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NASH BARGAINING AND RISK AVERSION

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Abstract. It is widely accepted among axiomatic bargaining theorists that if one bargainer is more risk averse than a second, the second will be a tougher bargaining opponent than the first against all opponents. We argue that this relationship between risk aversion and bargaining toughness is both highly fragile, and more nuanced than previously articulated. In the Nash and Kalai-Smorodinsky bargaining frameworks, we establish that when a bargainer is compared with a second who is “almost globally” more risk averse than the first, the supposedly immutable relationship between bargaining effectiveness and risk aversion evaporates. Specifically, we identify an upper-hemicontinuity failure of a correspondence which maps the power set of all lotteries to those utility pairs that satisfy our “almost global” comparative risk aversion relation on these subsets. We trace the consensus view that tougher bargainers are less risk-averse to an exclusive focus on precisely the point at which this correspondence implodes.

Keywords: Bargaining theory, Nash Bargaining, Kalai-Smorodinsky, Risk aversion.

JEL classification: C78, D1

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In the extensive literature on axiomatic bargaining theory, it is widely accepted that bargainers who are less risk-averse make tougher bargaining opponents. This connection has been identified as “one of the results most frequently quoted in the bargaining literature” (Volij and Winter, 2002, p. 120). The first formal statements of the result are in the seminal works by Roth (1979) and Kihlstrom, Roth and Schmeidler (1981) (henceforth KRS). The theme is further developed in Roth and Rothblum (1982) (RR) and Safra, Zhou and Zilcha (1990) (SZZ). In particular, KRS’s Theorems 1 and 2 relate, respectively, to the two best-known axiomatic bargaining models, developed by Nash and Kalai-Smorodinsky (KS). These theorems compare bargaining situations in which a given opponent with utility \( v \) bargains either against a benchmark player with utility \( u_0 \), or against another, globally more risk-averse player with utility \( u \).

For brevity, we shall henceforth identify players with their utility functions: “\( u \) bargains with \( v \)” will serve as shorthand for “the player with utility function \( u \) bargains against another with utility function \( v \).” We henceforth say that \( w \) is a tougher Nash-bargainer than \( w' \) against \( v \) if Nash-KS bargaining with \( w \) yields \( v \) less utility than bargaining with \( w' \). KRS’s results establish that under both solution concepts, if \( u \) is more risk-averse than \( u_0 \), then \( u_0 \) is a tougher bargainer than \( u \).

In this paper, we argue that the widely cited relationship between comparative global risk aversion and comparative bargaining toughness is in fact both highly fragile and more nuanced than has previously been articulated. It is fragile in the sense that if the notion of “global” is relaxed ever so slightly, in a particular direction, the relationship evaporates. It can be refined by distinguishing between risks that matter for bargaining toughness and those that don’t: in particular, in both frameworks, comparative aversion to risks involving a positive probability of negotiation breakdown—the worst possible outcome—plays a pivotal role. In the Nash framework, there is a link—in a sense to be made precise—between comparative bargaining toughness and comparative aversion to lotteries involving bad outcomes, but no relationship whatsoever if the lotteries involve only good outcomes. In the KS framework, we identify a necessary and sufficient condition for \( u \) to be a tougher bargainer than \( u_0 \) against \( v \); it can never be satisfied if \( u \) is globally more risk averse than \( u_0 \), but can be satisfied if \( u \) is “almost globally” more risk averse than \( u_0 \), in the sense we define. Finally, we establish that given any benchmark bargainer \( u_0 \), there is, in both frameworks,

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2 Throughout the paper, the relations “tougher than” and “more risk-averse than” will be irreflexive, i.e., the relevant inequalities will be strict rather than weak.
a bargainer \( u \) who is strictly tougher than \( u_0 \) against every bargaining opponent, yet is more risk averse than \( u_0 \) with respect all lotteries except those that assign positive probability either to negotiation breakdown, or to comparably bad negotiated outcomes. Since \( u_0 \) is a strictly tougher bargainer against all opponents than any \( u \) who is globally more risk averse than \( u_0 \), our result reveals a discontinuity—more precisely, an upper hemicontinuity failure—in the relation that maps comparative risk aversion to bargaining toughness.

To make these ideas precise, we formalize our notion of “almost globally more risk averse than...” in the following way. The standard definition of comparative risk aversion is:³ “\( u \) is more risk averse than \( u_0 \) if any risk that is undesirable for \( u \) is also undesirable for \( u_0 \).” If the universe of possible lottery outcomes is \( X \), then it is natural to assert that \( u \) is almost globally more risk averse than \( u_0 \) if for some subset \( X' \subset X \) that almost coincides with \( X \), any risk involving only outcomes in \( X' \) that is undesirable for \( u_0 \), is also undesirable for \( u \). This comparison leaves open the possibility of some remaining risks—necessarily ones which assign positive probability to realizations in \( X \setminus X' \)— that are undesirable to \( u_0 \) but acceptable to \( u \). If we apply our notion of “almost” to a sequence of subsets \( \{X^n\} \) that approach \( X \), while excluding the very worst outcomes in \( X \), the risks which remain exempt from comparison are concentrated in a vanishingly small subset of the universe of all possible risks. Yet for any \( n \), the set of utility pairs \( (u^n, u_0) \) which satisfy our “almost global” comparison criterion is extremely large, relative to the set of pairs \( (u, u_0) \) such that \( u \) is globally more risk averse than \( u_0 \). These much larger sets, that meet our comparative criterion for each \( n \), include pairs \( (u^n, u_0) \) such that \( u^n \) is a much tougher bargainer than \( u_0 \).

1. Preliminaries

In the classical formulation of axiomatic bargaining problems, two players, with utility functions \( u \) and \( v \) respectively, bargain over a set of possible feasible outcomes. If they fail to agree upon one of these, a disagreement outcome, \( D \), is implemented. (In general, outcomes are points in \( R^2 \), so utilities are multivariate. With one additional assumption, they can be represented by univariate functions. To reduce notation, we will use the same symbols for utilities and their univariate representations.) We assume that players’ utilities are defined on the simplex

\[
Z = \{ z \in R^2 : z_i \geq 0; \sum_{i} z_i \leq 1 \}
\]

and that \( D = (0, 0) \). Throughout, we assume that \( u \) derives

\[
(1) 3 (Eeckhoudt et al., 2005, Defn 1.4, p. 14). Eeckhoudt et. al. add the caveat that \( u_0 \) and \( u \) must have the same level of income. In the context of bargaining theory, there is no natural notion of “income.”
utility only from the first component of \( z \in Z \), while \( v \) derives utility only from the second. We will hold \( v \) constant and compare the “bargaining performance” of utility \( u \) against \( v \) relative to that of a benchmark utility \( u_0 \). Let \( S = \{ (u(z), v(z)) : z \in Z \} \subset R^2 \) denote the set of utility pairs that can be agreed upon, and let \( d = (u(D), v(D)) \in R^2 \) denote the utility pair that results from disagreement. Following Osborne and Rubinstein (1990, p. 10), we define a bargaining problem to be a pair \((S, d)\), where \( S \subset R^2 \) is compact and convex and there exists \( s \in S \) such that \( s \gg d \). Let \( B \) denote the set of all bargaining problems. A bargaining solution is a function \( f : B \rightarrow R^2 \) that assigns to each bargaining problem \((S, d) \in B \) a unique element of \( S \).

1.1. Bargainer utilities. In §§1-2, we impose two assumptions on bargainers’ utility functions:

Assumption A1): Utilities are von-Neumann Morgenstern, strictly concave and twice continuously differentiable on \( Z \). For all \( z \in [0, 1]^2 \), \( \frac{\partial u(z)}{\partial z_1} > 0 \) and \( \frac{\partial v(z)}{\partial z_2} > 0 \). An axiom invoked by both Nash and KS is that if either \( u \) or \( v \) is replaced by an affine transformation of itself, the bargaining outcome must remain unchanged. We therefore assume, w.l.o.g. Assumption A2): Each player derives utility 0 from \( D \) and a maximum utility of 1 on \( Z \).

A second axiom imposed in both frameworks requires that the bargaining solution must lie on the Pareto frontier, which is the north-east boundary of \( Z \), i.e., \( \{ (x, 1 - x) : x \in [0, 1] \} \). Invoking this axiom, we restrict our attention to this frontier, denoted \( P = [0, 1] \), and will henceforth treat all utility functions as if they were univariate functions defined on \( P \), with the understanding that the scalar \( x \in P \) represents the vector \((x, 1 - x) \in Z \). Thus, \( u(x) \) will denote the utility that derives from the share \( \frac{1 - x}{2} \). We can now rewrite

\[
\begin{align*}
\frac{\partial u(x)}{\partial x} > 0 > \frac{\partial v(x)}{\partial x}.
\end{align*}
\]

To streamline our presentation, we restrict attention to a compact family of utility functions:

Assumption A3): Bargainers’ utilities are drawn from a set \( U \) of functions satisfying assumptions A1-A2 that is compact in the sup norm topology.

1.2. Risk Aversion. The seminal papers that have considered comparative risk aversion in bargaining models compare players’ risk aversion over the entire domain of their utility functions. This paper augments these comparisons with analogous ones on subsets of the utility domain. Following Eeckhoudt et al. (2005), we will say that bargainer \( u \) is globally more risk-averse than \( u_0 \) (abbreviated to GMRA) if any risk that that is undesirable for \( u_0 \) is “even more”
undesirable for $u$. (Throughout this paper the term “more” will denote a strict relation.) Analogously, we say that $u$

4 For example, an affine transformation of $u$ is a function $w = a + bu$, for some $(a, b) \in \mathbb{R} \times \mathbb{R}^+$. 
5 For functions with domain $S$, a metric for the sup norm topology is: $\rho(f, g) = \sup_{s \in S} |f(s) - g(s)|$. 
is strictly risk-averse than $u_0$ on $X \subset P$ (abbreviated to MRA$^X$) if any risk involving outcomes in $X$ that is undesirable for $u_0$ is even more undesirable for $u$.

The literature has identified three equivalent ways to formalize the concept of GMRA. All three can be extended immediately to MRA$^X$. First, for each $X \subset P$, we define $u$ to be MRA$^X$ than $u_0$ if $^6$ $u|_X = \varphi(u_0|_X)$, where $\varphi$ is an increasing, strictly concave function.

Prop. 1 (adapted from Eeckhoudt et al. (2005, Prop 1.5)): For $X \subset P$, the following three statements are equivalent:

a) $u$ is MRA$^X$ than $u_0$ at $x$ if $u(x) > u_0(x)$;

b) $\forall x \in X$, $r^u(x) := -\frac{\partial^2}{\partial x^2} u(x) > r^0(x) := -\frac{\partial^2}{\partial x^2} u_0(x)$;

c) for any uncertain event $\tilde{z}$ with distribution $\mu$ whose support is contained in $X$, $u^{-1}(E_\mu[\tilde{z}]) > u_0^{-1}(E_\mu[\tilde{z}])$.

$r^u(\cdot)$ and $r^0(\cdot)$ in b) are the Arrow-Pratt coefficients of absolute risk aversion for $u$ and $u_0$, respectively (Pratt, 1964, p. 122); $u^{-1}(E_\mu[\tilde{z}])$ and $u_0^{-1}(E_\mu[\tilde{z}])$ in c) are the certainty equivalents$^7$ of the lottery $\tilde{z}$, for $u$ and $u_0$ respectively.

We focus primarily on subsets of $P$ of the form $[x, 1]$. For every $x > 0$, the restriction MRA$^{[x, 1]}$ is strictly weaker than GMRA; however, when $x = 0$, the two relations coincide, since $[0, 1] = P$. The condition that $u$ is MRA$^{[x, 1]}$ than $u_0$ imposes several restrictions on the relationship between $u$ and $u_0$, which are summarized in Lemma 1. Part e) of the lemma invokes some terminology: we say that $u$ intersects $u_0$ at $x$ if $u(x) = u_0(x)$. Further, we say that $u$ cuts $u_0$ below at $x$ if $u$ intersects $u_0$ at $x$ and $u'(x) > u'(y)$, with strict inequality if $x < 1$.

Lemma 1 (Implications of MRA$^{[x, 1]}$):

For $x \geq 0$, suppose that $u$ is MRA$^{[x, 1]}$. Then

$\forall y \in [x, 1] s.t. u(y) \geq u_0(y)$, then $u'(\cdot) > u_0'(\cdot)$ on $[x, y]$;

c) if $u(y) \geq u_0(y)$, then $u(y) > u_0(y)$;

d) if $u(x) = u_0(x)$, then $u'(\cdot) > u_0(\cdot)$ on $[x, 1]$;

e) if $u(y) < u_0(y)$ for some $y \in [x, 1]$, then $\exists y' \in (y, 1]$ s.t. $u$ cuts $u_0$ from below at $y'$.

In words, if $u$ is MRA$^{[x, 1]}$ than $u_0$ then: b) if $u$ is weakly steeper than $u_0$ at $y$, it is strictly steeper to the left of $y$; c) if $u$ is weakly flatter than $u_0$ at $y$, then it lies above $u_0$ at $y$;

d) if $u$ and $u_0$ agree at $x$, then $u$ dominates $u_0$ on the interior of $[x, 1]$. e) if $u$ is dominated by $u_0$ at $y$, then when it

$^6$ Given $f : Y \to R$, and $Y' \subset Y$, $f_{|Y'}$ denotes the restriction of $f$ to $Y'$, i.e., the function $g : Y' \to R$ such that for all $x \in Y'$,
$g(x) = f(x)$.

$^7$ The certainty equivalent for $u$ of a lottery $\tilde{z}$ is a certain outcome $y$ which yields $u$ the same utility as $\tilde{z}$.
intersects $u_0$ at some $y^* > y$—which it must, by assumption A2)—$u'$s slope at $y^*$ will strictly exceed $u_0'$s if $y^* < 1$, and weakly exceed $u_0'$s if $y^* = 1$.

Since by assumption A2, $u_0(0) = u(0) = 0$, an immediate implication of part d) of Lemma 1 is:

**Remark 1:** If $u$ is GMRA than $u_0$ then $u(\cdot) > u_0(\cdot)$ on the interior of $P$.

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2. Bargaining

We focus on the two best-known axiomatic bargaining solution concepts, $f^N$ and $f^{KS}$, formulated, respectively, by Nash (Nash, 1950; Nash, 1953) and Kalai and Smorodinsky (1975). Nash imposed four axioms: Symmetry, Pareto Efficiency, Invariance to Equivalent Utility representations, and Independence of Irrelevant Alternatives (IIA). He showed that these four axioms are satisfied by a unique element of $P$, which is the argmax, denoted $f^N$, of the function $N(\cdot\mid u, v)$ defined in (2) below. Having imposed assumption A2, we can define the Nash bargaining outcome as:

$$f^N(u, v) = \arg\max_{x \in P} N(x \mid u, v), \text{ where } N(x \mid u, v) := \log(u(x)) + \log(v(x))$$

(2)

$f^N(u, v)$ is the unique $x$ value that solves the first-order condition

$$\frac{\partial N(x \mid u, v)}{\partial x} = 0 \quad \frac{u(x)}{v(x)} = \frac{u(x)}{v(x)} \quad (3)$$

Because $N(\cdot \mid u, v)$ is continuous and strictly concave, and because (from A2) $N(x \mid u, v)$ approaches $-\infty$ as $x$ approaches either 0 or 1, a unique solution to (3) exists. Moreover, for all $u, v$ satisfying assumptions A1-A2, $f^N(u, v) \in (0, 1)$. (4)

KS's axiomatic framework replaces Nash's IIA axiom with a monotonicity axiom. KS's concept is defined in terms of the disagreement utility vector—in our case, $(0, 0)$—and the vector—in our case, $(1, 1)$—representing the highest utility that each player can obtain from some negotiated agreement. In words, the KS outcome is the intersection of the Pareto frontier of $S$ with the 45° line (in utility space) through the origin. Formally,

$$f^{KS}(u, v) = KS(\cdot\mid u, v)^{-1}(0) \text{ where } KS(x \mid u, v) := u(x) - v(x).$$

(5)

That is, $f^{KS}(u, v)$ is the (unique) root of KS($\cdot\mid u, v$). Once again, we have for all $u, v$ satisfying assumptions A1-A2, $f^{KS}(u, v) \in (0, 1)$. (6)

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8 Apart from the Pareto and Invariance axioms, the others invoked by Nash and KS play no role in the present paper beyond implying the solution concepts defined by (2) and (5). Accordingly, we do not define them here. For a presentation and detailed discussion of each axiom, see Osborne and Rubinstein (1990).

9 More generally, $N(x \mid u, v) := \log(u(x) - u(D)) + \log(v(x) - v(D))$. Assumption A2 imposes $u(D) = v(D) = 0$. 

Further, since the set $\mathcal{U}$ from which utilities are drawn is compact: $^{10}$ there exists $\varphi^+ > 0$ s.t. if $u, v \in \mathcal{U}$, then for $x \in [\mathcal{KS}, N]$ $f^{sc}(u, v) \in [\varphi^-, 1 - \varphi^+]$.

\[ (7) \]

Now fix $v \in \mathcal{U}$. Let $x^0$ denote the KS outcome, $x^w$ the Nash outcome, and let $x^k$ denote the KS outcome, $x^{w_k}$ the Nash outcome, when $u$ bargains against $v$. Since $v$ prefers lower $x$-values to higher ones (see (1)), the classical result is that if $u$ is GMRA than $u_0$, then $x^N > x^N_0$ and $x^K > x^K_0$. We will identify conditions under which these inequalities are reversed. We say that $u$ is a tougher Nash-bargainer than $u_0$ against $v$ if $x^N > x^N_0$. $^{\text{KS}}$

Both comparisons in (8) are defined relative to specific solution concept and a specific bargaining partner. We also define a notion of “global relative toughness”: $u$ is a tougher bargaining opponent than $u_0$ if $u$ is a tougher Nash- and KS-bargainer than $u_0$.

\[ (8) \]

Using Lemma 1, Lemmas 2 and 4 below identify conditions under which, for any $x > 0$, $u$ is MRA$^{[x, 1]}$ than $u_0$ and a tougher KS- or Nash-bargainer than $u_0$ against $v$.

Lemma 2 (Necessary and sufficient condition: KS): For $x > 0$, if $u$ is MRA$^{[x, 1]}$ than $u_0$ and $x < x^K_0$, then $u$ is a tougher KS-bargainer than $u_0$ against $v$ iff $u$ cuts $u_0$ from below at some $x > x^K_0$.

Lemma 2, combined with Remark 1, provides an alternative proof of KRS’s result associating the GMRA relation to relative KS-bargaining toughness: for $u$ to be KS-tougher than $u_0$ against $v$, $u$ must cut $u_0$ from below; but from Remark 1, this cannot happen if $u$ is GMRA than $u_0$. Conclude:

Prop. 2 (KRS, Thm 2): If $u$ is GMRA than $u_0$, then $u_0$ is a tougher KS-bargaining opponent than $u$.

Turning to Nash, Lemma 3 below provides a useful characterization of when $u$ is a tougher Nash bargainer than $u_0$ against $v$, although the condition does not depend on $u$ being MRA$^X$ than $u_0$.

Lemma 3 (Necessary and sufficient conditions: Nash): $u$ is a tougher Nash-bargainer than $u_0$ against $v$ iff $u(x_0^{N}) > u(x_0^{K})_0$. $^{\text{Nash}}$

Lemma 3 has an appealing intuitive interpretation: $u$ is a tougher Nash-bargainer than $u_0$ against $v$ iff, at the solution to the bargaining problem between $u_0$ and $v$, the elasticity$^{11}$ of $u$'s utility w.r.t. $u$'s negotiated share is greater than the corresponding elasticity for $u_0$. 
The relationship between comparative risk aversion and toughness is less clearcut in Nash’s framework than in KS’s. Lemma 4 is a partial analog of Lemma 2, but it is not a characterization.

10 Clearly, $f^N$ is a continuous function of $u$ and $v$. Since $U$ is compact, $f^N$ attains a maximum and minimum on $U \times U$ (Weierstrass). From (4), both maximum and minimum belong to $(0, 1)$. We can now define $\phi^N = \min_{u,v \in U} (f^N(u,v), 1 - f^N(u,v))$. Define $\phi^{KS}$ analogously, and let $\phi^* = \min(\phi^N, \phi^{KS})$.

11 Given a function $f$ of $x$, the elasticity of $f$ w.r.t. $x$ is defined as \( \frac{x \partial f(x)}{\partial x} \).
Lemma 4 (Sufficient conditions: Nash): For $x > 0$, if $u$ is $\text{MRA}^{[x,1]}$ than $u_0$, and $u$ cuts $u_0$ from below at some $x_0 \geq x^N$, then $u$ is a tougher Nash-bargainer than $u_0$ against $v$.

Since the inequality in the condition of Lemma 4 is weak—($u$ cuts $u_0$ from below at $x \geq x^N$)—but the inequality that the condition implies is strict—($x^N > x^N$)—it follows from continuity that the condition cannot also be necessary; there must exist an open neighborhood $X$ containing $x^N$ such that if $u$ cuts $u_0$ from below at any $x \in X$, then $x^N > x^N$. Indeed, in contrast to the KS formulation, it is not necessary for comparative Nash-toughness that $u$ cuts $u_0$ from below at any point: as Fig. 1 illustrates, $u$ can be Nash-tougher than $u_0$ against $v$, even when $u$ lies everywhere above $u_0$. (For heuristic clarity, we have drawn $u$ and $u_0$ in the figure as piecewise linear, so that neither function satisfies assumption A1; obviously, there are smooth perturbations of both functions which do satisfy assumption A1, while preserving the salient properties of the figure.) When $u_0$ bargains against $v = u_0$, by symmetry the Nash solution is $x^N = 0.5$. But since $u_0$ kinks at $x^N - \varphi$, while $u$ kinks just to the right of $x^N$, we have $u(x^N) \gg u'(x^N)$, while $u(x^N) = u_0(x^N)$.

It follows from Lemma 3, therefore, that when $\varphi > 0$ is sufficiently small, $x^N > x^N$. Note in this
example that for $x = x^N - \varphi$, $u$ is $\text{MRA}^{[x,1]}$ than $u_0$. (Clearly, $u$ is a concave transformation of $u_0|_{[x,1]}$.). However, $u$ is not GMRA than $u_0$: for example, as the figure illustrates, there is a lottery $z^\sim$, which realizes $x$ with probability $p$ and zero otherwise, and a sure outcome $y$, such that

$$Eu(z^\sim) = pu(x) > u(y) = u_0(y) > pu_0(x) = Eu_0(z^\sim)$$

so that $u$ prefers $z^\sim$ to the certain outcome $y$, but $u_0$ prefers $y$ to $z^\sim$.

Our next result, Prop. 3, is closely related to, but distinct from, the classical result that if $u$ is GMRA than $u_0$, then $u_0$ will be a tougher Nash-bargaining opponent than $u$ against $v$. To establish that $u_0$ is Nash-tougher than a specific opponent $v$, it suffices to assume only that $u$ is more averse than $u_0$ with respect to risks involving outcomes that are less satisfactory to $u$ than the solution obtained when $u_0$ Nash-bargains against $v$. In is in precisely this sense that what matters when it comes to Nash bargaining is relative aversiveness to risk over bad outcomes.

**then $u$: If $u$ is $\text{MRA}^{[x,1]}$ than $u$, then $u_0$ is a tougher Nash bargainer than $u$ against $v$.**

For completeness, we restate KRS’s result for Nash-bargaining, which is the analog of Prop. 2 above: it can be rederived as an immediate corollary of Prop. 3

**Prop. 4** (KRS, Thm 1): *If $u$ is GMRA than $u_0$, then $u_0$ is a tougher Nash-bargaining opponent than $u$.*

We now come to the main result of this paper: for any $x > 0$ no implication can be drawn about $u$’s bargaining toughness relative to $u_0$’s from the fact that $u$ is $\text{MRA}^{[x,1]}$ than $u_0$. Recalling that $\varphi^-$ was defined in (7):

**Prop. 5:** *$\forall x \in (0, \varphi^-)$ and $u_0 \in \text{int}(U)$, $\exists u \in U$ who is $\text{MRA}^{[x,1]}$ than $u_0$ and a tougher bargaining opponent than $u_0$.***

Now let $\mathcal{R} : [0, 1] : U \times U$ denote the correspondence mapping selected subsets of $P$ to the utility pairs that satisfy our comparative risk aversion relation:

$$\mathcal{R}(x) = \{(u, u_0) \in U \times U : u \text{ is } \text{MRA}^{[x,1]} \text{ than } u_0\}.$$

(10) For each $n$, $\text{MRA}^{[x,1]}$ is a strictly weaker comparative concept than GMRA, but in the limit, the distinction disappears, since $[0, 1] = P$. In the informal language of our introductory section, when

12 Since $\mathcal{R}()$ is defined on $\mathcal{R}$, we endow its domain with the Euclidean metric. We endow its co-domain with the metric $\gamma(f, g) = \max(\rho(f_1, g_1), \rho(f_2, g_2))$, where $\rho$ is the sup norm metric (footnote 5).
n is very large, \(R(1/n)\) consists of utility pairs \((u^n, u^n)\) such that \(u^n\) is “almost globally” more risk averse than \(u^n\), while \(R(0)\) consists of pairs \((u, u_0)\) such that \(u\) is globally more risk averse than \(u_0\). The disjunction between Prop. 5 and Props. 2 & 4 is a consequence of the fact that \(R(\cdot)\) is not upper hemicontinuous\(^{13}\) at \(x = 0\).

Prop. 5 establishes that for each \(n > 1/\varphi\), there exists a pair \((u^n, u^n) \in R(1/n)\) such that \(u^n\) is a tougher bargaining opponent than \(u^n\). Yet from Props. 2 & 4, if \(u\) is GMRA than \(u_0\), then \(u_0\) is a tougher bargaining opponent than \(u\). Since “tougher than” has been defined as a strict relation, and both the Nash and KS solution concepts are clearly continuous when their domains are endowed with the metric \(y\), defined in footnote 12, these propositions, taken together, necessarily imply that \(R(\cdot)\) is not upper hemicontinuous at zero.\(^{14}\)

\[ \text{Figure 2. } u \text{ is more risk averse than } u_0 \text{ w.r.t. almost all risks, but a much tougher bargainer} \]

Fig. 2 above graphs a pair of utilities, \((u^-, u_0)\), which illustrate two key messages of this paper. First, it demonstrates that \(u^-\) may be a tougher bargainer than \(u_0\), even though \(u^-\) is more aversive than \(u_0\) to all risks except ones involving a positive probability of an extremely undesirable outcome. Second, it reveals the magnitude of the implosion (upper hemicontinuity failure) of the correspondence \(R(\cdot)\)

\[^{13}\text{A correspondence } \Psi : S \to T \text{ is upper hemicontinuous if for every } s \in S \text{ and every open neighborhood } \Psi \text{ of } \Psi(s) \text{ there is an open neighborhood } S \text{ of } s \text{ such that } \Psi(s) \subset \Psi \text{ for every } s \in S.} \]

\[^{14}\text{While for Nash-bargaining, this implication is a little opaque, for KS-bargaining, it is transparent: from Remark 1, } (u, u_0) \in R(0) \text{ implies } u(\cdot) > u_0(\cdot) \text{ on } (0, 1). \text{ Since from assumption A2), } u \in U \implies u(0) = 0 \text{ and } u(1) = 1, \text{ it follows that } \Psi := \{ (u, u_0) \in U \times U : u(\cdot) > u_0(\cdot) \text{ on } (0, 1) \} \text{ is an open neighborhood of } R(0). \text{ Now consider an arbitrary open neighborhood } S \text{ of } 0. \text{ For } n \text{ sufficiently large, } 1/n \in S. \text{ From Prop. 5, there exists } (u^n, u^n) \in R(1/n) \text{ such that } u^n \text{ is a tougher KS-bargaining opponent than } u^n. \text{ From Lemma 2, } u^n \text{ must cut } u^n \text{ from below. Hence } (u^n, u^n) \not\in \Psi.} \]
at zero. Since the figure serves only as an heuristic example, we do not require that $u_0$ belongs to the compact set $U$ specified in assumption A3). Moreover, in the figure, $u_0$ is drawn for clarity as piecewise linear, with a kink at $x$; as in Fig. 1, it can obviously be smoothened and made strictly concave without changing any salient features of the example.

If $x < \phi \leq \min ff^{sc}(u^-, v) : v \in U, sc \in \{KS, N\}$, then since $u^-$ lies everywhere below $u_0$, $u^-$ will be a tougher bargainer than $u_0$ against every opponent $v \in U$ (Lemmas 2 and 4). On the other hand, $u_0$ is risk neutral, and hence less risk-averse than $u^-$ with respect to all lotteries except ones that assign positive probabilities to outcomes on either side of $x$. The figure illustrates one such exception: $u^-$ prefers the illustrated certain outcome $y$ to the lottery $z$ realizing $x \leq x < x_2$ with equal probability, while $u_0$ prefers $z$ to $y$. Indeed when $x = 0$, $u_0$ is risk neutral with respect to all risks except ones involving a positive probability of either negotiation breakdown or negotiated outcomes that are barely preferred to breakdown. To summarize, the example challenges the conventional wisdom, since: (a) $u^-$ is more risk averse than $u_0$ with respect to “virtually all” risks except for ones effectively involve negotiation breakdown; but (b) is (much) a tougher bargainer than $u_0$ against all opponents in $U$, in both the Nash and KS frameworks. On the other hand, the properties that the example exhibits are intuitive: $u_0$ desperately wants to achieve an agreement, but cares very little about which particular agreement is obtained. Unsurprisingly, against almost all $v$’s, the agreement predicted by both the Nash and KS frameworks will overwhelmingly favor $v$, delivering $u_0$ a share in a neighborhood of $x$.

The second purpose of Fig. 2 is to illustrate the magnitude of the correspondence $R(\cdot)$’s implosion at zero. For each $n \in N$, there is a pair $(u^-, u^0) \in R(1/n)$, having the same form with $u^0$

$$u^0(x) = \begin{cases} 0 & \text{if } x < 1/n \\ \frac{(n-1)x}{n} & \text{if } x \leq 1/n \\ \frac{(n-x^{-2n})/n - 1}{n-1} & \text{otherwise} \end{cases}$$

Clearly, much of its domain—for example, on $(x, 1/2)$—we have $y^{-\cdot} \ll u^0(\cdot)$. On the other hand, from Remark 1, $(u^-, u_0) \in R(0)$ implies $u^-(\cdot) \geq u_0(\cdot)$. Thus, for any $n$, $R(0)$ is a much smaller set than $R(1/n)$. Indeed, for each $\phi > 0$, there exists $\delta > 0$ and $N$ such that for $n > N$, $u^0(\delta) > 1 - 0.5\phi$ while $(u^-, u_0) \in R(0)$ implies $u_0(\delta) < 0.5\phi$. Hence, in the metric on function pairs defined in footnote 12, $\gamma((u^-, u^0), (u^-, u_0)) > 1 - \phi$. To summarize, we have established that there is a sequence $\gamma^{-\cdot}$ such that for all $n$, $(u^{-n}, u_0) \in R(1/n)$, and $\lim_n\gamma((u^{-n}, u_0))$ is of $\gamma$-distance 1 from any pair

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15 For any given compact set $U$, if $x$ is sufficiently small, $u_0$ will no longer belong to $U$. However, neither of the heuristic messages conveyed by the example depend on this inclusion.
\((u^-, u_0) \in R(0)\). The significance of this implosion will be immediately apparent: for any \(n\), the set
\(R(1/n)\) is huge relative to \(R(0)\); there is room in the former set for a very diverse array of different kinds of bargainer pairs, including pairs such as \((u^-, u^0)\), where \(u^-(\cdot) \ll u^0(\cdot)\), and is much a tougher
\[
\begin{pmatrix}
0 & 0
\end{pmatrix}
\]
bargainer than \(u^0\). But at the point of implosion, all bargaining pairs \((u^-, u_0)\) are eliminated unless they satisfy \(u^-(\cdot) \geq u_0(\cdot)\).

3. Conclusion

The goal of this paper has been to challenge the virtual consensus among bargaining theorists that in axiomatic bargaining theory, comparative bargaining toughness and comparative global risk aversion are inextricably linked. We have argued that the relationship between these two comparisons is in fact quite fragile. To demonstrate this fragility, we weaken the concept of “globally more risk averse than,” and compare the relative bargaining toughness of two agents, one of whom is more averse than the other to “almost all” but not all risks. We show that in this context, the almost globally more risk averse agent may be tougher than the other agent against all opponents. More abstractly, we argue that the consensus view regarding the relationship between risk aversion and bargaining toughness results from an exclusive focus on an implosion point of a correspondence that is not upper hemicontinuous.

The discontinuity we have identified can be interpreted in one of two ways. If one considers our notion of “almost globally more risk averse than...” to be closely comparable to “globally more risk averse than...,” then one might conclude that the literature has placed excessive emphasis on results such as Props. 2 & 4. Alternatively, one could take the view that “almost globally more risk averse than...” and “globally more risk averse than...” are really quite different concepts, because there is something qualitatively different about, on the one hand, risk aversion to lotteries that exclusively involve negotiated outcomes, and, on the other, lotteries which assign a positive probability to negotiation breakdown. An implication of this latter view would be that more attention should be devoted to developing a more nuanced understanding of this distinction.
Proof of Lemma 1. Part a): For $z \in [x, 1]$, 
\[
\frac{u'(z)u_0(z) - u''(z)u_0}{(z)^2} = -u'' - u_0''(z) < 0 \quad (11)
\]
\[
dz u_0' (z) = \frac{u_0'(z) - u_0'(z)}{z} = \frac{u_0''(z)}{z} < 0
\]
Part b) follows immediately from part a). Part a) also implies
If $\exists y \in [x, 1] \text{ s.t. } u''(y) = u_0(0)$ then $u''(\cdot) < u_0(\cdot)$ on $[y, 1]$. From assumption A2), $u''(1) = u_0(1)$. Hence,
\[
u'(y) = \frac{u''(1)}{y} u'(y) \frac{1}{y} > u_0(1) \frac{1}{y} u_0(0) = u_0(y)
\]
Part d): Since $u''(1) = u_0(1)$, $u''(x) = u_0(x) \text{ implies } \int (u''(y) - u_0(0))dy = 0$. Hence, $\exists y^* \in [x, 1] \text{ s.t. } u'(y^*) = u_0(y^*)$. From part b), $u''(\cdot) > u_0(\cdot)$ on $[x, y^*)$, so that $u''(\cdot) > u_0(\cdot)$ on $[y, 1]$. From (12),
\[
u''(\cdot) > u_0(\cdot) \text{ on } [y^*, 1]. \text{ From part c), } u''(\cdot) > u_0(\cdot) \text{ on } [y^*, 1].
\]
Part e): Assume that $u'(y) < u_0(y)$, for some $y \in [x, 1]$. Since $u''(1) = u_0(1) = 1$, there exists at least one $y \in (y, 1)$ such that $u''(y) = u_0(y)$. Let $y^*$ denote the smallest such number. If $y^* = 1$, then $u''(y^*) > u_0(y^*)$ (since otherwise $\exists \theta > 0$ such that $u''(\cdot) > u_0(\cdot)$ on $(1 - \theta, 1)$ and hence some $y < y^*$ such that $u'(y) = u_0(y)$). If $y^* < 1$, then from part c), $u''(y^*) > u_0(y^*)$. In either case, $u''(\cdot)$ cuts $u_0$ from below at $y^*$. Proof of Lemma 2. Sufficiency: Fix $x > x^{KS}$ s.t.

**References**


**Appendix: Proofs**
Lemma 1. $u^-(\cdot) > u_0^-(\cdot)$ on $(x_{KS}^-, x)$, so that $u^-(x_{KS}^+) < u_0(x_{KS}^+)$, and hence, $KS(x_{KS}^+ | u^-, v) < KS(x_{KS}^+ | u_0, v) = 0$. From (1) and (5), $rac{\partial KS(u^-, v)}{\partial x} > 0$. Hence, $x_{KS}^+ > x_{KS}^-$. 

Necessity: Assume that $x_{KS}^+ < x_{KS}^-$. In this case, $u^-(x_{KS}^+) = v(x_{KS}^+) < v(x_{KS}^-) = u_0(x_{KS}^-) < u_0(x_{KS}^+)$.

The two inequalities follow from (1); the two equalities follow from (5). It now follows from part e) of Lemma 1 that $u^-$ cuts $u_0$ from below at some $x \in (x_{KS}^-, 1]$. 

$u^-$ cuts $u_0$ from below at $x$. From part b) of Lemma 1, $u^-(\cdot) > u_0^-(\cdot)$ on $(x_{KS}^-, x)$, so that $u^-(x_{KS}^+) < u_0(x_{KS}^+)$, and hence, $KS(x_{KS}^+ | u^-, v) < KS(x_{KS}^+ | u_0, v) = 0$. From (1) and (5), $\frac{\partial KS(u^-, v)}{\partial x} > 0$. Hence, $x_{KS}^+ > x_{KS}^-$. 

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Proof of Lemma 3. Since $\mathcal{N}(\cdot|u^+,v)\) is strictly concave, $f^N(u^-,v) = \mathcal{N}(\cdot|u_0^-,v)$ by definition:

$$\frac{\partial^2\mathcal{N}(x_0^\eta)}{\partial x} < 0.$$ 

Proof of Lemma 4. Fix $x \in (0,1)$ s.t. $u^\eta(x)$ cuts $u_0$ from below at $x$. By definition, $u^\eta(x)$ is strictly concave, hence $u^\eta(x_0) < u(x_0)$. From (3),

$$\frac{\partial^2 \mathcal{N}(x_0^\eta)}{\partial x^2} < 0.$$ 

Proof of Proposition 3. From (11), we have that $f^\eta(x_0) < 0$ on $\text{MRA}^{[0,1]}$. That is, there exists a strictly decreasing function $a(\cdot)$ on $[0,1]$ such that for all $x \in [0,1]$, $a(x)u^\eta(x)$ is strictly decreasing, and $u^\eta(x) > 0$.

Theorem for integration (Wikipedia, 2015), there exists $\lambda \in [0,x,N]a(y)u^\eta(y)dy = \int_0^\lambda a(\lambda)u^\eta(\lambda)dy$.

Proof of Proposition 5. Fix $u_0 \in \text{int}(U)$. Since $r^\eta(\cdot)$ (defined in part b) of Prop. 1 is continuous and takes values on the compact set $P$, it attains a maximum on $P$. Hence $\exists c \in \mathbb{R}$ s.t. $c > r^\eta(\cdot)$ on $P$. For each $z \in Z$ s.t. $z_1 \in (\gamma/2,1)$, with $\psi(z) = \exp(\exp(-cz_1))$, and define $f(z) = \log(\psi(1,0) \backslash \log(\psi(z))) \leq 0$. Let $g$ be a smooth extension of $f$ to $Z$ satisfying $f(D) = 0$. For $\lambda \in \mathbb{R}$, let $u^{-\lambda} = u_0 + \lambda g$ (so that $u^{-\lambda} = u_0$). Clearly, the set of functions satisfying assumption A1 is open (in the sup norm topology) $u^{-\lambda}$ will also satisfy assumption A1), if $\lambda$ is sufficiently small. Moreover, since $u_0 \in \text{int}(U)$, $u^{-\lambda} \in U$ if $\lambda$ is sufficiently small. We now have, $\forall \lambda$:

$$\forall x \in [0,1], \quad \frac{\partial}{\partial \lambda} u^{-\lambda}(x) = f(x) < 0.$$ 

(13a)

and $\forall x \in [x_1,1)$:

$$f'(x) = c \quad \partial^\lambda \exp(cx).$$
\[ f''(x) = -c \exp(cx) \frac{\partial^2 u^{-\lambda}(x)}{\partial \lambda} \]

\[ \frac{\partial}{\partial \lambda} \left( \frac{\partial^2 u^{-\lambda}(x)}{\partial x^2} \right) = f''(x) < 0 \] (13b)

\[ \frac{\partial}{\partial \lambda} u^{-\lambda}(x) = f'(x)u_0(x) - f(x)u_0' \] (13c)

\[ \frac{\partial}{\partial \lambda} u^{-\lambda}(x) = \frac{u(x)^2}{\partial x^2} > 0 \] (since \( f(x) < 0 \)) (13d)

\[ \frac{\partial^2 u^{-\lambda}(x)}{\partial \lambda \partial x} = \frac{u^{-\lambda}(x)\dot{f}(x)}{\partial x} > 0 \] (13e)

Inequality (13e), together with Prop. 1:b) implies that \( \forall \lambda > 0, \) \( u^{-\lambda} \) is more risk averse than \( u_0 \)
on $[x, 1]$, and hence that $(u^{-\lambda}, u_0) \in \mathbb{R}(x)$. Moreover, (13a) implies that for all $\lambda$, $u^{-\lambda} < u_0$. Necessarily, therefore, $u^{-\lambda}$ cuts $u_0$ from below at 1. Moreover, since $u_0, v \in \mathbb{U}$, we have from (7) that $\hat{\sigma} \leq \min(u^{-\lambda}, y_{\hat{\sigma}})$. Since by assumption, $x < \hat{\sigma}$, Lemmas 2 and 4 are applicable and the result now follows.