.

**Theorem 5.6.** *Let  $v$  be a fixed vertex of a graph  $G$  and let  $\{C_1, \dots, C_r\}$  be a minimal generating set for  $\pi(G, v)$ . Then  $\{f_{C_1}, \dots, f_{C_r}\}$  is a linearly independent basis for the space of eigenfunctions of the directed edge matrix of  $G$  corresponding to the eigenvalue 1.*

*Proof.* Let  $\{B_1, \dots, B_r\}$  be the minimal generating set for  $\pi(G, v)$  described in Lemma 5.2. Then by Lemma 5.2,  $\{f_{B_1}, \dots, f_{B_r}\}$  is a linearly independent basis for the eigenfunctions of the directed edge matrix of  $G$  corresponding to the eigenvalue 1. Since  $\{C_1, \dots, C_r\}$  is also a generating set for  $\pi(G, v)$ ,

$$B_1 = \prod_{i=1}^j C_{\alpha_i}^{k_i}$$

for some integers  $k_1, \dots, k_j$  and where  $1 \leq \alpha_i \leq r$ . By Lemma 5.3 then,

$$f_{B_1} = \sum_{i=1}^j k_i f_{C_{\alpha_i}} = \sum_{i=1}^r \sum_{\substack{j \\ \alpha_j=i}} k_j f_{C_{\alpha_j}} = \sum_{i=1}^r \left( f_{C_i} \sum_{\substack{j \\ \alpha_j=i}} k_j \right).$$

So,  $f_{B_1}$  is a linear combination of  $f_{C_1}, \dots, f_{C_r}$ . Since the indexing of  $B_1, \dots, B_r$  was arbitrary, this implies that every element of  $\{B_1, \dots, B_r\}$  is a linear combination of  $f_{C_1}, \dots, f_{C_r}$ . Thus,  $\{f_{C_1}, \dots, f_{C_r}\}$  too is a basis for the eigenfunctions of the directed edge matrix of  $G$  corresponding to the eigenvalue 1. The eigenfunctions  $f_{C_1}, \dots, f_{C_r}$  must also be linearly independent since again the dimension of the space of eigenfunctions corresponding to the eigenvalue 1 is  $r$ .  $\square$

Due to the relationship described in the theorem, it is possible for properties of the eigenfunctions of the directed edge operator to inform us of properties of a minimal generating set for the fundamental group as illustrated by the following corollary:

**Corollary 5.7.** *If  $C$  is an element of a minimal generating set for  $\pi(G, v)$ , then  $C$  crosses some edge of  $G$  in one direction more often than in the other.*

*Proof.* Suppose  $C$  is an element of a minimal generating set for  $\pi(G, v)$  yet does not cross some edge of  $G$  in one direction more often than the other. Then  $f_C(e) = 0$  for all  $e \in E_d(G)$ . This is a contradiction however since by Theorem 5.6,  $f_C$  is an element of a linearly independent basis for the eigenfunctions of the directed edge operator of  $G$  corresponding to the eigenvalue 1. Therefore, the result holds.  $\square$

*Remark 5.8.* Note that the condition given in Corollary 5.7 is necessary but not sufficient for a closed walk through the vertex  $v$  to be an element of some minimal generating set for  $\pi(G, v)$ .

We include the following corollary for purposes of comparison to Theorem 5.1.

**Corollary 5.9.** *Let  $G$  be a graph with directed edge operator  $M$ . Let  $f$  be an eigenfunction of  $M$  with eigenvalue 1. Then  $f(e) = -f(e^{-1})$ .*

*Proof.* Let  $\{f_{C_1}, \dots, f_{C_r}\}$  be as in the statement of Theorem 5.6. By Lemma 5.3,  $f_{C_i}(e) = -f_{C_i}(e^{-1})$  for each  $i$ ,  $1 \leq i \leq r$  and each  $e \in E_d(G)$ . So,  $f(e) = -f(e^{-1})$  since  $f$  is a linear combination of  $f_{C_1}, \dots, f_{C_r}$  by Theorem 5.6.  $\square$



# 6

## Specializations of the multiedge zeta function

### 6.1 Multiedge zeta functions

In [13], Stark and Terras described an extremely versatile generalization of the Ihara zeta function called the multiedge zeta function.

**Definition 6.1.** Arbitrarily orient the edges  $e_1, e_2, \dots, e_{|E|}$  of an undirected graph  $G$  and let  $e_{|E|+i} = e_i^{-1}$  for all  $i$ ,  $1 \leq i \leq |E|$ . Then the  $2|E| \times 2|E|$  matrix  $W$  defined by

$$(W)_{ij} = \begin{cases} w_{ij} & \text{if } t(e_i) = s(e_j) \text{ and } s(e_i) \neq t(e_j), \\ 0 & \text{otherwise.} \end{cases}$$

where the  $w_{ij}$  are complex variables is a *multiedge zeta function matrix* of  $G$ .

**Definition 6.2.** For a prime  $[C]$  where  $C = f_1, f_2, \dots, f_k$  and the  $f_i$  are directed edges, the *multiedge norm* of  $C$  is

$$\mathbb{N}_E(C) = w(f_k, f_1) \prod_{i=1}^{s-1} w(f_i, f_{i+1}).$$

**Definition 6.3.** The *multiedge zeta function* of a graph  $G$  is

$$\zeta_E(W, G) = \prod_{[C]} (1 - \mathbb{N}_E(C))^{-1}.$$

As noted in Stark and Terras [13], in the special case where each of the variables  $w_{ij}$  is  $u$ , the multiedge zeta function reduces to the Ihara zeta function. Also, when each of the variables  $w_{ij}$  is 1, the multiedge zeta function is the directed edge matrix of the graph  $G$ . In the next section, we will define the directed edge matrix of a directed graph by making slightly different choices for our  $w_{ij}$ . Another useful specialization of the multiedge zeta function will be used in the proof of Theorem 6.6.

## 6.2 Ihara zeta functions of digraphs

In [6], Kotani and Sunada define the Ihara zeta function of a digraph and then define the Ihara zeta function of an undirected graph as the Ihara zeta function of its oriented line graph (which is itself a digraph). The multiedge zeta function can also be specialized (as in Mizuno and Sato [9]) to produce the Ihara zeta functions of digraphs which are not necessarily the oriented line graphs of undirected graphs.

We begin by extending what we mean by a *prime*  $[C]$ . Let  $G$  be a directed graph and let  $C$  be a closed walk  $\{e_1, e_2, \dots, e_n\}$  in  $G$  such that  $C^2$  (that is, the product  $PP$ ) contains no backtracks and  $C$  is not  $C_0^j$  for any closed walk  $C_0$  and integer  $j \geq 2$ . (A *backtrack* now is defined to be edges  $e_i, e_{i+1}$  of the walk  $C$  such that  $e_i = e_{i+1}^{-1}$ .) Then the prime  $[C]$  is the equivalence class  $\{\{e_1, e_2, \dots, e_n\}, \{e_2, e_3, \dots, e_n, e_1\}, \{e_3, e_4, \dots, e_n, e_1, e_2\}, \dots, \{e_n, e_1, \dots, e_{n-2}, e_{n-1}\}\}$ .

Note that if we think of an undirected graph as a directed graph by replacing each of its undirected edges with a bidirected edge, then the primes in this extended definition correspond to primes of the same length under the original definition. So, as our preliminary extended definition of the Ihara zeta function, we simply use the product in Definition 1.1.

Now we will extend our definition of a directed edge matrix:

**Definition 6.4.** Let  $G$  be a directed graph with underlying graph  $H$ . Let  $W$  be the multiedge zeta function matrix of  $H$ . Then the *directed edge matrix*  $M$  of  $G$  is the matrix  $W$  where we take

$$w_{ij} = \begin{cases} 1 & \text{if } e_i \text{ feeds into } e_j \neq e_i^{-1} \text{ and } e_i, e_j \in E_d(G) \\ 0 & \text{otherwise.} \end{cases}$$

Note that all directed edge matrices under the original definition are still directed edge matrices of their corresponding undirected graphs (when these graphs are viewed as directed graphs). However, for directed graphs in general, we have lost some of the symmetry of the sub-matrices in Remark 1.5. Also note that for computational purposes, it may be more efficient to consider a modified directed edge matrix (which is another step away from the special properties listed in Remark 1.5) as discussed in Section 9.3.

We also need to revisit the simplifying assumption that our graphs contain no vertices of degree one. Our new requirement (which agrees with the original when restricted to undirected graphs) is that every directed edge in a digraph is contained in some prime. This is equivalent to requiring that for each directed edge  $e \in E_d(G)$ , there is a directed walk from the the terminal vertex of  $e$  to the start vertex of  $e$  which neither starts nor ends with  $e^{-1}$ .

As with the Ihara zeta functions of undirected graphs, we would like something other than the (usually poorly behaved and infinite) product in our preliminary definition to work with.

Fortunately, by Stark and Terras (see comment on page 134 of [13] referring to proof in [12]), Theorem 1.4 holds for both directed and undirected graphs with our new definitions. We simply specialize the variables in the multiedge zeta function matrix  $W$  so that  $W = Mu$  (where  $M$ , formed from  $W$  as indicated above, is a directed edge matrix of a directed or undirected graph  $G$  with underlying graph

$H$ ) and note, for any prime  $[C]$  in the underlying graph  $H$ ,

$$\mathbb{N}_E(C) = \begin{cases} u^{\nu(C)} & \text{if } [C] \text{ is also a prime in } G \\ 0 & \text{otherwise.} \end{cases}$$

That is, the norm selects for primes in the underlying graph  $H$  in which edges are traversed in accordance with the directions of the directed edges of  $G$ . So,

$$\prod_{[C] \text{ in } H} (1 - \mathbb{N}_E(C))^{-1} = \prod_{[C] \text{ in } G} (1 - \mathbb{N}_E(C))^{-1} = \prod_{[C] \text{ in } G} (1 - u^{\nu(C)})^{-1}$$

Thus, the preliminary product definition of the Ihara zeta function of a directed or undirected graph is equal to  $\det(I - Mu)^{-1}$  for  $u$  sufficiently small. Therefore, we will define the Ihara zeta function of a directed or undirected graph to be  $\det(I - Mu)^{-1}$ .

Since  $u(\log \zeta)' = \sum_{m=1}^{\infty} N_m u^m = \sum_{m=g}^{\infty} N_m u^m$  where  $N_m$  now is the number of closed directed walks of length  $m$  without backtracking or tails (where closed walks of the same equivalence class which have different start vertices are counted as different) still holds for digraphs (see Kotani and Sunada [6]), so does Theorem 3.1 and its corollary (where girth in a digraph is defined to be the length of the shortest directed cycle).

Unfortunately, for digraphs in general, we do not have anything like Theorem 1.2 (which holds for undirected graphs or, equivalently, directed graphs in which every edge is bidirected). By Mizuno and Sato [8] however, in the special case in which  $G$  is a directed graph with no bidirected edges,  $\zeta_G(u)^{-1} = \det(I - Au)$  where  $A$  is the adjacency matrix of  $G$ . Here is an alternate proof of this result:

*Proof.* Let  $G$  be a directed graph with no bidirected edges. Let  $H$  be the graph underlying  $G$ . Let vertices  $v_1, v_2, \dots, v_{|V|}$  be the vertices of  $G$  and let  $e_1, e_2, \dots, e_{|E(H)|}$  be the edges used in forming a directed edge matrix of  $H$ . Define  $|V(H)| \times |E(H)|$  matrices  $S, T$  such that

$$(S)_{ij} = \begin{cases} 1 & v_i \text{ is the start vertex of } e_j \text{ in } G, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$(T)_{ij} = \begin{cases} 1 & v_i \text{ is the terminal vertex of } e_j \text{ in } G, \\ 0 & \text{otherwise.} \end{cases}$$

Note then that if  $A, M$  are the adjacency matrix and directed edge matrix respectively of  $G$ , then  $A = ST^T$  and  $M = TS^T$ . (If we allowed  $G$  to contain bidirected edges, then  $A$  would still be  $ST^T$ , but  $M$  would not be  $TS^T$ .) Thus,

$$\begin{pmatrix} I & 0 \\ T^T & I \end{pmatrix} \begin{pmatrix} I & Su \\ 0 & I - Mu \end{pmatrix} = \begin{pmatrix} I & Su \\ T^T & I \end{pmatrix} = \begin{pmatrix} I - Au & Su \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ T^T & I \end{pmatrix}$$

which implies that  $\det(I - Mu) = \det(I - Au)$ .  $\square$

This gives us a means of bounding the degree of the reciprocal of the Ihara zeta function in this special case:

**Theorem 6.5.** *Let  $G$  be a directed graph with no bidirected edges. Then the  $g \leq \deg(\zeta_G(u)^{-1}) \leq |V|$  where  $g$  is the girth of  $G$ .*

*Proof.* Note that degree of the polynomial  $\det(I - Au)$  is at most  $|V|$  (since  $A$  is a  $|V| \times |V|$  matrix). So,  $\deg(\zeta_G(u)^{-1}) \leq |V|$ . By Theorem 3.1 (which, as previously mentioned, extends to digraphs),  $\deg(\zeta_G(u)^{-1}) \geq g$  where  $g$  is the girth of  $G$ .  $\square$

Figure 6.1 shows that the bounds given in the theorem are achievable.

### 6.3 Revisiting Theorem 4.1

Recall that Theorem 4.1 states that if we replace an edge  $e$  of an undirected graph  $X$  with a walk of length  $k$ , then as  $k \rightarrow \infty$ , the limit of the Ihara zeta functions of the resulting graphs is the Ihara zeta function of  $X - e$  within a tragically small neighborhood of the origin. In this section, we not only extend this result to include the promised gloriously large region of radius one, but also consider directed graphs (which may contain bidirected edges) in which the edge we wish to delete may or may not be bidirected.

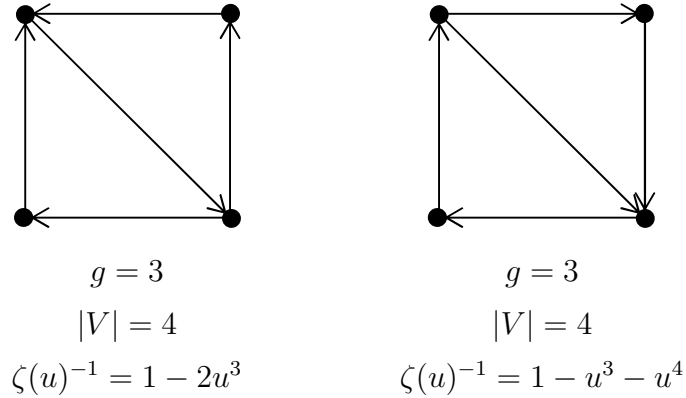


Figure 6.1 Examples illustrating that both the lower and upper bounds given in Theorem 6.5 for the degree of the Ihara zeta function of a digraph are achievable

**Theorem 6.6.** *Let  $Y$  be a directed graph possibly with bidirected edges (also, allow  $Y$  to break all simplifying assumptions except that  $Y$  must be finite) and let  $e \in E_d(Y)$ . Define  $X$  to be the graph such that  $V(X) = V(Y)$  and  $E_d(X) = E_d(Y) - \{e, e^{-1}\}$ . Define  $X_k$  to be the graph obtained by adding a walk  $\mathcal{W}$  of length  $k+1$  to  $X$  from  $s(e)$  to  $t(e)$ . If  $e^{-1} \notin E_d(Y)$  then take  $\mathcal{W}$  to be a directed walk; otherwise, take  $\mathcal{W}$  to be an undirected walk (that is, a walk composed of bidirected edges). Then*

$$\zeta_{X_k}(u)^{-1} = \zeta_X(u)^{-1} - p_1(u)u^k + p_2(u)u^{2k}$$

where

$$p_1(u) = 2\zeta_X^{-1} - \zeta_{Y-\{e\}}^{-1} - \zeta_{Y-\{e^{-1}\}}^{-1}$$

and

$$p_2(u) = \zeta_X^{-1} - \zeta_{Y-\{e\}}^{-1} - \zeta_{Y-\{e^{-1}\}}^{-1} + \zeta_Y^{-1}.$$

*Proof.* Let  $X$ ,  $Y$ ,  $e$ ,  $X_k$  be as in the theorem. Let  $W_k$  be the multiedge zeta function of  $Y$  specialized such that

$$(W_k)_{ij} = \begin{cases} u & \text{if } e_i \text{ feeds into } e_j \neq e_i^{-1}, e \neq e_i, \text{ and } e_i, e_j \in E_d(G); \\ u^{k+1} & \text{if } e_i \text{ feeds into } e_j \neq e_i^{-1}, e_i \in \{e, e^{-1}\}, \text{ and } e_j \in E_d(G); \\ 0 & \text{otherwise.} \end{cases}$$

Note that  $\zeta_{X_k}(u)^{-1} = \det(I - W_k)$ .

Define  $P$  to be a permutation matrix such that the penultimate column and row of  $P^T W_k P$  correspond to the directed edge  $e$  and the last column and row correspond to  $e^{-1}$ . Note then that  $P^T W_k P - I$  has the form

$$\begin{pmatrix} Mu - I & xu & yu \\ a^T u^{k+1} & -1 & 0 \\ b^T u^{k+1} & 0 & -1 \end{pmatrix}$$

where the zero-one vectors  $a, b, x, y$  and the matrix  $M$  are the same for every  $k$ .

In particular,  $M$  is a directed edge matrix for  $X$  and

$$P \begin{pmatrix} M & x & y \\ a^T & 0 & 0 \\ b^T & 0 & 0 \end{pmatrix} P^T$$

is a directed edge matrix for  $Y$ .

Thus,

$$\begin{aligned} \zeta_{X_k}(u)^{-1} &= \det(W_k - I) \\ &= \det(P^T(W_k - I)P) \\ &= \det(P^T W_k P - I) \\ &= \det \begin{pmatrix} Mu - I & xu & yu \\ a^T u^{k+1} & -1 & 0 \\ b^T u^{k+1} & 0 & -1 \end{pmatrix} \\ &= \det \begin{pmatrix} Mu - I & xu & yu \\ a^T u^{k+1} & -1 & 0 \\ b^T u^{k+1} & 0 & 0 \end{pmatrix} - \det \begin{pmatrix} Mu - I & xu \\ a^T u^{k+1} & -1 \end{pmatrix} \\ &= \det \begin{pmatrix} Mu - I & xu & yu \\ a^T u^{k+1} & 0 & 0 \\ b^T u^{k+1} & 0 & 0 \end{pmatrix} - \det \begin{pmatrix} Mu - I & yu \\ b^T u^{k+1} & 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
& - \det \begin{pmatrix} Mu - I & xu \\ a^T u^{k+1} & 0 \end{pmatrix} + \det ( Mu - I ) \\
& = u^{2k} \det \begin{pmatrix} Mu - I & xu & yu \\ a^T u & 0 & 0 \\ b^T u & 0 & 0 \end{pmatrix} - u^k \det \begin{pmatrix} Mu - I & yu \\ b^T u & 0 \end{pmatrix} \\
& - u^k \det \begin{pmatrix} Mu - I & xu \\ a^T u & 0 \end{pmatrix} + \det ( Mu - I ).
\end{aligned}$$

We will now compute each of these determinants separately. The last determinant is just  $\zeta_X(u)^{-1}$  since  $M$  is the directed edge matrix of  $X$ . Next we compute the  $u^k$  determinants:

$$\begin{aligned}
\det \begin{pmatrix} Mu - I & xu \\ a^T u & 0 \end{pmatrix} & = \det \begin{pmatrix} Mu - I & xu \\ a^T u & -1 \end{pmatrix} + \det ( Mu - I ) \\
& = - \det \begin{pmatrix} Mu - I & xu & 0 \\ a^T u & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \det ( Mu - I ) \\
& = - \zeta_{Y - \{e^{-1}\}}(u)^{-1} + \zeta_X(u)^{-1}.
\end{aligned}$$

Similarly,

$$\det \begin{pmatrix} Mu - I & yu \\ b^T u & 0 \end{pmatrix} = - \zeta_{Y - \{e\}}(u)^{-1} + \zeta_X(u)^{-1}.$$



Finally, we compute the  $u^{2k}$  determinant:

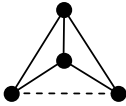
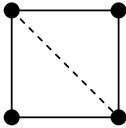
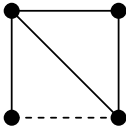
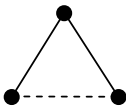
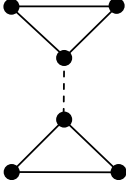
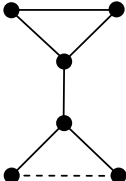
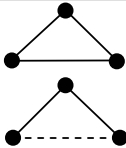
$$\begin{aligned}
\det \begin{pmatrix} Mu - I & xu & yu \\ a^T u & 0 & 0 \\ b^T u & 0 & 0 \end{pmatrix} &= \det \begin{pmatrix} Mu - I & xu & yu \\ a^T u^{k+1} & -1 & 0 \\ b^T u^{k+1} & 0 & -1 \end{pmatrix} + \det \begin{pmatrix} Mu - I & yu \\ b^T u^{k+1} & 0 \end{pmatrix} \\
&+ \det \begin{pmatrix} Mu - I & xu \\ a^T u^{k+1} & 0 \end{pmatrix} - \det (Mu - I) \\
&= \zeta_Y(u)^{-1} - \zeta_{Y-\{e^{-1}\}}(u)^{-1} - \zeta_X(u)^{-1} \\
&\quad - \zeta_{Y-\{e\}}(u)^{-1} - \zeta_X(u)^{-1} + \zeta_X(u)^{-1} \\
&= \zeta_Y(u)^{-1} - \zeta_{Y-\{e^{-1}\}}(u)^{-1} - \zeta_{Y-\{e\}}(u)^{-1} - \zeta_X(u)^{-1}.
\end{aligned}$$

So the result holds. □

A few examples of the polynomials  $p_1, p_2$  contained in this theorem are presented in Table 6.1.

Recognizing  $p_1, p_2$  as linear combinations of Ihara zeta functions allows us to find meaning in whether or not these polynomials are identically zero if  $Y$  is an undirected graph, as will be demonstrated by the following corollary.

Table 6.1 Some illustrative examples of  $\zeta_X, p_1, p_2$  from Theorem 6.6

The Graph $X$ (with $f_1$ drawn as a dashed edge)	Associated Polynomials
	$\zeta_X(u)^{-1} = 1 - 4u^3 - 2u^4 + 4u^6 + 4u^7 + u^8 - 4u^{10}$ $p_1(u) = 4u^3 + 4u^4 - 8u^6 - 12u^7 + 4u^8 + 8u^9$ $p_2(u) = 4u^6 + 8u^7 - 8u^9 - 20u^{10} + 16u^{12}$
	$\zeta_X(u)^{-1} = 1 - 2u^4 + u^8$ $p_1(u) = 4u^3 - 4u^7$ $p_2(u) = 4u^6 - 4u^{10}$
	$\zeta_X(u)^{-1} = 1 - 2u^3 + u^6$ $p_1(u) = 2u^3 + 2u^4 - 2u^6 - 2u^7$ $p_2(u) = u^6 + 2u^7 + u^8 - 4u^{10}$
	$\zeta_X(u)^{-1} = 1$ $p_1(u) = 2u^3$ $p_2(u) = u^6$
	$\zeta_X(u)^{-1} = 1 - 4u^3 + 6u^6 - 4u^9 + u^{12}$ $p_1(u) = 0$ $p_2(u) = -4u^8 + 8u^{11} - 4u^{14}$
	$\zeta_X(u)^{-1} = 1 - 2u^3 + u^6$ $p_1(u) = 2u^3 - 4u^6 + 4u^8 + 2u^9 - 4u^{11}$ $p_2(u) = u^6 - 2u^9 + 4u^{11} + u^{12} - 4u^{14}$
	$\zeta_X(u)^{-1} = 1 - 2u^3 + u^6$ $p_1(u) = 2u^3 - 4u^6 + 2u^9$ $p_2(u) = u^6 - 2u^9 + u^{12}$

**Corollary 6.7.** *Let  $Y, X, e, p_1, p_2$  be as in Theorem 6.6. Additionally, suppose  $Y$  is an undirected graph. Define  $\hat{e}$  to be the undirected version of the directed edge  $e$ . Then*

- (i) *the edge  $\hat{e}$  is a bridge in  $Y$  if and only if  $p_1(u)$  is identically zero,*
- (ii) *the edge  $\hat{e}$  is a bridge to a tree (that is, one of the connected components formed by the removal of  $\hat{e}$  is a tree) in  $Y$  if and only if  $p_2(u)$  is identically zero.*

*Proof.* We will first prove (i). So, suppose  $p_1(u)$  is identically zero. Then by Theorem 6.6,  $\zeta_X(u)^{-1} = \zeta_Y(u)^{-1}$ . But the primes counted in the product definition of the Ihara zeta function for  $X$  are a subset of the primes counted in the product definition of the Ihara zeta function for  $Y$ . So, for  $|u|$  small,

$$\prod_{[P]} (1 - u^{\nu(P)})^{-1} = 1$$

where the product is restricted to only those primes  $[P]$  which are contained in  $Y$  but not in  $X$ . This of course implies that there are no such primes  $[P]$ , which means there are no primes in  $Y$  which pass through the edge  $e$ . So, there is no walk from the terminal vertex of  $e$  to the start vertex of  $e$ . Thus, the only walk in  $Y$  from one vertex of  $\hat{e}$  to the other contains  $\hat{e}$  itself. Therefore,  $\hat{e}$  is a bridge. The other direction of the implication (i) is proved simply by following the proof for this direction in reverse.

Now will we prove (ii). So, suppose  $p_2(u)$  is identically zero (and that we have no foreknowledge concerning whether or not  $p_1(u)$  is identically zero). Then by Theorem 6.6,  $\zeta_{X_{2|E(X)|+1}}(u)^{-1} = \zeta_X(u)^{-1} - p_1(u)u^{2|E(X)|+1}$  and  $\zeta_{X_{2|E(X)|+2}}(u)^{-1} = \zeta_X(u)^{-1} - p_1(u)u^{2|E(X)|+2}$  where  $X_i$  are defined as in Theorem 6.6. Suppose  $p_1(u)$  is not identically zero. Note then that

$$\deg(p_1(u)u^{2|E(X)|+1}) > 2|E(X)| = \deg(\zeta_X(u)^{-1}).$$

So,  $\deg \left( \zeta_{X_{2|E(X)|+k}}(u)^{-1} \right) = \deg \left( p_1(u)u^{2|E(X)|+k} \right)$  for  $k = 1, 2$ . This is a contradiction however, since the  $\deg \left( p_1(u)u^{2|E(X)|+1} \right) \neq \deg \left( p_1(u)u^{2|E(X)|+2} \right)$ . Thus,  $p_1(u)$  is also identically zero. Since both  $p_1(u), p_2(u)$  are identically zero, all the primes in  $Y$  are also contained in  $X$  by Theorem 6.6. So, there are no primes which contain  $\hat{e}$ . Thus,  $\hat{e}$  is a bridge to a tree. Note that this tree may in fact be trivial (that is, a single vertex). For the other direction of the implication (ii), just note that if  $\hat{e}$  is a bridge to a tree, then  $\zeta_X(u) = \zeta_{X_k}(u)$  and apply Theorem 6.6 for two different values of  $k$ .  $\square$

Corollary 6.7 (i) then provides as with a simplification of Theorem 6.6 in the case in which the edge we wish to delete is a bridge:

**Corollary 6.8.** *Let  $Y, X, X_k, e$  be as in Theorem 6.6. Additionally, suppose  $Y$  is undirected and the undirected version of the directed edge  $e$  is a bridge. Then*

$$\zeta_{X_k}^{-1} = \zeta_X^{-1} + (\zeta_Y^{-1} - \zeta_X^{-1}) u^{2k}.$$

*Proof.* By Corollary 6.7,  $p_1(u)$  is identically 0 which implies that

$$-2\zeta_X^{-1} = -\zeta_{Y-\{e\}}^{-1} - \zeta_{Y-\{e^{-1}\}}^{-1}.$$

Thus,

$$\begin{aligned} p_2(u) &= \zeta_X^{-1} - \zeta_{Y-\{e\}}^{-1} - \zeta_{Y-\{e^{-1}\}}^{-1} + \zeta_Y^{-1} \\ &= \zeta_X^{-1} - 2\zeta_X^{-1} + \zeta_Y^{-1} \\ &= \zeta_Y^{-1} - \zeta_X^{-1} \end{aligned}$$

and the corollary holds.  $\square$

Using Corollary 6.8 as a guide, we constructed the graph in Figure 6.2 so that the reciprocal of the radius of convergence of its Ihara zeta function is approximately  $\pi$ . Note that no Ihara zeta function has radius of convergence exactly  $\pi^{-1}$  since Ihara zeta functions are reciprocals of polynomials with integer coefficients

and  $\pi$  (and thus,  $\pi^{-1}$ ) is transcendental. The program given in Section 9.9 attempts to construct a graph such that the reciprocal of the radius of convergence of its Ihara zeta function is within some  $\epsilon$  of a target value greater than 1.

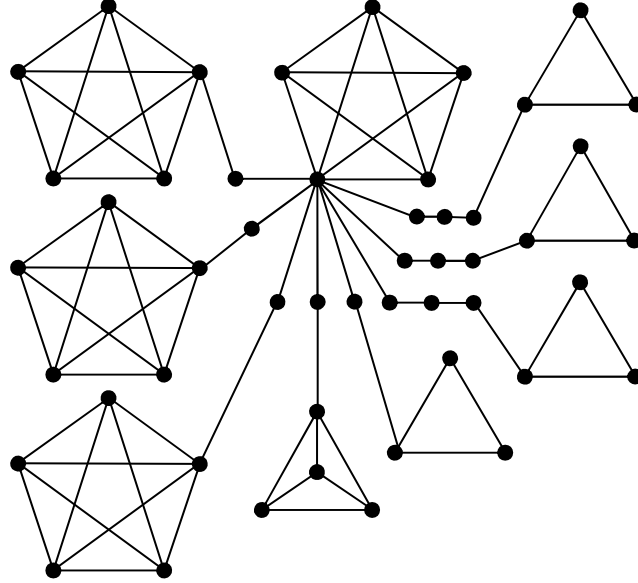


Figure 6.2 A graph whose Ihara zeta function has radius of convergence  $R$  where  $R^{-1} \approx 3.141593 \approx \pi$  (that is, both  $R^{-1}$  and  $\pi$  are 3.141593 when rounded to six places beyond the decimal)

We can also use the relations in Theorem 6.6 to create relations among  $X$  and the  $X_k$  as illustrated by the following corollary.

**Corollary 6.9.** For  $X, X_k$  as in the theorem and  $|u| < 1$ ,

$$\zeta_X(u)^{-1} = \frac{u^3 \zeta_{X_k}(u)^{-1} - (u^2 + u) \zeta_{X_{k+1}}(u)^{-1} + \zeta_{X_{k+2}}(u)^{-1}}{u^3 - (u^2 + u) + 1}.$$

for all integers  $k \geq 0$ .

*Proof.* By the theorem, there exist finite polynomials  $p_1(u)$  and  $p_2(u)$  independent of  $k$  such that  $\zeta_{X_k}(u)^{-1} = \zeta_X(u)^{-1} - p_1(u)u^k + p_2(u)u^{2k}$  for all integers  $k \geq 0$ . So, in particular, we have the following:

$$\zeta_{X_k}(u)^{-1} = \zeta_X(u)^{-1} - p_1(u)u^k + p_2(u)u^{2k} \quad (6.1)$$

$$\zeta_{X_{k+1}}(u)^{-1} = \zeta_X(u)^{-1} - p_1(u)u^{k+1} + p_2(u)u^{2k+2} \quad (6.2)$$

Multiplying both sides of (6.1) by  $u$  and subtracting using (6.2), we find

$$u\zeta_{X_k}(u)^{-1} - \zeta_{X_{k+1}}(u)^{-1} = (u-1)\zeta_X(u)^{-1} + p_2(u)(u^{2k} - u^{2k+2}) \quad (6.3)$$

which is true for all integers  $k \geq 0$ . So, in particular, we have

$$u\zeta_{X_{k+1}}(u)^{-1} - \zeta_{X_{k+2}}(u)^{-1} = (u-1)\zeta_X(u)^{-1} + p_2(u)(u^{2k+2} - u^{2k+4}) \quad (6.4)$$

Multiplying both sides of (6.3) by  $u^2$  and subtracting using (6.4), we find

$$u^3\zeta_{X_k}(u)^{-1} - (u^2 + u)\zeta_{X_{k+1}}(u)^{-1} + \zeta_{X_{k+2}}(u)^{-1} = (u^3 - (u^2 + u) + 1)\zeta_X(u)^{-1}.$$

□

In the next corollary, we extend Theorem 6.6 by allowing for the addition of vertices to more than one edge.

**Corollary 6.10.** *Let  $Y$  be directed graph (possibly with bidirected edges) and let  $F = \{f_1, \dots, f_{|F|}\} \subseteq E_d(Y)$  such that  $f_i \neq f_j^{-1}$  for  $1 \leq i, j \leq |F|$ . Define  $X$  to be the graph such that  $V(X) = V(Y)$  and  $E_d(X) = E_d(Y) - F - \{e : e^{-1} \in F\}$ . Define  $X(k_1, \dots, k_{|F|})$  to be the graph obtained by adding a walk  $\mathcal{W}_i$  of length  $k_i + 1$  to  $X$  from  $s(f_i)$  to  $t(f_i)$  where  $\mathcal{W}_i$  is directed if  $f_i^{-1} \notin E_d(Y)$  and undirected otherwise. Define  $Y(A, B, C)$  to be the graph such that  $V(Y(A, B, C)) = V(Y)$  and*

$$E_d(Y(A, B, C)) = E_d(Y) - (A \cup B \cup \{e \in E_d(Y) : e^{-1} \in A \cup C\}).$$

Then

$$\zeta_{X(k_1, \dots, k_{|F|})}(u)^{-1} = \sum_{\substack{A, B, C \subseteq F \\ A, B, C \text{ disjoint}}} \zeta_{Y(A, B, C)}(u)^{-1} p_{A, B, C, F}(u)$$

where

$$p_{A, B, C, F}(u) = \prod_{\substack{i \\ f_i \in A}} (1 - 2u^{k_i} + u^{2k_i}) \prod_{\substack{i \\ f_i \in B}} (u^{k_i} - u^{2k_i}) \prod_{\substack{i \\ f_i \in C}} (u^{k_i} - u^{2k_i}) \prod_{\substack{i \\ f_i \in F - (A \cup B \cup C)}} (u^{2k_i}).$$

*Remark 6.11.* The degree of  $p_{A,B,C,F}(u)$  is  $2 \sum_{i=1}^{|F|} k_i$ .

*Proof.* We induct on  $|F|$ . If  $|F| = 1$ , then the corollary follows immediate from the theorem. So, suppose  $|F| \geq 2$  and the corollary holds in the case in which we take  $F - \{f_1\}$  to be the set of edges to be deleted. Then by the theorem,

$$\begin{aligned}
\zeta_{X(k_1, \dots, k_{|F|})}^{-1} &= \zeta_{X(0, k_2, \dots, k_{|F|}) - \{f_1, f_1^{-1}\}}^{-1} - \left( 2\zeta_{X(0, k_2, \dots, k_{|F|}) - \{f_1, f_1^{-1}\}}^{-1} \right. \\
&\quad \left. - \zeta_{X(0, k_2, \dots, k_{|F|}) - \{f_1\}}^{-1} - \zeta_{X(0, k_2, \dots, k_{|F|}) - \{f_1^{-1}\}}^{-1} \right) u^{k_1} \\
&\quad + \left( \zeta_{X(0, k_2, \dots, k_{|F|}) - \{f_1, f_1^{-1}\}}^{-1} - \zeta_{X(0, k_2, \dots, k_{|F|}) - \{f_1\}}^{-1} \right. \\
&\quad \left. - \zeta_{X(0, k_2, \dots, k_{|F|}) - \{f_1^{-1}\}}^{-1} + \zeta_{X(0, k_2, \dots, k_{|F|})}^{-1} \right) u^{2k_1} \\
&= (1 - 2u^{k_1} + u^{2k_1}) \zeta_{X(0, k_2, \dots, k_{|F|}) - \{f_1, f_1^{-1}\}}^{-1} \\
&\quad + (u^{k_1} - u^{2k_1}) \zeta_{X(0, k_2, \dots, k_{|F|}) - \{f_1\}}^{-1} \\
&\quad + (u^{k_1} - u^{2k_1}) \zeta_{X(0, k_2, \dots, k_{|F|}) - \{f_1^{-1}\}}^{-1} \\
&\quad + (u^{2k_1}) \zeta_{X(0, k_2, \dots, k_{|F|})}^{-1}
\end{aligned}$$

(by the induction hypothesis)

$$\begin{aligned}
&= (1 - 2u^{k_1} + u^{2k_1}) \left( \sum_{\substack{A, B, C \subseteq F - \{f_1\} \\ A, B, C \text{ disjoint}}} \zeta_{Y(A \cup \{f_1\}, B, C)}^{-1} p_{A, B, C, (F - \{f_1\})}(u) \right) \\
&\quad + (u^{k_1} - u^{2k_1}) \left( \sum_{\substack{A, B, C \subseteq F - \{f_1\} \\ A, B, C \text{ disjoint}}} \zeta_{Y(A, B \cup \{f_1\}, C)}^{-1} p_{A, B, C, (F - \{f_1\})}(u) \right) \\
&\quad + (u^{k_1} - u^{2k_1}) \left( \sum_{\substack{A, B, C \subseteq F - \{f_1\} \\ A, B, C \text{ disjoint}}} \zeta_{Y(A, B, C \cup \{f_1\})}^{-1} p_{A, B, C, (F - \{f_1\})}(u) \right) \\
&\quad + (u^{2k_1}) \left( \sum_{\substack{A, B, C \subseteq F - \{f_1\} \\ A, B, C \text{ disjoint}}} \zeta_{Y(A, B, C)}^{-1} p_{A, B, C, (F - \{f_1\})}(u) \right)
\end{aligned}$$

$$\begin{aligned}
&= \left( \sum_{\substack{f_1 \in A \\ A, B, C \subseteq F \\ A, B, C \text{ disjoint}}} \zeta_{Y(A, B, C)}^{-1} p_{A, B, C, F}(u) \right) + \left( \sum_{\substack{f_1 \in B \\ A, B, C \subseteq F \\ A, B, C \text{ disjoint}}} \zeta_{Y(A, B, C)}^{-1} p_{A, B, C, F}(u) \right) \\
&+ \left( \sum_{\substack{f_1 \in C \\ A, B, C \subseteq F \\ A, B, C \text{ disjoint}}} \zeta_{Y(A, B, C)}^{-1} p_{A, B, C, F}(u) \right) + \left( \sum_{\substack{f_1 \in F - (A \cup B \cup C) \\ A, B, C \subseteq F \\ A, B, C \text{ disjoint}}} \zeta_{Y(A, B, C)}^{-1} p_{A, B, C, F}(u) \right) \\
&= \sum_{\substack{A, B, C \subseteq F \\ A, B, C \text{ disjoint}}} \zeta_{Y(A, B, C)}^{-1} p_{A, B, C, F}(u).
\end{aligned}$$

Therefore, the result holds.  $\square$

An example illustrating the application of Corollary 6.10 is shown in Figure 6.3. If we add the same number of vertices to each edge in  $F$  (that is, if  $k_1 = k_2 = \dots = k_{|F|}$  in Corollary 6.10), then we obtain the following:

**Corollary 6.12.** *If everything is as in the previous corollary, then*

$$\zeta_{X_k}(u)^{-1} = \sum \zeta_{Y(A, B, C)}(u)^{-1} (1 - u^k)^{2|A|+|B|+|C|} (u^k)^{2|F|-2|A|-|B|-|C|}$$

where the sum is over disjoint subsets  $A, B, C$  of  $F$  and  $X_k = X(\underbrace{k, \dots, k}_{|F| \text{ times}})$ .

*Proof.* From the previous corollary,  $\zeta_{X_k}(u)^{-1} =$

$$\sum \zeta_{Y(A, B, C)}(u)^{-1} (1 - 2u^k + u^{2k})^{|A|} (u^k - u^{2k})^{|B|+|C|} (u^{2k})^{|F|-|A|-|B|-|C|}.$$

Note that

$$\begin{aligned}
&(1 - 2u^k + u^{2k})^{|A|} (u^k - u^{2k})^{|B|+|C|} (u^{2k})^{|F|-|A|-|B|-|C|} \\
&= (1 - u^k)^{2|A|} (1 - u^k)^{|B|+|C|} (u^k)^{|B|+|C|} (u^k)^{2|F|-2|A|-2|B|-2|C|} \\
&= (1 - u^k)^{2|A|+|B|+|C|} (u^k)^{2|F|-2|A|-|B|-|C|}.
\end{aligned}$$

So, the result holds.  $\square$



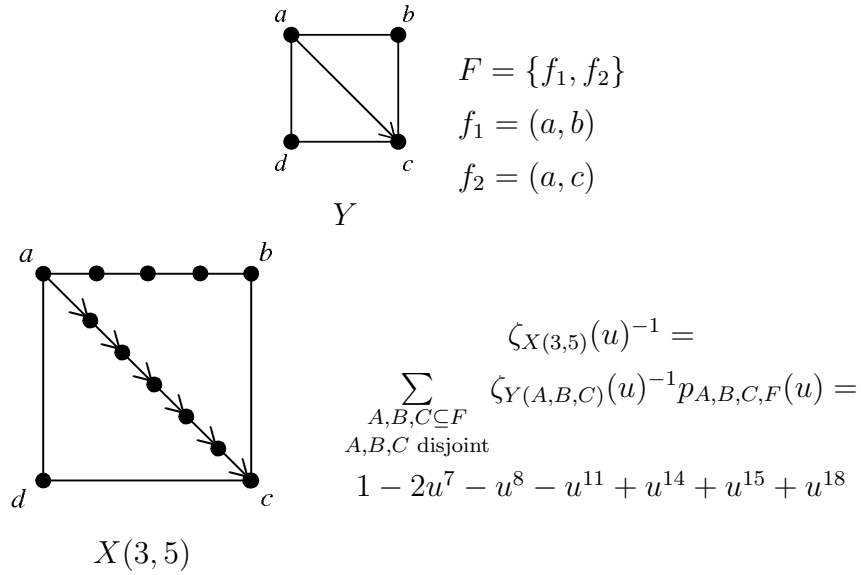


Figure 6.3 Example for Corollary 6.10 (Note: Table 6.2 explicitly lists all possible  $\zeta_{Y(A, B, C)}(u)^{-1}$  for this example.)

Now, finally, we expand Theorem 4.1 to the entire open neighborhood of radius one about the origin:

**Corollary 6.13.** *Let  $G$  be an undirected graph. Let  $F_0 \subseteq E$  and for each non-negative integer  $k$  define  $G(F_0, k)$  to be the graph  $G$  but with every edge contained in  $F_0$  divided into  $k + 1$  edges. Then for each  $u$  such that  $|u| < 1$ ,*

$$\lim_{k \rightarrow \infty} \zeta_{G(F_0, k)}(u) = \zeta_{(G - F_0)}(u).$$

*Proof.* Take  $Y = G$  and  $F$  to be a set of directed edges formed by arbitrary orienting the edges in  $F_0$ . Then apply the previous corollary and note that each term in the sum for which  $A \neq F$  contains  $u^k$  as a factor. Thus, the only term which does not go to zero as  $k$  goes to infinity is the term in which  $A = F$  and  $B = C = \emptyset$ . This term is

$$\zeta_{Y(F, \emptyset, \emptyset)}(u)^{-1} (1 - 2u^k + u^{2k})^{|F|}$$

which goes to  $\zeta_{Y(F,\emptyset,\emptyset)}(u)^{-1}$  as  $k \rightarrow \infty$ . Since  $Y(F,\emptyset,\emptyset) = G - F_0$ , the result holds.  $\square$

Table 6.2 Explicit listing of  $\zeta_{Y(A,B,C)}(u)^{-1}$  corresponding to the choices of the graph  $Y$  and the set of directed edges  $F = \{f_1, f_2\}$  given in Figure 6.3

$A$	$B$	$C$	$\zeta_{Y(A,B,C)}(u)^{-1}$
$\emptyset$	$\emptyset$	$\emptyset$	$1 - 2u^3 - 2u^4 + 2u^7 + u^8$
$\{f_1\}$	$\emptyset$	$\emptyset$	$1 - u^3$
$\emptyset$	$\{f_1\}$	$\emptyset$	$1 - 2u^3 - u^4 + u^7$
$\emptyset$	$\emptyset$	$\{f_1\}$	$1 - u^3 - u^4$
$\{f_2\}$	$\emptyset$	$\emptyset$	$1 - 2u^4 + u^8$
$\{f_1, f_2\}$	$\emptyset$	$\emptyset$	1
$\{f_2\}$	$\{f_1\}$	$\emptyset$	$1 - u^4$
$\{f_2\}$	$\emptyset$	$\{f_1\}$	$1 - u^4$
$\emptyset$	$\{f_2\}$	$\emptyset$	$1 - 2u^4 + u^8$
$\{f_1\}$	$\{f_2\}$	$\emptyset$	1
$\emptyset$	$\{f_1, f_2\}$	$\emptyset$	$1 - u^4$
$\emptyset$	$\{f_2\}$	$\{f_1\}$	$1 - u^4$
$\emptyset$	$\emptyset$	$\{f_2\}$	$1 - 2u^3 - 2u^4 + 2u^7 + u^8$
$\{f_1\}$	$\emptyset$	$\{f_2\}$	$1 - u^3$
$\emptyset$	$\{f_1\}$	$\{f_2\}$	$1 - 2u^3 - u^4 + u^7$
$\emptyset$	$\emptyset$	$\{f_1, f_2\}$	$1 - u^3 - u^4$

# 7

## Improving bounds on the poles

### 7.1 Pole bounds for digraphs

Next, we revisit Theorem 2.2 and its corollary which used the SVD of a directed edge matrix to bound the poles of Ihara zeta functions of undirected graphs. Here we apply the same approach to digraphs with no bidirected edges:

**Theorem 7.1.** *Suppose  $G$  is a digraph which satisfies our simplifying assumptions and also contains no bidirected edges. Then the singular values of a directed edge matrix  $M$  of  $G$  are*

$$\{c_1, c_2, \dots, c_n, 0, \dots, 0\}$$

where  $c_i = \sqrt{id(v_i)od(v_i)}$  (that is, the geometric mean of the indegree and outdegree of  $v_i$ ) and  $v_1, v_2, \dots, v_n$  are the  $n$  vertices of  $G$ .

*Proof.* Let  $G$  be as in the theorem. Let  $\{e_1, e_2, \dots, e_{2m}\}$  be the directed edges of  $H$  where  $H$  is the graph underlying  $G$ . Choose the indexing of the  $e_i$  such that edges ending at the same vertex are listed together. That is, if  $t(e_i) = t(e_j)$  for some  $i < j$  then  $t_k = t(e_i)$  for all  $k$ ,  $i < k < j$ . Let  $M$  be the  $2m \times 2m$  matrix defined by

$$(M)_{ij} = \begin{cases} 1 & \text{if } t(e_i) = s(e_j) \text{ and } s(e_i) \neq t(e_j), \\ 0 & \text{otherwise.} \end{cases}$$

Then for each  $i$  and  $j$ ,  $(MM^T)_{ij}$  is a count of the number of edges in  $E_d(G) - \{e_i^{-1}, e_j^{-1}\}$  whose start vertex is the end vertex of both  $e_i$  and  $e_j$ . So,

$$(MM^T)_{ij} = \begin{cases} 0 & \text{if } t(e_i) \neq t(e_j), \\ od(t(e_i)) & \text{if } t(e_i) = t(e_j). \end{cases}$$

Now choose  $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$  such that  $1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_n \leq 2m$  and  $t_{\alpha_i} = v_i$  for each  $i$ ,  $1 \leq i \leq n$ . Define  $c_i = \sqrt{id(v_i)od(v_i)}$  and note  $MM^T = \text{diag}(A_1, A_2, \dots, A_n)$  where  $A_i$  is a  $id(v_i) \times id(v_i)$  matrix with  $c_i^2$  in every entry.

For each  $i$ , define  $id(v_i) \times id(v_i)$  matrixes  $B_i = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & -1 & 0 \\ 1 & 0 & \cdots & 0 & -1 \end{pmatrix}$ , which

is nonsingular (since the product of its diagonal entries is nonzero), and  $C_i = \text{diag}(c_i^2, 0, \dots, 0)$ . Then  $A_i B_i = B_i C_i$ .

Now define  $X = \text{diag}(B_1, B_2, \dots, B_n)$ . Note  $X$  is nonsingular and  $(MM^T)X = \text{diag}(A_1, A_2, \dots, A_n) \cdot \text{diag}(B_1, B_2, \dots, B_n) = \text{diag}(B_1, B_2, \dots, B_n) \cdot \text{diag}(C_1, C_2, \dots, C_n) = X \cdot \text{diag}(C_1, C_2, \dots, C_n)$ . Thus, the columns of  $X$  are eigenvectors of  $MM^T$  with corresponding eigenvalues appearing on the diagonal of the matrix  $\text{diag}(C_1, C_2, \dots, C_n)$ . By the argument used in the proof of Theorem 2.2 then, the singular values for any directed edge matrix of  $G$  are  $\{c_1, c_2, \dots, c_n, 0, \dots, 0\}$ .  $\square$

As before, we will use these singular values to bound the poles of Ihara zeta functions. However, since the smallest singular value is zero, the theorem only provides us with a pole-free region about the origin:

**Corollary 7.2.** *Suppose  $G$  is a digraph which satisfies our simplifying assumptions and also contains no bidirected edges. Then no poles of the Ihara zeta function of  $G$  are contained within  $\{u : |u| < \frac{1}{c}\}$  where  $c = \max_{v \in V(G)} \sqrt{id(v)od(v)}$ .*

*Proof.* Let  $M$  be a directed edge matrix of the graph  $G$ . By the theorem, the largest singular value of  $M$  is  $c = \max_{v \in V(G)} \sqrt{id(v)od(v)}$ . So, by the argument used in the proof of Corollary 2.3, the result holds.  $\square$

In the following section, we will introduce directed edge matrix-induced graphs which will allow us to improve upon this corollary both by extending the set of graphs to which it applies and (in some cases) by expanding the pole-free region. We will then be able to compare the bound given by this corollary to the bound from Corollary 2.3.

## 7.2 Directed Edge Matrix-Induced Graphs

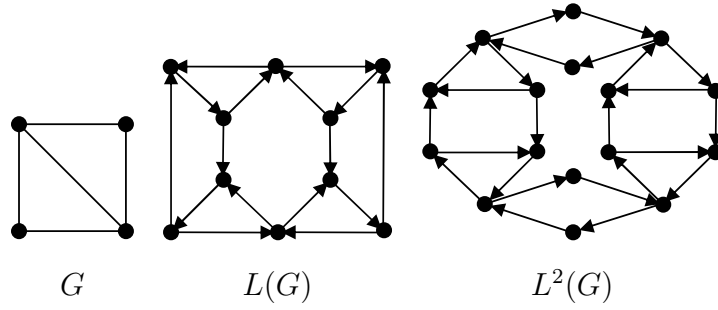
If  $G$  is a directed or undirected graph, then the directed edge matrix of  $G$  is an adjacency matrix of a directed graph. We will call the graph created in this way (less any isolated vertices) the *directed edge matrix-induced graph* of  $G$  and denote this graph by  $L(G)$ . Now define  $L^0(G) = G$  and  $L^{k+1}(G) = L(L^k(G))$  for  $k \geq 0$ . Figure 7.1 shows  $L^k(K_4 - e)$  for  $k = 0, 1, 2$ . If  $G$  is a directed cycle, then  $L^k(G) = G$  for all  $k$ . (In the special case where  $G$  is a directed graph with no bidirected edges,  $L(G)$  is known as the *line digraph*  $G$ . In the special case where  $G$  is an undirected graph,  $L(G)$  is known as the *oriented line graph* of  $G$ .)

**Theorem 7.3.** *Let  $G$  be a directed or undirected graph. Then  $\zeta_G(u) = \zeta_{L^k(G)}(u)$  for all non-negative integers  $k$ .*

*Remark 7.4.* The case in which  $G$  is undirected and  $k = 1$  is contained in Kotani and Sunada [6].

*Proof.* Let  $G$  be a directed or undirected graph. Let  $\mathbb{P}, \mathbb{P}_L$  be the set of primes in  $G, L(G)$  respectively. We will prove the theorem by showing that we can define a one-to-one mapping from  $\mathbb{P}$  onto  $\mathbb{P}_L$  which preserves lengths of primes.

Label the vertices of  $L(G)$  with the directed edges of  $G$  they correspond to according to the directed edge matrix of  $G$ . Note then that by the relationship



$$\zeta_G(u) = \zeta_{L(G)}(u) = \zeta_{L^2(G)}(u) = -(-1+u)^2(1+u)(1+u^2)(1+u+2u^2)(-1+u^2+2u^3)$$

Figure 7.1 Examples of directed edge matrix-induced graphs illustrating the operation  $L$  on  $K_4 - e$

between the directed edge matrix of  $G$  and the adjacency matrix of  $L(G)$ , directed edge  $e$  feeds into directed edge  $f$  in  $G$  if and only if the directed edge  $(e, f)$  is contained in  $E_d(L(G))$ . Define a function  $\phi$  from  $\{C : [C] \in \mathbb{P}\}$  to  $\{C : [C] \in \mathbb{P}_L\}$  such that  $\phi(C_0) = \{(e_1, e_2), (e_2, e_3), \dots, (e_j, e_1)\}$  where the  $e_i$  are directed edges of  $G$  such that  $C_0 = \{e_1, e_2, \dots, e_j\}$ .

We will show that  $\phi$  is well defined. Since  $[C_0]$  is a prime in  $G$ ,  $e_i$  feeds into  $e_{i+1}$  for  $i, 1 \leq i < j$ , and  $e_j$  feeds into  $e_1$ . Thus, the directed edges  $(e_j, e_1)$  and  $(e_i, e_{i+1})$  for  $i, 1 \leq i < j$ , are contained in  $E_d(L(G))$  and so  $\phi(C_0)$  is a closed walk in  $L(G)$ . Now we will show  $\phi(C_0)$  is backtrackless. Suppose  $\phi(C_0)$  contains a backtrack and, without loss of generality, assume this backtrack is  $(e_1, e_2)(e_2, e_1)$ . Then  $e_1$  feeds into  $e_2$  and  $e_2$  feeds into  $e_1$  which implies  $e_1 = e_2^{-1}$ . This is a contradiction since  $C_0 = \{e_1, e_2, \dots, e_j\}$  and  $C_0$  is backtrackless. So,  $\phi(C_0)$  too is backtrackless. A similar argument shows  $\phi(C_0)$  is also tailless. Now suppose there exists some integer  $k \geq 2$  which divides  $j$  and  $\{(e_1, e_2), (e_2, e_3), \dots, (e_j, e_1)\} = \{(e_1, e_2), (e_2, e_3), \dots, (e_{j/k}, e_1)\}^k$ . Then  $C_0 = \{e_1, e_2, \dots, e_{j/k}\}^k$ , again a contradiction since  $[C_0]$  is a prime. Therefore,  $[\phi(C_0)] \in \mathbb{P}_L$ . The function  $\phi$  preserves lengths since  $\nu(C_0) = j = \nu(\phi(C_0))$ . It is also invertible since for any

$\{(f_1, f_2), (f_2, f_3), \dots, (f_j, f_1)\} \in \{C : [C] \in \mathbb{P}_L\}$ ,  $\phi(\{f_1, f_2, f_3, \dots, f_j\}) = \{(f_1, f_2), (f_2, f_3), \dots, (f_j, f_1)\}$ .

We can show that  $\{f_1, f_2, f_3, \dots, f_j\} \in \{C : [C] \in \mathbb{P}\}$  by simply reversing the argument which demonstrated  $\phi(C_0) \in \{C : [C] \in \mathbb{P}_L\}$ . Note also that  $[\phi(C_0)]$  does not depend on our choice of representative  $C_0$  for the prime  $[C_0]$ . Thus, the function  $\psi$  from  $\mathbb{P}$  to  $\mathbb{P}_L$  defined by  $\psi([C]) = [\phi(C)]$  is one-to-one, onto, and length preserving. Therefore, for  $u$  sufficiently small,

$$\prod_{[C] \in \mathbb{P}} (1 - u^{\nu(C)})^{-1} = \prod_{[\phi(C)] \in \mathbb{P}_L} (1 - u^{\nu(\phi(C))})^{-1} = \prod_{[C] \in \mathbb{P}_L} (1 - u^{\nu(C)})^{-1}$$

which implies  $\zeta_G(u)^{-1} - \zeta_{L(G)}(u)^{-1} = 0$  in some neighborhood of the origin. Since both  $\zeta_G(u)^{-1}, \zeta_{L(G)}(u)^{-1}$  are finite polynomials, this implies  $\zeta_G(u)^{-1} = \zeta_{L(G)}(u)^{-1}$ . Since  $G$  was an arbitrary directed or undirected graph, the result holds by induction.  $\square$

The following results tell us how large we can expect the graph  $L^k(G)$  to be.

**Theorem 7.5.** *Let  $G$  be a directed or undirected graph with a directed edge matrix  $M$ . Then  $|E_d(L^k(G))| = \sum_{i,j} (M^k)_{ij}$  for all positive integers  $k$ .*

*Proof.* For each  $k$ , let  $\phi_{L^k(G)}$  be the one-to-one mapping from backtrackless directed walks of length at least one in  $L^k(G)$  onto backtrackless directed walks (of length greater than or equal to zero) in  $L^{k+1}(G)$  as defined in the lemma. Recall that  $\nu(C_0) = \nu(\phi_{L^k(G)}(C_0)) + 1$ .

Now define  $\psi_k$  to be the one-to-one mapping from backtrackless directed walks of length  $k + 1$  in  $G$  onto the directed edges of  $L^k(G)$  (which are the walks of length one in  $L^k(G)$ ) by  $\psi_k(C_0) = \phi_{L^{k-1}(G)} \circ \phi_{L^{k-2}(G)} \circ \phi_{L^{k-3}(G)} \circ \dots \circ \phi_{L^1(G)} \circ \phi_G$  for  $k > 0$ . So, the number of edges of  $L^k(G)$  is exactly the number of backtrackless directed walks of length  $k + 1$  in  $G$ . Since the number of backtrackless directed walks of length  $k + 1$  in  $G$  is  $\sum_{i,j} (M^k)_{ij}$  where  $M$  is a directed edge matrix of  $G$ ,  $|E_d(L^k(G))| = \sum_{i,j} (M^k)_{ij}$  for all positive integers  $k$   $\square$



**Corollary 7.6.** *Let  $G$  be a directed or undirected graph with a directed edge matrix  $M$ . Then  $|V(L^{k+1}(G))| = \sum_{i,j} (M^k)_{ij}$  for all positive integers  $k$ .*

*Proof.* Note that  $|E_d(L^k(G))| = |V(L^{k+1}(G))|$  (since the adjacency matrix for  $L^k(G)$  is formed from the directed edge matrix of  $L^{k+1}(G)$ ) for all positive integers  $k$  and apply the theorem.  $\square$

*Remark 7.7.* If we allow our simplifying assumption that every directed edge be contained in some prime to be violated, then the equality in the corollary must be replaced by  $|V(L^{k+1}(G))| \leq \sum_{i,j} (M^k)_{ij}$ . This is due to our elimination of isolated vertices when forming  $L^{k+1}(G)$ .

### 7.3 Indegree/outdegree bound on the poles

As promised, we will now reexamine Corollary 7.2 in light of what we have learned about directed edge matrix-induced graphs to obtain the following result:

**Corollary 7.8.** *Suppose  $G$  is a digraph which satisfies our simplifying assumptions and also contains no bidirected edges. Then no poles of the Ihara zeta function of  $G$  are contained within  $\{u : |u| < \frac{1}{c}\}$  where*

$$c = \min_{k \geq 0} \max_{\substack{x,y \in V(G) \\ (x,y) \in \mathbb{P}_k}} \sqrt{id(x)od(y)}$$

and  $\mathbb{P}_k = \{(x,y) : \text{there exists a backtrackless directed walk from vertex } x \text{ to vertex } y \text{ of length } k \text{ in } G\}$ .

*Proof.* Let  $G$  be as described in the corollary. By Corollary 7.2 and Theorem 7.3, no poles of the Ihara zeta function of  $G$  are contained within  $\{u : |u| < \frac{1}{c}\}$  where

$$c = \min_{k \geq 0} \max_{v \in V(L^k(G))} \sqrt{id(u)od(v)}.$$

So, we need only show that for each  $k \geq 0$ ,

$$\max_{v \in V(L^k(G))} \sqrt{id(v)od(v)} = \max_{\substack{u, v \in V(G) \\ (u, v) \in \mathbb{P}_k}} \sqrt{id(u)od(v)}.$$

For any directed walk  $\mathcal{W}$ , define  $s(\mathcal{W}), t(\mathcal{W})$  to be the start and terminal vertices respectively of  $\mathcal{W}$ . Note for any  $v \in V(L^{k+1}(G))$ ,  $id(v) = id(s(v))$  since  $v \in E_d(L^k(G))$  (that is,  $v$  is a directed walk of length one in  $L^k(G)$ ) and the vertices connected to  $v$  by a directed edge terminating at the vertex  $v$  in  $L^{k+1}(G)$  are in one-to-one correspondence with the edges feeding into the edge  $v$  in  $L^k(G)$ . Thus, for any  $v \in V(L^{k+1}(G))$ ,  $id(v) = id(s(v)) = id(s^2(v)) = \dots = id(s^{k+1}(v))$ . Similarly,  $od(v) = od(t^{k+1}(v))$ .

Now let  $\psi_k$  be the one-to-one mapping defined in the proof of Theorem 7.5 from backtrackless directed walks of length  $k+1$  in  $G$  onto  $V(L^{k+1}(G))$  (or equivalently,  $E_d(L^k(G))$ ) for  $k > 0$ . Also, define  $\psi_0$  to be the identity mapping on  $E_d(G)$  which we will think of as a mapping from the set of backtrackless directed walks of length one in  $G$  to  $V(L^0(G))$ . So,  $\psi_k^{-1}(V(L^{k+1}(G)))$  are the backtrackless directed walks of length  $k+1$  in  $G$ . Also,  $s(\psi_k^{-1}(v)) = s^{k+1}(v)$  and  $t(\psi_k^{-1}(v)) = t^{k+1}(v)$ . This is just due to the fact that if for example  $\psi_3^{-1}(v) = \{f_1, f_2, \dots, f_4\}$  where the  $f_i$  are directed edges of  $G$  then  $v = (((f_1, f_2), (f_2, f_3)), ((f_2, f_3), (f_3, f_4)))$  and so  $s^4(v) = s^3(((f_1, f_2), (f_2, f_3))) = s^2((f_1, f_2)) = s(f_1) = s(\{f_1, f_2, \dots, f_4\}) = s(\psi_3^{-1}(v))$ . Thus, for each  $k \geq 0$ ,

$$\begin{aligned} \max_{\substack{u, v \in V(G) \\ (u, v) \in \mathbb{P}_k}} \sqrt{id(u)od(v)} &= \max_{\mathcal{W} \in \psi_k^{-1}(V(L^{k+1}(G)))} \sqrt{id(s(\mathcal{W}))od(t(\mathcal{W}))} \\ &= \max_{v \in V(L^{k+1}(G))} \sqrt{id(s(\psi_k^{-1}(v)))od(t(\psi_k^{-1}(v)))} \\ &= \max_{v \in V(L^k(G))} \sqrt{id(s^{k+1}(v))od(t^{k+1}(v))} \\ &= \max_{v \in V(L^k(G))} \sqrt{id(v)od(v)} \end{aligned}$$

as desired. □

Note that Corollary 7.8 also holds if  $G$  contains bidirected edges (due to the relation between the in/outdegrees of  $G$  and  $L(G)$  and the fact that  $L(G)$  is a directed graph with no bidirected edges even if  $G$  itself contains bidirected edges). Also note however that in the case where  $G$  is an undirected graph,  $id(v) = id(s(v)) - 1 = deg(s(v)) - 1$  for any  $v \in V(L(G))$  where  $s$  is as in the proof of Corollary 7.8. This fact combined with Corollary 7.8 give us the following result:

**Corollary 7.9.** *Suppose  $G$  is an undirected graph. Then no poles of the Ihara zeta function of  $G$  are contained within  $\{u : |u| < \frac{1}{c}\}$  where*

$$c = \min_{k \geq 0} \max_{\substack{x, y \in V(G) \\ (x, y) \in \mathbb{P}_k}} \sqrt{(deg(x) - 1)(deg(y) - 1)}$$

and  $\mathbb{P}_k = \{(x, y) : \text{there exists a backtrackless walk from vertex } x \text{ to vertex } y \text{ of length } k \text{ in } G\}$ .

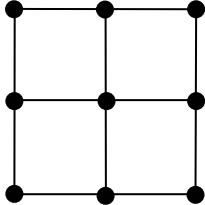
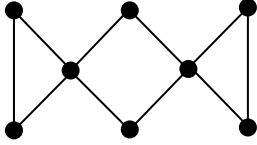
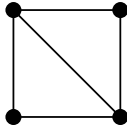
At worst, Corollary 7.9 will just give us the lower bound on the poles guaranteed by Theorem 2.2. The set of graphs for which this occurs is  $\mathbb{B} = \{X : \text{for all } k, \text{ there exists a (not necessarily closed) backtrackless walk of length } k \text{ in the graph } X \text{ which begins and ends with a vertex of degree } \max_{v \in V(G)} (deg(v))\}$ .

At best, Corollary 7.9 will give us a lower bound which is the square root of the lower bound from Theorem 2.2. Call the set of graphs for which this occurs  $\mathbb{G}$ . A necessary but not sufficient condition for a graph  $X$  to be in  $\mathbb{G}$  is that there exists an integer  $k$  such that every backtrackless walk of length  $k$  in the graph  $X$  which begins with a vertex of degree  $\max_{v \in V(X)} (deg(v))$  ends with a vertex of degree 2.

The application of Corollary 7.9 to a small yet interesting and instructive set of graphs is presented in Table 7.1. Note that the first graph is neither in  $\mathbb{G}$  nor  $\mathbb{B}$ , the second is in  $\mathbb{G}$ , and the third is in  $\mathbb{B}$ . Also, note that the first graph demonstrates that the necessary condition given above for a graph to be in  $\mathbb{G}$  is not sufficient.

There are classes of graphs for which the pole-free neighborhood of the origin given in Corollary 7.9 is the largest possible (that is,  $c^{-1}$  is the radius of convergence of  $\zeta$ ). Regular graphs, for instance, have this property. More interestingly, so do bi-regular bipartite graphs (since by Hashimoto [4] the radius of convergence of the Ihara zeta function of a bi-regular bipartite graph  $G$  is  $\sqrt{pq}$  where  $p+1, q+1$  are the two degrees of the vertices of  $G$ ).

Table 7.1 Some interesting examples of applying Corollary 7.9 by using the values of  $c_k = \max_{\substack{x,y \in V(G) \\ (x,y) \in \mathbb{P}_k}} \sqrt{(\deg(x) - 1)(\deg(y) - 1)}$  to find a lower bound on the poles of  $\zeta$

The Graph $X$ (with $R$ , the radius of convergence of $\zeta_X$ )	Values of $c_k$ and $\frac{1}{c}$
 <p data-bbox="521 877 789 926"><math>R = \frac{1}{\sqrt{3}} \approx 0.577350</math></p>	$c_k = \begin{cases} 2 & \text{if } k = 2, \\ \sqrt{6} & \text{if } k \text{ is odd,} \\ 3 & \text{otherwise.} \end{cases}$ <p data-bbox="1068 877 1230 919">So, <math>\frac{1}{c} = 0.5</math>.</p>
 <p data-bbox="391 1199 919 1283"><math>R = \text{the real root of } 3u^5 + u^3 + u^2 - 1 \approx 0.641217</math></p>	$c_k = \begin{cases} \sqrt{3} & \text{if } k = 1, \\ 3 & \text{otherwise.} \end{cases}$ <p data-bbox="987 1220 1312 1272">So, <math>\frac{1}{c} = \frac{1}{\sqrt{3}} \approx 0.577350</math>.</p>
 <p data-bbox="428 1556 886 1640"><math>R = \text{the real root of } 2u^3 + u^2 - 1 \approx 1.521378</math></p>	<p data-bbox="1013 1461 1284 1493"><math>c_k = 2</math> for all <math>k \geq 0</math>.</p> <p data-bbox="1068 1556 1230 1598">So, <math>\frac{1}{c} = 0.5</math>.</p>

# 8

## Multipath zeta functions

### 8.1 Specialization to Ihara zeta function

In [12] and [13], Stark and Terras introduce and develop the path zeta function:

**Definition 8.1.** Let  $r$  be the rank of the fundamental group of an undirected graph  $G$ . Then the  $2r \times 2r$  matrix  $Z$  defined by

$$(Z)_{ij} = \begin{cases} z_{ij} & \text{if } |i - j| \neq r, \\ 0 & \text{otherwise.} \end{cases}$$

where the  $z_{ij}$  are complex variables is the *multipath matrix* of  $G$ .

**Definition 8.2.** Let  $\{C_1, \dots, C_r\}$  be a minimal generating set for the fundamental group of an undirected graph  $G$ . For  $i$ ,  $1 \leq i \leq r$ , let  $C_{r+i} = C_i^{-1}$ . Let  $[C]$  be a prime in  $G$  and choose integers  $\alpha_1, \dots, \alpha_k \in \{1, \dots, r\}$  such that  $\prod_{i=1}^k C_{\alpha_i}$  is equivalent to  $C$  after removing any tail or backtrackings. Then the *multipath norm* of  $C$  is

$$\mathbb{N}_P(C) = z_{\alpha_k \alpha_1} \prod_{i=1}^{k-1} z_{\alpha_i \alpha_{i+1}}.$$

**Definition 8.3.** The *multipath zeta function* of an undirected graph  $G$  is

$$\zeta_P(Z, G) = \prod_{[C]} (1 - \mathbb{N}_P(C))^{-1}.$$

By Theorem 4 in [12],  $\zeta_P(Z, G)^{-1} = \det(I - Z)$ . Also, if we define  $S$ ,  $\{e_1, \dots, e_r\}$ , and  $\{C_1, \dots, C_r\}$  as in Lemma 5.2 and define  $Z^{Ihara}$  to be  $Z$  with variables specialized such that

$$z_{ij} = \begin{cases} d_S(t(e_i), s(e_j)) + 1 & \text{if } |i - j| \neq r, \\ 0 & \text{otherwise,} \end{cases}$$

where  $d_S$  is the vertex distance measure in  $S$ , then by Theorem 5 in [12],  $\zeta_P(Z^{Ihara}, G)$  is the Ihara zeta function of  $G$ . The procedure given in Section 9.4 calculates this specialized multipath matrix.

## 8.2 Specialization for digraphs

We will now extend the definitions of a multipath matrix, norm, and zeta function to directed graphs (which may contain bidirected edges).

**Definition 8.4.** Let  $G$  be a directed graph with underlying graph  $H$ . Let  $Z_H$  be a multipath matrix of  $H$ . Define  $Z_G$  to be  $Z_H$  specialized such that  $z_{ij} = 0$  if  $\{e_i, e_j\} \not\subseteq E_d(G)$ . Then  $Z_G$  is a *multipath matrix* of  $G$ .

**Definition 8.5.** Let  $G$  be a directed graph with underlying graph  $H$ . Let  $\mathbb{N}_P^H$  be the multipath norm associated with  $H$ . Let  $[C]$  be a prime in  $G$ . Then  $\mathbb{N}_P^G(C)$ , the *multipath norm* associated with  $G$  of  $C$  is just  $\mathbb{N}_P^H(C)$ .

The multipath zeta function of a directed graph  $G$  then is defined as in the case of undirected graphs.

Note that  $\zeta_P(Z_G, G) = \zeta_P(Z_G, H)$  (that is, the multipath zeta function of  $G$  is just the multipath zeta function of the underlying graph  $H$  with the specialization of variables described in the definition of the multipath matrix of  $G$ ) since the specialization causes the norm to select for primes in  $H$  which are also primes in  $G$ . So, by Theorem 4 in [12],  $\zeta_P(Z_G, G)^{-1} = \det(I - Z_G)$ .

Now let  $S$  be the spanning tree of  $H$  used to create  $Z_H^{Ihara}$ , the specialization of the multipath matrix of  $H$  used to obtain the Ihara zeta function of  $H$ . Define

$Z_G^{Ihara}$  to be  $Z_H^{Ihara}$  specialized such that  $z_{ij} = 0$  if either  $\{e_i, e_j\} \not\subseteq E_d(G)$  or the walk from  $t(e_i)$  to  $s(e_j)$  in  $S$  is not contained in  $G$ . Then  $\zeta_P(Z_G^{Ihara}, G)$  is the Ihara zeta function of  $G$ . The procedure given in Section 9.5 calculates  $Z_G^{Ihara}$ .



# 9

## *Mathematica* programs

In this chapter, we include our *Mathematica 5.1* programs which should be run after including the *Combinatorica* package with the following command:

```
<< DiscreteMath 'Combinatorica'
```

Also, note that comment lines are bracketed by (\* \*).

### 9.1 Calculating $\zeta^{-1}$ using Theorem 1.2

The following code creates a procedure which returns the reciprocal of the Ihara zeta function of an undirected simple graph  $G$  using the method suggested by Theorem 1.2:

```
(* Procedure for calculating the reciprocal of the Ihara zeta function  
of a graph G *)  
Ihara[G_] := Module[{A, Q, r},  
  A = ToAdjacencyMatrix[G];  
  Q = DiagonalMatrix[Total[A]] - IdentityMatrix[V[G]];  
  r = M[G] - V[G] + 1;
```

```
(1 - u^2)^(r - 1)* Det[IdentityMatrix[V[G]]
  - ToAdjacencyMatrix[G]*u + Q*u^2]
];
```

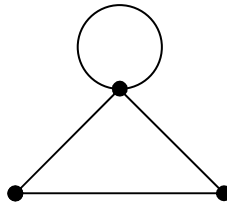


Figure 9.1 A graph with a loop which could be created in *Mathematica* with the command `G=FromUnorderedPairs[{{1,1},{1,2},{1,3},{2,3}}`]

So, after executing this procedure, `1/Ihara[CompleteGraph[4]]` for example would be the Ihara zeta function of  $K_4$ .

Note that the `ToAdjacencyMatrix[ ]` function differs slightly from our definition of an adjacency matrix. According to our definition, each loop is to be counted twice while `ToAdjacencyMatrix[ ]` counts each loop only once. So, for instance, if  $G$  is the graph in Figure 9.1, then our adjacency matrix might be

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

while `ToAdjacencyMatrix[G]` (with the same ordering of vertices used to determine our adjacency matrix) would be the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

So, if we wanted to calculate the Ihara zeta function of a graph which may contain loops in *Mathematica* using the method suggested by Theorem 1.2, we

would want to include following line (which doubles the diagonal entries of the matrix A) immediately after the `A = ToAdjacencyMatrix[G]` command:

```
A = A + DiagonalMatrix[Table[A[[i, i]], {i, 1, Length[A]}]];
```

## 9.2 Creating a directed edge matrix

The following code creates a procedure which returns a directed edge matrix of  $G$  where  $G$  is a simple undirected graph or a directed graph (possibly with bidirected edges) whose underlying graph is simple:

```
(* Procedure for calculating a directed edge matrix of a graph G *)
DEM[G_] := Module[{Ed, Elabels, A},
  Ed = ToOrderedPairs[G];

  (* Use the Union[ ] function to eliminate repeats in the labeling
  scheme which may be introduced if for instance both (a,b) and (b,a)
  are in the directed edge set of G. *)
  Elabels = Union[ToUnorderedPairs[G]];

  (* Use RotateLeft[ ] to create the reversed edges for the column/row
  labeling. *)
  Elabels = Join[Elabels, RotateLeft[Elabels, 0, 1]];

  A = ToAdjacencyMatrix[G];

  Table[ If[(Part[Elabels, i, 2] == Part[Elabels, j, 1]) &&
    (2*Abs[i - j] != Length[Elabels]) &&
    MemberQ[Ed, Elabels[[i]]] && MemberQ[Ed, Elabels[[j]]], 1, 0],
    i, Length[Elabels], j, Length[Elabels]
  ];
```

So, after executing this procedure, we could find the reciprocal of the Ihara zeta function of a graph  $G$  with the following commands:

```
MG=DEM[G];
Det[IdentityMatrix[Length[MG]-MG*u]
```

### 9.3 Creating a modified directed edge matrix

Note that a great deal of the machinery in the procedure for creating a directed edge matrix is to account the ordering of edges such that  $e_{|E|+i} = e_i^{-1}$  (which gives the directed edge matrix the structure described in Remark 1.5). Here we describe a modified directed edge matrix which has less structure but also requires less machinery to construct.

**Definition 9.1.** Let  $E_d(G) = \{e_1, e_2, \dots, e_{|E_d(G)|}\}$  where  $G$  is a (directed or undirected) graph. The  $2|E_d(G)| \times 2|E_d(G)|$  matrix  $\tilde{M}$  given by

$$\left(\tilde{M}\right)_{ij} = \begin{cases} 1 & \text{if } t(e_i) = s(e_j) \text{ and } s(e_i) \neq t(e_j), \\ 0 & \text{otherwise.} \end{cases}$$

is defined to be a *modified directed edge matrix* of  $G$ .

Note that if  $M$  and  $\tilde{M}$  are a directed edge matrix and modified directed edge matrix of a (possibly directed) graph  $G$  respectively, then for some permutation matrix  $P$ ,

$$P^T M P = \begin{pmatrix} \tilde{M} & 0 \\ 0 & 0_k \end{pmatrix}$$

where  $0_k$  is a  $k \times k$  matrix of zeros and  $k = 2|E| - |E_d|$ . Thus, Theorem 1.4 also holds (for both directed and undirected graphs) if the directed edge matrix  $M$  is

replaced by a modified directed edge matrix  $\tilde{M}$  since

$$\begin{aligned}
 \zeta_G(u)^{-1} &= \det(I - Mu) \\
 &= \det(I - P\tilde{M}P^T u) \\
 &= \det(P(I - \tilde{M}u)P^T) \\
 &= \det(P) \det(I - \tilde{M}u) \det(P^T) \\
 &= \det(I - \tilde{M}u) \det(PP^T) \\
 &= \det(I - \tilde{M}u).
 \end{aligned}$$

The following code creates a procedure which returns a modified directed edge matrix of  $G$  where  $G$  is a simple undirected graph or a directed graph (possibly with bidirected edges) whose underlying graph is simple:

```
(* Procedure for calculating a modified directed edge matrix of a graph
G *)
MDEM[G_] := Module[{Ed},
  Ed=ToOrderedPairs[G];
  Table[If[(Part[Ed,i,2]==Part[Ed,j,1]) &&
    (Part[Ed,i,1]!=Part[Ed,j,2]),1,0],i,Length[Ed],j,Length[Ed]]
];
```

So, after executing this procedure, we could find the reciprocal of the Ihara zeta function of a graph  $G$  with the following commands:

```
MG=MDEM[G];
Det[IdentityMatrix[Length[MG]]-MG*u]
```

Also, the directed edge matrix-induced graph  $L(G)$  of  $G$  can be produced using MDEM[ ] with the following command:

```
LG=FromAdjacencyMatrix[MDEM[G], Type -> Directed];
```

## 9.4 Creating a specialized multipath matrix for undirected graphs

The following code creates a procedure which returns a specialized multipath matrix of  $G$  (as discussed in Section 8.1) where  $G$  is a simple undirected graph:

```
(* Procedure for calculating a specialized multipath matrix of an undi-
rected graph G *)
Multipath[G_] := Module[{S, Egen, r},
  S = MinimumSpanningTree[G];
  Egen = Complement[ToUnorderedPairs[G], ToUnorderedPairs[S]];
  r = Length[Egen];
  Egen = Join[Egen, RotateLeft[Egen, {0, 1}]];
  Table[If[Abs[i - j] == r, 0,
    u ^ Length[ShortestPath[S, Egen[[i, 2]], Egen[[j, 1]]]],
    {i, 1, 2*r}, {j, 1, 2*r}
  ];
```

So, after executing this procedure, we could find the reciprocal of the Ihara zeta function of a graph  $G$  with the following commands:

```
Z=Multipath[G];
Det[IdentityMatrix[Length[Z]]-Z]
```

## 9.5 Creating a specialized multipath matrix for (un)directed graphs

The following code creates a procedure which returns a specialized multipath matrix of  $G$  (as discussed in Section 8.2) where  $G$  is a simple undirected graph or a directed graph (possibly with bidirected edges) whose underlying graph is simple:

(\* Procedure for calculating a multipath zeta function matrix of a simple (un)directed graph G \*)

```
DMultipath[G_] := Module[{H, Ed, S, Egen, r},
  H = G;
  Ed = ToOrderedPairs[H];
  H = FromUnorderedPairs[Union[ToUnorderedPairs[H]]];
  S = MinimumSpanningTree[H];
  Egen = Complement[ToUnorderedPairs[H], ToUnorderedPairs[S]];
  r = Length[Egen];
  Egen = Join[Egen, RotateLeft[Egen, {0, 1}]];
  Table[If[Abs[i - j] == r, 0,
    If[Complement[Partition[
      ShortestPath[S, Egen[[i, 2]], Egen[[j, 1]], 2, 1], Ed] == {},
      u ^ Length[ShortestPath[S, Egen[[i, 2]], Egen[[j, 1]]], 0],
    {i, 1, 2*r}, {j, 1, 2*r}
  ];
```

So, after executing this procedure, we could find the reciprocal of the Ihara zeta function of a graph  $G$  with the following commands:

```
Z=DMultipath[G];
Det[IdentityMatrix[Length[Z]]-Z]
```

## 9.6 Calculating poles

Assuming that we have already included the procedure `Ihara[ ]`, the following commands store the poles of the Ihara zeta function of a simple undirected graph  $G$  as a list in the variable `AllPoles` and then stores the real poles as a list in the variable `Realpoles`:

```
AllPoles=u /. Solve[Ihara[G]==0,u];
RealPoles=Select[AllPoles,Element[#,Reals]&]]];
```

The radius of convergence  $R$  of  $\zeta_G(u)$  can then be found using the following command:

```
R=Min[Abs[RealPoles]];
```

The following commands (which should be issued after including the procedure DEM[ ]) find an approximation to  $R$  (stored in the variable ApproxR) using Corollary 4.16 :

```
d=5000; (* d here is as in the corollary. *)
ApproxR=1/Total[MatrixPower[DEM[G],d][[1]]^(1/(d-1))];
```

## 9.7 Adding vertices to edges as in Theorem 6.6

The following code creates a procedure which adds vertices to an edge of a graph (that is, it replaces an edge with a walk of length  $k + 1$ ) as in Theorem 6.6 or contracts the edge if  $k = -1$ .

```
(* Procedure which adds k vertices to edge e of a graph G or contracts
the edge e if k = -1*)
Stretch[G_, e_, k_] := Module[{H, f, i, EE},
  H = G; f = e;
  For[i = 1, i <= k, i++,
    H = AddVertex[DeleteEdge[H, f]];
    H = AddEdge[AddEdge[H, {f[[1]], V[H]}], {V[H], f[[2]]}];
```



```

      f[[2]] = V[H];
];
If[k == -1,
  EE = ToUnorderedPairs[H];
  H = DeleteVertex[DeleteEdge[FromUnorderedPairs[
    Table[{If[EE[[i, 1]] == f[[2]], f[[1]], EE[[i, 1]], If[EE[[i, 2]] == f[[2]],
      f[[1]], EE[[i, 2]]}], {i, 1, Length[EE]}], {f[[1]], f[[1]]}, f[[2]]];
];
H
];

```

So, after executing this procedure, we could for example create a graph  $G$  which is  $K_5$  with 5 vertices added to the edge from vertex 1 to vertex 2 with the following command:

```
G = Stretch[CompleteGraph[5], {1, 2}, 5];
```

## 9.8 Calculating the polynomials $p_{A,B,C,F}$ from Corollary 6.10

For every allowable  $A, B, C$ , the following program outputs the directed graph  $Y(A, B, C)$ , the reciprocal of it Ihara zeta functions, and the polynomial  $p_{A,B,C,F}(u)$  as in Corollary 6.10. It also stores the product  $\zeta_{Y(A,B,C)}(u)^{-1}p_{A,B,C,F}(u)$  in a list called AllTogether and outputs the sum of this list before terminating. By the corollary, this sum is of course the Ihara zeta function of  $X(k_1, k_2, \dots, k_{|F|})$ .

(\* EdY is the set of directed edges of the graph Y. \*)

```
EdY={{1,2},{2,3},{3,4},{4,1},{2,4},{2,1},{1,4},{4,3},{3,2}};
```

(\* F and k are as in the corollary. \*)

```

F={{1,2},{2,4}};
k={3,5};

pA[n.]:=1-2*u^n+u^(2*n);
pB[n.]:=u^n-u^(2*n);
pC[n.]:=u^n-u^(2*n);
pF[n.]:=u^(2*n);

AllTogether={};
For[i=0,i<4^Length[F],
  EdYABC=EdY;
  p=1;
  For[j=0,j<Length[F],
    b=IntegerPart[Mod[i,4^(j+1)]/(4^j)];
    If[(b==0) || (b==1),EdYABC=Complement[EdYABC,{F[[j+1]]}];
    If[(b==0) || (b==2),
      EdYABC=Complement[EdYABC,{Reverse[F[[j+1]]]}];
    If[(b==0),p=p*pA[k[[j+1]]];
    If[(b==1),p=p*pB[k[[j+1]]];
    If[(b==2),p=p*pC[k[[j+1]]];
    If[(b==3),p=p*pF[k[[j+1]]];
  j++];
W=Table[If[(Part[EdYABC,i,2]==Part[EdYABC,j,1]) &&
  (Part[EdYABC,i,1]!=Part[EdYABC,j,2]),1,0],
  {i,Length[EdYABC]},{j,Length[EdYABC]}];
Z=Det[IdentityMatrix[Length[W]]-W*u];
Print["-----"];
ShowLabeledGraph[FromOrderedPairs[EYABC,Type->Directed]];
AllTogether=Append[AllTogether,Z*p];
Print["1/Z=",Factor[Z],"=",Expand[Z] ];
Print["pABCF=",p];

```

```

i++];

Print["-----"];
Print[Expand[Total[AllTogether]]];

```

## 9.9 Targeting a specific radius of convergence

The following program attempts to create a graph  $G$  such that the reciprocal of the radius of convergence of  $\zeta_G(u)$  is within  $\epsilon$  of a target value greater than 1. Here we have chosen  $\epsilon$  to be 0.0000001 and the target value to be the number  $e$ . The procedures `Stretch[ ]` and `Ihara[ ]` are used and should be included prior to executing this program.

The program also displays its progress by showing the graph it is constructing after each addition along with the reciprocal of the radius of convergence of its Ihara zeta function. It also provides a written description of the additions it has made.

```

target = E;
epsilon = 0.0000001;

k = Floor[target] + 2;
d = -1;
H = CompleteGraph[k];
R = 1/(k - 2);
Print["Starting with K_", k, "."];
G = H;
While[(Abs[1/R - target] > epsilon),
  H = Stretch[ AddEdge[GraphUnion[G, CompleteGraph[k]],
    1, V[G] + 1], 1, V[G] + 1, d];
  Z = Ihara[H];

```

```

allzeros = u /. Solve[Z == 0, u];
R = Min[Abs[1/N[1/allzeros, 10]]];
If[(1/R > target) && ((1/R - target) > epsilon),
  d = d + 1; If[(d > 1) && (k > 3), k = k - 1; d = -1],
  Print["-----"];
  Print["Adding K_", k, " attached by walk of length ", d + 1, " :"];
  GraphPlot[H, VertexStyleFunction -> Automatic];
  Print["target=", N[target, 10], "..."];
  Print[" 1/R=", N[1/R, 10], "..."];
  G = H
];
];

```

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