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Khovanov homology and smooth 4-manifolds

by

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Professor Ian Agol, Chair
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Professor Michael Hutchings

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Abstract

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The traditional way to study smooth 4-manifolds up to diffeomorphism, after the groundbreaking work of S. Donaldson, is to employ gauge-theoretic or Floer-theoretic invariants, including Donaldson invariants [Don90], Seiberg–Witten invariants [SW94], and Heegaard Floer invariants [OS04]. These tools, powerful as they are, typically come with constraints on the topology of the 4-manifolds, and there is a sense in which their applicability to new phenomena is approaching their natural limits. This thesis presents several studies on smooth 4-manifolds via Khovanov homology [Kho00] and the associated Rasmussen s-invariants [Ras10], tools that originate in representation theory and are combinatorial in nature.

In the first chapter, we establish (relative) adjunction-type inequalities for surfaces in $\#^k\overline{\mathbb{CP}^2}\setminus int(B^4)$, with a correction term coming from the s-invariant of the boundary link [Ren24a; Ren25]. Compared to similar inequalities in Heegaard Floer theory, the adjunction inequality for the s-invariant has the advantage of potentially detecting an exotic copy of $\#^k\overline{\mathbb{CP}^2}$, whose existence is not currently known. The inequality is established as a consequence of a full computation of the filtration structure of the Lee homology [Lee05] of all torus links. Of independent interest, we also propose a conjectural formula for the rational Khovanov homology of torus links $T(n,n)$.

In the second chapter, we review the construction of skein lasagna modules, a package of smooth 4-manifold invariants introduced by Morrison–Walker–Wedrich [MWW22]. We review some formal properties, make some general observations, and introduce some natural variants of skein lasagna modules, including skein lasagna modules with 1-*dimensional inputs*. Then we specialize to the theory of Khovanov and Lee skein lasagna modules.

In the third chapter, we present joint work with Michael Willis [RW24] on detecting pairs of exotic 4-manifolds (with boundary) using Khovanov and Lee skein lasagna modules. We introduce a lasagna analogue of Rasmussen's s-invariants. We present some vanishing and

nonvanishing results for Khovanov skein lasagna modules and lasagna s-invariants. The techniques developed in the first chapter are crucial for the main nonvanishing results.

In the last chapter, we present joint work with Ian Sullivan, Paul Wedrich, Michael Willis, and Melissa Zhang [Ren+25] on using Khovanov skein lasagna modules to prove excellent functoriality properties for the theory of Khovanov homology in connected sums of copies of $S^1 \times S^2$ as defined by Rozansky [Roz10] and Willis [Wil21]. This provides an interesting example of the theory of skein lasagna modules with 1-dimensional inputs; this theory admits a forgetful map from the usual theory of Khovanov skein lasagna modules.

Compared with the original sources [Ren24a; Ren25; RW24; Ren+25], this thesis contains significant additions in Sections 1.6.2, 2.4, 2.5, 2.6.2.2, 2.6.4, 3.4.4, 3.5.1, 3.5.2, 4.1.1, 4.1.3, 4.2.3, 4.3.1; simplifications or significant expositional changes have been made in Section 1.4, Chapter 2, Sections 3.3.1, 3.3.2, 3.4.1, 3.4.3, 3.4.4, 3.5.5, 3.5.9, 4.2.4, and Appendix D.

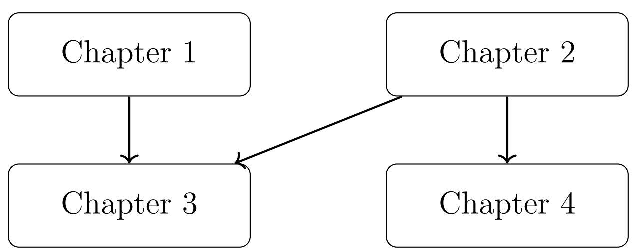

Figure 0.1: Logical dependence of chapters.

To my mother

Contents

Acknowledgments

The years have flown by. I am deeply indebted to my advisor, Ian Agol, for his guidance and support. Ian has been my role model as a mathematician, and it is hard to overstate his influence on me throughout these years. In the realm of low-dimensional topology, I learned from him to go both as deeply as possible into individual problems and as broadly as possible across the field. I learned to be ambitious about beautiful mathematical problems, and at the same time to remain humble as a mathematical researcher. Needless to say, my research has constantly benefited from his deep insights.

After I began my career as a PhD student, my mathematical interests gradually grew and branched out in several directions. I regret, in a certain sense, that this thesis represents a part of my mathematical work that does not lie directly under the shadow of Ian's extraordinarily broad expertise. I want to thank Professor Ciprian Manolescu and his collaborators, whose work helped me make the greatest leap into the area presented here. I also thank Professor Paul Wedrich, who has had the strongest influence on the later parts of my work, both through our communications and through his research work and research group. I also wish to acknowledge the lectures delivered by Tomasz Mrowka at MIT in 2020–2021, which partly shaped my taste in low-dimensional topology.

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Finally, I thank my mother for her unwavering love and wisdom throughout my entire life. Every achievement of mine is also yours.

Chapter 1

Lee filtration structure of torus links

1.1 Introduction

Khovanov homology is a link invariant introduced by Khovanov [Kho00], which categorifies the Jones polynomial. Despite its computability for every given link diagram, few families of links have had their Khovanov homology fully determined. Notable examples where the complete answer is known include alternating links [Lee05] (more generally, quasi-alternating links [MO08]), and the torus links $T(2, m)$ [Kho00], $T(3, m)$ [Tur08; Sto09; Gil12; Ben17].

Specifically, in the case of positive torus links $T(n, m)$, although the Jones polynomials have a well-known closed form, this is far from true for Khovanov homology. The investigation of the Khovanov homology of torus links dates back at least to Stošić [Sto07; Sto09], who computed, for example, some low ($h \leq 4$) homological degree parts of $Kh(T(n, m))$ and the highest ($h = 2kn^2$) homological degree part of $Kh(T(2n, 2kn))$. More interestingly, he showed the existence of the "stable Khovanov homology groups" of $T(n, m)$ as $m \to \infty$, which has since been extensively investigated; see, for example, [GOR13]. It is also worth remarking that the Khovanov–Rozansky triply graded homology of $T(n, m)$ was completely determined [HM19]. However, a comprehensive understanding of the ordinary Khovanov homology for torus links remains elusive.

Nevertheless, many useful knot or link invariants that are more computable and possess desirable properties can be derived from Khovanov homology or its variants. A notable example is Rasmussen's s-invariant for knots, extracted from Lee homology [Lee05], whose values on torus knots were computed and played a crucial role in providing the first gauge-theory-free proof of Milnor's conjecture on the slice genus of torus knots [Ras10]. In the case of links, the quantum filtration structure of the Lee homology can be considered as a natural generalization of the s-invariant for knots, encompassing the generalization proposed by Beliakova–Wehrli [BW08] and Pardon [Par12] as special cases. In this chapter, we completely determine the quantum filtration structure of the Lee homology of all torus links.

In the statements below, let n, m be positive integers, $d = \gcd(n, m)$, $n_1 = n/d$, $m_1 = m/d$. For $p, q \geq 0$ with $p + q = d$, let $T(n, m)_{p,q}$ denote the torus link $T(n, m)$ equipped with an orientation in which p of the components are oriented oppositely to the other q components.

Theorem 1.1.1. *The s-invariant of $T(n, m)_{p,q}$, over any coefficient field, is given by*

$$s(T(n, m)_{p,q}) = (n_1|p - q| - 1)(m_1|p - q| - 1) - 2\min(p, q).$$

Here, the s-invariant of an oriented link over any field with characteristic not equal to 2 is defined in the same way as in [BW08]. In characteristic 2, one should use the Bar-Natan deformation (cf. [Bar05]) of Khovanov homology instead of the Lee deformation.

By construction of the Khovanov/Lee complex, the Lee homology of $T(n, m)_{p,q}$, as a homologically graded and quantum filtered vector space, is equal to that of the positive torus link $T(n, m) := T(n, m)_{d,0}$ up to a bidegree shift. Thus we will only state the structure of $Kh_{Lee}(T(n, m))$. By Lee [Lee05, Proposition 4.3], $Kh_{Lee}(T(n, m))$ as a homologically graded vector space is determined by the linking matrix of $T(n, m)$. Explicitly,

$$\dim Kh^{2n_1m_1pq}_{Lee}(T(n, m)) = \begin{cases} 2\binom{d}{q}, & p \neq q \\ \binom{d}{q}, & p = q, \end{cases}$$

for every pair of nonnegative integers (p, q) with $p+q = d$, with other graded components being zero. The following theorem determines (the isomorphism type of) its quantum filtration structure.

Theorem 1.1.2. *The isomorphism type of the Lee homology (over $\mathbb{Q}$) of the positive torus link $T(n, m)$ is determined, as a homologically graded and quantum filtered S_d-representation, by*

$$\begin{aligned} &gr^{6n_1m_1pq+s(T(n,m)_{p,q})+2r-1}(Kh^{2n_1m_1pq}_{Lee}(T(n, m))) \\ &= \begin{cases} (d - r, r), & r = 0,\ p \neq q \\ (d - r, r) \oplus (d - r + 1, r - 1), & 0 < r < \min(p, q) + 1,\ p \neq q \\ (d - r + 1, r - 1), & r = \min(p, q) + 1,\ p \neq q \\ (d - r, r), & 0 \leq r \leq \min(p, q),\ p = q, \end{cases} \end{aligned}$$

for every pair of nonnegative integers (p, q) with $p+q = d$, with all other bigraded components being zero. Here, the symmetric group S_d acts by permuting the strands in $T(n, m)$ as a d-cable, $gr^\bullet V$ denotes the associated graded vector space of a filtered vector space V, and (a, b) denotes the irreducible representation of S_{a+b} given by the two-row Young diagram (a, b).

In particular, one recovers the graded dimension of the associated graded vector space of $Kh_{Lee}(T(n, m))$ from the formula

$$\dim(d - r, r) = \binom{d}{r} - \binom{d}{r - 1},\ 0 \leq r \leq \frac{d}{2}.$$

In the special case $m = n$, this confirms a conjecture of Pardon [Par12, Section 5.2].

Since taking the mirror image of a link has the effect of taking the dual on the Lee homology, the quantum filtration structure of $T(n, -m)$ is determined by that of $T(n, m)$. We can also read off its s-invariants as follows (cf. the second paragraph in the proof of Theorem 1.1.1).

Corollary 1.1.3. *The s-invariant, over $\mathbb{Q}$, of $T(n, -m)_{p,q}$ is given by*

$$s(T(n,-m)_{p,q}) = \begin{cases} -(n_1|p-q|-1)(m_1|p-q|-1), & p \neq q \\ 1, & p = q. \end{cases} \qquad \square$$

As observed by Manolescu–Marengon–Sarkar–Willis [Man+23], Theorem 1.1.1 in the special case $m = n$ implies the following corollary, which is an adjunction-type inequality for s-invariants of null-homologous oriented links in a connected sum of $(S^1 \times S^2)$'s, as defined in their paper. We remark that, however, one cannot deduce an adjunction-type inequality from Corollary 1.1.3 using the same proof.

Corollary 1.1.4. *If $\Sigma \subset Z = (I \times \ell(S^1 \times S^2))\#k\overline{\mathbb{CP}^2}$ is an oriented cobordism between two null-homologous oriented links $L_0, L_1 \subset \ell(S^1 \times S^2)$ with $\pi_0(L_1) \to \pi_0(\Sigma)$ surjective, then*

$$s(L_1) \leq s(L_0) - \chi(\Sigma) - [\Sigma]^2 - |[\Sigma]|'.$$

Here $|[\Sigma]|'$ is defined as the sum $\sum_{i=1}^k |[\Sigma] \cdot z_i|$ where $z_1, \cdots, z_k \in H_2(Z) \cong \mathbb{Z}^{\ell+k}$ are the generators coming from the $\overline{\mathbb{CP}^2}$ factors.

Corollary 1.1.4 holds over any coefficient field as long as $\ell = 0$. For $\ell > 0$, in [Man+23], the s-invariant is only defined over fields with characteristic not equal to 2, although we expect everything to hold in characteristic 2 as well.

The term $[\Sigma]^2$ above is well-defined, and the term $|[\Sigma]|'$ is independent of the choice of the decomposition $Z = (I \times \ell(S^1 \times S^2))\#k\overline{\mathbb{CP}^2}$, both thanks to the links L_i being null-homologous. Of course, one may also dualize and obtain a similar adjunction inequality in $(I \times \ell(S^1 \times S^2))\#k\mathbb{CP}^2$ (cf. Section 1.2.3).

In practice, the special case $\ell = 0$ might be the most useful, where s reduces to Beliakova–Wehrli's generalization of classical Rasmussen's s-invariant (defined in Section 1.2.2). In particular, this opens a new approach to detect exotic $k\overline{\mathbb{CP}^2}$ whose existence is not yet known, for example by modifying the constructions in [MP23]. When $\ell = 0$, $L_0 = \emptyset$, and $L_1 = K$ is a knot, Corollary 1.1.4 takes the following form.

Corollary 1.1.5 ([Man+23, Conjecture 9.8]). *If $(\Sigma, K) \subset ((k\overline{\mathbb{CP}^2})\backslash B^4, S^3)$ is a connected orientable properly embedded surface with boundary a knot K, then*

$$s(K) \leq 1 - \chi(\Sigma) - [\Sigma]^2 - |[\Sigma]|.$$

Here $\|[\Sigma]\|$ *is the* L^1*-norm of* $[\Sigma]$ *(denoted as* $|[\Sigma]|'$ *in Corollary 1.1.4).* □

The adjunction-type inequality parallel to Corollary 1.1.5 for the τ-invariant in knot Floer homology was established by Ozsváth and Szabó two decades ago [OS03]. The inequality parallel to the more general Corollary 1.1.4 appeared recently (for knots, and in some special cases for links) in the work of Hedden and Raoux [HR23]. In fact, their inequalities were stated for more general smooth 4-manifolds. This naturally leads us to the question of whether Corollary 1.1.4 and Corollary 1.1.5 can be generalized to those settings, which could be challenging since it (in its full generality) entails generalizing the s-invariant to rationally null-homologous links in arbitrary closed oriented 3-manifolds. It is worth remarking, however, that in order to successfully construct exotic $k\overline{\mathbb{CP}^2}$'s detectable by the s-invariant using a modified version of the construction in [MP23], as explored in [Qin25], one should hope that Corollary 1.1.5 does not have a generalization applicable to arbitrary simply connected negative definite 4-manifolds.

Ciprian Manolescu informed us that the adjunction inequality may fail for spatially refined s-invariants defined by Lipshitz–Sarkar [LS14], Sarkar–Scaduto–Stoffregen [SSS20] and Schuetz [Sch25]. We analyze the failure, and prove the following on the positive side.

Theorem 1.1.6. *Theorem 1.1.1, Corollary 1.1.4 with* $\ell = 0$*, and Corollary 1.1.5 all apply with* s *replaced by* $s_+^{Sq^1}$*, the* s*-version of the even* Sq^1*-refinement of* s *(defined for knots in [LS14], extended to all oriented links as in Section 1.2.4).*

Of independent interest, we showed in [Ren24b], using techniques similar to those in Section 1.5 of this chapter, that the s-invariant for knots in $\mathbb{RP}^3$ defined by Manolescu–Willis [MW25] satisfies an adjunction-type inequality for surfaces in DTS^2, the unit disk bundle of the tangent bundle over S^2. As observed by I. So [So25], this can be used to establish a refinement of the slice-Bennequin inequality for the slice genus, in the Stein manifold DTS^2, of transverse knots in $\mathbb{RP}^3$ with its unique tight contact structure. As this topic is peripheral to the main theme, we do not include it in the thesis.

We summarize the structure of this chapter and briefly describe the proofs of the main results as follows.

In Section 1.2.1, we state a "graphical lower bound", Theorem 1.2.1 (see also Figure 1.1 and Figure 1.2), for the Khovanov homology of the torus links $T(n,n)$ and the torus knots $T(n+1,n)$. In the case of $T(n,n)$, in homological degrees $2pq$ with $p+q=n$, the bound is sharp, and the nonvanishing groups with the lowest quantum degrees are $\mathbb{Z}$, which also give rise to Lee homology generators. As we shall see in Section 1.2.2, this implies Theorem 1.1.1 in the special case $m=n$. The general case is then proved in Section 1.2.3 inductively using the adjunction inequality (Corollary 1.1.4). The proof of Theorem 1.2.1, which follows an induction scheme set up by Stošić [Sto07; Sto09], is a cumbersome verification that is not illuminating and is deferred to Section 1.5.

As further illustrations of the power of the adjunction inequality, we give two applications. In Section 1.3.1, we provide an optimal bound on the change of the s-invariant when full twists are applied to an oriented link. In particular, the s-invariant grows linearly when the twist number is sufficiently large, answering positively a question in [Man+23]. In Section 1.3.2, we show there exist knots in S^3 with simultaneously large $\mathbb{CP}^2$-genus and $\overline{\mathbb{CP}^2}$-genus.

Section 1.4 is independent of Section 1.3. We exploit the S_d-symmetry on the homology of d-cables established in [GLW18]. By working equivariantly, we are able to deduce Theorem 1.1.2 inductively from Theorem 1.1.1.

Finally, in Section 1.6, we state some numerical observations as open questions. In particular, we propose a conjectural (recursive) formula for the rational Khovanov homology of $T(n,n)$.

1.2 The s-invariants of $T(n,m)$

1.2.1 A graphical lower bound for $Kh(T(n,n))$ and $Kh(T(n+1,n))$

In this section we state the core technical result of this chapter, which gives a "graphical lower bound" on the Khovanov homology of $T(n,n)$ and $T(n+1,n)$, whose proof is postponed to Section 1.5. We assume the reader is familiar with Khovanov homology and Lee homology, and refer to the literature mentioned in the introduction for background. See also [Bar02] for a short introduction.

Before stating the theorem, we introduce two families of auxiliary functions $q_{n,n}, q_{n+1,n} \colon \mathbb{Z} \to \mathbb{Z} \cup \{+\infty\}$, which serve as quantum lower bounds for the Khovanov homology of $T(n,n), T(n+1,n)$ in various homological degrees, respectively. Let

$$h_{max}(T(n,n)) := \lfloor n^2/2 \rfloor,\ h_{max}(T(n+1,n)) := \lfloor n^2/2 \rfloor + \lfloor n/2 \rfloor.$$

The functions $q_{n,n}$ are defined by:

(a) $q_{n,n}(h) = +\infty$ for $h < 0$ or $h > h_{max}(T(n,n))$;

(b) $q_{n,n}(0) = n^2 - 2n$;

(c) If $p + q = n$, $p \geq q > 0$, then for $2(p+1)(q-1) < h \leq 2pq$,

$$q_{n,n}(h) = n^2 + 2\left\lceil \frac{h}{2} \right\rceil - 2p.$$

The functions $q_{n+1,n}$ are defined by:

q \ h	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
54																			$\mathbb{Z}^5$
52																			$\mathbb{Z}^9$
50																$\mathbb{Z}^5$	$\mathbb{Z}^9$		$\mathbb{Z}^5$
48															$\mathbb{Z}_3$	$\mathbb{Z}^6$	$\mathbb{Z}^{14}$		$\mathbb{Z}$
46													$\mathbb{Z}$	$\mathbb{Z}^6$	$\mathbb{Z}^5$	$\mathbb{Z}$	$\mathbb{Z}^6$		
44													$\mathbb{Z}_2 \oplus \mathbb{Z}_5$	$\mathbb{Z}^2 \oplus (\mathbb{Z}_2)^5$	$\mathbb{Z}^6$		$\mathbb{Z}$		
42											$\mathbb{Z}_2$	$\mathbb{Z}^2$	$\mathbb{Z}^6$	$\mathbb{Z}_2$	$\mathbb{Z}$				
40										$\mathbb{Z}^2$	$\mathbb{Z}^5 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_5$	$\mathbb{Z}^2$						
38								$\mathbb{Z}$		$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}^7$								
36						$\mathbb{Z}$		$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}^2$		$\mathbb{Z}$								
34						$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}$										
32				$\mathbb{Z}$	$\mathbb{Z}$		$\mathbb{Z}$												
30				$\mathbb{Z}_2$	$\mathbb{Z}$														
28			$\mathbb{Z}$																
26	$\mathbb{Z}$																		
24	$\mathbb{Z}$																		

Figure 1.1: The Khovanov homology of $T(6,6)$. The lower bound given by $q_{6,6}$ is drawn red. The homology groups $Kh^{2pq,q_{6,6}(2pq)}$ are colored green.

(a) $q_{n+1,n}(h) = +\infty$ for $h < 0$ or $h > h_{max}(T(n+1,n))$;

(b) If $p+q=n$, $p-2 \geq q > 0$, then

$$q_{n+1,n}(2pq+1) = q_{n,n}(2pq+1) + n - 3;$$

(c) For other $0 \leq h \leq h_{max}(T(n,n))$,

$$q_{n+1,n}(h) = q_{n,n}(h) + n - 1;$$

(d) For $h_{max}(T(n,n)) \leq h \leq h_{max}(T(n+1,n))$,

$$q_{n+1,n}(h) = \left\lfloor \frac{n^2}{2} \right\rfloor + 2h - 1.$$

Note the two definitions of $q_{n+1,n}(h_{max}(T(n,n)))$ via (c) and (d) above agree.

We now state the main technical theorem. See Figure 1.1 and Figure 1.2 for an illustration of the case $n = 6$. The reader is warned that the same letter q is (unfortunately) used for two different purposes: the quantum degree or filtration degree and an integer between 0 and n that is complementary to p. It should be clear from context which of these is referred to.

Theorem 1.2.1. *(1) $Kh^{h,q}(T(n,n)) = 0$ for $q < q_{n,n}(h)$. Moreover, for every $p+q=n$, $p,q > 0$, the saddle cobordism $T(n-2,n-2) \sqcup U \to T(n,n)$ induces an isomorphism*

$$Kh^{2(p-1)(q-1),q_{n-2,n-2}(2(p-1)(q-1))-1}(T(n-2,n-2) \sqcup U) \xrightarrow{\cong} Kh^{2pq,q_{n,n}(2pq)}(T(n,n)).$$

$q \backslash h$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
61																					$\mathbb{Z}_3$	$\mathbb{Z}_2$
59																					$\mathbb{Z}_2 \oplus \mathbb{Z}_3$	$\mathbb{Z}_2$
57																	$\mathbb{Z}$		$\mathbb{Z}_4$	$\mathbb{Z}$	$\mathbb{Z}_2$	
55																	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}$	$(\mathbb{Z}_2)^2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3$		
53															$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$			
51													$\mathbb{Z}$	$\mathbb{Z}$	$(\mathbb{Z}_2)^2$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$				
49													$\mathbb{Z}_2 \oplus \mathbb{Z}_5$	$\mathbb{Z}^3 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$					
47											$\mathbb{Z}_2$	$\mathbb{Z}^3$	$\mathbb{Z}$	$(\mathbb{Z}_2)^2$	$\mathbb{Z}$							
45										$\mathbb{Z}^2$	$\mathbb{Z}_2$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^2$	$\mathbb{Z}^2$									
43								$\mathbb{Z}$		$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}^2$	$\mathbb{Z}_2$										
41						$\mathbb{Z}$		$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}^2$		$\mathbb{Z}$											
39						$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}$													
37				$\mathbb{Z}$	$\mathbb{Z}$		$\mathbb{Z}$															
35				$\mathbb{Z}_2$	$\mathbb{Z}$																	
33			$\mathbb{Z}$																			
31	$\mathbb{Z}$																					
29	$\mathbb{Z}$																					

Figure 1.2: The Khovanov homology of $T(7,6)$. The lower bound given by $q_{7,6}$ is drawn red. The homology group $Kh^{2\cdot 6-1, q_{7,6}(2\cdot 6-1)}$ is colored yellow.

(2) $Kh^{h,q}(T(n+1,n)) = 0$ *for* $q < q_{n+1,n}(h)$. *Moreover,* $Kh^{2n-1, q_{n+1,n}(2n-1)}(T(n+1,n))$ *is torsion.*

Corollary 1.2.2. *For any* $p+q=n$, $p,q \geq 0$, *we have* $Kh^{2pq, q_{n,n}(2pq)}(T(n,n)) = \mathbb{Z}$.

Proof. For $pq = 0$, this follows from Stošić's computation [Sto07, Theorem 3.4] that $Kh^{0,*}(T(n,n)) = \mathbb{Z}$ for $* = (n-1)^2 \pm 1$ and 0 otherwise. The general case then follows by induction using the last statement in Theorem 1.2.1(1). □

The case $p = n-1$, $q = 1$ of Corollary 1.2.2 confirms a conjecture of Stošić [Sto07, Conjecture 3.8].

1.2.2 Proof of Theorem 1.1.1 for $m = n$

We prove Theorem 1.1.1 in the special case $m = n$, assuming Theorem 1.2.1. By symmetry we may assume $p \geq q$. The statement then takes the form $s(T(n,n)_{p,q}) = (p-q-1)^2 - 2q$.

We induct on n. The base cases $n = 1, 2$ are easily checked. Assume $n \geq 3$. If $q = 0$, the result follows either from the sharpness of Kawamura–Lobb's inequality on s-invariants for nonsplit positive links [AT17, Corollary 2.4], or Stošić's computation of $Kh^{0,*}(T(n,n))$ mentioned above, or a link version of [Kho03, Proposition 6.1]. Assume from now on $q > 0$.

The linking matrix of $T(n,n)$ is $(\ell_{ij})_{1 \leq i,j \leq n}$ where $\ell_{ij} = 0$ if $i = j$ and 1 otherwise. Thus by Lee [Lee05, Proposition 4.3], $Kh^{2pq}_{Lee}(T(n,n))$ is the vector space spanned by the canonical generators $[s_{\mathfrak{o}}]$, one for each orientation $\mathfrak{o}$ of $T(n,n)$ realizing $T(n,n)_{p,q}$ (see also [Ras10]). Moreover, the Lee homology of $T(n,n)$ is that of $T(n,n)_{p,q}$ with a bidegree shift $t^{2pq}q^{6pq}$

[Kho00, Proposition 28][1]. Here, t, q denote homological and quantum degree shifts, respectively. We hope that this new use of the letter q does not lead to confusion.

According to Beliakova–Wehrli's definition of the s-invariant [BW08, Definition 7.1], $s(T(n,n)_{p,q})$ over $\mathbb{Q}$ equals the quantum filtration degree of $[s_{\mathfrak{o}}] \in Kh_{Lee}(T(n,n)_{p,q})$ plus 1, where $\mathfrak{o}$ is any orientation that realizes $T(n,n)_{p,q}$. Taking into account the degree shift $6pq$, we need to prove $[s_{\mathfrak{o}}] \in Kh_{Lee}(T(n,n))$ has quantum filtration degree $(p-q-1)^2 - 2q - 1 + 6pq = q_{n,n}(2pq)$.

Every element of $Kh^{2pq}_{Lee}(T(n,n))$ is a linear combination of these $[s_{\mathfrak{o}}]$'s, and thus has quantum filtration degree no less than that of these $[s_{\mathfrak{o}}]$'s. It remains to prove the lowest filtration level of $Kh^{2pq}_{Lee}(T(n,n))$ is $q_{n,n}(2pq)$.

We prove this by showing the equivalent statement that the homology group $Kh^{2pq,q_{n,n}(2pq)}(T(n,n))\otimes \mathbb{Q} \cong \mathbb{Q}$ (cf. Corollary 1.2.2) survives to the E_∞ page in the Lee spectral sequence from $E_1 = Kh(T(n,n)) \otimes \mathbb{Q}$ to $Kh_{Lee}(T(n,n))$. The differential on the E_r page has bidegree $(1, 4r)$. By Theorem 1.2.1, we have $Kh^{2pq-1,*}(T(n,n)) = 0$ for $* < q_{n,n}(2pq-1) = q_{n,n}(2pq)$, so $Kh(T(n,n)) \otimes \mathbb{Q}$ cannot be annihilated by differentials mapping into it. To see that all differentials out of it are zero, we observe that the naturality of the Lee spectral sequence applied to the cobordism $T(n-2,n-2) \to T(n,n)$ gives the following commutative diagram on page E_r:

$$\begin{tikzcd}
Kh^{2(p-1)(q-1),q_{n-2,n-2}(2(p-1)(q-1))}(T(n-2,n-2)) \otimes \mathbb{Q} \arrow[r, "\cong"] \arrow[d, "d_r=0"] & Kh^{2pq,q_{n,n}(2pq)}(T(n,n)) \otimes \mathbb{Q} \arrow[d, "d_r"] \\
E_r^{2(p-1)(q-1)+1,q_{n-2,n-2}(2(p-1)(q-1))+4r}(T(n-2,n-2)) \arrow[r] & E_r^{2pq+1,q_{n,n}(2pq)+4r}(T(n,n)).
\end{tikzcd}$$

Here the vertical map on the left is zero by induction hypothesis and the horizontal map on the top is an isomorphism by Theorem 1.2.1(1). Consequently, the vertical map on the right is also zero.

The proof above works with $\mathbb{Q}$ replaced by any coefficient field with characteristic not equal to 2. For characteristic 2, one should replace the Lee homology with Bar-Natan homology, and Lee spectral sequence with Bar-Natan/Turner spectral sequence [Bar05; Tur06] (whose r^{th} differential has bidegree $(1, 2r)$), but everything else goes through. Thus the theorem is proved in the special case $m = n$. □

1.2.3 Adjunction inequality and proof of Theorem 1.1.1

Manolescu–Marengon–Sarkar–Willis [Man+23, Theorem 6.10] proved the adjunction inequality (Corollary 1.1.4) in the special case of null-homologous cobordisms, using the computation of $s(T(2p,2p)_{p,p})$ in their paper. Having computed all $s(T(n,n)_{p,q})$, Corollary 1.1.4

[1] Note that the proposition contains a typo.

in its full generality is proved exactly the same way. For completeness, we sketch their proof here. Then we apply this inequality to prove Theorem 1.1.1 in its full generality.

Proof of Corollary 1.1.4. Turning the cobordism upside down and reversing the ambient orientation, we obtain a cobordism Σ^t in $Z^t = (I \times \ell(S^1 \times S^2))\#k\mathbb{CP}^2$ from L_1 to L_0 with $\pi_0(L_1) \to \pi_0(\Sigma^t)$ surjective. Choose embedded 2-spheres $S_1, \cdots, S_k$ representing generators $\overline{z_1}, \cdots, \overline{z_k} \in H_2(Z^t)$ coming from the $\mathbb{CP}^2$ factors. We may assume Σ^t intersects each S_i transversely, in some p_i points positively and q_i points negatively. Since S_i has self-intersection 1, a tubular neighborhood $\nu(S_i)$ has boundary S^3, and the projection to the core $\partial\nu(S_i) \to S_i$ is the Hopf fibration. Therefore, removing all $\nu(S_i)$ and tubing each $\partial\nu(S_i)$ to $\{1\} \times (S^1 \times S^2) \subset Z^t$ gives a cobordism Σ_0^t from L_1 to $L_0 \sqcup (\sqcup_i T(n_i, n_i)_{p_i,q_i})$ in $I \times \ell(S^1 \times S^2)$, where $n_i = p_i + q_i$. Topologically, Σ_0^t is obtained by deleting $\sum_i (p_i + q_i)$ disks in the interior of $\Sigma^t \cong \Sigma$. Now by [Man+23, Theorem 1.5, Proposition 3.7] we have

$$\left(s(L_0) + \sum_{i=1}^{k} s(T(n_i, n_i)_{p_i,q_i}) - k\right) - s(L_1) \geq \chi(\Sigma_0^t) = \chi(\Sigma) - \sum_{i=1}^{k}(p_i + q_i).$$

By Theorem 1.1.1 with $m = n = n_i$, this simplifies to

$$\begin{aligned} s(L_0) - s(L_1) &\geq \chi(\Sigma) - \sum_{i=1}^{k}\big((|p_i - q_i| - 1)^2 - 2\min(p_i, q_i) - 1 + p_i + q_i\big) \\ &= \chi(\Sigma) - \sum_{i=1}^{k} |p_i - q_i|^2 + \sum_{i=1}^{k} |p_i - q_i| = \chi(\Sigma) + [\Sigma]^2 + |[\Sigma]|'. \end{aligned}$$

□

Proof of Theorem 1.1.1. We induct on $m + n$. The base cases are $m = n$, which we have already addressed. To perform the induction step, by symmetry we may assume $n < m$. If $pq = 0$, we conclude as before. Assume $p, q > 0$; thus $d \geq 2$. There is a cobordism Σ from $T(n, m - n)_{p,q}$ to $T(n, m)_{p,q}$ in $\overline{\mathbb{CP}^2}$ obtained by adding a positive full twist. The surface Σ is a disjoint union of d annuli, which intersects a copy of $\overline{\mathbb{CP}^1} \subset \overline{\mathbb{CP}^2}$ transversely at $n_1 p$ points positively and $n_1 q$ points negatively. Applying Corollary 1.1.4 to Σ, we obtain

$$s(T(n, m)_{p,q}) \leq s(T(n, m-n)_{p,q}) + n_1^2|p-q|^2 - n_1|p-q| = (n_1|p-q|-1)(m_1|p-q|-1) - 2\min(p, q).$$

On the other hand, since $T(n, m)$ is a d-cable on $T(n_1, m_1)$, there is a saddle cobordism $T(n - 2n_1, m - 2m_1)_{p-1,q-1} \sqcup U \to T(n, m)_{p,q}$. Thus

$$s(T(n, m)_{p,q}) \geq s(T(n-2n_1, m-2m_1)_{p-1,q-1} \sqcup U) - 1 = (n_1|p-q|-1)(m_1|p-q|-1) - 2\min(p, q),$$

which completes the proof. □

1.2.4 Spatially refined s-invariants

Using spatial refinements of Khovanov homology and odd Khovanov homology, Lipshitz–Sarkar [LS14], Sarkar–Scaduto–Stoffregen [SSS20] and Schuetz [Sch25] defined refinements of Rasmussen's s-invariants for knots. Ciprian Manolescu informed us that the adjunction inequality, in the form of Corollary 1.1.5, may fail for some of these refined s-invariants. The counterexample he found is the knot 9_{42} which bounds a disk in the punctured $\overline{\mathbb{CP}^2}$ with homology class $[\overline{\mathbb{CP}^1}]$ whose geometric intersection number with $\overline{\mathbb{CP}^1}$ is 3 (see Figure 1.3), but for which some versions of the odd Sq^1-, Sq^2- and even Sq^2-refined s-invariants are equal to 2, violating Corollary 1.1.5.

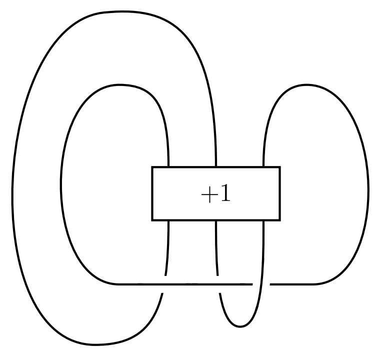


Figure 1.3: A diagram of the knot 9_{42}. Untwisting the twist region results in the unknot.

Let us see what may go wrong with the proof of Corollary 1.1.5, or more generally of Corollary 1.1.4 with $\ell = 0$, if it were applied to these refined s-invariants. The proof relies on the following three facts about s-invariants.

(a) s is defined for all oriented links in S^3. If $\Sigma\colon L_0 \to L_1$ is an oriented cobordism in $I \times S^3$ with $\pi_0(L_0) \to \pi_0(\Sigma)$ surjective, then $s(L_1) - s(L_0) \geq \chi(\Sigma)$.

(b) If $T(n,n)_{p,q}$ is the torus link $T(n,n)$, equipped with an orientation where p strands are oriented against the other q strands, $p \geq q$, $p+q = n$, then $s(T(n,n)_{p,q}) = (p-q)^2 - 2p + 1$.

(c) $s(L \sqcup T(n,n)_{p,q}) = s(L) + s(T(n,n)_{p,q}) - 1$ for any oriented link L.

Lipshitz–Sarkar's refined s-invariants are only defined for knots, but it is not hard to extend the definition to links and address item (a), which we will do in a while. For this natural generalization, however, one can verify using KnotJob [Sch] that $s_+^{Sq^2}(T(3,3)_{2,1}) = s(T(3,3)_{2,1}) + 2$, violating item (b). The same violation for $T(3,3)_{2,1}$ appears for the odd Sq^1-refinement and the s-version of one of Schuetz's two odd Sq^2-refinements (once appropriately defined). These contribute to the failure of the corresponding refined s-invariants to satisfy the adjunction inequality for the example of 9_{42}.

To show Theorem 1.1.6, it suffices to justify (a)(b)(c) for $s_+^{Sq^1}$, for then the relevant statements would follow from the same arguments as in Section 1.2.3. We address each of them in the following three subsections, respectively.

1.2.4.1 Spatially refined s-invariants for links

We recall the construction and some properties of even spatial refinements of Rasmussen's s-invariant from Lipshitz–Sarkar [LS14], with the relevant definitions extended from knots to oriented links in the natural way.

Let θ be a stable cohomology operation of degree m over a field $\mathbb{F}$ and L be a nonempty oriented link in S^3 with orientation $\mathfrak{o}$. Let $W(L) = \mathrm{span}\{[\mathfrak{s}_{\mathfrak{o}}], [\mathfrak{s}_{\bar{\mathfrak{o}}}]\} \subset Kh^0_{Lee}(L;\mathbb{F})$ (use Bar-Natan homology Kh_{BN} [Bar05; Tur06] instead of Lee homology Kh_{Lee} [Lee05] if $\mathrm{char}(\mathbb{F}) = 2$, same below) denote the 2-dimensional subspace spanned by the canonical generators corresponding to the orientation of L and its reverse. For an integer $q \equiv \#\pi_0(L) \pmod 2$, consider the maps

$$Kh^{-m,q}(L;\mathbb{F}) \xrightarrow{\theta} Kh^{0,q}(L;\mathbb{F}) \xleftarrow{p} H^0(CKh^{\geq q}_{Lee}(L;\mathbb{F})) \xrightarrow{j} Kh^0_{Lee}(L;\mathbb{F}), \tag{1.1}$$

where Kh is the Khovanov homology [Kho00], CKh_{Lee} is the Lee chain complex, and $CKh^{\geq q}_{Lee}$ is its subcomplex consisting of elements with filtered degree at least q.

With respect to L, the integer q is *θ-half-full* (resp. *θ-full*) if the subspace $V^q(L) := j(p^{-1}(\mathrm{im}(\theta))) \cap W(L)$ of $W(L)$ is at least 1-dimensional (resp. 2-dimensional); it is *half-full* (resp. *full*) if the subspace $\mathrm{im}(j) \cap W(L)$ of $W(L)$ is at least 1-dimensional (resp. 2-dimensional). Hence

- if q is θ-half-full (resp. θ-full), then q is half-full (resp. full);
- if q is half-full (resp. full), then $q-2$ is θ-half-full (resp. θ-full).

Note that in this language, the classical s-invariant of L over $\mathbb{F}$ is

$$s^{\mathbb{F}}(L) = \max\{q\colon q \text{ is half-full}\} - 1 = \max\{q\colon q \text{ is full}\} + 1.$$

Define the θ-refined s-invariants of L by

$$r_+^{\theta}(L) := \max\{q\colon q \text{ is } \theta\text{-half-full}\} + 1, \quad s_+^{\theta}(L) := \max\{q\colon q \text{ is } \theta\text{-full}\} + 3.$$

Thus $r_+^{\theta}(L), s_+^{\theta}(L) \in \{s^{\mathbb{F}}(L), s^{\mathbb{F}}(L)+2\}$. If 0 denotes the zero stable cohomology operation over $\mathbb{F}$, then $s^{\mathbb{F}} = r_+^0 = s_+^0$.

For computational purposes, we note that $s_+^{\theta}(L) = s^{\mathbb{F}}(L) + 2$ if and only if there exists $x \in H^0(CKh^{\geq s^{\mathbb{F}}(L)-1}_{Lee}(L;\mathbb{F}))$ with $j(x) = [\mathfrak{s}_{\mathfrak{o}}]$ and $p(x) \in \mathrm{im}(\theta)$. Similarly, $r_+^{\theta}(L) = s^{\mathbb{F}}(L) + 2$

if and only if there exists $x \in H^0(CKh_{Lee}^{\geq s^{\mathbb{F}}(L)+1}(L;\mathbb{F}))$ with $j(x) = [\mathfrak{s}_{\mathfrak{o}}] \pm [\mathfrak{s}_{\bar{\mathfrak{o}}}]$ and $p(x) \in \mathrm{im}(\theta)$ for some sign $\pm$.

If $\Sigma\colon L_0 \to L_1$ is an oriented link cobordism in $I \times S^3$ between nonempty links with $\pi_0(L_0) \to \pi_0(\Sigma)$ surjective, then $Kh_{Lee}(\Sigma)$ maps $W(L_0)$ isomorphically onto $W(L_1)$, which restricts to monomorphisms $V^q(L_0) \to V^{q+\chi(\Sigma)}(L_1)$. It follows that

$$r_+^\theta(L_1) - r_+^\theta(L_0) \geq \chi(\Sigma), \quad s_+^\theta(L_1) - s_+^\theta(L_0) \geq \chi(\Sigma). \tag{1.2}$$

Finally, we define $r_+^\theta(\emptyset) = s_+^\theta(\emptyset) = 1$. It is straightforward to check that (1.2) still holds when L_0, L_1 are allowed to be the empty link.

The minus versions of the spatial refinements of s are defined by $r_-^\theta(L) := -r_+^\theta(-L)$, $s_-^\theta(L) := -s_+^\theta(-L)$, where $-L$ denotes the reverse mirror of L. We will not be using them.

1.2.4.2 Sq^1-refined s-invariants for $T(n,n)$

Proposition 1.2.3. *For $p \geq q$, $p+q = n$, we have*

$$s_+^{Sq^1}(T(n,n)_{p,q}) = s^{\mathbb{F}_2}(T(n,n)_{p,q}) = (p-q)^2 - 2p + 1.$$

The same holds for $r_+^{Sq^1}$ when $p = q$.

Proof. Write for short $T = T(n,n)_{p,q}$ and $s(\cdot) = s^{\mathbb{F}_2}(\cdot)$. By Theorem 1.1.1 and Theorem 1.2.1, we know $Kh^{0,s(T)-1}(T) = \mathbb{Z}$ and $Kh^{1,s(T)-1}(T) = 0$. Since Sq^1 is the Bockstein homomorphism, it follows that $Sq^1\colon Kh^{-1,s(T)-1}(T;\mathbb{F}_2) \to Kh^{0,s(T)-1}(T;\mathbb{F}_2)$ is the zero map, hence $s_+^{Sq^1}(T) = s(T)$. Finally, since we will not be using this, we only remark that when $p = q$, the induction scheme in Section 1.5 easily implies that $Kh^{-1,*}(T) = 0$ for all $*$, hence $r_+^{Sq^1}(T) = s(T)$. □

1.2.4.3 $s_+^{Sq^1}$ under disjoint union with nice links

Proposition 1.2.4. *Let T be an oriented link in S^3 such that*

$$Sq^1\colon Kh^{i-1,s^{\mathbb{F}_2}(T)-1}(T;\mathbb{F}_2) \to Kh^{i,s^{\mathbb{F}_2}(T)-1}(T;\mathbb{F}_2)$$

is zero for $i = 0, 1$. Then, for any oriented link L in S^3, we have

$$s_+^{Sq^1}(L \sqcup T) = s_+^{Sq^1}(L) + s_+^{Sq^1}(T) - 1.$$

By arguments in the proof of Proposition 1.2.3 above, we know all torus links $T = T(n,n)_{p,q}$ satisfy the conditions in Proposition 1.2.4.

Proof. Write for short $s(\cdot) = s^{\mathbb{F}_2}(\cdot)$. When $L = \emptyset$ or $T = \emptyset$, the statement is trivial. Below we assume that both T, L are nonempty. Since $Sq^1\colon Kh^{-1,s(T)-1}(T;\mathbb{F}_2) \to Kh^{0,s(T)-1}(T;\mathbb{F}_2)$ is zero, we know $s_+^{Sq^1}(T) = s(T)$.

Since $s(L \sqcup T) = s(L) + s(T) - 1$, it suffices to show that $s_+^{Sq^1}(L \sqcup T) = s(L \sqcup T) + 2$ if and only if $s_+^{Sq^1}(L) = s(L) + 2$. Although the "if" direction is simpler, we prove the two directions together.

We first observe that any filtered cochain complex C^* over a field $\mathbb{F}$ is filtered isomorphic to a filtered complex of the form

$$C^* \cong H^*(C) \oplus (\oplus_\alpha E_\alpha^*) \oplus (\oplus_\beta A_\beta^*)$$

where each E_α (resp. A_β) is a 2-term filtered cochain complex $\mathbb{F} \xrightarrow{=} \mathbb{F}$ in some homological degree, which is not (resp. is) filtration-preserving. Thus, C^* is filtered chain homotopy equivalent to $H^*(C) \oplus (\oplus_\alpha E_\alpha^*)$.

In particular, for each $X \in \{L, T\}$, we can choose such a filtered chain homotopy equivalence (which we simply denote as $=$) $CKh_{BN}^*(X) = Kh_{BN}^*(X) \oplus (\oplus_\alpha E_{X,\alpha}^*)$. Then $Kh^{*,*}(X;\mathbb{F}_2) = gr^*(Kh_{BN}^*(X)) \oplus (\oplus_\alpha gr^*(E_{X,\alpha}^*))$, and $p_X(j_X^{-1}([\mathfrak{s}_{\mathfrak{o}_X}])) \subset Kh^{0,s(X)-1}(X;\mathbb{F}_2)$ (see (1.1) for notation; here $q = s(X)-1$ and subscripts indicate dependence on the link) is the affine subspace modeled on

$$V_X := p_X(\ker(j_X)) = \oplus_\alpha \mathrm{im}(H^0(E_{X,\alpha}^{*,\geq s(X)-1}) \to gr^{s(X)-1}(E_{X,\alpha}^0))$$

that contains $x_X := gr^{s(X)-1}([\mathfrak{s}_{\mathfrak{o}_X}])$.

For the disjoint union, we have

$$\begin{aligned} CKh_{BN}^*(L \sqcup T) = Kh_{BN}^*(L) \otimes Kh_{BN}^*(T) \, &\oplus (\oplus_\alpha E_{L,\alpha}^* \otimes Kh_{BN}^*(T)) \\ &\oplus (\oplus_\beta Kh_{BN}^*(L) \otimes E_{T,\beta}^*) \oplus (\oplus_{\alpha,\beta} E_{L,\alpha}^* \otimes E_{T,\beta}^*) \end{aligned}$$

and we will use the decomposition

$$\begin{aligned} Kh^{0,s(L\sqcup T)-1}(L \sqcup T;\mathbb{F}_2) &= \oplus_{h,q\in\mathbb{Z}} \left(Kh^{h,s(L)-1+q}(L;\mathbb{F}_2) \otimes Kh^{-h,s(T)-1-q}(T;\mathbb{F}_2)\right) \\ &= Kh^{0,s(L)-1}(L;\mathbb{F}_2) \otimes gr^{s(T)-1}(Kh_{BN}^0(T)) \oplus H_{skew} \end{aligned}$$

where

$$\begin{aligned} H_{skew} = \, &Kh^{0,s(L)-1}(L;\mathbb{F}_2) \otimes (\oplus_\beta gr^{s(T)-1}(E_{T,\beta}^0)) \\ &\oplus (\oplus_{(h,q)\neq(0,0)} Kh^{h,s(L)-1+q}(L;\mathbb{F}_2) \otimes Kh^{-h,s(T)-1-q}(T;\mathbb{F}_2)). \end{aligned}$$

We see that $p_{L\sqcup T}(j_{L\sqcup T}^{-1}([\mathfrak{s}_{\mathfrak{o}_L}] \otimes [\mathfrak{s}_{\mathfrak{o}_T}])) \subset Kh^{0,s(L\sqcup T)-1}(L \sqcup T;\mathbb{F}_2)$ is the affine subspace containing $x_L \otimes x_T$ modeled on

$$V_{L\sqcup T} := p_{L\sqcup T}(\ker(j_{L\sqcup T})) = V_L \otimes gr^{s(T)-1}(Kh_{BN}^0(T)) \oplus W$$

for some subspace $W \subset H_{skew}$.

On the other hand, since $Sq^1_{L\sqcup T} = Sq^1_L \otimes 1 + 1 \otimes Sq^1_T$, by our assumption on T, the image $\mathrm{im}(Sq^1_{L\sqcup T}) \subset Kh^{0,s(L\sqcup T)-1}(L \sqcup T; \mathbb{F}_2)$ is equal to

$$\mathrm{im}(Sq^1_L\colon Kh^{-1,s(L)-1}(L;\mathbb{F}_2) \to Kh^{0,s(L)-1}(L;\mathbb{F}_2)) \otimes gr^{s(T)-1}(Kh^0_{BN}(T)) \oplus W'$$

for some subspace $W' \subset H_{skew}$. It follows that

$$\begin{aligned} & s_+^{Sq^1}(L \sqcup T) = s(L \sqcup T) + 2 \\ \iff & x_L \otimes x_T \in V_{L\sqcup T} + \mathrm{im}(Sq^1_{L\sqcup T}) \\ \iff & x_L \in V_L + \mathrm{im}(Sq^1_L) \\ \iff & s_+^{Sq^1}(L) = s(L) + 2. \end{aligned}$$

□

1.3 Applications of the adjunction inequality

1.3.1 Eventual linearity of the s-invariant under full twists

Let L be an oriented link in S^3. A full twist can be performed to L along any (unoriented) 2-disk in S^3 that intersects L transversely in the interior. More generally, if $D_1, \cdots, D_\ell \subset S^3$ are ℓ disjoint such 2-disks, we can independently perform any number of full twists along these disks. Given $\vec{n} = (n_1, \cdots, n_\ell)$, let $L(D_1, D_2, \cdots, D_\ell; \vec{n}) \subset S^3$ denote the oriented link obtained by performing n_i full twists (n_i positive ones if $n_i \geq 0$; $-n_i$ negative ones if $n_i < 0$) to L along D_i. Let d_i denote the algebraic intersection number of D_i and L (which is well-defined up to sign).

Proposition 1.3.1. *For $n_1, \cdots, n_\ell$ large, the number*

$$s(L; D_1, \cdots, D_\ell; \vec{n}) := s(L(D_1, \cdots, D_\ell; \vec{n})) - \sum_{i=1}^{\ell} n_i |d_i|(|d_i| - 1)$$

is independent of $n_1, \cdots, n_\ell$.

Therefore, the stable number, denoted $s(L; D_1, \cdots, D_\ell)$, is an isotopy invariant of the oriented link L together with 2-disks $D_1, \cdots, D_\ell$ in S^3. In the special case $n_1 = n_2 = \cdots = n_\ell$, this answers positively a question of Manolescu–Marengon–Sarkar–Willis [Man+23, Question 9.1].

Remark 1.3.2. (1) By performing 0-surgeries on each $\partial D_i \subset S^3$, L can be regarded as an oriented link in $\ell(S^1 \times S^2)$. Manolescu–Marengon–Sarkar–Willis [Man+23, Theorem 1.4] proved Proposition 1.3.1 in the special case $d_1 = \cdots = d_\ell = 0$, $n_1 = \cdots = n_\ell$, and further showed that in this case the stable number $s(L; D_1, \cdots, D_\ell)$ is an invariant of the null-homologous oriented link L in $\ell(S^1 \times S^2)$, which can then be defined as the s-invariant of $L \subset \ell(S^1 \times S^2)$. However, as follows from their Remark 9.6, the stable number in the general case does not define an invariant of $L \subset \ell(S^1 \times S^2)$. In fact, sliding the strands intersecting D_i over ∂D_i changes the stable number by $\pm 2|d_i|(|d_i|-1)$. It would be interesting if one can use Proposition 1.3.1 to define s-invariants of links in some other 3-manifolds, or to define s-invariants for links in $\ell(S^1 \times S^2)$ valued in $\mathbb{Z}/gcd(d_1, \cdots, d_\ell)$.

(2) Manolescu–Marengon–Sarkar–Willis [Man+23, Conjecture 8.31] conjectured that when $d_1 = \cdots = d_\ell = 0$, if there is a generic projection of $L, D_1, \cdots, D_\ell$ to $\mathbb{R}^2$ which consists of ℓ disjoint 2-disks corresponding to D_i's and a positive link diagram corresponding to L, then the number $s(L; D_1, \cdots, D_\ell; \vec{n})$ already stabilizes when $\vec{n} = \vec{0}$, i.e. it is independent of $\vec{n}$ for $n_1, \cdots, n_\ell \geq 0$. This is not true, because adding a positive full twist appropriately to two oppositely oriented strands in the right-handed trefoil unknots it, but the right-handed trefoil and the unknot have s-invariants 2 and 0, respectively.

The proof of Proposition 1.3.1 is an easy consequence of the following bound on the behavior of the s-invariant under twists.

Proposition 1.3.3. *Let $L \subset S^3$ be an oriented link, and $D \subset S^3$ a 2-disk intersecting it transversely in the interior, in p points positively and q points negatively, where $p \geq q$. Then for $m' > m$,*

$$s(L; D; m') - s(L; D; m) \in \begin{cases} [-2p+2, 0], & p > q \\ [-2p, 0], & p = q. \end{cases}$$

Remark 1.3.4. By considering $T(n, -2n)_{p,q}$, $T(n, -n)_{p,q}$, and a disjoint union of n unknots, we see the bounds in Proposition 1.3.3 are sharp (cf. Corollary 1.1.3).

Proof. The lower bounds are exactly those in [Rob11, Theorem 1.2]. We prove the upper bounds. Adding $m' - m$ full twists along D gives a cobordism $L(D; m) \to L(D; m')$ in $(m'-m)\overline{\mathbb{CP}^2}$ with Euler characteristic 0 and homology class $(p-q, \cdots, p-q)$. Thus Corollary 1.1.4 gives

$$s(L(D; m')) \leq s(L(D; m)) + (m - m')((p-q)^2 - (p-q)),$$

or equivalently $s(L; D; m') \leq s(L; D; m)$. □

Proof of Proposition 1.3.1. Proposition 1.3.3 implies $s(L; D_1, \cdots, D_\ell; \vec{n})$ is nonincreasing in the coordinates of $\vec{n}$, and has a lower bound independent of $\vec{n}$; consequently, it is constant for large $\vec{n}$. □

1.3.2 Knots with large $\mathbb{CP}^2$- and $\overline{\mathbb{CP}^2}$-genus

For a closed oriented smooth 4-manifold M and a knot K in S^3, the M-genus of K is the minimal genus of a smooth orientable surface in $M \backslash B^4$ that bounds K. Marco Marengon informed us of the following consequence of Corollary 1.1.5 (see also [Mar+24, Proposition 2.1]).

Proposition 1.3.5. *There exist knots with simultaneously arbitrarily large $\mathbb{CP}^2$-genus and $\overline{\mathbb{CP}^2}$-genus.*

Proof. Using Corollary 1.1.5 and following the argument in [Mar+24, Proposition 2.1], it suffices to construct a knot with large s-invariant, small τ-invariant, and vanishing Levine–Tristram signature function. For example, such a knot can be taken to be a connected sum of some copies of the untwisted negative Whitehead double of $T(2,-3)$ (which has vanishing Levine–Tristram signature [Lit79, Theorem 2] and $\tau = -1$ [HO08, Theorem 1.2]) with some copies of Piccirillo's companion to the Conway knot, denoted as K' in [Pic20] (which has vanishing Levine–Tristram signature and $\tau = 0$ as it is topologically slice, and $s = 2$). □

This is in contrast to the fact that the $(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2})$-genus (and the $(S^2 \times S^2)$-genus) of any knot is 0 [Nor69, Corollary 3 and Remark], and the fact that every knot has topological $\mathbb{CP}^2$-genus and $\overline{\mathbb{CP}^2}$-genus at most 1 [Kas+24, Corollary 1.15]. We remark that the proof does not extend to $k\mathbb{CP}^2$ for $k > 1$, and to our knowledge no knot is known to be non-slice in $2\mathbb{CP}^2$. In fact, all knots are topologically slice in $2\mathbb{CP}^2$ [Kas+24, Corollary 1.15].

1.4 The Lee filtration structure of $T(n, m)$

In this section we prove Theorem 1.1.2. The discussion here is a slightly simplified version compared to [Ren24a, Section 4]. Most of our discussion here is presented for a general d-cable of a framed knot.

1.4.1 An S_d-symmetry

The torus link $T(n, m)$ can be seen as the d-cable of the torus knot $T(n_1, m_1)$, taken with framing $n_1 m_1$. Thus the braid group B_d acts on $T(n, m)$ by permuting the strands. More generally, whenever $K \subset S^3$ is a framed oriented knot, its d-cable $K(d)$ carries a B_d-action.

Theorem 1.4.1 ([GLW18]). *The B_d-action on $K(d)$ descends to an S_d-action on $H(K(d))$, where H denotes either the Khovanov homology or the Lee homology. Furthermore, the S_d-action descends to an action of the Temperley–Lieb algebra $TL_d(1)$.*

Here, $TL_d(1)$ is the Temperley–Lieb algebra on d strands, with a closed circle evaluating to $-(1+1^{-1})=-2$, which comes with a map $\mathbb{Z}[S_d]\to TL_d(1)$ sending $(i\ i+1)\in S_d$ to $\mathrm{id}+e_i\in TL_d(1)$, where e_i is the cap-cup diagram between the i-th and $(i+1)$-th strands. Over $\mathbb{Q}$, the irreducible S_d-representations are classified by the Young diagrams of size d, and the ones pulled back from $TL_d(1)$ are exactly those given by a two-row Young diagram $(d-r,r)$, $0\le r\le\lfloor d/2\rfloor$. In particular, the irreducible S_d-representations that appear in $Kh_{Lee}(K(d))$ or $Kh(K(d);\mathbb{Q})$ must be of the form $(d-r,r)$.

Note that the Bar-Natan formalism of Khovanov homology that we are using in this chapter has an intrinsic sign ambiguity on cobordism maps; thus, the braid group action on $K(d)$ only gives an action on $H(K(d))$ up to sign. Part of the content of Theorem 1.4.1 is to establish a consistent convention for the choice of signs, which we will recall and use in the next section.

1.4.2 Equivariance

The unknot U has Lee homology $Kh_{Lee}(U)=q^1(\mathbb{Q}[X]/(X^2-1))$, where X has homological degree 0 and quantum degree -2. Flipping U induces an involution ι on $Kh_{Lee}(U)$ up to sign, which we fix by requiring it to send $1,X$ to $-1,X$, respectively. Thus, $Kh_{Lee}(U)\cong 1\oplus\epsilon$ as S_2-representations. Abusing notation, we also use $1,\epsilon$ to also denote the corresponding subrepresentations of $Kh_{Lee}(U)$, then the quantum filtration structure of $Kh_{Lee}(U)$ is determined by

$$q(\epsilon)=1,\ q(1)=-1. \tag{1.3}$$

Let $\mathfrak{o}_U$ denote the orientation of U and $\overline{\mathfrak{o}_U}$ its reverse. Rasmussen's rescaled Lee canonical generators $[\widetilde{\mathfrak{s}}]$ [Ras05] associated to the orientation $\mathfrak{o}_U$ of U and the reverse orientation $\overline{\mathfrak{o}_U}$ are defined up to signs, which we fix by requiring $[\widetilde{\mathfrak{s}_{\mathfrak{o}_U}}]=(X+1)/2$ and $[\widetilde{\mathfrak{s}_{\overline{\mathfrak{o}_U}}}]=(X-1)/2$. Thus the involution ι on $Kh_{Lee}(U)$ exchanges $[\widetilde{\mathfrak{s}_{\mathfrak{o}_U}}]$ and $[\widetilde{\mathfrak{s}_{\overline{\mathfrak{o}_U}}}]$.

A similar description applies to the Khovanov homology of U.

Proposition 1.4.2. *Let $d\ge 2$, and let $f\in\mathbb{Z}$ denote the framing coefficient of a framed oriented knot $K\subset S^3$. For H equal to Lee homology or Khovanov homology, the induced map*

$$t^{2f(d-1)}q^{6f(d-1)-1}H(K(d-2)\sqcup U)\to H(K(d)) \tag{1.4}$$

by the saddle cobordism is $(S_{d-2}\times S_2)$-equivariant. Here $S_{d-2}\times S_2\subset S_d$ via the natural inclusion. Moreover, upon negating the involution on $H(U)$, the induced map

$$H(K(d))\to t^{2f(d-1)}q^{6f(d-1)+1}H(K(d-2)\sqcup U) \tag{1.5}$$

by the (backward) saddle cobordism is $(S_{d-2}\times S_2)$-equivariant.

Proof. We only prove the equivariance of (1.4). The equivariance of (1.5) is proved similarly. Note that the induced maps (1.4) and (1.5) each have a sign ambiguity, but this does not affect the statement.

Let Σ denote the saddle cobordism $K(d-2)\sqcup U \to K(d)$. For a braid $\alpha \in B_d$, let Σ_α denote the corresponding self-isotopy of $K(d)$. Let Σ_ι denote the self-isotopy of U that flips itself around, and let $\sigma_i \in B_d$ denote the usual Artin generators. Now observe that the following pairs of cobordisms are isotopic rel boundary:

$$\Sigma \circ \Sigma_{\sigma_i} \sim \Sigma_{\sigma_i} \circ \Sigma,\ i = 1, 2, \cdots, d-3;\ \Sigma \circ \Sigma_\iota \sim \Sigma_{\sigma_{d-1}} \circ \Sigma.$$

These show the equivariance of (1.4) up to sign. Here $\sigma_1, \cdots, \sigma_{d-1}$ are the usual braid group generators.

To remove the sign ambiguity, we pass to Lee homology and look into the sign convention in [GLW18, Section 7.2] which we adopted. Let $\mathfrak{o}_p$ denote the parallel orientation on $K(d)$, $\mathfrak{o}_a$ denote the alternating orientation on $K(d)$, obtained from $\mathfrak{o}_p$ by reversing the evenly labeled strands, and $\mathfrak{o}_{a,i}$ denote the orientation obtained from $\mathfrak{o}_a$ by reversing the i-th and $(i+1)$-th strands.

The sign of $\varphi_i := Kh_{Lee}(\Sigma_{\sigma_i})$ is chosen so that $\varphi_i([\widetilde{s_{\mathfrak{o}_p}}]) = [\widetilde{s_{\mathfrak{o}_p}}]$. Grigsby–Licata–Wehrli [GLW18] also considered maps ψ_i on $Kh_{Lee}(K(d))$ corresponding to $e_i \in TL_d(1)$, namely it is the map induced by the annulus cobordism $K(d) \to K(d-2)$ that annihilates the components labeled $i, i+1$, followed by the annulus cobordism $K(d-2) \to K(d)$ that recreates two components with labels $i, i+1$. This defines ψ_i up to sign. We have $\psi_i([\widetilde{s_{\mathfrak{o}_a}}]) = \epsilon_1[\widetilde{s_{\mathfrak{o}_a}}] + \epsilon_2[\widetilde{s_{\mathfrak{o}_{a,i}}}]$ for some signs $\epsilon_1, \epsilon_2 \in \{\pm 1\}$, and the sign of ψ_i is fixed by requiring

$$\psi_i([\widetilde{s_{\mathfrak{o}_a}}]) = -[\widetilde{s_{\mathfrak{o}_a}}] \pm [\widetilde{s_{\mathfrak{o}_{a,i}}}].$$

Under these two sign conventions, they showed that

$$\varphi_i = \mathrm{id} + \psi_i. \tag{1.6}$$

Note that these do not fix the signs for the rescaled Rasmussen generators.

Since we have shown that (1.4) is equivariant up to sign, to prove the full equivariance, it now suffices to check on particular Lee generators.

First we check $Kh_{Lee}(\Sigma)\varphi_i = \varphi_i Kh_{Lee}(\Sigma)$, $1 \le i \le d-3$. Since ψ_i vanishes on any $[\widetilde{s_{\mathfrak{o}}}]$ where the $i, i+1$ components are parallel in $\mathfrak{o}$, we see $\varphi_i = \mathrm{id}$ on such generators. Therefore $Kh_{Lee}(\Sigma)\varphi_i([\widetilde{s_{\mathfrak{o}}}] \otimes [\widetilde{s_{\mathfrak{o}_U}}]) = Kh_{Lee}(\Sigma)([\widetilde{s_{\mathfrak{o}}}] \otimes [\widetilde{s_{\mathfrak{o}_U}}]) = \varphi_i Kh_{Lee}(\Sigma)([\widetilde{s_{\mathfrak{o}}}] \otimes [\widetilde{s_{\mathfrak{o}_U}}])$ (which is nonzero) for any such $\mathfrak{o}$.

Next we check $Kh_{Lee}(\Sigma)\iota = \varphi_{d-1} Kh_{Lee}(\Sigma)$. We have not been precise about how the orientation of U is reflected in the saddle map $K(d-2)\sqcup U \to K(d)$, so for concreteness let's assume the saddle map can be realized as an oriented link cobordism $(K(d-2)\sqcup U, \mathfrak{o}_a \sqcup \mathfrak{o}_U) \to (K(d), \mathfrak{o}_a)$, although the other choice of orientation leads to exactly the same computation. We observe that the composite cobordism $K(d) \to K(d-2) \to K(d)$ inducing ψ_{d-1} factors through the saddle $\Sigma\colon K(d-2) \sqcup U \to K(d)$, from which we see, for some sign choice for

$Kh_{Lee}(\Sigma)$, that we have $Kh_{Lee}(\Sigma)([\widetilde{s_{\mathfrak{o}_a}}]\otimes 1) = \psi_{d-1}([\widetilde{s_{\mathfrak{o}_a}}]) = -[\widetilde{s_{\mathfrak{o}_a}}] \pm [\widetilde{s_{\mathfrak{o}_{a,d-1}}}]$. It follows from $1 = [\widetilde{s_{\mathfrak{o}_U}}] - [\widetilde{s_{\overline{\mathfrak{o}_U}}}]$ and our assumption on orientations that $Kh_{Lee}(\Sigma)([\widetilde{s_{\mathfrak{o}_a}}]\otimes[\widetilde{s_{\mathfrak{o}_U}}]) = -[\widetilde{s_{\mathfrak{o}_a}}]$ and $Kh_{Lee}(\Sigma)([\widetilde{s_{\mathfrak{o}_a}}]\otimes[\widetilde{s_{\overline{\mathfrak{o}_U}}}]) = \mp[\widetilde{s_{\mathfrak{o}_{a,d-1}}}]$. Now one can check, using (1.6), that $Kh_{Lee}(\Sigma)\iota([\widetilde{s_{\mathfrak{o}_a}}]\otimes [\widetilde{s_{\mathfrak{o}_U}}]) = \mp[\widetilde{s_{\mathfrak{o}_{a,d-1}}}] = \varphi_{d-1}Kh_{Lee}(\Sigma)([\widetilde{s_{\mathfrak{o}_a}}]\otimes[\widetilde{s_{\mathfrak{o}_U}}])$. □

1.4.3 Structure of symmetry on homology of cables

By exploiting the equivariance of the saddle maps studied in Proposition 1.4.2, we obtain the following structural result on the homology of cables as representations of symmetric groups.

Proposition 1.4.3. *Let K, d, f be as in Proposition 1.4.2. For any $1 \le r \le d/2$, the annulus cobordisms $K(d-2) \to K(d)$ and $K(d) \to K(d-2)$ induce isomorphisms*

$$t^{2f(d-1)}q^{6f(d-1)}\mathrm{Hom}_{S_{d-2}}((d-r-1,r-1), H(K(d-2))) \rightleftarrows \mathrm{Hom}_{S_d}((d-r,r), H(K(d)))$$

that are inverse to each other up to a scalar. Here, H denotes the Lee homology or the rational Khovanov homology.

Proof. We suppress the grading shifts. By Proposition 1.4.2, the annulus cobordisms $K(d-2) \to K(d) \to K(d-2)$ induce maps

$$\begin{aligned}
&\mathrm{Hom}_{S_{d-2}}((d-r-1,r-1), H(K(d-2)))\\
\xrightarrow{\cong}\;&\mathrm{Hom}_{S_{d-2}\times S_2}((d-r-1,r-1)\otimes\epsilon, H(K(d-2))\otimes H(U))\\
\to\;&\mathrm{Hom}_{S_{d-2}\times S_2}((d-r-1,r-1)\otimes\epsilon, \mathrm{Res}^{S_d}_{S_{d-2}\times S_2}H(K(d)))\\
\to\;&\mathrm{Hom}_{S_{d-2}\times S_2}((d-r-1,r-1)\otimes 1, H(K(d-2))\otimes H(U))\\
\xrightarrow{\cong}\;&\mathrm{Hom}_{S_{d-2}}((d-r-1,r-1), H(K(d-2))),
\end{aligned}$$

where the composition is an isomorphism since the composition $K(d-2) \to K(d) \to K(d-2)$ is a torus which evaluates to ± 2 in H, as seen by a neck-cutting (by the sign convention in [GLW18], it evaluates to -2). Composing the other way, $K(d) \to K(d-2) \to K(d)$ induces an endomorphism of $\mathrm{Hom}_{S_{d-2}\times S_2}((d-r-1,r-1)\otimes\epsilon, \mathrm{Res}^{S_d}_{S_{d-2}\times S_2}H(K(d)))$ which is also an isomorphism: this is because (1.6) (which holds for Kh as well as for Kh_{Lee}) shows that the composition $H(K(d)) \to H(K(d))$ is equal to the S_{d-2}-equivariant map $\varphi_{d-1} - \mathrm{id}$ where φ_{d-1} is the map induced by exchanging the last two strands in $K(d)$. The statement now follows from the isomorphism

$$\mathrm{Hom}_{S_{d-2}\times S_2}((d-r-1,r-1)\otimes\epsilon, \mathrm{Res}^{S_d}_{S_{d-2}\times S_2}H(K(d))) \cong \mathrm{Hom}_{S_d}((d-r,r), H(K(d))),$$

as seen from Frobenius reciprocity, Theorem 1.4.1 about the descent to $TL_d(1)$, and

$$\mathrm{Ind}^{S_d}_{S_{d-2}\times S_2}((d-r-1,r-1)\otimes\epsilon) = (d-r,r)\oplus(d-r,r-1,1)\oplus(d-r-1,r,1)\oplus(d-r-1,r-1,1,1)$$

(non-Young diagrams are understood to be zero). □

Corollary 1.4.4. *Let K, f, H be as in Proposition 1.4.3, $d \geq 0$. As graded/filtered S_d-representations, we have*

$$H(K(d)) \cong \bigoplus_{r=0}^{\lfloor d/2 \rfloor} t^{2fr(d-r)} q^{6fr(d-r)} H(K(d-2r))^{S_{d-2r}} \otimes (d-r, r). \quad \square \tag{1.7}$$

A version of Corollary 1.4.4 was also obtained by Ritter von Merkl [Rit25, Corollary 1.2].

Remark 1.4.5. In the discussions in this section up to this point, all statements have natural generalizations to the setup where $K \cup L$ is an oriented link with K a framed knot, and where we consider the S_d-action on the homology of $K(d) \cup L$. One can also formulate the analogous statements for annular Khovanov homology, as in [GLW18].

1.4.4 The invariant part of Lee homology of cables

Before returning to the case of torus links, we establish a last general result for cables regarding the invariant part of their Lee homology.

Proposition 1.4.6. *Let $K \subset S^3$ be a framed oriented knot, with framing $f \neq 0$, and let $d \geq 0$. Then for each (p, q) with $p \geq q \geq 0$, $p + q = d$, we have*

$$Kh_{Lee}^{2fpq}(K(d))^{S_d} \cong \begin{cases} q^{6fpq+s_{p,q}(K)}\mathbb{Q} \otimes V, & p > q \\ q^{6fpq+s_{p,q}(K)}\mathbb{Q}, & p = q \end{cases}$$

for some $s_{p,q}(K) \in \mathbb{Z}$, where V denotes the filtered vector space with $gr(V) \cong \mathbb{Q} \oplus q^2\mathbb{Q}$. At other homological degrees, the Lee homology of $K(d)$ is zero.

Proof. For an orientation $\mathfrak{o}$ of $K(d)$, let $k_+(\mathfrak{o})$ denote the number of strands in $K(d)$ on which the orientation $\mathfrak{o}$ agrees with the one induced from K. For each p, q as in the statement, the Lee homology of $K(d)$ in homological degree $2fpq$ is generated by Rasmussen's rescaled canonical generators associated to orientations $\mathfrak{o}$ with $k_+(\mathfrak{o}) = p$ or q. By the description of the S_d-action on these canonical generators, we see that the invariant part has dimension 1 if $p = q$, and dimension 2 if $p > q$. It remains to prove that, when $p > q$, this 2-dimensional vector space is of the form as described in the proposition.

Let $\mathfrak{s}_p := \sum_{k_+(\mathfrak{o})=p} [\widetilde{\mathfrak{s}_\mathfrak{o}}]$ and similarly $\mathfrak{s}_q$, so that $Kh_{Lee}^{2fpq}(K(d))^{S_d}$ is spanned by $\mathfrak{s}_p$ and $\mathfrak{s}_q$. Then $\mathfrak{s}_p$ and $\mathfrak{s}_q$ are conjugate to each other, in the sense that they agree in one quantum $\mathbb{Z}/4$-degree where $Kh_{Lee}(K(d))$ is supported, and negatives of each other in the other quantum $\mathbb{Z}/4$-degree. This implies that $|q(\mathfrak{s}_p + \mathfrak{s}_q) - q(\mathfrak{s}_p - \mathfrak{s}_q)| \geq 2$. On the other hand, consider the operator $L := \sum_{i=1}^d X_i$ on $Kh_{Lee}(K(d))$, where X_i is a dot operation on the i-th strand of $K(d)$, with sign chosen so that $X_i[\widetilde{\mathfrak{s}_\mathfrak{o}}] = \epsilon[\widetilde{\mathfrak{s}_\mathfrak{o}}]$ for $\epsilon = 1$ if the orientation $\mathfrak{o}$ on the i-th strand agrees with the orientation induced from K, and $\epsilon = -1$ otherwise. We see L is of filtered degree -2 exchanging $\mathfrak{s}_p \pm \mathfrak{s}_q$ up to nonzero scalars, hence $|q(\mathfrak{s}_p + \mathfrak{s}_q) - q(\mathfrak{s}_p - \mathfrak{s}_q)| \leq 2$. The statement follows, with $s_{p,q}(K) = q(\mathfrak{s}_p) = q(\mathfrak{s}_q)$. $\square$

1.4.5 The case of torus links

We return to the torus link $T(n,m)$, the d-cable on the n_1m_1-framed torus knot $T(n_1,m_1)$, $n_1,m_1>0$. By Corollary 1.4.4, it suffices to determine the structure of $Kh_{Lee}(T(n,m))^{S_d}$, since they recover the structure of the Lee homology of all torus links.

Lemma 1.4.7. *In the notation of Proposition 1.4.6, we have $s_{p,q}(T(n_1,m_1)) = s(T(n,m)_{p,q})-1$.*

Proof. We induct on d. When $d=0$, the statement follows from $s(\emptyset)=1$. When $d=1$, the statement follows from the definition of s-invariants for knots.

Assume $d\geq 2$. The contribution to $Kh_{Lee}^{2n_1m_1pq}(T(n,m))$ in (1.7) coming from $T(n_1(d-2r),m_1(d-2r))$ is zero if $r>q$, and is supported in quantum degree at least

$$6n_1m_1r(d-r)+6n_1m_1(p-r)(q-r)+s_{p-r,q-r}(K) = 6n_1m_1pq+(n_1|p-q|-1)(m_1|p-q|-1)-2q+2r-1$$

for $1\leq r\leq q$ by induction hypothesis and Theorem 1.1.1. On the other hand, if $\mathfrak{o}$ is an orientation of $T(n,m)$ with p strands oriented against the other q strands, then

$$q([\widetilde{\mathfrak{s}_{\mathfrak{o}}}]) = 6n_1m_1pq+s(T(n,m)_{p,q})-1 = 6n_1m_1pq+(n_1|p-q|-1)(m_1|p-q|-1)-2q-1$$

is strictly smaller than the lower bound given by any term above. Since every element in $Kh_{Lee}^{2n_1m_1pq}(T(n,m))$ is a linear combination of such canonical generators, all of which have the same quantum filtration degree, we see no element can have filtration degree strictly less than $q([\widetilde{\mathfrak{s}_{\mathfrak{o}}}])$. Thus, the $r=0$ term in (1.7) must contribute $q([\widetilde{\mathfrak{s}_{\mathfrak{o}}}])$ as the bottom degree, implying that $s_{p,q}(T(n_1,m_1)) = s(T(n,m)_{p,q})-1$, as desired. □

Proof of Theorem 1.1.2. This is a consequence of Corollary 1.4.4, Proposition 1.4.6, Lemma 1.4.7, and Theorem 1.1.1. □

1.5 Proof of Theorem 1.2.1

We follow the induction scheme set up by Stošić [Sto07; Sto09]. For ease of notation, in this section, we write $T_{n,m}$ for the torus link $T(n,m)$. Define an auxiliary family of links $D_{n,m}^i$, $m,n\geq 0$, $0\leq i\leq n-1$, as the braid closure of the braid $(\sigma_1\cdots\sigma_{n-1})^m\sigma_1\cdots\sigma_i\in B_n$. Thus, $T_{n,m}=D_{n,m}^0=D_{n,m-1}^{n-1}$. For $i>0$, performing a 0-resolution at the crossing of $D_{n,m}^i$ corresponding to the last letter σ_i gives the link $D_{n,m}^{i-1}$, while performing a 1-resolution gives another link which we denote by $E_{n,m}^{i-1}$. The reader is warned that these notations do not agree with those in [Sto07; Sto09].

For our purposes, the cases $m=n,n-1$ will be useful. The following statements are easily checked. The first two items appeared in [Sto09, Proof of Theorem 1]. See Figure 1.4 for an illustration of the third item.

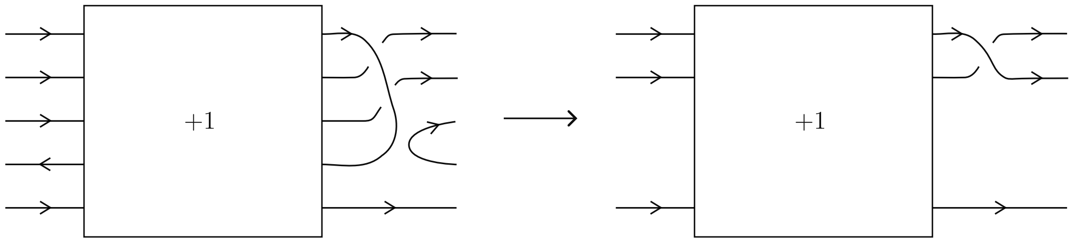


Figure 1.4: The oriented link $E^3_{5,5} \simeq D^1_{3,3}$

Lemma 1.5.1.

- $E^{n-2}_{n,n-1} \simeq D^{n-3}_{n-2,n-3} \sqcup U$;
- $E^{i}_{n,n-1} \simeq D^{i}_{n-2,n-3}$, $i = 0, 1, \cdots, n-3$;
- $E^{i}_{n,n} \simeq D^{i-1}_{n-2,n-2}$, $i = 1, 2, \cdots, n-2$;
- $E^{0}_{n,n} \simeq D^{0}_{n-2,n-2} \sqcup U$. □

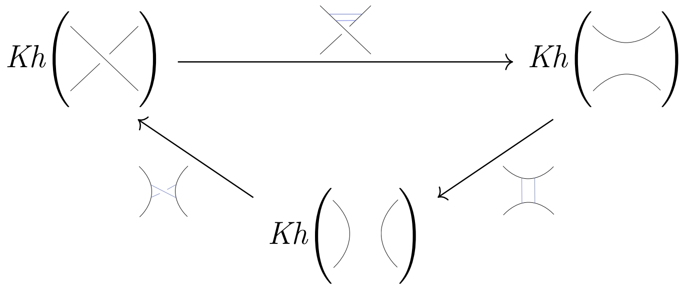
Equip $D^i_{n,m}$ with the orientation where all components are oriented in the same direction. Equip $E^i_{n,m}$ with the orientation coming from the right-hand sides of Lemma 1.5.1 for $m = n, n-1$. Then all crossings in $D^i_{n,m}$ are positive, while $E^i_{n,n-1}$ has $2n-3$ negative crossings and $E^i_{n,n}$ has $2n-2$ negative crossings. Keeping track of the degree shifts (see e.g. [Tur17, Section 3.1 Case II]), the skein long exact sequences on Khovanov homology corresponding to the resolution at the last letter σ_i in $D^i_{n,m}$ read

$$\cdots \to Kh^{h-2n+2,q-6n+7}(E^{i-1}_{n,n-1}) \to Kh^{h,q}(D^{i}_{n,n-1}) \to Kh^{h,q-1}(D^{i-1}_{n,n-1}) \to \cdots, \tag{1.8}$$

$$\cdots \to Kh^{h-2n+1,q-6n+4}(E^{i-1}_{n,n}) \to Kh^{h,q}(D^{i}_{n,n}) \to Kh^{h,q-1}(D^{i-1}_{n,n}) \to \cdots. \tag{1.9}$$

Moreover, the maps in the above long exact sequences are the maps induced by the obvious saddle cobordisms between the relevant links. Since the author is not aware of this last claim on cobordisms in the existing literature, we take a short detour and record this into the following lemma.

Lemma 1.5.2. *At a crossing of a link diagram, we have the following skein exact triangle in Khovanov homology, where the morphisms are induced by the saddles as indicated. (For* *simplicity, we have suppressed all grading shifts.)*



Proof. Let CKh denote the Khovanov chain complex. Then by definition

$$CKh(\;) = Cone(CKh(\;) \longrightarrow CKh(\;)).$$

This gives rise to an exact triangle as stated, except that we still have to check the top and left morphisms in the exact triangle agree with the morphisms induced by the saddle cobordisms given in the statement.

First we check the top morphism. Identify $CKh(\;) = CKh(\;) \oplus CKh(\;)$ as modules, we see that the top morphism in the exact triangle is induced by the projection of $CKh(\;)$ onto $CKh(\;)$. On the other hand, the morphism given in the statement is induced by the chain map

$$CKh(\;) \longrightarrow CKh(\;) \xrightarrow{(R1^+)^{-1}} CKh(\;) \tag{1.10}$$

where $(R1^+)^{-1}$ denotes the chain homotopy equivalence induced by undoing the positive twist. See [HS24, Table 1, Table 3] for succinct descriptions of the induced maps by saddle cobordisms and Reidemeister I moves. On the direct summand $CKh(\;)$, (1.10) equals

$$CKh(\;) \to CKh(\;) \to CKh(\;) \tag{1.11}$$

where the first map is induced by the splitting saddle cobordism $\Delta\colon 1 \mapsto 1 \otimes X + X \otimes 1, X \mapsto X \otimes X$ on the top strand, and the second map is induced by the death cobordism $\epsilon\colon 1 \mapsto 0, X \mapsto 1$ on the middle circle. Since $(1 \otimes \epsilon) \circ \Delta = 1$, the composition (1.11) is the identity. On the direct summand $CKh(\;)$, (1.10) equals

$$CKh(\;) \to CKh(\;) \to CKh(\;),$$

where the second map is identically zero. We have thus shown that the top morphism in the exact triangle equals the stated morphism.

Next we check the left cobordism. The morphism in the exact triangle is induced by the inclusion of $CKh(\;)$ into $CKh(\;)$ as a direct summand. On the other hand, the morphism given in the statement is induced by the chain map

$$CKh(\;) \xrightarrow{R1^-} CKh(\;) \longrightarrow CKh(\;) \tag{1.12}$$

where $R1^-$ denotes the chain homotopy equivalence induced by creating the negative twist, which is equal to the map $CKh()\;()\to CKh()\circ())$ induced by the birth cobordism $\iota\colon 1\mapsto 1$ creating the middle circle, followed by the inclusion $CKh()\circ()) \to CKh()\!\!\times\!\!\circ())$ as a direct summand. On the direct summand $CKh()\circ())$, the second map in (1.12) equals the merging saddle cobordism $m\colon 1\otimes 1\mapsto 1, 1\otimes X\mapsto X, X\otimes 1\mapsto X, X\otimes X\mapsto 0$ on the right two components. Since $m\circ(\iota\otimes 1)=1$, the composition (1.12) equals the inclusion map, and the proof is complete. $\square$

Next we introduce some notations that will be convenient. For two functions $f,g\colon \mathbb{Z}\to\mathbb{Z}\sqcup\{+\infty\}$, define $\min(f,g)$ to be the function $\mathbb{Z}\to\mathbb{Z}\sqcup\{+\infty\}$ with $\min(f,g)(h)=\min(f(h),g(h))$. We write $f>g$ if $f(h)>g(h)$ for all h with $f(h)<+\infty$, and write $f<g$, $f\geq g$, $f\leq g$ analogously. For $h,q\in\mathbb{Z}$, we define $f[h]\{q\}\colon\mathbb{Z}\to\mathbb{Z}\sqcup\{+\infty\}$ by $(f[h]\{q\})(h')=f(h'-h)+q$. For a function defined on a subset of $\mathbb{Z}$ that takes values in $\mathbb{Z}\sqcup\{+\infty\}$, we abuse notation and use the same expression to denote its extension by $+\infty$ to all of $\mathbb{Z}$. Finally, define $t_{n,n}(h):=\inf\{q\colon Kh^{h,q}(T_{n,n})\neq 0\}$ to be the quantum infimum function for $Kh(T_{n,n})$, and similarly $t_{n+1,n}, d^i_{n,m}, e^i_{n,m}$ to be the quantum infimum functions for $Kh(T_{n+1,n})$, $Kh(D^i_{n,m})$, $Kh(E^i_{n,m})$, respectively.

Thus, for example, the lower bound of Theorem 1.2.1(1) says $t_{n,n}\geq q_{n,n}$.

Before launching into the proof of Theorem 1.2.1, we note the following relations among the functions $q_{n,n}$ and $q_{n+1,n}$ defined in Section 1.2.1.

Lemma 1.5.3.

$$q_{n-2,n-2}[2n-2]\{6n-8\}=q_{n,n}|_{h\geq 2n-2}\geq q_{n,n}, \tag{1.13a}$$
$$q_{n-1,n-2}[2n-2]\{6n-6\}\geq q_{n+1,n}|_{h\leq h_{max}(T_{n+1,n})-1}, \tag{1.13b}$$
$$q_{n,n}\{n-1\}\geq q_{n+1,n}, \tag{1.13c}$$
$$q_{n,n-1}\{n-1\}\geq q_{n,n}, \tag{1.13d}$$
$$q_{n,n-1}[1]\{2\}|_{1\neq h\leq h_{max}(T_{n,n-1})}\geq q_{n,n-1}, \tag{1.13e}$$
$$q_{n,n-1}[2]\{4\}|_{h\leq h_{max}(T_{n,n-1})}\geq q_{n,n-1}. \tag{1.13f}$$

Moreover, (1.13c) is strict at $h=2n-1$. (1.13d) is strict at $h=2pq$ for any $p+q=n$, $p\geq q>0$; it is strict at $h=2pq-1$ for any $p+q=n$, $p\geq q>1$.

These are elementary. We sketch mostly geometric proofs of them.

Sketch proof. The functions $q_{n,n}$ are thought of as staircases as indicated by the red segments in Figure 1.1. The height of a step is usually 2, except 4 right after homological degrees $2pq$. Thus to check the equality in (1.13a), it suffices to note that the two sets $\{2pq\colon p+q=n, p\geq q>0\}$ and $\{2pq\colon p+q=n-2, p\geq q\geq 0\}$ are identical up to a shift of $2n-2$, and that $q_{n,n}(2n-2)=q_{n-2,n-2}(0)+6n-8$.

The function $q_{n+1,n}$ is thought of as a shift of $q_{n,n}$ by $\{n-1\}$ with all but the first big step flattened (by decreasing the height by 2 at $2pq+1$, $p \geq q > 0$), plus $\lfloor n/2 \rfloor$ short small steps as tail, see Figure 1.2. This description immediately gives (1.13c)(1.13e)(1.13f). Together with (1.13a), this also proves (1.13b) (we can apply the truncation in the end because the tail of $q_{n+1,n}$ is one longer than that of $q_{n-1,n-2}$). Since $2n-1 = 2(n-1)\cdot 1 + 1$, (1.13c) is strict at $h = 2n-1$.

Finally, (1.13d) is more contrived, so we first calculate algebraically. At $h = 0$, (1.13d) is an equality. For $0 < h \leq h_{max}(T_{n-1,n-1})$, choose $p+q = n, p \geq q > 0$ and $p'+q' = n-1, p' \geq q' > 0$ with $h \in (2(p+1)(q-1), 2pq] \cap (2(p'+1)(q'-1), 2p'q']$. Then

$$q_{n-1,n-1}(h) + 2n - 3 - q_{n,n}(h) = (n-1)^2 + 2\left\lceil \frac{h}{2} \right\rceil - 2p' + 2n - 3 - n^2 - 2\left\lceil \frac{h}{2} \right\rceil + 2p = 2(q'-q). \tag{1.14}$$

If $h \neq 2(p'+1)(q'-1)+1$ or $q' = 1$, (1.14) gives $q_{n,n-1}\{n-1\}(h) = q_{n-1,n-1}(h) + 2n - 3 \geq q_{n,n}(h)$ since $q' \geq q$. If $h = 2(p'+1)(q'-1)+1$ and $q' \neq 1$, (1.14) gives $q_{n,n-1}\{n-1\}(h) = q_{n-1,n-1}(h) + 2n - 5 = q_{n,n}(h)$ since $q' > q$.

To prove (1.13d) for $h_{max}(T_{n-1,n-1}) \leq h \leq h_{max}(T_{n,n-1})$, we note that in this range $q_{n,n-1}\{n-1\}$ is a tail of short small steps while $q_{n,n}$ is a combination of long small and long big steps. Thus (1.13d) follows from its validity at $h = h_{max}(T_{n-1,n-1})$, unless $q_{n,n}$ has a big step right after $h = h_{max}(T_{n-1,n-1})$. In this exceptional case, $h_{max}(T_{n-1,n-1}) = 2pq$ for some $p+q = n$, $p \geq q > 0$. This implies $q' > q$ in (1.14), so (1.13d) is strict at $h = h_{max}(T_{n-1,n-1})$, and (1.13d) also follows.

Finally we prove the addendum about the strictness of (1.13d). For $0 < h \leq h_{max}(T_{n-1,n-1})$, $h \in (2(p+1)(q-1), 2pq] \cap (2(p'+1)(q'-1), 2p'q']$, (1.13d) is strict if and only if $q' > q+1$ or $q' = q+1$ and $h \neq 2(p'+1)(q'-1)+1$. This is the case for $h = 2pq$ if $p \geq q > 0$, and for $h = 2pq-1$ if $p \geq q > 1$.

Now assume $h = 2pq, 2pq-1$, $h > h_{max}(T_{n-1,n-1})$. The case for $h = 2pq$ is trivial because

$$q_{n,n-1}\{n-1\}(2pq) = q_{n,n-1}\{n-1\}(2pq-1) + 2 \geq q_{n,n}(2pq-1) + 2 = q_{n,n}(2pq) + 2.$$

For $h = 2pq-1$, we divide into three cases.

Case 1: If $2pq-1 > \max(2(p+1)(q-1)+1, h_{max}(T_{n-1,n-1})+1)$, then

$$q_{n,n-1}\{n-1\}(2pq-1) = q_{n,n-1}\{n-1\}(2pq-3) + 4 \geq q_{n,n}(2pq-3) + 4 = q_{n,n}(2pq-1) + 2.$$

Case 2: If $2pq-1 = 2(p+1)(q-1)+1$, then $p = q+1$, so $h_{max}(T_{n,n-1}) = 2q^2 + q$ is less than $2pq-1$ unless $q = 1$, which is excluded in our hypothesis.

Case 3: If $2pq-1 = h_{max}(T_{n-1,n-1}) + 1 > 2(p+1)(q-1)+1$, then the strictness of (1.13d) at $2pq-1$ is equivalent to that at $h_{max}(T_{n-1,n-1})$. The latter is true because $q' > q$ in (1.14) unless $q = 1$. □

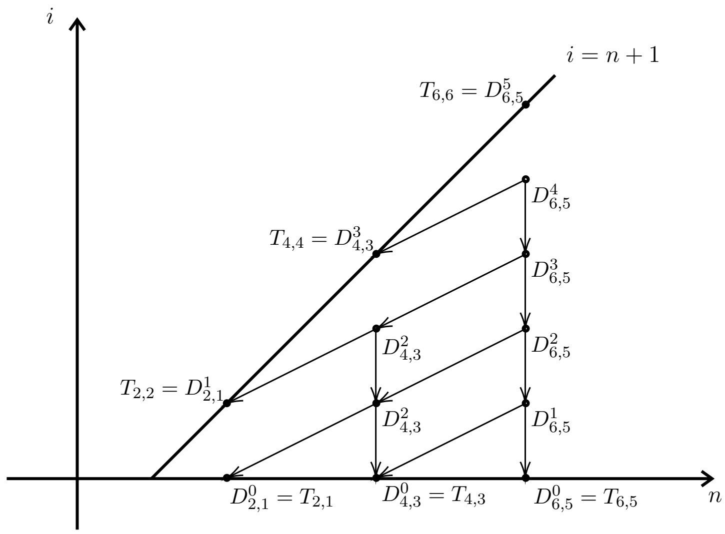


Figure 1.5: Flowchart of applying (1.15) to $D^4_{6,5}$

Proof of Theorem 1.2.1. With the notation above, we need to prove $t_{n,n} \geq q_{n,n}$, $t_{n+1,n} \geq q_{n+1,n}$, and the two addenda to (1)(2).

We induct on n. The base cases $n = 1, 2$ are easily checked. Below we assume $n \geq 3$.

(1) The statements for $T_{n,n}$.

The long exact sequence (1.8) gives

$$d^i_{n,n-1} \geq \min(e^{i-1}_{n,n-1}[2n-2]\{6n-7\}, d^{i-1}_{n,n-1}\{1\}). \tag{1.15}$$

Using Lemma 1.5.1, we thus have

$$t_{n,n} \geq \min(t_{n-2,n-2}[2n-2]\{6n-8\}, d^{n-2}_{n,n-1}\{1\}) =: \min(A, B),$$

where $A \geq q_{n-2,n-2}[2n-2]\{6n-8\} \geq q_{n,n}$ by induction hypothesis and (1.13a). Inductively applying (1.15) (cf. Figure 1.5) we obtain

$$\begin{aligned} B \geq \min\Bigg(&\min_{\substack{n-m=2r\geq 0\\ m\geq 2}} t_{m,m-1}\left[{\sum}'(2i-2)\right]\left\{{\sum}'(6i-8)+n-1\right\}, \\ &\min_{\substack{n-m=2r>0\\ m\geq 1}} t_{m,m}\left[{\sum}'(2i-2)\right]\left\{{\sum}'(6i-8)+n-m\right\}\Bigg). \end{aligned} \tag{1.16}$$

Here and henceforth, each $\sum'$ is a sum over $i \in (m, n]$ with the same parity as m, n. By induction hypothesis and (1.13a), each term in the second minimum in (1.16) is bounded below by

$$q_{m,m}\left[{\sum}'(2i-2)\right]\left\{{\sum}'(6i-8)+n-m\right\} \geq q_{n,n}\{n-m\} > q_{n,n}.$$

Similarly, by induction hypothesis and (1.13b)(1.13d)(1.13f), each term in the first minimum in (1.16) is bounded below by

$$\begin{aligned} &q_{m,m-1}\left[{\sum}'(2i-4)+2r\right]\left\{{\sum}'(6i-12)+4r+n-1\right\} \\ &\geq q_{n,n-1}[2r]\{4r+n-1\} \geq \min(q_{n,n-1}\{n-1\}, tail) \geq q_{n,n} \end{aligned}$$

where $tail$ is the function defined on $[h_{max}(T_{n,n-1}), h_{max}(T_{n,n-1})+2r]$ which consists of short small steps (in the sense of the proof of Lemma 1.5.3) and agrees with $q_{n,n-1}\{n-1\}$ at $h = h_{max}(T_{n,n-1})$.

Now it remains to prove the statement about the saddle cobordism. The relevant induced map α fits in the long exact sequence (1.8)

$$\begin{aligned} \cdots \to Kh^{h-1,q-1}(D^{n-2}_{n,n-1}) &\xrightarrow{\beta} Kh^{h-2n+2,q-6n+7}(T_{n-2,n-2}\sqcup U) \\ &\xrightarrow{\alpha} Kh^{h,q}(T_{n,n}) \to Kh^{h,q-1}(D^{n-2}_{n,n-1}) \to \cdots \end{aligned} \tag{1.17}$$

for $h = 2pq > 0$, $q = q_{n,n}(h)$. The map α would be an isomorphism if $Kh^{h-1,q-1}(D^{n-2}_{n,n-1}) = Kh^{h,q-1}(D^{n-2}_{n,n-1}) = 0$, or equivalently, using the notation above, that $B(h-1), B(h) > q(= q_{n,n}(h) = q_{n,n}(h-1))$.

We re-examine the estimate $B \geq q_{n,n}$ above. The contribution from the second term in (1.16) is always strict. The contribution from the first term is strict for h, and is strict for $h-1$ if $q > 1$, because (1.13d) is strict for such homological degrees by the second addendum of Lemma 1.5.3. If $q = 1$, however, the inequality is strict at $h-1 = 2n-3$ for $r > 0$ (as the left-hand side is $+\infty$), but not for $r = 0$. In this exceptional case, the addendum in the induction hypothesis for $T_{n,n-1}$ states that the relevant homology group $Kh^{2n-3,q_{n,n-1}(2n-3)}(T_{n,n-1})$ is torsion, thus so is every $Kh^{2n-3,q_{n,n-1}(2n-3)+i}(D^i_{n,n-1})$, in view of (1.9). This group for $i = n-2$ and the group $Kh^{0,(n-3)^2-2}(T_{n-2,n-2}\sqcup U) = \mathbb{Z}$ appear as the first two terms in (1.17), which implies β in (1.17) is zero, thus α is an isomorphism.

(2) The statements for $T_{n+1,n}$.

The long exact sequence (1.9) gives

$$d^i_{n,n} \geq \min(e^{i-1}_{n,n}[2n-1]\{6n-4\}, d^{i-1}_{n,n}\{1\}). \tag{1.18}$$

Using Lemma 1.5.1, we obtain

$$t_{n+1,n} \geq \min(t_{n-1,n-2}[2n-1]\{6n-4\}, d^{n-2}_{n,n}\{1\}) =: \min(A, B)$$

where $A \geq q_{n-1,n-2}[2n-1]\{6n-4\} \geq q_{n+1,n}[1]\{2\}|_{2n-1\leq h\leq h_{max}(T_{n+1,n})} \geq q_{n+1,n}$ by induction hypothesis and (1.13b)(1.13e), and

$$B \geq \min_{n-m=2r\geq 0} t_{m,m}\left[\sum{}'(2i-1)\right]\left\{\sum{}'(6i-6)+n-1\right\}. \tag{1.19}$$

By induction hypothesis and (1.13a)(1.13c)(1.13e), the $r=0$ term in the summation is bounded below by $q_{n,n}\{n-1\} \geq q_{n+1,n}$, and each of the $r>0$ terms is bounded below by

$$\begin{aligned} q_{m,m}\left[\sum{}'(2i-1)\right]\left\{\sum{}'(6i-6)+n-1\right\} &\geq q_{n,n}[r]\{2r+n-1\}|_{h\geq 2n-1} \\ \geq q_{n+1,n}[r]\{2r\}|_{2n-1\leq h\leq h_{max}(T_{n,n})+r} &\geq q_{n+1,n}. \end{aligned}$$

It remains to show $Kh^{2n-1,q_{n+1,n}(2n-1)}(T_{n+1,n})$ is torsion. This group sits in the long exact sequence (1.9)

$$\begin{aligned} \cdots \to Kh^{2n-2,q_{n+1,n}(2n-1)-1}(D^{n-2}_{n,n}) &\xrightarrow{\gamma} Kh^{0,q_{n-1,n-2}(0)}(T_{n-1,n-2}) \\ \to Kh^{2n-1,q_{n+1,n}(2n-1)}(T_{n+1,n}) &\to Kh^{2n-1,q_{n+1,n}(2n-1)-1}(D^{n-2}_{n,n}) \to \cdots. \end{aligned}$$

Thus it suffices to show, upon tensoring $\mathbb{Q}$, that γ is surjective and $Kh^{2n-1,q_{n+1,n}(2n-1)-1}(D^{n-2}_{n,n}) = 0$.

By Lemma 1.5.2, γ is induced by a saddle cobordism $D^{n-2}_{n,n} \to T_{n-1,n-2}$. Let $\theta\colon Kh^{0,q_{n-1,n-2}(0)+2}(T_{n-1,n-2}) \to Kh^{2n-2,q_{n+1,n}(2n-1)-1}(D^{n-2}_{n,n})$ be the map induced by the backward saddle cobordism. Then by the neck-cutting relation [Bar05, Section 11.2] we have $\gamma\circ\theta = 2X$, where

$$X\colon Kh^{0,q_{n-1,n-2}(0)+2}(T_{n-1,n-2}) \to Kh^{0,q_{n-1,n-2}(0)}(T_{n-1,n-2}) \tag{1.20}$$

is the map induced by the dotted cobordism on $T_{n-1,n-2}$. By Stošić [Sto07, Theorem 3.4], $Kh^{0,*}(T_{n-1,n-2}) = \mathbb{Z}$ for $* = q_{n-1,n-2}(0)+1\pm 1$ and 0 otherwise, and $Kh^{1,*}(T_{n-1,n-2}) = 0$. Hence the reduced Khovanov homology of $T_{n-1,n-2}$ has $\widetilde{Kh}^{1,*}(T_{n-1,n-2}) = 0$, and thus $\widetilde{Kh}^{0,*}(T_{n-1,n-2}) = \mathbb{Z}$ for $* = q_{n-1,n-2}(0)+1$ and 0 otherwise, in view of the long exact sequence

$$\cdots \to \widetilde{Kh}^{-1,*-1} \to \widetilde{Kh}^{0,*+1} \to Kh^{0,*} \to \widetilde{Kh}^{0,*-1} \to \widetilde{Kh}^{1,*+1} \to \cdots$$

applied to $T_{n-1,n-2}$. Consequently, (1.20) is an isomorphism. This proves the surjectivity of $\gamma\otimes\mathbb{Q}$.

Next we show $Kh^{2n-1,q_{n+1,n}(2n-1)-1}(D^{n-2}_{n,n})\otimes\mathbb{Q} = 0$. Let $d^i_{n,m,\mathbb{Q}}$ denote the quantum infimum function for $Kh(D^i_{n,m})\otimes\mathbb{Q}$. The above estimates apply equally well with $d^i_{n,m}$ replaced by $d^i_{n,m,\mathbb{Q}}$. We now need to show the inequality $d^{n-2}_{n,n,\mathbb{Q}}\{1\} \geq q_{n+1,n}$ is strict at homological degree $2n-1$. We re-examine the estimate for B above. The contribution from the $r=0$

term in (1.19) is strict at $2n-1$, because (1.13c) is strict at $2n-1$ by the first addendum of Lemma 1.5.3. The $r \geq 2$ terms are also strict because the left-hand sides are $+\infty$. For the $r = 1$ term, the contribution is not necessarily strict, but it would be if we could improve the contribution coming from the first term in (1.18) at homological degree $2n-1$ by an extra positive quantum shift, for each i. Upon tensoring $\mathbb{Q}$, we can indeed make this improvement, by showing that the relevant maps $Kh^{2n-2,q_{n-2,n-2}(0)+i-2+6n-5}(D^{i-1}_{n,n}) \otimes \mathbb{Q} \to Kh^{0,q_{n-2,n-2}(0)+i-2}(E^{i-1}_{n,n}) \otimes \mathbb{Q}$ in (1.9)$\otimes\mathbb{Q}$ are surjective. Note that for $i = n-1$ this map is exactly $\gamma \otimes \mathbb{Q}$, whose surjectivity has just been shown. The $i < n-1$ cases follow from exactly the same argument, using the fact that, by an induction on i using (1.9), $Kh^{0,*}(D^i_{n-2,n-2}) \otimes \mathbb{Q} = \mathbb{Q}$ for $* = q_{n-2,n-2}(0) + i + 1 \pm 1$ and 0 otherwise, and $Kh^{1,*}(D^i_{n-2,n-2}) = 0$ (the base case for $D^0_{n-2,n-2} = T_{n-2,n-2}$ follows again from [Sto07, Theorem 3.4]). $\square$

Remark 1.5.4. The argument above carries through with $\mathbb{Q}$ replaced by any $\mathbb{F}_p$, $p \neq 2$. This shows the order of every element in $Kh^{2n-1,q_{n+1,n}(2n-1)}(T_{n+1,n})$ is a power of 2.

1.6 Expectations

We make some comments and observations mostly related to Theorem 1.2.1. First, we remark that the statements in Theorem 1.2.1 are almost designed minimally so that the $m = n$ case of Theorem 1.1.1 can be proved. One may try to prove more about $Kh(T(n,n))$ and $Kh(T(n+1,n))$ of independent interest using the same induction scheme. For example, it seems true that the bound $q_{n,n}$ is sharp not only for $h = 2pq$, but for any even h; not only $Kh^{2n-1,q_{n+1,n}(2n-1)}(T(n+1,n))$ but also any $Kh^{2pq+1,q_{n+1,n}(2pq+1)}(T(n+1,n))$ is torsion, which equals 0 if n is odd. Moreover, one can similarly expect to obtain a "graphical upper bound" for $Kh(T(n,n))$ and $Kh(T(n+1,n))$. This might lead to a proof of Corollary 1.1.3 (over any coefficient field) without resorting to Theorem 1.1.2. One may also try to prove similar results on Khovanov homology of more general torus links $T(n,m)$.

By pushing the representation theory techniques further, one may try to prove a version of Theorem 1.1.2 for more general coefficient fields.

In the rest of the section, we turn to some speculation on the rational Khovanov homology of $T(n,n), T(n+1,n)$, and some more general cable knots and links.

1.6.1 Rational Khovanov homology of $T(n,n)$

Let $D^i_{n,m}$ and $E^i_{n,m}$ be defined as in Section 1.5.

Conjecture 1.6.1. *The saddle cobordism $D^i_{n,n-1} \to D^{i-1}_{n,n-1}$ at the crossing of $D^i_{n,n-1}$ corresponding to the last letter σ_i induces a surjection in rational Khovanov homology.*

Equivalently, the conjecture states that the exact triangle

$$\begin{tikzcd}
t^{2n-2}q^{6n-7}Kh(E^{i-1}_{n,n-1}) \arrow[rr] & & Kh(D^{i}_{n,n-1}) \arrow[dl] \\
& q^1 Kh(D^{i-1}_{n,n-1}) \arrow[ul, "{[1]}"] &
\end{tikzcd}$$

splits over $\mathbb{Q}$ into a short exact sequence

$$0 \to t^{2n-2}q^{6n-7}Kh(E^{i-1}_{n,n-1};\mathbb{Q}) \to Kh(D^{i}_{n,n-1};\mathbb{Q}) \to q^1 Kh(D^{i-1}_{n,n-1};\mathbb{Q}) \to 0.$$

An affirmative answer to Conjecture 1.6.1 would enable one to express the rational Khovanov homology of the torus links $T(n,n)$ entirely in terms of those of the torus knots $T(n'+1,n')$. Explicitly, let $K_n(t,q)$ denote the Poincaré polynomial of the Khovanov homology of $T(n+1,n)$ and $L_n(t,q)$ denote that of $T(n,n)$, then Conjecture 1.6.1 is equivalent to the following.

Conjecture 1.6.2. *The Poincaré polynomials of $Kh(T(n,n))$ are recursively defined by*

$$L_0 = 1,\ \ L_1 = q^{-1}+q,$$

$$\begin{aligned} L_n =& t^{2n-2}(q^{6n-8}+q^{6n-6})L_{n-2} + \sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor} C_{i-1}t^{2i(n-i)}q^{6i(n-i)}L_{n-2i} \\ &+ \sum_{i=0}^{\lfloor\frac{n-2}{2}\rfloor}\left(\binom{n-2}{i}-\binom{n-2}{i-1}\right)t^{2i(n-i)}q^{6i(n-i)+n-2i-1}K_{n-2i-1}, \quad n\geq 2. \end{aligned}$$

Here $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n^{th} Catalan number.

On the other hand, Shumakovitch and Turner have the following conjecture.

Conjecture 1.6.3 ([GOR13, Conjecture 1.8]). *The Poincaré polynomials of $Kh(T(n,n-1))$ are recursively defined by*

$$K_1 = K_2 = q^{-1}+q,\ \ K_3 = q+q^3+t^2q^5+t^3q^9,$$

$$K_n = q^{2n-4}K_{n-1} + t^{2n-4}q^{6n-12}K_{n-2} + t^{2n-3}q^{8n-16}K_{n-3}, \quad n\geq 4.$$

Conjecture 1.6.2 and Conjecture 1.6.3 together establish a recursive formula for the rational Khovanov homology of $T(n,n)$.

With the help of the computer program KnotJob [Sch], we have verified Conjecture 1.6.1 for the cases $n\leq 8$. The integral version of the conjecture is not true, and the first counterexample is found at $n=7$, $i=6$. We also verified Conjecture 1.6.3 for $n\leq 8$.

One can also consider the (rational) annular Khovanov homology version of Conjecture 1.6.1. With the help of the program implemented in [Hun+15], we are able to verify the cases $n \leq 4$ for the annular version. Due to the existence of a spectral sequence from annular Khovanov homology to the usual Khovanov homology, an affirmative answer to the annular version implies the version as stated. We remark that, however, the recursion in Conjecture 1.6.3 does not hold in the annular version.

1.6.2 Invariant rational Khovanov homology of some cables

Corollary 1.4.4 implies that $Kh(T(n,n);\mathbb{Q})$ can be recovered from $Kh(T(n',n');\mathbb{Q})^{S_{n'}}$ for various n'. It is thus reasonable to study the latter group. Instead of reformulating Conjecture 1.6.2 for the S_n-invariants, we propose the following alternative conjectures which imply both Conjecture 1.6.2 and Conjecture 1.6.3.

To start, as in the setup of Corollary 1.4.4, let $K \subset S^3$ be a framed oriented knot and let $K(n)$ denote its n-cable, or more precisely its $(n,0)$-cable, taken with respect to its framing. Let $K(n,-1)$ denote its $(n,-1)$-cable. Let $f \in \mathbb{Z}$ denote the framing coefficient of K. Paul Wedrich informed us of the following result, which follows from evaluating the computation in [GW23, Theorem 5.1] using a cabling functor coming from [GW23, Corollary 6.4] with $N = 2$.

Theorem 1.6.4 ([GW23, Theorem 5.1, Corollary 6.4]). *There is an exact triangle*

$$\begin{tikzcd} t^{2f(n-1)}q^{6f(n-1)-2}Kh(K(n-2);\mathbb{Q})^{S_{n-2}} \arrow[rr] & & Kh(K(n);\mathbb{Q})^{S_n} \arrow[dl] \\ & q^{n-1}Kh(K(n,-1);\mathbb{Q}) \arrow[ul, "{[1]}"] & \end{tikzcd} , \tag{1.21}$$

where the horizontal arrow is the dotted annulus creation map post-composed with symmetrization.

Note that (1.21) is independent of the orientation of K.

Remark 1.6.5. In the same spirit as Remark 1.4.5, Theorem 1.6.4 can also be stated for a link $K \cup L$ where cables are taken along the knot component K.

Conjecture 1.6.6. *(1) For any (unframed, unoriented) knot $K \subset S^3$, there exist integers $N_\pm(K)$ so that (1.21) splits if K is equipped with a framing f with $f \notin (N_-(K), N_+(K))$. More precisely, for $f \geq N_+(K)$, the map starting from the bottom row is zero; for $f \leq N_-(K)$, the horizontal map is zero.*

(2) For the unknot U, one can take $N_+(U) = 0$ and $N_-(U) = -1$ in (1).

Using (1.7), one can show that Conjecture 1.6.2 is equivalent to the claim in Conjecture 1.6.6 about the 1-framed unknot. Conjecture 1.6.3 is a consequence of Conjecture 1.6.6 for the (± 1)-framed unknots.

Conjecture 1.6.6 for 2-cables should be easy to establish. Using KnotJob [Sch], we have verified Conjecture 1.6.6 for various cables of the unknot, as well as some 3-cables of the trefoil knots.

Chapter 2

Skein lasagna modules and variants

Skein lasagna modules are smooth 4-manifold invariants introduced by Morrison–Walker–Wedrich [MWW22]. Roughly speaking, for a functorial link homology theory in S^3 satisfying some additional mild properties, the skein lasagna module construction associates a module for each oriented smooth 4-manifold, which can be thought of as an upgrade of the input link homology theory. In this chapter, we review the general construction in [MWW22] of skein lasagna modules and some of their formal properties, as well as present some natural variants of the construction, including skein lasagna modules with 1-*dimensional inputs*. We also compute these variants of skein lasagna modules for all 4-manifolds, for the universal trivial link homology theory. Then, we specialize to the case of Khovanov and Lee skein lasagna modules, the main tools of Chapter 3, and present some background results. See also [Wed25] for a recent survey.

2.1 Link homology theories

We abstract the allowed types of link homology theory for the purpose of constructing skein lasagna modules. See also [MWW22; MWW24].

Define the cobordism category of admissible links in $\mathbb{R}^3$, denoted $\mathbf{Links}_{\mathbb{R}^3}$, to be the category with objects given by (framed, oriented) admissible links in $\mathbb{R}^3$, and morphisms given by (framed, oriented) cobordisms in $I \times \mathbb{R}^3$ up to isotopy rel boundary. Here, admissibility can be any suitable condition imposed on the space of all links, so that each link type has an admissible representative; for instance, for the purpose of defining Khovanov–Rozansky $\mathfrak{gl}_N$ homologies, one can take links with a generic projection onto the xy-coordinate plane to be admissible. Endow $\mathbf{Links}_{\mathbb{R}^3}$ with a monoidal structure with the monoidal unit given by the empty link $\emptyset$ and the monoidal product given by disjoint unions of links, concretely realized by rescaling the y-coordinates and juxtaposition. The higher structural maps are also given by rescaling the y-coordinates. The admissibility condition is assumed to be closed under disjoint unions. From now on in this thesis, we always work with framed oriented links and cobordisms between them respecting these structures, where the framing means a normal 1-framing, although one can equally consider other settings.

A *link homology theory in* $\mathbb{R}^3$ is a lax monoidal functor

$$Z\colon \mathbf{Links}_{\mathbb{R}^3} \to \mathcal{D},$$

where $\mathcal{D} = (\mathcal{D}, 1_{\mathcal{D}}, \otimes)$ is a symmetric monoidal additive (R-)linear category, for some commutative ring R. Here, the lax monoidality condition means Z comes with a morphism $1_{\mathcal{D}} \to Z(\emptyset)$ and morphisms $Z(L_1) \otimes Z(L_2) \to Z(L_1 \sqcup L_2)$ that are natural in L_1, L_2, with some higher structural maps satisfying the natural unitality and associativity axioms. Note that the isomorphism type of $Z(L)$ for an admissible link L is invariant under isotoping L (not necessarily through admissible links), so one may talk about the "homology" (i.e. value under Z) of a link type rather than a concrete admissible link.

For us, $\mathcal{D}$ will usually be the category of (graded and/or filtered) R-modules. Our convention for ($\mathbb{Z}$-)filtered R-modules is that the zero element has $-\infty$ filtration degree, and we allow nonzero elements to have filtration degree $-\infty$ so that the category is cocomplete. Occasionally, we consider the case when $\mathcal{D} = \mathbf{K}^b(R)$ is the bounded homotopy category of R-modules. In many examples relevant to us, Z will be *strictly monoidal*, in the sense that the structural maps $1_{\mathcal{D}} \to Z(\emptyset)$, $Z(L_1) \otimes Z(L_2) \to Z(L_1 \sqcup L_2)$ are isomorphisms. This is the case, for example, for the Khovanov–Rozansky $\mathfrak{gl}_N$ chain complex or its deformations, or the Khovanov–Rozansky $\mathfrak{gl}_N$ homology over a field $R = \mathbb{F}$ or its deformations.

For the purpose of defining skein lasagna modules, we will need link homology theories with less strict ambient spaces. Let $\mathbf{Links}_0$ denote the category whose objects are pairs (S, L), where S is an oriented 3-manifold diffeomorphic to a finite disjoint union of 3-spheres and $L \subset S$ is a (framed, oriented) link, and whose morphisms from (S_0, L_0) to (S_1, L_1) are cobordisms (W, Σ) from (S_0, L_0) to (S_1, L_1), considered up to diffeomorphisms rel boundary of the pair, such that when capping off S_0 by 4-balls, W becomes a disjoint union of 4-balls. The identity morphism at $(S, L) \in \mathbf{Links}_0$ is $(I \times S, I \times L)$, and morphisms in $\mathbf{Links}_0$ compose as cobordisms. The category $\mathbf{Links}_0$ is monoidal under taking disjoint unions, with monoidal unit $(\emptyset, \emptyset)$.

Definition 2.1.1. A *lasagna* $(3, 1, 0)$*-link homology theory* is a lax monoidal functor $Z\colon \mathbf{Links}_0 \to \mathcal{D}$ for some monoidal category $\mathcal{D}$.

Although this will always be the case for us, we do not require $\mathcal{D}$ to be additive or linear in Definition 2.1.1. More importantly, we do not require Z to be strictly monoidal, in contrast to the situation in Proposition 2.1.2 below.

Morrison–Walker–Wedrich [MWW22, Section 4] carefully established the following "concrete-to-abstract upgrade" result.

Proposition 2.1.2. *If $Z\colon \mathbf{Links}_{\mathbb{R}^3} \to \mathcal{D}$ is a link homology theory in $\mathbb{R}^3$ that satisfies the following additional conditions, then Z can be upgraded to a strictly monoidal lasagna $(3,1,0)$-link homology theory $Z\colon \mathbf{Links}_0 \to \mathcal{D}$.*

(a) For any admissible link L, the isotopy from L to itself given by sweeping a strand on L around the 2-sphere at infinity induces the identity map on $Z(L)$.

(b) For any admissible link L, the isotopy from L to itself given by a 2π-rotation in $\mathbb{R}^3$ induces the identity map on $Z(L)$. □

As we will carry out a much more involved analogue of this "upgrade" in Chapter 4 when we discuss skein lasagna modules with 1-dimensional inputs, we only give an incomplete sketch proof of Proposition 2.1.2 here, following [MWW22].

Sketch proof. Call a link $L \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$ *admissible* if it is an admissible link in $\mathbb{R}^3$. Condition (a) is equivalent to saying that Z descends to a functor from the cobordism category of admissible links in S^3. For a link L in an oriented 3-manifold S diffeomorphic to S^3, an *L-admissible* parametrization ϕ of S is an orientation-preserving parametrization $\phi\colon S \xrightarrow{\cong} S^3$ so that $\phi(L)$ is admissible.

We "define" $Z(S,L)$ to be $Z(\phi(L))$. Since $\mathrm{Diff}^+(S^3)$ is connected, any two such parametrizations ϕ_0, ϕ_1 are connected by a smooth path of parametrizations ϕ_t, $t \in I$. One uses $\phi_t(L)$ to identify $Z(\phi_0(L))$ and $Z(\phi_1(L))$. The identification depends on the homotopy class of the path (ϕ_t), and there is a priori $\mathbb{Z}/2$ ambiguity coming from $\pi_1(\mathrm{Diff}^+(S^3)) = \mathbb{Z}/2$. But this ambiguity is killed by (b). Formally, if we let $I(S,L)$ denote the indiscrete category of L-admissible parametrizations of S, then $Z(\phi(L))$ is defined as the limit of the functor $I(S,L) \to \mathcal{D}$ sending ϕ to $Z(\phi(L))$.

For a link L in an oriented 3-manifold $S = \sqcup_{i=1}^m S_i$ where each S_i is diffeomorphic to S^3, we define $Z(S,L)$ to be $\otimes_{i=1}^m Z(S_i, L \cap S_i)$.

Finally, we assign a map $Z(W,\Sigma) = Z([(W,\Sigma)])$ functorially to a morphism represented by $(W,\Sigma)\colon (S_0,L_0) \to (S_1,L_1)$. Assume S_1 is diffeomorphic to a single 3-sphere; in general, take a tensor product over connected components of S_1 (equivalently, of W). Delete a small 4-ball B_0 in W away from Σ, pick a collection of arcs γ in $W \backslash int(B_0)$ away from Σ, one for each component of S_0, that connects ∂B_0 to components of S_0. Then, the outer boundary of a tubular neighborhood $\nu(B_0 \cup \gamma \cup S_0)$ of $B_0 \cup \gamma \cup S_0$, denoted $S_0^{\#}$, is diffeomorphic to a 3-sphere, whose intersection with Σ, denoted $L_0^{\sqcup}$, is the disjoint union of parts of L_0 on various components of S_0. The exterior of $\nu(B_0 \cup \gamma \cup S_0)$ in W is diffeomorphic to $I \times S^3$, in which Σ is a cobordism from $L_0^{\sqcup}$ to L_1. We define $Z(W,\Sigma)$ to be the composition

$$Z(S_0,L_0) = \bigotimes_{[S_{0,i}] \in \pi_0(S_0)} Z(S_{0,i}, L_0 \cap S_{0,i}) \to Z(S_0^{\#}, L_0^{\sqcup}) \to Z(S_1,L_1),$$

where the middle map is defined via the structural maps coming from lax monoidality, and the last map is defined by picking a parametrization of $W \backslash \nu(B_0 \cup \gamma \cup S_0) \cong I \times S^3$ and applying the functoriality of Z in concretely parametrized 3-spheres. One checks that this map is independent of the choices made, again using (a) and (b). One also checks this assignment is functorial. □

A *birth morphism* (unique up to isomorphism) in $\mathbf{Links}_0$ is a morphism of the form $(B, \emptyset)\colon (\emptyset, \emptyset) \to (S, \emptyset)$ where B is diffeomorphic to a 4-ball. A *pants morphism* in $\mathbf{Links}_0$ is a morphism of the form $(P, \Sigma)\colon (S_0 \sqcup S_1, L_0 \sqcup L_1) \to (S_0 \# S_1, L_0 \sqcup L_1)$ where S_0, S_1 each have a single component, $L_0 \subset S_0, L_1 \subset S_1$, such that there is an arc $\gamma \subset P$ connecting S_0 to S_1 disjoint from $L_0 \sqcup L_1$ so that (P, Σ) is a slight thickening of $(S_0 \cup \gamma \cup S_1, L_0 \sqcup L_1)$. We shall say a lasagna $(3, 1, 0)$-link homology theory Z is *multiplicative* if the birth morphism and any pants morphism induce an isomorphism on Z. This is not to be confused with the strict monoidality of Z under disjoint unions as in the conclusion of Proposition 2.1.2.

If Z is a lasagna $(3, 1, 0)$-link homology theory induced from a link homology theory Z in $\mathbb{R}^3$ verifying the conditions in Proposition 2.1.2, then $Z(B, \emptyset)$ for the birth morphism and $Z(P, \Sigma)$ for pants morphisms are defined via the structure maps for the lax monoidality of Z as a link homology theory in $\mathbb{R}^3$.

It follows from the construction in Proposition 2.1.2 that

Lemma 2.1.3. *In the setup of Proposition 2.1.2, Z (as a lasagna $(3, 1, 0)$-link homology theory) is multiplicative if and only if Z (as a link homology theory in $\mathbb{R}^3$) is strictly monoidal.* □

Remark 2.1.4. One may use an operad-theoretic way to redefine the notion of a strictly monoidal lasagna $(3, 1, 0)$-link homology theory; see [MWW22, Section 5.1].

2.2 Skein lasagna modules

In this section, we fix a lasagna $(3, 1, 0)$-link homology theory Z whose target category $\mathcal{D}$ is cocomplete[1], and define the *Z-input skein lasagna modules*. These are skein-theoretic invariants defined for every pair (X, L), where X is a smooth oriented 4-manifold, possibly noncompact and/or with boundary, and $L \subset \partial X$ is a (possibly empty) link on its boundary. The definition we present is partly motivated by a conversation with Sungkyung Kang.

Let (X, L) be a pair as above. For the purpose of defining skein lasagna modules with 0-dimensional inputs, or simply skein lasagna modules, a *skein in X rel L* is a pair (B, Σ), sometimes abbreviated as Σ, where $B \subset int(X)$ is a finite disjoint union of embedded 4-balls in X, and $\Sigma \subset X \backslash int(B)$ is a properly smoothly embedded framed compact oriented surface

[1]So, $\mathcal{D}$ is not allowed to be $\mathbf{K}^b(R)$ here.

with $\Sigma|_{\partial X} = L$. The components of B are called the *input balls* of the skein Σ, and the restriction of $-\partial\Sigma|_{\partial B}$ to various components of ∂B are called the *input links* of Σ. Here, the negative sign denotes orientation-reversal.

Definition 2.2.1. The *category of skeins* in X rel L is the poset category $\mathcal{C}(X;L) = \mathcal{C}_{0,\geq 0}(X;L)$ whose objects are skeins in X rel L, and whose morphisms are $(B,\Sigma) \to (B',\Sigma')$ for $B \subset B'$, $\Sigma' = \Sigma \backslash int(B')$.

A lasagna $(3,1,0)$-link homology theory Z induces a functor, still denoted

$$Z\colon \mathcal{C}(X;L) \to \mathcal{D},$$

by

$$Z(B,\Sigma) = Z(\Sigma) := Z(\partial B, -\partial\Sigma|_{\partial B}) \tag{2.1}$$

on objects and

$$Z((B,\Sigma) \to (B',\Sigma')) := Z(B' \backslash int(B), \Sigma \cap B')$$

on morphisms[2].

Definition 2.2.2. The *Z-input skein lasagna module* of (X,L) is

$$\mathcal{S}_0^Z(X;L) := \mathrm{colim}(Z\colon \mathcal{C}(X;L) \to \mathcal{D}). \tag{2.2}$$

The *Z-input skein lasagna module* of X is $\mathcal{S}_0^Z(X) := \mathcal{S}_0^Z(X;\emptyset)$.

More explicitly, when $\mathcal{D}$ is the category of R-modules (perhaps graded or filtered), which is the only case we will be considering in the rest of this thesis when it comes to skein lasagna modules (as opposed to lasagna link homology theories), we have

$$\mathcal{S}_0^Z(X;L) = R\{(\Sigma,v)\colon \Sigma \text{ is a skein in } X \text{ rel } L,\ v \in Z(\Sigma)\}/\sim$$

where $\sim$ is the equivalence relation generated by R-linearity in v and $(\Sigma,v) \sim (\Sigma', Z(e)(v))$ for every morphism $e\colon \Sigma \to \Sigma'$ in $\mathcal{C}(X;L)$. Here $R\bullet$ denotes the R-linear span of a set $\bullet$. The pair (Σ,v) is called a *(Z-input) lasagna filling* of X rel L, and v is called the *decoration* on the input links of Σ. If the lasagna $(3,1,0)$-link homology theory Z is strictly monoidal, v can be decomposed into a linear combination of tensor products of decorations on the individual input links of Σ.

It is instructive to see why our definition implies that if (B,Σ) and (B',Σ') are isotopic skeins, via an isotopy that carries a decoration v to a decoration v', then the lasagna fillings (Σ,v) and (Σ',v') define the same class in the skein lasagna module, a feature apparent in the more classical definitions of skein modules.

[2]To avoid B touching B' on its boundary, one should shrink B slightly in order to define the assignment on morphisms.

Remark 2.2.3. One can consider various levels of "strictness" for defining the category of skeins $\mathcal{C}(X;L)$. Our definition is the strictest possible, while the definition presented in [RW24, Section 2.1] is more lax. The choice of strictness does not affect the resulting skein lasagna modules, as can be seen from a compactness argument. The definition presented here admits a cleaner comparison to the 1-input-ball version of skein lasagna modules presented in Section 2.4.1.

2.3 Properties of skein lasagna modules

We present some properties of the skein lasagna modules, partly following Manolescu–Neithalath [MN22]. Although their paper dealt with the case where the link homology theory Z is the Khovanov–Rozansky $\mathfrak{gl}_N$ homology, most proofs remain valid in a more general setup. For simplicity, in this section, we assume the target category of Z is the category of R-modules (possibly graded and/or filtered), so we can talk about lasagna fillings. Many results we discuss should hold in a more general setting.

2.3.1 Gluing formulas

We mention some results for special types of gluings before presenting the general construction.

Proposition 2.3.1 ($3,4$**-handle attachments, [MN22, Proposition 2.1]).** *If (X',L) is obtained from (X,L) by attaching a 3-handle (resp. 4-handle) away from L, then there is a natural surjection (resp. isomorphism) $\mathcal{S}_0^Z(X;L)\to\mathcal{S}_0^Z(X';L)$.* □

Proposition 2.3.2 (Sum formulas, [MN22, Theorem 1.4, Corollary 7.3]). *Let (X_1,L_1) and (X_2,L_2) be two pairs, $(X_1*X_2,L_1\sqcup L_2)$ be their boundary connected sum or connected sum, where $*\in\{\natural,\#\}$ and where the operation is performed along any chosen components or boundary components of X_1 and X_2 away from L_1 and L_2. There are natural maps*

$$\mathcal{S}_0^Z(X_1;L_1)\otimes\mathcal{S}_0^Z(X_2;L_2)\to\mathcal{S}_0^Z(X_1\sqcup X_2;L_1\sqcup L_2)\to\mathcal{S}_0^Z(X_1*X_2;L_1\sqcup L_2),$$

where the first map is an isomorphism if Z is strictly monoidal, and the second map is an isomorphism if Z is multiplicative.

Proof. The second map is defined by a gluing construction explained later in the section. The claim on the isomorphism follows from [MN22, Theorem 1.4, Corollary 7.3].

If (Σ_i,v_i) are lasagna fillings of X_i rel L_i, $i=1,2$, then $(\Sigma_1\sqcup\Sigma_2,\rho(v_1\otimes v_2))$ is a lasagna filling of $X_1\sqcup X_2$ rel $L_1\sqcup L_2$, where ρ denotes a structure map for the lax monoidality of Z. The first map in the statement is defined by sending $[(\Sigma_1,v_1)]\otimes[(\Sigma_2,v_2)]$ to $[(\Sigma_1\sqcup\Sigma_2,\rho(v_1\otimes v_2))]$. When Z is strictly monoidal, the functor $\mathcal{C}(X_1\sqcup X_2;L_1\sqcup L_2)\to\mathcal{D}$ whose colimit defines

$\mathcal{S}_0^Z(X_1 \sqcup X_2; L_1 \sqcup L_2)$ is a tensor product of the defining functors for $\mathcal{S}_0^Z(X_i; L_i)$, $i = 1, 2$, so the map is an isomorphism. $\square$

Remark 2.3.3. More generally, if $\mathcal{D}$ is a closed monoidal category, the first map in Proposition 2.3.2 exists and is an isomorphism when Z is strictly monoidal.

We now explain the setup for general gluing. Suppose (X, L) is a pair and $Y \subset X$ is a properly embedded oriented 3-manifold, with (possibly empty) transverse intersection with L in some point set $P \subset \partial Y$ with framing and orientation. Then Y cuts (X, L) into a pair (X', T'), where $\partial X' = (\partial X \backslash\backslash \partial Y) \cup (Y \sqcup -Y)$, and $T' \subset \partial X \backslash\backslash \partial Y$ is a framed oriented tangle with $\partial T' = P \sqcup -P$.

If $T_0 \subset Y$ is a framed oriented tangle with $\partial T_0 = P$, then $T' \cup (T_0 \sqcup -T_0)$ is a framed oriented link in $\partial X'$. A lasagna filling of $(X', T' \cup (T_0 \sqcup -T_0))$ glues along Y to a lasagna filling of (X, L) linearly, inducing a map

$$\mathcal{S}_0^Z(X'; T' \cup (T_0 \sqcup -T_0)) \to \mathcal{S}_0^Z(X; L). \tag{2.3}$$

Proposition 2.3.4. *The gluing map*

$$\bigoplus_{\substack{T_0 \subset Y \\ \partial T_0 = P}} \mathcal{S}_0^Z(X'; T' \cup (T_0 \sqcup -T_0)) \to \mathcal{S}_0^Z(X; L)$$

is a surjection.

Proof. Every lasagna filling of (X, L) can be isotoped so that the input balls miss Y and the skein intersects Y transversely in some tangle T_0. $\square$

More generally, it is possible to write down a gluing formula to recover $\mathcal{S}_0^Z(X; L)$ from $\mathcal{S}_0^Z(X'; T' \cup (\bullet \sqcup -\bullet))$ in terms of a coend. Since such a formula does not give too much insight into explicit computations, we refrain from spelling it out.

We consider some specializations of gluings. When Y is separating, we may write $(X', T') = (X_1, T_1) \sqcup (X_2, T_2)$ where the boundary of X_1 (resp. X_2) contains Y (resp. $-Y$). Write $L_1 = T_1 \cup T_0$ and $L_2 = -T_0 \cup T_2$. A framed surface $S \subset X_2$ with $\partial S = L_2$ is a lasagna filling of $(X_2; L_2)$ (with empty input balls and input links, and decoration given by the image of 1 under the monoidal structure map $R \to Z(\emptyset)$), and plugging in its class in the second tensorial summand of

$$\mathcal{S}_0^Z(X_1; L_1) \otimes \mathcal{S}_0^Z(X_2; L_2) \to \mathcal{S}_0^Z(X'; L_1 \sqcup L_2) \to \mathcal{S}_0^Z(X; L) \tag{2.4}$$

defines a gluing map

$$\mathcal{S}_0^Z(X_2; S) \colon \mathcal{S}_0^Z(X_1; L_1) \to \mathcal{S}_0^Z(X; L). \tag{2.5}$$

Here, the first map in (2.4) is defined by the monoidal structural maps of Z as explained in the proof of Proposition 2.3.2.

We specialize (2.5) further by assuming that Y is closed, $T_0 = T_2 = \emptyset$, and $S = \emptyset$. Thus, $X = X_1 \cup_Y X_2$ is obtained from X_1 by gluing X_2 to it along some boundary components of X_1 disjoint from $L_1 = L$. In this case, the inclusion $i\colon X_1 \hookrightarrow X$ induces a map

$$\mathcal{S}_0^Z(i)\colon \mathcal{S}_0^Z(X_1; L) \to \mathcal{S}_0^Z(X; L). \tag{2.6}$$

One type of inclusion map not exactly included in (2.6) is the following.

Proposition 2.3.5. *Let (X, L) be a pair and X' be X with a union of some boundary components or collar neighborhoods of boundary components not touching L removed. Then the inclusion map $X' \subset X$ induces an isomorphism $\mathcal{S}_0^Z(X'; L) \cong \mathcal{S}_0^Z(X; L)$.*

Proof. All skeins in X rel L and morphisms between them can be isotoped to be contained in X'. □

Corollary 2.3.6. *The skein lasagna module $\mathcal{S}_0^Z(X)$ is an invariant of int(X), the interior of X.* □

We discuss the natural maps in Proposition 2.3.1 and Proposition 2.3.2.

The map in Proposition 2.3.1 is defined using (2.6). For Proposition 2.3.2, the first map is defined using the lax monoidality of Z; the second map for $* = \natural$ is a special case of the second map in (2.4) with $Y = B^3$, and the second map for $* = \#$ is defined by first deleting two 4-balls using the inverse of the map in Proposition 2.3.1, then using the second map in (2.4) with $Y = S^3$.

2.3.2 4D neck-cutting

The following 4-dimensional neck-cutting operation is often useful. For example, it is used in the proof of Proposition 2.3.2 concerning the second map being an isomorphism.

Lemma 2.3.7 (4-dimensional neck-cutting [MN22, Lemma 7.2]). *Suppose Z is multiplicative. Let (Σ, v) be a Z-input lasagna filling of X rel L. Suppose $B_0 \subset$ int(X) is a 4-ball not meeting the input balls of Σ, inside which Σ is a neck: it is diffeomorphic to $[-1,1] \times K \subset [-1,1] \times B^3 = B_0$ for some framed oriented link $K \subset B^3$ under some identification $[-1,1] \times B^3 = B_0$. Then (Σ, v) is equivalent to a lasagna filling (Σ', v') where $\Sigma = \Sigma'$ outside B_0 and $\Sigma' = ([-1,-3/4] \sqcup [3/4,1]) \times K$ with two extra input balls $([-3/4,-1/2] \sqcup [1/2,3/4]) \times (1-\epsilon)B^3$ inside B_0, for some small $\epsilon > 0$.*

Proof. Let $B_1 := [-0.9, 0.9] \times (1-\epsilon/2)B^3 \subset B_0$ and $B_2 := ([-3/4, -1/2] \sqcup [1/2, 3/4]) \times (1-\epsilon)B^3 \subset B_0$, for some small $\epsilon > 0$ so that $K \subset (1-2\epsilon)B^3$. Let B denote the input balls of Σ. Then there are morphisms $e\colon (B, \Sigma) \to (B \sqcup B_1, \Sigma \backslash int(B_1))$ and $f\colon (B \sqcup B_2, \Sigma') := (B \sqcup B_2, \Sigma \backslash (-3/4, 3/4) \times (1-\epsilon)B^3) \to (B \sqcup B_1, \Sigma \backslash int(B_1))$ in $\mathcal{C}(X; L)$. Since f is induced by a pants morphism, $Z(f)$ is an isomorphism by the assumption on Z. Consequently (Σ, v) is equivalent to $(\Sigma', Z(f)^{-1}Z(e)(v))$, as desired. □

2.3.3 Recover the link homology theory

The skein lasagna modules $\mathcal{S}_0^Z(\bullet; \bullet)$ upgrade the lasagna $(3,1,0)$-link homology theory Z in the following sense.

Proposition 2.3.8. *(1) Let (S, L) be an object in* $\mathbf{Links}_0$*, and B be a disjoint union of 4-balls with $\partial B = S$. There is a natural isomorphism $\mathcal{S}_0^Z(B; L) \cong Z(S, L)$.*

(2) Let $[(W, \Sigma)]\colon (S_0, L_0) \to (S_1, L_1)$ be a morphism in $\mathbf{Links}_0$*, and B_0 be a disjoint union of 4-balls with $\partial B_0 = S_0$. Let $B_1 = B_0 \cup_{S_0} W$, a disjoint union of 4-balls with $\partial B_1 = S_1$. Under the natural isomorphisms in (1), the map $Z(W, \Sigma)$ is given by the gluing map $\mathcal{S}_0^Z(W; \Sigma)\colon \mathcal{S}_0^Z(B_0; L_0) \to \mathcal{S}_0^Z(B_1; L_1)$ as defined in (2.5).*

Proof. The statement on objects is proved in [MWW22, Example 5.6] and the statement on morphisms is a simple exercise. □

2.3.4 Gradings

Every object Σ of $\mathcal{C}(X; L)$ represents a homology class in $H_2(X, L)$, whose image under the boundary homomorphism ∂ in the long exact sequence

$$0 \to H_2(X) \to H_2(X, L) \xrightarrow{\partial} H_1(L) \to H_1(X) \to \cdots$$

is $[L]$, the fundamental class of L. If $[S]\colon \Sigma \to \Sigma'$ is a morphism, then $[\Sigma] = [\Sigma'] \in H_2(X, L)$. This means $\mathcal{S}_0^Z(X; L)$ admits a grading by $H_2^L(X) := \partial^{-1}([L])$, which we write as

$$\mathcal{S}_0^Z(X; L) = \bigoplus_{\alpha \in H_2^L(X)} \mathcal{S}_0^Z(X; L; \alpha).$$

We call the grading of $\mathcal{S}_0^Z(X; L)$ by $H_2^L(X)$ the *skein grading*, with more justification given in Section 2.3.5. Of course, $H_2^L(X)$ is nonempty if and only if the image of $[L]$ in $H_1(X)$ is zero, i.e., L is null-homologous in X; in particular, $\mathcal{S}_0^Z(X; L)$ is zero if L is not null-homologous.

Extra grading/filtration structures on the link homology theory Z usually induce grading/filtration structures on the skein lasagna module $\mathcal{S}_0^Z$. To make this precise, let's say the target category $\mathcal{D}$ of Z is the category of A-graded/filtered R-modules for some abelian group

A, where the morphisms are all R-linear maps, not necessarily grading-preserving. We note that the category $\mathbf{Links}_0$ is a $\mathbb{Z}$-graded category, with the grading given on the morphisms by $|[(W,\Sigma)]| := \chi(\Sigma)$. If the lasagna $(3,1,0)$-link homology theory $Z\colon \mathbf{Links}_0 \to \mathcal{D}$ is a graded/filtered functor with respect to a homomorphism $\mathbb{Z} \to A$ sending 1 to $a \in A$, in the sense that for every morphism $[(W,\Sigma)]$, $Z(W,\Sigma)$ is a graded/filtered R-linear map of degree $|[(W,\Sigma)]|a$, then the skein lasagna module $\mathcal{S}_0^Z(X;L)$ is an A-graded/filtered R-module, with $Z(\Sigma)$ in (2.1) shifted by $\chi(\Sigma)a$.

The results discussed in Section 2.3.1 and Section 2.3.3 respect the extra grading/filtration structures. With the exception of (2.5), all natural maps that appeared are grading/filtration-preserving, and isomorphisms are as graded/filtered R-modules, while (2.5) shifts the grading/filtration by $\chi(\Sigma)a$ where a is as in the previous paragraph. The skein grading is also respected in the natural sense; for instance, the self-gluing map (2.3) respects the gluing map $H_2^{T'\cup(T_0\sqcup -T_0)}(X') \to H_2^L(X)$ on the relative second homology.

2.3.5 The bordism set of skeins

The skein lasagna module $\mathcal{S}_0^Z(X;L)$ decomposes as a direct sum over $\Omega_{0,\geq 0}(X;L) := \pi_0(\mathcal{C}(X;L))$, the set of connected components of the category $\mathcal{C}(X;L) = \mathcal{C}_{0,\geq 0}(X;L)$. We call $\Omega_{0,\geq 0}(X;L)$ the *bordism set of skeins* in X rel L. In this section, we show that this decomposition is the same as the skein grading by $H_2^L(X)$ defined in Section 2.3.4.

Theorem 2.3.9. *Taking homology defines a bijection*

$$\Omega_{0,\geq 0}(X;L) \xrightarrow{\cong} H_2^L(X). \tag{2.7}$$

By definition, $\Omega_{0,\geq 0}(X;L)$ is the set of (properly embedded, compact, oriented) framed surfaces Σ in X outside finitely many open balls with $\partial\Sigma|_{\partial X} = L$ (such Σ is called a skein in X rel L) up to

(a) enclosing some small balls by some large balls and deleting the interior of the large balls, or its inverse.

Although

(b) isotopies of Σ rel boundary;

(c) isotopies of input balls in X (that drag Σ along)

are not directly allowed in the imposed relations along morphisms, they can be recovered from (a), as can be seen from a compactness argument.

In the context of studying skeins in X rel L, it is natural to consider the following cobordism relation. We say two skeins Σ_0, Σ_1 in X rel L with input balls $\sqcup_i int(B_i^{(0)}), \sqcup_i int(B_i^{(1)})$ are *framed cobordant rel L* if there is a properly embedded framed compact oriented 3-manifold $Y \subset (I \times X)\backslash\nu(\Gamma)$ for some tubular neighborhood $\nu(\Gamma)$ of some 1-complex $\Gamma \subset I \times int(X)$, such that $\partial Y|_{\{k\}\times X} = \{k\} \times \Sigma_k$, $k = 0, 1$, and $\partial Y|_{I\times\partial X} = I \times L$. It is understood that $\nu(\Gamma) \cap (\{k\} \times X) = \sqcup_i \{k\} \times int(B_i^{(k)})$, $k = 0, 1$, and that Y can have boundary on $\partial\nu(\Gamma)$.

Lemma 2.3.10. *Two skeins are equal in $\Omega_{0,\geq 0}(X; L)$ if and only if they are framed cobordant.*

Proof. One direction is clear: (a) is realized by taking the identity cobordism but gradually enlarging the small deleted balls (with creations of new balls if necessary) so that they merge with each other and become the large balls.

For the converse, after an isotopy we can decompose every framed cobordism of skeins in X rel L into the following elementary moves:

(i) Proper isotopy of the surface and deleted balls rel ∂X;

(ii) Morse move away from deleted balls;

(iii) Pushing a Morse critical point into one deleted ball;

(iv) Deleting a local ball away from the surface, or its inverse;

(v) Choosing a path γ between two deleted balls B_1, B_2 disjoint from the surface and replacing B_1, B_2 by a small tubular neighborhood of $B_1 \cup B_2 \cup \gamma$, or its inverse.

That these moves are sufficient can be proved by using (i)(iv)(v) to account for the topology change of the 1-complex Γ as one moves along the time direction (the I-direction), and (i)(ii)(iii) to account for the topology change of the cobordism 3-manifold, assuming Γ is constant in time.

We see immediately that (i) is realized by (b) and (c), while (iii),(iv), and (v) are realized by (a). Since a Morse move is local, we can use (a) twice to delete a local ball and reglue it back to realize (ii). $\square$

Proof of Theorem 2.3.9. First we show the map is surjective. Any class $\alpha \in H_2^L(X)$ can be represented by an oriented embedded surface $\Sigma \subset X$ with $\partial\Sigma = L$. The obstruction to putting a framing on Σ lies in $H^2(\Sigma, L)$. Once we delete local 4-balls at points on each component of Σ, this obstruction vanishes and we get a skein Σ with $[\Sigma] = \alpha$.

Next we show the map is injective. Without loss of generality, assume X is connected. Suppose two skeins Σ_0, Σ_1 have the same homology class. By enclosing all input balls by one large ball, we may assume $\Sigma_0, \Sigma_1 \in X^\circ := X\backslash int(B^4)$ for some fixed ball B^4 in X. Since $[\Sigma_1] - [\Sigma_0] = 0 \in H_2(I \times X^\circ, I \times \partial X)$, we can find a properly immersed compact oriented 3-manifold $Y \looparrowright I \times X^\circ$ that cobounds Σ_0, Σ_1. By general position we may assume that the self-intersection locus of Y is an embedded 1-manifold in $I \times X^\circ$ with endpoints on $\partial(I \times X^\circ)$ and interior singularities of the immersion modeled on the Whitney umbrella [Whi44]. Deleting a tubular neighborhood along the self intersection makes Y an embedded cobordism between Σ_0 and Σ_1 in $(I \times X)\backslash\nu(\Gamma)$ for some 1-complex Γ. The obstruction to putting a compatible framing on Y lies in $H^2(Y, \partial Y|_{(\partial I \times X^\circ)\cup(I\times\partial X)}) \cong H_1(Y, \partial Y|_{\partial\nu(\Gamma)})$, which can be made zero upon a further deletion of the tubular neighborhood of a 1-complex. This yields a framed cobordism between Σ_0, Σ_1, showing that they are equal in $\Omega_{0,\geq 0}(X; L)$ by Lemma 2.3.10. ☐

Remark 2.3.11. An alternative proof of Theorem 2.3.9 can be deduced from a relative and noncompact version of [KMT12, Theorem 2] applied to (X°, L) together with the fact that a framed link translation (as defined in their paper) can be undone in $\Omega_{0,\geq 0}(X; L)$ by a neck-cutting and regluing (Lemma 2.3.7). However, for our purpose in Section 3.3, we need a similar result for the space of a pair of skeins, which can be proved in the same way as we demonstrated, but is inaccessible by directly applying [KMT12, Theorem 2].

2.4 Variants of skein lasagna modules

2.4.1 1 input ball version

In this section, we assume that X is connected.

In the definition of skeins, instead of allowing arbitrarily many input balls in X, we may instead require that there is exactly one input ball, yielding an alternative definition of skein lasagna modules. More formally, consider the full subcategory $\mathbf{Links}_{0,1}$ of $\mathbf{Links}_0$ consisting of (S, L) where S is diffeomorphic to S^3. The corresponding category of skeins in X rel L, denoted $\mathcal{C}_{0,1}(X; L)$, is the full subcategory of $\mathcal{C}_{0,\geq 0}(X; L)$ consisting of skeins with exactly one input ball. The restriction of a lasagna $(3, 1, 0)$-link homology theory Z with a cocomplete target to $\mathbf{Links}_{0,1}$ induces a functor

$$Z\colon \mathcal{C}_{0,1}(X; L) \to \mathcal{D}. \tag{2.8}$$

Equivalently, (2.8) is the restriction of $Z\colon \mathcal{C}_{0,\geq 0}(X; L) \to \mathcal{D}$ induced by Z.

Definition 2.4.1. The 1*-ball Z-input skein lasagna module* of $(X; L)$ is

$$\mathcal{S}_0^{Z,\text{1-ball}}(X; L) := \operatorname{colim}(Z\colon \mathcal{C}_{0,1}(X; L) \to \mathcal{D}).$$

The inclusion $\mathcal{C}_{0,1}(X; L) \subset \mathcal{C}_{0,\geq 0}(X; L)$ induces a forgetful map

$$\mathcal{S}_0^{Z,\text{1-ball}}(X; L) \to \mathcal{S}_0^Z(X; L). \tag{2.9}$$

Proposition 2.4.2. *The forgetful map (2.9) is surjective. It is an isomorphism when X is simply-connected.*

Proof. The map (2.9) is surjective because every object in $\mathcal{C}_{0,\geq 0}(X;L)$ has a successor in $\mathcal{C}_{0,1}(X;L)$.

When X is simply-connected, we show that $\mathcal{C}_{0,1}(X;L)$ is cofinal in $\mathcal{C}_{0,\geq 0}(X;L)$ in the sense of [The25, Tag 04E6], which implies (2.9) is an isomorphism. Let $x \leftarrow y \rightarrow x'$ be a diagram in $\mathcal{C}_{0,\geq 0}$ where $x, x' \in \mathcal{C}_{0,1}$. We need to show that there is a zigzag in $\mathcal{C}_{0,1}$ connecting x and x' under y. We shrink the input balls of x and x' to assume that they are each a tubular neighborhood of a tree connecting the input balls of y. Thus, the union of the input balls of x and x' is a tubular neighborhood of a graph connecting the input balls of y, which is by assumption contained in a larger ball in X. Thus we may assume $x = x'$ and the statement is plain. $\square$

However, when $H_1(X) \neq 0$, (2.9) is usually not an injection. In fact, a relative and non-compact version of [KMT12, Theorem 2] or a relative version of [Tay12, Theorem 6.6] shows that the relevant bordism set of skeins $\Omega_{0,1}(X;L) := \pi_0(\mathcal{C}_{0,1}(X;L))$ admits a surjection onto $H_2^L(X) \cong \Omega_{0,\geq 0}(X;L)$ by taking homology, whose fiber over $\alpha \in H_2^L(X)$ is a torsor over the cokernel of the map $\bullet \cap 2\alpha \colon H_3(X) \to H_1(X)$, which is in general nontrivial. Thus, $\mathcal{S}_0^{Z,\text{1-ball}}$ is often a strict refinement of $\mathcal{S}_0^Z$.

Example 2.4.3. Take $Z = KhR_2$, the Khovanov–Rozansky $\mathfrak{gl}_2$ homology, in which case we abbreviate $\mathcal{S}_0^{KhR_2}$ as $\mathcal{S}_0^2$. Let $K \subset S^3$ be an unframed, oriented knot. By neck-cutting (Lemma 2.3.7), unwinding and regluing, we see that the lasagna filling $(K \times S^1, 1)$ of $S^3 \times S^1$ (with any framing on $K \times S^1$) is equivalent to some lasagna filling $(T, 1)$ where T is a local torus in $S^3 \times S^1$. Since every torus in B^4 evaluates to 2 in KhR_2 (see [Ras05] for a proof in the Kh case, up to sign), $(T, 1)$, and hence $(K \times S^1, 1)$, is equivalent to $(\emptyset, 2)$. However, the neck-cutting relation no longer holds for $\mathcal{S}_0^{2,\text{1-ball}}$, so $(K \times S^1, 1)$ cannot be simplified there in general. In fact, if the framing on $K \times S^1$ has twists along the K factor, then $K \times S^1$ and $\emptyset$ do not represent the same element in $\Omega_{0,1}(S^3 \times S^1)$, so $(K \times S^1, 1)$ is not equivalent to $(\emptyset, 2)$ in $\mathcal{S}_0^{2,\text{1-ball}}(S^3 \times S^1)$, as the latter represents a class that is nonzero in the quotient $\mathcal{S}_0^2(S^3 \times S^1)$ [MWW23, Corollary 4.2].

As noted in Example 2.4.3, in the 1-ball theory, we cannot perform a neck-cutting to separate a surface locally of the form $(-1,1) \times L$ for some link $L \subset B^3$. Since the proof of the connected and boundary sum formulas (Proposition 2.3.2) requires a neck-cutting, we also lose them for $\mathcal{S}_0^{Z,\text{1-ball}}$. Thus, the computation of the refinement $\mathcal{S}_0^{Z,\text{1-ball}}$ could be more challenging than $\mathcal{S}_0^Z$.

2.4.2 1-dimensional input versions

Embedded 4-balls in a 4-manifold X are essentially 0-dimensional; they are tubular neighborhoods of points in X. Deleting 4-balls from X does not destroy the smooth topology of X, as there is a unique way to refill the balls since $\mathrm{Diff}^+(S^3)$ is connected [Cer68]. Hence, allowing skeins to live in the complement of balls in X does not lose information about the diffeomorphism type of X, giving some justification of the definition of skein lasagna modules.

In this respect, it is natural to allow the "input balls" of skeins to take up more space in X, namely allowing them to be tubular neighborhoods of embedded finite graphs (or, 1-dimensional CW complexes) in X: One can show $\mathrm{Diff}^+(\natural^n(S^1\times B^3)) \xrightarrow{\partial} \mathrm{Diff}^+(\#^n(S^1\times S^2))$ is surjective on π_0, hence there is a unique way to refill these neighborhoods in. This leads to a version of skein lasagna modules with 1-*dimensional inputs*, whose definition is justified in that the construction does not destroy the topology of the 4-manifold in an obvious way. The rest of this section is bookkeeping mimicking the discussions for the usual skein lasagna modules. This discussion remains vacuous until we construct an interesting example of a lasagna $(3,1,1)$-link homology theory in Chapter 4.

A 4-*dimensional* 1-*handlebody* is a compact oriented 4-manifold admitting a handle decomposition with only $0,1$-handles. Equivalently, it is an oriented manifold diffeomorphic to $\sqcup_{i=1}^k \natural^{m_i}(S^1\times B^3)$ for some $k\ge 0$, $m_1,\cdots,m_k\ge 0$. A 4-*dimensional relative* 1-*handlebody complement* W is an oriented 4-dimensional cobordism, i.e. an oriented 4-manifold whose boundary comes with a partition $\partial W=(-\partial_-W)\sqcup\partial_+W$, such that $W\cong_\varphi X_1\backslash int(X_0)$ for some 4-dimensional 1-handlebodies X_0,X_1 with $X_0\subset int(X_1)$, so that the diffeomorphism φ (usually dropped from the notation) restricts to $\partial_-W\cong\partial X_0$, $\partial_+W\cong\partial X_1$.

If W is a 4-dimensional relative 1-handlebody complement, then X_0,X_1 are uniquely determined by W up to diffeomorphisms. If W_0,W_1 are 4-dimensional relative 1-handlebody complements and $\phi\colon \partial_+W_0\to\partial_-W_1$ is an orientation-preserving diffeomorphism, then $W_0\cup_\phi W_1$ is also a 4-dimensional relative 1-handlebody complement; moreover, $W_i\cong X_{i+1}\backslash int(X_i)$, $i=0,1$, and $W_0\cup_\phi W_1\cong X_2\backslash int(X_0)$, for some 4-dimensional 1-handlebodies $X_0\subset X_1\subset X_2$.

Let $\mathbf{Links}_1$ denote the category whose objects are pairs (S,L), where S is an oriented 3-manifold diffeomorphic to $\sqcup_{i=1}^k\#^{m_i}(S^1\times S^2)$ for some $k\ge 0$, $m_1,\cdots,m_k\ge 0$, and $L\subset S$ is a framed oriented link, and whose morphisms from (S_0,L_0) to (S_1,L_1) are cobordisms (W,Σ) from (S_0,L_0) to (S_1,L_1), considered up to diffeomorphisms rel boundary of the pair, such that W is a relative 1-handlebody complement from S_0 to S_1 (as a cobordism). As in the case of $\mathbf{Links}_0$, identity morphisms and compositions in $\mathbf{Links}_1$ are naturally defined, and $\mathbf{Links}_1$ is monoidal under taking disjoint unions.

A *lasagna* $(3,1,1)$-*link homology theory* is a lax monoidal functor $Z\colon \mathbf{Links}_1\to\mathcal{D}$ for some monoidal category $\mathcal{D}$.

A *birth morphism* in $\mathbf{Links}_1$ is a birth morphism in $\mathbf{Links}_0$[3], and a *pants morphism* in $\mathbf{Links}_1$ is defined verbatim as in the case of $\mathbf{Links}_0$ (although it does not have to be a morphism in $\mathbf{Links}_0$). A lasagna $(3,1,1)$-link homology theory Z is *multiplicative* if the birth morphism and any pants morphism induce an isomorphism on Z.

Let (X, L) be a pair as before, namely X is an oriented 4-manifold, and $L \subset \partial X$ is a framed oriented link. A *skein in X rel L with 1-dimensional inputs* is a pair (B, Σ), sometimes abbreviated as Σ, where $B \subset int(X)$ is a 4-dimensional 1-handlebody, and $\Sigma \subset X \backslash int(B)$ is a properly smoothly embedded framed compact oriented surface with $\Sigma|_{\partial X} = L$. The manifold B is called the *input manifold* of the skein Σ, and the link $-\partial\Sigma|_{\partial B}$ in ∂B is called the *input link* of Σ.

The *category of skeins in X rel L with 1-dimensional inputs* is the poset category $\mathcal{C}_{1,\geq 0}(X; L)$ whose objects are skeins in X rel L with 1-dimensional inputs, and whose morphisms are containments as in Definition 2.2.1.

A lasagna $(3,1,1)$-link homology theory $Z\colon \mathbf{Links}_1 \to \mathcal{D}$ induces a functor $Z\colon \mathcal{C}_{1,\geq 0}(X; L) \to \mathcal{D}$ as before, and when $\mathcal{D}$ is cocomplete, the *Z-input skein lasagna module of* (X, L) is

$$\mathcal{S}_0^Z(X; L) := \operatorname{colim}(Z\colon \mathcal{C}_{1,\geq 0}(X; L) \to \mathcal{D}).$$

Our current discussion is vacuous in that we have not given an example of a lasagna $(3,1,1)$-link homology theory. A trivial example will be given in Section 2.5. Constructing an interesting such example is the goal of Chapter 4. For now, we remark on some properties and variants of the 1-dimensional-input theory $\mathcal{S}_0^Z(X; L)$.

2.4.2.1 Properties

The gluing map (2.3) and its specializations are defined in exactly the same way for the 1-dimensional-input theory $\mathcal{S}_0^Z$, although Proposition 2.3.4 may fail to hold. Most other properties carry through.

Proposition 2.4.4. *The statements on $3,4$-handle attachments (Proposition 2.3.1), sum formulas (Proposition 2.3.2), and recovery of the lasagna link homology theory (Proposition 2.3.8) all hold for the 1-dimensional-input skein lasagna module theory $\mathcal{S}_0^Z$. Here, for the last statement on recovering Z, one should replace the appearance of 4-balls in Proposition 2.3.8 with 4-dimensional 1-handlebodies.*

Proof. These are proved in the same way as the 0-dimensional-input theory. We only remark that to construct an inverse of the map $\mathcal{S}_0^Z(X_1 \sqcup X_2; L_1 \sqcup L_2) \to \mathcal{S}_0^Z(X_1 \natural X_2; L_1 \sqcup L_2)$ when

[3] It is also natural to consider the variation where one regards all $(B, \emptyset)\colon (\emptyset, \emptyset) \to (S, \emptyset)$ with B having a single component to be birth morphisms. On the level of multiplicative lasagna $(3,1,1)$-link homology theories, this amounts to requiring that the empty link in $S^1 \times S^2$ evaluates to the monoidal unit.

Z is multiplicative, unlike in [MN22, Proof of Theorem 1.4], one may not be able to isotope the input manifold of a lasagna filling to miss the gluing region. Instead, one directly deletes a large ball around the gluing region, enclosing all of the skein (with the input manifolds) within this region, and uses the multiplicativity assumption on Z to cut the neck. □

The category $\mathcal{C}_{1,\geq 0}(X;L)$ is connected; hence, unlike the 0-dimensional-input case, we do not get a nontrivial skein grading on $\mathcal{S}_0^Z(X;L)$.

2.4.2.2 Variants

A lasagna $(3,1,1)$-link homology theory Z (with a cocomplete target) can be restricted to various full subcategories of $\mathbf{Links}_1$, giving many variants of the Z-input skein lasagna modules, as in Section 2.4.1, related to $\mathcal{S}_0^Z$ via forgetful maps. For instance:

(1) Restricting to the full subcategory consisting of (S,L) with L having 2-divisible homology class, as will be a natural setup in Chapter 4, we obtain a theory $\mathcal{S}_0^{Z,\text{2-div}}$ with a skein grading $H_2^L(X;\mathbb{Z}/2)$ on $\mathcal{S}_0^{Z,\text{2-div}}(X;L)$. Here, $H_2^L(X;\mathbb{Z}/2)$ is the preimage of $[L]$ under the connecting homomorphism $H_2(X,L;\mathbb{Z}/2)\to H_1(L;\mathbb{Z}/2)$.

(2) Restricting to the full subcategory consisting of (S,L) with L null-homologous, we obtain a theory $\mathcal{S}_0^{Z,\text{0-div}}$ with a skein grading $H_2^L(X)$ on $\mathcal{S}_0^{Z,\text{0-div}}(X;L)$;

(3) Restricting to the full subcategory $\mathbf{Links}_0$, we get a lasagna $(3,1,0)$-link homology theory $Z|_0$, and we recover a theory $\mathcal{S}_0^{Z|_0}$ of skein lasagna modules with 0-dimensional inputs.

(4) Restricting to the full subcategory $\mathbf{Links}_{1,1}$ consisting of (S,L) with S connected and nonempty, we obtain a "1-ball" version theory $\mathcal{S}_0^{Z,\text{1-ball}}$.

(5) Analogously, we obtain theories $\mathcal{S}_0^{Z,\text{2-div},\text{1-ball}}$ and $\mathcal{S}_0^{Z,\text{0-div},\text{1-ball}}$.

(6) Restricting to the full subcategory $\mathbf{Links}_{0,1}$, we obtain the 1-ball version theory $\mathcal{S}_0^{Z|_0,\text{1-ball}}$ for the theory (3).

These theories are related to one another by forgetful maps. More precisely, we have the following.

Proposition 2.4.5. *Suppose X is connected. The variants of $\mathcal{S}_0^Z(X;L)$ considered above are related to each other by forgetful maps which fit into the following commutative diagram:*

$$\begin{tikzcd}
\mathcal{S}_0^{Z|_0,\text{1-ball}}(X;L) \arrow[r] \arrow[d, two heads] & \mathcal{S}_0^{Z,\text{0-div},\text{1-ball}}(X;L) \arrow[r] \arrow[d, "\cong"] & \mathcal{S}_0^{Z,\text{2-div},\text{1-ball}}(X;L) \arrow[r] \arrow[d, "\cong"] & \mathcal{S}_0^{Z,\text{1-ball}}(X;L) \arrow[d, "\cong"] \\
\mathcal{S}_0^{Z|_0}(X;L) \arrow[r] & \mathcal{S}_0^{Z,\text{0-div}}(X;L) \arrow[r] & \mathcal{S}_0^{Z,\text{2-div}}(X;L) \arrow[r] & \mathcal{S}_0^{Z}(X;L).
\end{tikzcd} \tag{2.10}$$

If X is simply-connected, then the horizontal maps in (2.10) are surjective, and the first vertical map is an isomorphism.

Thus, unlike the 0-dimensional-input case, the "1-ball" versions ((4) and (5) above) of the 1-dimensional-input theory are not essentially new theories. When X is simply-connected, we get four successive refinements of $\mathcal{S}_0^Z(X;L)$ in (2.10).

Proof. When X is simply-connected, every object in $\mathcal{C}_{1,\geq 0}(X;L)$ has a successor in $\mathcal{C}_{0,\geq 0}(X;L)$, so $\mathcal{S}_0^{Z|_0}(X;L) \to \mathcal{S}_0^Z(X;L)$ is surjective. Similarly, $\mathcal{S}_0^{Z|_0}(X;L) \to \mathcal{S}_0^{Z,\text{i-div}}(X;L)$ is surjective, $i = 0, 2$, so the entire bottom row consists of surjective maps. The same argument applies to the first row. The statements concerning the first vertical arrow are proved in Proposition 2.4.2, and the statement concerning the other vertical arrows can be proved in an analogous way, except that simple connectivity is no longer necessary for the cofinality of $\mathcal{C}_{1,1}(X;L)$ in $\mathcal{C}_{1,\geq 0}(X;L)$ and variants of this statement with the divisibility conditions. $\square$

2.5 A universal trivial example

Before considering more interesting examples, we compute various versions of skein lasagna modules for the universal trivial lasagna link homology theory where the target is symmetric monoidal and additive.

For any monoidal categories $\mathcal{C}, \mathcal{D}$, the *trivial functor* from $\mathcal{C}$ to $\mathcal{D}$ is the functor that sends every object to the monoidal unit $1_{\mathcal{D}}$ and every morphism to $\mathrm{id}_{1_{\mathcal{D}}}$. Let **Ab** denote the category of abelian groups, the initial cocomplete symmetric monoidal additive category. The universal trivial lasagna $(3,1,1)$-link homology theory (to a cocomplete symmetric monoidal additive category) is given by $H\colon \mathbf{Links}_1 \to \mathbf{Ab}$ sending each object to $\mathbb{Z}$ and each morphism to $\mathrm{id}_{\mathbb{Z}}$, and the universal trivial lasagna $(3,1,0)$-link homology theory is given by the restriction $H|_0$ of H to $\mathbf{Links}_0$, which is isomorphic to the theory induced by the Khovanov–Rozansky $\mathfrak{gl}_1$ homology with grading information forgotten. See [MWW24, Section 3.2] for an account of the $\mathfrak{gl}_1$ theory.

To exemplify, we compute the diagram (2.10) for $Z = H$.

Proposition 2.5.1. *For $Z = H$, (2.10) (for X connected) becomes*

$$\begin{tikzcd}
\mathbb{Z}[\Omega_{0,1}(X;L)] \arrow[r, two heads] \arrow[d, two heads] & \mathbb{Z}[H_2^L(X)] \arrow[r] \arrow[d, "="] & \mathbb{Z}[H_2^L(X;\mathbb{Z}/2)] \arrow[r] \arrow[d, "="] & \mathbb{Z} \arrow[d, "="] \\
\mathbb{Z}[H_2^L(X)] \arrow[r, "="] & \mathbb{Z}[H_2^L(X)] \arrow[r] & \mathbb{Z}[H_2^L(X;\mathbb{Z}/2)] \arrow[r] & \mathbb{Z},
\end{tikzcd}$$

where $\Omega_{0,1}(X;L)$ admits a natural map onto $H_2^L(X)$ whose fiber over $\alpha \in H_2^L(X)$ is a torsor over $\operatorname{coker}(\bullet \cap 2\alpha\colon H_3(X) \to H_1(X))$*. Here, $\mathbb{Z}[S]$ denotes the abelian group freely generated by elements in a set S.*

Proof. If $\Omega_{1,\geq 0}(X;L)=\pi_0(\mathcal{C}_{1,\geq 0}(X;L))$ denotes the bordism set of skeins in X rel L with 1-dimensional inputs, then there is a natural isomorphism

$$\mathcal{S}_0^H(X;L)\cong\mathbb{Z}[\Omega_{1,\geq 0}(X;L)]=\mathbb{Z}\{e_\alpha\colon \alpha\in\Omega_{1,\geq 0}(X;L)\}$$

sending the class represented by a lasagna filling $(\Sigma,1)$ to $e_{[\Sigma]}$, where $[\Sigma]\in\Omega_{1,\geq 0}(X;L)$ is the class represented by Σ. Similarly, to compute other variants of H-input skein lasagna modules, it suffices to compute the corresponding bordism set of skeins. The statement for $\Omega_{0,1}(X;L)$ follows from [KMT12; Tay12] as observed in Section 2.4.1, the statement for $\Omega_{0,\geq 0}(X;L)$ follows from Theorem 2.3.9. It remains to prove $\Omega_{1,\geq 0}(X;L)=*$ is a singleton, $\Omega_{1,\geq 0}^{\text{2-div}}=H_2^L(X;\mathbb{Z}/2)$ and $\Omega_{1,\geq 0}^{\text{0-div}}(X;L)\cong H_2^L(X)$.

We show any skein (B,Σ) in X rel L with 1-dimensional inputs can be related to one that is standard. By deleting extra balls, we may assume Σ has no closed components. Let $K=-\partial\Sigma|_{\partial B}$ denote the input link of Σ. We may replace B by a tubular neighborhood of K pushed slightly into B touching its boundary along K. Next, we may replace B by a tubular neighborhood of $\Sigma\backslash((-2\epsilon,0]\times L)$ and delete the part of Σ inside it. Finally, we may replace B by a tubular neighborhood of a slight push-in of L, and we have $\Sigma=[-\epsilon,0]\times L$ is standard. This proves $\Omega_{1,\geq 0}(X;L)=*$.

Taking homology gives a map $\Phi\colon\Omega_{1,\geq 0}^{\text{2-div}}(X;L)\to H_2^L(X;\mathbb{Z}/2)$. We show Φ is an isomorphism using an argument analogous to that in the proof of Theorem 2.3.9. We show Φ is surjective. For any $\alpha\in H_2^L(X;\mathbb{Z}/2)$, we may represent it by an unoriented immersed compact surface $\Sigma_0\looparrowright X$ with $\partial\Sigma_0=L$, whose singularities are transverse double points. Delete a tubular neighborhood, denoted B, of the union of the singularities of Σ_0 with a push-in copy of L along Σ_0 and with some additional curves on Σ_0, so that $\Sigma:=\Sigma_0\backslash int(B)$ has no closed components and is orientable. Orienting and framing Σ in a way compatible with L, we see that (B,Σ) is a skein in X rel L with 1-dimensional inputs that represents $\alpha\in H_2^L(X;\mathbb{Z}/2)$. We show Φ is injective. Let (B_0,Σ_0) and (B_1,Σ_1) be skeins in X rel L with 1-dimensional inputs with $\mathbb{Z}/2$-null-homologous input links, that represent the same class in $H_2^L(X;\mathbb{Z}/2)$. We show that they are related to one another. We may assume $B_0=B_1=B$. Find an unoriented immersed compact 3-manifold Y in $I\times(X\backslash int(B))$ in general position cobounding $\{0\}\times\Sigma_0$ and $\{1\}\times\Sigma_1$. We delete from $I\times(X\backslash int(B))$ the tubular neighborhood of a 2-complex so that the complement of Y has no singularity and has an orientation and a framing both compatible with Σ_0 and Σ_1. Putting the 2-complex in general position, we see that a movie along the I-direction gives a sequence of skeins in X rel L with 1-dimensional inputs with $\mathbb{Z}/2$-null-homologous input links that relates Σ_0 to Σ_1 in $\mathcal{C}_{1,\geq 0}^{\text{2-div}}(X;L)$.

Using an analogous but simpler argument, one shows that the map $\Omega_{1,\geq 0}^{\text{0-div}}(X;L)\to H_2^L(X)$ is an isomorphism. Alternatively, this follows from the fact that $\Omega_{0,\geq 0}(X;L)\to\Omega_{1,\geq 0}^{\text{0-div}}(X;L)$ is surjective as every object in $\mathcal{C}_{1,\geq 0}^{\text{0-div}}(X;L)$ has a predecessor in $\mathcal{C}_{0,\geq 0}(X;L)$. $\square$

2.6 Khovanov and Lee skein lasagna modules

In the majority of the thesis, we will be employing the lasagna $(3,1,0)$-link homology theories KhR_2 and KhR_{Lee} induced via Proposition 2.1.2 from the Khovanov–Rozansky $\mathfrak{gl}_2$ homology and its Lee deformation, which are renormalizations of the usual Khovanov homology [Kho00] and its Lee deformation [Lee05]. In this section, we review some features of these link homology theories, partly to establish renormalization conventions and notation. Then we review some formal properties of the resulting Khovanov and Lee skein lasagna modules, again partly following [MN22].

2.6.1 The link homology theories

We follow the convention in [MWW22]. Khovanov–Rozansky $\mathfrak{gl}_2$ homology is a renormalization of the Khovanov–Rozansky $\mathfrak{sl}_2$ homology defined in [KR08] (and extended to $\mathbb{Z}$-coefficients in [Bla10]) that is sensitive to the framing of a link. Since conventionally most computations have been done using the classical Khovanov homology [Kho00], we write down explicitly the renormalization differences between it and the $\mathfrak{gl}_2$ theory.

Let KhR_2 and Kh denote the Khovanov–Rozansky $\mathfrak{gl}_2$ homology and the usual Khovanov homology, respectively. These theories are each $\mathbb{Z}^2$-graded, with the first grading the *homological degree* and the second grading the *quantum degree*. For a framed oriented admissible link $L \subset \mathbb{R}^3$, they are related by [Bel+23]

$$KhR_2^{h,q}(L) \cong Kh^{h,-q-w(L)}(-L), \tag{2.11}$$

where $-L$ is the reverse mirror of L, and $w(L)$ is the writhe of L, defined as the linking number between L and a slight pushoff of L in the framing direction. Admissibility of L means that its projection to the first two coordinates is generic. When we want to emphasize that we are working over a commutative ring R, write $Kh(L;R)$ and $KhR_2(L;R)$ for the corresponding homology theories.

A (possibly dotted) framed oriented cobordism $S\colon L_0 \to L_1$ in $I \times \mathbb{R}^3$ induces a map $KhR_2(S)\colon KhR_2(L_0) \to KhR_2(L_1)$ functorially in S, of bidegree $(0, -\chi(S) + 2\#(\text{dots}))$, which only depends on the isotopy class of S rel boundary [Bla10]. Here, a "dot" marks a multiplication-by-X operation in the associated Frobenius algebra (see the paragraph below). Let $\bar{S}\colon -L_0 \to -L_1$ denote the mirror image of S. Then the identification (2.11) is functorial in the sense that $Kh(\bar{S})\colon Kh(-L_0) \to Kh(-L_1)$ is equal to $KhR_2(S)$ under (2.11) up to sign.

Morrison–Walker–Wedrich [MWW22, Theorem 1.1] showed that the sweep-around move induces the identity map on KhR_2, so KhR_2 defines a link homology theory in S^3. We will note an alternative simple proof of this result in Section 2.6.4. Since the classical Khovanov homology is only known to be functorial in $\mathbb{R}^3$ up to sign [Jac04], KhR_2 can be thought of as a sign fix and a functorial upgrade of Kh.

Both the constructions of KhR_2 and Kh depend on the input of the Frobenius algebra $V = \mathbb{Z}[X]/(X^2)$[4] with comultiplication and counit given by

$$\Delta 1 = X \otimes 1 + 1 \otimes X,\ \Delta X = X \otimes X,\ \epsilon 1 = 0,\ \epsilon X = 1.$$

Here X has quantum degree 2 in the KhR_2 case and -2 in the Kh case.

The Lee deformation of V over $\mathbb{Q}$ [Lee05] is the Frobenius algebra $V_{Lee} = \mathbb{Q}[X]/(X^2-1)$ with

$$\Delta 1 = X \otimes 1 + 1 \otimes X,\ \Delta X = X \otimes X + 1 \otimes 1,\ \epsilon 1 = 0,\ \epsilon X = 1.$$

Using V_{Lee} instead of V defines the Khovanov–Rozansky $\mathfrak{gl}_2$ Lee homology and the classical Lee homology, denoted KhR_{Lee} and Kh_{Lee}, respectively, which are related by an equation analogous to (2.11). Since V_{Lee} is no longer graded but only filtered, the resulting homology theories carry quantum filtration structures, instead of quantum gradings (although a compatible quantum $\mathbb{Z}/4$-grading is still present). We warn the reader that $0 \in KhR_{Lee}$ has quantum filtration degree $-\infty$ while $0 \in Kh_{Lee}$ has quantum filtration degree $+\infty$, since the two filtrations go in different directions.

Either KhR_2 or KhR_{Lee} changes only by a renormalization upon changing the orientation of the link L. More precisely, if $(L, \mathfrak{o})$ denotes the link L equipped with a possibly different orientation $\mathfrak{o}$, then for $\bullet \in \{2, Lee\}$,

$$t^{(w(L)-w(L,\mathfrak{o}))/2} q^{(w(L,\mathfrak{o})-w(L))/2} KhR_\bullet(L, \mathfrak{o}) \cong KhR_\bullet(L). \tag{2.12}$$

Here, t, q indicate shifts in homological degree and quantum (filtration) degree, respectively.

It is sometimes more convenient to change from the basis $\{1, X\}$ of V_{Lee} to the basis $\{\mathbf{a} = (1+X)/2,\ \mathbf{b} = (1-X)/2\}$. In this new basis,

$$\mathbf{ab} = 0,\ \mathbf{a}^2 = \mathbf{a},\ \mathbf{b}^2 = \mathbf{b},\ 1 = \mathbf{a}+\mathbf{b},\ X = \mathbf{a}-\mathbf{b},\ \Delta\mathbf{a} = 2\mathbf{a}\otimes\mathbf{a},\ \Delta\mathbf{b} = -2\mathbf{b}\otimes\mathbf{b},\ \epsilon\mathbf{a} = \tfrac{1}{2},\ \epsilon\mathbf{b} = -\tfrac{1}{2}.$$

Like the classical story in [Ras10], the Khovanov–Rozansky $\mathfrak{gl}_2$ Lee homology of a framed oriented link L has a basis consisting of canonical generators $x_\mathfrak{o}$, one for each orientation $\mathfrak{o}$ of L as an unoriented link. The explicit renormalization convention of our choice of generators, together with the proof of the following properties, can be found in Appendix A.

Proposition 2.6.1.

(1) The unique canonical generator of the empty link $\emptyset$ is $1 \in \mathbb{Q} \cong KhR_{Lee}(\emptyset)$.

(2) The $\mathfrak{gl}_2$ Lee homology of the n-framed oriented unknot $(U^n, \mathfrak{o})$ is canonically isomorphic to $q^{-n-1}V_{Lee}$, with canonical generators given by $x_\mathfrak{o} = \mathbf{a}$, $x_{\bar{\mathfrak{o}}} = \mathbf{b}$, renormalized by q^{-n-1}.

[4] Strictly speaking, the construction of KhR_2 uses the formalism of foams, but for simplicity, our exposition remains in the classical Bar-Natan formalism throughout.

Let L be a framed oriented link and $\mathfrak{o}$ be an orientation of L as an unoriented link. Write $L = L_+(\mathfrak{o}) \cup L_-(\mathfrak{o})$ as a union of sublinks on which $\mathfrak{o}$ agrees (resp. differs) with the orientation of L.

(3) $x_{\mathfrak{o}}$ is homogeneous with homological degree

$$h(x_{\mathfrak{o}}) = -2\ell k(L_+(\mathfrak{o}), L_-(\mathfrak{o})) = (w(L, \mathfrak{o}) - w(L))/2.$$

(4) $x_{\mathfrak{o}} \pm x_{\bar{\mathfrak{o}}}$ is homogeneous with $\mathbb{Z}/4$ quantum degrees

$$q_{\mathbb{Z}/4}(x_{\mathfrak{o}} \pm x_{\bar{\mathfrak{o}}}) = (-w(L) - \#L - 1 \pm 1) \bmod 4.$$

(5) If $S\colon L \to L'$ is a framed oriented link cobordism, then

$$KhR_{Lee}(S)(x_{\mathfrak{o}}) = 2^{n(S)} \sum_{\mathfrak{O}|_L = \mathfrak{o}} (-1)^{n(S_-(\mathfrak{O}))} x_{\mathfrak{O}|_{L'}}, \tag{2.13}$$

where $\mathfrak{O}$ runs over orientations on S that are compatible with the orientation $\mathfrak{o}$ (thus each $(S, \mathfrak{O})$ is an oriented cobordism from $(L, \mathfrak{o})$ to $(L', \mathfrak{O}|_{L'})$), $S_-(\mathfrak{O})$ is the union of components of S whose orientations disagree with $\mathfrak{O}$, and

$$n(\Sigma) = \frac{-\chi(\Sigma) + \#\partial_+\Sigma - \#\partial_-\Sigma}{2} \in \mathbb{Z} \tag{2.14}$$

for a surface Σ regarded as a cobordism from the negative boundary $\partial_-\Sigma$ to the positive boundary $\partial_+\Sigma$.

Note that the function $n(\cdot)$ in (2.14) is additive under gluing cobordisms.

The $\mathfrak{gl}_2$ s-invariant of L is defined to be

$$s_{\mathfrak{gl}_2}(L) := q(x_{\mathfrak{o}_L}) - 1$$

where $\mathfrak{o}_L$ is the orientation of L and $q\colon KhR_{Lee}(L) \to \mathbb{Z} \sqcup \{-\infty\}$ is the quantum filtration function. It relates to the classical s-invariant of oriented links (as defined in [BW08], after [Ras10]) by

$$s_{\mathfrak{gl}_2}(L) = -s(-L) - w(L). \tag{2.15}$$

From now on, we will sometimes drop the $\mathfrak{gl}_2$ terminology and simply call KhR_2 and KhR_{Lee} Khovanov homology and Lee homology, respectively.

2.6.2 Definition

The link homology theories KhR_2 and KhR_{Lee} for links in $\mathbb{R}^3$ both satisfy the conditions in Proposition 2.1.2, so we obtain upgrades to lasagna $(3,1,0)$-link homology theories. Definition 2.2.2 applied to these two theories then yields two packages of skein lasagna modules, which we abbreviate as $\mathcal{S}_0^2$ and $\mathcal{S}_0^{Lee}$, called *Khovanov skein lasagna modules* and *Lee skein lasagna modules.*

By the discussion on gradings in Section 2.3.4, if Σ is a skein in X rel L, with input balls $\sqcup_{i=1}^k B_i$ and input links $\sqcup_{i=1}^k K_i$, then for $\bullet \in \{2, Lee\}$, the space of $KhR_\bullet$-lasagna fillings with underlying skein Σ is

$$KhR_\bullet(\Sigma) = q^{-\chi(\Sigma)} \otimes_{i=1}^k KhR_\bullet(K_i). \tag{2.16}$$

2.6.2.1 Dots and (interior) decorations

Since the category of links $\mathbf{Links}_0$ in our setup in Section 2.1 has morphisms given by undotted cobordisms, in the construction of $\mathcal{S}_0^2$ and $\mathcal{S}_0^{Lee}$, we have to first restrict to undotted cobordisms and run the construction. If one instead modifies the setup to allow dotted cobordisms, this would yield an equivalent theory of skein lasagna modules, but where the skeins are allowed to carry dots on them.

We do not formally pursue this approach of generalization. Instead, we sometimes allow dots and more generally dots labeled by elements in $q^{-1}V$ or $q^{-1}V_{Lee}$ on skeins, by interpreting a v-labeled dot as having the effect of deleting a small input ball around the dot, and putting a v decoration on the resulting 0-framed unknot on its boundary (a dot is interpreted as an X-labeled dot).

2.6.2.2 Monoidality and a multiplicative variant

The Khovanov homology KhR_2, as a link homology theory in $\mathbb{R}^3$, is not strictly monoidal over a general commutative ring. Indeed, if $L_0, L_1 \subset \mathbb{R}^3$ are admissible links, we have, on the level of Khovanov chain complexes, $CKhR_2(L_0 \sqcup L_1) = CKhR_2(L_0) \otimes CKhR_2(L_1)$, so unless we work over a field, the monoidal structure map $KhR_2(L_0) \otimes KhR_2(L_1) \to KhR_2(L_0 \sqcup L_1)$ may not be an isomorphism. Consequently, the resulting lasagna $(3,1,0)$-link homology theory $KhR_2 = KhR_2(\bullet; R)\colon \mathbf{Links}_0 \to \mathbf{Mod}_R^{\mathbb{Z}^2}$ is not multiplicative for a general commutative ring R.

One could consider a multiplicative variant of the lasagna $(3,1,0)$-link homology theory KhR_2 as follows. We first consider the strictly monoidal link homology theory in $\mathbb{R}^3$ given by the Khovanov chain complex functor $CKhR_2 = CKhR_2(\bullet; \mathbb{Z})\colon \mathbf{Links}_{\mathbb{R}^3} \to \mathbf{K}^b(\mathbb{Z})^{\mathbb{Z}}$. By the proof of the sweep-around move in [MWW22], $CKhR_2$ actually satisfies property (a) in Proposition 2.1.2; see also Section 2.6.4 for an alternative proof. Property (b) is also easily checked, so Proposition 2.1.2 upgrades $CKhR_2$ to a strictly monoidal lasagna $(3,1,0)$-link homology theory $CKhR_2\colon \mathbf{Links}_0 \to \mathbf{K}^b(\mathbb{Z})^{\mathbb{Z}}$ that is multiplicative by Lemma 2.1.3. Taking the homology with R-coefficients, we recover a lasagna $(3,1,0)$-link homology theory, denoted by

$$KhR_{2,\mathrm{mul}} = KhR_{2,\mathrm{mul}}(\bullet; R)\colon \mathbf{Links}_0 \to \mathbf{Mod}_R^{\mathbb{Z}^2},$$

where the target category is now cocomplete. We may then apply the skein lasagna module construction to define the associated skein lasagna modules, denoted $\mathcal{S}_0^{2,\mathrm{mul}} = \mathcal{S}_0^{2,\mathrm{mul}}(\bullet;\bullet;R)$. It shares all properties enjoyed by $\mathcal{S}_0^2$ presented in Section 2.6.3.

The strict monoidality and multiplicativity properties of the three Khovanov lasagna $(3,1,0)$-link homology theories over a general commutative ring R are summarized as follows:

Lasagna link homology theory	Strictly monoidal	Multiplicative
$CKhR_2$	Yes	Yes
KhR_2	Yes	Not necessarily
$KhR_{2,\mathrm{mul}}$	Not necessarily	Yes

When $R = \mathbb{F}$ is a field, by the Künneth isomorphism, the lasagna $(3,1,0)$-link homology theories KhR_2 and $KhR_{2,\mathrm{mul}}$ are naturally isomorphic, thus so are the resulting skein lasagna modules. When R is a general commutative ring, however, the two theories need not agree, and the multiplicative version seems to enjoy better formal properties. For instance, the 4-dimensional neck-cutting operation (Lemma 2.3.7) is available for $\mathcal{S}_0^{2,\mathrm{mul}}$ but not necessarily for $\mathcal{S}_0^2$. For the two natural maps in the sum formula, Proposition 2.3.2, the first is an isomorphism for $\mathcal{S}_0^2$, while the second is an isomorphism for $\mathcal{S}_0^{2,\mathrm{mul}}$.

Example 2.6.2. By Proposition 2.3.8, $\mathcal{S}_0^2(B^4) = KhR_2(\emptyset) = R$. Take embeddings $B^4 \subset S^1 \times B^3 \subset B^4$. The composition of these two embeddings is a collar thickening, so the induced map $\mathcal{S}_0^2(B^4) \to \mathcal{S}_0^2(B^4)$ is the identity map id_R by Proposition 2.3.5. The same description applies to $\mathcal{S}_0^{2,\mathrm{mul}}$. In the case of the latter theory, the neck-cutting operation (Lemma 2.3.7) shows that the map $\mathcal{S}_0^{2,\mathrm{mul}}(B^4) \to \mathcal{S}_0^{2,\mathrm{mul}}(S^1 \times B^3)$ is surjective, which shows $\mathcal{S}_0^{2,\mathrm{mul}}(S^1 \times B^3) = R$. In the case of $\mathcal{S}_0^2$, however, we are unable to compute $\mathcal{S}_0^2(S^1 \times B^3)$, unless R is a field in which case the theory is isomorphic to $\mathcal{S}_0^{2,\mathrm{mul}}$.

Letting $\mathbb{Z}[E_1, E_2] = H^*(BGL(2,\mathbb{C}))$ as a graded ring, the universal, or $GL(2,\mathbb{C})$-equivariant, Khovanov chain complex functor on $\mathbf{Links}_{\mathbb{R}^3}$ upgrades to a multiplicative lasagna $(3,1,0)$-link homology theory

$$CKhR_2^{univ}\colon \mathbf{Links}_0 \to \mathbf{K}^b(\mathbf{Mod}_{\mathbb{Z}[E_1,E_2]})^{\mathbb{Z}}$$

via Proposition 2.1.2, which specializes to the lasagna $(3,1,0)$-link homology theories $CKhR_2$, $CKhR_{Lee}$ via base change maps $\mathbb{Z}[E_1,E_2] \to R,\ E_1, E_2 \mapsto 0$ and $\mathbb{Z}[E_1,E_2] \to \mathbb{Q},\ E_1 \mapsto 0, E_2 \mapsto -1$, respectively, which then gives rises to lasagna $(3,1,0)$-link homology theories

$KhR_{2,\mathrm{mul}}$ and KhR_{Lee}. In this sense, it appears that the classical construction of Khovanov skein lasagna modules $\mathcal{S}_0^2(\bullet; R)$ takes homology "too early."

Most of our computations in Chapter 3 will be carried out over the field $\mathbb{Q}$, so we do not dwell on the differences between $\mathcal{S}_0^2$ and $\mathcal{S}_0^{2,\mathrm{mul}}$, and will refer only to $\mathcal{S}_0^2$ for the rest of this chapter and the next. On the other hand, to state our construction in Chapter 4 in its full generality, it will be convenient to use the "mul" versions of the relevant invariants.

2.6.3 Properties

It follows from our general discussion in Section 2.3.4 that for a pair (X, L) of a 4-manifold X and a framed oriented link $L \subset \partial X$, the Khovanov skein lasagna module

$$\mathcal{S}_0^2(X; L) = \bigoplus \mathcal{S}_{0,h,q}^2(X; L; \alpha)$$

is an R-module with a homological grading by $h \in \mathbb{Z}$, a quantum grading by $q \in \mathbb{Z}$, and a skein grading by $\alpha \in H_2^L(X)$. Similarly, the Lee skein lasagna module $\mathcal{S}_0^{Lee}(X; L)$ is a $\mathbb{Q}$-vector space with a homological $\mathbb{Z}$-grading, a quantum filtration, a compatible quantum $\mathbb{Z}/4$-grading, and a skein grading by $H_2^L(X)$. A Khovanov/Lee lasagna filling of (X, L) is of the form (Σ, v), where Σ is a skein in X rel L with some input balls B_i, possibly with interior decorations, and $v \in \otimes_i KhR_\bullet(-\partial\Sigma|_{\partial B_i})$ is a decoration on the input links, $\bullet \in \{2, Lee\}$. If v is homogeneous with homological degree h and quantum degree (or filtration degree) q, and only dots appear as interior decorations on Σ, then the class represented by (Σ, v) in $\mathcal{S}_0^\bullet(X; L)$ has homological degree h, quantum degree (or filtration degree at most) $q - \chi(\Sigma) + 2\#(dots)$, and skein degree $[\Sigma]$. When we want to emphasize the base ring R, write $\mathcal{S}_0^2(X; L; R) = \oplus_\alpha \mathcal{S}_0^2(X; L; \alpha; R)$.

Proposition 2.6.3 (Sum formulas). *For any $* \in \{\sqcup, \natural, \#\}$, we have*

$$\mathcal{S}_0^2(X_1 * X_2; L_1 \sqcup L_2; \mathbb{F}) \cong \mathcal{S}_0^2(X_1; L_1; \mathbb{F}) \otimes \mathcal{S}_0^2(X_2; L_2; \mathbb{F})$$

for any field $\mathbb{F}$, and

$$\mathcal{S}_0^{Lee}(X_1 * X_2; L_1 \sqcup L_2) \cong \mathcal{S}_0^{Lee}(X_1; L_1) \otimes \mathcal{S}_0^{Lee}(X_2; L_2),$$

respecting all grading/filtration structures.

Proof. See the discussions about strict monoidality in Section 2.6.2.2. The statement follows from Proposition 2.3.2. □

Attaching 4-handles induces isomorphisms on skein lasagna modules, as stated in Proposition 2.3.1. In the case of 3-handles, one can be more explicit than Proposition 2.3.1.

Proposition 2.6.4 (3-handle attachments). *If (X', L) is obtained from (X, L) by attaching a 3-handle along a 2-sphere $S \subset \partial X$ away from L, then for $\bullet \in \{2, Lee\}$, the inclusion map induces a surjection $\mathcal{S}_0^\bullet(X; L) \twoheadrightarrow \mathcal{S}_0^\bullet(X'; L)$ whose kernel is generated by two relations on the level of lasagna fillings: An undotted sphere S^- evaluates to zero, and a one-dotted sphere S^- evaluates to 1. Here S^- denotes a slight push-in of S to the interior of X, which is one component of the skein of the lasagna filling under discussion.*

Proof. The two relations are clearly in the kernel. Conversely, applying a Bar-Natan neck-cutting relation in the proof of Proposition 2.3.1 shows that the kernel is contained in the span of these relations. □

Next, we describe a formula of Khovanov and Lee skein lasagna modules for 2-handlebodies, developed in [MN22]. From now until the end of this section, suppose X is a 2-handlebody obtained by attaching 2-handles along a framed oriented link $K = K_1 \cup \cdots \cup K_m \subset S^3 = \partial B^4$ disjoint from a framed oriented link $L \subset S^3$. Then L can be regarded as a link in ∂X. A class $\alpha \in H_2^L(X)$ intersects the cocore of the 2-handle on K_i algebraically some α_i times, and the assignment $\alpha \mapsto (\alpha_i)_i$ gives an isomorphism $H_2^L(X) \cong \mathbb{Z}^m$. Write $\alpha_i^+ = \max(0, \alpha_i)$ and $\alpha_i^- = \max(0, -\alpha_i)$. Write $|\cdot|$ for the L^1-norm on $\mathbb{Z}^m$.

For $k^+, k^- \in \mathbb{Z}^m_{\geq 0}$, let $K(k^+, k^-) \cup L \subset S^3$ denote the framed oriented link obtained from $K \cup L$ by replacing each K_i by $k_i^+ + k_i^-$ parallel copies of itself, with orientations on k_i^- of them reversed.

Proposition 2.6.5 (2-handlebody formula, [MN22, Theorem 1.1, Proposition 3.8][5]). *Let (X, L) and α be given as above. Then*

$$\mathcal{S}_0^\bullet(X; L; \alpha) \cong \left(\bigoplus_{r \in \mathbb{Z}^m_{\geq 0}} q^{-|\alpha| - 2|r|} KhR_\bullet(K(\alpha^+ + r, \alpha^- + r) \cup L) \right) \Big/ \sim \tag{2.17}$$

where $\bullet \in \{2, Lee\}$ and $\sim$ is the linear equivalence relation generated by

(i) $\sigma_i \cdot x \sim x$ for $x \in q^{-|\alpha|-2|r|} KhR_\bullet(K(\alpha^+ + r, \alpha^- + r) \cup L)$ and $\sigma_i \in S_{|\alpha_i| + 2r_i}$, where σ_i acts by permuting the cables for K_i (cf. [GLW18])[6];

(ii) $\phi_i(x) \sim x$ for $x \in q^{-|\alpha|-2|r|} KhR_\bullet(K(\alpha^+ + r, \alpha^- + r) \cup L)$ where

$$\phi_i \colon q^{-|\alpha|-2|r|} KhR_\bullet(K(\alpha^+ + r, \alpha^- + r) \cup L) \to q^{-|\alpha|-2|r|-2} KhR_\bullet(K(\alpha^+ + r + e_i, \alpha^- + r + e_i) \cup L)$$

[5] The Kh in Proposition 3.8 in [MN22] should be KhR_2 instead.

[6] Strictly speaking, elements in $S_{|\alpha_i|+2r_i}$ may permute cables of K_i with different orientations, so it does not act on $q^{-|\alpha|-2|r|} KhR_\bullet(K(\alpha^+ + r, \alpha^- + r) \cup L)$. Nevertheless, since changing orientation only affects $KhR_\bullet$ by a renormalization, we can first work with $K(\alpha^+ + \alpha^- + 2r, 0) \cup L$ and then renormalize. For the symmetric group action on homology of cables, we follow the sign convention in [GLW18, Section 7.2].

is the map induced by the dotted annulus cobordism that creates two parallel, oppositely oriented cables of K_i, and where e_i is the i^{th} coordinate vector.

(iii) $\varphi_i(x) \sim 0$ *for* $x \in q^{-|\alpha|-2|r|-2}KhR_\bullet(K(\alpha^+ + r, \alpha^- + r) \cup L)$ *where*

$$\varphi_i\colon q^{-|\alpha|-2|r|-2}KhR_\bullet(K(\alpha^+ + r,\alpha^- + r) \cup L) \to \\ q^{-|\alpha|-2|r|-2}KhR_\bullet(K(\alpha^+ + r + e_i, \alpha^- + r + e_i) \cup L)$$

is the map induced by the undotted annulus cobordism that creates two parallel, oppositely oriented cables of K_i.

Moreover, the relation (iii) is a consequence of (i) if 2 *is invertible in the base ring.* □

Proposition 2.6.5 is proved using a general position argument together with some results from [GLW18] on symmetric group action on cables. More generally, one can write down a formula for the change of skein lasagna modules under a 4-dimensional 2-handle attachment; see [MWW23] for details. We remark that [MWW23] also wrote down a formula for the Khovanov skein lasagna module of 1-handlebodies (with some boundary links), but it is often too complicated to be practical. One of our motivations for constructing the theory of Khovanov skein lasagna modules with 1-dimensional inputs in Chapter 4 is to circumvent the complication involving 4-dimensional 1-handles.

In the case of Lee skein lasagna modules, or Khovanov skein lasagna modules over $\mathbb{Q}$, it will be convenient to rewrite (2.17) as follows.

Proposition 2.6.6 (2-handlebody formula in characteristic zero). *In the same notation as in Proposition 2.6.5, assuming we work over $\mathbb{Q}$ if $\bullet = 2$, then*

$$\mathcal{S}_0^\bullet(X;L;\alpha) \cong \mathrm{colim}_{r\in\mathbb{Z}^m_{\geq 0}} q^{-|\alpha|-2|r|}KhR_\bullet(K(\alpha^+ + r, \alpha^- + r) \cup L)^{\prod_{i=1}^m S_{|\alpha_i|+2r_i}}. \tag{2.18}$$

Here, the colimit is taken along $r \to r + e_i$, $i = 1, \cdots, m$, where the corresponding map is given by ϕ_i composed with symmetrization, the symmetrized dotted annulus creation map.

Proof. Relation (iii) in Proposition 2.6.5 is redundant. Relation (ii) expresses (2.17) as a colimit as in our setup. Relation (i) acts termwise by taking the $(\prod_i S_{|\alpha_i|+2r_i})$-coinvariants. In characteristic zero, coinvariants agree with invariants (this holds equally for filtered vector spaces), so we obtain (2.18). □

Changing the orientation of a link only affects $KhR_\bullet$ by a grading shift, given by (2.12). Therefore for 2-handlebodies X, by (2.17), $\mathcal{S}_0^\bullet(X;L)$ is independent of the orientation of L (as long as it is null-homologous) up to renormalizations, and $\mathcal{S}_0^\bullet(X;L;\alpha)$ only depends on the mod 2 reduction of $\alpha \in H_2^L(X) \subset H_2(X,L)$ up to renormalizations. Explicitly, suppose $\beta \in H_2(X,L)$ has the same mod 2 reduction as $\alpha \in H_2^L(X)$ such that $\partial\beta \in H_1(L)$ is the fundamental class of L with a possibly different orientation. Let $(L, \partial\beta)$ denote the link L with orientation given by $\partial\beta$. Then $\beta \in H_2^{(L,\partial\beta)}(X)$, and we have

$$\mathcal{S}_0^\bullet(X;(L,\partial\beta);\beta) \cong (tq^{-1})^{(\alpha^2-\beta^2)/2}\mathcal{S}_0^\bullet(X;L;\alpha). \tag{2.19}$$

This holds for any 2-handlebody X.

The bilinear product on $H_2^L(X)$ in (2.19) (similarly for $H_2^{(L,\partial\beta)}(X)$) is defined by counting intersections of two surfaces in X with boundary L, one copy of which is pushed off L via its framing. In particular, it is independent of the choice of $K \cup L \subset S^3$.

Example 2.6.7. The 2-handlebody $X = D^2 \times S^2$ is the trace on the 0-framed unknot. It follows from Proposition 2.6.5 that $\mathcal{S}_0^\bullet(D^2 \times S^2)$ can be computed from the Khovanov or Lee homology of unlinks and dotted cobordism maps between them. Explicitly, [MN22, Theorem 1.2] computed that $\mathcal{S}_0^2(D^2\times S^2) = R[A_0^\pm, A_1]$ as tri-graded R-modules, where A_0 has tri-degree $(0,0,1)$ and A_1 has tri-degree $(0,-2,1)$. More explicitly, for every $\alpha \in H_2(D^2\times S^2)$,

$$\mathcal{S}_{0,h,q}^2(D^2 \times S^2;\alpha) = \begin{cases} R, & h=0,\ q\le 0,\ 2|q \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, $gr_q\mathcal{S}_{0,h}^{Lee}(D^2 \times S^2;\alpha)$ is given by the same formula with R replaced by $\mathbb{Q}$.

Using Proposition 2.6.5, [MN22, Theorem 1.3] also obtained partial information on Khovanov skein lasagna modules for D^2-bundles over S^2 with Euler number $n \neq 0$, denoted $D(n)$, namely that

$$\mathcal{S}_{0,0,*}^2(D(n);0) = \begin{cases} \mathbb{Z}, & n<0,\ *=0 \\ 0, & \text{otherwise.} \end{cases}$$

For $n = \pm 1$, this also yields the corresponding computation for $\mathbb{CP}^2$, $\overline{\mathbb{CP}^2}$, in view of Proposition 2.3.1. In Chapter 3, these results will be improved for $n > 0$ in Example 3.2.2, and for $n < 0$ in Example 3.5.1, Section 3.5.8, and Section 3.5.9.

If L', L are two framed links in an oriented 3-manifold with the same underlying unframed link, their framings differ by a tuple of integers, one for each link component. The total framing difference between them, denoted $w(L', L)$, is the sum of these integers. For example, if U^n denotes the n-framed unknot in S^3, then $w(U^n, U^m) = n - m$.

Proposition 2.6.8 (Framing change). *If $L, L' \subset \partial X$ are framed oriented links that are equal as unframed oriented links, then for $\bullet \in \{2, Lee\}$,*

$$\mathcal{S}_0^\bullet(X;L') \cong q^{-w(L',L)}\mathcal{S}_0^\bullet(X;L).$$

Proof. For a given lasagna filling of (X, L), create new input balls near each component of L that intersect the skein in disks. Change the framing of L to L' and push the framing change into the intersection disks. Then the sum of framings on the input unknots is $w(L', L)$. Assigning these unknots the Khovanov or Lee class 1 (the element which corresponds to $1 \in KhR_\bullet(U)$ under the Reidemeister I induced isomorphisms, where U is the 0-framed unknot) produces a lasagna filling of (X, L'). This assignment induces the claimed graded isomorphism. □

Finally, in view of the trick for framing changes in the proof of Proposition 2.6.8, we can state a slight improvement of the gluing construction (2.5). In the general setup when gluing (X_2, S) to (X_1, L_1) to obtain (X, L), in order to get an induced map on skein lasagna modules, we needed to assume that S was framed, so that it could be regarded as a skein in X_2 rel L_2. In the Khovanov or Lee skein lasagna module setup, however, we can drop the framing assumption, at the expense of deleting extra input balls to compensate. Thus, for any unframed surface $S \subset X_2$ with the previous assumptions, there is a gluing morphism

$$\mathcal{S}_0^\bullet(X_2; S)\colon \mathcal{S}_0^\bullet(X_1; L_1) \to \mathcal{S}_0^\bullet(X; L), \tag{2.20}$$

which has homological degree 0 and quantum (filtration) degree $-\chi(S) + \chi(T_0) - [S]^2$. If S is already framed, this map agrees with the previous construction.

2.6.4 An alternative proof of sweep-around move

The main technical result of [MWW22] is to show that the sweep-around move induces the identity chain map up to chain homotopy on the Khovanov–Rozansky $\mathfrak{gl}_N$ chain complex $CKhR_N$ for any N. In this short section, we note that the statement for $N = 2$ is a consequence of the following result of Gujral–Levine [GL22]. Part of this trick was also noted and used in our joint work with H. Yang [RY25, Section 3].

Theorem 2.6.9 (Gujral–Levine [GL22, Theorem 1.2, Remark 7.6]). *Suppose $L_1 \sqcup L_2$ and $L_1' \sqcup L_2'$ are split framed oriented links in S^3, and $C_1 \cup C_2\colon L_1 \sqcup L_2 \to L_1' \sqcup L_2'$ is a link cobordism which restricts to link cobordisms $C_i\colon L_i \to L_i'$, $i = 1, 2$. Then the induced chain maps $CKhR_2(C_1 \cup C_2)$ and $CKhR_2(C_1 \sqcup C_2)$ are chain homotopic.* □

Their result was stated in CKh with a sign ambiguity, but one can transfer it to $CKhR_2$ using the natural isomorphism (up to sign) between them, and fix the sign using the Lee homology.

To fix notation, let T be a framed oriented admissible $(1,1)$-tangle in B^3 and $\hat{T} \subset \mathbb{R}^3$ its closure, which is a framed oriented admissible link in $\mathbb{R}^3$, where the projection of the segment $\gamma = \hat{T} \backslash T$ to $\mathbb{R}^2$ is a simple arc disjoint from B^2. The sweep-around move for $\hat{T}$ is the isotopy, denoted sw_T, from $\hat{T}$ to itself that sweeps the strand γ around the S^2 at infinity.

Proof of $CKhR_2(sw_T) \simeq \mathrm{id}$. Let U be the unknot, and $s\colon \hat{T} \sqcup U \to \hat{T}$ be the saddle cobordism that joins U with $\hat{T}$. Let sw_T' denote the isotopy from $\hat{T} \sqcup U$ to itself that keeps $\hat{T}$

fixed and loops U around $\hat{T}$ once. Then, provided the looping direction of sw'_T is chosen compatibly with the sweeping direction of sw_T, the two cobordisms $sw_T \circ s$ and $s \circ sw'_T$ are isotopic rel boundary. It follows that $CKhR_2(sw_T) \circ CKhR_2(s) \simeq CKhR_2(s) \circ CKhR_2(sw'_T)$. Since $CKhR_2(s)$ admits a right inverse and $CKhR_2(sw'_T) \simeq \mathrm{id}$ by Theorem 2.6.9, we have $CKhR_2(sw_T) \simeq \mathrm{id}$. $\square$

Chapter 3

Khovanov skein lasagna modules and exotic 4-manifolds

In this chapter, we present some explicit computations for the Khovanov skein lasagna modules $\mathcal{S}_0^2$ and Lee skein lasagna modules $\mathcal{S}_0^{Lee}$ of smooth 4-manifolds, introduced in Section 2.6 in Chapter 2, and show that they can be used to detect exotic pairs of smooth 4-manifolds. Throughout this chapter, the base commutative ring R, if not indicated in the notation, is assumed to be $\mathbb{Z}$, although most of the time we will be working over $\mathbb{Q}$.

3.1 Introduction

The revolutionary work of Donaldson and Freedman in the 1980s revealed a drastic dichotomy between the topological and smooth categories of manifolds in dimension 4. Since then, a central focus of smooth 4–manifold topology has been the discovery of new exotic phenomena and the resulting refinement of our understanding of the geography of smooth 4–manifolds, largely through the development of increasingly sophisticated smooth invariants.

A pair of smooth 4-manifolds X_1 and X_2 is said to be *exotic* if they are homeomorphic but not diffeomorphic. Under favorable conditions (typically involving the fundamental group), showing two 4-manifolds are homeomorphic reduces to checking some algebraic topological invariants agree, thanks to Freedman's fundamental theorems [Fre82]. Showing that they are non-diffeomorphic usually involves showing some gauge-theoretic or Floer-theoretic smooth invariants (Donaldson invariants [Don90], Seiberg–Witten invariants [SW94; Wit94], Heegaard Floer type invariants [OS04], and various variations/generalizations) differ. These tools, powerful as they are, typically come with constraints on the topology of the 4–manifolds (for instance, hypotheses on b_2^+ or related structural assumptions), and there is a sense in which their applicability to new phenomena is approaching its natural limit.

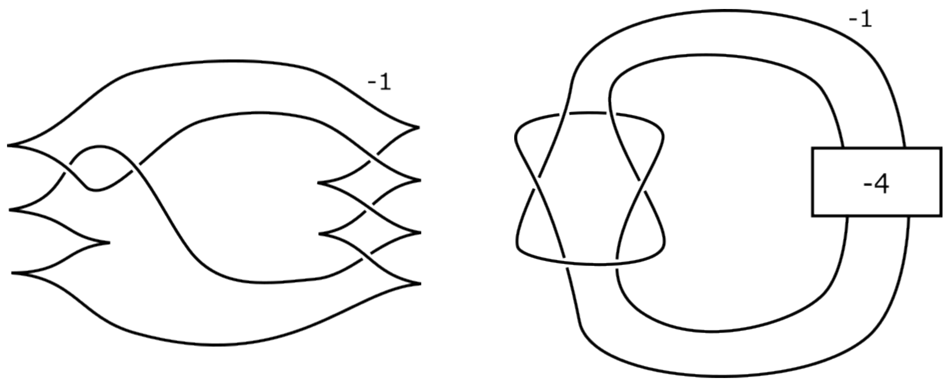


Figure 3.1: Kirby diagrams of X_1, X_2 in Theorem 3.1.2.

The main result of this chapter is to present one of the first analysis-free detections of compact orientable exotic 4–manifolds, using the theory of Khovanov and Lee skein lasagna modules, introduced in Section 2.6.

Remark 3.1.1. To put our work in a broader historical context, some exemplary analysis-free methods to detect exotic 4-manifolds include the following:

(1) For nonorientable examples, using Rohlin-type obstructions, exotic $\mathbb{RP}^4$'s were constructed by Cappell–Shaneson [CS76] (shown to be homeomorphic to $\mathbb{RP}^4$ later; see [Rub84, Page 221] and [HKT94]), and $K3\#\mathbb{RP}^4$ and $11(S^2 \times S^2)\#\mathbb{RP}^4$ were shown to be an exotic pair by Kreck [Kre84].

(2) For noncompact examples, exotic $\mathbb{R}^4$'s can be constructed from knots that are topologically slice but not smoothly slice [Gom85]. The existence of such knots can be shown by Freedman's result [Fre82] and Rasmussen's s-invariant [Ras10] from Khovanov homology.

(3) Exotic pairs of surfaces in B^4 with boundary were also known to be detectable by Khovanov homology [HS24]. As noted in [Nah25], this can be used to construct compact orientable exotic pairs of 4-manifolds.

The n-trace on a knot K, denoted $X_n(K)$, is the 4-manifold obtained by attaching an n-framed 2-handle to B^4 along $K \subset S^3 = \partial B^4$. We first state a special case of our main exotica detection theorem, namely that the (-1)-traces on the mirror image of the knot 5_2 and the pretzel knot $P(3, -3, -8)$ form an exotic pair.

Theorem 3.1.2. *Let*

$$X_1 := X_{-1}(-5_2), \quad X_2 := X_{-1}(P(3, -3, -8)).$$

Then X_1 and X_2 form an exotic pair of 4-manifolds. In fact, even their interiors are exotic. The graded modules $\mathcal{S}_0^2(X_1)$ and $\mathcal{S}_0^2(X_2)$ are not isomorphic.

The exotic pair X_1, X_2 was first discovered by Akbulut [Akb91b; Akb91a]. The fact that X_1, X_2 are homeomorphic is standard after Freedman: a sequence of Kirby moves shows ∂X_1 and ∂X_2 are homeomorphic and a suitable extension of Freedman's Theorem [Fre82; Boy86] then implies X_1 and X_2 are homeomorphic. On the other hand, the fact that X_1, X_2 are not diffeomorphic was detected by an intricate computation of Donaldson invariants. Later, the result was reproved by Akbulut–Matveyev [AM97, Theorem 3.2] using a simplified argument which essentially says $X_{-1}(-5_2)$ is Stein (since the maximal Thurston–Bennequin number of -5_2 is 1) while $X_{-1}(P(3,-3,-8))$ is not (since $P(3,-3,-8)$ is slice). This latter proof is elegant, yet still depends heavily on the computation of Seiberg–Witten invariants of Stein surfaces, as well as deep results of Eliashberg [Eli90] and Lisca–Matić [LM97]. In comparison, our proof of Theorem 3.1.2 depends on the much lighter package of Khovanov homology, which is combinatorial in nature.

Since over a field, the Khovanov skein lasagna module of a connected sum or boundary connected sum behaves multiplicatively (Proposition 2.6.3), X_1 and X_2 remain exotic after (boundary) connected sums with any 4-manifolds W with nonvanishing skein lasagna module (at least when the support of $\mathcal{S}_0^2(W)$ is small in a suitable sense). Such tricks lead to many new exotic pairs. We write down some explicit examples in the following corollary to give the reader a flavor of what can be proven. For more choices of such candidates for connected summands, one may look into Section 3.5.

Corollary 3.1.3. *Let X_1, X_2 be as in Theorem 3.1.2.*

(1) The 4-manifolds $X_1^{\natural a}\natural X_2^{\natural b}$ and $X_1^{\natural a'}\natural X_2^{\natural b'}$ form an exotic pair if $a+b=a'+b'$ and $(a,b)\neq(a',b')$. The exotica remain under connected sums or boundary sums with any copies of $S^1\times S^3$, $S^2\times D^2$, $\overline{\mathbb{CP}^2}$, and the negative smooth $E8$ manifold. Moreover, the interiors of such pairs are also exotic.

(2) Whenever $a,b>0$, the first manifold in the pair in (1), namely $X_1^{\natural a}\natural X_2^{\natural b}$ possibly with extra connected or boundary connected summands as chosen above, admits a self-homeomorphism rel boundary that is not homotopic to a diffeomorphism.

Mukherjee [Muk23] informed us that most of these variations also appear to be achievable via gauge/Floer-theoretic methods and topological arguments. Below, we present a family of exotic knot traces generalizing Theorem 3.1.2 that does not seem to be directly recoverable by other methods, in particular because Rasmussen's s-invariant $s(K)$ appears in the hypothesis of the theorem. Let $P_{n,m}$ and $Q_{n,m}$ be satellite patterns drawn as in Figure 3.2, and $P_{n,m}(K)$, $Q_{n,m}(K)$ be the corresponding satellite knots with companion knot K. These satellite patterns were studied by Yasui [Yas15].

Theorem 3.1.4. *If a knot $K\subset S^3$ has a slice disk Σ in $k\mathbb{CP}^2\backslash int(B^4)$ and n,m are integers satisfying*

$$s(K)=|[\Sigma]|-[\Sigma]^2,\ n<-[\Sigma]^2,\ m\geq 0,$$

then $X_n(P_{n,m}(K))$ and $X_n(Q_{n,m}(K))$ form an exotic pair. Their interiors are also exotic.

In the special case when $K = U$ is the unknot, $k = 0$, Σ is the trivial slice disk of K in B^4, and $n = -1$, $m = 0$, Theorem 3.1.4 recovers Theorem 3.1.2. One can construct more exotic pairs in the spirit of Corollary 3.1.3 by taking connected sums of these examples and other standard manifolds appearing in that corollary. One may also deduce the existence of exotic pairs of compact contractible 4-manifolds (see Remark 3.5.10) from Theorem 3.1.4.

Now, we turn to some explicit computations of $\mathcal{S}_0^2$, $\mathcal{S}_0^{Lee}$, and other applications, some of which build towards a proof of Theorem 3.1.2 presented in Section 3.1.5. In the current section, many statements are presented in a specialized form. In such cases, we indicate the corresponding generalizations in parentheses after the statement numbers.

Convention. *Throughout this chapter, all 4-manifolds are assumed to be smooth, compact, and oriented. All surfaces are oriented and smoothly and properly embedded.*

3.1.1 A vanishing result and slice obstruction

For a knot $K \subset S^3$, let $TB(K)$ denote its maximal Thurston–Bennequin number. We have the following vanishing criterion for the Khovanov skein lasagna module.

Theorem 3.1.5. *If the knot trace $X_n(K)$ for some knot $K \subset S^3$ and framing $n \geq -TB(-K)$ embeds into a 4-manifold X, then $\mathcal{S}_0^2(X; L) = 0$ for any $L \subset \partial X$.*

For example, since $S^2 \times S^2$ contains an embedded sphere with self-intersection 2, we see the 2-trace of the unknot embeds into $S^2 \times S^2$, implying $\mathcal{S}_0^2(S^2 \times S^2) = 0$. This recovers one of the main results of [SZ24].

The folklore trace embedding lemma says that $X_n(K)$ embeds into X if and only if $-K$ is $(-n)$-slice in X, i.e. there is a framed slice disk in $X \backslash int(B^4)$ of the $(-n)$-framed knot $-K \subset S^3 = -\partial B^4$. Therefore, the contrapositive of Theorem 3.1.5 can be phrased as follows.

Corollary 3.1.6. *Suppose $\mathcal{S}_0^2(X; L) \neq 0$ for some $L \subset \partial X$. If a knot $K \subset S^3$ is n-slice in X, then $n > TB(K)$.* $\square$

3.1.2 Lee skein lasagna modules and lasagna s-invariants

We show in Section 3.3 that the structure of $\mathcal{S}_0^{Lee}(X; L)$, except for the quantum filtration, is determined by simple algebraic data of (X, L). Moreover, we extract numerical invariants from the quantum filtration structure. For the purpose of exposition, in the current section we assume L is the empty link. Let $q \colon \mathcal{S}_0^{Lee}(X) \to \mathbb{Z} \cup \{-\infty\}$ denote the quantum filtration function.

Theorem 3.1.7. *(Theorem 3.3.1) The Lee skein lasagna module of a 4-manifold X is $\mathcal{S}_0^{Lee}(X) \cong \mathbb{Q}[H_2(X)^2]$, with a basis consisting of canonical generators x_{α_+,α_-}, one for each pair $\alpha_+, \alpha_- \in H_2(X)$.*

Here, $\mathbb{Q}[S]$ denotes the $\mathbb{Q}$-vector space with a basis given by elements in a set S. The isomorphism $\mathcal{S}_0^{Lee}(X) \cong \mathbb{Q}[H_2(X)^2]$ is as $(\mathbb{Z} \times \mathbb{Z}/4 \times H_2^L(X))$-graded vector spaces (recall $H_2^L(X)$ is an $H_2(X)$-torsor, introduced in Section 2.3.4); see Theorem 3.3.1 for grading information.

Definition 3.1.8. (Definition 3.3.7) The *lasagna s-invariant* of X at a homology class $\alpha \in H_2(X)$ is

$$s(X; \alpha) := q(x_{\alpha,0}).$$

Remark 3.1.9. Morrison–Walker–Wedrich [MWW24] gave an independent proof of the structural theorem for Lee skein lasagna modules, stated in a more general form for deformed $\mathfrak{gl}_N$ homologies. They extracted slightly different invariants from the filtration structure, which gives somewhat less useful information than ours in the $\mathfrak{gl}_2$ case. Our invariants generalize Rasmussen's s-invariants for links and enjoy nice functorial properties.

The lasagna s-invariants of X thus define a smooth invariant $s(X; \cdot) : H_2(X) \to \mathbb{Z} \cup \{-\infty\}$.

The genus function of a 4-manifold X, defined by

$$g(X; \alpha) := \min\{g(\Sigma) \colon \Sigma \subset X, [\Sigma] = \alpha\},$$

captures subtle information about the smooth topology of X. Like the classical s-invariant, which provides a lower bound on the slice genus of links, the lasagna s-invariants provide a lower bound on the genus functions of 4-manifolds.

Theorem 3.1.10. *(Theorem 3.3.8(6) and subsequent corollaries) The genus function of a 4-manifold X has a lower bound given by*

$$g(X; \alpha) \geq \frac{s(X; \alpha) + \alpha^2}{2}.$$

Here, $\alpha^2 = \alpha \cdot \alpha$ is computed via the intersection form on $H_2(X)$.

Unlike the classical s-invariant, the lasagna s-invariants are not directly computable from link diagrams. However, we are able to compute them in some special cases. We indicate one special case where Theorem 3.1.10 does provide useful information.

The *n-shake genus* of a knot K is defined by $g_{sh}^n(K) := g(X_n(K); 1)$, where 1 is a generator of $H_2(X_n(K)) \cong \mathbb{Z}$. Clearly, $g_{sh}^n(K) \leq g_4(K)$, but the inequality need not be strict [Akb77; Pic19]. Theorem 3.1.10 specializes to a shake genus bound

$$g_{sh}^n(K) \geq \frac{s(X_n(K); 1) + n}{2}. \tag{3.1}$$

Theorem 3.1.11. *(1) If K is concordant to a positive knot, then for $n \leq 0$ we have $s(X_n(K);1) = s(K) - n$, where $s(K)$ is the Rasmussen s-invariant of K. Thus*

$$g_{sh}^n(K) = g_4(K) = \frac{s(K)}{2},\ n \leq 0.$$

(2) If $\Sigma \subset k\overline{\mathbb{CP}^2}\backslash(int(B^4) \sqcup int(B^4))$ is a concordance from K to a positive knot K', then for $n \leq [\Sigma]^2$ we have $s(X_n(K);1) \geq s(K') + [\Sigma]^2 + |[\Sigma]| - n$. Thus

$$g_{sh}^n(K) \geq \frac{s(K') + [\Sigma]^2 + |[\Sigma]|}{2},\ n \leq [\Sigma]^2.$$

Remark 3.1.12. Theorem 3.1.11(1) gives a refinement of Rasmussen [Ras10, Theorem 4] which was used to provide the first analysis-free proof of the Milnor conjecture about the slice genus of torus knots. The computation of the shake genus in Theorem 3.1.11(1) can also be carried out using Seiberg–Witten theory:

(i) The adjunction inequality for Stein manifolds [LM97] implies

$$g_{sh}^n(K) \geq (tb(\mathcal{K}) + |rot(\mathcal{K})| + 1)/2 \tag{3.2}$$

for any Legendrian representative $\mathcal{K}$ of K with $tb(\mathcal{K}) = n+1$. When $n < TB(K)$, take $\mathcal{K}$ to be a stabilization of a maximal-tb representative of K with $tb(\mathcal{K}) = n+1$ and $|rot(\mathcal{K})|$ maximized. Then the right-hand side of (3.2) agrees with $s(K)/2$ for positive knots [Tan99, Theorem 2]. For example, Figure 3.1 (left) shows a Legendrian representative of -5_2 with $tb = 0$ and $|rot| = 1$, which implies that $g_{sh}^{-1}(-5_2) \geq 1$. This was used in the reproof of Theorem 3.1.2 in [AM97]; see also Section 3.1.5.2.

(ii) Two concordant knots have the same shake genus (by an easy topological argument).

3.1.3 Comparison results for 2-handlebodies

The Khovanov homology of a link admits a spectral sequence to its Lee homology. In particular, in every bidegree, the rank of the Khovanov homology is bounded below by the dimension of the associated graded vector space of the Lee homology. Although we are not able to define a spectral sequence from $\mathcal{S}_0^2$ to $\mathcal{S}_0^{Lee}$, we do have a rank inequality when X is a 2-handlebody. Let $gr_\bullet V$ denote the associated graded vector space of a filtered vector space V.

Theorem 3.1.13. *If X is a 2-handlebody, then in any tri-degree $(h,q,\alpha) \in \mathbb{Z}^2 \times H_2^L(X)$,*

$$\operatorname{rank}(\mathcal{S}_{0,h,q}^2(X;L;\alpha)) \geq \dim(gr_q(\mathcal{S}_{0,h}^{Lee}(X;L;\alpha))).$$

In particular, if the quantum filtration on $\mathcal{S}_0^{Lee}(X;L)$ is not identically $-\infty$, then $\mathcal{S}_0^2(X;L) \neq 0$.

We state a comparison result for Lee skein lasagna modules of 2-handlebodies, which can sometimes be used as a nonvanishing criterion for Khovanov skein lasagna modules in view of Theorem 3.1.13; see e.g. Section 3.1.5.1.

Proposition 3.1.14. *(Proposition 3.4.3) Suppose X (resp. X') is a 2-handlebody obtained by attaching 2-handles to B^4 along a framed oriented link $K \subset S^3$ (resp. $K' \subset S^3$). If K is framed concordant to K', then $\mathcal{S}_0^{Lee}(X) \cong \mathcal{S}_0^{Lee}(X')$ as tri-graded, quantum filtered vector spaces. Under the identification $H_2(X) \cong H_2(X')$ induced by a link concordance, X and X' have equal lasagna s-invariants.*

3.1.4 A nonvanishing result for 2-handlebodies from link diagrams

Using an argument that relies on choosing a link diagram, in Section 3.4.4, we prove a nonvanishing result for 2-handlebodies with sufficiently negative framings. For ease of exposition, here we only state the case for knot traces. Let $n_-(K)$ be the minimal number of negative crossings among all diagrams of a knot K.

Theorem 3.1.15. *(Theorem 3.4.5) Let K be a knot and $X = X_n(K)$ be its n-trace.*

(1) If $n < -2n_-(K)$, then $s(X;0) = 0$ and $\mathcal{S}^2_{0,0,q}(X;0;\mathbb{Q}) = gr_q(\mathcal{S}^{Lee}_{0,0}(X;0)) = \begin{cases} \mathbb{Q}, & q = 0 \\ 0, & q \neq 0. \end{cases}$

(2) If K is positive, then for $n \le 0$, $s(X;0) = 0$, $s(X;1) = s(K) - n$. For $n < 0$,

$$\mathcal{S}^2_{0,0,q}(X;1;\mathbb{Q}) = gr_q(\mathcal{S}^{Lee}_{0,0}(X;1)) = \begin{cases} \mathbb{Q}, & q = s(X;1), s(X;1) - 2 \\ 0, & \textit{otherwise.} \end{cases}$$

3.1.5 Proofs of Theorem 3.1.2

We give two proofs that $int(X_1)$ and $int(X_2)$ are not diffeomorphic. The first proof distinguishes them on the level of Khovanov/Lee skein lasagna modules, while the second proof distinguishes their smooth genus functions. The following facts will be used:

- -5_2 is a positive knot, with $s(-5_2) = 2$.
- $P(3, -3, -8)$ is a slice knot.

3.1.5.1 First proof

Applying Theorem 3.1.15(2) to -5_2, we obtain $s(X_1;1) = 3$,

$$\mathcal{S}^2_{0,0,q}(X_1;1;\mathbb{Q}) = gr_q(\mathcal{S}^{Lee}_{0,0}(X_1;1)) = \begin{cases} \mathbb{Q}, & q = 1, 3 \\ 0, & \text{otherwise.} \end{cases}$$

Applying Theorem 3.1.15(2) to the unknot, and then Proposition 3.1.14 to $P(3,-3,-8)$ and the unknot, we obtain $s(X_2;1)=1$, and

$$\mathcal{S}^2_{0,0,q}(X_2;1;\mathbb{Q}) \supset gr_q(\mathcal{S}^{Lee}_{0,0}(X_2;1)) = \begin{cases} \mathbb{Q}, & q=\pm 1 \\ 0, & \text{otherwise.} \end{cases}$$

where the first containment follows from Theorem 3.1.13. In particular, $\mathcal{S}^2_{0,0,-1}(X_1;1;\mathbb{Q}) \neq \mathcal{S}^2_{0,0,-1}(X_2;1;\mathbb{Q})$, hence X_1 and X_2 are not diffeomorphic. In fact, since skein lasagna modules are smooth invariants of the interiors of 4-manifolds (cf. Corollary 2.3.6), $int(X_1)$ and $int(X_2)$ are not diffeomorphic. □

3.1.5.2 Second proof

Applying Theorem 3.1.11(1) to -5_2, we obtain $g_{sh}^{-1}(-5_2)=1$. On the other hand, capping off a slice disk of $P(3,-3,-8)$ by the core of the 2-handle in $X_{-1}(P(3,-3,-8))$ shows that $g_{sh}^{-1}(P(3,-3,-8))=0$. This proves $int(X_1)$ and $int(X_2)$ are non-diffeomorphic, without actually computing $\mathcal{S}_0^\bullet(X_i)$. □

3.1.6 Example: Khovanov and Lee skein lasagna module of $\overline{\mathbb{CP}^2}$

The results in Chapter 1 of this thesis have the following implications for the Lee and Khovanov skein lasagna modules of $\overline{\mathbb{CP}^2}$. Note the products α^2, β^2 below utilize the intersection form on $H_2(\overline{\mathbb{CP}^2})$.

Proposition 3.1.16. *The lasagna s-invariants of $\overline{\mathbb{CP}^2}$ are given by $s(\overline{\mathbb{CP}^2};\alpha)=|\alpha|$. The associated graded vector space of the Lee skein lasagna module of $\overline{\mathbb{CP}^2}$ is given by $gr_q\mathcal{S}^{Lee}_{0,(\beta^2-\alpha^2)/2}(\overline{\mathbb{CP}^2};\alpha)=\mathbb{Q}$ for*

$$\beta \equiv \alpha \pmod 2,\ q = \begin{cases} \frac{\alpha^2-\beta^2}{2}+|\beta|-1\pm 1, & \beta\neq 0 \\ \frac{\alpha^2}{2}, & \beta = 0, \end{cases}$$

and 0 in other tri-degrees. Moreover, for $\beta\equiv\alpha \pmod 2$,

$$\mathcal{S}^2_{0,(\beta^2-\alpha^2)/2,(\alpha^2-\beta^2)/2+|\beta|}(\overline{\mathbb{CP}^2};\alpha)=\mathbb{Z}.$$

In fact, in Conjecture 3.5.15, we formulate an expected complete determination of $\mathcal{S}^2_0(\overline{\mathbb{CP}^2};\mathbb{Q})$.

A consequence of Proposition 3.1.16 is the following.

Corollary 3.1.17. *If a 4-manifold X admits an embedding $i\colon X\hookrightarrow k\overline{\mathbb{CP}^2}$ for some k, then $s(X;\alpha)\geq |i_*\alpha|$, with equality achieved if $\alpha=0$. If in addition, X is a 2-handlebody, then $\mathcal{S}^2_{0,0,q}(X;\alpha)\neq 0$ for $q=s(X;\alpha)$, and for $q=s(X;\alpha)-2$ if $\alpha\neq 0\in H_2(X)$.*

If $\Sigma \subset k\mathbb{CP}^2 \backslash (int(B^4) \sqcup int(B^4))$ is an oriented cobordism in $k\mathbb{CP}^2$ between two oriented links L_0, L_1, the last part of Proposition 3.1.16 enables us to define an induced map

$$Kh(\Sigma)\colon Kh(L_0) \to Kh(L_1)$$

between ordinary Khovanov homology of L_0 and L_1, at least over a field and up to sign, with bidegree $(0, \chi(\Sigma) - [\Sigma]^2 + |[\Sigma]|)$. We carry out this construction in more detail in Section 3.5.11.

3.1.7 Organization of the chapter

In the short Section 3.2, we prove Theorem 3.1.5 by applying a bound on Khovanov homology by Ng [Ng05], and we give some examples of 4-manifolds with vanishing Khovanov skein lasagna module.

In Section 3.3, we develop the theory of Lee skein lasagna modules and lasagna s-invariants, which can be thought of as a lasagna generalization of the classical theories of Lee [Lee05] and Rasmussen [Ras10] for links in S^3. A generalization of Theorem 3.1.7, in particular the definition of canonical Lee lasagna generators, is presented in Section 3.3.1. The behavior of canonical Lee lasagna generators under morphisms is given in Theorem 3.3.6 in Section 3.3.2. Building on these results, in Section 3.3.3 we define and examine the lasagna s-invariants, generalizing Definition 3.1.8 and Theorem 3.1.10. We have also included a discussion about the coefficient field $\mathbb{Q}$ at the end of Section 3.3.

In Section 3.4, we make use of a 2-handlebody formula by Manolescu–Neithalath [MN22] to prove nonvanishing results for Khovanov and Lee skein lasagna modules of 2-handlebodies. In Section 3.4.1, we prove Theorem 3.1.13 which shows that finiteness results for the quantum filtration of Lee skein lasagna modules imply nonvanishing results for Khovanov skein lasagna modules. In Section 3.4.2 we prove a generalization of Proposition 3.1.14 and in Section 3.4.3 we show that lasagna s-invariants can be expressed in terms of classical s-invariants of some cable links. Finally, in the somewhat technical Section 3.4.4, we prove a vast generalization of the diagrammatic nonvanishing criterion Theorem 3.1.15.

Finally, in Section 3.5, we give examples and applications, mostly to the theory and tools developed in Section 3.3 and Section 3.4. This includes proofs of Corollary 3.1.3, Theorem 3.1.4, Theorem 3.1.11, and an expanded discussion of Section 3.1.6. Other discussions include a proof that s-invariants of cables of the Conway knot can obstruct its sliceness, that the Khovanov skein lasagna module over $\mathbb{Q}$ does not detect potential exotica arising from Gluck twists in S^4, and a conjecture about skein lasagna modules of Stein 2-handlebodies.

Most of our developments are independent of proving Theorem 3.1.2. Readers only interested in Theorem 3.1.2 may want to look at its proof in Section 3.1.5 and delve into the corresponding sections. In particular, to distinguish the smooth structures of the pair X_1, X_2 in Theorem 3.1.2, one can read certain statements in Section 3.3 (mainly Definition 3.3.7

and Corollary 3.3.14), Section 3.4.4, and Section 3.5.4. The minimal amount of theory needed to actually distinguish their Khovanov skein lasagna modules depends moreover on Section 3.4.1 and Section 3.4.2 (but not really on Corollary 3.3.14 and Section 3.5.4). Still, the statements in those sections are often much more general than needed for Theorem 3.1.2, which at times complicates the proofs.

3.2 Vanishing criterion and examples

In this section we prove Theorem 3.1.5 and give a few examples where the theorem applies. The proof is a straightforward application of Ng's maximal Thurston–Bennequin bound from Khovanov homology [Ng05] and the 2-handlebody formula (2.17).

Lemma 3.2.1 ([Ng05, Theorem 1]). *For an oriented link L, the Khovanov homology $Kh^{h,q}(L)$ is supported in the region*

$$TB(L) \leq q-h \leq -TB(-L).$$

Proof of Theorem 3.1.5. It suffices to prove the case $X = X_n(K)$, $n \geq -TB(-K)$, since the general case follows from it and Proposition 2.3.4 applied to $X_1 = X_n(K)$, $X_2 = X \backslash int(X_1)$.

We give K framing n and an arbitrary orientation. Any $L \subset \partial X$ can be isotoped to miss the 2-handle, hence we may henceforth assume L is a framed oriented link in S^3 missing K. Since $-n \leq TB(-K)$, we can find a Legendrian representative $\mathcal{K} \cup \mathcal{L}$ of $-K \cup -L$ with $\mathcal{K}$ having Thurston–Bennequin number $tb(\mathcal{K}) = -n$. In view of Proposition 2.6.8, we may assume without loss of generality that the framing of $-L$ induced from L agrees with the contact framing of $\mathcal{L}$.

Let $\mathcal{K}(k^+, k^-)$ denote the Legendrian $(k^+ + k^-)$-cable of $\mathcal{K}$ with the orientation on k^- of the strands reversed; in other words, $\mathcal{K}(k^+, k^-)$ is the link obtained by taking $k^+ + k^-$ pushoffs of $\mathcal{K}$ by the Reeb flow associated to a compatible contact form on S^3, and reversing the orientation on k^- of the components. The underlying link type of $\mathcal{K}(k^+, k^-) \cup \mathcal{L}$ is $-(K(k^+, k^-) \cup L)$, and the contact framing on the former agrees with the framing on the latter induced from K and L. We thus have

$$TB(-(K(k^+,k^-) \cup L)) \geq tb(\mathcal{K}(k^+,k^-) \cup \mathcal{L}) = w(-(K(k^+,k^-) \cup L)) = -w(K(k^+,k^-) \cup L). \tag{3.3}$$

By Lemma 3.2.1 and (3.3), $Kh(-(K(k^+,k^-) \cup L))$ is supported in $q-h \geq -w(K(k^+,k^-) \cup L)$. By our convention on KhR_2 (see (2.11)), $KhR_2(K(k^+,k^-) \cup L)$ is supported in $q+h \leq w(K(k^+,k^-) \cup L) - w(K(k^+,k^-) \cup L) = 0$. Now Proposition 2.6.5 implies that $\mathcal{S}_0^2(X; L; \alpha) = 0$ for any α, as $q^{-|\alpha|-2|r|} KhR_2(K(r+\alpha^+, r+\alpha^-) \cup L)$ is supported in $q+h \leq -|\alpha| - 2r \to -\infty$ as $r \to \infty$ yet the map ϕ in Proposition 2.6.5(ii) preserves the grading $q+h$. $\square$

Example 3.2.2. The unknot $K = U$ has maximal tb number -1. The n-trace on U is $D(n)$, the D^2-bundle over S^2 with Euler number n, which is also a tubular neighborhood of an embedded sphere in a 4-manifold with self-intersection n. Therefore in this case, Theorem 3.1.5 says $\mathcal{S}_0^2(X; L) = 0$ whenever X contains an embedded sphere with positive self-intersection. This is an adjunction-type obstruction for $\mathcal{S}_0^2$ to be nontrivial.

For example, $D(n)$ with $n > 0$, $\mathbb{CP}^2$, $S^2 \times S^2$, or the orientation-reversal of the $K3$ surface have vanishing Khovanov skein lasagna modules.

Example 3.2.3. The right-handed trefoil -3_1 has $TB(-3_1) = 1$. Therefore, the ±1-traces on 3_1 both have vanishing Khovanov skein lasagna modules. The boundaries of these 4-manifolds are the Poincaré homology sphere $\Sigma(2,3,5)$ for -1, and the orientation-reversal of the Brieskorn sphere $\Sigma(2,3,7)$ for $+1$.

One way to see embedded traces is through a Kirby diagram of the 4-manifold X. If there is an n-framed 2-handle attached along some knot K not going over 1-handles, then the 2-handle together with the 4-ball is a copy of $X_n(K)$ inside X. Consequently, if we see such a 2-handle with $n \geq -TB(-K)$, then $\mathcal{S}_0^2(X; L) = 0$ for any L.

Remark 3.2.4. As proved in [MN22, Theorem 1.3] (and will be reproved more generally in Section 3.5.9), $k\overline{\mathbb{CP}^2}$ has nonvanishing skein lasagna module. Therefore, we obtain a slice obstruction from Corollary 3.1.6, which states that if Σ is a slice disk in $k\overline{\mathbb{CP}^2}\backslash int(B^4)$ for a knot K, then

$$TB(K) + [\Sigma]^2 < 0.$$

In this special case, however, the obstruction is somewhat weaker than the obstruction coming from the adjunction inequality for Rasmussen s-invariant, Corollary 1.1.5, which in our case implies $TB(K) + [\Sigma]^2 + |[\Sigma]| < 0$ since $TB(K) < s(K)$ [Pla06, Proposition 4].

3.3 Lee skein lasagna modules and lasagna s-invariants

In this section, we explore the structure of the Lee skein lasagna module for an arbitrary pair (X, L) of a 4-manifold X and a framed oriented link $L \subset \partial X$. In particular, we prove the statements in Section 3.1.2 (except Theorem 3.1.11) in a more general setup.

Recall from Section 2.6.3 in Chapter 2 that the Lee skein lasagna module is defined to be the skein lasagna module with KhR_{Lee} as the link homology theory input. For a pair (X, L), its Lee skein lasagna module $\mathcal{S}_0^{Lee}(X; L)$ is a vector space over $\mathbb{Q}$ with a homological $\mathbb{Z}$-grading, a quantum $\mathbb{Z}/4$-grading, a homology class grading by $H_2^L(X)$, and an (increasing) quantum filtration (so that 0 has filtration degree $-\infty$). Since the filtration degree of an element can decrease under a filtered map, a nonzero element in the colimit $\mathcal{S}_0^{Lee}(X; L)$ may also have filtration degree $-\infty$. Let $q\colon \mathcal{S}_0^{Lee}(X;L) \to \mathbb{Z} \sqcup \{-\infty\}$ denote the quantum filtration function.

We refer the reader back to Section 2.6.1, especially to Proposition 2.6.1, for our convention on the Lee homology KhR_{Lee}.

3.3.1 Structure of Lee skein lasagna module

For a pair (X, L), let L_i be the components of L and $[L_i] \in H_1(L)$ be the fundamental class of L_i. Let $\partial\colon H_2(X, L) \to H_1(L)$ be the boundary homomorphism. Define the set of *double classes* in X rel L to be

$$H_2^{L,\times 2}(X) := \left\{ (\alpha_+, \alpha_-) \in H_2(X, L)^2 \colon \partial\alpha_\pm = \sum_i \epsilon_{i,\pm}[L_i],\ \epsilon_{i,\pm} \in \{0, 1\},\ \epsilon_{i,+} + \epsilon_{i,-} = 1 \right\}.$$

Double classes are precisely those pairs of second homology classes which can be represented by double skeins, to be defined below. First, though, we present the following generalization of Theorem 3.1.7, the proof of which will occupy the remainder of this section.

Theorem 3.3.1. *The Lee skein lasagna module of (X, L) is $\mathcal{S}_0^{Lee}(X; L) \cong \mathbb{Q}[H_2^{L,\times 2}(X)]$, with a basis consisting of canonical generators x_{α_+,α_-}, one for each double class $(\alpha_+, \alpha_-) \in H_2^{L,\times 2}(X)$. Moreover,*

(1) x_{α_+,α_-} has homological degree $-2\alpha_+ \cdot \alpha_-$;

(2) x_{α_+,α_-} has homology class degree $\alpha := \alpha_+ + \alpha_-$;

(3) $x_{\alpha_+,\alpha_-} \pm x_{\alpha_-,\alpha_+}$ has quantum $\mathbb{Z}/4$ degrees $-\alpha^2 - \#L - 1 \pm 1$;

(4) $q(x_{\alpha_+,\alpha_-}) = q(x_{\alpha_-,\alpha_+}) = \max(q(x_{\alpha_+,\alpha_-} \pm x_{\alpha_-,\alpha_+}))$, which also equals $\min(q(x_{\alpha_+,\alpha_-} \pm x_{\alpha_-,\alpha_+})) + 2$ if $\alpha_+ \neq \alpha_-$ in $H_2(X, L; \mathbb{Q})$ (in particular if $L \neq \emptyset$).

Here, the bilinear product on $H_2(X, L)$ is defined using the framing of L as the boundary condition. When $X = B^4$, the bilinear product computes the linking numbers of the boundary links and thus, in view of Proposition 2.3.8, Theorem 3.3.1 recovers classical structural results on $KhR_{Lee}(L)$ by Lee [Lee05, Proposition 4.3] and Rasmussen [Ras10, Proposition 2.3, 3.3, Lemma 3.5] [BW08, Section 6.1].

Before starting the proof, for ease of notation, instead of allowing all skeins in X rel L and morphisms between them, we will only allow the ones in the full subcategory $\mathcal{C}^\circ(X; L) \subset \mathcal{C}(X; L)$ of skeins Σ such that each component of Σ touches at least one input ball. One can redefine skein lasagna modules as a colimit indexed by the subcategory $\mathcal{C}^\circ(X; L)$; this is a definition equivalent to Definition 2.2.2 since one can show that $\mathcal{C}^\circ(X; L)$ is cofinal in

$\mathcal{C}(X;L)$ in the sense of [The25, Tag 04E6]. To distinguish the terminology, we call an object in $\mathcal{C}^\circ(X;L)$ a *punctured skein* in X rel L.

We define a *(punctured) double skein* in X rel L to be a punctured skein Σ together with a partition of its components $\Sigma = \Sigma_+ \cup \Sigma_-$. Equivalently, a double skein is a pair $(\Sigma, \mathfrak{O})$ where Σ is a punctured skein (which is already oriented) and $\mathfrak{O}$ is an orientation of Σ as an unoriented surface. The equivalence is given by defining Σ_+ to be the union of components of Σ on which $\mathfrak{O}$ matches the given orientation of Σ. A double skein $\Sigma = \Sigma_+ \cup \Sigma_-$ represents a double class $([\Sigma_+], [\Sigma_-]) \in H_2^{L,\times 2}(X)$.

Definition 3.3.2. Let $(\Sigma, \mathfrak{O})$ be a double skein in X rel L, with input balls B_i and input links K_i. The *canonical Lee lasagna filling* of $(\Sigma, \mathfrak{O})$ is the lasagna filling of (X, L) with skein Σ and decoration $2^{-n(\Sigma)}(-1)^{-n(\Sigma_-)} \otimes_i x_{\mathfrak{O}|_{K_i}}$, with a filtration degree shift by $-\chi(\Sigma)$ as in (2.16) (suppressed from the notation).

Recall the relevant definitions in Section 2.6.1. In particular, $n(\cdot)$ is defined as in (2.14), with Σ regarded as a cobordism from its input links to $L \subset \partial X$.

If Σ is a punctured skein in X rel L with input links K_i, then every element in $KhR_{Lee}(\Sigma)$ (recall the definition (2.16)) can be written as a linear combination of *pure Lee lasagna fillings*, defined as the ones of the form $\otimes_i x_{\mathfrak{o}_i}$, for some orientation $\mathfrak{o}_i$ of K_i as an unoriented link. A pure Lee lasagna filling with punctured skein Σ is *incompatible* if it is not a canonical Lee lasagna filling for some double skein structure on Σ.

Lemma 3.3.3. *An incompatible pure Lee lasagna filling defines the zero class in* $\mathcal{S}_0^{Lee}(X;L)$.

Proof. Let Σ be a punctured skein with input links K_i. If a pure Lee lasagna filling with skein Σ does not come from a double skein structure on Σ, then there are two different knot components K_{ij} and $K_{i'j'}$ among the input links whose orientations are not compatible on some component Σ_0 of Σ. Delete a small input ball on Σ_0 and put the decoration 1 on the resulting unknot $(U, \mathfrak{o})$ on its boundary; this does not affect the class in $\mathcal{S}_0^{Lee}(X;L)$. If we delete a tubular neighborhood of an arc on Σ_0 from U to K_{ij} and evaluate the lasagna filling, either the decoration $\mathbf{a}$ or $\mathbf{b}$ on U dies; the other decoration dies under such an evaluation connecting U to $K_{i'j'}$. This proves the lemma. □

If $\Sigma \to \Sigma'$ is a morphism in $\mathcal{C}^\circ(X;L)$, a double skein structure $\mathfrak{O}$ on Σ induces one on Σ' by restricting the orientation, and $([\Sigma_+], [\Sigma_-]) = ([\Sigma'_+], [\Sigma'_-]) \in H_2^{L,\times 2}(X)$.

Lemma 3.3.4. *If* $\Sigma = \Sigma_+ \cup \Sigma_-$ *is a double skein in* X *rel* L *and* $e\colon \Sigma \to \Sigma'$ *is a morphism in* $\mathcal{C}^\circ(X;L)$ *inducing the double skein structure* $\Sigma'_+ \cup \Sigma'_-$ *on* Σ'*, then*

$$KhR_{Lee}(e)(x(\Sigma_+, \Sigma_-)) = x(\Sigma'_+, \Sigma'_-) + \cdots$$

where the term $\cdots$ is a linear combination of incompatible pure Lee lasagna fillings. With the same notation, we have $KhR_{Lee}(e)(\cdots) = \cdots$.

Proof. This is a consequence of the behavior of Lee canonical generators under cobordisms described in (2.13). For example, for the first statement, we have

$$\begin{aligned} & KhR_{Lee}(e)(x(\Sigma_+, \Sigma_-)) \\ = & 2^{-n(\Sigma)}(-1)^{-n(\Sigma_-)}2^{n(S)}(-1)^{n(S_-)}2^{n(\Sigma')}(-1)^{n(\Sigma'_-)}x(\Sigma'_+, \Sigma'_-) + \cdots \\ = & x(\Sigma'_+, \Sigma'_-) + \cdots . \end{aligned}$$

□

Now we are ready to prove Theorem 3.3.1.

Proof of the isomorphism in Theorem 3.3.1. We first prove the statement concerning the isomorphism type of $\mathcal{S}_0^{Lee}(X; L)$, then the four addenda. We define an augmentation map $\varepsilon\colon \mathcal{S}_0^{Lee}(X; L) \to \mathbb{Q}[H_2^{L,\times 2}(X)]$. For any punctured skein Σ in X rel L, we define $\varepsilon\colon KhR_{Lee}(\Sigma) \to \mathbb{Q}[H_2^{L,\times 2}(X)]$ by mapping the class represented by a canonical Lee lasagna filling $x(\Sigma_+, \Sigma_-)$ to $e_{([\Sigma_+],[\Sigma_-])}$, the generator of the $([\Sigma_+], [\Sigma_-])$-th coordinate of the codomain, and an incompatible pure Lee lasagna filling to zero. Lemma 3.3.3 and Lemma 3.3.4 guarantee that this descends to a well-defined map ε on $\mathcal{S}_0^{Lee}(X; L)$.

We show ε is an isomorphism. Any pair $(\alpha_+, \alpha_-) \in H_2^{L,\times 2}(X)$ can be represented by properly immersed surfaces $\Sigma_\pm \subset X$ with $\partial\Sigma_\pm$ forming a partition of $L \subset X$ as an oriented unframed link. By transversality we may assume all singularities of $\Sigma = \Sigma_+ \cup \Sigma_-$ are transverse double points. Delete balls around these singularities to make Σ embedded in X with some balls deleted, and delete more local balls to make Σ a punctured skein, so we can put a framing on Σ compatibly with L. Now the class represented by the canonical Lee lasagna filling $x(\Sigma_+, \Sigma_-)$ is mapped to $e_{(\alpha_+,\alpha_-)}$, proving the surjectivity of ε.

We remark that the above proof for surjectivity is a replica of that of (2.7) about the bordism set of skeins, with minor changes. We do the same for injectivity but skip the details. It suffices to show any $x(\Sigma_+, \Sigma_-)$, $x(\Sigma'_+, \Sigma'_-)$ with $([\Sigma_+], [\Sigma_-]) = ([\Sigma'_+], [\Sigma'_-]) \in H_2^{L,\times 2}(X)$ represent the same element in $\mathcal{S}_0^{Lee}(X; L)$. By mimicking the proof of Theorem 2.3.9, we see that $\Sigma = \Sigma_+ \cup \Sigma_-$ and $\Sigma' = \Sigma'_+ \cup \Sigma'_-$ are related by morphisms in $\mathcal{C}^\circ(X; L)$ (or their reverses) compatible with the double skein structures. Namely, we can find a zigzag of morphisms in $\mathcal{C}^\circ(X; L)$

$$\Sigma = \Sigma_0 \leftarrow \Sigma_1 \rightarrow \Sigma_2 \leftarrow \cdots \rightarrow \Sigma_{2k} = \Sigma'$$

and double skein structures on each Σ_i compatible with Σ, Σ' and the morphisms. Taking the corresponding canonical Lee lasagna fillings and applying Lemma 3.3.4, we obtain a zigzag

$$x(\Sigma_+, \Sigma_-) = x(\Sigma_{0+}, \Sigma_{0-}) \leftarrow\!\shortmid x(\Sigma_{1+}, \Sigma_{1-}) \mapsto \cdots \mapsto x(\Sigma_{2k+}, \Sigma_{2k-}) = x(\Sigma'_+, \Sigma'_-)$$

up to lasagna fillings that are zero in $\mathcal{S}_0^{Lee}(X; L)$. This proves the injectivity of ε. □

Definition 3.3.5. For any $(\alpha_+, \alpha_-) \in H_2^{L,\times 2}(X)$, the *Lee canonical generator* x_{α_+,α_-} of $\mathcal{S}_0^{Lee}(X;L)$ is the class $[x(\Sigma_+,\Sigma_-)]$ for any double skein $\Sigma = \Sigma_+ \cup \Sigma_-$ representing (α_+,α_-).

Proof of the additional items in Theorem 3.3.1. Let (Σ_+,Σ_-) be a double skein as in Definition 3.3.5. (2) is clear since $[\Sigma] = [\Sigma_+] + [\Sigma_-] = \alpha_+ + \alpha_- = \alpha$. The proofs of (1) and (3) will make use of Proposition 2.6.1(3)(4) about gradings of Lee canonical generators in KhR_{Lee}.

Denote by K_i's the input links of Σ, which also admit decompositions $K_{i+}\cup K_{i-}$ induced from the decomposition $\Sigma_+ \cup \Sigma_-$. The Lee canonical generator $x_{\mathfrak{O}|_{K_i}}$ assigned to K_i in the filling $x(\Sigma_+,\Sigma_-)$ has homological degree $-2\ell k(K_{i+},K_{i-})$. This proves (1) since $\sum \ell k(K_{i+},K_{i-}) = [\Sigma_+]\cdot[\Sigma_-]$. To prove (3), note that the class x_{α_-,α_+} is represented by $x(\Sigma_-,\Sigma_+)$ associated to the reverse double skein $\Sigma = \Sigma_- \cup \Sigma_+$, which assigns canonical generators $x_{\overline{\mathfrak{O}|_{K_i}}}$ to each input link of Σ, which themselves are conjugates of $x_{\mathfrak{O}|_{K_i}}$ (meaning that they are equal in quantum $\mathbb{Z}/4$ degree $-w(K_i) - \#K_i$ and negatives of each other in degree $-w(K_i)-\#K_i-2$). Taking into account the extra renormalization factor $(-1)^{n(\cdot)}$ and the quantum shift $-\chi(\Sigma)$ as in Definition 3.3.2, we see that $x(\Sigma_+,\Sigma_-)$ and $x(\Sigma_-,\Sigma_+)$ agree in quantum $\mathbb{Z}/4$ degree

$$
\begin{aligned}
&\sum(-w(K_i) - \#K_i) - \chi(\Sigma) - 2(n(\Sigma_+) + n(\Sigma_-))\\
=&\,(-\alpha^2 - \#\partial_-\Sigma) - \chi(\Sigma) + \chi(\Sigma) - \#L + \#\partial_-\Sigma\\
=&\,-\alpha^2 - \#L,
\end{aligned}
$$

and are negatives of each other in quantum $\mathbb{Z}/4$ degree $-\alpha^2 - \#L - 2$, proving (3).

Finally we prove (4) which is more subtle. If $q(x_{\alpha_+,\alpha_-}) = q > -\infty$, we can find Lee lasagna fillings $(\Sigma^{(j)}, v^{(j)})$, $j = 1,\cdots,M$, where $\Sigma^{(j)}$ is a punctured skein with $[\Sigma^{(j)}] = \alpha_+ + \alpha_-$, $h(v^{(j)}) = h(x_{\alpha_+,\alpha_-}) = -2\alpha_+\cdot\alpha_-$ and $q(v^{(j)}) \le q$ for all j, so that $\sum_j [(\Sigma^{(j)}, v^{(j)})] = x_{\alpha_+,\alpha_-}$. Define a conjugation action ι on $\oplus_j KhR_{Lee}(\Sigma^{(j)})$ by 1 on the quantum $\mathbb{Z}/4$ degree $-\alpha^2 - \#L$ part and -1 on the other part.

Consider $v := \sum_j v^{(j)} = \sum_j \sum_{\mathfrak{o}} v_{\mathfrak{o}} = \sum_j \sum_{\mathfrak{o}} \lambda_{\mathfrak{o}}(\otimes_i x_{\mathfrak{o}_i}) \in \oplus_j KhR_{Lee}(\Sigma^{(j)})$, where in the j-th summand, $\mathfrak{o} = (\mathfrak{o}_i)$ runs over orientations of the input links $K_i^{(j)}$ of $\Sigma^{(j)}$. Each term $v_{\mathfrak{o}}$ represents $c(\mathfrak{o})x_{[\Sigma_+^{(j)}(\mathfrak{O})],[\Sigma_-^{(j)}(\mathfrak{O})]}$ in $\mathcal{S}_0^{Lee}(X;L)$ for some constant $c(\mathfrak{o})$ if $\mathfrak{o}$ comes from a double skein structure $\mathfrak{O}$ on $\Sigma^{(j)}$, and zero otherwise. In the first case, $[\iota v_{\mathfrak{o}}] = c(\mathfrak{o})x_{[\Sigma_-^{(j)}(\mathfrak{O})],[\Sigma_+^{(j)}(\mathfrak{O})]}$ by (3). In the second case $[\iota v_{\mathfrak{o}}] = 0$. Summing up, we have $[\iota v] = x_{\alpha_-,\alpha_+}$, thus $q(x_{\alpha_-,\alpha_+}) \le q(\iota v) = q(v) \le q(x_{\alpha_+,\alpha_-})$. By symmetry, $q(x_{\alpha_+,\alpha_-}) = q(x_{\alpha_-,\alpha_+})$. By the triangle inequality, this also equals $\max(q(x_{\alpha_+,\alpha_-} \pm x_{\alpha_-,\alpha_+}))$.

Finally, suppose $\alpha_+ \neq \alpha_-$ in $H_2(X,L;\mathbb{Q})$ and $q(x_{\alpha_+,\alpha_-}) = q > -\infty$. We prove $\min(q(x_{\alpha_+,\alpha_-} \pm x_{\alpha_-,\alpha_+})) = q-2$. Since $x_{\alpha_+,\alpha_-} \pm x_{\alpha_-,\alpha_+}$ have different $\mathbb{Z}/4$ grading, we know $\min(q(x_{\alpha_+,\alpha_-} \pm x_{\alpha_-,\alpha_+})) \le q-2$. We show the reverse inequality using an idea borrowed from the proof of Proposition 1.4.6 in Chapter 1.

Let $\pm = +$ or $-$ be the sign that minimizes $q(x_{\alpha_+,\alpha_-} \pm x_{\alpha_-,\alpha_+})$ (the minimum a priori could be $-\infty$), and let q_0 be any integer greater than or equal to this minimal value. Choose lasagna fillings $(\Sigma^{(j)}, v^{(j)})$, $j = 1, \cdots, M'$, the sum of which represents $x_{\alpha_+,\alpha_-} \pm x_{\alpha_-,\alpha_+}$, with each $\Sigma^{(j)}$ punctured, $[\Sigma^{(j)}] = \alpha_+ + \alpha_-$, $h(v^{(j)}) = -2\alpha_+ \cdot \alpha_-$ and $q(v^{(j)}) \le q_0$. Then, like before, we can rewrite

$$v := \sum_j v^{(j)} = \sum_j \sum_{\mathfrak{O}} \lambda_{\mathfrak{O}} x_{\mathfrak{O}} + \cdots , \tag{3.4}$$

where $\mathfrak{O}$ runs over orientations of $\Sigma^{(j)}$ with $[\Sigma_+^{(j)}(\mathfrak{O})] \cdot [\Sigma_-^{(j)}(\mathfrak{O})] = \alpha_+ \cdot \alpha_-$, $x_{\mathfrak{O}}$ is a shorthand for the canonical Lee lasagna filling $x(\Sigma_+^{(j)}(\mathfrak{O}), \Sigma_-^{(j)}(\mathfrak{O}))$, and $\cdots$ denotes a linear combination of incompatible pure Lee lasagna fillings.

Let $H_2^{L,\times 2}(X;\mathbb{Q})$ be the subset of $H_2(X;L;\mathbb{Q})^2$ defined analogously to $H_2^{L,\times 2}(X)$, but with $\mathbb{Q}$-coefficients, and let $\Delta\colon H_2^{L,\times 2}(X;\mathbb{Q}) \to H_2(X;L;\mathbb{Q})$ denote the map that takes the difference of the two coordinates. By assumption, $\Delta(\alpha_+,\alpha_-) \neq 0$, so we may pick a linear functional ℓ on $H_2(X;L;\mathbb{Q})$ with $\ell(\Delta(\alpha_+,\alpha_-)) = 1$.

For each $j = 1, \cdots, M'$, the set of double skein structures on $\Sigma^{(j)}$ can be identified with $\{\pm 1\}^{\pi_0(\Sigma^{(j)})}$ in the natural way, which comes with a map to $H_2^{L,\times 2}(X;\mathbb{Q})$ by taking the rational homology. The composition

$$\{\pm 1\}^{\pi_0(\Sigma^{(j)})} \to H_2^{L,\times 2}(X;\mathbb{Q}) \xrightarrow{\Delta} H_2(X;L;\mathbb{Q}) \xrightarrow{\ell} \mathbb{Q}$$

extends to a linear map on $\mathbb{Q}^{\pi_0(\Sigma^{(j)})}$, thus is defined by taking the inner product with some vector $(a_k^{(j)})_{k\in\pi_0(\Sigma^{(j)})} \in \mathbb{Q}^{\pi_0(\Sigma^{(j)})}$.

Define an operator $L^{(j)} = \sum_{k\in\pi_0(\Sigma^{(j)})} a_k^{(j)} X_k^{(j)}$ on $KhR_{Lee}(\Sigma^{(j)})$, where $X_k^{(j)}$ is a dot operation on a chosen interior boundary component of the k-component of $\Sigma^{(j)}$; as such, for any $\mathfrak{O} \in \{\pm 1\}^{\pi_0(\Sigma^{(j)})}$, $L^{(j)} x_{\mathfrak{O}} = \sum_{k\in\pi_0(\Sigma^{(j)})} a_k^{(j)} \mathfrak{O}_k x_{\mathfrak{O}} = \ell([\Sigma_+^{(j)}(\mathfrak{O})] - [\Sigma_-^{(j)}(\mathfrak{O})]) x_{\mathfrak{O}}$. Let L be the operator $\sum_j L^{(j)}$ on $\oplus_j KhR_{Lee}(\Sigma^{(j)})$. Then the sum of the lasagna fillings on $\Sigma^{(1)}, \cdots, \Sigma^{(M')}$ with decorations given by components of

$$Lv = \sum_j L^{(j)}\Big(\sum_{\mathfrak{O}} \lambda_{\mathfrak{O}} x_{\mathfrak{O}} + \cdots\Big) = \sum_{j,\mathfrak{O}} \ell([\Sigma_+^{(j)}(\mathfrak{O})] - [\Sigma_-^{(j)}(\mathfrak{O})]) \lambda_{\mathfrak{O}} x_{\mathfrak{O}} + \cdots$$

represents the class $x_{\alpha_+,\alpha_-} \mp x_{\alpha_-,\alpha_+}$ by the choice of ℓ.

Since the operator L has filtered degree 2, we conclude that

$$q = q(x_{\alpha_+,\alpha_-} \mp x_{\alpha_-,\alpha_+}) \le q_0 + 2.$$

Since $q_0 \ge q(x_{\alpha_+,\alpha_-} \pm x_{\alpha_-,\alpha_+})$ is arbitrary, we have $q(x_{\alpha_+,\alpha_-} \pm x_{\alpha_-,\alpha_+}) \ge q - 2$, as desired. $\square$

3.3.2 Canonical Lee lasagna generators under morphisms

The induced map on Lee homology of a link cobordism is determined by its assignment on Lee canonical generators, given by (2.13). In this section, we prove a lasagna generalization of this formula. In the most general form, the relevant cobordism map we examine is (2.20).

We recall the notation and explain the setup before stating the generalization. The 4-manifold $X = X_1 \cup_Y X_2$ is obtained by gluing X_1, X_2 along some common part $Y, -Y$ of their boundaries, where Y is a compact 3-manifold possibly with boundary. Two framed oriented tangles $T_1 \subset \partial X_1 \backslash int(Y)$ and $T_0 \subset Y$ glue to a framed oriented link $L_1 \subset \partial X_1$, and $S \subset X_2$ is a (not necessarily framed) surface with an oriented link boundary $\partial S = -T_0 \cup T_2 = L_2 \subset \partial X_2$ where T_2 is a tangle in $\partial X_2 \backslash int(-Y)$. Then $L = T_1 \cup T_2$ is a framed oriented link in ∂X. The gluing of (X_2, S) to (X_1, L_1) along (Y, T_0) induces the cobordism map

$$\mathcal{S}_0^{Lee}(X_2; S)\colon \mathcal{S}_0^{Lee}(X_1; L_1) \to \mathcal{S}_0^{Lee}(X; L) \tag{3.5}$$

as in (2.20), which has homological degree 0 and quantum filtration degree $-\chi(S) + \chi(T_0) - [S]^2$. We recall that by deleting local balls on S (one for each component) and putting a compatible framing, the punctured surface S° is a skein in X_2 rel L_2 with a natural lasagna filling by assigning 1 to all boundary unknots, and (3.5) is defined by putting the class of this lasagna filling in the second tensor summand of the more general gluing map (2.5).

A class $\alpha \in H_2^{L_1}(X_1)$ and a class $\beta \in H_2^{L_2}(X_2)$ glue to a class $\alpha * \beta \in H_2^{L}(X)$. Similarly, a double class $(\alpha_+, \alpha_-) \in H_2^{L_1,\times 2}(X_1)$ and a double class $(\beta_+, \beta_-) \in H_2^{L_2,\times 2}(X_2)$ glue to a double class $(\alpha_+, \alpha_-) * (\beta_+, \beta_-) = (\alpha_+ * \beta_+, \alpha_- * \beta_-) \in H_2^{L,\times 2}(X)$ if they are compatible, i.e. if $\partial\alpha_+, \partial\beta_+$ (and thus $\partial\alpha_-, \partial\beta_-$) agree on T_0, and do not glue if they are incompatible.

The punctured surface S° represents the class $[S] \in H_2^{L_2}(X_2)$. Thus the cobordism map (3.5) decomposes into direct sums of maps $\mathcal{S}_0^{Lee}(X_1; L_1; \alpha) \to \mathcal{S}_0^{Lee}(X; L; \alpha * [S])$.

Theorem 3.3.6. *The cobordism map* (3.5) *is determined by*

$$\mathcal{S}_0^{Lee}(X_2; S)(x_{\alpha_+,\alpha_-}) = 2^{m(S,L_1)} \sum_{\mathfrak{D}} (-1)^{m(S_-(\mathfrak{D}),L_{1-})} x_{\alpha_+ * [S_+(\mathfrak{D})], \alpha_- * [S_-(\mathfrak{D})]}. \tag{3.6}$$

Here the sum runs over double skein structures $\mathfrak{D}$ on S compatible with (α_+, α_-), $m(S, L_1)$ is an integral constant depending only on the topological type of the input pair, which equals $n(S)$ if $S \cap L_1$ is a link, thought of as the negative boundary of S, and L_{1-} is the sublink of L_1 with fundamental class $\partial\alpha_-$.

Proof. This is almost tautological from the construction. Let $\Sigma = \Sigma_+ \cup \Sigma_-$ be a double skein in X_1 rel L_1 representing (α_+, α_-). By Definition 3.3.5, x_{α_+,α_-} is represented by the canonical Lee lasagna filling $x(\Sigma_+, \Sigma_-)$ of the double skein Σ, as defined in Definition 3.3.2. After gluing (X_2, S°) to (X_1, Σ), we decompose each label 1 on the unknots on S into $\mathbf{a} + \mathbf{b}$. Then we obtain a linear combination of pure lasagna fillings with underlying punctured skein $\Sigma\cup S^\circ$, the class represented by which is a linear combination of canonical Lee lasagna generators over double skein structures $\mathfrak{D}$ on S compatible with (Σ_+,Σ_-), as in the summation of the theorem. Keeping track of the constants, we see that the theorem holds with

$$m(S,L_1)=n(\Sigma\cup S^\circ)-n(\Sigma)=n(\Sigma\cup S)-n(\Sigma)=\frac{-\chi(S)+\chi(\partial S\cap L_1)+\#(\partial S\Delta L_1)-\#L_1}{2}$$

which is an invariant of the pair (S,L_1), and which equals $n(S)$ if $S\cap L_1$ is a link by the additivity of $n(\cdot)$. □

3.3.3 Lasagna s-invariants and genus bounds

Let (X,L) be a pair as before. In view of Theorem 3.3.1, we make the following definition.

Definition 3.3.7. The *lasagna s-invariant* of (X,L) at a double class $(\alpha_+,\alpha_-)\in H_2^{L,\times 2}(X)$ is

$$s(X;L;\alpha_+,\alpha_-):=q(x_{\alpha_+,\alpha_-})-2\alpha_+\cdot\alpha_-\in\mathbb{Z}\sqcup\{-\infty\}.$$

The *lasagna s-invariant* of (X,L) at a class $\alpha\in H_2^L(X)$ is

$$s(X;L;\alpha):=s(X;L;\alpha,0)=q(x_{\alpha,0})\in\mathbb{Z}\sqcup\{-\infty\}.$$

When $L=\emptyset$, we often drop it from the notation of lasagna s-invariants.

When X is a 2-handlebody, the isomorphism (2.19) respects the canonical Lee lasagna generators in the sense that $x_{\beta_+,\beta_-}\in\mathcal{S}_0^{Lee}(X;(L,\partial\beta);\beta)$ is identified (up to renormalizations) with $x_{\alpha_+,\alpha_-}\in\mathcal{S}_0^{Lee}(X;L;\alpha)$ for the unique pair $(\alpha_+,\alpha_-)\in H_2^{L,\times 2}(X)$ with $\alpha_+-\alpha_-=\beta_+-\beta_-$. In particular,

$$s(X;L;\alpha_+,\alpha_-)=s(X;(L,\partial(\alpha_+-\alpha_-));\alpha_+-\alpha_-). \tag{3.7}$$

Thus, the double class version of lasagna s-invariants in Definition 3.3.7 can be recovered from the single class version. When X is not a 2-handlebody, however, it is not clear that (2.19) holds in a filtered sense for Lee skein lasagna modules, so we do not have this conclusion.

We prove a few properties of lasagna s-invariants.

Theorem 3.3.8. *The lasagna s-invariants satisfy the following properties.*

(0) (Empty manifold) $s(\emptyset;\emptyset;0)=0$.

(1) (Recover classical s) *If $*$ denotes the unique element in $H_2^L(B^4)$, then*

$$s(B^4;L;*)=s_{\mathfrak{gl}_2}(L)+1=-s(-L)-w(L)+1,$$

where s is the classical s-invariant defined by Rasmussen [Ras10] and Beliakova–Wehrli [BW08].

(2) (Symmetries) $s(X;L;\alpha_+,\alpha_-) = s(X;L;\alpha_-,\alpha_+) = s(X;L^r;-\alpha_+,-\alpha_-)$, *where L^r denotes the orientation reversal of L.*

(3) (Parity) $s(X;L;\alpha_+,\alpha_-) \equiv (\alpha_+ + \alpha_-)^2 + \#L \pmod 2$ *(by convention $-\infty$ has arbitrary parity).*

(4) (Connected sums) *If $(X, L = L_1 \sqcup L_2)$ is a boundary sum, a connected sum, or the disjoint union of (X_1, L_1) and (X_2, L_2), and $(\alpha_{i,+}, \alpha_{i,-}) \in H_2^{L_i,\times 2}(X_i)$, $i = 1, 2$, then*

$$s(X;L;\alpha_{1,+}+\alpha_{2,+},\alpha_{1,-}+\alpha_{2,-}) = s(X_1;L_1;\alpha_{1,+},\alpha_{1,-}) + s(X_2;L_2;\alpha_{2,+},\alpha_{2,-}).$$

(5) (Reduced connected sum) *If $(X, L = L_1\#_{K_1,K_2}L_2)$ is a (reduced) boundary sum of (X_1, L_1) and (X_2, L_2) performed along framed oriented intervals on components $K_i \subset L_i$, $i = 1, 2$, and $(\alpha_{i,+}, \alpha_{i,-}) \in H_2^{L_i,\times 2}(X_i)$ are double classes compatible with the boundary sum, then*

$$s(X;L_1\#_{K_1,K_2}L_2;\alpha_{1,+}*\alpha_{2,+},\alpha_{1,-}*\alpha_{2,-}) = s(X_1;L_1;\alpha_{1,+},\alpha_{1,-})+s(X_2;L_2;\alpha_{2,+},\alpha_{2,-})-1.$$

(6) (Genus bound) *If (X_2, L_2) is glued to (X_1, L_1) along some (Y, T_0) to obtain (X, L), and (unframed) $S \subset X_2$ is properly embedded with $\partial S = L_2$, as in the setup of (3.5), such that every component of S has a boundary on T_0, then for any compatible partition $S = S_+ \cup S_-$ and double class $(\alpha_+, \alpha_-) \in H_2^{L_1,\times 2}(X_1)$,*

$$s(X;L;\alpha_+ * [S_+],\alpha_- * [S_-]) \le s(X_1;L_1;\alpha_+,\alpha_-) - \chi(S) + \chi(T_0) - [S]^2.$$

The special case of Theorem 3.3.8(5) when $X_1 = X_2 = B^4$, in view of (1), recovers the connected sum formula

$$s(L_1\#_{K_1,K_2}L_2) = s(L_1) + s(L_2)$$

for the s-invariant of links in S^3, a fact that was proved only recently [Man+23, Theorem 7.1].

Since we performed gluing in its most general form, Theorem 3.3.8(6) has a rather complicated look. We state some special cases as corollaries, roughly in decreasing order of generality. Note that for 2-handlebodies, by (3.7), the double class versions of the genus bounds are equivalent to the single class versions.

Corollary 3.3.9. *Suppose $(X_1, L_1 \sqcup L_0)$, $(X_2, -L_0 \sqcup L_2)$ are two pairs that can be glued along some common boundary (Y, L_0) where Y is a closed 3-manifold (in particular $L_1, L_2 \not\subset Y$). Suppose (unframed) $S \subset X_2$ has $\partial S = -L_0 \sqcup L_2$ and each component of S has a boundary component on $-L_0$, then for any compatible partition $S = S_+ \cup S_-$ and double class $(\alpha_+, \alpha_-) \in H_2^{L_1\sqcup L_0,\times 2}(X_1)$,*

$$s(X_1 \cup_Y X_2; L_1 \sqcup L_2; \alpha_+ * [S_+], \alpha_- * [S_-]) \le s(X_1; L_1 \sqcup L_0; \alpha_+, \alpha_-) - \chi(S) - [S]^2.$$

In particular, for any class $\alpha \in H_2^{L_1\sqcup L_0}(X_1)$,

$$s(X_1 \cup_Y X_2; L_1 \sqcup L_2; \alpha * [S]) \le s(X_1; L_1 \sqcup L_0; \alpha) - \chi(S) - [S]^2.$$

Proof. This is the special case of Theorem 3.3.8(6) when Y is closed, so $T_0 = L_0$ with $\chi(T_0) = 0$. □

Remark 3.3.10. Although Theorem 3.3.8(6) is more general than Corollary 3.3.9, it is implied by the latter. This is because we can replace (X_2, S) by its union with a collar neighborhood of $(\partial X_1, L_1)$. Nevertheless, the proof of Theorem 3.3.8(6) is only notationally more complicated. The same remark applies to Theorem 3.3.6.

Corollary 3.3.11. *Suppose $(W, S)\colon (Y_1, L_1) \to (Y_2, L_2)$ is an unframed cobordism between framed oriented links in closed oriented 3-manifolds such that each component of S has a boundary on L_1, and X is a 4-manifold that bounds Y_1. Then for any compatible partition $S = S_+ \cup S_-$ and double class $(\alpha_+, \alpha_-) \in H_2^{L_1,\times 2}(X)$,*

$$s(X \cup_{Y_1} W; L_2; \alpha_+ * [S_+], \alpha_- * [S_-]) \leq s(X; L_1; \alpha_+, \alpha_-) - \chi(S) - [S]^2.$$

In particular, for any class $\alpha \in H_2^{L_1}(X)$,

$$s(X \cup_Y W; L_2; \alpha * [S]) \leq s(X; L_1; \alpha) - \chi(S) - [S]^2.$$

Proof. This is the special case of Corollary 3.3.9 when $\partial X_1 = Y$. □

In the special case of Corollary 3.3.11 when $S \subset I \times S^3$ is a link cobordism between links $L_1, L_2 \subset S^3$, each component of which has a boundary on L_1, and when $X = B^4$, in view of Proposition 2.3.8 and Theorem 3.3.8(1), we have $s(-L_1) \leq s(-L_2) - \chi(S)$. Of course we can reverse the orientations to obtain

$$s(L_1) \leq s(L_2) - \chi(S),$$

which recovers the classical genus bound for link cobordisms from s-invariants [BW08, (7)][1].

Corollary 3.3.12. *Suppose (X, L) is a pair and $S_+, S_- \subset X$ are disjoint properly embedded surfaces with $\partial S_+ \cup \partial S_- = L$, then*

$$2g(S_+) + 2g(S_-) \geq s(X; L; [S_+], [S_-]) + [S_+]^2 + [S_-]^2 - \#L.$$

In particular, if $S \subset X$ is properly embedded with $\partial S = L$, then

$$2g(S) \geq s(X; L; [S]) + [S]^2 - \#L.$$

Proof. Delete a local ball on each component of S_+ and S_-, regarding the rest of X and $S_\pm$ as a cobordism and applying Corollary 3.3.11, we get

$$s(X; L; [S_+], [S_-]) \leq s((B^4)^{\sqcup(\#S_+ + \#S_-)}; U^{\sqcup(\#S_+ + \#S_-)}; *) - \chi(S^\circ) - [S]^2,$$

where $s((B^4)^{\sqcup(\#S_+ + \#S_-)}; U^{\sqcup(\#S_+ + \#S_-)}; *) = (\#S_+ + \#S_-)s(B^4; U; *) = \#S_+ + \#S_-$ by Theorem 3.3.8(1)(4). The statement follows by simplification. □

[1] Note (7) in [BW08] incorrectly missed the condition about components of S having nonempty boundary on one end.

In the special case $X = B^4$ and $L = K \subset S^3$ is a knot, the single class version of Corollary 3.3.12 recovers Rasmussen's slice genus bound for knots [Ras10, Theorem 1].

Corollary 3.3.13. *Suppose X is a 4-manifold, and $S_+, S_- \subset X$ are disjoint closed embedded surfaces, then*
$$2g(S_+) + 2g(S_-) \geq s(X; [S_+], [S_-]) + [S_+]^2 + [S_-]^2.$$
In particular, if $S \subset X$ is an embedded surface, then
$$2g(S) \geq s(X; [S]) + [S]^2.$$

Proof. This is the special case of Corollary 3.3.12 when $L = \emptyset$. □

The *genus function* of a 4-manifold X, $g(X; \cdot) \colon H_2(X) \to \mathbb{Z}_{\geq 0}$, is defined by
$$g(X; \alpha) = \min\{g(\Sigma) \colon \Sigma \subset X \text{ is a closed embedded surface with } [\Sigma] = \alpha\}.$$

Corollary 3.3.14. *(Theorem 3.1.10) The genus function of a 4-manifold X has a lower bound given by*
$$g(X; \alpha) \geq \frac{s(X; \alpha) + \alpha^2}{2}.$$

Proof. This is a reformulation of the second part of Corollary 3.3.13. □

Lasagna s-invariants obstruct embeddings between 4-manifolds in the following sense.

Corollary 3.3.15. *If $i \colon X \hookrightarrow X'$ is an inclusion between two 4-manifolds, then for any class $\alpha \in H_2(X)$,*
$$s(X'; i_*\alpha) \leq s(X; \alpha).$$

Proof. This is the special case of the single class version of Corollary 3.3.11 when $L_1 = L_2 = S = \emptyset$. □

Corollary 3.3.16. *For any compact oriented 4-manifold X, we have $s(X; 0) \leq 0$.*

Proof. This is a consequence of Corollary 3.3.15 and Theorem 3.3.8(0). □

We return to Theorem 3.3.8. It is an easy consequence of the hard work that we have done in previous sections.

Proof of Theorem 3.3.8. (0) $x_{0,0} \in \mathcal{S}_0^{Lee}(\emptyset)$ is represented by the Lee lasagna filling $(\emptyset, 1)$ on the empty skein, which has filtration degree 0.

(1) $x_{*,0} \in \mathcal{S}_0^{Lee}(B^4; L)$ is represented by the Lee lasagna filling on the skein $I \times L \subset I \times S^3 = B^4 \backslash int(\frac{1}{2}B^4)$ with decoration $x_{\mathfrak{o}_L}$ on the input link L. Under the identification $\mathcal{S}_0^{Lee}(B^4; L) \cong KhR_{Lee}(L)$ given by Proposition 2.3.8, $x_{*,0}$ is equal to the Lee canonical generator $x_{\mathfrak{o}_L}$, which has filtration degree $s_{\mathfrak{gl}_2}(L) + 1 = -s(-L) - w(L) + 1$ by (2.15).

(2) The first equality is a consequence of Theorem 3.3.1(4). Reversing the orientation of all skeins defines an isomorphism $\mathcal{S}_0^{Lee}(X; L) \cong \mathcal{S}_0^{Lee}(X; L^r)$ which negates the homology class grading but preserves other gradings and the quantum filtration, and exchanges the canonical Lee lasagna generators. This proves the second equality.

(3) This is a consequence of Theorem 3.3.1(3).

(4) This is a consequence of Proposition 2.3.2 and the fact that the canonical Lee lasagna generators behave tensorially under the various sum operations (as can be seen on the canonical Lee lasagna filling level).

(6) This is a consequence of Theorem 3.3.6. Under the condition on the components of S and the compatibility assumption, the right-hand side of (3.6) has exactly one term, which has $S_\pm(\mathfrak{O}) = S_\pm$. The statement follows from the fact that $\mathcal{S}_0^{Lee}(X_2; S)$ has filtered degree $-\chi(S) + \chi(T_0) - [S]^2$ (cf. the comment after (3.5)).

(5) There is a framed saddle cobordism $(I \times \partial X, S)$ from $(\partial X, L = L_1 \#_{K_1,K_2} L_2)$ to $(\partial X, L_1 \sqcup L_2)$, where ∂X is bounded by $X = X_1 \natural X_2$. Thus Corollary 3.3.11 (which is a corollary of (6)) and (4) imply

$$s(X; L; \alpha_{1,+} * \alpha_{2,+}, \alpha_{1,-} * \alpha_{2,-}) \geq s(X_1; L_1; \alpha_{1,+}, \alpha_{1,-}) + s(X_2; L_2; \alpha_{2,+}, \alpha_{2,-}) - 1.$$

To prove the reverse inequality, by Theorem 3.3.1(4), we can choose signs $\epsilon_i = \pm 1$, $i = 1, 2$, such that

$$q(x_{\alpha_{i,+},\alpha_{i,-}} + \epsilon_i x_{\alpha_{i,-},\alpha_{i,+}}) = s(X_i; L_i; \alpha_{i,+}, \alpha_{i,-}) - 2,\ i = 1, 2.$$

By Theorem 3.3.6, the reverse S^r of S induces $\mathcal{S}_0^{Lee}(I \times \partial X; S^r) \colon \mathcal{S}_0^{Lee}(X; L_1 \sqcup L_2) \to \mathcal{S}_0^{Lee}(X; L)$ that maps $(x_{\alpha_{1,+},\alpha_{1,-}} + \epsilon_1 x_{\alpha_{1,-},\alpha_{1,+}}) \otimes (x_{\alpha_{2,+},\alpha_{2,-}} + \epsilon_2 x_{\alpha_{2,-},\alpha_{2,+}})$ to a nonzero multiple of $x_{\alpha_{1,+}*\alpha_{2,+},\alpha_{1,-}*\alpha_{2,-}} \pm x_{\alpha_{1,-}*\alpha_{2,-},\alpha_{1,+}*\alpha_{2,+}}$. Since $\mathcal{S}_0^{Lee}(I \times \partial X; S^r)$ has filtered degree 1, it follows that

$$\begin{aligned}
&s(X; L; \alpha_{1,+} * \alpha_{2,+}, \alpha_{1,-} * \alpha_{2,-}) \\
\leq\,& q(x_{\alpha_{1,+}*\alpha_{2,+},\alpha_{1,-}*\alpha_{2,-}} \pm x_{\alpha_{1,-}*\alpha_{2,-},\alpha_{1,+}*\alpha_{2,+}}) + 2 \\
\leq\,& q(x_{\alpha_{1,+},\alpha_{1,-}} + \epsilon_1 x_{\alpha_{1,-},\alpha_{1,+}}) + q(x_{\alpha_{2,+},\alpha_{2,-}} + \epsilon_2 x_{\alpha_{2,-},\alpha_{2,+}}) + 3 \\
=\,& s(X_1; L_1; \alpha_{1,+}, \alpha_{1,-}) + s(X_2; L_2; \alpha_{2,+}, \alpha_{2,-}) - 1.
\end{aligned}$$

□

3.4 Nonvanishing criteria for 2-handlebodies

The Lee skein lasagna module of a pair (X, L) is never 0 by Theorem 3.3.1. However, in our setup, it is appropriate to set the following convention.

Convention. *The Lee skein lasagna module $\mathcal{S}_0^{Lee}(X; L)$ is* vanishing *if the filtration function on it is identically $-\infty$; equivalently, it is vanishing if the lasagna s-invariants of (X, L) are all $-\infty$. It is* nonvanishing *if it is not vanishing.*

When X is a 2-handlebody, the Khovanov or Lee skein lasagna module of a pair (X, L) has a nice formula (2.17). From now on, we will also mostly restrict to characteristic zero, so the 2-handlebody formula takes the form (2.18) in terms of a colimit along symmetrized dotted annulus creation maps between symmetrized cables. In this section, we make use of this 2-handlebody formula to provide some explicit criteria for $\mathcal{S}_0^2(X; L)$ or $\mathcal{S}_0^{Lee}(X; L)$ to be nonvanishing. By our discussions in Section 3.2 and Section 3.3, this leads to smooth genus bounds for second homology classes of 4-manifolds. This section is an expansion of Section 3.1.3 and Section 3.1.4.

Throughout the section, we assume that X is the 2-handlebody obtained by attaching 2-handles to B^4 along a framed link $K = K_1 \cup \cdots \cup K_m \subset S^3$, and $L \subset \partial X$ is given by a framed oriented link in S^3 disjoint from K, as in the setup of (2.17) and (2.18).

3.4.1 Rank inequality between Khovanov and Lee skein lasagna modules

In this section, we prove Theorem 3.1.13, which states that the rank of $\mathcal{S}_0^2(X; L)$ is bounded below by the dimension of $gr(\mathcal{S}_0^{Lee}(X; L))$ in every tri-degree (h, q, α). In particular, if $\mathcal{S}_0^{Lee}(X; L)$ is nonvanishing, so is $\mathcal{S}_0^2(X; L)$. By Theorem 3.1.5, this has the following consequences.

Corollary 3.4.1. *If a knot trace $X_n(K)$ embeds into a 4-manifold X for some knot K and framing $n \geq -TB(-K)$, then $\mathcal{S}_0^{Lee}(X; L)$ is vanishing for any $L \subset \partial X$. In particular, the lasagna s-invariants of (X, L) are all $-\infty$.*

Proof. This is a consequence of Theorem 3.1.5 and Theorem 3.1.13 when $X = X_n(K)$. The general case follows from this and Proposition 2.3.4. □

Recall that a knot $K \subset S^3$ is n-slice in a 4-manifold X if there exists a framed slice disk in $X \backslash int(B^4)$ of the n-framed knot $K \subset S^3 = -\partial B^4$.

Corollary 3.4.2. *Suppose some class in $\mathcal{S}_0^{Lee}(X; L)$ has finite filtration degree. If $K \subset S^3$ is n-slice in X, then $n > TB(K)$.*

Proof. The folklore trace embedding lemma states that K is n-slice in X if and only if $X_{-n}(-K)$ embeds in X. Therefore, the statement is the contrapositive of the first part of Corollary 3.4.1 for $(-K,-n)$ in place of (K,n). □

We turn to Theorem 3.1.13. We will only use the 2-handlebody condition in a mild way, namely that the 2-handlebody formula (2.18) expresses $\mathcal{S}_0^\bullet(X;L)$ as a filtered colimit instead of an arbitrary colimit (2.2).

Proof of Theorem 3.1.13. As shorthands, write $S(r) := S_{|\alpha_1|+2r_1}\times\cdots\times S_{|\alpha_m|+2r_m}$, $K(r) := K(\alpha^+ + r, \alpha^- + r)\cup L$. Then (2.18) for $\bullet = Lee$ is rewritten as

$$\mathcal{S}_0^{Lee}(X;L;\alpha) \cong \operatorname{colim}_{r\in\mathbb{Z}^m_{\geq 0}} q^{-|\alpha|-2|r|} KhR_{Lee}(K(r))^{S(r)}, \tag{3.8}$$

as h-graded and q-filtered vector spaces. Similarly for $\mathcal{S}_0^2(X;L;\alpha;\mathbb{Q})$.

The expression (3.8) can be rewritten as a filtered colimit with terms given by $KhR_{Lee}(K(r))$. More explicitly, consider the filtered index category $\mathcal{I}$ with object set given by $\mathbb{Z}^m_{\geq 0}$ and with $\operatorname{Hom}(r,r') = \{s\in(\mathbb{Z}_{\geq 0}\sqcup\{-\infty\})^m\colon r_i\leq r_i' \text{ and } (r_i\leq s_i\leq r_i' \text{ or } s_i=-\infty),\ \forall i\}$. Identity morphisms are $\mathrm{id}_r = (-\infty,\cdots,-\infty)$, and compositions are defined by $s'\circ s = \max(s,s')$ taken componentwise. Then, $\mathcal{S}_0^{Lee}(X;L;\alpha)$ is the colimit of the functor

$$F_{KhR_{Lee}} := q^{-|\alpha|-2|\bullet|}KhR_{Lee}(K(\bullet))\colon \mathcal{I}\to \mathrm{Filt}(\mathbf{Vect}_{\mathbb{Q}})^{\mathbb{Z}},$$

where a morphism $s\colon r\to r'$ is sent to the map induced by a sequence of dotted annulus cobordisms from $K(r)$ to $K(r')$ post-composed with symmetrization by $S(s)$ on the sublink $K(s)\subset K(r')$, where $-\infty$ components of s are understood to contribute no cable component in $K(s)$. Analogously, $\mathcal{S}_0^2(X;L;\alpha;\mathbb{Q})$ is the colimit of

$$F_{KhR_2} := q^{-|\alpha|-2|\bullet|}KhR_2(K(\bullet);\mathbb{Q})\colon \mathcal{I}\to\mathbf{Vect}_{\mathbb{Q}}^{\mathbb{Z}^2}.$$

For every morphism $s\in\mathcal{I}$, there is a formal linear combination of dotted cobordism maps that induces both $F_{KhR_{Lee}}(s)$ and $F_{KhR_2}(s)$. If $\{(E_r,d_r)\}_{r\geq 1}$ denotes the Lee spectral sequence with $E_1 = KhR_2$ that converges to KhR_{Lee}, we can use the same cobordism maps to define functors $F_{E_r}\colon\mathcal{I}\to\mathbf{Vect}_{\mathbb{Q}}^{\mathbb{Z}^2}$ for $r\in\mathbb{Z}_{>0}\cup\{\infty\}$. By naturality of the Lee spectral sequence, each d_r for $r<\infty$ defines a natural transformation from F_{E_r} to itself, which induces a differential $\operatorname{colim} d_r$ on $\operatorname{colim} F_{E_r}$. Since filtered colimits are exact in vector spaces, objectwise convergence gives

$$\operatorname{colim} F_{E_\infty} \cong \operatorname{colim}(gr\circ F_{KhR_{Lee}}) \cong gr(\operatorname{colim} F_{KhR_{Lee}})/gr_{-\infty}(\operatorname{colim} F_{KhR_{Lee}}).$$

Moreover, for every $r<\infty$,

$$\operatorname{colim} F_{E_{r+1}} \cong H_*(\operatorname{colim} F_{E_r}, \operatorname{colim} d_r).$$

Since we do not have a priori uniform control on convergence, the spectral sequence $\{(\operatorname{colim} E_r, \operatorname{colim} d_r)\}_{r\geq 1}$ need not converge to $\operatorname{colim} E_\infty$. Nevertheless, E_∞ is a functorial subquotient of each E_r, a property preserved under filtered colimit, so $\operatorname{colim} E_\infty$ is a subquotient of every $\operatorname{colim} E_r$. Taking $r=1$ recovers the theorem. □

3.4.2 Comparison results for Lee skein lasagna modules

In this section, we prove two sample comparison results for Lee skein lasagna modules of 2-handlebodies, as an easy consequence of the structure theorem of Lee skein lasagna modules (Theorem 3.3.1) and the behavior of the canonical generators under gluing (Theorem 3.3.6).

Recall that (X, L) is the pair that arises from $K \cup L \subset S^3$ after attaching 2-handles along K. Let (X', L') be another such pair from $K' \cup L'$. Suppose there is a concordance $C\colon K \to K'$, that is, a genus 0 component-preserving framed link cobordism in $I \times S^3$ (thus topologically C is a disjoint union of annuli). Then the surgery on C gives a homology cobordism W between the surgery on K and the surgery on K'. In particular, W (and thus C) induces an isomorphism $\varphi_C\colon H_*(\partial X) \cong H_*(\partial X')$. Moreover, $X' = X \cup_{\partial X} W$. If $A \subset W$ is a concordance between L and L', then (W, A) induces an isomorphism $H_2(X, L) \cong H_2(X', L')$ that restricts to $\phi_A\colon H_2^L(X) \cong H_2^{L'}(X')$.

Proposition 3.4.3. *Let (X, L) and (X', L') be obtained from $K \cup L, K' \cup L' \subset S^3$ as above, and let $C\colon K \to K'$ be a concordance that induces a homology cobordism $W\colon \partial X \to \partial X'$.*

(1) If $A \subset W$ is a framed concordance between L, L', then $\mathcal{S}_0^{Lee}(X; L) \cong \mathcal{S}_0^{Lee}(X'; L')$ as tri-graded (by $\mathbb{Z} \times \mathbb{Z}/4 \times H_2^L(X)$, where $H_2^L(X) \cong H_2^{L'}(X')$ via ϕ_A), quantum filtered vector spaces. Moreover, $s(X; L; \alpha) = s(X'; L'; \phi_A(\alpha))$ for all $\alpha \in H_2^L(X)$.

(2) If L has components $L_1, \cdots, L_k$, L' has components $L_1', \cdots, L_k'$, and $\varphi_C([L_i]) = [L_i']$ for all i, then $\mathcal{S}_0^{Lee}(X; L)$ is nonvanishing if and only if $\mathcal{S}_0^{Lee}(X'; L')$ is nonvanishing.

Proof. (1) Gluing (W, A) to (X, L) defines a filtered (degree 0) map $\mathcal{S}_0^{Lee}(W; A)\colon \mathcal{S}_0^{Lee}(X; L) \to \mathcal{S}_0^{Lee}(X'; L')$. Turning (W, A) upside down gives another filtered map $\mathcal{S}_0^{Lee}(X'; L') \to \mathcal{S}_0^{Lee}(X; L)$. By Theorem 3.3.6, the two maps interchange the corresponding canonical Lee lasagna generators, and thus are inverses to each other. The statement follows.

(2) By symmetry, it suffices to assume the filtration of $\mathcal{S}_0^{Lee}(X; L)$ is identically $-\infty$ and prove the same for $\mathcal{S}_0^{Lee}(X'; L')$.
Under the given condition, we can find immersed cobordisms $S_i\colon L_i \to L_i'$ in W that intersect each other transversely. After deleting some local balls, we can regard $S = S_1 \cup \cdots \cup S_k$ as a punctured skein in W rel $-L \sqcup L'$ and put on it a compatible framing. For any double class $(\alpha_+, \alpha_-) \in H_2^{L,\times 2}(X)$, there is a unique compatible double skein structure $S = S_+ \cup S_-$. Gluing (W, S) with the canonical Lee lasagna filling $x(S_+, S_-)$ to (X, L) maps $x_{\alpha_+,\alpha_-} \in \mathcal{S}_0^{Lee}(X; L)$ to a nonzero multiple of $x_{\alpha_+ * [S_+], \alpha_- * [S_-]} \in \mathcal{S}_0^{Lee}(X'; L')$. Since $q(x_{\alpha_+,\alpha_-}) = -\infty$, we see $q(x_{\alpha_+ * [S_+], \alpha_- * [S_-]}) = -\infty$ as well. Since all canonical Lee lasagna generators of (X', L') arise this way, we conclude that the filtration function on $\mathcal{S}_0^{Lee}(X'; L')$ is identically $-\infty$. ∎

3.4.3 Lasagna s-invariants and s-invariants of cables

In this section we show that the lasagna s-invariants of (X, L) are determined by the classical s-invariants of cables of $K \cup L$. Recall that as in the setup of (2.17) and (2.18), the choice $K \cup L \subset S^3$ gives an identification $H_2^L(X) \cong \mathbb{Z}^m$.

Proposition 3.4.4. *For any $\alpha \in H_2^L(X) \cong \mathbb{Z}^m$,*

$$\begin{aligned} s(X; L; \alpha) &= \lim_{r \to \infty^m} (s_{\mathfrak{gl}_2}(K(\alpha^+ + r, \alpha^- + r) \cup L) - 2|r|) - |\alpha| + 1 \\ &= - \lim_{r \to \infty^m} (s(-(K(\alpha^+ + r, \alpha^- + r) \cup L)) + 2|r|) - w(K(\alpha^+, \alpha^-) \cup L) - |\alpha| + 1. \end{aligned}$$

Moreover, the quantity within the limit sign in the first (resp. second) line is nonincreasing (resp. nondecreasing) in r.

Proof. The $\mathfrak{gl}_2$ s-invariant relates to the classical s-invariant via (2.15). Since $w(K(\alpha^+ + r, \alpha^- + r) \cup L)$ is independent of r, the claims on the second row follow from the first. The existence of the annulus creation cobordism from $K(\alpha^+ + r, \alpha^- + r) \cup L$ to $K(\alpha^+ + r + e_i, \alpha^- + r + e_i) \cup L$ implies that the quantity within the limit sign in the first row is nonincreasing in r. In particular, the limit exists (which could be $-\infty$). Thus, it remains to prove the first equality.

We describe explicitly the isomorphism (2.17). Delete a 4-ball B^4 slightly smaller than the 0-handle of X and put $K(r) := K(\alpha^+ + r, \alpha^- + r) \cup L$ on its boundary. Cap off the sublink $K(\alpha^+ + r, \alpha^- + r)$ by some parallel copies of cores of the 2-handles of various orientations corresponding to $(\alpha^+ + r, \alpha^- + r)$, and put a cylindrical surface $I \times L$ connecting the inner link $L \subset \partial B^4$ and the outer link $L \subset \partial X$. This gives a punctured skein in (X, L) with input link $K(r)$. Putting decorations on $K(r)$ defines $KhR_{Lee}(K(r)) \to \mathcal{S}_0^{Lee}(X; L)$. Taking the direct sum over r defines a map that descends to the isomorphism (2.17). It follows that the canonical Lee lasagna filling on this punctured skein as a double skein (double skein structure given by its own orientation) is given up to a scalar by the Lee canonical generator of $K(r)$ (with its own orientation) decorated on the input ball.

As in the proof of Theorem 3.1.13, (2.18) for $\bullet = Lee$ is rewritten in our shorthand notation as (3.8). For an orientation $\mathfrak{o}$ of $K(r)$, let $k_{+,i}(\mathfrak{o})$ denote the number of strands in $K(r)$ parallel to K_i on which the orientation $\mathfrak{o}$ agrees with the one induced from K_i, $i = 1, \cdots, m$. We will be concerned with orientations $\mathfrak{o}$ of $K(r)$ that are compatible with the orientation of L, with $k_+(\mathfrak{o}) = \alpha^+ + r$ (componentwise). Let $\mathfrak{O}(r)$ denote the set of such orientations, and let $KhR_{Lee,\mathfrak{O}(r)}(K(r))$ denote the subspace of $KhR_{Lee}(K(r))$ spanned by Lee canonical generators associated to such orientations. By the description of the $S(r)$-action on the canonical generators, we know $S(r)$ acts on $KhR_{Lee,\mathfrak{O}(r)}(K(r))$, with a 1-dimensional invariant subspace $KhR_{Lee,\mathfrak{O}(r)}(K(r))^{S(r)}$ spanned by the element $x^{inv}(r) := c_r \sum_{\mathfrak{o} \in \mathfrak{O}(r)} x_{\mathfrak{o}}$ shifted by $q^{-|\alpha|-2|r|}$, where the constant $c_r \neq 0$ is chosen so that $x^{inv}(r)$ represents the canonical Lee lasagna generator $x_{\alpha,0} \in \mathcal{S}_0^{Lee}(X; L; \alpha)$ under (3.8). In fact, $x^{inv}(r)$ is the unique element in

$KhR_{Lee}(K(r))^{S(r)}$ that represents $x_{\alpha,0}$ under (3.8). Define $s_{\mathfrak{gl}_2}^{inv}(K(r)) := q(x^{inv}(r))$. Since (3.8) holds as filtered vector spaces, we have

$$s(X;L;\alpha) = \lim_{r\to\infty^m}(s_{\mathfrak{gl}_2}^{inv}(K(r)) - |\alpha| - 2|r|). \tag{3.9}$$

It remains to show that the right-hand side of (3.9) is unchanged if we replace $s_{\mathfrak{gl}_2}^{inv}(K(r))$ by $s_{\mathfrak{gl}_2}(K(r))+1$, the latter of which is equal to $q(x_{\mathfrak{o}})$ where $\mathfrak{o}$ is any element in $\mathfrak{O}(r)$.

Since $x^{inv}(r)$ is a sum of $x_{\mathfrak{o}}$ and some $S(r)$-translates of it (all of which have the same filtration degree), we have $q(x^{inv}(r)) \le q(x_{\mathfrak{o}})$, thus

$$\lim_{r\to\infty^m}(s_{\mathfrak{gl}_2}^{inv}(K(r)) - |\alpha| - 2|r|) \le \lim_{r\to\infty^m}(s_{\mathfrak{gl}_2}(K(r)) + 1 - |\alpha| - 2|r|). \tag{3.10}$$

Now suppose the right-hand side of (3.10) is not $-\infty$, so the value stabilizes to some limit M for r sufficiently large. We claim that for r sufficiently large, in the sense that the right-hand side of (3.10) stabilizes to M at $r-(1,\cdots,1)$, we have $q(x^{inv}(r)) = q(x_{\mathfrak{o}}) = M + |\alpha| + 2|r|$, from which the statement follows.

To this end, we follow the strategy in Section 1.4 in Chapter 1, see in particular the proof of Lemma 1.4.7. The point is that, by exploiting the maps $q^{-|\alpha|-2|r|}KhR_{Lee,\mathfrak{O}(r-e_i)}(K(r-e_i)) \to q^{-|\alpha|-2|r|}KhR_{Lee,\mathfrak{O}(r)}(K(r))$ induced by the undotted annulus creations, and applying Proposition 1.4.3 (really, a multi-component version of it; see Remark 1.4.5), we see that all irreducible $S(r)$-subrepresentations of $q^{-|\alpha|-2|r|}KhR_{Lee,\mathfrak{O}(r)}(K(r))$ other than the invariant subspace have a filtration level bounded from above by $M-2$. Since the invariant subspace is spanned by $x^{inv}(r)$ and r is in the stable range of the right-hand side of (3.10), we must have $q(x^{inv}(r)) - |\alpha| - 2|r| = M$, as desired. □

3.4.4 Diagrammatic nonvanishing result

For simplicity, in this section we assume $L=\emptyset$. We give a diagrammatic criterion to guarantee the nonvanishing of $\mathcal{S}_0^2(X)$ and $\mathcal{S}_0^{Lee}(X)$. The results presented here contain a small improvement over those in [RW24, Section 5.4]. See also [Nah26] for a slightly simplified approach to some nonvanishing results presented in this section in a simplified setup.

We explain our setup. As before, assume X is a 2-handlebody obtained by attaching 2-handles to B^4 along a framed oriented link $K\subset S^3$ with components $K_1,\cdots,K_m$ and framings $p_1,\cdots,p_m$. Let $p=(p_1,\cdots,p_m)$ and P be the diagonal matrix with entries $p_1,\cdots,p_m$. Choose a link diagram D of K as an unframed link. Let N denote the crossing matrix of D, where N_{ii} is the number of crossings of K_i in diagram D, and N_{ij} is half the number of crossings between K_i and K_j in D for $i\ne j$. Let $w=(w_1,\cdots,w_m)$ be the writhe vector of the diagram D, where w_i is the writhe of the i-th component of D, and let W be the diagonal matrix with entries $w_1,\cdots,w_m$. Finally, for $f,n\in\mathbb{Z}^m$, let K^f denote the link K

with framing f, and $K^f(n)$ denote the n-cable of K with framing f, obtained by replacing K_i by an f_i-framed $|n_i|$-cable, with orientation reversed if $n_i < 0$. Thus, any $K^w(n)$ has a natural diagram induced from D by taking blackboard-framed cables. (Here and below, unless explicitly stated otherwise, the blackboard framing of a diagram of a framed link is equal to the framing of the link.)

Finally, fix a class $\epsilon \in H_2(X;\mathbb{Z}/2) \cong (\mathbb{Z}/2)^m$, and let $\mathbb{Z}^m_\epsilon$ denote the subset of $\mathbb{Z}^m$ of elements with mod 2 reduction ϵ. Define

$$h\colon \mathbb{Z}^m_\epsilon \to \mathbb{Z},\ \alpha \mapsto \alpha^T(P-W+N)\alpha.$$

Theorem 3.4.5. *In the notation above, suppose*

(i) $P-W+N$ is negative semidefinite.

(ii) For any $n \in \mathbb{Z}^m_\epsilon \cap \mathbb{Z}^m_{\geq 0}$ with $h(n) = \max(h)$, the link diagram of $K^w(n)$ induced from D is positive.

Then,

(0) A class $\alpha \in \mathbb{Z}^m_\epsilon$ maximizes the self-intersection number α^2 if and only if it maximizes $h(\alpha)$, in which case $\alpha^2 = h(\alpha) = \max(h)$.

For every class $\alpha \in \mathbb{Z}^m_\epsilon$ that maximizes α^2, or equivalently $h(\alpha)$, we have

(1) For $\bullet \in \{2, Lee\}$, $\mathcal{S}^\bullet_{0,>0}(X;\alpha) = 0$.

(2) We have $\mathcal{S}^2_{0,0}(X;\alpha;\mathbb{Q}) = gr(\mathcal{S}^{Lee}_{0,0}(X;\alpha))$ as graded vector spaces, with total dimension given by the number of maximal points of h on $\mathbb{Z}^m_\epsilon$. If in addition, $w_i > p_i$ for all i, and no knot component of D is a crossingless unknot, then

$$\mathcal{S}^2_{0,0}(X;\alpha;\mathbb{Q}) \cong \bigoplus_{\substack{n\in\mathbb{Z}^m_\epsilon\cap\mathbb{Z}^m_{\geq 0}\\ h(n)=\max(h)}} q^{\langle w-p,n\rangle-|n|} KhR^0_2(K^w(n);\mathbb{Q}), \tag{3.11}$$

where $\langle\cdot,\cdot\rangle$ is the inner product on $\mathbb{Z}^m$.

(3) $s(X;\alpha) = -s(-K^p(\alpha)) - \alpha^2 - |\alpha| + 1$.

Remark 3.4.6. (1) Since every $K^w(n)$ in the summation in (3.11) is positive, each summand can be easily computed using algebraic data of the link diagram D, as observed in [Kho03, Proposition 6.1] in the case of knots. Explicitly, for a framed link L with a nonsplit positive diagram D where each D_i is nonsplit, we have

$$KhR^0_2(L) \cong q^{-O(D)}\mathbb{Z} \oplus q^{-O(D)+2}\mathbb{Z}, \tag{3.12}$$

where $O(D)$ denotes the number of Seifert circles of D.

(2) For every $\alpha \in \mathbb{Z}_\epsilon^m$, not necessarily maximizing α^2, we can deduce the structure of $\mathcal{S}_0^2(X;\alpha;\mathbb{Q})$ and $\mathcal{S}_0^{Lee}(X;\alpha)$ at the highest nontrivial homological degree, namely $\frac{1}{2}(\max(h) - \alpha^2)$, from the special case in Theorem 3.4.5(2) by the renormalization formula (2.19).

(3) It is possible to allow a boundary link $L \subset \partial X$ in the statement of Theorem 3.4.5, but we refrained from doing so to avoid making the statement too complicated.

Corollary 3.4.7. *With the same notation as before, if $P - W + N$ is negative definite, then $s(X;0) = 0$, and both $\mathcal{S}_{0,0}^2(X;0;\mathbb{Q})$ and $\mathcal{S}_{0,0}^{Lee}(X;0)$ are 1-dimensional, concentrated at (filtration) degree 0. More generally, for any $\alpha \in H_2(X)$, the highest homological degree part of $\mathcal{S}_0^2(X;\alpha;\mathbb{Q})$ is equal to the associated graded vector space of the corresponding part of $\mathcal{S}_0^{Lee}(X;\alpha;\mathbb{Q})$, whose graded dimension is finite, and computable from the input data with time complexity at most polynomial in the size of D and exponential in m.*

Proof. By adding positive kinks to D if necessary, we may assume it does not contain a crossingless unknot component. Apply Theorem 3.4.5(2)(3) with $\epsilon = 0$, where (ii) is satisfied because the maximum of h is achieved at the unique point $n = 0$, and any diagram of $K^w(0) = \emptyset$ is positive. So the right-hand side of (3.11) has a single term $KhR_2^0(\emptyset) = \mathbb{Q}$ at degree 0. Also, $s(X;0) = -s(\emptyset) + 1 = 0$.

For an arbitrary α, we may use (2.19) to reduce to the case when α maximizes α^2 within its mod 2 homology class. Since $P - W + N$ is negative definite, (3.11) is a finite sum where the number of summands is at most exponential in m, and each term can be computed using (3.12) in polynomial time. □

Proof of Theorem 3.1.15. Let D be any diagram of K. In the notation of Theorem 3.4.5, we have $m = 1$, $p = n$, $P - W + N = n + 2n_-(D)$. Now (1) follows from Corollary 3.4.7.

For (2), take D to be a positive diagram of K with at least one crossing. Then for $n \leq 0$, the conditions in Theorem 3.4.5 are satisfied for both $\epsilon = 0, 1$, and we obtain $s(X;0) = -s(\emptyset) + 1 = 0$, $s(X;1) = -s(-K) - n - 1 + 1 = s(K) - n$ by Theorem 3.4.5(3). For $n < 0$, by tracking the grading shifts, the claim on $\mathcal{S}_{0,0}^\bullet(X;1)$ follows from Theorem 3.4.5(2), where the summation has only one term $1 \in \mathbb{Z}_\epsilon^m \cap \mathbb{Z}_{\geq 0}^m$. □

The rest of the section is devoted to proving Theorem 3.4.5.

3.4.4.1 Proof Strategy

We recall the 2-handlebody formula for X and $\alpha \in \mathbb{Z}_{\geq 0}^m$ (2.18):

$$\mathcal{S}_0^\bullet(X;\alpha) = \operatorname{colim}_{r \in \mathbb{Z}_{\geq 0}^m} q^{-|\alpha|-2|r|} KhR_\bullet(K^p(\alpha + r, r))^{S_{\alpha+2r}}, \tag{3.13}$$

where when $\bullet = 2$ we assume to work over $\mathbb{Q}$. Here for $a \in \mathbb{Z}_{\geq 0}^m$, $S_a := \prod_j S_{a_j}$.

Step 1: For a maximizer $\alpha \in \mathbb{Z}_\epsilon^m$ of h, give an upper bound on the rank of $KhR_2(K^p(\alpha+r,r))$ in the highest expected homological degree, namely homological degree zero. In particular, this will show $KhR_2^{>0}(K^p(\alpha+r,r)) = 0$, proving (1). The key technical input will be a comparison lemma that provides an upper bound of $KhR_2(K^p(n))$ in terms of $KhR_2(K^q(n'))$ for $q \geq p$, $0 \leq n' \leq n$, in a spirit similar to that of Section 1.5 in Chapter 1.

Step 2: For a maximizer α of h, give a lower bound on the dimension of $KhR_{Lee}(K^p(\alpha+r,r))$ in homological degree zero. This part is easy, because KhR_{Lee} (disregarding the filtration structure) is determined by algebraic data.

The two bounds in the first two steps will be equal to each other, thus we can determine $KhR_2^0(K^p(\alpha+r,r);\mathbb{Q}) = gr(KhR_{Lee}^0(K^p(\alpha+r,r)))$. The equality also forces (0) (of course, one can also prove (0) using easy algebra).

Step 3: Compute $KhR_2(K^p(\alpha+r,r);\mathbb{Q})^{S_{\alpha+2r}}$ in homological degree 0. Argue that the symmetrized dotted annulus creation maps in (3.13) are all injective in homological degree 0, thereby computing $\mathcal{S}_{0,0}^2(X;\alpha;\mathbb{Q}) = gr(\mathcal{S}_{0,0}^{Lee}(X;\alpha))$, proving (2). By tracking Lee canonical generators as in the proof of Proposition 3.4.4, we compute $s(X;\alpha)$, proving (3). The technical tools in this step are (multi-component versions of) Corollary 1.4.4 and Theorem 1.6.4 in Chapter 1. The injectivity will follow from the splitting of (1.21) in the relevant homological grading, which is in turn deduced from a dimension count.

3.4.4.2 Setup and preliminaries

Every condition and conclusion of Theorem 3.4.5, with the possible exception of (2), is not affected by adding positive kinks to D. Therefore, we may assume without loss of generality that the extra assumptions in (2) hold throughout, namely

(iii) $w_i > p_i$ for all i.

(iv) No knot component of D is a crossingless unknot.

We also assume $\alpha \in \mathbb{Z}_\epsilon^m \cap \mathbb{Z}_{\geq 0}^m$, as the general case is only notationally more complicated (and can indeed be deduced from the special case by reversing the orientations of some K_i's).

In the setup of Theorem 3.4.5, it is convenient to note the following:

- $P - W + N_+ - N_-$ is the intersection form of X on $H_2(X) \cong \mathbb{Z}^m$.
- $w(K^p(\alpha)) = \alpha^2 = \alpha^T(P - W + N_+ - N_-)\alpha$ for any $\alpha \in \mathbb{Z}_\epsilon^m$. If α is a maximizer for h, then $w(K^p(\alpha)) = \alpha^2 = h(\alpha) = \max(h)$.
- If $\alpha \in \mathbb{Z}_\epsilon^m$ is a maximizer of h, $(|\alpha_1|, \cdots, |\alpha_m|)$ is also a maximizer.

- If $n \in \mathbb{Z}^m_\epsilon \cap \mathbb{Z}^m_{\geq 0}$, then $w(K^w(n)) \leq n^T N n$, with equality achieved if n is a maximizer of h.

Our bounds on the rank of homologies will be stated in terms of their Poincaré polynomials. For a link L, let $P(L)(t,q)$ be the Poincaré polynomial of $KhR_2(L)$. For convenience, we will also consider the renormalized homology of L, defined to be

$$\widetilde{KhR}_2(L) := t^{w(L)/2} q^{-w(L)/2} KhR_2(L),$$

which is independent of the orientation of L (cf. (2.12)). Let $\tilde{P}(L)(t,q)$ be the Poincaré polynomial of $\widetilde{KhR}_2(L)$, which is a Laurent polynomial in $t^{1/2}, q^{1/2}$.

For two (Laurent) polynomials P, Q in variables $t^{1/2}, q^{1/2}$, we write $P \leq Q$ if every coefficient of P is less than or equal to that of Q. For two vectors $a, b \in \mathbb{Z}^m$, write $a \leq b$ if $a_i \leq b_i$ for all i.

We will also make use of the following easy lemma. For a (Laurent) polynomial P in $t^{1/2}, q^{1/2}$, $t_{max}(P)$ denotes the maximal t-degree where it is nonzero.

Lemma 3.4.8. *For $n \in \mathbb{Z}^m_{\geq 0}$, we have $t_{max}(\tilde{P}(K^w(n))) \leq \frac{1}{2} n^T N n$.*

Proof. For any framed link diagram D of L, $t_{max}(P(L))$ is at most the number of negative crossings of D, and thus $t_{max}(\tilde{P}(L))$ is at most one half the total number of crossings of D. For D the standard diagram of $K^w(n)$, this number is $\frac{1}{2} n^T N n$. □

3.4.4.3 Step 1: Upper bound for KhR_2 and proof of (1)

For $a \geq b$, $a, b \in \mathbb{Z}^k$, let (a,b) denote the irreducible S_{a+b}-representation $(a,b) := \otimes_j (a_j, b_j)$, where (a_j, b_j) is the irreducible $S_{a_j+b_j}$-representation over $\mathbb{Q}$ given by the two-row Young diagram (a_j, b_j). In the following lemma, for $p, w \in \mathbb{Z}^m$, P, W will denote the diagonal matrices with entries given by p, w, as in the setup of Theorem 3.4.5. For $j = 1, \cdots, m$, $K^{p+e_j/n_j}(n)$ will denote the framed link obtained from $K^p(n)$ by inserting a $1/n_j$-twist along the components parallel to K_j; in particular, its writhe is n_j more than that of $K^p(n)$, and it is identical to $K^p(n)$ if $n_j = 0$.

For the reader's convenience, we note that for $r, n \in \mathbb{Z}_{\geq 0}$, $r \leq n/2$,

$$\dim(n-r, r) = \binom{n}{r} - \binom{n}{r-1}.$$

Lemma 3.4.9 (Comparison lemma).

(1) For $n \in \mathbb{Z}^m_{\geq 0}$, $p, w \in \mathbb{Z}^m$, with $p \leq w$, we have

$$\tilde{P}(K^p(n)) \leq \sum_{\substack{0 \leq n' \leq n \\ 2 | n - n'}} R_{n,n',w-p} \tilde{P}(K^w(n')) \tag{3.14}$$

for some polynomials $R_{n,n',w-p} \in \mathbb{Z}[t^{\pm 1/2}, q^{\pm 1/2}]$ independent of K with

$$\begin{aligned}
&R_{n,n',w-p}(t,q) \\
&= t^{-\sum_i \frac{(w_i-p_i)n_i'^2}{2}} q^{\sum_i \frac{(w_i-p_i)(n_i'^2+2n_i')}{2}} \prod_{j=1}^{m} \left(\sum_{r_j=0}^{\frac{n_j-n_j'}{2}} \dim \left(\frac{n_j+n_j'}{2} + r_j, \frac{n_j-n_j'}{2} - r_j \right) q^{2r_j} \right) + \cdots \\
&= (t^{-1}q)^{\frac{1}{2}n'^T(W-P)n'} q^{\langle w-p,n' \rangle} \sum_{\substack{n' \le n'' \le n \\ 2 | n-n''}} \dim \left(\frac{n+n''}{2}, \frac{n-n''}{2} \right) q^{|n''|-|n'|} + \cdots ,
\end{aligned}$$

where $\cdots$ are terms with lower t-degrees.

(2) For $n \in \mathbb{Z}^m_{\ge 0}$, $p, w \in \mathbb{Z}^m$, $j \in \{1, \cdots, m\}$, with $p + e_j \le w$, we have

$$(t^{-1}q)^{n_j/2} q \tilde{P}(K^{p+e_j/n_j}(n)) \le \sum_{\substack{0 \le n' \le n \\ 2|n-n' \\ n_j' = n_j}} R_{n,n',w-p} \tilde{P}(K^w(n')). \tag{3.15}$$

Lemma 3.4.9 is a consequence of an inductive argument pioneered by Stošić [Sto09], which we have employed in Section 1.5 in Chapter 1. Although it might have been implicit in [Sto09, Proof of Theorem 1], or in more details in [Tag13], for completeness we sketch a proof of Lemma 3.4.9 in Appendix B.

By assumption (iii), we may apply Lemma 3.4.9(1) to our setup using w, p given in the statement of Theorem 3.4.5. By Lemma 3.4.8, we find that each summand on the right-hand side of (3.14) has $t_{max} \le -\frac{1}{2}n'^T(W-P)n' + \frac{1}{2}n'^T N n = h(n') \le \frac{1}{2}\max(h)$. Therefore,

$$\tilde{P}(K^p(n)) \le \sum_{\substack{0 \le n' \le n \\ 2|n-n' \\ h(n') = \max(h)}} R_{n,n',w-p} \tilde{P}(K^w(n')) + \cdots , \tag{3.16}$$

with the highest t-degree shown to be $\frac{1}{2}\max(h)$. If $n = \alpha + 2r$, where $\alpha \in \mathbb{Z}^m_\epsilon \cap \mathbb{Z}^m_{\ge 0}$ is a maximizer of h and $r \ge 0$, renormalizing gives

$$P(K^p(\alpha + r, r)) \le \sum_{\substack{0 \le n' \le n \\ 2|n-n' \\ h(n') = \max(h)}} q^{\langle w-p,n' \rangle} P(K^w(n')) \sum_{\substack{n' \le n'' \le n \\ 2|n-n''}} \dim \left(\frac{n+n''}{2}, \frac{n-n''}{2} \right) q^{|n''|-|n'|} + \cdots . \tag{3.17}$$

Here, each $K^w(n')$ in the summation is a positive link by assumption (ii), so it is supported in nonpositive homological degree. In particular, $KhR_2^{>0}(K^p(\alpha + r, r)) = 0$, and $\mathcal{S}^2_{0,>0}(X; \alpha) = 0$ in view of the 2-handlebody formula (2.17) (this holds over $\mathbb{Z}$, as the comparison lemma 3.4.9

holds true over any coefficient field). Since $gr(KhR_{Lee})$ is bounded from above by KhR_2, we also have $\mathcal{S}_{0,>0}^{Lee}(X;\alpha)=0$. This proves (1).

For each term n' in the summation in (3.17), let $s(n')$ denote the number of split components of the link $K^w(n')$, or equivalently the number of connected components of its diagram as a 4-valent graph; note that by assumption (iv), $s(n')$ depends only on the set of nonzero indices in n'. Now by (3.12) and (3.17), we obtain

$$\dim(KhR_2^0(K^p(\alpha+r,r);\mathbb{Q}))\le\sum_{\substack{0\le n'\le n\\ 2|n-n'\\ h(n')=\max(h)}}2^{s(n')}\binom{n}{\frac{n-n'}{2}},\tag{3.18}$$

where as usual, $\binom{a}{b}$ for $a,b\in\mathbb{Z}_{\ge0}^m$ is a shorthand for $\prod_j\binom{a_j}{b_j}$.

3.4.4.4 Step 2: Matching lower bound for KhR_{Lee}

For a link L, the dimension of $KhR_{Lee}^0(L)$ is given by the number of orientations $\mathfrak{o}$ of L as an unoriented link so that $w(L,\mathfrak{o})=w(L)$. In our setup, if $n=\alpha+2r$, $\alpha\in\mathbb{Z}_\epsilon^m\cap\mathbb{Z}_{\ge0}^m$ a maximizer of h, $r\ge0$, and $\mathfrak{o}$ is an orientation of $K^p(\alpha+r,r)$ making it an algebraic n'-cable of K^p, then $-n\le n'\le n$, $2|n-n'$, and $w(K^p(\alpha+r,r),\mathfrak{o})=w(K^p(n'))=n'\cdot n'=n'^T(P-W+N_+-N_-)n'$. In particular, $w(K^p(\alpha+r,r))=w(K^p(\alpha))=\max(h)$.

For any $n'\in\mathbb{Z}_\epsilon^m\cap\mathbb{Z}_{\ge0}^m$ with $h(n')=\max(h)$, by (ii) and Step 1, the link $K^p(n')$ has a positive diagram with a $p-w$ framing change with $s(n')$ connected components. In particular, the $2^{s(n')}$ orientations on $K^p(n')$ obtained from the one on $K^p(n')$ by flipping the orientations in blocks, one for each connected component of the diagram, all have writhe $w(K^p(n'))=h(n')=\max(h)$. Thus, by breaking the count of orientations into algebraic cable types, we obtain

$$\dim(KhR_{Lee}^0(K^p(\alpha+r,r)))=\sum_{\substack{-n\le n'\le n\\ 2|n-n'\\ (n')^2=\max(h)}}\binom{n}{\frac{n-n'}{2}}\ge\sum_{\substack{0\le n'\le n\\ 2|n-n'\\ h(n')=\max(h)}}2^{s(n')}\binom{n}{\frac{n-n'}{2}}.\tag{3.19}$$

Comparing (3.18) and (3.19), we conclude that all inequalities are equalities. In particular, for $\alpha\in\mathbb{Z}_\epsilon^m$, we have $\alpha^2=\max(h)$ if and only if $h(\alpha)=\max(h)$. Since $\alpha^2\le h(\alpha)$ always holds, (0) is proved. Furthermore, we know that $KhR_2^0(K^p(\alpha+r,r);\mathbb{Q})=gr(KhR_{Lee}^0(K^p(\alpha+r,r)))$ have graded dimension given by the right-hand side of (3.17).

3.4.4.5 Step 3: Compute $(KhR_2^0)^{inv}$, injectivity, and proofs of (2)(3)

Let α,r and $n=\alpha+2r$ be as before. From Step 2, we know that $gr(\mathcal{S}_{0,0}^{Lee}(X;\alpha))/gr_{-\infty}(\mathcal{S}_{0,0}^{Lee}(X;\alpha))=\mathcal{S}_{0,0}^2(X;\alpha;\mathbb{Q})$. We will compute the symmetrized terms in the colimit formula (3.13) for $\bullet=2$ over $\mathbb{Q}$ in homological degree 0, and show that all morphisms between them are injective, which will compute $\mathcal{S}^2_{0,0}(X;\alpha;\mathbb{Q})$ and show $gr_{-\infty}(\mathcal{S}^{Lee}_{0,0}(X;\alpha)) = 0$, proving (2) (the statement on the dimension follows from the discussion in Step 2).

We first prove that

$$q^{-|\alpha|-2|r|}KhR^0_2(K^p(\alpha+r,r);\mathbb{Q})^{S_{\alpha+2r}} \cong \bigoplus_{\substack{0\le n'\le n\\ 2|n-n'\\ h(n')=\max(h)}} q^{\langle w-p,n'\rangle-|n'|}KhR^0_2(K^w(n');\mathbb{Q}). \tag{3.20}$$

We convert back to Poincaré polynomials for this computation. After renormalizing, this amounts to showing

$$q^{-|n|}\tilde{P}_{Sym}(K^p(n)) = \sum_{\substack{0\le n'\le n\\ 2|n-n'\\ h(n')=h(\alpha)}} (t^{-1}q)^{\frac{1}{2}n'^T(W-P)n'}q^{\langle w-p,n'\rangle-|n'|}\tilde{P}(K^w(n')) + \cdots, \tag{3.21}$$

where $\tilde{P}_{Sym}(K^p(n))$ denotes the Poincaré polynomial of $\widetilde{KhR}_2(K^p(n);\mathbb{Q})^{S_n}$.

Rewrite (3.16) (which we know to be an equality for any $n \in \mathbb{Z}^m_\epsilon \cap \mathbb{Z}^m_{\ge 0}$; Step 1 and 2 only showed this for $n \ge \alpha$, but the same proof goes regardless) as

$$\begin{aligned}\tilde{P}(K^p(n)) = &\sum_{\substack{0\le n''\le n\\ 2|n-n''}} \dim\left(\frac{n+n''}{2},\frac{n-n''}{2}\right)\\ &\times q^{|n''|}\sum_{\substack{0\le n'\le n''\\ 2|n-n'\\ h(n')=h(\alpha)}} (t^{-1}q)^{\frac{1}{2}n'^T(W-P)n'}q^{\langle w-p,n'\rangle-|n'|}\tilde{P}(K^w(n')) + \cdots.\end{aligned} \tag{3.22}$$

On the other hand, a multi-component version of Corollary 1.4.4 (see Remark 1.4.5) gives

$$\tilde{P}(K^p(n)) = \sum_{\substack{0\le n''\le n\\ 2|n-n''}} \dim\left(\frac{n+n''}{2},\frac{n-n''}{2}\right)\tilde{P}_{Sym}(K^p(n'')). \tag{3.23}$$

We obtain (3.21) inductively in $n \in \mathbb{Z}^m_\epsilon \cap \mathbb{Z}^m_{\ge 0}$ from comparing (3.22) and (3.23).

Next, we show that the symmetrized dotted annulus creation maps in the colimit system (3.13) are all injective. In view of (3.20), this will prove (2).

A multi-component version of Theorem 1.6.4 (see Remark 1.6.5), deduced from [GW23, Theorem 5.1], after renormalizing to our setup, gives exact triangles

$$\begin{tikzcd}[column sep=small] q^{-|n|+2}\widetilde{KhR}_2(K^p(n-2e_j);\mathbb{Q})^{S_{n-2e_j}} \arrow[rr] & & q^{-|n|}\widetilde{KhR}_2(K^p(n);\mathbb{Q})^{S_n} \arrow[dl] \\ & (t^{-1}q)^{n_j/2}q^{1-|n|}\widetilde{KhR}_2(K^{p+e_j/n_j}(n);\mathbb{Q})^{S_{n_{\hat{j}}}} \arrow[ul, "{[1]}"] & \end{tikzcd} \tag{3.24}$$

,

where the horizontal map is induced by the symmetrized dotted annulus creation, and where $n_{\hat{j}} \in \mathbb{Z}_{\geq 0}^{m-1}$ is obtained from n by removing the j-th coordinate. We prove that the horizontal map in (3.24) is injective in homological degree $\frac{1}{2}\max(h)$, which amounts to the injectivity of the morphisms in (3.13) in homological degree 0. This is done by a dimension count, namely we will show that at homological degree $\frac{1}{2}\max(h)$, the dimension of $\widetilde{KhR}_2(K^p(n);\mathbb{Q})^{S_n}$ is greater than or equal to the sum of those of the other two terms in (3.24).

At homological degree $\frac{1}{2}\max(h)$, from (3.20), the dimension of the middle term is

$$\dim(\widetilde{KhR}_2^{\frac{1}{2}\max(h)}(K^p(n);\mathbb{Q})^{S_n}) = \sum_{\substack{0\leq n'\leq n\\ 2|n-n'\\ h(n')=\max(h)}} \dim(KhR_2^0(K^w(n');\mathbb{Q})),$$

and similarly for the term on the left. By the exactness of (3.24), it follows that the term on the bottom in this homological degree has dimension at least the difference, namely

$$\dim(\widetilde{KhR}_2^{\frac{1}{2}\max(h)+n_j/2}(K^{p+e_j/n_j}(n);\mathbb{Q})^{S_{n_{\hat{j}}}}) \geq \sum_{\substack{0\leq n'\leq n\\ 2|n-n'\\ h(n')=\max(h)\\ n_j'=n_j}} \dim(KhR_2^0(K^w(n');\mathbb{Q})). \tag{3.25}$$

By Corollary 1.4.4, we then have

$$\dim(\widetilde{KhR}_2^{\frac{1}{2}\max(h)+n_j/2}(K^{p+e_j/n_j}(n);\mathbb{Q})) \tag{3.26}$$
$$\geq \sum_{\substack{0\leq n'\leq n''\leq n\\ n\equiv n'\equiv n'' \pmod 2\\ n_j'=n_j''=n_j\\ h(n')=\max(h)}} \dim\left(\frac{n+n''}{2},\frac{n-n''}{2}\right)\dim(KhR_2^0(K^w(n');\mathbb{Q})). \tag{3.27}$$

On the other hand, part (2) of the comparison lemma 3.4.9 shows that (3.26) is bounded from above by (3.27). Thus, the inequalities (3.27) and (3.25) must both be equalities. In particular, the horizontal map (3.24) at homological degree $\frac{1}{2}\max(h)$ is injective, proving (2).

To prove (3), we note that (3.9) noted in the proof of Proposition 3.4.4 says that $s(X;\alpha)$ can be computed as the limit of the filtration levels of the invariant Lee canonical generators in the colimit formula (3.13). Since $gr(KhR_{Lee}^0(K^p(\alpha+r,r))^{S_{\alpha+2r}}) = KhR_2^0(K^p(\alpha+r,r);\mathbb{Q})^{S_{\alpha+2r}}$ and the symmetrized dotted annulus creation maps between the symmetrized Khovanov homology groups are injective, we know that the corresponding maps on the symmetrized Lee homology groups are filtration-degree preserving. Consequently, we have $s(X;\alpha) = s_{\mathfrak{gl}_2}^{inv}(K^p(\alpha)) - |\alpha|$ in the notation of (3.9), where $s_{\mathfrak{gl}_2}^{inv}(K^p(\alpha))$ is the filtration degree of the symmetrization of the Lee canonical generator $x_{\mathfrak{o}}$ of $K^p(\alpha)$ associated to its own orientation $\mathfrak{o}$, under the S_α-action. Since all parallel strands in $K^p(\alpha)$ have the same orientation, the S_α-action acts trivially on $x_{\mathfrak{o}}$. Thus,

$$s(X;\alpha) = q(x_{\mathfrak{o}}) - |\alpha| = s_{\mathfrak{gl}_2}(K^p(\alpha)) + 1 - |\alpha| = -s(K^p(\alpha)) - \alpha^2 - |\alpha| + 1,$$

as desired. □

Remark 3.4.10. A multi-component version of Conjecture 1.6.6 in Chapter 1 concerning the injectivity of the morphisms in the colimit formula (3.13) for a 2-handlebody would imply a much stronger nonvanishing result than that stated in Theorem 3.4.5 whenever the framings of the 2-handles are small enough. We will come back to this speculation in Section 3.5.

3.5 Nonvanishing examples and applications

In this section we present some explicit computations and applications for Khovanov and Lee skein lasagna modules and lasagna s-invariants of some 2-handlebodies (with empty boundary link). Roughly, the first half of this section utilizes the tools developed in Section 3.4, while the second half utilizes results in Chapter 1.

3.5.1 Knot traces of some small knots

Knot \ Framing	-4	-3	-2	-1	0	1	2	3	4	5	6
U	N	N	N	N	N	V	V	V	V	V	V
3_1	N	?	?	V	V	V	V	V	V	V	V
-3_1	N	N	N	N	N	?	?	?	?	?V	V
4_1	N	N	N	N	N	?V	?V	V	V	V	V

Table 3.1: Vanishing/nonvanishing of Khovanov skein lasagna modules of some knot traces. An entry V/N means the corresponding knot trace has vanishing/nonvanishing Khovanov skein lasagna module. An entry ?V means it has vanishing Lee skein lasagna module (meaning the filtration function is identically $-\infty$).

Example 3.5.1 (Unknot)**.** By Theorem 3.1.15, for $n \le 0$, the D^2-bundle $D(n)$ over S^2 with Euler number n has $s(D(n);0) = 0$, $s(D(n);1) = -n$. In particular, $\mathcal{S}^2_{0,0}(D(n);\alpha) \neq 0$ for $\alpha = 0, 1$.

For $n < 0$, more explicitly we have $\mathcal{S}^2_{0,0}(D(n);0;\mathbb{Q}) = \mathbb{Q}$ concentrated in degree 0 and $\mathcal{S}^2_{0,0}(D(n);1;\mathbb{Q}) = \mathbb{Q}^2$ concentrated in degrees $-n, -n-2$.

For $n = 0$, a direct computation using the 2-handlebody formula yields (see Manolescu–Neithalath [MN22, Theorem 1.2] or Example 2.6.7) $\mathcal{S}_0^2(D(0)) \cong \mathbb{Z}[A_0^{\pm}, A_1]$ as tri-graded abelian groups, where A_0 has tri-degree $(0,0,1)$ and A_1 has tri-degree $(0,-2,1)$. A similar computation gives $s(D(0);\alpha) = 0$ for all α. Alternatively, this follows from a direct application of Proposition 3.4.4 or Theorem 3.4.5.

On the other hand, for $n > 0$, Example 3.2.2 shows $\mathcal{S}_0^2(D(n)) = 0$ and consequently all lasagna s-invariants of $D(n)$ are $-\infty$.

For other examples, we mostly only mention the vanishing/nonvanishing of the Khovanov skein lasagna modules of the knot traces. For small knots, Corollary 3.1.17 (which will be proved in Section 3.5.9) usually provides more effective nonvanishing bounds than Theorem 3.1.15. We emphasize however that Corollary 3.1.17 does not provide an explicit s-invariant in most cases.

Example 3.5.2 (Trefoils)**.** The left-handed trefoil, or 3_1, has 3 negative crossings. Therefore, Theorem 3.1.15(1) shows $X_n(3_1)$ has nonvanishing Khovanov skein lasagna module for $n < -6$.

We can do better as follows. The knot 3_1 can be obtained from the unknot by a crossing change that is realized by a -1 twist on two parallel strands with the same orientation. By Kirby calculus, the twist is realized by a $(+1)$-framed 2-handle attached to B^4, with the effect that the 0-framed unknot on ∂B^4 becomes the (-4)-framed unknot on $\partial(B^4 \cup \text{2-handle})$. This implies that 3_1 is (-4)-slice in $\mathbb{CP}^2$, thus is n-slice in some $k\mathbb{CP}^2$ for every $n \leq -4$. It follows that $X_n(3_1)$ embeds in some $k\overline{\mathbb{CP}^2}$ for $n \leq -4$. By Corollary 3.1.17, $X_n(3_1)$ has nonvanishing Khovanov skein lasagna module for $n \leq -4$.

On the other hand, $TB(-3_1) = 1$, so $X_n(3_1)$ has vanishing Khovanov skein lasagna module for $n \geq -1$ by Theorem 3.1.5. It would be interesting to know whether $X_n(3_1)$ has nonvanishing Khovanov skein lasagna module for $n = -3, -2$.

The right-handed trefoil -3_1 is positive. By either Theorem 3.1.15 or Corollary 3.1.17 we know $X_n(-3_1)$ has nonvanishing Khovanov skein lasagna module for $n \leq 0$. On the other hand, since $TB(3_1) = -6$, $X_n(-3_1)$ has vanishing Khovanov skein lasagna module for $n \geq 6$. In this case we have a larger gap of unknown cases. We remark that Proposition 3.5.9 will imply that $X_5(-3_1)$ has vanishing Lee skein lasagna module, as there is a framed concordance in $\overline{\mathbb{CP}^2}$ from the 1-framed unknot to the 5-framed -3_1.

In the case of $X_0(-3_1)$, take D to be the standard 3 crossing diagram of -3_1, $w = 3$, $p = 0$ in the notation of Theorem 3.4.5. Then the n-cable of D has $2n$ Seifert circles, so (3.11) and (3.12) imply that $\mathcal{S}_{0,0}^2(X_0(-3_1);\alpha;\mathbb{Q})$ is infinite-dimensional in quantum degrees 0 and 2, and zero elsewhere, for any $\alpha \in H_2(X_0(-3_1))$. This answers [MWW23, Question 1.6] in the negative.

Example 3.5.3 (Figure 8)**.** The standard diagram of the figure 8 knot, or 4_1, has 2 negative crossings. Theorem 3.1.15(1) shows $X_n(4_1)$ has nonvanishing Khovanov skein lasagna module for $n < -4$, while Corollary 3.1.17 shows so for $n \leq 0$ as 4_1 is 0-slice in $\mathbb{CP}^2$. On the other hand, since $-4_1 = 4_1$ and $TB(4_1) = -3$, we know $X_n(4_1)$ has vanishing Khovanov skein lasagna module for $n \geq 3$ by Theorem 3.1.5.

It would be interesting to know whether $X_n(4_1)$ has nonvanishing Khovanov skein lasagna module for $n = 1, 2$. Again, Proposition 3.5.9 will imply they both have vanishing Lee skein lasagna modules.

3.5.2 Negative semidefinite plumbings and generalizations

Example 3.5.4 (Negative semidefinite plumbing tree)**.** Let X be a negative semidefinite plumbing tree of spheres, i.e. a plumbing of D^2-bundles over 2-spheres along a tree, whose intersection form is negative semidefinite. Then we can orient the unknots in the underlying Kirby diagram so that all crossings are positive. In particular, in the notation of Theorem 3.4.5, $P - W + N = P - W + N_+ - N_-$ is the intersection form of X, which is negative semidefinite. Therefore, Theorem 3.4.5 implies $s(X; 0) = 0$ and X has nonvanishing Khovanov skein lasagna module.

Example 3.5.5 (The E8 plumbing manifold)**.** The negative smooth $E8$ manifold, denoted $E8$, obtained by plumbing eight (-2)-disk bundles over S^2 according to the negative $E8$ matrix, belongs to the family described in Example 3.5.4. In fact, since $P - W + N$ is negative-definite, we get a finiteness result. We give a more precise account as follows.

By the properties of the E8 lattice, we know that every element in $H_2(E8) \cong \mathbb{Z}^8$ has non-positive even self-intersection. The zero element is the unique element with self-intersection 0. There are 240 elements with self-intersection -2, and two different such elements have the same mod 2 reduction if and only if they are negatives of each other. There are 2160 elements with self-intersection -4 that come in 135 groups of 16 elements each according to their mod 2 reductions. In total, elements in $\mathbb{Z}^8$ with self-intersection at least -4 account for all the $2^8 = 1 + 120 + 135$ elements in $H_2(E8; \mathbb{Z}/2)$. We claim that

(a) $\mathcal{S}_{0,0}^2(E8; 0; \mathbb{Q}) = \mathbb{Q}$ concentrated in degree 0; $s(E8; 0) = 0$.

(b) For any $\alpha \in H_2(E8)$ with $\alpha^2 = -2$, $\mathcal{S}_{0,0}^2(E8; \alpha; \mathbb{Q}) = \mathbb{Q}^2$ concentrated in degrees 0 and 2; $s(E8; \alpha) = 2$.

(c) For any $\alpha \in H_2(E8)$ with $\alpha^2 = -4$, $\mathcal{S}_{0,0}^2(E8; \alpha; \mathbb{Q}) = \mathbb{Q}^{16}$ concentrated in degrees $0, 2, 4$ with dimensions $7, 8, 1$, respectively; $s(E8; \alpha) = 4$.

Statement (a) follows directly from Theorem 3.4.5.

To prove (b)(c), we first make the following observation. The Weyl group of the $E8$ lattice, generated by reflections along the roots of $E8$, acts transitively on the set of norm 2, norm 4 vectors in the $E8$ lattice, respectively. Since every root in the $E8$ lattice is represented by a (-2)-sphere in the $E8$ manifold, every reflection can be realized on the $E8$ manifold by a Seidel twist. Thus, the orientation-preserving diffeomorphism group of $E8$ acts transitively on the set of second homology classes with self-intersection -2 and -4, respectively. We may therefore prove the claim on any preferred choice of α. We also add positive kinks, one for each unknot in the standard plumbing Kirby diagram of $E8$, so that in the notation of Theorem 3.4.5, $w = (1, \cdots, 1) > (-2, \cdots, -2) = p$ and the condition about no crossingless unknot in Theorem 3.4.5(2) is fulfilled.

For (b), we pick $\alpha = (1, 0, \cdots, 0)$. Then we get from Theorem 3.4.5(2) that $\mathcal{S}_{0,0}^2(E8; \alpha; \mathbb{Q}) = q^{3-1} KhR_2^0(U^1; \mathbb{Q}) = q^1 KhR_2(U; \mathbb{Q}) = \mathbb{Q}^2$ concentrated in degrees $0, 2$, and from Theorem 3.4.5(3) that $s(E8; \alpha) = -s(U) - \alpha^2 - |\alpha| + 1 = 0 + 2 - 1 + 1 = 2$.

For (c), to be explicit, we order the vertices in the plumbing tree so that the intersection form on $H_2(E8) \cong \mathbb{Z}^8$ is represented by

$$A = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{pmatrix}.$$

We pick $\alpha = (0, 0, 0, 0, 0, 1, 0, 1)$. By Theorem 3.4.5(3) we have $s(E8; \alpha) = -s(U \sqcup U) - \alpha^2 - |\alpha| + 1 = 1 + 4 - 2 + 1 = 4$. By Theorem 3.4.5(2) we know $\mathcal{S}_{0,0}^2(E8; \alpha; \mathbb{Q}) = \mathbb{Q}^{16}$, although to verify the claim on the grading we have to work harder. The 16 norm-4 elements in $\mathbb{Z}^8$ having the same mod 2 reduction as α are given by

$$\begin{aligned} v_1 &= (0, 0, 0, 0, 0, 1, 0, -1), \\ v_2 &= (0, 0, 0, 0, 0, 1, 0, 1), \\ v_3 &= (0, 0, 0, 0, 2, 1, 0, 1), \\ v_4 &= (0, 0, 0, 2, 2, 1, 0, 1), \\ v_5 &= (0, 0, 2, 2, 2, 1, 0, 1), \\ v_6 &= (0, 2, 2, 2, 2, 1, 0, 1), \\ v_7 &= (2, 2, 2, 2, 2, 1, 0, 1), \\ v_8 &= (2, 4, 6, 8, 10, 7, 4, 5) \end{aligned}$$

and their negatives. In (3.11), we have 7 direct summands, coming from $n = v_2, \cdots, v_8$, each summand being $q^{2|n|} KhR_2^0(K^{(1, \cdots, 1)}(n); \mathbb{Q})$, for K the underlying link of the plumbing diagram. The term $n = v_2$ gives a contribution $q^4 KhR_2^0(U^1 \sqcup U^1; \mathbb{Q}) = q^4(1+q^{-2})^2\mathbb{Q} = (1+2q^2+q^4)\mathbb{Q}$. For each $i = 3, \cdots, 8$, the diagram of $K^{(1,\cdots,1)}(v_i)$ is positive with $2|v_i|$ Seifert circles, hence it gives a contribution $q^{2|v_i|}KhR_2^0(K^{(1,\cdots,1)}(v_i); \mathbb{Q}) = (1+q^2)\mathbb{Q}$ by (3.12). The claim follows.

It could be interesting to exploit the symmetry of $E8$ and the results above to obtain computations of s-invariants of some cables of the plumbing link of $E8$ that might otherwise be difficult to access (see also the proof of Theorem 3.4.5(3)).

Example 3.5.6 (Traces on alternating chainmail links)**.** Let G be a chainmail graph without loops and with positive edge weights and nonpositive vertex weights, and K be its associated chainmail link, in the sense of [Ago23, Section 2]. Then K has a standard positive diagram (as an unframed link), and a negative semidefinite framing matrix which is equal to $P-W+N$ for its standard diagram. This means the trace X on K has nonvanishing Khovanov skein lasagna module. This generalizes Example 3.5.4.

Example 3.5.7 (Branched double cover of definite surfaces of alternating links)**.** Suppose $L \subset S^3$ is a nonsplit alternating link, and $S \subset B^4$ is an unoriented surface bounding L with negative-definite Gordon–Litherland pairing [GL78]. Greene [Gre17] showed that every such S is isotopic rel boundary to a checkerboard surface of some alternating diagram of L. Conversely, one of the checkerboard surfaces of any alternating diagram is negative definite, a fact that is much easier.

The branched double cover of S is a 4-manifold $\Sigma(S)$ bounding the branched double cover $\Sigma(L)$ of L. It is also considered by Ozsváth–Szabó [OS05] (denoted X_L in Section 3 there), and from the description there (see also [Gre13, Section 3.1]) we realize that $\Sigma(S)$ is a special case of Example 3.5.6, hence it has nonvanishing Khovanov skein lasagna module. In particular, $\Sigma(L)$ bounds a 2-handlebody with nonvanishing Khovanov skein lasagna module. Since $\overline{\Sigma(L)} = \Sigma(-L)$ we see $\Sigma(L)$ bounds such 2-handlebodies on both sides.

Example 3.5.8. One can further generalize Example 3.5.6 by connect-summing some components of K with 0-framed positive knots. The same argument shows the resulting trace has nonvanishing Khovanov skein lasagna module.

3.5.3 More exotica from connected sums and non-smoothable homeomorphisms

In this section we prove Corollary 3.1.3. One can distinguish the relevant pairs of manifolds by their Khovanov skein lasagna modules, but we give a simpler argument by showing that they have different lasagna s-invariants.

Proof of Corollary 3.1.3. Write $W_1 := X_1^{\natural a} \natural X_2^{\natural b}$ and $W_2 := X_1^{\natural a'} \natural X_2^{\natural b'}$, where $a+b = a'+b' = n$ and $(a,b) \neq (a',b')$.

Recall from Section 3.1.5 that $s(X_1;0) = s(X_2;0) = 0$, $s(X_1;1) = 3$, $s(X_2;1) = 1$, where 1 is a generator of $H_2(X_i)$. Since we did not specify the sign of 1, we equally have $s(X_1;-1) = 3$, $s(X_2;-1) = 1$.

We first prove (1). By the connected sum formula (Theorem 3.3.8(4)),

$$s(W_1;(\epsilon_1,\cdots,\epsilon_n)) = 3a+b$$

for any $\epsilon_i = \pm 1$. If there were a diffeomorphism $W_1 \cong W_2$, then by considering the intersection form we see it sends each class $(\epsilon_1,\cdots,\epsilon_n)$ to some class $(\epsilon_1',\cdots,\epsilon_n')$, $\epsilon_i' = \pm 1$. However, $s(W_2;(\epsilon_1',\cdots,\epsilon_n')) = 3a'+b' \neq 3a+b$, a contradiction.

By (a Lee version of) [MWW23, Corollary 4.2] we know $s(S^1 \times S^3;0) = 0$. Since $H_2(S^1 \times S^3) = 0$ the above argument remains valid under connected sums with $S^1 \times S^3$. Since $s(\overline{\mathbb{CP}^2};1) = 1$ by Example 3.5.1 for $n = -1$ and Proposition 2.3.1, the argument remains valid under further connected sums with $\overline{\mathbb{CP}^2}$.

By the $n = 0$ case of Example 3.5.1, we know $s(S^2 \times D^2;\alpha) = 0$ for all α. Any diffeomorphism $k(S^1 \times S^3)\#m\overline{\mathbb{CP}^2}\#W_1\natural(S^2 \times D^2)^{\natural c} \cong k(S^1 \times S^3)\#m\overline{\mathbb{CP}^2}\#W_2\natural(S^2 \times D^2)^{\natural c}$ sends each class $(\epsilon_1,\cdots,\epsilon_{m+n},0^c)$ to some class $(\epsilon_1',\cdots,\epsilon_{m+n}',\alpha_1,\cdots,\alpha_c)$. Again, the s-invariant of the latter class is not equal to the s-invariant of the former class, a contradiction.

By Example 3.5.5, we know the negative $E8$ manifold has $s(E8;0) = 0$. Any diffeomorphism between the above pair with d further copies of $E8$ added sends each $(\epsilon_1,\cdots,\epsilon_{m+n},0^c,0^d)$ to $(\epsilon_1',\cdots,\epsilon_{m+n}',\alpha_1,\cdots,\alpha_c,0^d)$, whose s-invariants still differ, leading to a contradiction.

Finally, by Corollary 2.3.6, the interiors of any of the above pairs are not diffeomorphic. This finishes the proof of (1).

Now we prove (2). If $a,b > 0$, by the connected sum formula, $s(W_1;(1,0,\cdots,0)) = 3 > 1 = s(W_1;(0,\cdots,0,1))$, so there is no diffeomorphism of W_1 whose action on $H_2(W_1)$ sends e_1 to e_{a+b}. On the other hand, by applying a result of Boyer [Boy86] following Freedman [Fre82], any isometry of $(H_2(W_1),\cdot)$ can be realized by a homeomorphism of W_1 rel boundary, so there are certain homeomorphisms of W_1 rel boundary whose action on $H_2(W_1)$ cannot be realized by a diffeomorphism. Such homeomorphisms can be extended by the identity after connect-summing W_1 with copies of the standard manifolds $S^1 \times S^3, \overline{\mathbb{CP}^2}, S^2 \times D^2, E8$ above, but the same argument shows that they still cannot be homotoped to diffeomorphisms. □

3.5.4 Nonpositive shake genus of positive knots

Proof of Theorem 3.1.11(1). By Theorem 3.1.15(2), if K' is a positive knot and $n \leq 0$, then $s(X_n(K');1) = s(K') - n$. By Proposition 3.1.14, if K is a knot concordant to K', then $s(X_n(K);1) = s(X_n(K');1)$. Thus (3.1) implies $g_{sh}^n(K) \geq s(K')/2$. On the other hand, since K' is positive, Rasmussen's bound $s(K')/2 \leq g_4(K')$ achieves equality. Since $g_4(K') = g_4(K) \geq g_{sh}^n(K)$, all these quantities are equal. $\square$

3.5.5 Another comparison result for lasagna s-invariants

We deduce a comparison result for lasagna s-invariants in the spirit of Proposition 3.4.3 in Section 3.4.2. As in the setup there, assume X is a 2-handlebody obtained by attaching 2-handles to a framed link $K = K_1 \cup \cdots \cup K_m \subset S^3 = \partial B^4$, and X' is one obtained by attaching handles to $K' = K_1' \cup \cdots \cup K_m'$. For simplicity, the boundary link L is taken to be empty.

Proposition 3.5.9. *Suppose $C\colon K \to K'$ is a framed concordance in $k\overline{\mathbb{CP}^2}$ that restricts to concordances $C_i\colon K_i \to K_i'$. Then*

$$s(X';\alpha) \leq s(X;\alpha) - |[C(\alpha)]|$$

for any $\alpha \in H_2(X) \cong H_2(X') \cong \mathbb{Z}^m$, where $[C(\alpha)] = \sum_i \alpha_i [C_i] \in H_2(k\overline{\mathbb{CP}^2}) \cong \mathbb{Z}^k$.

One can formulate a proof of Proposition 3.5.9 using the adjunction inequality for the s-invariant, Corollary 1.1.4 in Chapter 1, and the limit formula for lasagna s-invariants of 2-handlebodies, Proposition 3.4.4, as was done in [RW24]. We present a better proof below, assuming the part of Proposition 3.1.16 about the lasagna s-invariants of $\overline{\mathbb{CP}^2}$. Since the latter depends on the computation of the s-invariants of the torus links $T(n,n)_{p,q}$, Theorem 1.1.1 in Chapter 1, it is not completely independent of the other approach.

Proof. By assumption, there is an embedding $i\colon X \hookrightarrow X' \# k\overline{\mathbb{CP}^2}$, under which $i_*\alpha = \alpha + C(\alpha)$. It follows from Corollary 3.3.15 and Theorem 3.3.8(4) that $s(X;\alpha) \geq s(X';\alpha) + s(k\overline{\mathbb{CP}^2};[C(\alpha)]) = s(X';\alpha) + |[C(\alpha)]|$. $\square$

We now deduce Theorem 3.1.11(2).

Proof of Theorem 3.1.11(2). Putting a framing on Σ makes it a framed cobordism from (K,n) to $(K', n-[\Sigma]^2)$. It follows from Proposition 3.5.9 that $s(X_n(K);1) \geq s(X_{n-[\Sigma]^2}(K');1) + |[\Sigma]|$. If $n \leq [\Sigma]^2$, by Theorem 3.1.15 we know $s(X_{n-[\Sigma]^2}(K');1) = s(K') - n + [\Sigma]^2$, thus the statement follows. $\square$

3.5.6 Yasui's family of knot trace pairs

Yasui [Yas15, Figure 10] defined two families of satellite patterns $P_{n,m}$, $Q_{n,m}$, $n,m \in \mathbb{Z}$ (see Figure 3.2). The patterns $P_{n,0}$ are also known as the (twisted) Mazur pattern [Maz61,

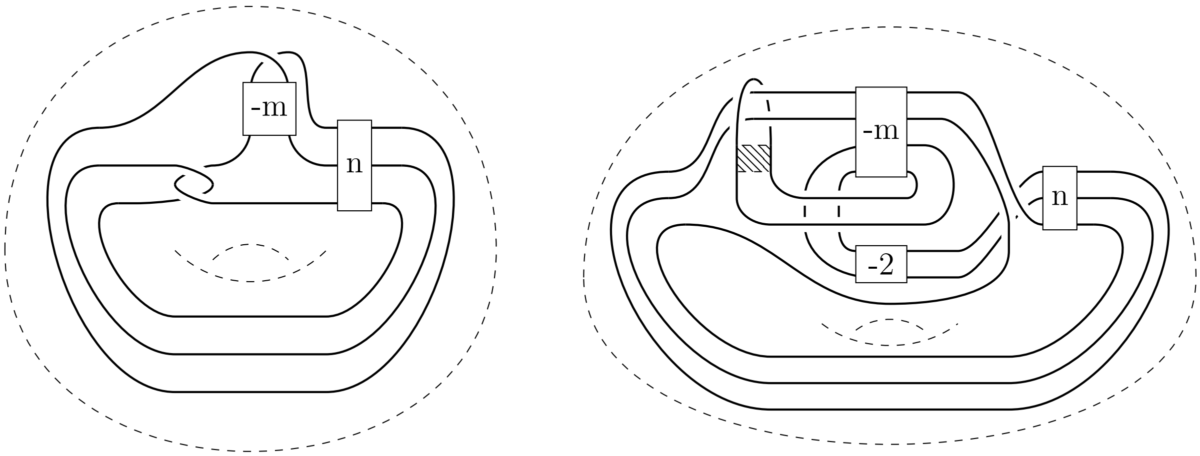


Figure 3.2: Satellite patterns $P_{n,m}$ (left) and $Q_{n,m}$ (right). The boxes denote full twists. The shaded band gives rise to a concordance between $Q_{n,m}$ and the identity pattern.

Fig. 1], and have been the subject of study in many papers. Yasui showed that for any n, m and any knot $K \subset S^3$, the two manifolds $X_n(P_{n,m}(K))$ and $X_n(Q_{n,m}(K))$, namely the n-traces on the two corresponding satellite knots, are homeomorphic. Moreover, he showed that if there exists a Legendrian representative $\mathcal{K}$ of K with

$$tb(\mathcal{K}) + |rot(\mathcal{K})| - 1 = 2g_4(K) - 2, \; n \leq tb(\mathcal{K}), \; m \geq 0, \tag{3.28}$$

then $X_n(P_{n,m}(K))$ and $X_n(Q_{n,m}(K))$ are not diffeomorphic, thus form an exotic pair [Yas15, Theorem 4.1]. In fact, under these conditions, the exotica were detected by a discrepancy in shake slice genera: $g^n_{sh}(P_{n,m}(K)) = g_4(K) + 1 > g^n_{sh}(Q_{n,m}(K))$. Thus, the interiors of such pairs are also exotic. The case $K = U$, $n = -1$, $m = 0$ yields the exotic pair $X_{-1}(-5_2)$ and $X_{-1}(P(3, -3, -8))$ in Theorem 3.1.2.

Using lasagna s-invariants, we present a different family (as stated in Theorem 3.1.4) of Yasui's knot trace pairs that are exotic. We recall that our hypothesis on (K, n, m) is that there exists a slice disk Σ of K in some $k\mathbb{CP}^2$ such that

$$s(K) = |[\Sigma]| - [\Sigma]^2, \; n < -[\Sigma]^2, \; m \geq 0. \tag{3.29}$$

Before giving the proof, we compare and discuss the conditions imposed on K by Yasui (3.28) and by us (3.29).

The adjunction inequality for the s-invariant (Corollary 1.1.5) applied to the slice disk Σ implies that $s(K) \geq |[\Sigma]| - [\Sigma]^2$. Therefore the first condition in (3.29) is that this adjunction inequality attains equality for Σ. Similarly, the first condition in (3.28) is that the slice-Bennequin inequality attains equality for $\mathcal{K}$. These two conditions are satisfied on somewhat different families of knots. For example, as remarked by Yasui, his condition on K is satisfied for all positive torus knots. On the other hand, our condition on K is satisfied for all negative torus knots (cf. Lemma 3.5.11). Our condition also has the advantage of being closed under concordances and connected sums of knots.

Proof of Theorem 3.1.4. As observed by Yasui, adding a band as indicated in Figure 3.2 and capping off the unknot component gives a concordance between $Q_{n,m}$ and the identity satellite pattern. Therefore, $Q_{n,m}(K)$ is concordant to K, thus by Proposition 3.1.14 and Proposition 3.4.4 we have

$$\begin{aligned} &s(X_n(Q_{n,m}(K));1) = s(X_n(K);1) \\ =&-\lim_{r\to\infty}(s(-K(r+1,r))+2r)-n-1+1 \leq -s(-K(1,0))-n = s(K)-n. \end{aligned} \tag{3.30}$$

Deleting a local ball in B^4 near Σ, turning the cobordism upside down and reversing the ambient orientation, we obtain a concordance $-\Sigma^\circ\colon K \to U$ in $k\overline{\mathbb{CP}^2}$. Applying the satellite operation $P_{n,m}$ gives a framed concordance $C_1 = P_{n,m}(-\Sigma^\circ)\colon (P_{n,m}(K), n) \to (P_{n+[\Sigma]^2,m}(U), n + [\Sigma]^2)$ in $k\overline{\mathbb{CP}^2}$ with $[C_1] = [-\Sigma^\circ]$ since $P_{n,m}$ has winding number 1. Since $n < -[\Sigma]^2$, there is a framed concordance $C_2\colon (P_{n+[\Sigma]^2,m}(U), n+[\Sigma]^2) \to (P_{-1,m}(U), -1)$ in $k_1\overline{\mathbb{CP}^2}$ obtained by adding k_1 positive twists along the meridian of the satellite solid torus, which has $[C_2] = (1,\cdots,1) \in \mathbb{Z}^{k_1} = H_2(k_1\overline{\mathbb{CP}^2})$, where $k_1 = -1-n-[\Sigma]^2$. Similarly, there is a null-homologous framed concordance $C_3\colon (P_{-1,m}(U), -1) \to (P_{-1,0}(U), -1) = (-5_2, -1)$ in $k_2\overline{\mathbb{CP}^2}$ where $k_2 = m$ (note that the two strands being twisted $-m$ times in $P_{n,m}$ are oppositely oriented). Now Proposition 3.5.9 applied to the concordance $C_1 \cup C_2 \cup C_3\colon (P_{n,m}(K), n) \to (-5_2, -1)$ yields

$$\begin{aligned} &s(X_n(P_{n,m}(K));1) \geq s(X_{-1}(-5_2);1) + |[C_1]| + |[C_2]| + |[C_3]| \\ =&\, 3 + |[\Sigma]| + k_1 + 0 = |[\Sigma]| - [\Sigma]^2 - n + 2 = s(K) - n + 2. \end{aligned} \tag{3.31}$$

Together, (3.30) and (3.31) imply that $s(X_n(P_{n,m}(K));1) > s(X_n(Q_{n,m}(K));1)$, proving the theorem. □

Remark 3.5.10. Let $C \cup K$ be a two-component link in S^3, where C is an unknot and $\ell k(C,K) = 1$. Regarding C as a dotted 1-handle and attaching an n-framed 2-handle along $P_{n,m}(K)$ (resp. $Q_{n,m}(K)$) yields a Mazur manifold Z_P (resp. Z_Q), namely a contractible 4-manifold with one handle of each index $0,1,2$. The manifolds Z_P and Z_Q are homeomorphic, and Hayden–Mark–Piccirillo [HMP21, Theorem 2.7] showed that if $C \cup K$ satisfies the (very mild) assumptions that

- K is not the meridian of C,
- every self-homeomorphism of ∂Z_P preserves the JSJ torus (with orientation) given by the image of the satellite torus of $P_{n,m}(K)$ in ∂Z_P,

then Z_P being diffeomorphic to Z_Q implies that $X_n(P_{n,m}(K))$ is diffeomorphic to $X_n(Q_{n,m}(K))$. Therefore, by choosing suitable $C \cup K$, Theorem 3.1.4 also yields various families of exotic Mazur manifolds. A more explicit detection of exotic Mazur manifolds using Khovanov homology appears in [Nah25], based on ideas in Section 3.5.11.

As an example, we present a family of knots K for which Theorem 3.1.4 applies.

Lemma 3.5.11. *For every negative torus knot $-T(p,q)$, there exists a slice disk Σ in some $k\mathbb{CP}^2$ with $|[\Sigma]| - [\Sigma]^2 = -(p-1)(q-1) = s(-T(p,q))$.*

Proof. We induct on $p+q$. For $p = q = 1$, $-T(p,q) = U$ is the unknot and we take Σ to be the trivial slice disk in B^4.

For the induction step, by symmetry, we may assume $p > q \geq 1$. There is a positive twist along q strands in $-T(p,q)$ that converts it into $-T(p-q,q)$. This gives a concordance $C\colon -T(p-q,q) \to -T(p,q)$ in $\mathbb{CP}^2$ with $[C] = q[\mathbb{CP}^1]$. By induction hypothesis, there is a slice disk Σ' of $-T(p-q,q)$ in $k'\mathbb{CP}^2$ with $|[\Sigma']| - [\Sigma']^2 = -(p-q-1)(q-1)$. Then $\Sigma = \Sigma' \cup C$ is a slice disk of $-T(p,q)$ in $(k'+1)\mathbb{CP}^2$ with $|[\Sigma]| - [\Sigma]^2 = -(p-1)(q-1)$. $\square$

3.5.7 Conway knot cables can obstruct its sliceness

Recently, Piccirillo [Pic20] famously proved that the Conway knot, denoted *Conway*, is not slice. Building on her proof, we show the following.

Proposition 3.5.12. *Let $Conway(n^+, n^-)$ denote the 0-framed $(n^+ + n^-)$-cable of the Conway knot, with the orientation on n^- of the strands reversed. Then for some $n > 0$ we have*

$$s(Conway(n+1,n)) > -2n = s(U^{\sqcup(2n+1)}).$$

Here $U^{\sqcup m}$ is the m-component unlink.

If *Conway* were slice, then any cable of it is concordant to the corresponding cable of the unknot, implying that they have the same s-invariant. Therefore, Proposition 3.5.12 shows that, if one had enough computational power, we could have obstructed the sliceness of the Conway knot by computing the s-invariants of its cables. In practice, however, the computation of $s(Conway(2,1))$ seems already out of reach.

Proof. Piccirillo [Pic20] found a knot K' with $X_0(K') = X_0(Conway)$ and $s(K') = 2$. Let X be the orientation-reversal of this common trace, i.e. $X = X_0(-Conway) = X_0(-K')$. By Proposition 3.4.4, we have

$$s(X;1) = -\lim_{n\to\infty}(s(K'(n+1,n)) + 2n) \leq -s(K'(1,0)) = -s(K') = -2.$$

Again, by Proposition 3.4.4 we have $\lim_{n\to\infty}(s(Conway(n+1,n))+2n) = -s(X;1) \geq 2$ and hence $s(Conway(n+1,n)) + 2n \geq 2 > 0$ for all sufficiently large n. □

3.5.8 Knot traces of torus knots

In this section, we use the computation of the s-invariants of torus links with various orientations presented in Chapter 1, namely Theorem 1.1.1, to determine the complete structure of the Lee skein lasagna modules of large negative traces on negative torus links.

Recall the notation for torus links from Chapter 1: for $d = r + s > 0$ and coprime $p, q \neq 0$, $T(dp, dq)_{r,s}$ denote the torus link $T(dp, dq)$ equipped with an orientation where r of the strands are oriented against the other s strands.

We study the s-invariant of $T(p,q)^k(r,s)$, the k-framed $(r+s)$-cable of $T(p,q)$, where s of the strands have their orientation reversed.

Lemma 3.5.13. *For any $p, q > 0$, $k \geq pq$, $r, s \geq 0$, we have*

$$s(T(p,q)^k(r,s)) = k|r-s|^2 - (k - pq + p + q)|r-s| - 2\min(r,s) + 1.$$

Proof. When $d := r + s = 0$, the statement is true as $s(\emptyset) = 1$. Suppose $d > 0$. Twisting $k - pq$ times along the d parallel strands in $T(dp,dq)_{r,s} = T(p,q)^{pq}(r,s)$ yields $T(p,q)^k(r,s)$; this gives rise to a cobordism in $(k-pq)\overline{\mathbb{CP}^2}$ from $T(dp,dq)_{r,s}$ to $T(p,q)^k(r,s)$. The adjunction inequality for the s-invariant (Corollary 1.1.4) applied to this cobordism, plus the computation of the s-invariants of positive torus links with various orientations (Theorem 1.1.1) implies that

$$\begin{aligned} s(T(p,q)^k(r,s)) &\leq s(T(dp,dq)_{r,s}) + (k-pq)(|r-s|^2 - |r-s|) \\ &= k|r-s|^2 - (k-pq+p+q)|r-s| - 2\min(r,s) + 1. \end{aligned}$$

On the other hand, since one can cap off $\min(r,s)$ pairs of oppositely oriented components in $T(p,q)^k(r,s)$ by annuli to obtain $T(p,q)^k(|r-s|,0)$, we have

$$\begin{aligned} s(T(p,q)^k(r,s)) &\geq s(T(p,q)^k(|r-s|,0)) - 2\min(r,s) \\ &= k|r-s|^2 - (k-pq+p+q)|r-s| - 2\min(r,s) + 1 \end{aligned}$$

where the last equality holds as one can find a positive diagram of $T(p,q)^k(|r-s|,0)$, say obtained by adding $k-pq$ full twists to a standard positive diagram of $T(p|r-s|, q|r-s|)$, and then using the fact that the s-invariant of a link with a positive diagram D is equal to the writhe of D minus twice the number of Seifert circles of D plus 1. □

It follows from Lemma 3.5.13 and Proposition 3.4.4 that for any $p, q > 0$, $k \geq pq$ and $\alpha \in H_2(X_{-k}(-T(p,q))) \cong \mathbb{Z}$, we have

$$\begin{aligned}
&s(X_{-k}(-T(p,q));\alpha)\\
=&-\lim_{r\to\infty}(s(T(p,q)^k(\alpha^+ + r, \alpha^- + r)) + 2r) + w(T(p,q)^k(\alpha^+ + r, \alpha^- + r)) - |\alpha| + 1\\
=&-\lim_{r\to\infty}(k|\alpha|^2 - (k - pq + p + q)|\alpha| - 2r + 1 + 2r) + k|\alpha|^2 - |\alpha| + 1\\
=&(k - pq + p + q - 1)|\alpha|.
\end{aligned} \tag{3.32}$$

We remark that (3.32) only yields vacuous lower bounds for the smooth genus function of $X_{-k}(-T(p,q))$ via Corollary 3.3.14.

Theorem 3.3.1 states that the Lee skein lasagna module of $X_{-k}(-T(p,q))$ is generated by Lee canonical generators x_{α^+,α^-}, $\alpha^+, \alpha^- \in H_2(X_{-k}(-T(p,q)))$. The generator x_{α^+,α^-} has homological degree $-2\alpha^+ \cdot \alpha^-$ ($\cdot$ is the intersection form on H_2 instead of the product on $\mathbb{Z}$) and homology class degree $\alpha^+ + \alpha^-$. Moreover, (3.32) and (3.7) imply it has quantum filtration degree

$$s(X_{-k}(-T(p,q)); \alpha^+ - \alpha^-) + 2\alpha^+ \cdot \alpha^- = (k - pq + p + q - 1)|\alpha^+ - \alpha^-| + 2\alpha^+ \cdot \alpha^-.$$

These data (together with Theorem 3.3.1(4)) completely determine the structure of $\mathcal{S}_0^{Lee}(X_{-k}(-T(p,q)))$. More explicitly, we have the following.

Proposition 3.5.14. *If $p, q > 0$ are coprime and $k \geq pq$, then $s(X_{-k}(-T(p,q));\alpha) = (k - pq + p + q - 1)|\alpha|$. Therefore, the associated graded vector space of $\mathcal{S}_0^{Lee}(X_{-k}(-T(p,q)))$ is given by*

$$gr_*(\mathcal{S}^{Lee}_{0,(\beta^2-\alpha^2)/2}(X_{-k}(-T(p,q));\alpha)) = \mathbb{Q}$$

for

$$\beta \equiv \alpha \pmod 2, \quad * = \begin{cases} \frac{\alpha^2-\beta^2}{2} + (k - pq + p + q - 1)|\beta| - 1 \pm 1, & \beta \neq 0 \\ \frac{\alpha^2}{2}, & \beta = 0, \end{cases}$$

and 0 in other tri-degrees. □

In the standard diagram, the number of negative crossings of $-T(p,q)$ is $pq - p$ or $pq - q$. Thus the nonvanishing bound on k given by Proposition 3.5.14 is better than the one given by Theorem 3.1.15. However, Corollary 3.1.17 gives an even better bound. For example, $-T(n, n+1)$ is n^2-slice in $\mathbb{CP}^2$, thus $X_{-k}(-T(n, n+1))$ has nonvanishing Lee and Khovanov skein lasagna module for $k \geq n^2$.

Finally we note that the computation of the s-invariants for negative torus links with various orientations, Corollary 1.1.3, implies that for $p, q > 0$, $s(X_{pq}(T(p,q));\alpha) = -\infty$ for all α. However, since $TB(-T(p,q)) = -pq$ [EH01], we already know from Corollary 3.4.1 that $s(X_k(T(p,q));\alpha) = -\infty$ for all $k \geq pq$ and all α.

3.5.9 Expectations and computations for $\overline{\mathbb{CP}^2}$

In this section, we present a conjectural formula for the full structure of the Khovanov skein lasagna module of $\overline{\mathbb{CP}^2}$ over $\mathbb{Q}$, before proving Proposition 3.1.16 which addresses the cases that we can currently compute.

Since $\overline{\mathbb{CP}^2}$ is the (-1)-trace on the unknot with an additional 4-handle attached, the 2-handlebody formula (2.18) and Proposition 2.3.1 express the Khovanov skein lasagna module of $\overline{\mathbb{CP}^2}$ as a colimit

$$\mathcal{S}_0^2(\overline{\mathbb{CP}^2};\alpha;\mathbb{Q}) = \mathrm{colim}_{r\to\infty} q^{-|\alpha|-2r} KhR_2(-T(|\alpha|+2r,|\alpha|+2r)_{\alpha_++r,\alpha_-+r};\mathbb{Q})^{S_{|\alpha|+2r}} \quad (3.33)$$

along the symmetrized dotted annulus creation maps. Conjecture 1.6.6 in Chapter 1 for the 1-framed unknot, or equivalently Conjecture 1.6.2 or Conjecture 1.6.1, would imply that these morphisms are injective, implying the following conjectural formula for $\mathcal{S}_0^2(\overline{\mathbb{CP}^2};\mathbb{Q})$. Note that by (2.19), all $\mathcal{S}_0^2(\overline{\mathbb{CP}^2};\alpha;\mathbb{Q})$ are determined from those with $\alpha = 0, 1$.

Conjecture 3.5.15. *Let $K_n(t,q)$ denote the Poincaré polynomial of $Kh(T(n,n-1);\mathbb{Q})$ for $n>0$. Then $\mathcal{S}_0^2(\overline{\mathbb{CP}^2};0;\mathbb{Q})$ has Poincaré polynomial*

$$1 + \sum_{n=1}^{\infty} t^{-2n^2} q^{6n^2-4n+1} K_{2n}(t,q^{-1}) \quad (3.34)$$

and $\mathcal{S}_0^2(\overline{\mathbb{CP}^2};1;\mathbb{Q})$ has Poincaré polynomial

$$\sum_{n=1}^{\infty} t^{-2n^2+2n} q^{6n^2-10n+4} K_{2n-1}(t,q^{-1}). \quad (3.35)$$

More generally, Conjecture 1.6.6 implies similar conjectural formulas for knot traces with small enough framings, but the case of $\overline{\mathbb{CP}^2}$ is more explicit in that $K_n(t,q)$ also admits a conjectural recursive formula due to Shumakovitch and Turner; see Conjecture 1.6.3. We have verified that this conjecture as well as Conjecture 3.5.15 are consistent with the data of $Kh(T(n,n))$ for $n \le 8$. Ritter von Merkl [Rit25, Conjecture 4.3] converted Conjecture 3.5.15 into a closed formula for $\mathcal{S}_{0,0}^2(\overline{\mathbb{CP}^2};\mathbb{Q})$.

In the rest of the section, we present some computations available for $\overline{\mathbb{CP}^2}$.

The $p=q=k=1$ case of Proposition 3.5.14 gives

$$gr_{2p^2+2p}\mathcal{S}^{Lee}_{0,-2p^2}(\overline{\mathbb{CP}^2};0) = \mathbb{Q},\ \ p \ge 0,$$

which by Theorem 3.1.13 implies that $\mathrm{rank}(\mathcal{S}^2_{0,-2p^2,2p^2+2p}(\overline{\mathbb{CP}^2};0)) \ge 1$.

On the other hand, one can check that the only contribution to the term $t^{-2p^2}q^{2p^2+2p}$ in (3.34) comes from the $n = p$ term in which $K_{2p}(t, q^{-1}) = q^{-(4p^2-6p+1)} + q^{-(4p^2-6p+3)} + \cdots$ where $\cdots$ are terms with higher t-degrees (cf. [Kho03, Proposition 6.1], Theorem 1.2.1(2)). Thus Conjecture 3.5.15 predicts $\mathcal{S}^2_{0,-2p^2,2p^2+2p}(\overline{\mathbb{CP}^2}; 0; \mathbb{Q}) = \mathbb{Q}$.

In fact, we can do better by directly resorting to Theorem 1.2.1(1) to see that after renormalization, the generators of $Kh^{2(n+p)(n-p),q_{2n,2n}(2(n+p)(n-p))}(T(2n, 2n)) \cong \mathbb{Z}$ in the notation there are identified along the dotted annulus cobordism maps in the 2-handlebody formula, Proposition 2.6.5 (they are acted on trivially by the symmetric group, by comparing to the statement over $\mathbb{Q}$), thus descend to generators of

$$\mathcal{S}^2_{0,-2p^2,2p^2+2p}(\overline{\mathbb{CP}^2}; 0) \cong \mathbb{Z},\ p \geq 0. \tag{3.36}$$

Similarly,

$$\mathcal{S}^2_{0,-2p^2+2p,2p^2-1}(\overline{\mathbb{CP}^2}; 1) \cong \mathbb{Z},\ p \geq 1, \tag{3.37}$$

and a generator comes from generators of $Kh^{2(n+p-1)(n-p),q_{2n-1,2n-1}(2(n+p-1)(n-p))}(T(2n-1, 2n-1)) \cong \mathbb{Z}$. This verifies Conjecture 3.5.15 in these tri-gradings.

Proof of Proposition 3.1.16. The statements for lasagna s-invariants and Lee skein lasagna modules follow from the $p = q = k = 1$ case of Proposition 3.5.14 and Proposition 2.3.1. The statement for Khovanov skein lasagna modules follows from (3.36) and (3.37) for $\alpha = 0, 1$, and (2.19) in general. $\square$

Proof of Corollary 3.1.17. By the connected sum formula for lasagna s-invariants (Theorem 3.3.8(4)) and Proposition 3.1.16, $s(k\overline{\mathbb{CP}^2}; \alpha) = |\alpha|$. Thus by Corollary 3.3.15 we have $s(X; \alpha) \geq |i_*\alpha|$. The equality when $\alpha = 0$ follows from Corollary 3.3.16.

We have $gr_q(\mathcal{S}^{Lee}_{0,0}(X; \alpha)) \neq 0$ for $q = s(X; \alpha)$ by Definition 3.3.7, and for $q = s(X; \alpha) - 2$ if the image of α in $H_2(X; \mathbb{Q})$ is nonzero by Theorem 3.3.1(4). When X is a 2-handlebody, the second part of the statement thus follows from Theorem 3.1.13. $\square$

For the purpose of the next section, we record the following local finiteness result for $\mathcal{S}^2_0(\overline{\mathbb{CP}^2}; \mathbb{Q})$. By using the exact triangle (1.21) from [GW23], we are able to simplify the proof in [RW24].

Proposition 3.5.16. *The Poincaré polynomials of $\mathcal{S}^2_0(\overline{\mathbb{CP}^2}; \alpha; \mathbb{Q})$ for $\alpha = 0, 1$ are bounded above by (3.34) and (3.35), respectively. In particular, the total dimension of $\mathcal{S}^2_{0,\geq h_0,*}(\overline{\mathbb{CP}^2}; \alpha; \mathbb{Q})$ is finite for every $h_0 \in \mathbb{Z}$, $\alpha \in H_2(\overline{\mathbb{CP}^2}) \cong \mathbb{Z}$.*

Proof. The upper bounds follow from the colimit formula (3.33) and the exact triangle (1.21). The claim on the finiteness then follows from the upper bound on the homological degree of $Kh(T(n, n-1))$ given by Theorem 1.2.1(2) in Chapter 1 for $\alpha = 0, 1$, hence for all α by (2.19). $\square$

3.5.10 $\overline{\mathbb{CP}^2}$-stably standard homotopy spheres

We show that the Khovanov skein lasagna module over $\mathbb{Q}$ is unable to detect potential exotic 4-spheres that dissolve after taking connected sums with copies of $\overline{\mathbb{CP}^2}$. Note that homotopy 4-spheres coming from Gluck twists dissolve after a single connected sum with $\overline{\mathbb{CP}^2}$.

Proposition 3.5.17. *If Σ is a homotopy 4-sphere with $\Sigma \# k\overline{\mathbb{CP}^2} = k\overline{\mathbb{CP}^2}$ for some k, then $\mathcal{S}_0^2(\Sigma; \mathbb{Q}) = \mathbb{Q}$, concentrated in tri-degree $(0, 0, 0)$.*

Proof. Applying $\mathcal{S}_0^2$ to $\Sigma \# k\overline{\mathbb{CP}^2} = k\overline{\mathbb{CP}^2}$ and using the connected sum formula (Proposition 2.3.2), we obtain $\mathcal{S}_0^2(\Sigma; \mathbb{Q}) \otimes \mathcal{S}_0^2(\overline{\mathbb{CP}^2}; \mathbb{Q})^{\otimes k} \cong \mathcal{S}_0^2(\overline{\mathbb{CP}^2}; \mathbb{Q})^{\otimes k}$. Since $\mathcal{S}_0^2(\overline{\mathbb{CP}^2}; \mathbb{Q})$ is nonzero and satisfies a suitable finiteness condition given by Proposition 3.5.16, we conclude that $\mathcal{S}_0^2(\Sigma; \mathbb{Q}) = \mathbb{Q}$. $\square$

In Section 4.7 in Chapter 4, we will give a more direct proof that Gluck twists induce isomorphisms on Khovanov skein lasagna modules.

3.5.11 Induced map of link cobordisms in $k\overline{\mathbb{CP}^2}$

Unless otherwise stated, in this section we work over a field $\mathbb{F}$, which will be dropped from the notation throughout.

We begin with a general construction. Suppose X is a closed 4-manifold and $L_0, L_1 \subset S^3$ are two framed oriented links. A cobordism from L_0 to L_1 in X is a properly embedded framed oriented surface $\Sigma \subset X^{\circ\circ} := X \backslash (int(B^4) \sqcup int(B^4))$ with $\partial\Sigma = -L_0 \sqcup L_1 \subset S^3 \sqcup S^3$. Thus Σ is a skein in $X^{\circ\circ}$ rel $-L_0 \sqcup L_1$ without input balls, which itself defines a lasagna filling with homological degree 0, quantum degree $-\chi(\Sigma)$, and homology class degree $[\Sigma]$, which represents a class $x_\Sigma \in \mathcal{S}^2_{0,0,-\chi(\Sigma)}(X^{\circ\circ}; -L_0 \sqcup L_1; [\Sigma])$.

On the other hand, since $(X^{\circ\circ}, -L_0 \sqcup L_1) = (X, \emptyset) \# (B^4, -L_0) \# (B^4, L_1)$, the connected sum formula (Proposition 2.3.2) and Proposition 2.3.8 imply that over a field $\mathbb{F}$,

$$\begin{aligned}
\mathcal{S}_0^2(X^{\circ\circ}; -L_0 \sqcup L_1; [\Sigma]) &\cong \mathcal{S}_0^2(X; [\Sigma]) \otimes \mathcal{S}_0^2(B^4; -L_0) \otimes \mathcal{S}_0^2(B^4; L_1) \\
&\cong \mathcal{S}_0^2(X; [\Sigma]) \otimes KhR_2(-L_0) \otimes KhR_2(L_1) \\
&\cong \mathcal{S}_0^2(X; [\Sigma]) \otimes \mathrm{Hom}(KhR_2(L_0), KhR_2(L_1)).
\end{aligned}$$

If $\theta \in (\mathcal{S}_0^2(X; [\Sigma]))^*_{h_0, q_0}$ is a dual Khovanov skein lasagna class, then we obtain a homomorphism

$$KhR_2(\Sigma, \theta) := \theta(x_\Sigma) \colon KhR_2^{*,*}(L_0) \to KhR_2^{*+h_0, *+q_0-\chi(\Sigma)}(L_1).$$

In the case when $X = \overline{\mathbb{CP}^2}$, there are natural dual lasagna classes, one for each homology class $\alpha \in H_2(X)$. In fact, by Proposition 3.1.16, for each $\alpha \in H_2(\overline{\mathbb{CP}^2})$, we have $\mathcal{S}^2_{0,0,|\alpha|}(\overline{\mathbb{CP}^2};\alpha;\mathbb{Z}) = \mathbb{Z}$. Therefore over a field $\mathbb{F}$ and up to sign, $(\mathcal{S}^2_0(\overline{\mathbb{CP}^2};\alpha))^*_{0,-|\alpha|} \cong \mathbb{F}$ has a canonical dual lasagna generator θ_α. Thus for a cobordism $\Sigma\colon L_0 \to L_1$ in $\overline{\mathbb{CP}^2}$, there is an induced map

$$KhR_2(\Sigma) := KhR_2(\Sigma,\theta_{[\Sigma]})\colon KhR_2^{*,*}(L_0) \to KhR_2^{*,*-\chi(\Sigma)-|[\Sigma]|}(L_1)$$

defined up to sign. Translate back to the more familiar Khovanov homology and reverse the ambient orientation, we see if $\Sigma\colon L_0 \to L_1$ is a cobordism in $\mathbb{CP}^2$ between unframed oriented links, then there is an induced map $Kh(\Sigma)\colon Kh(L_0) \to Kh(L_1)$ defined up to sign, with bidegree $(0,\chi(\Sigma)-[\Sigma]^2+|[\Sigma]|)$, at least over a field. This map only depends on the isotopy class of Σ rel boundary.

In fact, we can also give the following more direct construction of $Kh(\Sigma)\colon Kh(L_0) \to Kh(L_1)$ over $\mathbb{Z}$, which was suggested to us by Hongjian Yang. By transversality, suppose Σ intersects the core $\mathbb{CP}^1$ of $\mathbb{CP}^2$ transversely, in some p points positively and q points negatively. A tubular neighborhood of $\mathbb{CP}^1$ has boundary S^3 on which Σ is the torus link $-T(p+q,p+q)_{p,q} \subset S^3$ (the torus link $-T(p+q,p+q)$ equipped with an orientation where p of the strands are oriented against the other q strands). Tubing this boundary S^3 with the negative boundary of $(\mathbb{CP}^2)^{\circ\circ}$ changes Σ into a cobordism $\Sigma^\circ\colon L_0 \sqcup (-T(p+q,p+q)_{p,q}) \to L_1$ in $I \times S^3$, inducing a map

$$Kh(\Sigma^\circ)\colon Kh(L_0)\otimes Kh(-T(p+q,p+q)_{p,q}) \to Kh(L_1) \tag{3.38}$$

up to sign with bidegree $(0,\chi(\Sigma^\circ))$ where $\chi(\Sigma^\circ) = \chi(\Sigma)-p-q$. On the other hand, Corollary 1.2.2 in Chapter 1 implies that $Kh^{0,-(p-q)^2+2\max(p,q)}(-T(p+q,p+q)_{p,q};\mathbb{Z}) \cong \mathbb{Z}$. Plugging a generator of it into (3.38) defines a map

$$Kh(\Sigma)\colon Kh(L_0) \to Kh(L_1)$$

over $\mathbb{Z}$ up to sign with bidegree $(0,\chi(\Sigma^\circ)-(p-q)^2+2\max(p,q)) = (0,\chi(\Sigma)-[\Sigma]^2+|[\Sigma]|)$ as $[\Sigma] = \pm(p-q)[\mathbb{CP}^1]$.

We argue briefly that this direct construction agrees with the previous one over a field. We begin with the lasagna construction. Realize $(\overline{\mathbb{CP}^2})^{\circ\circ}$ as $I \times S^3 \# B^4$ with a 2-handle attached to ∂B^4 along a (-1)-framed unknot and another 4-handle attached. Then a suitable generalization of the 2-handlebody formula (see [MWW23, Theorem 3.2]) gives

$$\begin{aligned}&\mathcal{S}^2_0((\overline{\mathbb{CP}^2})^{\circ\circ};-L_0\sqcup L_1;\alpha)\\ \cong& \left(\bigoplus_r q^{-|\alpha|-2r}\mathcal{S}^2_0(I\times S^3\#B^4;-L_0\sqcup L_1\sqcup -T(\alpha^++r,\alpha^-+r))\right)\Big/\sim\end{aligned} \tag{3.39}$$

for some equivalence relation $\sim$, and the element x_Σ for a skein Σ with $[\Sigma]=\alpha$ is represented by the class of Σ° as a lasagna filling in the right-hand side of (3.39) (Σ° is a skein without input balls in $I\times S^3\#B^4$ rel some $-L_0\sqcup L_1\sqcup -T(p+q,p+q)_{p,q}$ obtained by cutting out a tubular neighborhood of a core $\overline{\mathbb{CP}^1}\subset\overline{\mathbb{CP}^2}$). The evaluation of x_Σ by the dual lasagna class $\theta_{[\Sigma]}\in\mathcal{S}_0^2(\overline{\mathbb{CP}^2})^*$ is realized by regarding Σ° as a skein in $I\times S^3$ rel $-L_0\sqcup L_1$, where $-T(p+q,p+q)_{p,q}\subset\partial B^4$ is regarded as an input link on an input ball (so its orientation is reversed) with decoration given by a generator of $KhR_2^{0,-2\max(p,q)}(T(p+q,p+q)_{p,q})\cong\mathbb{F}$. Tubing this input ball to the negative boundary, then Proposition 2.3.8 shows the equivalence to the direct construction.

In the case when a cobordism $\Sigma\colon L_0\to L_1$ in $\mathbb{CP}^2$ is given by a negative full twist along some $p+q$ strands of L (p of which oriented against the other q), the induced map $Kh(\Sigma)\colon Kh(L_0)\to Kh(L_1)$ can be described by putting a local torus link $-T(p+q,p+q)_{p,q}$ near L_0, decorated by a generator of $Kh^{0,-(p-q)^2+2\max(p,q)}(-T(p+q,p+q)_{p,q})$, and performing $p+q$ saddles between $-T(p+q,p+q)_{p,q}$ and L_0 to realize the twist.

We note that the same constructions of induced maps on Kh—both the lasagna approach and the direct approach—also apply to cobordisms in $k\overline{\mathbb{CP}^2}$. For the lasagna approach, this is because the preferred dual lasagna classes θ_α of $\overline{\mathbb{CP}^2}$ give rise to preferred dual lasagna classes $\otimes_{i=1}^k\theta_{\alpha_i}$ of $k\overline{\mathbb{CP}^2}$. For the direct approach, we simply cut out neighborhoods of every core $\mathbb{CP}^1$ in $k\mathbb{CP}^2$. We summarize as follows.

Proposition 3.5.18. *For an oriented link cobordism $\Sigma\colon L_0\to L_1$ in $k\mathbb{CP}^2$, there is an induced map*

$$Kh(\Sigma)\colon Kh(L_0)\to Kh(L_1)$$

defined over any coefficient field up to sign, with bidegree $(0,\chi(\Sigma)-[\Sigma]^2+|[\Sigma]|)$, which only depends on the isotopy class of Σ rel boundary. It agrees with the classical induced map for link cobordisms in $I\times S^3$ when $k=0$. Moreover, if $\Sigma'\colon L_1\to L_2$ is another oriented link cobordism in some $k'\mathbb{CP}^2$, then $Kh(\Sigma'\circ\Sigma)=Kh(\Sigma')\circ Kh(\Sigma)$ up to sign. $\square$

The two perspectives complement each other. The direct construction has the advantage that the induced map is computable from diagrams; however, it is not a priori clear that it is independent of the choices of the core $\mathbb{CP}^1$'s and the paths used to tube the extra boundaries to the negative boundary. The lasagna construction is manifestly well-defined and satisfies the functoriality statement in Proposition 3.5.18; yet it is not directly computable.

Our construction has been exploited by Nahm [Nah25] to detect an exotic pair of Mazur manifolds.

If $L\subset S^3$ and $D=\sqcup_{i=1}^k D_i$ is a disjoint union of 2-disks intersecting L transversely in the interior, such that the algebraic intersection between each D_i and L is zero, then twisting along the various D_i induces maps whose colimit recovers the Khovanov homology of L

regarded as a link in $\#^k(S^1\times S^2)$, the 0-surgery of S^3 along ∂D. This homology theory for links in $\#^k(S^1\times S^2)$ is defined by Rozansky [Roz10] and Willis [Wil21], and will be the main focus in Chapter 4.

3.5.12 A conjecture for Stein manifolds

Some similarities between skein lasagna modules and other gauge/Floer-theoretic smooth invariants appear in our study. For example, the skein lasagna modules are sensitive to orientations, and there is an adjunction-type inequality (Theorem 3.1.10) for lasagna s-invariants. However, a serious constraint on our computation is that we were not able to produce any 4-manifold with $b_2^+>0$ that has finite lasagna s-invariants or nonvanishing Khovanov skein lasagna module. In fact, if one were able to find a simply-connected closed 4-manifold X with $b_2^+(X)>0$ that has $\mathcal{S}_0^2(X)\neq 0$, then $X\#\overline{\mathbb{CP}^2}$ is an exotic closed 4-manifold, as it has nonvanishing $\mathcal{S}_0^2$ while its standard counterpart (some $k\mathbb{CP}^2\#\ell\overline{\mathbb{CP}^2}$) has vanishing $\mathcal{S}_0^2$.

To pursue this thread, we propose the following conjecture, which seems plausible and desirable.

Conjecture 3.5.19. *If K is a knot and $X=X_n(K)$ for some $n<TB(K)$ (hence X is Stein [Eli90; Gom98]), then $s(X;0)=0$ and $s(X;1)=s(K)-n$. In particular, $\mathcal{S}_0^2(X)\neq 0$. More generally, any Stein 2-handlebody X has $s(X;0)>-\infty$ and nonvanishing Khovanov skein lasagna module.*

It might not be completely unreasonable to hope that one can take the constants $N_+(K),N_-(K)$ in Conjecture 1.6.6 in Chapter 1 to be $TB(-K),TB(K)-1$, respectively. This would imply Conjecture 3.5.19 for the case of knot traces in a very strong sense.

The part concerning $s(X;1)$ in Conjecture 3.5.19 implies the n-shake genus of K for $n<TB(K)$ (or more generally $n<TB(K')$ for some K' concordant to K) is bounded below by $s(K)/2$, generalizing Theorem 3.1.11(1). This seems to be the proper refinement of the bound from the Stein adjunction inequality (3.2).

By Proposition 3.4.4, Conjecture 3.5.19 is equivalent to an assertion on the s-invariants of cables of K (or some link in the 2-handlebody case).

Chapter 4

Khovanov skein lasagna modules with 1-dimensional inputs

In Section 2.4.2 in Chapter 2, we defined skein lasagna modules with input given by a lasagna $(3, 1, 1)$-link homology theory, but we have not provided an example of such a link homology theory, except for a universally trivial one presented in Section 2.5. It is the purpose of this chapter to present an interesting such theory, built on the theory of Khovanov homology for (admissible) links in $\#^m(S^1 \times S^2)$ defined by Rozansky [Roz10] and Willis [Wil21]. This chapter contains some small modifications compared with [Ren+25], which allow us to prove the main functoriality result on the chain level over the integers up to homotopy.

4.1 Introduction

4.1.1 Main results and constructions of skein lasagna modules

For simplicity, we first state the main functoriality result and carry out the main construction of this chapter in the special case of null-homologous links. Recall from Section 2.4.2 that $\mathbf{Links}_1$ is the category whose objects are framed oriented links in oriented manifolds diffeomorphic to some $\sqcup_{i=1}^k \#^{m_i}(S^1 \times S^2)$ and whose morphisms are link cobordisms in relative 4-dimensional 1-handlebody complements. Let $\mathbf{Links}_{1,0\text{-div}} \subset \mathbf{Links}_1$ be the full subcategory of null-homologous links. Let $\mathbf{K}^-(\mathrm{Free}_{fg}(\mathbb{Z}))^{\mathbb{Z}}$ be the homotopy category of bounded above cochain complexes of finitely generated free graded abelian groups. The additional grading is referred to as the *quantum grading*.

Theorem 4.1.1. *There is a strictly monoidal multiplicative lasagna* $(3, 1, 1)$*-link homology theory*

$$CKhR^-_{2,0\text{-div}} \colon \mathbf{Links}_1 \to \mathbf{K}^-(\mathrm{Free}_{fg}(\mathbb{Z}))^{\mathbb{Z}} \tag{4.1}$$

satisfying:

(1) If $L \subset \#^m(S^1 \times S^2)$ is a null-homologous admissible link, $CKhR^-_{2,\text{0-div}}(\#^m(S^1 \times S^2), L)$ is naturally isomorphic to the Rozansky–Willis chain complex of L.

(2) It recovers the lasagna $(3,1,0)$-link homology theory $CKhR_2$ (defined in Section 2.6.2.2) when restricting to $\mathbf{Links}_0$.

(3) $CKhR^-_{2,\text{0-div}}(S,L) = 0$ for any object $(S,L) \in \mathbf{Links}_1$ not contained in $\mathbf{Links}_{1,\text{0-div}}$.

(4) $CKhR^-_{2,\text{0-div}}(W,\Sigma)$ is homogeneous of quantum degree $-\chi(\Sigma)$ for any morphism (W,Σ).

Consequently, by taking homology over a commutative ring R and passing to the cocompletion, we obtain a multiplicative (but not necessarily strictly monoidal) lasagna $(3,1,1)$-link homology theory, denoted

$$KhR^-_{2,\text{mul,0-div}} = KhR^-_{2,\text{mul,0-div}}(\bullet; R) \colon \mathbf{Links}_1 \to \mathbf{Mod}_R^{\mathbb{Z}^2}.$$

The construction in Section 2.4.2 yields a theory of 1-dimensional-input skein lasagna modules with input $KhR^-_{2,\text{mul,0-div}}$, denoted $\mathcal{S}_0^{2,\text{1d,mul,0-div}} = \mathcal{S}_0^{2,\text{1d,mul,0-div}}(\bullet;\bullet;R)$. See Section 2.6.2.2 for a discussion about the subscript and superscript "mul" in the setting of 0-dimensional input Khovanov skein lasagna modules.

In the general case without assuming links in $\#^m(S^1 \times S^2)$ to be null-homologous, the unrenormalized Rozansky–Willis homology has a grading ambiguity, so we work with a renormalized version of it, which assigns to every admissible link in $\#^m(S^1 \times S^2)$ a cochain complex of finitely generated free graded abelian groups, with integral homological and quantum gradings either both in $\mathbb{Z}$, or both in $\mathbb{Z} + \frac{1}{2}$, and with an additional $\mathbb{Z}/2$-grading called the Koszul grading that controls the sign convention in homological algebra, such that the homological grading and the Koszul grading determine the same relative $\mathbb{Z}/2$-grading. With this in mind, let $\mathbf{K}^{-,1/2}(\mathrm{Free}_{fg}(\mathbb{Z}))^{\mathbb{Z}}$ denote the homotopy category of bounded above cochain complexes of finitely generated free graded abelian groups, with a quantum $\frac{1}{2}\mathbb{Z}$-grading and a Koszul $\mathbb{Z}/2$-grading, in addition to a homological $\frac{1}{2}\mathbb{Z}$-grading, where the three gradings of each object are subject to the conditions described above. Morphisms are homogeneous chain maps that preserve the homological and Koszul degrees.

Theorem 4.1.2. *There is a strictly monoidal multiplicative lasagna $(3,1,1)$-link homology theory*

$$\widetilde{CKhR}_2^- : \mathbf{Links}_1 \to \mathbf{K}^{-,1/2}(\mathrm{Free}_{fg}(\mathbb{Z}))^{\mathbb{Z}} \tag{4.2}$$

that extends renormalized Rozansky–Willis chain complexes for admissible links in $\#^m(S^1 \times S^2)$. For a morphism (W,Σ), $\widetilde{CKhR}_2^-(W,\Sigma)$ is homogeneous of quantum degree $-\chi(\Sigma)$.

See Appendix D for a complete list of elementary moves and their induced maps on $\widetilde{CKhR}_2^-$.

By analogy with the preceding discussion in the null-homologous case, after taking homology over a commutative ring R, we obtain a multiplicative lasagna $(3,1,1)$-link homology theory

$$\widetilde{KhR}_{2,\mathrm{mul}}^{-}\colon \mathbf{Links}_1 \to \mathbf{Mod}_R^{(\frac{1}{2}\mathbb{Z})^2\times\mathbb{Z}/2}$$

and consequently a theory of skein lasagna modules with 1-dimensional inputs, denoted $\mathcal{S}_0^{\tilde{2},\mathrm{1d,mul}}$, called *Khovanov skein lasagna modules with* 1*-dimensional inputs*.

The theory of Khovanov skein lasagna modules with 1-dimensional inputs enjoys formal properties analogous to the 0-dimensional-input case, including the $3,4$-handle attachment formulas, the recovery of the lasagna $(3,1,1)$-link homology theories, and the (boundary) connected sum formulas, as described in Proposition 2.4.4, where in the case of 3-handle attachment, the explicit formula given by Proposition 2.6.4 also holds.

As discussed in Section 2.4.2.2, see (2.10), we also obtain four potentially different variants of $\mathcal{S}_0^{\tilde{2},\mathrm{1d,mul}}$ by successively restricting $\widetilde{KhR}_{2,\mathrm{mul}}^{-}$ to subcategories of $\mathbf{Links}_1$. The Rozansky–Willis chain complex is zero on links that are not $\mathbb{Z}/2$-null-homologous, so $\mathcal{S}_0^{\tilde{2},\mathrm{1d,mul,2\text{-}div}}$ is canonically isomorphic to $\mathcal{S}_0^{\tilde{2},\mathrm{1d,mul}}$. On null-homologous links, $\widetilde{CKhR}_2^{-}$ is isomorphic to $CKhR_2^-$ up to a grading shift, so $\mathcal{S}_0^{\tilde{2},\mathrm{1d,mul,0\text{-}div}}$ is canonically isomorphic to $\mathcal{S}_0^{2,\mathrm{1d,mul,0\text{-}div}}$ up to a renormalization, and (4.1) is obtained from (4.2) by restricting to $\mathbf{Links}_{1,\text{0-div}}$, renormalizing, and extending by zero, noting that $\mathbf{Links}_{1,\text{0-div}}$ is a cosieve in $\mathbf{Links}_1$. Finally, since the restriction of $CKhR_{2,\text{0-div}}^{-}$ to $\mathbf{Links}_0$ is the lasagna $(3,1,0)$-link homology theory $CKhR_2$ by Theorem 4.1.1, the 0-dimensional input skein lasagna modules induced from $\widetilde{CKhR}_2^{-}$ recover (up to renormalizations) the 0-dimensional-input Khovanov skein lasagna modules $\mathcal{S}_0^{2,\mathrm{mul}}$ and $\mathcal{S}_0^{2,\text{1-ball}}$ that were previously defined in Chapter 2.

4.1.2 Key ingredients

Our proof of the excellent functoriality of $\widetilde{CKhR}_2^{-}$ is rather unconventional, and provides a great instance of an application of the theory of Khovanov skein lasagna modules with 0-dimensional inputs. Sullivan–Zhang [SZ24] proved that the Rozansky–Willis homology can be recovered from the Khovanov skein lasagna module of $\natural^m(D^2\times S^2)$ with boundary links, in an appropriate sense (see Theorem 4.2.2). To check the functoriality of Rozansky–Willis homology in the strong sense we need, instead of checking all movie moves ("second order" moves) relating sequences of elementary cobordisms (we are unaware of how to obtain a complete set of movie moves in our setup), we employ Sullivan–Zhang's result as a bypass. As lasagna gluing operations are manifestly functorial, this alternative viewpoint significantly simplifies the task, allowing us to perform checks only on the elementary cobordisms ("first order" moves) themselves.

We remark that even the very special consequence of Theorem 4.1.1 and Theorem 4.1.2 that the Rozansky–Willis homology is functorial for link cobordisms in $I \times \#^m(S^1 \times S^2)$ was previously unknown, and would be rather involved to prove if one proceeds otherwise.

We highlight three key ingredients, in addition to Theorem 4.2.2, in the proof of Theorem 4.1.2 that are of independent interest.

Proposition 4.1.3 (Proposition 4.5.7). *Let $k \geq 1$, $m_1, \cdots, m_k \geq 0$. The diffeomorphism group of $D_{std} := \#_{i=1}^k \natural^{m_i}(D^2 \times S^2)$ rel boundary fits into an exact sequence*

$$1 \to \mathrm{Diff}_{\partial,loc}(D_{std}) \to \mathrm{Diff}_\partial(D_{std}) \to \mathbb{Z}^{m(m-1)/2} \times (\mathbb{Z}/2)^{k-1} \to 1.$$

Here, $\mathrm{Diff}_{\partial,loc}(D_{std})$ denotes the subgroup consisting of diffeomorphisms isotopic rel boundary to one supported in a local 4-ball, and $m = \sum_{i=1}^k m_i$.

When $D_{std} = D^2 \times S^2$, this is a version of Gabai's 4-dimensional lightbulb theorem [Gab20, Corollary 1.7]. We deduce Proposition 4.1.3 as a consequence of Gabai's result. The cokernel of $\mathrm{Diff}_{\partial,loc}(D_{std}) \to \mathrm{Diff}_\partial(D_{std})$ in Proposition 4.1.3 is generated by Dehn twists along embedded 3-spheres, as well as implanted barbell diffeomorphisms defined by Budney–Gabai [BG19]. We rediscovered the barbell diffeomorphism during this work and will present an alternative description of it in the proof of Proposition 4.5.7. Since D_{std} embeds into B^4, the map $\pi_0(\mathrm{Diff}_\partial(B^4)) \xrightarrow{\cong} \pi_0(\mathrm{Diff}_{\partial,loc}(D_{std}))$ induced by a local embedding $B^4 \subset D_{std}$ is an isomorphism. By comparing to the work of Orson–Powell [OP25, Theorem A(1)], the cokernel can also be identified with the topological mapping class group.

Corollary 4.1.4. *The natural map*

$$\mathrm{Diff}_\partial(D_{std})/\mathrm{Diff}_{\partial,loc}(D_{std}) \cong \pi_0(\mathrm{Diff}_\partial(D_{std}))/\pi_0(\mathrm{Diff}_\partial(B^4)) \to \pi_0(\mathrm{Homeo}_\partial(D_{std}))$$

is an isomorphism. □

In other words, modulo diffeomorphisms contained in B^4, $D_{std} = \#_{i=1}^k \natural^{m_i}(D^2 \times S^2)$ does not admit exotic diffeomorphisms. This is in contrast to the existence of such exotic diffeomorphisms on many 4-manifolds (including contractible ones) detected using gauge theory; see [Rub98; KM21; KMT23; Qiu24] and references therein. We learned from Slava Krushkal that Proposition 4.1.3 and Corollary 4.1.4 for the case $k = 1$ have been obtained in [Kru+24, Theorem 1.8], so we do not claim originality.

Theorem 4.1.5 (Theorem 4.7.1). *Gluck twists induce isomorphisms on Khovanov skein lasagna modules over any commutative ring.*

In particular, Khovanov skein lasagna modules cannot detect exotica arising from Gluck twists (to our knowledge, no such phenomenon has ever been detected on compact orientable 4-manifolds). This was already hinted at in Section 3.5.10 in Chapter 3. See Theorem 4.7.1 for a more precise statement as well as some formal properties enjoyed by the induced map.

Theorem 4.1.6 (Theorem C.2.1). *The universal, or* $\mathrm{GL}(2,\mathbb{C})$*-equivariant, Khovanov–Rozansky* $\mathfrak{gl}_2$ *homology for* $\mathfrak{gl}_2$ *webs in* S^3 *and* $\mathfrak{gl}_2$ *foams in* $I \times S^3$ *between them is functorial on the chain level up to chain homotopy, over* $H^*(BGL(2,\mathbb{C})) \cong \mathbb{Z}[E_1, E_2]$*. Moreover, it can be extended to singular* $\mathfrak{gl}_2$ *foams, and the map induced by such a foam is independent of the embedding of the interior of* 2*-labeled faces.*

See Appendix C for the precise setup we are using in Theorem 4.1.6, in particular our definition of singular $\mathfrak{gl}_2$ foams. It suffices to say here that we are allowing transverse double points between 1- and 2-labeled faces, or between 2- and 2-labeled faces, as well as singular points on faces that introduce framing changes. The functoriality of $\mathfrak{gl}_2$ homology for links and link cobordisms in S^3 was proved by Morrison–Walker–Wedrich [MWW22], following the work of Jacobsson [Jac04], Blanchet [Bla10], and others; see also Section 2.6.4 in Chapter 2 for an alternative proof of [MWW22]. The functoriality of $\mathfrak{gl}_2$ homology for $\mathfrak{gl}_2$ webs in $\mathbb{R}^3$ and $\mathfrak{gl}_2$ foams between them was proved by Queffelec [Que22]. Theorem 4.1.6 is a simultaneous generalization of these results.

After some preliminaries in Section 4.2, we will be able to give a more comprehensive overview of the rest of this chapter in Section 4.3. Throughout the chapter, we use the simpler Bar-Natan formalism for Khovanov homology as opposed to the $\mathfrak{gl}_2$ webs and foams formalism. Consequently, some constructions, definitions, and arguments will carry sign ambiguities at various stages. We resolve the sign issues in Appendix C.

4.1.3 Lee version

In this section, in the form of an expanded remark, we explain what can be done for the Lee deformation of Rozansky–Willis homology, as studied by [Man+23]. Our proof of Theorem 4.1.2 does not work verbatim to establish functoriality for the Lee version of Rozansky–Willis homology. Nevertheless, one may directly identify the Lee canonical generators and check on these generators, using the moves in Appendix D, that the Lee homology is functorial, giving rise to lasagna $(3,1,1)$-link homology theories $KhR^-_{Lee,0\text{-div}}\colon \mathbf{Links}_1 \to \mathrm{Filt}(\mathbf{Vect}_{\mathbb{Q}}^{\mathbb{Z}})$ and $\widetilde{KhR}^-_{Lee}\colon \mathbf{Links}_1 \to \mathrm{Filt}^{\frac{1}{2}\mathbb{Z}}(\mathbf{Vect}_{\mathbb{Q}}^{\frac{1}{2}\mathbb{Z}\times\mathbb{Z}/2})$. Keeping track of the Lee canonical generators, one obtains smooth genus bounds for cobordisms in $\mathbf{Links}_1$ in terms of the s-invariants of the boundary links as defined in [Man+23], in the spirit of Rasmussen [Ras10]. We do not claim, however, to have a proof of the functoriality of an analogue of the Lee spectral sequence.

We obtain the associated Lee skein lasagna modules with 1-dimensional inputs, denoted $\mathcal{S}_0^{Lee,1\mathrm{d},0\text{-div}}$ for the unrenormalized and null-homologous-input version and $\mathcal{S}_0^{\widetilde{Lee},1\mathrm{d}}$ for the general version. We drop the "mul" superscripts since we work over the field $\mathbb{Q}$.

For any pair (X, L), the forgetful map $\mathcal{S}_0^{Lee}(X;L) \to \mathcal{S}_0^{Lee,1\mathrm{d},0\text{-div}}(X;L)$ is an isomorphism of graded vector spaces and is filtered; in fact, it is filtration-preserving if X is simply connected.

The images of the Lee canonical generators of $\mathcal{S}_0^{Lee}(X;L)$, defined in Definition 3.3.5, give Lee canonical generators of $\mathcal{S}_0^{Lee,\text{1d,0-div}}(X;L)$, and we obtain 1-dimensional-input lasagna s-invariants of (X,L) for double classes in X rel L; these invariants are bounded above by the ordinary lasagna s-invariants defined in Definition 3.3.7. When X is simply connected, the two versions of the lasagna s-invariants agree, and hence $\mathcal{S}_0^{Lee,\text{1d,0-div}}$ detects the exotic 4-manifold pairs in Theorems 3.1.2 and 3.1.4 in Chapter 3 via the s-invariants.

On the other hand, there is a canonical isomorphism $\mathcal{S}_0^{\widetilde{Lee},\text{1d}}(X;L) \cong \mathbb{Q}[H_2^{u(L)}(X)]$ of vector spaces, where $H_2^{u(L)}(X) := \sqcup_{\mathfrak{o}} H_2^{(L,\mathfrak{o})}(X)$, $\mathfrak{o}$ ranges over orientations of L as an unoriented link, and $(L,\mathfrak{o})$ denotes L equipped with the orientation $\mathfrak{o}$. We obtain another version of lasagna s-invariants, defined for each class in $H_2^{u(L)}(X)$.

4.2 Preliminaries

Convention. *Throughout the rest of Chapter 4, unless otherwise indicated, chain complexes are defined over $\mathbb{Z}$ and follow the cohomological convention. The homology of a chain complex is taken over a commutative R, which is often suppressed from the notation. Khovanov skein lasagna modules are also assumed to take coefficients in R. Homology groups of topological spaces are taken with $\mathbb{Z}$-coefficients unless otherwise indicated. The Khovanov(–Rozansky $\mathfrak{gl}_2$) homology of the unknot is standardly identified as a unital algebra with $R[x]/(x^2)$.*

Caveat. *Throughout the rest of Chapter 4, we will exclusively work with the "mul" version of Khovanov homology in the sense of Section 2.6.2.2, and omit the "mul" subscript from the notation of homology groups like $KhR_{2,\text{mul}}^-(L)$. Thus, on an admissible link L in a disjoint union of S^3, for instance, unlike in Chapter 2 or Chapter 3, over a general commutative ring, $KhR_2(L)$ will* not *be the tensor product of its restriction on the individual S^3 components. For the "mul" version of Khovanov skein lasagna modules, to avoid confusion, we still keep the "mul" superscript.*

4.2.1 Rozansky–Willis homology for framed links

Fix an integer $m \geq 0$. Rozansky [Roz10] and Willis [Wil21] defined Khovanov homology for links in $\#^m(S^1\times S^2)$. For our purposes, we carefully describe the topological setup as follows.

Fix once and for all a surgery diagram for $\#^m(S^1\times S^2)$ as the 0-surgery on an m-component unlink, drawn in a standard way on the plane. Explicitly, this means that we have the following data:

- a decomposition $\#^m(S^1\times S^2) = M\cup N$, where N is a disjoint union of m solid tori;
- a diffeomorphism $M = S^3\backslash\nu(U_m)$ for an m-component unlink U_m;

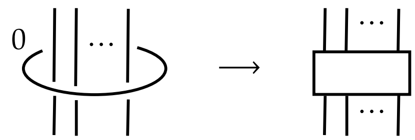


Figure 4.1: Left: Diagram of an admissible link near a surgery region. Right: A Rozansky projector.

- a point $\infty \in int(M) \subset S^3$;
- a diffeomorphism $S^3 \backslash \{\infty\} = \mathbb{R}^3$ that sends $\nu(U_m)$ to the set of points within distance 0.01 to $\sqcup_{i=1}^m \{(x, 0, z)\colon (x-i)^2 + z^2 = 0.01\}$, and sends U_m to $\sqcup_{i=1}^m \{(x, -0.01z, z)\colon (x-i)^2 + z^2 = 0.01\}$.

A framed oriented link $L \subset \#^m(S^1 \times S^2)$ is *admissible* if $L \subset int(M)\backslash\{\infty\}$ and the orthogonal projection of $L \cup U_m$ onto $\mathbb{R}^2 \times \{0\}$ is standard near the (projection of the) surgery regions (say $\sqcup_{i=1}^m (i-0.2, i+0.2) \times (-0.02, 0.02) \subset \mathbb{R}^2$) in the sense shown on the left of Figure 4.1. The *writhe* of an admissible link L, denoted $w(L)$, is the writhe of the framed oriented link L regarded as a link in S^3 via $M \subset S^3$.

For an admissible link $L \subset \#^m(S^1 \times S^2)$, its *unrenormalized bounded below Rozansky–Willis chain complex*, denoted $CKhR_2^+(L)$, is obtained by inserting Rozansky projectors in the link diagram of L at each surgery region (shown pictorially in Figure 4.1), resolving all other crossings and evaluating using the Khovanov functor. See Section 4.2.2 for a discussion of Rozansky projectors, which are bounded below chain complexes in Bar-Natan's tangle category. Consequently, $CKhR_2^+(L)$ is a bounded below chain complex of finitely generated free abelian groups.

The *unrenormalized bounded above Rozansky–Willis chain complex* of an admissible link L is

$$CKhR_2^-(L) := \mathrm{Hom}(CKhR_2^+(-L), \mathbb{Z}).$$

Here, $-L$ denotes the reverse mirror of L, defined as the orientation-reversal of the image of L under an orientation-reversing involution on $\#^m(S^1 \times S^2)$ that preserves the decomposition $\#^m(S^1 \times S^2) = M \cup N$ and acts on M by inverting the z-coordinate in $\mathbb{R}^3$. After slightly readjusting the position of $U_m \subset \mathbb{R}^3$, the involution respects the admissibility of links. On the level of link diagrams, $-L$ is obtained from L by switching each crossing. Alternatively, $CKhR_2^-(L)$ is the chain complex obtained by inserting dual Rozansky projectors in the link diagram of $-L$ at each surgery region, resolving all crossings and evaluating using the Khovanov functor.

More generally, fix $k \geq 0$, $m_1, \cdots, m_k \geq 0$, and write $S_{std} := \sqcup_{i=1}^{k} \#^{m_i}(S^1 \times S^2)$. A link $L \subset S_{std}$ is *admissible* if its restrictions L_i, $i = 1, \cdots, k$, to the components of S_{std} are all admissible. In this case, its *unrenormalized bounded above/below Rozansky–Willis chain complex* is $CKhR_2^{\pm}(L) := \otimes_{i=1}^{k} CKhR_2^{\pm}(L_i)$. The homology of $CKhR_2^{\pm}(L) \otimes R$, denoted $KhR_2^{\pm}(L) = KhR_2^{\pm}(L; R)$, is the *unrenormalized bounded above/below Rozansky–Willis homology* of L (with R-coefficients). These homology groups are bigraded R-modules, bounded from above/below in the homological grading, that are finitely generated in every homological degree.

If an admissible link $L \subset S_{std}$ is not $\mathbb{Z}/2$-null-homologous, then $CKhR_2^{\pm}(L) = 0$. If L is an admissible link in S^3, no Rozansky projector is inserted, and we have $CKhR_2^{-}(L) = CKhR_2^{+}(L) = CKhR_2(L)$.

By our $\mathfrak{gl}_2$ convention, for admissible links in $\#^m(S^1 \times S^2)$, KhR_2^{-} is related to the homology Kh in [Wil21] by (compare (2.11))

$$KhR_2^{-,h,q}(L; \mathbb{Z}) = Kh^{h,-q-w(L)}(-L).$$

In the rest of this chapter, we will work with the *renormalized Rozansky–Willis chain complexes*, defined for any admissible link $L \subset S_{std}$ as

$$\widetilde{CKhR}_2^{\pm}(L) = (tq^{-1})^{w(L)/2} CKhR_2^{\pm}(L),$$

where t, q denote homological, quantum degree shifts, respectively. Here, the *writhe* $w(L)$ is defined as the sum of the writhes of the restriction of L to the components of S_{std}. The bounded above/below versions are still dual to each other upon taking the reverse mirror. Note that the gradings of $\widetilde{CKhR}_2^{\pm}(L)$ take values in half-integers when $w(L)$ is odd. Technically, an integral, or at least a mod 2 homological grading is required for applying the Koszul sign convention in homological algebra. For this purpose, it is better to regard $\widetilde{CKhR}_2^{+}(L)$ as $\frac{1}{2}\mathbb{Z}_h \times \frac{1}{2}\mathbb{Z}_q \times (\mathbb{Z}/2)_k$-graded, where h, q are homological, quantum gradings taking values in half-integers as defined above, and k is the Koszul grading, defined as the mod 2 homological grading of $CKhR_2(L)$, which controls the Koszul sign convention, although most of the time we will not be dwelling on this difference and will often omit the Koszul grading in our discussion.

The isomorphism type of $KhR_2^{\pm}(L)$ (in fact, the chain homotopy type of $CKhR_2^{\pm}(L)$) is shown in [Roz10; Wil21] to be an invariant of L up to isotopy in S_{std}, up to global grading shifts by multiples of t^2q^{-2}. The renormalized homology $\widetilde{KhR}_2^{\pm}(L)$ has no such grading ambiguity. Since every framed oriented link can be isotoped to be admissible, $\widetilde{KhR}_2^{\pm}$ defines

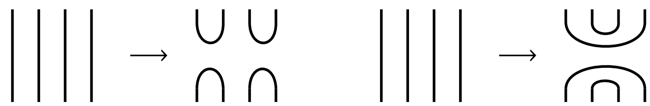


Figure 4.2: The family of maps that constructs $P^{\vee}_{4,0}$ via the cobar construction in [Hog20] (grading shifts suppressed).

a bigraded R-module invariant up to isomorphism of isotopy classes of framed oriented links in S_{std}. However, $\widetilde{KhR}_2^{\pm}$ is not known to be functorial with respect to link cobordisms in $I \times S_{std}$. We will address its functoriality in Section 4.4.

In later sections, we will drop the adjectives "renormalized" and "bounded above/below," and refer to both $\widetilde{KhR}_2^{\pm}$ for admissible links in S_{std} simply as the (classical) Rozansky–Willis homology.

4.2.2 Properties of Rozansky projectors

We collect some properties enjoyed by the Rozansky projectors that will be useful for us. This section is not required for reading Section 4.3. We work with the Bar-Natan formalism, and this is the source of many sign ambiguities we will encounter later. We refer to Appendix C.3 for a construction of Rozansky projectors using $\mathfrak{gl}_2$ webs and foams, which removes the sign ambiguity.

Fix a nonnegative integer ℓ. The Rozansky projector on ℓ strands, denoted $P^{\vee}_{\ell,0}$, is a certain infinite complex in the Bar-Natan category of the disk with 2ℓ endpoints, bounded from below. One way to construct $P^{\vee}_{\ell,0}$ is to take the family of maps

$$\{1_\ell \to q^{\ell/2}\delta \otimes \delta^t\},$$

apply the cobar construction of Hogancamp [Hog20, Section 3], and simplify by delooping the circle components. Here, 1_ℓ is the identity tangle on ℓ strands, δ runs over all crossingless matchings of the ℓ points on the top, δ^t is the vertical reflection of δ, $\otimes$ denotes vertical composition of tangles going from top to bottom, and each map $1_\ell \to q^{\ell/2}\delta \otimes \delta^t$ is given by the $\ell/2$ natural saddles. See Figure 4.2 for an illustration for $\ell = 4$. For ℓ odd, there is no such δ, so the projector $P^{\vee}_{\ell,0}$ is zero.

The precise definition will not play a role in our proof. Instead, we make a list of properties enjoyed by the Rozansky projectors $P^{\vee}_{\ell,0}$. These will be justified in Section C.3 in the $\mathfrak{gl}_2$ webs and foams setup.

Proposition 4.2.1. *The Rozansky projectors $P^{\vee}_{\ell,0}$ satisfy the following properties.*

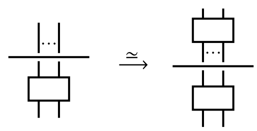
(1) Each $P^\vee_{\ell,0}$ is a chain complex, where each term is a direct sum of (ℓ,ℓ) Temperley-Lieb diagrams of through-degree zero, i.e. composites of the form $\alpha \otimes \beta^t$, where α is an $(\ell,0)$ diagram and β^t a $(0,\ell)$ diagram. Differentials between different terms are linear combinations of dotted cobordisms of through-degree zero, i.e. of the form $\sum(f \otimes g^t \colon \alpha \otimes \beta^t \to \alpha' \otimes \beta'^t)$ for some dotted cobordisms $f \colon \alpha \to \alpha'$, $g \colon \beta \to \beta'$. Informally, this means that the projectors have an "empty region in the middle."

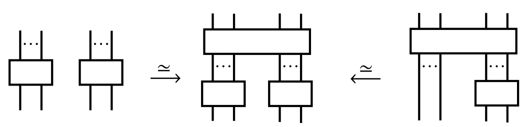
(2) $P^\vee_{\ell,0}$ is unital: it comes with a chain map $\iota_\ell \colon 1_\ell \to P^\vee_{\ell,0}$, called the unit map.

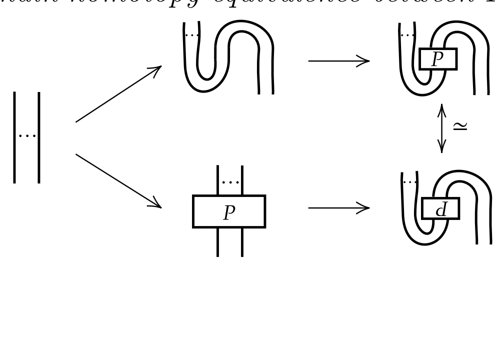
(3) The Rozansky projector on 0 strands is the empty diagram. The unit map $\iota_0 \colon 1_0 \to P^\vee_{0,0}$ is the identity map.

(4) The maps $\iota_\ell \otimes \mathrm{id}, \mathrm{id} \otimes \iota_\ell \colon P^\vee_{\ell,0} \to P^\vee_{\ell,0} \otimes P^\vee_{\ell,0}$ are chain homotopic.

(5) If T is an (ℓ_0,ℓ_1)-tangle, then $P^\vee_{\ell_0,0} \otimes T \xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes\iota_{\ell_1}} P^\vee_{\ell_0,0} \otimes T \otimes P^\vee_{\ell_1,0}$ and $T \otimes P^\vee_{\ell_1,0} \xrightarrow{\iota_{\ell_0}\otimes\mathrm{id}\otimes\mathrm{id}} P^\vee_{\ell_0,0} \otimes T \otimes P^\vee_{\ell_1,0}$ are chain homotopy equivalences. In particular, $\iota_\ell \otimes \mathrm{id}$ and $\mathrm{id} \otimes \iota_\ell$ in (4) are chain homotopy equivalences.

(6) The unit map  *(as well as its mirrored version) is a chain homotopy equivalence.*

(7) The unit maps in  *are chain homotopy equivalences.*

(8) $P^\vee_{\ell,0}$ is symmetric: there is a canonical chain homotopy equivalence between $P^\vee_{\ell,0}$ and the planar rotation of $P^\vee_{\ell,0}$ by π that makes  *commute up to homotopy.*

4.2.3 A lasagna interpretation of Rozansky–Willis homology

Fix $k \geq 0$, $m_1, \cdots, m_k \geq 0$. The manifold $S_{std} := \sqcup_{i=1}^k \#^{m_i}(S^1 \times S^2)$ is naturally the boundary of $D_{std} := \#_{i=1}^k \natural^{m_i}(D^2 \times S^2)$. Here, D_{std} is thought of as equipped with a standard handle decomposition with one 0-handle, $\sum_{i=1}^k m_i$ 0-framed 2-handles, k 3-handles, and

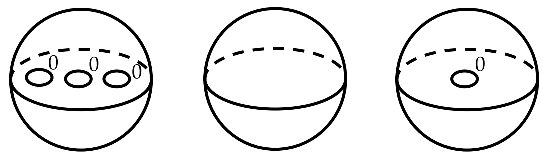


Figure 4.3: The Kirby diagram for the concrete $\#_{i=1}^{3}\natural^{m_i}(D^2 \times S^2)$ with 3-handles drawn, $(m_1, m_2, m_3) = (3, 0, 1)$. The 4-handle is the one touching the outer belt point of each 3-handle.

one 4-handle, so that the 3-handles are attached along $S^2(1) + (10i, 0, 0) \subset \mathbb{R}^3 \subset S^3 = \partial B^4$, $i = 1, 2, \cdots, k$, and the 2-handles in the i-th connected summand are attached inside $B^3(1/2) + (10i, 0, 0)$, along an unlink $U_{m_i} \subset S_i^3$ in standard position. Here, S_i^3 denotes the S^3 boundary component of $B^4 \cup$ (3-handles) containing $(10i, 0, 0)$, whose ∞ point is the inner belt point of the corresponding 3-handle. In this description, $S^2(R)$ and $B^3(R)$ are the 2-sphere and 3-ball of radius R centered at $0 \in \mathbb{R}^3$, respectively. See Figure 4.3.

An admissible link $L \subset S_{std} = \partial D_{std}$ determines a canonical class $\alpha_L \in H_2^L(D_{std})$, characterized by having trivial algebraic intersections with the 2-cocores of D_{std}. In [SZ24], Sullivan–Zhang showed that one can recover Rozansky–Willis homology from Khovanov skein lasagna modules in the following sense. Their paper proves the theorem over $\mathbb{Q}$ for the case $k = 1$, but essentially the same proof applies more generally with a small change, as we will point out below.

Theorem 4.2.2 ([SZ24, Remark 1.6]). *For every admissible link $L \subset S_{std} = \sqcup_{i=1}^{k}\#^{m_i}(S^1 \times S^2)$, there is a canonical isomorphism of R-modules*

$$\mathcal{S}_0^{2,\mathrm{mul}}(D_{std}; L) \cong \widetilde{KhR}_2^{+}(L) \otimes \mathcal{S}_0^{2,\mathrm{mul}}(D_{std}) \tag{SZ}$$

Moreover, in each skein grading $\alpha \in H_2^L(D_{std})$, the isomorphism in the forward direction is homogeneous with (homological, quantum, skein) tridegree shift $(\alpha^2/2, -\alpha^2/2, -\alpha_L)$.

The intersection pairing on $H_2^L(D_{std})$ appearing in the grading shifts above is defined by using the framing of L as the boundary condition. The term $\widetilde{KhR}_2^{+}(L)$ in (SZ) is set to concentrate in skein grading 0.

Recall from Section 2.6.2.2 that $\mathcal{S}_0^{2,\mathrm{mul}}$ is the skein lasagna module associated with the lasagna $(3,1,0)$-link homology theory obtained from the lasagna $(3,1,0)$-link homology theory $CKhR_2$ by passing to homology over R, which is isomorphic to $\mathcal{S}_0^2$ when R is a field. It enjoys all properties presented for $\mathcal{S}_0^2$ in Section 2.6.3, in addition to the general properties presented in Section 2.3. The "non-mul" version of Theorem 4.2.2 does not seem to hold for $k > 1$ over a general commutative ring R.

As it will be of importance, in the rest of this section, we examine the isomorphism (SZ) in more detail.

Write $m = \sum_{i=1}^{k} m_i$. For $n \in \mathbb{Z}^m$, let $n_+, n_- \in \mathbb{Z}^m$ be defined by $(n_+)_j = \max(n_j, 0)$, $(n_-)_j = \max(-n_j, 0)$. Let $|n| = \sum_{j=1}^{m} |n_j|$ be the L^1-norm of n.

Fix a homology class $\alpha \in H_2^L(D_{std})$. Write $\alpha = \alpha_L + n$ for some $n \in \mathbb{Z}^m \cong H_2(D_{std})$. At the skein class α, the isomorphism (SZ) is the composition of the following sequence of isomorphisms.

$$\begin{aligned}
&\mathcal{S}_0^{2,\mathrm{mul}}(D_{std}; L; \alpha) \\
\xleftarrow{\cong}\ & \oplus_{r\in\mathbb{Z}^m_{\geq 0}} q^{-|n|-2|r|} KhR_2(L \cup (n_+ + r, n_- + r) \text{ belts})/\sim && \text{(SZ1)}\\
=\ & \oplus_{r\in\mathbb{Z}^m_{\geq 0}} q^{-|n|-2|r|} (t^{-1}q)^{\alpha^2/2} \widetilde{KhR}_2^+(L \cup (n_+ + r, n_- + r) \text{ belts})/\sim && \text{(SZ2)}\\
\xrightarrow{\cong}\ & (t^{-1}q)^{\alpha^2/2} \oplus_{r\in\mathbb{Z}^m_{\geq 0}} q^{-|n|-2|r|} \widetilde{KhR}_2^+(L^\circ \otimes (\otimes_{j=1}^m P^\vee_{\ell_j,0}) \cup (n_+ + r, n_- + r) \text{ belts})/\sim && \text{(SZ3)}\\
\xrightarrow{\cong}\ & (t^{-1}q)^{\alpha^2/2} \oplus_{r\in\mathbb{Z}^m_{\geq 0}} q^{-|n|-2|r|} \widetilde{KhR}_2^+(L^\circ \otimes (\otimes_{j=1}^m P^\vee_{\ell_j,0}) \sqcup U^{n_+ + r, n_- + r})/\sim && \text{(SZ4)}\\
\xrightarrow{\cong}\ & (t^{-1}q)^{\alpha^2/2} \widetilde{KhR}_2^+(L) \otimes \mathcal{S}_0^{2,\mathrm{mul}}(D_{std}; \alpha - \alpha_L). && \text{(SZ5)}
\end{aligned}$$

The individual steps are explained as follows.

(SZ1): This is almost the 2-handlebody formula (2.17), except that one also has to note that the additional 3-handle attachments induce isomorphisms, by the 3-handle attachment formula, Proposition 2.6.4, since each attaching sphere bounds a 3-ball in $\natural^m(D^2 \times S^2)$ which can be assumed to miss any given lasagna filling after a neck-cutting (Lemma 2.3.7), and that the additional 4-handle attachment induces an isomorphism by Proposition 2.3.1. Alternatively, one can resort to the connected sum formula, Proposition 2.3.2, to reduce to the case $k = 1$.

Explicitly, the isomorphism is given as follows. Let $v \in KhR_2(L \cup (n_+ + r, n_- + r) \text{ belts})$. Then the class in the right-hand side represented by v is sent to the class in the left-hand side represented by the lasagna filling $((I \times L) \cup (n_+ + r, n_- + r) \text{ cores}, v)$, where the skein has k input balls, the i-th of which is given by a slightly shrunken collar neighborhood of $B^3(1) + (10i, 0, 0) \subset S^3 = \partial B^4$ in the 0-handle B^4, and the cores refer to parallel copies of cores of 2-handles, slightly extended to the interior of B^4. The inverse of this isomorphism is given as follows. Any lasagna filling representing an element in the left-hand side is equivalent to one of the form $((I \times L) \cup (n_+ + r, n_- + r) \text{ cores}, v)$ by first pushing the skein down the 3,4-handles and then applying general position as in [MN22]. The class represented by such a lasagna filling is sent to the class on the right-hand side represented by $v \in KhR_2(L \cup (n_+ + r, n_- + r) \text{ belts})$.

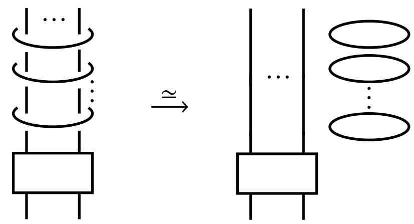


Figure 4.4: The belt-slide isomorphism (SZ4), shown near one surgery region.

(SZ2): This is the renormalization by the writhe of $L \cup (n_+ + r, n_- + r)$ belts, which is equal to α^2.

(SZ3): Here, L° denotes L regarded in $\sqcup_{i=1}^k S^3$, but with a neighborhood of the surgery regions removed, and ℓ_j denotes the geometric intersection number between L and the disk in $\mathbb{R}^2 \times \{0\}$ bounded by the j-th component of $\sqcup_{i=1}^k U_{m_i}$. The second $\otimes$ symbol in this line indicates disjoint union, while the first represents insertion of each Rozansky projector $P^\vee_{\ell_j,0}$ at the deleted neighborhood of the j-th surgery region, placed immediately below the j-th collection of belts, as shown on the left of Figure 4.4. The map is induced by the unit maps $1_{\ell_j} \to P^\vee_{\ell_j,0}$, and is shown to be an isomorphism, over $\mathbb{Q}$ and after symmetrization, by Sullivan–Zhang [SZ24]. Their proof uses a degree argument relying on results of Hogancamp [Hog18] concerning the categorified Jones–Wenzl projectors, which implies that the dotted annulus cobordism map is nilpotent in every finite range. Utilizing [SZ24, Corollary 5.7] rather than [SZ24, (5.2)], the same proof applies over any commutative ring R.

(SZ4): This follows from [Wil21, Lemma 3.12]. We sketch the proof. Each $P^\vee_{\ell_j,0}$ is a chain complex whose terms are direct sums of through-degree 0 Temperley-Lieb diagrams with some quantum degree shifts. One can use simplifying Reidemeister II moves to slide the belts off each term in the complex through the "empty region." These sliding-off homotopy equivalences (each well-defined up to sign), plus some higher differentials between cross terms contributing to $P^\vee_{\ell_j,0}$, can be patched together to form a chain homotopy equivalence that slides the belts off the projectors, say to the right of the strands as shown in Figure 4.4, after some compatible sign choices. Here $U^{a,b}$ denotes the $(a+b,0)$-cable of (a slightly shifted copy of) $U_m = \sqcup_{i=1}^k U_{m_i}$, with the orientation on b strands reversed, $a, b \in \mathbb{Z}^m_{\geq 0}$. This isomorphism has an overall sign indeterminacy. To fix the sign, see Appendix C.4.1.

(SZ5): Since the homology of an unlink is a finitely generated free R-module, the split disjoint unions give rise to tensor products on $\widetilde{KhR}_2^+$. By definition, the first tensorial factor becomes $\widetilde{KhR}_2^+(L)$, the Rozansky–Willis homology of the admissible link $L \subset S_{std}$. By the 2-handlebody formula (2.18), the second tensorial factor (under the direct sum, modulo the equivalence relation) becomes $\mathcal{S}_0^{2,\mathrm{mul}}(\natural^m(D^2\times S^2);\alpha-\alpha_L)$, which is isomorphic to $\mathcal{S}_0^{2,\mathrm{mul}}(D_{std};\alpha-\alpha_L)$ via the map induced by $3,4$-handle attachments.

4.2.4 Khovanov skein lasagna modules of $\#\natural(D^2\times S^2)$

As in Section 4.2.3, let $k\geq 0$, $m_1,\cdots,m_k\geq 0$, and write $D_{std}:=\#_{i=1}^k\natural^{m_i}(D^2\times S^2)$, thought of as concretely built out of handles as indicated in Figure 4.3. We study the Khovanov skein lasagna module of D_{std}. This section is not required for reading Section 4.3.

The computation in this section holds true for both $\mathcal{S}_0^2$ and $\mathcal{S}_0^{2,\mathrm{mul}}$.

Example 4.2.3. The Khovanov skein lasagna module of $D^2\times S^2$ was studied in [MN22, Theorem 1.2]; see also Example 2.6.7. The same computation holds true for the "mul" version and gives

$$\mathcal{S}_0^{2,\mathrm{mul}}(D^2\times S^2)\cong R[A_0^{\pm},A_1]$$

where A_0 is of tridegree $(0,0,[S^2])$, represented by a one-dotted core 2-sphere, and A_1 is of tridegree $(0,-2,[S^2])$, represented by an undotted core 2-sphere. Applying Bar-Natan's neck-cutting relation, we see that the orientation-reversal of the one-dotted (resp. undotted) core 2-sphere represents A_0^{-1} (resp. $-A_0^{-2}A_1$). More generally, any monic monomial in $R[A_0^{\pm},A_1]$ is represented by a disjoint union of core 2-spheres and their orientation-reversal, possibly with some dots.

An analogous computation, using first the 2-handlebody formula, Proposition 2.6.5, then the 3-handle attachment formula, Proposition 2.6.4, and finally the 4-handle attachment formula, Proposition 2.3.1, shows that

$$\mathcal{S}_0^{2,\mathrm{mul}}(D_{std})\cong R[A_{i,j,0}^{\pm},A_{i,j,1}\colon 1\leq i\leq k,1\leq j\leq m_i] \tag{4.3}$$

where $A_{i,j,0}$ has tridegree $(0,0,e_{i,j})$ and $A_{i,j,1}$ has tridegree $(0,-2,e_{i,j})$, with explicit representatives described analogously as before. Here $e_{i,j}\in H_2(D_{std})$ is the class represented by the j-th core 2-sphere of the i-th connected summand in D_{std}.

We rewrite (4.3) more invariantly as

$$\mathcal{S}_0^{2,\mathrm{mul}}(D_{std})\cong R[H_2(D_{std})]\otimes_{\mathbb{Z}}\mathrm{Sym}_{\mathbb{Z}}(H_2(D_{std})), \tag{4.4}$$

where the first factor is the group ring of $H_2(D_{std})$ over R, in which each $\alpha\in H_2(D_{std})$ has tridegree $(0,0,\alpha)$, and where the second factor is the symmetric algebra of $H_2(D_{std})$, in which each $\beta\in H_2(D_{std})$ has tridegree $(0,-2,0)$. Comparing (4.3) and (4.4), $A_{i,j,0}$ corresponds to $[e_{i,j}]\otimes 1$, and $A_{i,j,1}$ corresponds to $[e_{i,j}]\otimes e_{i,j}$.

We claim that (4.4) is independent of the explicit parametrization of D_{std} in terms of handles. We prove this by giving an intrinsic description of the isomorphism from the right-hand side to the left-hand side. Let a basis element $f:=[\alpha]\otimes\beta_1\cdots\beta_n$ in the right-hand side be given, where we may assume all $\beta_r \in H_2(D_{std})$ are primitive. We pick a disjoint union of one-dotted oriented 2-spheres, denoted $\dot{S}$, in ∂D_{std}, that represents $\alpha - \sum_r \beta_r \in H_2(D_{std}) \cong H_2(\partial D_{std})$. Similarly, for each r, we pick an undotted oriented 2-sphere S_r in ∂D_{std} representing β_r. We claim that under (4.4), f corresponds to the element in $\mathcal{S}_0^{2,\mathrm{mul}}(D_{std})$ represented by the union $\dot{S} \sqcup (\sqcup_r S_r)$, where the copies of spheres are pushed slightly into the interior of D_{std} with different depths so that they become disjoint. To show this, let $S_\# \subset \partial D_{std}$ be the collection of $m = \sum_i m_i$ 2-spheres that are shown in Figure 4.3 as small spheres encircling each of the 0-framed 2-handles. We perform levelwise Bar-Natan neck-cutting along push-in copies of the spheres in $S_\#$ and evaluate all the resulting local spheres. After these operations, all remaining 2-spheres are parallel to core 2-spheres of D_{std}, and one may easily check the claim using the description for (4.3).

Thus, $\mathcal{S}_0^{2,\mathrm{mul}}(D_{std})$ is naturally an R-algebra, and in fact an $R[H_2(D_{std})]$-algebra. The subalgebra $R[H_2(D_{std})] \subset \mathcal{S}_0^{2,\mathrm{mul}}(D_{std})$ consists of exactly the quantum degree 0 elements.

Definition 4.2.4. The *shifting automorphism* by $\alpha \in H_2(D_{std})$ is the operator

$$\mathrm{id}_\alpha \colon \mathcal{S}_0^{2,\mathrm{mul}}(D_{std}) \to \mathcal{S}_0^{2,\mathrm{mul}}(D_{std})$$

defined by multiplication by $[\alpha] \otimes 1$, under the isomorphism (4.4).

Thus, the shifting automorphisms assemble to a group action $H_2(D_{std}) \to \mathrm{Aut}(\mathcal{S}_0^{2,\mathrm{mul}}(D_{std}))$ that is (homological, quantum) degree-preserving.

4.3 Proof outline

In this section, we sketch the proof strategy of our main theorem, Theorem 4.1.2. Theorem 4.1.1 about the null-homologous case will be recovered by restriction and extension by zero, as explained in the introduction.

We introduce some terminology. If X is a smooth oriented manifold, by an *abstract* X we mean a smooth oriented manifold that is diffeomorphic to X. If $(S, L) \in \mathbf{Links}_1$, then S is an abstract $S_{std} = \sqcup_{i=1}^k \#^{m_i}(S^1 \times S^2)$ for some $k \geq 0$, $m_1, \cdots, m_k \geq 0$, and an *L-admissible parametrization* of S is an orientation-preserving diffeomorphism $\phi \colon S \xrightarrow{\cong} S_{std}$.

4.3.1 Reduce to homology-level functoriality

The isomorphism (SZ) is on the level of homology, while Theorem 4.1.2 asserts the stronger functoriality on the chain level. The gap is bridged by the following homological algebra fact plus some special features of our construction.

Lemma 4.3.1. *If C, D are bounded above chain complexes of finitely generated free abelian groups, and $f, g\colon C \to D$ are chain maps that induce the same map on homology over any coefficient ring, then f and g are chain homotopic.* $\square$

We will first establish the functoriality of the Rozansky–Willis homology, obtaining lasagna $(3,1,1)$-link homology theories

$$\widetilde{KhR}_2^-(=\widetilde{KhR}_{2,\mathrm{mul}}^-)\colon \mathbf{Links}_1 \to \mathbf{Mod}_R^{(\frac{1}{2}\mathbb{Z})^2\times\mathbb{Z}/2}, \tag{4.5}$$

one for each commutative ring R. This construction has the following special features which allow the upgrade to the chain level.

First, for an object (S, L) in $\mathbf{Links}_1$ and an L-admissible parametrization ϕ of S, the homology $\widetilde{KhR}_2^-(S, L)$ is canonically isomorphic to the homology of $\widetilde{CKhR}_2^-(\phi(L))$ over R, and if ϕ' is another L-admissible parametrization of S, then the two canonical isomorphisms are related by the induced map on homology of a chain map $\iota_{\phi,\phi'}\colon \widetilde{CKhR}_2^-(\phi(L)) \to \widetilde{CKhR}_2^-(\phi'(L))$. Lemma 4.3.1 shows that $\iota_{\phi,\phi'}$ is well-defined up to homotopy, allowing us to define $\widetilde{CKhR}_2^-(S, L)$ by using any $\widetilde{CKhR}_2^-(\phi(L))$ as a model.

Second, for a morphism $(W, \Sigma)\colon (S_0, L_0) \to (S_1, L_1)$ in $\mathbf{Links}_1$ and L_j-admissible parametrizations ϕ_j of S_j, $j = 0, 1$, in terms of the chain models given by ϕ_0, ϕ_1, $\widetilde{KhR}_2^-(W, \Sigma)\colon \widetilde{KhR}_2^-(\phi_0(L_0)) \to \widetilde{KhR}_2^-(\phi_1(L_1))$ is well-defined and is equal to the induced map on homology by some chain map $\widetilde{CKhR}_2^-(\phi_0(L_0)) \to \widetilde{CKhR}_2^-(\phi_1(L_1))$. Lemma 4.3.1 shows that the chain map is well-defined up to homotopy, providing a model for $\widetilde{CKhR}_2^-(W, \Sigma)$.

4.3.2 Turning cobordisms inside out

The main idea of supplying the morphism data for (4.5), namely cobordism maps on Rozansky–Willis homology for morphisms in $\mathbf{Links}_1$, is to turn 1-handlebodies inside out and upgrade Theorem 4.2.2 to the level of morphisms to obtain a dual result. Let $\mathbf{Mod}_R^{+,(\frac{1}{2}\mathbb{Z})^2\times\mathbb{Z}/2}$ be the category of (homological, quantum, Koszul) graded R-modules in which the homological grading is bounded from below.

Theorem 4.3.2. *There is a monoidal functor*

$$\widetilde{KhR}_2^+\colon \mathbf{Links}_1^{op} \to \mathbf{Mod}_R^{+,(\frac{1}{2}\mathbb{Z})^2\times\mathbb{Z}/2}$$

that extends the renormalized Rozansky–Willis homology $\widetilde{KhR}_2^+$ for admissible links in $S_{std} = \sqcup\#(S^1 \times S^2)$. For a morphism (W, Σ), the linear map $\widetilde{KhR}_2^+(W, \Sigma)$ is homogeneous of *degree* $(0, -\chi(\Sigma), 0)$. *The birth morphism and any pants morphism in* $\mathbf{Links}_1$ *(as defined in Section 2.4.2) induce an isomorphism.*

Theorem 4.1.2 is recovered from Theorem 4.3.2 as follows. First, the special features of our construction of $\widetilde{KhR}_2^+$ in the dual sense explained in Section 4.3.1 allow us to upgrade Theorem 4.3.2 to a statement on the chain level that is exactly dual to Theorem 4.1.2; in particular, we obtain a strictly monoidal functor

$$\widetilde{CKhR}_2^+ : \mathbf{Links}_1^{op} \to \mathbf{K}^{+,1/2}(\mathrm{Free}_{fg}(\mathbb{Z}))^{\mathbb{Z}}$$

that is multiplicative in the dual sense and recovers $\widetilde{CKhR}_2^+$ on admissibly parametrized objects, where multiplicativity follows from the chain map assignments in Section 4.6.4 and Section 4.6.5. Then we dualize by taking

$$\widetilde{CKhR}_2^- := (\bullet)^\vee \circ (\widetilde{CKhR}_2^+)^{op} \circ (-),$$

where $(-)$ is the autoequivalence of $\mathbf{Links}_1$ sending an object (S, L) to its orientation-reversal $(-S, -L)$ and a morphism $(W, \Sigma)\colon (S_0, L_0) \to (S_1, L_1)$ to its orientation-reversal $(-W, -\Sigma)\colon (-S_0, -L_0) \to (-S_1, -L_1)$, and $(\bullet)^\vee\colon (\mathbf{K}^{+,1/2}(\mathrm{Free}_{fg}(\mathbb{Z}))^{\mathbb{Z}})^{op} \to \mathbf{K}^{-,1/2}(\mathrm{Free}_{fg}(\mathbb{Z}))^{\mathbb{Z}}$ is the dualization functor which is an equivalence of categories by the finiteness and freeness assumptions.

The rest of this chapter is devoted to proving Theorem 4.3.2.

The classical Rozansky–Willis homology, introduced in Section 4.2.1, is only defined for admissible links in concretely parametrized $\sqcup\#(S^1 \times S^2)$. Thus, the functor $\widetilde{KhR}_2^+$ described in Theorem 4.3.2 has not even been defined on objects. Nevertheless, in the rest of the section, we ignore parametrization issues, and sketch how the cobordism maps for $\widetilde{KhR}_2^+$ are defined.

Suppose $(W, \Sigma)\colon (S_0, L_0) \to (S_1, L_1)$ is a morphism in $\mathbf{Links}_1$. Turning (W, Σ) upside down and reversing the orientations of W and Σ, we obtain the transpose cobordism $(W^t, \Sigma^t)\colon (S_1, L_1) \to (S_0, L_0)$.

Write $W = X_1 \backslash int(X_0)$ for 4-dimensional 1-handlebodies X_0, X_1. Choose an orientation-preserving embedding $X_1 \hookrightarrow S^4$. Taking the complement and reversing the orientation, we get an embedding $-(S^4 \backslash int(X_1)) \hookrightarrow -(S^4 \backslash int(X_0))$, where each $-(S^4 \backslash int(X_j))$ is some abstract $\#_{i=1}^k \natural^{m_i}(D^2 \times S^2)$.

By construction of the skein lasagna modules, we have a gluing map

$$\mathcal{S}_0^{2,\mathrm{mul}}(W^t; \Sigma^t)\colon \mathcal{S}_0^{2,\mathrm{mul}}(-(S^4 \backslash int(X_1)); L_1) \to \mathcal{S}_0^{2,\mathrm{mul}}(-(S^4 \backslash int(X_0)); L_0). \tag{4.6}$$

Upon fixing L_j-admissible parametrizations $-(S^4\backslash int(X_j)) \cong \#_{i=1}^{k(j)}\natural^{m_i^{(j)}}(D^2\times S^2)$, $j=0,1$, and using the isomorphism (SZ), (4.6) can be regarded as a map

$$\widetilde{KhR}_2^+(L_1)\otimes\mathcal{S}_0^{2,\mathrm{mul}}(\#_{i=1}^{k(1)}\natural^{m_i^{(1)}}(D^2\times S^2))\to\widetilde{KhR}_2^+(L_0)\otimes\mathcal{S}_0^{2,\mathrm{mul}}(\#_{i=1}^{k(0)}\natural^{m_i^{(0)}}(D^2\times S^2)). \quad (4.7)$$

We will analyze the map (4.7) and extract the desired morphism $\widetilde{KhR}_2^+(W,\Sigma)\colon \widetilde{KhR}_2^+(S_1,L_1)\to \widetilde{KhR}_2^+(S_0,L_0)$ as the "first tensorial factor" of (4.7), in an appropriate sense. As the gluing map (4.6) is manifestly functorial under composition, the functoriality of $\widetilde{KhR}_2^+(W,\Sigma)$ will be a formal consequence of our construction.

4.3.3 Organization of the remaining sections

The sketch proof of Theorem 4.3.2 in Section 4.3.2 is rather imprecise, with two major omissions.

(A) We have to address the distinction between "links in an abstract $\sqcup\#(S^1\times S^2)$" and "admissible links in the concrete $\sqcup\#(S^1\times S^2)$."

(B) The choice of the embedding $X_1\hookrightarrow S^4$ up to isotopy corresponds to a spin structure on X_1, which is in a noncanonical one-to-one correspondence with $H^1(X_1;\mathbb{Z}/2)$. We have to remove this choice.

In order to address item (A), we first need to prove the functoriality of Rozansky–Willis homology for cobordisms in concrete $\#(S^1\times S^2)$. This will be carried out in Section 4.4. For a given cobordism between admissible links, we examine the map (4.7), whose definition entails no choice in this setup. It will be sufficient to examine (4.7) for a set of elementary cobordisms that generate all cobordisms.

In Section 4.5, we address (A), with the extra assumption that our abstract $\sqcup\#(S^1\times S^2)$ is equipped with a spin structure. This entails a study of the diffeomorphism group of $\#\natural(D^2\times S^2)$ rel boundary. After proving Proposition 4.1.3 using Gabai's 4-dimensional lightbulb theorem [Gab20], we reduce the problem mainly to understanding the action on $\mathcal{S}_0^{2,\mathrm{mul}}(\#\natural(D^2\times S^2);L)$ by barbell diffeomorphisms defined by Budney–Gabai [BG19], for admissible links $L\subset\sqcup\#(S^1\times S^2)$.

In Section 4.6, we carry out the heuristics in Section 4.3.2 carefully for morphisms that come with ambient spin structures, by decomposing an arbitrary such morphism into elementary ones, and examining (4.7) for each elementary morphism.

In Section 4.7, we remove the spin assumption and address (B). This requires a construction of maps on $\mathcal{S}_0^{2,\mathrm{mul}}$ induced by Gluck twists on framed embedded 2-spheres in 4-manifolds. Such induced maps will be isomorphisms that transform the grading in a specified way, proving Theorem 4.1.5.

Sign ambiguities will come up at various stages of our proofs. Appendix C employs the $\mathfrak{gl}_2$ webs and foams formalism to fix the signs.

A complete list of elementary moves for $\mathbf{Links}_1$ and their induced chain maps on $\widetilde{CKhR}_2^-$ is recorded in Appendix D.

4.4 Functoriality in concrete $\#(S^1 \times S^2)$

Fix $k \geq 0$, $m_1, \cdots, m_k \geq 0$. Write for short $S_{std} := \sqcup_{i=1}^k \#^{m_i}(S^1 \times S^2)$.

4.4.1 The statements

Let $\mathbf{Links}_{S_{std}}$ denote the category whose objects are admissible links in S_{std}, and whose morphisms are link cobordisms in $I \times S_{std}$ up to isotopy rel boundary. The identity morphisms and compositions are defined in the natural way. The goal of this section is to extend Rozansky–Willis homology to a functor on $\mathbf{Links}_{S_{std}}$.

Theorem 4.4.1. *There is a functor*

$$\widetilde{KhR}_2^+ : \mathbf{Links}_{S_{std}}^{op} \to \mathbf{Mod}_R^{+,(\frac{1}{2}\mathbb{Z})^2 \times \mathbb{Z}/2}$$

that extends the definition of $\widetilde{KhR}_2^+$ on objects as in Section 4.2.1. For a morphism Σ, $\widetilde{KhR}_2^+(\Sigma)$ is homogeneous of degree $(0, -\chi(\Sigma))$ (the Koszul grading is suppressed) and is induced by a chain map.

Write $D_{std} := \#_{i=1}^k \natural^{m_i}(D^2 \times S^2)$. Thus $\partial D_{std} = S_{std}$. The following concept plays an important role in this chapter.

Definition 4.4.2. For an admissible link $L \subset S_{std}$ and any $\alpha \in H_2^L(D_{std})$, the *lasagna quantum grading* on $\mathcal{S}_0^{2,\mathrm{mul}}(D_{std}; L; \alpha)$ is the contribution to the quantum grading coming from the second tensorial factor in the right-hand side of the bigrading-preserving isomorphism

$$\mathcal{S}_0^{2,\mathrm{mul}}(D_{std}; L; \alpha) \cong ((t^{-1}q)^{\alpha^2/2} \widetilde{KhR}_2^+(L)) \otimes \mathcal{S}_0^{2,\mathrm{mul}}(D_{std}; \alpha - \alpha_L)$$

from Theorem 4.2.2.

For a link cobordism $\Sigma\colon L_0 \to L_1$ in $I \times S_{std}$, let $\Sigma^t\colon L_1 \to L_0$ denote its transpose, obtained by turning Σ upside down, reversing its orientation and the ambient orientation.

Theorem 4.4.3. *For a link cobordism $\Sigma\colon L_0 \to L_1$ between admissible links in S_{std}, the gluing map*

$$\mathcal{S}_0^{2,\mathrm{mul}}(I \times S_{std}; \Sigma^t)\colon \mathcal{S}_0^{2,\mathrm{mul}}(D_{std}; L_1) \to \mathcal{S}_0^{2,\mathrm{mul}}(D_{std}; L_0)$$

is nonincreasing in the lasagna quantum grading. The corresponding associated graded map $gr\mathcal{S}_0^{2,\mathrm{mul}}(I\times S_{std};\Sigma^t)$*, under the isomorphism* (SZ)*, is of the form*

$$\widetilde{KhR}_2^+(\Sigma)\otimes gr(\mathrm{id}_{\alpha_{L_1}*[\Sigma^t]-\alpha_{L_0}})\colon \widetilde{KhR}_2^+(L_1)\otimes gr\mathcal{S}_0^{2,\mathrm{mul}}(D_{std})\to \widetilde{KhR}_2^+(L_0)\otimes gr\mathcal{S}_0^{2,\mathrm{mul}}(D_{std}) \tag{4.8}$$

for some map $\widetilde{KhR}_2^+(\Sigma)\colon \widetilde{KhR}_2^+(L_1)\to \widetilde{KhR}_2^+(L_0)$ *induced by a chain map.*

We recall that id_α in (4.8) denotes the shifting automorphism on $\mathcal{S}_0^{2,\mathrm{mul}}(D_{std})$ by α, defined in Definition 4.2.4. The star $*$ denotes the natural concatenation map on homology.

The proof of Theorem 4.4.3 takes up the bulk of Section 4.4. In Section 4.4.2 we decompose it into various cases, which are treated individually in Sections 4.4.3–4.4.7. Before going there, we deduce Theorem 4.4.1 as a consequence of Theorem 4.4.3.

Proof of Theorem 4.4.1 assuming Theorem 4.4.3. The functor $\widetilde{KhR}_2^+$ is defined on objects in Section 4.2.1. Let $\Sigma\colon L_0\to L_1$ be a morphism. If the map (4.8) is nonzero, $\widetilde{KhR}_2^+(\Sigma)$ is determined uniquely; if it is zero, define $\widetilde{KhR}_2^+(\Sigma)$ to be zero. Since $\mathcal{S}_0^{2,\mathrm{mul}}(I\times S_{std};\Sigma^t)$ is homogeneous with bidegree shift $(0,-\chi(\Sigma))$, the same is true for $\widetilde{KhR}_2^+(\Sigma)$.

It remains to check functoriality. If $\Sigma\colon L\to L$ is the identity morphism, then $\mathcal{S}_0^{2,\mathrm{mul}}(I\times S_{std};\Sigma^t)$, hence (4.8), is the identity map. It follows that $\widetilde{KhR}_2^+(\Sigma)$ is the identity map. If $\Sigma_i\colon L_i\to L_{i+1}$, $i=0,1$, are two morphisms in $\mathbf{Links}_{S_{std}}$, then $\mathcal{S}_0^{2,\mathrm{mul}}(I\times S_{std};(\Sigma_1\circ\Sigma_0)^t)=\mathcal{S}_0^{2,\mathrm{mul}}(I\times S_{std};\Sigma_0^t)\circ\mathcal{S}_0^{2,\mathrm{mul}}(I\times S_{std};\Sigma_1^t)$. After taking the associated graded maps, we see again from (4.8) that $\widetilde{KhR}_2^+(\Sigma_1\circ\Sigma_0)=\widetilde{KhR}_2^+(\Sigma_0)\circ\widetilde{KhR}_2^+(\Sigma_1)$. $\square$

We remark that taking associated graded maps is necessary for Theorem 4.4.3. If $\Sigma\colon\emptyset\to\emptyset$ is the cobordism between two empty links given by the j-th core 2-sphere in the i-th disjoint union summand in $\{1/2\}\times S$, the induced map $\mathcal{S}_0^{2,\mathrm{mul}}(D_{std})\to\mathcal{S}_0^{2,\mathrm{mul}}(D_{std})$ is nonzero (it is the multiplication map by $A_{i,j,1}$ in the notation of Example 4.2.3). However, since $\widetilde{KhR}_2^+(\emptyset)=R$, any map of degree $(0,-\chi(\Sigma))=(0,-2)$ is necessarily zero. We will see that taking associated graded maps, as well as introducing nontrivial shifting automorphisms id_α, is necessary only for one elementary move (handleslide) described in Proposition 4.4.4.

4.4.2 Decomposition into elementary cobordisms

If Theorem 4.4.3 holds for two composable morphisms, then it holds for their composition as well. Therefore, it suffices to decompose any cobordism into a composition of some elementary cobordisms, and check Theorem 4.4.3 for these elementary ones.

Proposition 4.4.4. *Every morphism in* $\mathbf{Links}_{S_{std}}$ *is a composition of some elementary morphisms of the following forms. See Figure 4.5.*

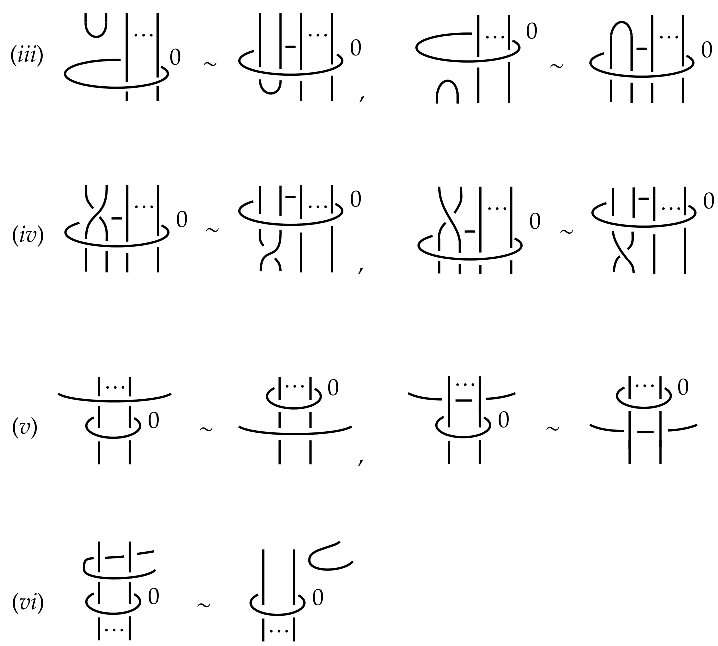


Figure 4.5: Type (iii)–(vi) moves in Proposition 4.4.4. Here, for (iv), we have only drawn, for simplicity, the case of pushing a crossing between the first two strands, but it is understood that similar pushing is allowed for crossings between any two adjacent strands.

(i) Isotopies via admissible links;

(ii) Reidemeister moves or a Morse move (birth, death, saddle) away from the surgery regions;

(iii) Finger moves, also in reverse;

(iv) Crossing moves, also in reverse;

(v) Overpass/underpass moves;

(vi) Handleslides.

Proof. This follows from a general position argument, see e.g. [Wil21, Proposition 3.2]. □

4.4.3 Moves away from surgery regions

We prove Theorem 4.4.3 for moves (i) and (ii) in Proposition 4.4.4.

By the description of the isomorphism (SZ1) in Section 4.2.3, at each $\alpha \in H_2^L(D_{std})$, the map $\mathcal{S}_0^{2,\mathrm{mul}}(I \times S_{std}; \Sigma^t)$ in terms of row (SZ1) is induced summand-wise by the map induced by Σ^t, namely

$$\begin{aligned} &KhR_2(\Sigma^t \cup (I \times (n_+ + r, n_- + r) \text{ belts})): \\ &\quad KhR_2(L_1 \cup (n_+ + r, n_- + r) \text{ belts}) \to KhR_2(L_0 \cup (n_+ + r, n_- + r) \text{ belts}) \end{aligned} \tag{4.9}$$

on each direct summand. Here $n = \alpha - \alpha_{L_1} \in H_2(D_{std}) = \mathbb{Z}^{\sum_{i=1}^k m_i}$. The same description applies for the map in terms of row (SZ2).

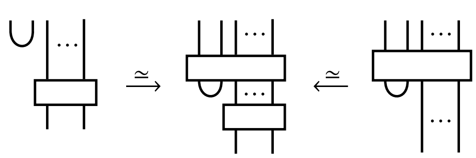
The isomorphisms (SZ3) and (SZ4) are local near the surgery regions, and thus intertwine with the maps on $\widetilde{KhR}_2^+$ induced by Σ^t on rows (SZ2)–(SZ4) summand-wise. It follows that $\mathcal{S}_0^{2,\mathrm{mul}}(I \times S_{std}; \Sigma^t) = \widetilde{KhR}_2^+(\Sigma) \otimes \mathrm{id}$ in terms of row (SZ5), where $\widetilde{KhR}_2^+(\Sigma) \colon \widetilde{KhR}_2^+(L_1) \to \widetilde{KhR}_2^+(L_0)$ is induced by an isotopy of the link diagram (for move (i)) or a Reidemeister/Morse-induced map on the link diagram (for move (ii)). In particular, it preserves the lasagna quantum grading, and the associated map is of the form (4.8), as desired (note $\alpha_{L_1} * [\Sigma^t] - \alpha_{L_0} = 0$ since Σ^t is disjoint from the 2-cocores of D_{std}).

4.4.4 Finger moves

We prove Theorem 4.4.3 for move (iii) in Proposition 4.4.4. We only consider the move pushing a downward-pointing finger up, as the reverse is similar. Thus, the reverse Σ^t is the move pushing a downward-pointing finger down, shown as the left to right direction in Figure 4.5(iii).

By the description of the isomorphism (SZ1) in Section 4.2.3, the map $\mathcal{S}_0^{2,\mathrm{mul}}(I \times S_{std}; \Sigma^t)$ in terms of rows (SZ1) and (SZ2) is each induced by summand-wise cobordism maps pushing the finger down across the belts, as shown in the first row of Figure 4.6. For simplicity, we only depict the case of 2 belts at the surgery region with some orientations that are omitted from the diagram. Similar simplifications in figures will not be further remarked upon.

We claim that in terms of row (SZ3), $\mathcal{S}_0^{2,\mathrm{mul}}(I \times S_{std}; \Sigma^t)$ is equal to a map that is summand-wise given by pushing the finger first down the relevant projector $P_{\ell,0}^\vee$, then down across the belts, as shown in the second row of Figure 4.6. Here, the push-down map across the projector, denoted f, is the "sandwich" composition



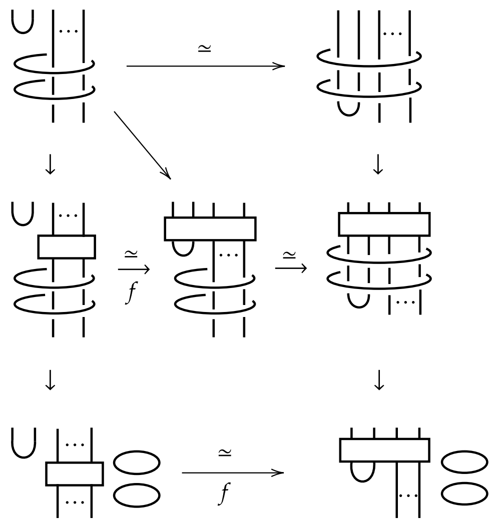
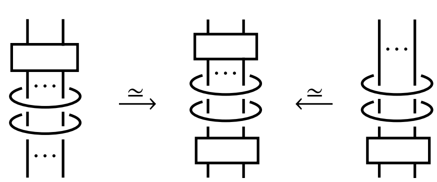


Figure 4.6: Rows (SZ2)–(SZ4) of a finger move.

(the two maps are isomorphisms on homology by Proposition 4.2.1(5)). We need to prove the upper half of Figure 4.6 commutes. This is straightforward: the quadrilateral commutes by locality, and the triangle commutes by definition of f and locality.

We claim that in terms of row (SZ4), $\mathcal{S}_0^{2,\mathrm{mul}}(I\times S_{std};\Sigma^t)$ is equal to a map that is summand-wise given by pushing the cup down across the belts, as shown in the third row of Figure 4.6. Namely, we need to prove the lower half of Figure 4.6 commutes. We fill this lower rectangle into Figure 4.7, where the middle map g is the "sandwich" map defined similarly to f, namely as the composition



.

For convenience, projectors in Figure 4.7 are labeled by 1 or 2, and the subscripts in the unit maps $\iota_\bullet$ and sliding maps $s_\bullet$ refer to the corresponding projectors. All regions in Figure 4.7 commute by definition or locality, except the middle rectangle labeled R. We fill in the rectangle R into the diagram of isomorphisms

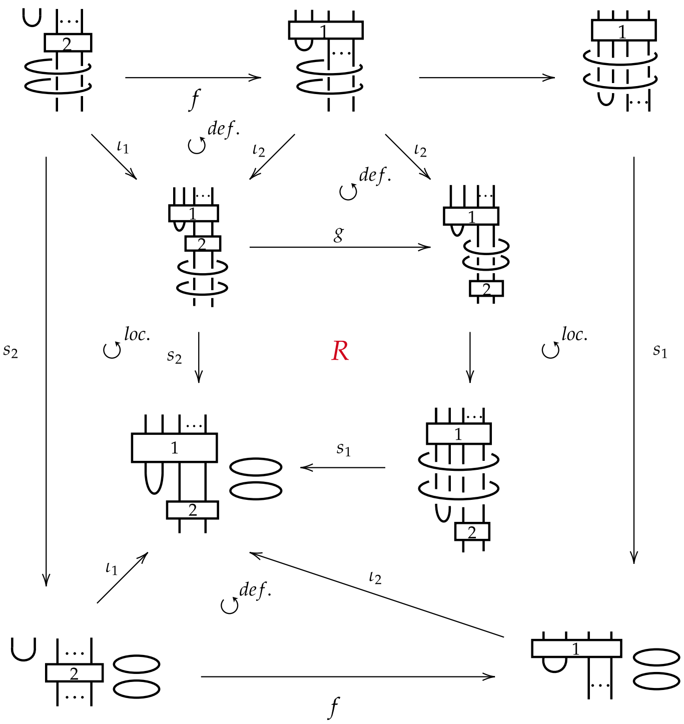
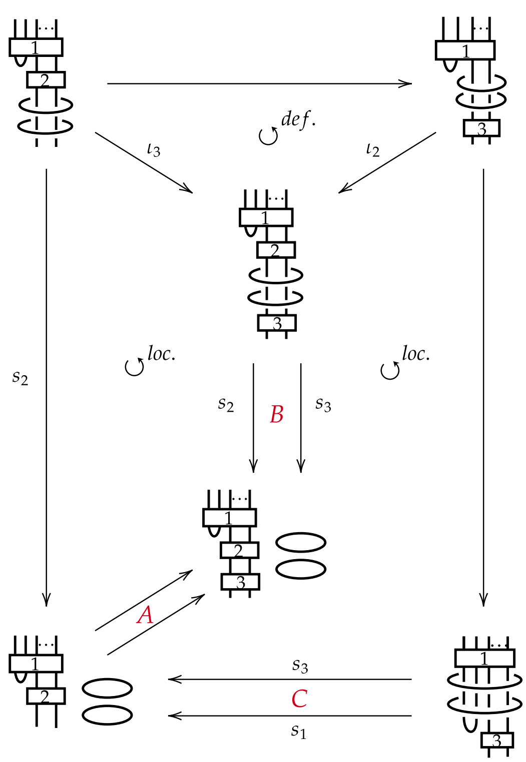
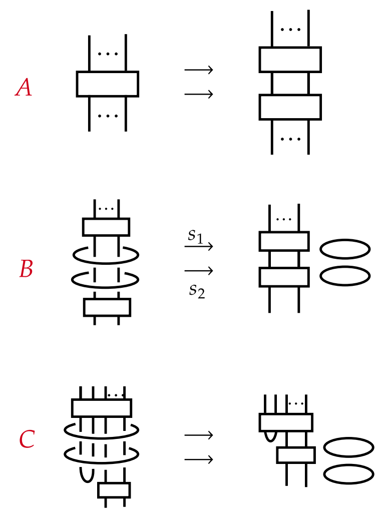


Figure 4.7: The lower rectangle in Figure 4.6.



,

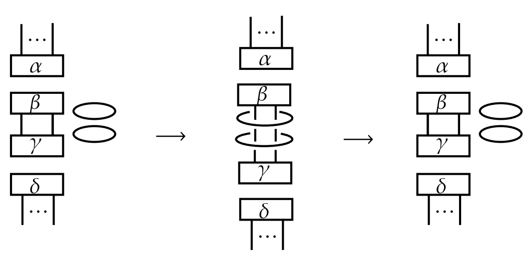
where all regions except A, B, C commute by definition or locality. We redraw regions A, B, C:



.

The commutativity of region A is exactly Proposition 4.2.1(4).

We consider region B. On the chain level, each projector is a chain complex of through-degree 0 Temperley-Lieb diagrams. Each of the two chain maps s_1, s_2 is given termwise by sliding the belts down the Temperley-Lieb diagram from above/below using simplifying Reidemeister II moves, with some higher differentials going between cross terms. Explicit chain homotopy inverses s_1^{-1}, s_2^{-1} can also be written down using [Gug72, Section 3]; see also [Wil21, Proof of Proposition 2.10] for an account in our situation, the relevant property used here being that simplifying Reidemeister II moves are *very strong deformation retracts* [Wil21, Definition 2.9]. As all terms in the complex of the chain model of the right-hand side of region B have internal homological degree 0, the formulas in [Gug72] show that the composition $s_2 \circ s_1^{-1}$ is given termwise by compositions of Reidemeister-induced maps, without any nontrivial cross term. Each such termwise composition takes the form



,

for some crossingless tangles $\alpha, \beta, \gamma, \delta$, where the first map slides the belts into the strands from the top, and the second map slides them out from the bottom. We claim this composition is the identity chain map. By locality, we may ignore the α, δ boxes. The composition is the identity on homology up to sign because the composite cobordism is isotopic to the identity cobordism in $I \times S^3$, and Khovanov homology is functorial in S^3 [MWW22, Theorem 1.1]. Since the diagram is crossingless, this implies that the composition is identity on the chain level up to sign. In Appendix C.4.2, we show that under the appropriate setup using webs and foams, the corresponding composition map is equal to the identity chain map, rather than its negation, proving the commutativity of region B.

The commutativity of region C is proved in exactly the same way.

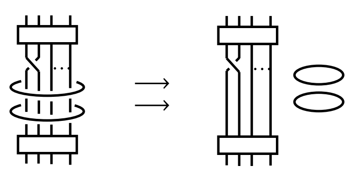
Now, it follows that the map $\mathcal{S}_0^{2,\mathrm{mul}}(I \times S_{std}; \Sigma^t) = \widetilde{KhR}_2^+(\Sigma) \otimes \mathrm{id}$ in terms of row (SZ5), where $\widetilde{KhR}_2^+(\Sigma)$ is the map on homology induced by pushing the finger down the projector. The statement follows.

4.4.5 Crossing moves

We prove Theorem 4.4.3 for move (iv) in Proposition 4.4.4. We only consider the case of pushing a positive crossing up the belts. The cases of pushing down or of a negative crossing are similar.

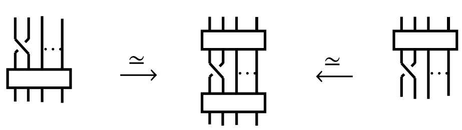
The proof is analogous to the case of finger moves. The only difference is that we need to prove the commutativity of the following analog of region C in Section 4.4.4

D



.

Assuming the commutativity of region D, we can conclude that $\mathcal{S}_0^{2,\mathrm{mul}}(I \times S_{std}; \Sigma^t) = \widetilde{KhR}_2^+(\Sigma) \otimes \mathrm{id}$ where $\widetilde{KhR}_2^+(\Sigma)$ is the map on homology induced by pushing the crossing down the projector, defined as the "sandwich" composition



.

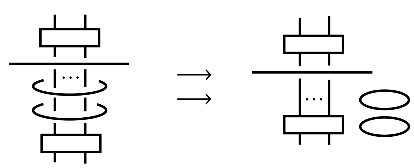
We prove the commutativity of region D. Let Φ denote the endomorphism of the right-hand side given by first sliding the belts into the strands from the top, then sliding them off from the bottom. We show Φ is the identity map on homology. The same argument for region B in Section 4.4.4 no longer applies, because the terms in the twisted complex of the chain model of the right-hand side of region D have nontrivial internal homological degrees, hence cross terms may exist when applying [Gug72]. Nevertheless, [Gug72] implies that Φ is nondecreasing in the external homological degree. By judiciously choosing Reidemeister III induced chain maps within their chain homotopy class when passing the belts across the middle crossing (see [MWW22, Section 3.5], in the $\mathfrak{gl}_2$ webs and foams context), we may assume that Φ is nondecreasing[1] in the internal homological degree, and its internal-degree-preserving part on the two resolutions of the middle crossing, within each term of the twisted complex, is each given by the Reidemeister-induced maps. Since Φ preserves the total homological degree, it must therefore preserve both the internal and external homological degrees. Consequently, by the same argument as before, Φ is given by a termwise chain map that is the identity on each term of the twisted complex up to termwise signs. In Appendix C.4.2, we remove the sign ambiguity and show that Φ is termwise the identity chain map, hence the identity map overall, proving the commutativity of region D.

4.4.6 Overpass/underpass moves

We prove Theorem 4.4.3 for move (v) in Proposition 4.4.4. We only consider the case of an overpass move, as the underpass case is similar.

The proof is again analogous to the case of finger moves. The only difference is that we need to prove the commutativity of the following analog of region C in Section 4.4.4

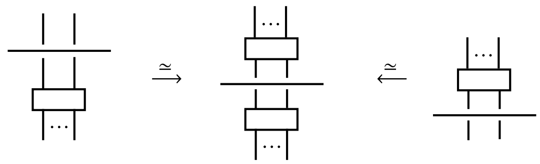
E



.

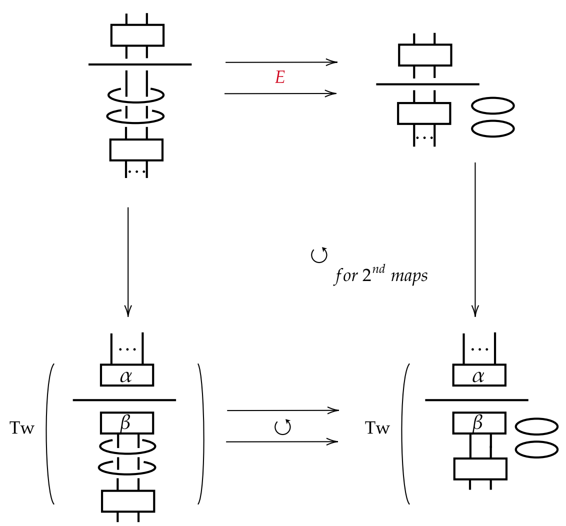
Assuming the commutativity of region E, we can conclude that $\mathcal{S}_0^{2,\mathrm{mul}}(I \times S_{std}; \Sigma^t) = \widetilde{KhR}_2^+(\Sigma) \otimes \mathrm{id}$ where $\widetilde{KhR}_2^+(\Sigma)$ is the map on homology induced by pushing the overpassing strand down the projector, defined as the "sandwich" composition (both maps are isomorphisms on homology by Proposition 4.2.1(6))

[1]Note that [MWW22] uses the homological convention instead of the cohomological convention that we adopt here. Therefore, the words "increasing" and "decreasing" in their Lemma 3.12 have the opposite meaning to ours.



.

We prove the commutativity of region E. Consider the diagram



where we expand the first projector to rewrite the diagrams as twisted complexes, and the vertical maps slide the overstrand into the empty region by termwise Reidemeister II simplifying maps, with some higher differentials. The two horizontal maps in the second row are equal by the same argument as for region B in Section 4.4.4. If we take the pair of horizontal maps that slide the belts off the second projector, one for each row in the diagram, then the square commutes by locality. Thus, to prove the commutativity of region E, it suffices to show the square also commutes for the other pair of horizontal maps.

The proof is similar to the one for region B in Section 4.4.4. We repeat the argument again. Redraw the relevant parts of this square termwise:

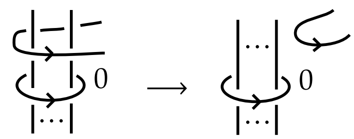
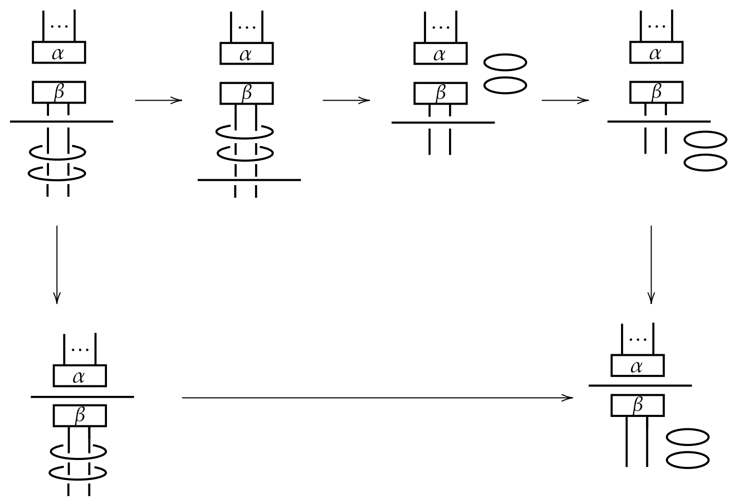


Figure 4.8: The reverse of a negative handleslide, with orientations on the 2-handle attaching curve and the sliding strand indicated.



The two total chain maps are assembled from these termwise maps, with some higher differentials. If we start from the lower-right corner and compose the chain maps or their explicit homotopy inverses (which are termwise Reidemeister-induced maps plus some higher differentials) in a loop, we obtain the termwise identity chain map up to individual signs, without higher differentials, because the termwise composite cobordisms are isotopic to identity and the source and target diagrams are crossingless. The sign ambiguity is removed in Appendix C.4.2. This proves the commutativity of region E.

4.4.7 Handleslides

We prove Theorem 4.4.3 for move (vi) in Proposition 4.4.4. We only consider the case where the sign of the handleslide is negative, namely where $\Sigma^t \subset I \times S_{std} \subset D_{std}$ intersects the 2-cocores positively at one point, see Figure 4.8. The positive case is similar. Suppose the strand is sliding over the j-th 2-handle of D_{std}.

We claim that the map $\mathcal{S}_0^{2,\mathrm{mul}}(I \times S_{std}; \Sigma^t)$ in terms of row (SZ1) is induced by the summand-

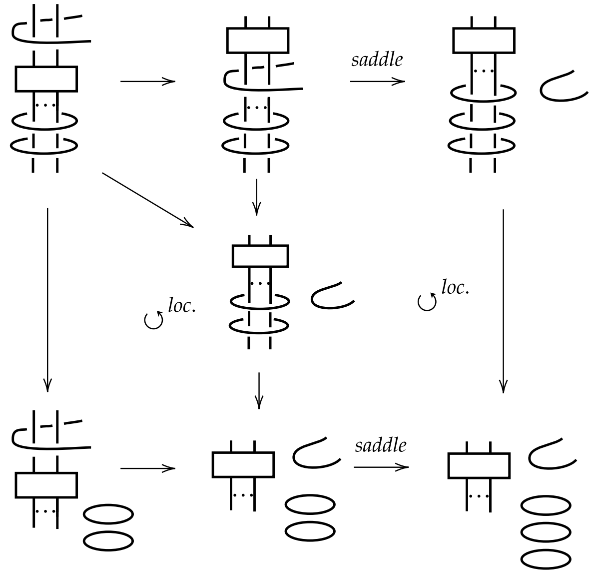
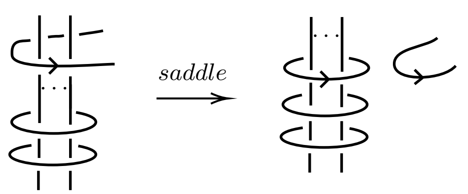


Figure 4.9: Rows (SZ3) and (SZ4) of a reverse handleslide move.

wise saddle map

$$KhR_2(L_1 \cup (n_+ + r, n_- + r) \text{ belts}) \xrightarrow{\text{saddle}} KhR_2(L_0 \cup (n_+ + e_j + r, n_- + r) \text{ belts}),$$

or in local diagram, similar to the annular model situation in [HRW25, Section 4], as



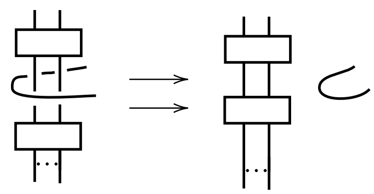
On the lasagna filling level, the map $\mathcal{S}_0^{2,\mathrm{mul}}(I \times S_{std}; \Sigma^t)$ attaches to the standard skein $(I \times L_1) \cup ((n_+ + r, n_- + r)$ cores) the cobordism Σ^t. If one enlarges the input ball a bit to make the new skein $\Sigma^t \cup (I \times L_1) \cup ((n_+ + r, n_- + r)$ cores) standard of the form $(I \times L_0) \cup ((n_+ + e_j + r, n_- + r)$ cores), the Khovanov decoration changes according to the cobordism between the two balls, which is exactly given by the claimed saddle. This proves the claim.

The same description of $\mathcal{S}_0^{2,\mathrm{mul}}(I \times S_{std}; \Sigma^t)$ applies to row (SZ2). By the same argument as in Section 4.4.4 using an analog of the upper half of Figure 4.6, the map $\mathcal{S}_0^{2,\mathrm{mul}}(I \times S_{std}; \Sigma^t)$

in terms of row (SZ3) is summand-wise given by the first row of Figure 4.9. Here the first map is the "sandwich" map as before.

We claim that Figure 4.9 commutes. This will imply $\mathcal{S}_0^{2,\mathrm{mul}}(I\times S_{std};\Sigma^t)$ in terms of row (SZ4) is given by the bottom row of Figure 4.9. The two rectangles commute by locality, while the commutativity of the triangle follows from the commutativity of region A in Section 4.4.4 and region F:

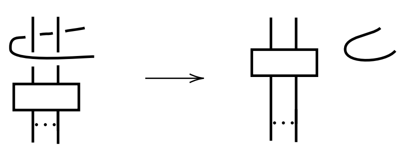
F



.

The commutativity of region F follows from the same argument as for region B in Section 4.4.4. This proves the claim.

Now, by a Bar-Natan neck-cutting, the saddle map in the bottom row of Figure 4.9 is equal to $(birth)\otimes(dot)+(dotted\ birth)\otimes 1$. In terms of row (SZ5), the first term decreases the lasagna quantum grading while the second term preserves it. This proves that $\mathcal{S}_0^{2,\mathrm{mul}}(I\times S_{std};\Sigma^t)$ is nonincreasing in the lasagna quantum grading. The associated graded map is given by $\widetilde{KhR}_2^+(\Sigma)\otimes gr(u)$, where $\widetilde{KhR}_2^+(\Sigma)\colon \widetilde{KhR}_2^+(L_1)\to\widetilde{KhR}_2^+(L_0)$ is the map on homology induced by the slide-off chain map



(4.10)

and $u\colon \mathcal{S}_0^{2,\mathrm{mul}}(D_{std})\to\mathcal{S}_0^{2,\mathrm{mul}}(D_{std})$ is the gluing map that attaches a collar of the boundary containing a copy of the dotted j-th core S^2 as skein. By the description of $\mathcal{S}_0^{2,\mathrm{mul}}(D_{std})$ in Section 4.2.4, $u=\mathrm{id}_{e_j}$. Since $e_j=\alpha_{L_1}*[\Sigma^t]-\alpha_{L_0}$, this finishes the proof of Theorem 4.4.3.

4.5 Homology in abstract spin $\#(S^1\times S^2)$

As in the previous section, fix $k\geq 0$, $m_1,\cdots,m_k\geq 0$, and write for short $S_{std}:=\sqcup_{i=1}^k\#^{m_i}(S^1\times S^2)$, $D_{std}:=\#_{i=1}^k\natural^{m_i}(D^2\times S^2)$. Equip D_{std} with its unique spin structure, and $S_{std}=\partial D_{std}$ with the induced boundary spin structure. When $k=0$, $S_{std}=\emptyset$, and the statements in Section 4.5.1 are trivial. Assume from now on $k\geq 1$.

4.5.1 The statements

If X is a spin manifold, an *abstract spin* X is a spin manifold that is spin diffeomorphic to X. The goal of this section is to construct the Rozansky–Willis homology for links in an abstract spin S_{std}, as stated in the following theorem.

Theorem 4.5.1. *Let S be an abstract spin S_{std} and $L \subset S$ be a framed oriented link. There is a bigraded R-module $\widetilde{KhR}_2^+(S, L)$, called the Rozansky–Willis homology of (S, L), which is an invariant of the pair (S, L) up to spin diffeomorphisms. When $S = S_{std}$ and L is admissible, $\widetilde{KhR}_2^+(S, L)$ is canonically isomorphic to $\widetilde{KhR}_2^+(L)$.*

Let (S, L) be as in Theorem 4.5.1. If $\rho\colon S \xrightarrow{\cong} S_{std}$ is a spin L-admissible parametrization of S, then $\widetilde{KhR}_2^+(\rho(L))$ would be a natural candidate for $\widetilde{KhR}_2^+(S, L)$. For this idea to work, if ρ' is another spin L-admissible parametrization of S, we need to identify $\widetilde{KhR}_2^+(\rho(L))$ with $\widetilde{KhR}_2^+(\rho'(L))$ in a canonical and functorial way. This is the content of the following proposition.

Proposition 4.5.2. *Let $L \subset S_{std}$ be an admissible link and $\phi \in \mathrm{Diff}^{spin}(S_{std})$ be an L-admissible spin diffeomorphism of S_{std}. There is a canonical isomorphism*

$$\widetilde{KhR}_2^+(\phi)\colon \widetilde{KhR}_2^+(\phi(L)) \xrightarrow{\cong} \widetilde{KhR}_2^+(L) \tag{4.11}$$

of graded R-modules induced by a chain map. If $\phi' \in \mathrm{Diff}^{spin}(S_{std})$ is $\phi(L)$-admissible, then

$$\widetilde{KhR}_2^+(\phi) \circ \widetilde{KhR}_2^+(\phi') = \widetilde{KhR}_2^+(\phi' \circ \phi)\colon \widetilde{KhR}_2^+(\phi'(\phi(L))) \xrightarrow{\cong} \widetilde{KhR}_2^+(L).$$

Again, one could state the theorem in the covariant way by inverting the diffeomorphisms.

Proof of Theorem 4.5.1 assuming Proposition 4.5.2. Let $P^{spin}(S, L)$ denote the set of L-admissible spin parametrizations of S. Then, $\widetilde{KhR}_2^+(S, L)$ can be defined as the "cross-section" submodule of $\prod_{\rho \in P^{spin}(S,L)} \widetilde{KhR}_2^+(\rho(L))$ consisting of elements $(v_\rho)_\rho$ with $v_{\rho'} = \widetilde{KhR}_2^+(\rho \circ (\rho')^{-1})(v_\rho)$ for all $\rho, \rho' \in P^{spin}(S, L)$. More formally, $\widetilde{KhR}_2^+(S, L)$ is the limit of $\widetilde{KhR}_2^+(\bullet(L))$ regarded as a functor from the indiscrete category with object set $P^{spin}(S, L)$ to $\mathbf{Mod}_R^{+,(\frac{1}{2}\mathbb{Z})^2 \times \mathbb{Z}/2}$, where morphisms are sent to the canonical identifications.

For any $\rho \in P^{spin}(S, L)$, the natural map $\widetilde{KhR}_2^+(S, L) \xrightarrow{\cong} \widetilde{KhR}_2^+(\rho(L))$ is an isomorphism of graded R-modules. The case $S = S_{std}$ and $\rho = \mathrm{id}$ gives the last statement of the theorem. $\square$

Every spin diffeomorphism $\phi \in \mathrm{Diff}^{spin}(S_{std})$ admits a lift $\tilde{\phi} \in \mathrm{Diff}^+(D_{std})$ in the orientation-preserving diffeomorphism group of D_{std}, as can be seen in Section 4.5.2. Proposition 4.5.2 is therefore a formal consequence of the following theorem.

Theorem 4.5.3. *Let L, ϕ be as in Proposition 4.5.2, and let $\tilde{\phi} \in \mathrm{Diff}^+(D_{std})$ be a lift of ϕ. The pushforward map*

$$\mathcal{S}_0^{2,\mathrm{mul}}(\tilde{\phi}^{-1})\colon \mathcal{S}_0^{2,\mathrm{mul}}(D_{std};\phi(L)) \to \mathcal{S}_0^{2,\mathrm{mul}}(D_{std};L)$$

induces a map $gr_0\mathcal{S}_0^{2,\mathrm{mul}}(D_{std};\phi(L)) \to gr_0\mathcal{S}_0^{2,\mathrm{mul}}(D_{std};L)$ on the 0-th associated graded spaces with respect to the lasagna quantum grading. Under the isomorphism (SZ), this induced map is uniquely of the form

$$\widetilde{KhR}_2^+(\phi) \otimes gr_0(\mathrm{id}_\alpha \circ \phi_*^{-1})\colon \widetilde{KhR}_2^+(\phi(L)) \otimes gr_0\mathcal{S}_0^{2,\mathrm{mul}}(D_{std}) \to \widetilde{KhR}_2^+(L) \otimes gr_0\mathcal{S}_0^{2,\mathrm{mul}}(D_{std}) \tag{4.12}$$

for some isomorphism $\widetilde{KhR}_2^+(\phi)$ independent of $\tilde{\phi}$ induced by a chain map, and some $\alpha \in H_2(D_{std})$ depending on $\tilde{\phi}$ and $[L] \in H_1(S_{std})$. Here, ϕ_* in (4.12) is defined as the pushforward isomorphism $\mathcal{S}_0^{2,\mathrm{mul}}(\tilde{\phi})\colon \mathcal{S}_0^{2,\mathrm{mul}}(D_{std}) \xrightarrow{\cong} \mathcal{S}_0^{2,\mathrm{mul}}(D_{std})$.

Note that by the description of $\mathcal{S}_0^{2,\mathrm{mul}}(D_{std})$ in Section 4.2.4, the map $\phi_* = \mathcal{S}_0^{2,\mathrm{mul}}(\tilde{\phi})\colon \mathcal{S}_0^{2,\mathrm{mul}}(D_{std}) \to \mathcal{S}_0^{2,\mathrm{mul}}(D_{std})$ only depends on the action of $\tilde{\phi}$ on $H_2(D_{std})$ (hence, in particular, is independent of the choice of $\tilde{\phi}$). Note also that the uniqueness statement in Theorem 4.5.3 is trivial.

The proof of Theorem 4.5.3 takes up the bulk of Section 4.5. In Section 4.5.2 we decompose it into various cases, which are treated individually in Sections 4.5.4–4.5.9. Proposition 4.5.2 is then a consequence.

Proof of Proposition 4.5.2 assuming Theorem 4.5.3. We check that the assignment $\phi \mapsto \widetilde{KhR}_2^+(\phi)$ given by Theorem 4.5.3 is functorial. If $\phi = \mathrm{id}$, taking $\tilde{\phi} = \mathrm{id}$ in Theorem 4.5.3 yields $\widetilde{KhR}_2^+(\phi) = \mathrm{id}$. If $\phi_j \in \mathrm{Diff}^{spin}(S_{std})$ with lifts $\tilde{\phi}_j \in \mathrm{Diff}^+(D_{std})$, $j = 0, 1$, where ϕ_0 is L-admissible and ϕ_1 is $\phi_0(L)$-admissible, then $\tilde{\phi}_1 \circ \tilde{\phi}_0$ is a lift of $\phi_1 \circ \phi_0$. Since $\mathcal{S}_0^{2,\mathrm{mul}}((\tilde{\phi}_1 \circ \tilde{\phi}_0)^{-1}) = \mathcal{S}_0^{2,\mathrm{mul}}(\tilde{\phi}_0^{-1}) \circ \mathcal{S}_0^{2,\mathrm{mul}}(\tilde{\phi}_1^{-1})$, we know from Theorem 4.5.3 that

$$\begin{aligned}&\widetilde{KhR}_2^+(\phi_1 \circ \phi_0) \otimes gr_0(\mathrm{id}_\alpha \circ \mathcal{S}_0^{2,\mathrm{mul}}((\tilde{\phi}_1 \circ \tilde{\phi}_0)^{-1}))\\ =&(\widetilde{KhR}_2^+(\phi_0) \circ \widetilde{KhR}_2^+(\phi_1)) \otimes gr_0(\mathrm{id}_{\alpha_0} \circ \mathcal{S}_0^{2,\mathrm{mul}}(\tilde{\phi}_0^{-1}) \circ \mathrm{id}_{\alpha_1} \circ \mathcal{S}_0^{2,\mathrm{mul}}(\tilde{\phi}_1^{-1}))\end{aligned}$$

for some $\alpha, \alpha_0, \alpha_1 \in H_2(D_{std})$. It follows that $\alpha = \alpha_0 + (\tilde{\phi}_0)_*^{-1}(\alpha_1)$ and $\widetilde{KhR}_2^+(\phi_1 \circ \phi_0) = \widetilde{KhR}_2^+(\phi_0) \circ \widetilde{KhR}_2^+(\phi_1)$. □

An important special case of Theorem 4.5.3 is when $\phi = \mathrm{id}$. We state this as a proposition by itself, which will in turn be an ingredient of the proof of Theorem 4.5.3.

Proposition 4.5.4. *Let $L \subset S_{std}$ be an admissible link and $\psi \in \mathrm{Diff}_\partial(D_{std})$ be a diffeomorphism of D_{std} rel boundary. The pushforward map*

$$\mathcal{S}_0^{2,\mathrm{mul}}(\psi)\colon \mathcal{S}_0^{2,\mathrm{mul}}(D_{std};L) \to \mathcal{S}_0^{2,\mathrm{mul}}(D_{std};L)$$

induces a map on gr_0, which under the isomorphism (SZ) takes the form

$$\mathrm{id} \otimes gr_0(\mathrm{id}_\alpha)\colon \widetilde{KhR}_2^+(L) \otimes gr_0\mathcal{S}_0^{2,\mathrm{mul}}(D_{std}) \to \widetilde{KhR}_2^+(L) \otimes gr_0\mathcal{S}_0^{2,\mathrm{mul}}(D_{std})$$

for some $\alpha \in H_2(D_{std})$ depending on ψ and $[L] \in H_1(S_{std})$.

4.5.2 Decomposition into elementary diffeomorphisms

The independence of the map $\widetilde{KhR}_2^+(\phi)$ in Theorem 4.5.3 on the choice of $\tilde{\phi}$ follows from Proposition 4.5.4. Thus, assuming Proposition 4.5.4, Theorem 4.5.3 can be regarded as a statement for the pair $(L, \tilde{\phi})$.

If Theorem 4.5.3 holds for $(L, \tilde{\phi})$ and $(\phi(L), \tilde{\phi}')$, then it holds for $(L, \tilde{\phi}' \circ \tilde{\phi})$ as well. Therefore, it suffices to decompose any $\tilde{\phi}$ with ϕ being L-admissible into a composition of elementary diffeomorphisms $\tilde{\phi}_m \circ \cdots \circ \tilde{\phi}_1$ with each ϕ_i being $\phi_{i-1} \circ \cdots \circ \phi_1(L)$-admissible, and prove Theorem 4.5.3 for such elementary diffeomorphisms.

Proposition 4.5.5. *The spin mapping class group of S_{std}, $\pi_0(\mathrm{Diff}^{spin}(S_{std}))$, is generated by mapping classes represented by the following spin diffeomorphisms:*

(i) Switching the i-th and i'-th connected component when $m_i = m_{i'}$, $1 \le i < i' \le k$;

(ii) Switching the j-th and $(j+1)$-th connected summands in the i-th connected component, when $1 \le j < j+1 \le m_i$;

(iii) Inverting the first connected summand in the i-th connected component by reflecting both the S^1 factor and the S^2 factor, when $m_i > 0$;

(iv) Sliding the first 0-framed surgery circle negatively over the second 0-framed surgery circle in the i-th connected component, when $m_i > 1$.

Proof. This is standard. See e.g. [Lau74, Theorem III.4.3]. □

A *strongly admissible link* in S_{std} is an admissible link so that on each connected component of S_{std} diffeomorphic to some $\#^m(S^1 \times S^2)$ with $m > 0$, the diagram of the link is standard near the (projection of the) collective surgery regions (say $(0.8, m+0.2) \times (-0.02, 0.02) \subset \mathbb{R}^2$) in that no strand passes through the regions between adjacent surgery regions; see Figure 4.10. A diffeomorphism ϕ of S_{std} is *L-strong-admissible* if $\phi(L)$ is strongly admissible.

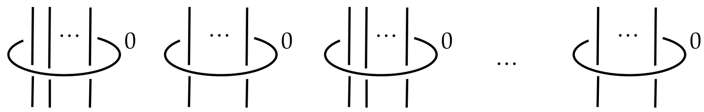


Figure 4.10: Diagram of a strongly admissible link near the surgery regions.

Proposition 4.5.6. *Every diffeomorphism $\tilde{\phi} \in \mathrm{Diff}^+(D_{std})$ can be decomposed into a composition of some elementary diffeomorphisms of the following forms.*

(i) Diffeomorphisms rel boundary;

(ii) Isotopy insertions: a diffeomorphism that is trivial outside a collar neighborhood $[-1,0] \times S_{std}$ of the boundary, and is an isotopy starting from $\mathrm{id}_{S_{std}}$ in this parametrized collar neighborhood;

(iii) Connected summand rearrangements: exchanging the i-th and i'-th connected summands, when $m_i = m_{i'}$, $1 \leq i < i' \leq k$;

(iv) Boundary summand rearrangements: exchanging the j-th and $(j+1)$-th boundary summands in the i-th connected summand, when $1 \leq j < j+1 \leq m_i$;

(v) Inversions: inverting the first boundary summand in the i-th connected summand by reflecting both the D^2 factor and the S^2 factor, when $m_i > 0$;

(vi) Negative handleslides: sliding the first 0-framed 2-handle negatively over the second 0-framed 2-handle in the i-th connected summand, when $m_i > 1$.

Moreover, we can arrange the following:

- *Type (iii)–(vi) diffeomorphisms can be taken to be of some standard forms: type (iii) ones exchange the i-th and i'-th 3-handles together with the two corresponding collections of 2-handles while preserving all other regions (cf. Figure 4.3); type (iv)–(vi) ones can be locally visualized on the boundary in the presence of a strongly admissible link as in Figure 4.11.*

- *If $\phi = \tilde{\phi}|_{S_{std}}$ is L-admissible for an admissible link $L \subset S_{std}$, then the decomposition $\tilde{\phi} = \tilde{\phi}_m \circ \cdots \circ \tilde{\phi}_1$ can be chosen so that each ϕ_i is $\phi_{i-1} \circ \cdots \circ \phi_1(L)$-admissible, and $\phi_{i-1} \circ \cdots \circ \phi_1(L)$-strong-admissible if ϕ_{i+1} is of type (iii)–(vi).*

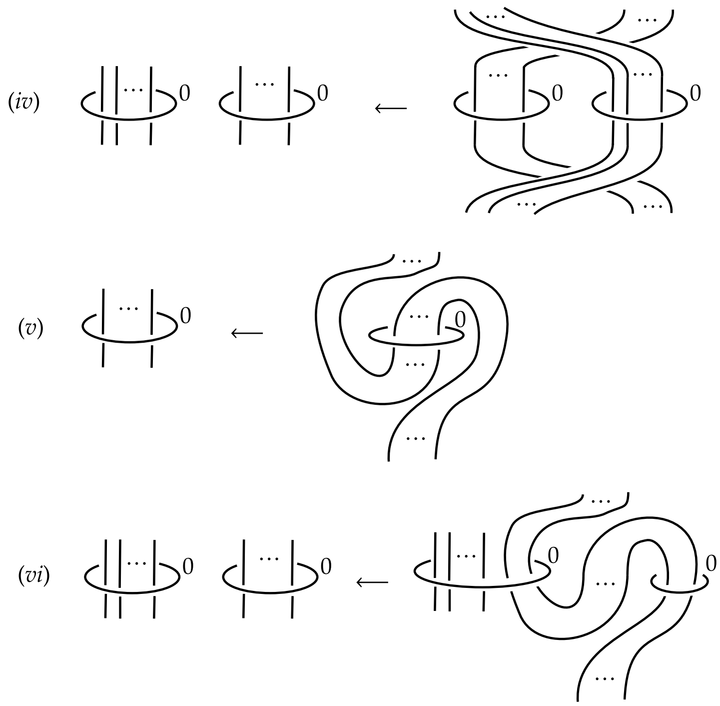


Figure 4.11: Elementary diffeomorphisms of types (iv)–(vi) in Proposition 4.5.6 carrying the strongly admissible links on right-hand sides to the admissible links on the left-hand sides.

Proof. Pick a sequence $\phi_1, \cdots, \phi_r \in \mathrm{Diff}^{spin}(S_{std})$, each of some standard form described in Proposition 4.5.5, taken to be compatible with Figure 4.11, such that $\phi_r \circ \cdots \circ \phi_1$ and ϕ represent the same mapping class. Pick standard lifts $\tilde{\phi}_1, \cdots, \tilde{\phi}_r \in \mathrm{Diff}^+(D_{std})$. Then $\tilde{\phi} = \tilde{\phi}_{r+2} \circ \cdots \circ \tilde{\phi}_1$ for some type (ii) diffeomorphism $\tilde{\phi}_{r+1}$ and some type (i) diffeomorphism $\tilde{\phi}_{r+2}$. The (strong) admissibility conditions on the decomposition in the presence of an admissible link can be achieved by inserting extra type (ii) diffeomorphisms between each pair of adjacent $\tilde{\phi}_i$'s as well as before $\tilde{\phi}_1$. $\square$

4.5.3 The diffeomorphism group of $\#\natural(D^2 \times S^2)$ rel boundary

In order to prove Proposition 4.5.4, we need to understand $\mathrm{Diff}_\partial(D_{std})$, the diffeomorphism group of D_{std} rel boundary.

Write $m = \sum_{i=1}^k m_i$, the second Betti number of D_{std}. Let S_j denote the sphere obtained from the j-th 2-core capped off by a standard disk in B^4, and C_j denote the j-th 2-cocore of D_{std}, for $j = 1, \cdots, m$. The boundary belt circles of the cocores, denoted $U_1, \cdots, U_m$, are equipped with the framings coming from the cocores. The intersection number can be defined between any two homology classes in the set $H_2(D_{std})\cup(\cup_{j=1}^m H_2^{U_j}(D_{std}))$, where $H_2^{U_j}(D_{std})\subset H_2(D_{std},U_j)$ denotes the preimage of $[U_j]\in H_1(U_j)$ under the connecting homomorphism $H_2(D_{std},U_j)\to H_1(U_j)$. When evaluating on the sequence $([S_1],\cdots,[S_m],[C_1],\cdots,[C_m])$, the intersection pairing takes the matrix form $\left(\begin{smallmatrix}0 & I_m\\ I_m & 0\end{smallmatrix}\right)$, where I_m is the identity $m\times m$ matrix.

If $\psi\in\mathrm{Diff}_\partial(D_{std})$, then $\psi_*=\mathrm{id}$ on $H_2(D_{std})$. However, this need not be the case on $H_2^{U_j}(D_{std})$. With respect to the sequence $[S_j],[C_j]$, ψ_* takes a matrix form $\left(\begin{smallmatrix}I_m & X\\ 0 & I_m\end{smallmatrix}\right)$ for some $m\times m$ integral matrix X. Since ψ_* respects the intersection pairing, $X\in\mathfrak{o}(m;\mathbb{Z})$ is an $m\times m$ integral skew-symmetric matrix. We have constructed a group homomorphism

$$h_1\colon \mathrm{Diff}_\partial(D_{std})\to\mathfrak{o}(m;\mathbb{Z}).$$

On the other hand, the set of spin structures on D_{std} rel the standard spin structure on S_{std}, denoted $Spin_\partial(D_{std})$, is affine over $H^1(D_{std},S_{std};\mathbb{Z}/2)\cong H_3(D_{std};\mathbb{Z}/2)\cong(\mathbb{Z}/2)^{k-1}$. Every element in $\mathrm{Diff}_\partial(D_{std})$ acts on $Spin_\partial(D_{std})$ as a translation by some class in $H_3(D_{std};\mathbb{Z}/2)$, giving rise to a map

$$h_2\colon \mathrm{Diff}_\partial(D_{std})\to H_3(D_{std};\mathbb{Z}/2),$$

which is a group homomorphism since $\mathrm{Diff}_\partial(D_{std})$ acts trivially on $H_3(D_{std};\mathbb{Z}/2)$, noting that $H_3(D_{std};\mathbb{Z}/2)$ is generated from the boundary.

Recall from the introduction that $\mathrm{Diff}_{\partial,loc}(D_{std})$ denotes the subgroup of $\mathrm{Diff}_\partial(D_{std})$ consisting of diffeomorphisms that can be isotoped rel boundary to be supported in a local 4-ball.

Proposition 4.5.7. *The group homomorphism $h=h_1\times h_2$ fits into the short exact sequence*

$$1\to\mathrm{Diff}_{\partial,loc}(D_{std})\to\mathrm{Diff}_\partial(D_{std})\xrightarrow{h}\mathfrak{o}(m;\mathbb{Z})\times H_3(D_{std};\mathbb{Z}/2)\to 1.$$

Proof. By choosing a local 4-ball disjoint from all S_j,C_j, we see that $\mathrm{Diff}_{\partial,loc}(D_{std})\subset\ker(h)$.

We show the exactness at the last term. The map h_2 is surjective because we can insert Dehn twists along the 3-core spheres of D_{std}. To prove the surjectivity of h_1, it suffices to exhibit for each $1\le i<j\le m$ a diffeomorphism $\beta_{i,j}\in\mathrm{Diff}_\partial(D_{std})$ that sends $[C_i]$ to $[C_i]+[S_j]$, $[C_j]$ to $[C_j]-[S_i]$, and each other $[C_k]$ to itself.

The union of the 0-handle, the i-th 2-handle and the j-th 2-handle in D_{std} forms a standard $\natural^2(D^2\times S^2)$. Thus it suffices to exhibit a diffeomorphism $\beta=\beta_{1,2}\in\mathrm{Diff}_\partial(\natural^2(D^2\times S^2))$ with $\beta_*[C_1]=[C_1]+[S_2]$, $\beta_*[C_2]=[C_2]-[S_1]$. This is provided by the barbell diffeomorphism defined by Budney–Gabai [BG19, Section 5]. For our purposes, we give an alternative description of this diffeomorphism below.

See Figure 4.12. Consider the positive Hopf link $U_1\cup U_2\subset B^3$. Let D_i be a spanning disk for U_i that intersects U_{i+1} transversely at one point, $i=1,2$, index modulo 2. The

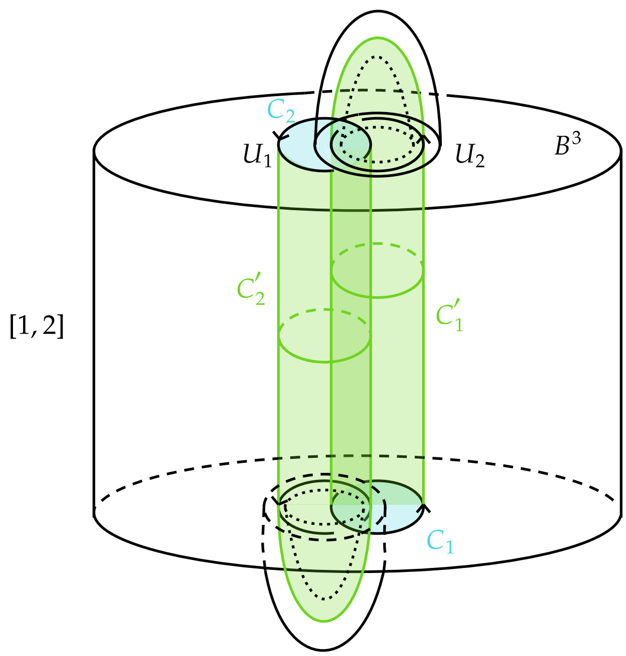


Figure 4.12: The barbell diffeomorphism of $\natural^2(D^2 \times S^2)$, determined by sending C_j to C_j', $j = 1, 2$.

manifold $\natural^2(D^2 \times S^2)$ is obtained from attaching 0-framed 2-handles along $\{i\} \times U_i$, $i = 1, 2$, in ∂B^4, where $B^4 = [1,2] \times B^3$. Let K_1, K_2 denote the cores of the 2-handles, with the usual orientation convention that $\partial K_i = -\{i\} \times U_i$, $i = 1, 2$. The cocores of $\natural^2(D^2 \times S^2)$ are given by $C_1 := -\{1\} \times D_2$, $C_2 := \{2\} \times D_1$ (interior slightly pushed into the interior of B^4). Define disks $C_1' := (I \times U_2) \cup K_2$, $C_2' := (I \times U_1) \cup (-K_1)$. Then $[C_1'] = [C_1] + [S_2]$, $[C_2'] = [C_2] - [S_1]$. One can check, for example by cutting along $\partial[1,2] \times B^3$ and regluing, that the complement of $C_1' \cup C_2'$ in $\natural^2(D^2 \times S^2)$ is diffeomorphic to B^4 via some diffeomorphism canonical up to isotopy rel boundary. Thus there is a diffeomorphism $\beta \in \mathrm{Diff}_\partial(\natural^2(D^2 \times S^2))$ that carries C_i to C_i', as desired. The mapping class of β is well-defined.

We show the exactness at the middle term. Supposing $\psi \in \ker(h)$, we show that ψ can be isotoped to be supported in a local 4-ball. By the relative Hurewicz theorem, each $\psi(C_j)$ is homotopic to C_j. Since the C_j's admit disjoint dual 2-spheres, and have trivial normal bundles rel boundary, Gabai's 4-dimensional lightbulb theorem [Gab20, Theorem 10.1] (see also [Gab21, Theorem 0.6]) implies that ψ is isotopic rel boundary to some ψ' that fixes each neighborhood of C_j pointwise. The complement of a neighborhood of $\partial D_{std} \cup (\cup_j C_j)$ is a 4-sphere with k open 4-balls removed. Choose $k - 1$ disjoint embedded arcs in this holed S^4 connecting the holes. Isotope ψ' to some ψ'' rel the exterior of this holed S^4 that is the

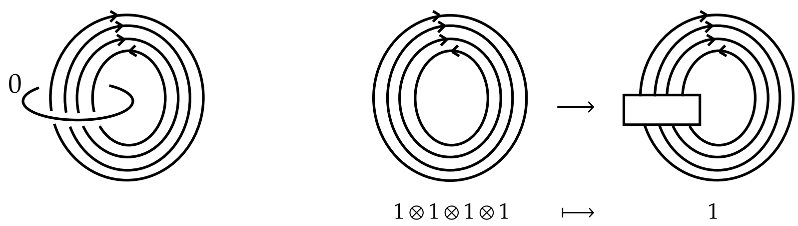


Figure 4.13: Left: Diagram of a standard belt link near a surgery region. Right: The element 1 in the Rozansky–Willis homology of a standard belt link.

identity on the $k-1$ arcs. Since ψ acts trivially on $Spin_\partial(D_{std})$, we may further assume that ψ'' preserves the framings of the arcs, hence fixes a neighborhood of the arcs. It follows that ψ'' is supported in a local 4-ball, as desired. $\square$

4.5.4 Diffeomorphisms rel boundary

In this section we prove Proposition 4.5.4, and consequently Theorem 4.5.3 for type (i) diffeomorphisms in Proposition 4.5.6. Since the statement respects compositions in ψ, it suffices to check on a set of generators of $\mathrm{Diff}_\partial(D_{std})$, which by the proof of Proposition 4.5.7 can be taken to consist of elements of $\mathrm{Diff}_{\partial,loc}(D_{std})$, Dehn twists along 3-spheres, and barbell diffeomorphisms implanted standardly in D_{std}.

If $\psi\in\mathrm{Diff}_{\partial,loc}(D_{std})$, then $\mathcal{S}_0^{2,\mathrm{mul}}(\psi)=\mathrm{id}$ because we can localize ψ to avoid any given skein. If ψ is a 3-sphere twist, by putting lasagna fillings in a general position with respect to the twisting sphere, neck-cutting using Lemma 2.3.7, twisting and regluing, we know that $\mathcal{S}_0^{2,\mathrm{mul}}(\psi)=\mathrm{id}$ (here, we used the fact that $\pi_1(\mathrm{Diff}(S^3))=\mathbb{Z}/2$ acts trivially on Khovanov–Rozansky $\mathfrak{gl}_2$ homology).

Let $\beta_{i,j}\in\mathrm{Diff}_\partial(D_{std})$ denote the barbell diffeomorphism implanted in the union of the 0-handle and the i,j-th 2-handles of D_{std}, $i<j$. It remains to check Proposition 4.5.4 for $\beta_{i,j}$.

We formulate the following special case of Proposition 4.5.4. A *standard belt link* is an admissible link in S_{std} which is a parallel cable of the union of the belt circles of the 2-cocores in D_{std}, with various orientations, that takes a standard form as indicated locally on the left of Figure 4.13. A *belt link* is a framed oriented link in S_{std} isotopic to a standard belt link. For a standard belt link U, define $1\in\widetilde{KhR}_2^+(U)$ to be the image of the class 1 ($=1\otimes\cdots\otimes1$) in KhR_2 of U viewed as an unlink in $\sqcup_{i=1}^k S^3$, under the unit maps that create Rozansky projectors at the surgery regions (see the right of Figure 4.13).

Lemma 4.5.8. *Let $U \subset S_{std}$ be a standard belt link. The pushforward map $\mathcal{S}_0^{2,\mathrm{mul}}(\beta_{i,j})\colon \mathcal{S}_0^{2,\mathrm{mul}}(D_{std};U) \to \mathcal{S}_0^{2,\mathrm{mul}}(D_{std};U)$, under the isomorphism (SZ), sends $1\otimes 1$ to $1\otimes \mathrm{id}_\alpha(1)$ plus terms with lower lasagna quantum gradings, for some α depending on i,j and $[U]\in H_1(S_{std})$.*

Note that by the description of (SZ) in Section 4.2.3, the element $1\otimes 1 \in \mathcal{S}_0^{2,\mathrm{mul}}(D_{std};U)$ is represented by the skein given by the collection of cocores that caps U off.

Proof. Let C denote the disjoint union of 2-cocores in D_{std} that cap U off. As observed, $1\otimes 1$ is represented by the skein C with empty decoration. The diffeomorphism $\beta_{i,j}$ preserves each cocore except those parallel to the i,j-th 2-cocores C_i, C_j of D_{std}, which are sent to disks parallel to C_i', C_j' shown as C_1', C_2' in Figure 4.12.

The element $\mathcal{S}_0^{2,\mathrm{mul}}(\beta_{i,j})(1\otimes 1)$ is represented by the skein $\beta_{i,j}(C)$ with empty decoration. We track this element in the sequence of isomorphisms relating $\mathcal{S}_0^{2,\mathrm{mul}}(D_{std};U)$ to $\widetilde{KhR}_2^+(U)\otimes \mathcal{S}_0^{2,\mathrm{mul}}(D_{std})$ in Section 4.2.3.

Let n_+, n_- denote the number of positively, negatively oriented i-th belt circles in U and m_+, m_- denote similarly those numbers of j-th belt circles. Thus, the homology class of $\beta_{i,j}(C)$ is $\alpha_U + (n_+ - n_-)[S_j] - (m_+ - m_-)[S_i]$.

Delete a slightly shrunken 0-handle from D_{std} and evaluate the skein $\beta_{i,j}(C)$ inside this ball. We see that in terms of row (SZ1)[2], $\mathcal{S}_0^{2,\mathrm{mul}}(\beta_{i,j})(1\otimes 1)\in \mathcal{S}_0^{2,\mathrm{mul}}(D_{std};U;\alpha_U + (m_- - m_+)[S_i] + (n_+ - n_-)[S_j])$ is represented by $1\otimes \mathrm{coev}(1) \in KhR_2(U')\otimes KhR_2(H_+^{(m,n)} \sqcup (-H_+^{(m,n)})) \cong KhR_2(U \cup (m_-e_i + n_+e_j, m_+e_i + n_-e_j)$ belts) in the direct summand $r = \min(m_+, m_-)e_i + \min(n_+, n_-)e_j$. Here $U' \subset S^3$ are components of U that are not parallel to the i,j-th belt circles, $H_+^{(m,n)}$ is the cable of the positive Hopf link defined as indicated in Figure 4.14, and $\mathrm{coev}\colon R = KhR_2(\emptyset) \to KhR_2(H_+^{(m,n)} \sqcup (-H_+^{(m,n)}))$ is the coevaluation map. The copy $H_+^{(m,n)}$ is thought of as the components of U that are parallel to the j-th belt circle together with (n_+, n_-) belts coming from components of $\beta_{i,j}(C)$ parallel to C_i', and a similar description applies to the reverse mirror copy $-H_+^{(m,n)}$. See Figure 4.15 for a sketch of $\beta_{i,j}(C)$, especially the induced orientation on the $H_+^{(m,n)} \sqcup (-H_+^{(m,n)})$ part of the input link.

Tracing $\mathcal{S}_0^{2,\mathrm{mul}}(\beta_{i,j})(1\otimes 1)$ further down (SZ2)–(SZ4), we see that the corresponding elements in these three rows are given by $1 \in \widetilde{KhR}_2^+(U')$ tensored with the image of $1 \in \widetilde{KhR}_2^+(\emptyset)$ under the composition of maps shown in Figure 4.16.

In order to prove the lemma, with $\alpha = (m_- - m_+)e_i + (n_+ - n_-)e_j$, in view of Example 4.2.3, Definition 4.2.4, and the proof of (SZ), it suffices to show that the image of 1 in the last row of Figure 4.16 is equal to $1\otimes 1\otimes x\otimes x \in \widetilde{KhR}_2^+(1_m)\otimes \widetilde{KhR}_2^+(1_n)\otimes \widetilde{KhR}_2^+(U^{n_+,n_-})\otimes$

[2] Technically, in the description of the isomorphism (SZ1), the standard skein comes with k input balls instead of a single input ball. This difference is insignificant.

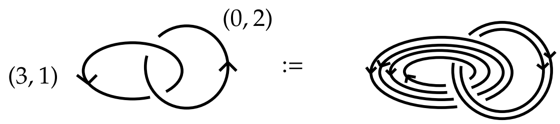


Figure 4.14: The link $H_+^{(m,n)} \subset S^3$ for $(m_+, m_-) = (3, 1)$, $(n_+, n_-) = (0, 2)$.

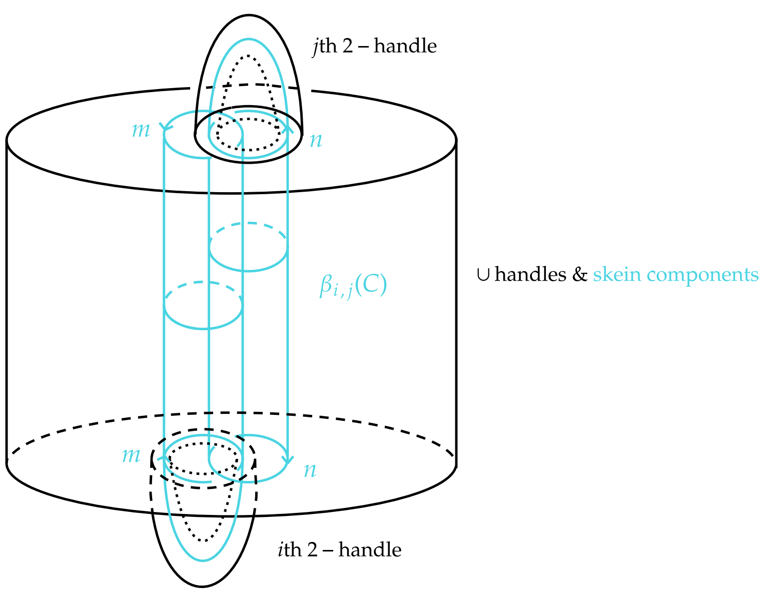


Figure 4.15: The skein $\beta_{i,j}(C)$ is the union of an n-cable on C'_i and an m-cable on C'_j, and other cocore components.

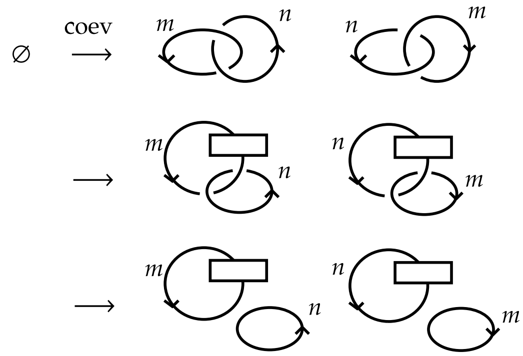


Figure 4.16: The barbell move check.

$\widetilde{KhR}_2^+(U^{m_-,m_+})$ plus terms with lower quantum gradings in the $\widetilde{KhR}_2^+(U^{n_+,n_-})\otimes\widetilde{KhR}_2^+(U^{m_-,m_+})$ factor. Here 1_ℓ denotes a standard belt link in $S^1\times S^2$ with ℓ_+ strands positively oriented and ℓ_- ones negatively oriented, $U^{a,b}$ has the same meaning as in (SZ4), and $x \in \widetilde{KhR}_2^+(U^{a,b})$ denotes the element represented by $x\otimes\cdots\otimes x$.

Following [Roz10; Wil21], Manolescu–Marengon–Sarkar–Willis [Man+23, Corollary 2.2] showed that the Rozansky projector can be approximated by full twists in a quantitative sense. For us, this implies that, for $|\ell|$ even,

$$\widetilde{KhR}_2^{+,0,-|\ell|}(1_\ell) \cong \widetilde{KhR}_2^{+,0,-|\ell|}(T(|\ell|,|\ell|)_{\ell_+,\ell_-}) \cong Kh^{-|\ell|^2/2,-3|\ell|^2/2+|\ell|}(-T(|\ell|,|\ell|)) \cong R,$$

where $T(|\ell|,|\ell|)_{\ell_+,\ell_-}$ denotes the positive torus link $T(|\ell|,|\ell|)$ equipped with an orientation where ℓ_- of the strands have the orientation reversed. Similarly,

$$\widetilde{KhR}_2^{+,0,<-|\ell|}(1_\ell) \cong Kh^{-|\ell|^2/2,>-3|\ell|^2/2+|\ell|}(-T(|\ell|,|\ell|)) = 0,$$

$$\widetilde{KhR}_2^{+,<0,*}(1_\ell) \cong Kh^{<-|\ell|^2/2,*}(-T(|\ell|,|\ell|)) = 0.$$

Here, the equalities about torus links $T(|\ell|,-|\ell|)$ follow from the calculation of Stošić [Sto09, Theorem 1, Theorem 3], suitably renormalized. Consequently, for degree reasons, the image of 1 under the maps in Figure 4.16 is a scalar c times $1\otimes 1\otimes x\otimes x$ plus lower terms. Note that we used the fact that $1\in\widetilde{KhR}_2^{+,0,-|\ell|}(1_\ell)$ is nonzero, which can be verified for example by tracking the element $1\otimes\cdots\otimes 1$ in the KhR_2 of the $|\ell|$-component unlink in the diagram in Figure 4.17, where the top right map is an equivalence by Proposition 4.2.1(3)(5).

Now we prove $c=1$. In each row of Figure 4.16, one may cap off the m strands in the left half picture by $|m|/2$ dotted annuli, and the n strands in the right half picture by $|n|/2$ dotted

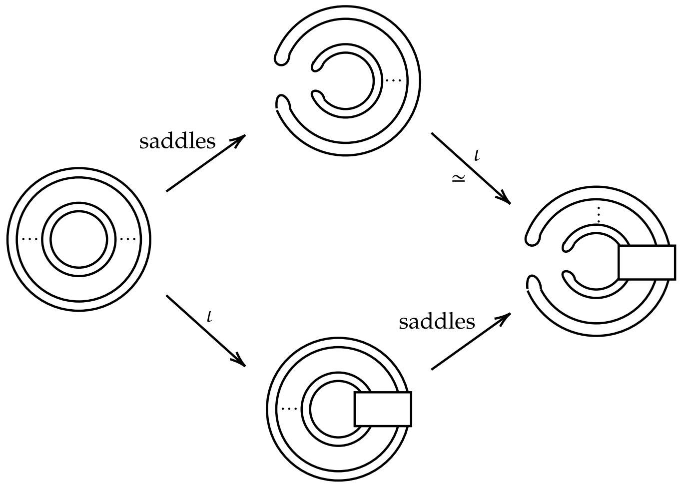


Figure 4.17: Tracking the element $1 \otimes \cdots \otimes 1$.

annuli. For the strands with Rozansky projectors, capping off means following the right half of Figure 4.17 to go from the bottom term to the top term, then capping circles off by dotted caps. This capping off procedure commutes (up to sign) with the maps of Figure 4.16 (passing from the middle row to the last row of Figure 4.16 requires a termwise check, and technically we have termwise sign ambiguities; this will be fixed in Appendix C.4.3), so that we can consider the image of $1 \in \widetilde{KhR}_2^+(\emptyset)$ in two ways. If we perform all of the maps in Figure 4.16 before capping off, we see the element $c(1 \otimes 1 \otimes x \otimes x)$ mapping to $\pm c(x \otimes x) \in \widetilde{KhR}_2^+(U^{n_+,n_-} \sqcup U^{m_-,m_+})$ after capping off. If instead we cap off immediately in the first row, the unit maps and belt pull-offs are identity maps and we see a cobordism $\emptyset \to H_+^{(m,n)} \sqcup (-H_+^{(m,n)}) \to U^{n_+,n_-} \sqcup U^{m_-,m_+}$ isotopic to a disjoint union of dotted annulus creations, so that $1 \in \widetilde{KhR}_2^+(\emptyset)$ maps to $\pm x \otimes x$, proving that $c = \pm 1$. We will fix the sign $c = 1$ in Appendix C.4.3. □

We are now ready to deduce Proposition 4.5.4 from Lemma 4.5.8.

Proof of Proposition 4.5.4. As already explained, it suffices to prove the statement for $\psi = \beta_{i,j}$.

Let $x \in \mathcal{S}_0^{2,\mathrm{mul}}(D_{std}; L)$ be represented by some lasagna filling (Σ, v). By an isotopy rel boundary, we may assume that $\beta_{i,j}$ is supported in D_{std}, a shrunk copy of D_{std}. By an isotopy, we may assume that the input balls of Σ are disjoint from D'_{std}. By neck-cutting (Lemma 2.3.7), we may assume that Σ is disjoint from the 3-handles of D_{std}. By general position and isotopy, we may assume that Σ intersects D'_{std} in a disjoint union of 2-cocores with standard framings and various orientations, and that $U := \Sigma \cap \partial D'_{std}$ is a standard belt link. By tubing and isotopy, we may assume that Σ has a single input ball, which is contained in a collar neighborhood $[0,1] \times S_{std}$ of $\partial D'_{std}$ outside D'_{std} in which Σ takes the form $([0,1] \times U) \cup ([1/2,1] \times L_0)$ where L_0 is the input link of Σ, identified with a local link in S_{std} away from U and the surgery regions. Let $D''_{std} = D'_{std} \cup ([0,1] \times S_{std})$ and let Σ° denote the part of Σ in $D_{std} \backslash int(D''_{std}) = [1,2] \times S_{std}$. We have the commutative diagram

$$\begin{tikzcd}
\mathcal{S}_0^{2,\mathrm{mul}}(D'_{std}; U) \arrow[r, "\mathcal{S}_0^{2,\mathrm{mul}}(\beta_{i,j})"] \arrow[d, "v\otimes\bullet"'] & \mathcal{S}_0^{2,\mathrm{mul}}(D'_{std}; U) \arrow[d, "v\otimes\bullet"] \\
\mathcal{S}_0^{2,\mathrm{mul}}(D''_{std}; U \cup L_0) \arrow[d, "\mathcal{S}_0^{2,\mathrm{mul}}([1,2]\times S_{std};\Sigma^\circ)"'] & \mathcal{S}_0^{2,\mathrm{mul}}(D''_{std}; U \cup L_0) \arrow[d, "\mathcal{S}_0^{2,\mathrm{mul}}([1,2]\times S_{std};\Sigma^\circ)"] \\
\mathcal{S}_0^{2,\mathrm{mul}}(D_{std}; L) \arrow[r, "\mathcal{S}_0^{2,\mathrm{mul}}(\beta_{i,j})"] & \mathcal{S}_0^{2,\mathrm{mul}}(D_{std}; L).
\end{tikzcd} \tag{4.13}$$

Tracing the element $1 \otimes 1$ in the top left corner of (4.13), by Theorem 4.4.3 and Lemma 4.5.8, we get

$$\begin{tikzcd}
1 \otimes 1 \arrow[r, mapsto] \arrow[d, mapsto] & 1 \otimes \mathrm{id}_\alpha(1) + \cdots \arrow[d, mapsto] \\
(v \otimes 1) \otimes 1 \arrow[d, mapsto] & (v \otimes 1) \otimes \mathrm{id}_\alpha(1) + \cdots \arrow[d, mapsto] \\
x = \widetilde{KhR}_2^+(\Sigma^\circ)(v \otimes 1) \otimes \mathrm{id}_{\alpha'}(1) + \cdots \arrow[r, mapsto] & \widetilde{KhR}_2^+(\Sigma^\circ)(v \otimes 1) \otimes \mathrm{id}_{\alpha+\alpha'}(1) + \cdots ,
\end{tikzcd}$$

where $\cdots$ are terms with negative lasagna quantum gradings. The statement follows. $\square$

4.5.5 Isotopy insertions

We prove Theorem 4.5.3 for type (ii) diffeomorphisms in Proposition 4.5.6.

By assumption, $\tilde{\phi}$ is supported in a collar neighborhood of the boundary, on which it takes the form

$$\Phi \colon [-1,0] \times S_{std} \to [-1,0] \times S_{std}, \ (t,x) \mapsto (t, \phi_t(x))$$

for some isotopy ϕ_t between id and ϕ. By considering the action on lasagna fillings, we see that $\mathcal{S}_0^{2,\mathrm{mul}}(\tilde{\phi}^{-1})$ is equal to the gluing map

$$\mathcal{S}_0^{2,\mathrm{mul}}(I \times S_{std}; \Phi^{-1}([-1,0] \times \phi(L))) \colon \mathcal{S}_0^{2,\mathrm{mul}}(D_{std}; \phi(L)) \to \mathcal{S}_0^{2,\mathrm{mul}}(D_{std}; L).$$

Now the statement follows from Theorem 4.4.3, with $\widetilde{KhR}_2^+(\phi) = \widetilde{KhR}_2^+((\Phi^{-1}([-1,0] \times \phi(L)))^t)$.

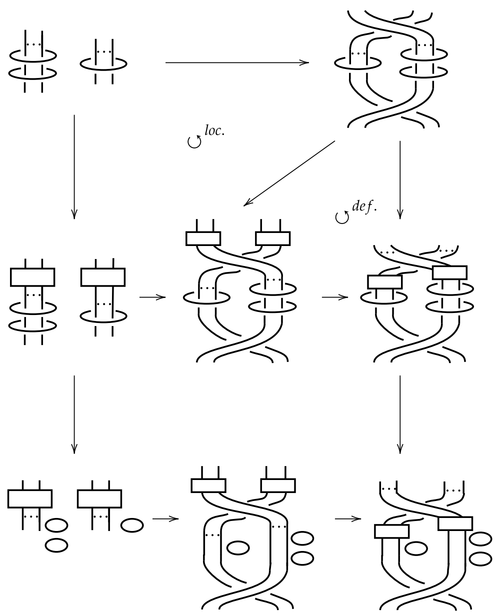


Figure 4.18: Rows (SZ2)–(SZ4) of a reverse standard boundary-summand-exchange diffeomorphism.

4.5.6 Connected summand exchanges

We prove Theorem 4.5.3 for type (iii) diffeomorphisms in Proposition 4.5.6.

A standard such diffeomorphism acts on a standard lasagna filling of L in the sense of the explanation of (SZ1) in Section 4.2.3 by exchanging the i-th and i'-th input balls, carrying together the i-th and i'-th collections of $2,3$-handles and skeins within.

Alternatively, in terms of row (SZ1), $\mathcal{S}_0^{2,\mathrm{mul}}(\tilde{\phi}^{-1})$ acts by exchanging the i-th and i'-th tensorial factors if one decomposes the KhR_2 of $L\cup$ belts $\subset\sqcup_{i=1}^k S^3$ into a tensor product. This description propagates along the sequence of isomorphisms (SZ2)–(SZ4). We conclude that the statement holds (even without taking 0-th associated graded maps) with $\widetilde{KhR}_2^+(\phi)\colon \widetilde{KhR}_2^+(\phi(L))\to\widetilde{KhR}_2^+(L)$ being the map induced on the chain level by exchanging the i-th and i'-th tensorial factors, and $\alpha=0$.

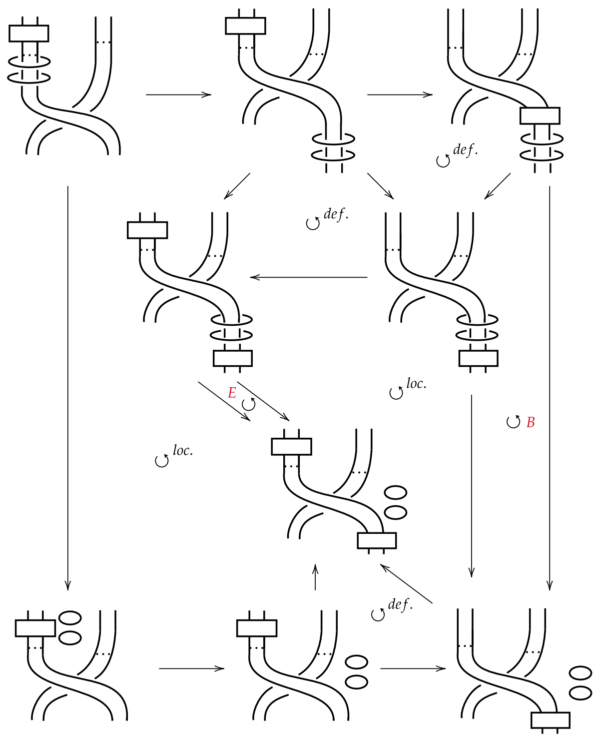
4.5.7 Boundary summand exchanges

We prove Theorem 4.5.3 for type (iv) diffeomorphisms in Proposition 4.5.6.

We follow the strategy in Section 4.4.4. We claim that the map $\mathcal{S}_0^{2,\mathrm{mul}}(\tilde{\phi}^{-1})$ in terms of rows

(SZ2)–(SZ4) is given by the rows in Figure 4.18, where the second maps on the second and third rows are the "sandwich" maps defined as before.

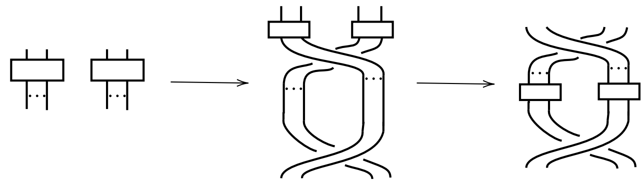
The claim for row (SZ2) comes from an easy examination of the isomorphism (SZ1) as before, and the claim for row (SZ3) follows because the upper rectangle of Figure 4.18 commutes. To show the claim for row (SZ4), it suffices to prove the lower rectangle of Figure 4.18 commutes. By conjugating and sliding one belt-projector combination at a time, it suffices to show the commutativity of the boundary of the following diagram of isomorphisms:



.

Here, all but two regions commute by definition and locality, and the remaining two of them commute by the commutativity of region B in Section 4.4.4 and a variant of region E in Section 4.4.6. The claim follows.

In particular, tracing through the isomorphism (SZ5), we see that Theorem 4.5.3 holds with $\widetilde{KhR}_2^+(\phi)$ induced by the composition



and $\alpha = 0$.

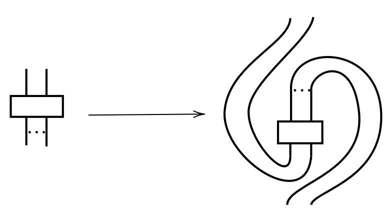
4.5.8 Inversions

We prove Theorem 4.5.3 for type (v) diffeomorphisms in Proposition 4.5.6.

We claim that the map $\mathcal{S}_0^{2,\mathrm{mul}}(\tilde{\phi}^{-1})$ in terms of rows (SZ2)–(SZ4) is given by the rows of Figure 4.19. Here the second map on the second row is given by the usual "sandwich" map, and the second map on the third row is induced by the chain map that moves, termwise, the unknotted circles through the empty region of the projector (see Proposition 4.2.1(1)).

It suffices to justify the commutativity of the diagram. We comment on the regions whose commutativity are not immediate. The rectangle region in the upper half commutes because the Rozansky projector is symmetric in the sense of Proposition 4.2.1(8). The lower right triangle region commutes by the commutativity of region B in Section 4.4.4, while the lower-most triangle region commutes by checking termwise on the chain level, starting and ending at one of the bottom terms (which is termwise crossingless).

We conclude that Theorem 4.5.3 holds with $\widetilde{KhR}_2^+(\phi)$ induced by the rotation map



and $\alpha = 0$ (note that the belt orientations get flipped after the first map in the second row of Figure 4.19, which reflects the action of $\tilde{\phi}^{-1}$ on $H_2(D_{std})$ appearing in the second factor of (4.12)).

4.5.9 Handleslides

We prove Theorem 4.5.3 for type (vi) diffeomorphisms in Proposition 4.5.6.

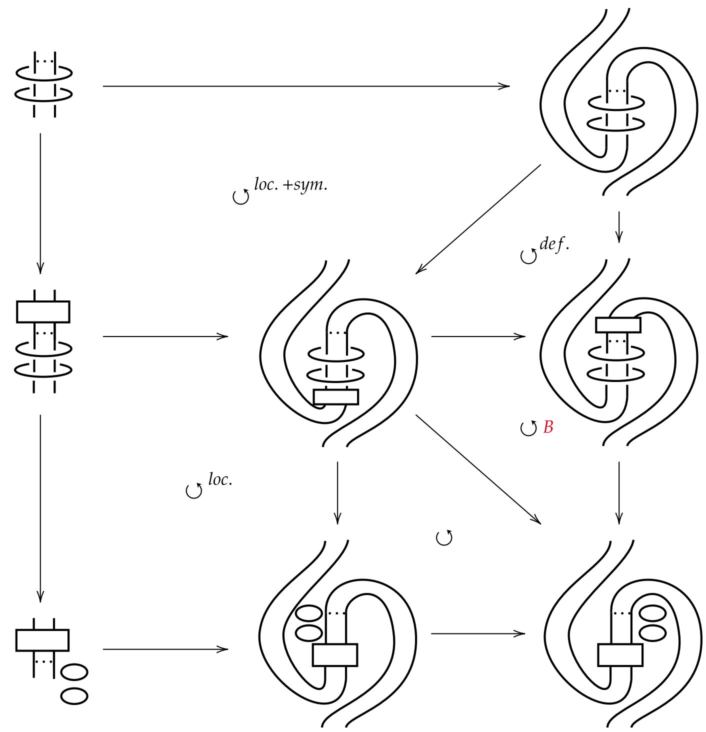
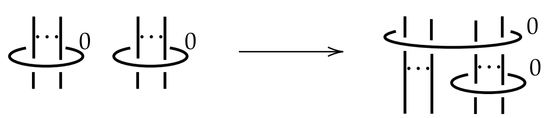


Figure 4.19: Rows (SZ2)–(SZ4) of a reverse standard inversion diffeomorphism.

For ease of drawing diagrams, we draw $\tilde{\phi}^{-1}$ near the handleslide region as



with the isotopy putting the first 2-handle back to its standard position omitted.

We claim that $\mathcal{S}_0^{2,\mathrm{mul}}(\tilde{\phi}^{-1})$ in terms of rows (SZ2)–(SZ4) is equal to rows $1, 2, 4$ of Figure 4.20. Here, the second map in the second row (and other appearances of this local picture) is given

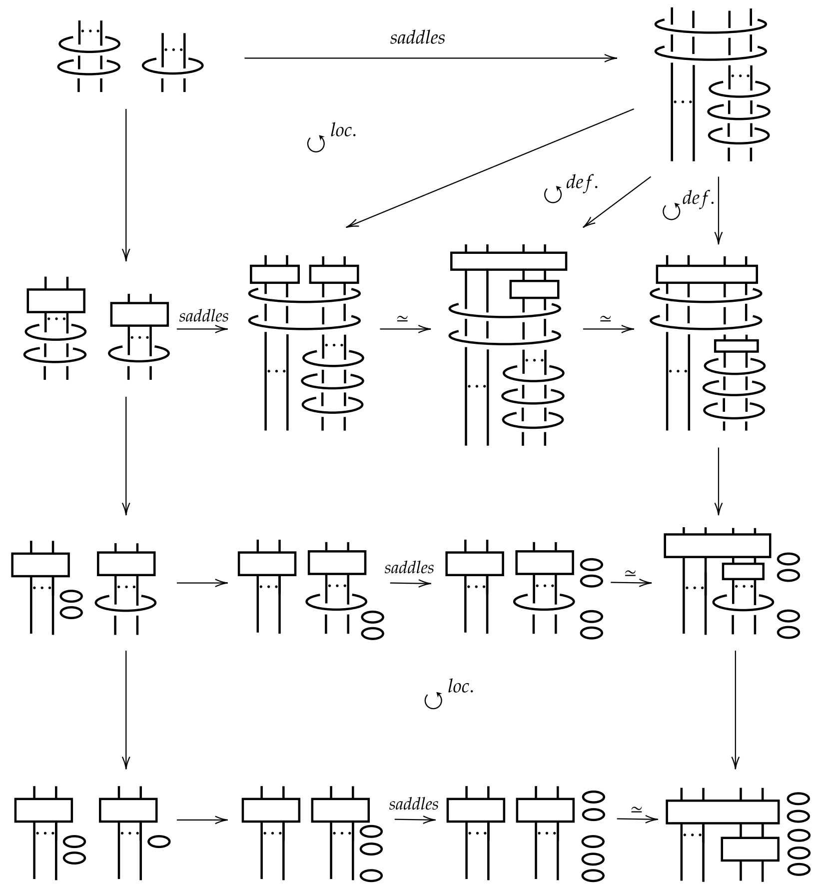
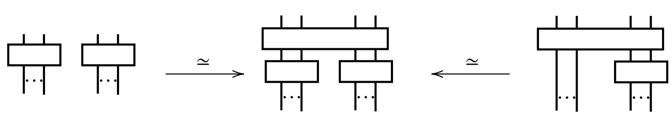


Figure 4.20: Rows (SZ2)–(SZ4) of a reverse standard negative handleslide diffeomorphism, shown as rows $1, 2, 4$ in the figure.

by the composition of isomorphisms (see Proposition 4.2.1(7))



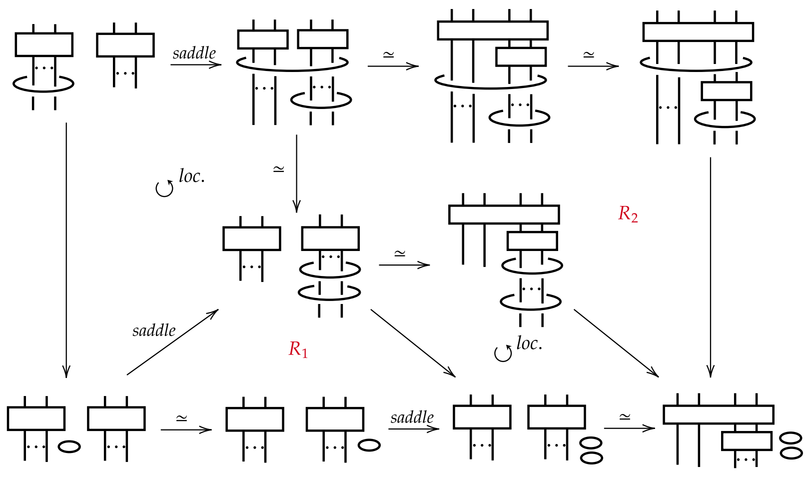
, (4.14)

and the third map in the second row is the usual "sandwich" map.

To see the claim for row (SZ2), start with a lasagna filling $(I \times L \cup (n_+ + r, n_- + r) \text{ cores}, v)$, standard in the sense of the explanation of (SZ1) in Section 4.2.3. When we slide the j-th 2-handle over the $(j+1)$-st using $\tilde{\phi}^{-1}$, keep the input balls invariant and wrap the j-th collection of cores in the skein over the $(j+1)$-st 2-core. After enlarging input balls of the skein, the new skein now has input link $L \cup (n_+ + r + (n_-)_j e_{j+1}, n_- + r + (n_+)_j e_{j+1})$ belts, and the evaluation map performed is given by the claimed saddles.

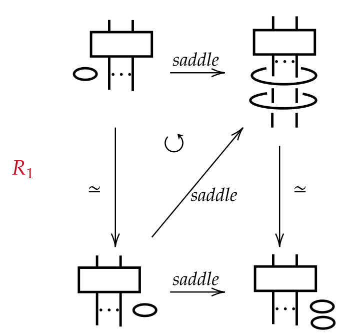
The claim for row (SZ3) follows by the commutativity of the upper rectangle of Figure 4.20.

To show the claim for row (SZ4), note that the lower rectangle of Figure 4.20 commutes, hence it remains to show the commutativity of the middle rectangle. We keep the relevant parts of this rectangle and focus on one belt at a time; thus, it suffices to show the commutativity of the boundary of the following diagram:

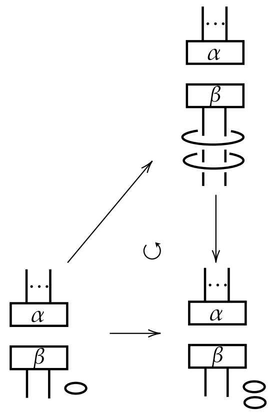


Here, two of the rectangles commute by locality. We justify the commutativity of regions R_1 and R_2 below.

The relevant parts of region R_1 are redrawn as

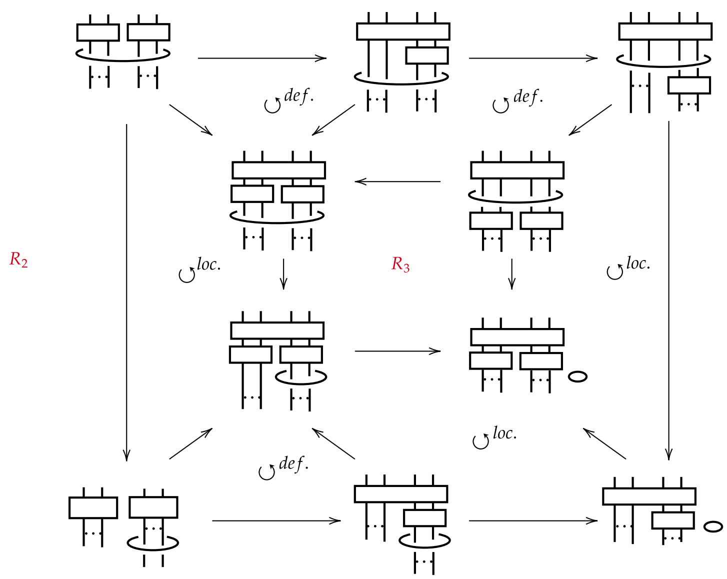


where the upper triangle commutes. The commutativity of the lower triangle up to sign follows from the fact that on the chain level, the termwise diagrams



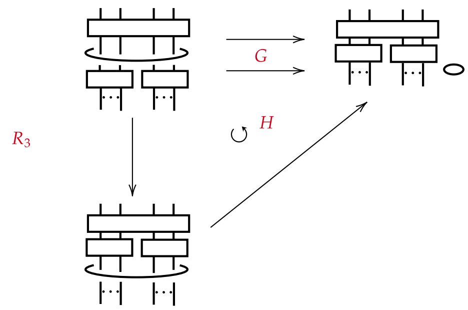
commute up to sign, and that the lower row of R_1 is crossingless, hence no cross term exists. The sign is fixed in Appendix C.4.2.

The relevant parts of region R_2 are redrawn as

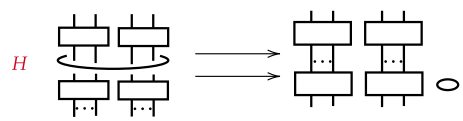


,

where all regions except R_3 commute by definition or locality. We fill in region R_3 (and omit one term) as



Here, region G commutes by the same argument as the commutativity of region B in Section 4.4.4, and the lower triangle commutes because one can prove the commutativity of the following region H using again the same proof as region B:



This proves the claim for row (SZ4).

At lasagna quantum degree 0, each unknotted circle appearing in row 4 of Figure 4.20 carries an x label, which is sent to $x \otimes x$ under a saddle map. We conclude that Theorem 4.5.3 holds for $\widetilde{KhR}_2^+(\phi)$ induced by the composition map (4.14) post-composed with an isotopy that puts the first projector in standard position, and $\alpha = 0$. This finishes the proof of Theorem 4.5.3.

4.6 Functoriality for spin morphisms

4.6.1 The statements

In this section, we prove a special case of Theorem 4.3.2, assuming all objects and morphisms are spin.

More precisely, let $\mathbf{Links}_1^{spin}$ denote the category defined similarly as $\mathbf{Links}_1$ in Section 2.4.2 in Chapter 2, except that each S in an object (S, L) is assumed to be an abstract spin $\sqcup_{i=1}^k \#^{m_i}(S^1 \times S^2)$, and each W in a morphism $(W, \Sigma)\colon (S_0, L_0) \to (S_1, L_1)$ comes with a spin structure making it a spin cobordism from S_0 to S_1. The goal of this section is to prove the following theorem.

Theorem 4.6.1. *There is a lax monoidal functor* $\widetilde{KhR}_2^+ : (\mathbf{Links}_1^{spin})^{op} \to \mathbf{Mod}_R^{+,(\frac{1}{2}\mathbb{Z})^2\times\mathbb{Z}/2}$ *that extends the definition of* $\widetilde{KhR}_2^+$ *on objects as in Section 4.5. For a morphism* (W,Σ), $\widetilde{KhR}_2^+(W,\Sigma)$ *is homogeneous of degree* $(0,-\chi(\Sigma))$.

Theorem 4.6.1 is a formal consequence of the "turning cobordisms inside out" trick described in Section 4.3.2, together with the following theorem.

Theorem 4.6.2. *Let* $D_{std,j} := \#_{i=1}^{k^{(j)}}\natural^{m_i^{(j)}}(D^2\times S^2)$, $S_{std,j} := \partial D_{std,j}$, *and let* $L_j \subset S_{std,j}$ *be an admissible link,* $j=0,1$. *If* $i\colon D_{std,1} \hookrightarrow int(D_{std,0})$ *is a smooth embedding such that* $W^t := D_{std,0}\backslash int(D_{std,1})$ *is a 4-dimensional relative 1-handlebody complement with* $\partial_- W^t = -S_{std,0}$, $\partial_+ W^t = -S_{std,1}$, *and* $\Sigma^t \subset W^t$ *is a cobordism between* L_1 *and* L_0, *then the map*

$$\mathcal{S}_0^{2,\mathrm{mul}}(W^t;\Sigma^t)\colon \mathcal{S}_0^{2,\mathrm{mul}}(D_{std,1};L_1) \to \mathcal{S}_0^{2,\mathrm{mul}}(D_{std,0};L_0)$$

induces a map on the 0*-th associated graded spaces with respect to the lasagna quantum grading. Under the isomorphism (SZ), this induced map is of the form*

$$\widetilde{KhR}_2^+(\Sigma)\otimes gr_0(\mathrm{id}_\alpha\circ i_*)\colon \widetilde{KhR}_2^+(L_1)\otimes gr_0\mathcal{S}_0^{2,\mathrm{mul}}(D_{std,1}) \to \widetilde{KhR}_2^+(L_0)\otimes gr_0\mathcal{S}_0^{2,\mathrm{mul}}(D_{std,0})$$

for some $\widetilde{KhR}_2^+(\Sigma) = \widetilde{KhR}_2^+(W,\Sigma)\colon \widetilde{KhR}_2^+(L_1) \to \widetilde{KhR}_2^+(L_0)$ *induced by a chain map and some* $\alpha \in H_2(D_{std,0})$. *Here,* i_* *denotes the pushforward map* $\mathcal{S}_0^{2,\mathrm{mul}}(i)\colon \mathcal{S}_0^{2,\mathrm{mul}}(D_{std,1}) \to \mathcal{S}_0^{2,\mathrm{mul}}(D_{std,0})$.

The proof of Theorem 4.6.2 takes up the bulk of Section 4.6. In Section 4.6.2 we decompose it into various cases, which are treated individually in Sections 4.6.3–4.6.7. Before going there, we deduce Theorem 4.6.1 as a consequence of Theorem 4.6.2.

Proof of Theorem 4.6.1 assuming Theorem 4.6.2. Let $(W,\Sigma)\colon (S_0,L_0)\to(S_1,L_1)$ be a morphism in $\mathbf{Links}_1^{spin}$. Write $W = X_1\backslash int(X_0)$ for spin 4-dimensional 1-handlebodies X_0, X_1. Choosing a spin embedding $X_1 \hookrightarrow S^4$, we get an embedding $-(S^4\backslash int(X_1)) \hookrightarrow -(S^4\backslash int(X_0))$. Choose parametrizations $\tilde\phi_j\colon -(S^4\backslash int(X_j)) \xrightarrow{\cong} D_{std,j} = \#_{i=1}^{k^{(j)}}\natural^{m_i^{(j)}}(D^2\times S^2)$ so that $\phi_j = \tilde\phi_j|_{S_j}$ is L_j-admissible, $j=0,1$. Then (W,Σ) gives rise to a map

$$\mathcal{S}_0^{2,\mathrm{mul}}(\tilde\phi_0(W^t);\tilde\phi_0(\Sigma^t))\colon \mathcal{S}_0^{2,\mathrm{mul}}(D_{std,1};\phi_1(L_1)) \to \mathcal{S}_0^{2,\mathrm{mul}}(D_{std,0};\phi_0(L_0)), \tag{4.15}$$

which determines a map $\widetilde{KhR}_2^+(\tilde\phi_0(W),\tilde\phi_0(\Sigma))\colon \widetilde{KhR}_2^+(\phi_1(L_1)) \to \widetilde{KhR}_2^+(\phi_0(L_0))$ by Theorem 4.6.2, uniquely so if we insist it to be zero whenever (4.15) is zero. This in turn uniquely determines a map $\widetilde{KhR}_2^+(W,\Sigma)$ making

$$\begin{tikzcd}
\widetilde{KhR}_2^+(\phi_1(L_1)) \arrow[r, "\widetilde{KhR}_2^+(\tilde\phi_0(W),\tilde\phi_0(\Sigma))"] \arrow[d, "\cong"] & \widetilde{KhR}_2^+(\phi_0(L_0)) \arrow[d, "\cong"] \\
\widetilde{KhR}_2^+(S_1,L_1) \arrow[r, "\widetilde{KhR}_2^+(W,\Sigma)"] & \widetilde{KhR}_2^+(S_0,L_0)
\end{tikzcd}$$

commute.

We check that $\widetilde{KhR}_2^+(W,\Sigma)$ is independent of $\tilde\phi_0,\tilde\phi_1$. For another choice $\tilde\phi_0',\tilde\phi_1'$, we have the commutative diagram

$$\begin{tikzcd}
\mathcal{S}_0^{2,\mathrm{mul}}(D_{std,1};\phi_1'(L_1)) \arrow[r, "\mathcal{S}_0^{2,\mathrm{mul}}(\tilde\phi_0'(W^t);\tilde\phi_0'(\Sigma^t))"] \arrow[d, "\mathcal{S}_0^{2,\mathrm{mul}}(\tilde\phi_1\circ\tilde\phi_1'^{-1})"', "\cong"] & \mathcal{S}_0^{2,\mathrm{mul}}(D_{std,0};\phi_0'(L_0)) \arrow[d, "\mathcal{S}_0^{2,\mathrm{mul}}(\tilde\phi_0\circ\tilde\phi_0'^{-1})"', "\cong"] \\
\mathcal{S}_0^{2,\mathrm{mul}}(D_{std,1};\phi_1(L_1)) \arrow[r, "\mathcal{S}_0^{2,\mathrm{mul}}(\tilde\phi_0(W^t);\tilde\phi_0(\Sigma^t))"] & \mathcal{S}_0^{2,\mathrm{mul}}(D_{std,0};\phi_0(L_0))
\end{tikzcd}$$

By Theorem 4.5.3 and Theorem 4.6.2, this diagram descends to a commutative diagram on gr_0, which under isomorphism (SZ) becomes a commutative diagram

$$\begin{tikzcd}
\widetilde{KhR}_2^+(\phi_1'(L_1))\otimes gr_0\mathcal{S}_0^{2,\mathrm{mul}}(D_{std,1}) \arrow[r, "\widetilde{KhR}_2^+(\tilde\phi_0'(W),\tilde\phi_0'(\Sigma))\otimes gr_0(\mathrm{id}_{\alpha_0}\circ i'_*)"] \arrow[d, "\substack{\widetilde{KhR}_2^+(\phi_1'\circ\phi_1^{-1})\otimes\\ gr_0(\mathrm{id}_{\alpha_2}\circ(\phi_1\circ\phi_1'^{-1})_*)}"', "\cong"] & \widetilde{KhR}_2^+(\phi_0'(L_0))\otimes gr_0\mathcal{S}_0^{2,\mathrm{mul}}(D_{std,0}) \arrow[d, "\substack{\widetilde{KhR}_2^+(\phi_0'\circ\phi_0^{-1})\otimes\\ gr_0(\mathrm{id}_{\alpha_3}\circ(\phi_0\circ\phi_0'^{-1})_*)}"', "\cong"] \\
\widetilde{KhR}_2^+(\phi_1(L_1))\otimes gr_0\mathcal{S}_0^{2,\mathrm{mul}}(D_{std,1}) \arrow[r, "\widetilde{KhR}_2^+(\tilde\phi_0(W),\tilde\phi_0(\Sigma))\otimes gr_0(\mathrm{id}_{\alpha_1}\circ i_*)"] & \widetilde{KhR}_2^+(\phi_0(L_0))\otimes gr_0\mathcal{S}_0^{2,\mathrm{mul}}(D_{std,0})
\end{tikzcd}$$

for some $\alpha_0,\alpha_1,\alpha_2,\alpha_3$, where $i,i'\colon D_{std,1}\hookrightarrow D_{std,0}$ are naturally determined by the parametrizations. The two composition maps on the second tensorial factors are both nonzero, as they each send the element $gr_0(1)$ (1 is the element represented by the empty skein) to a nonzero element. Therefore, the two composition maps $\widetilde{KhR}_2^+(\phi_0'\circ\phi_0^{-1})\circ\widetilde{KhR}_2^+(\tilde\phi_0'(W),\tilde\phi_0'(\Sigma))$ and $\widetilde{KhR}_2^+(\tilde\phi_0(W),\tilde\phi_0(\Sigma))\circ\widetilde{KhR}_2^+(\phi_1'\circ\phi_1^{-1})$ on the first tensorial factors are equal up to some scalar $\lambda\in R$. If they are nonzero, then the two compositions on the second tensorial factors are equal up to λ^{-1}. Since either composition on the second tensorial factors, after postcomposing with the map on $gr_0\mathcal{S}_0^{2,\mathrm{mul}}$ induced by an embedding $D_{std,0}\subset S^4$, sends $gr_0(1)\in gr_0\mathcal{S}_0^{2,\mathrm{mul}}(D_{std,1})$ to $gr_0(1)\in gr_0\mathcal{S}_0^{2,\mathrm{mul}}(S^4)\cong R$, we have $\lambda=1$. This shows that $\widetilde{KhR}_2^+(W,\Sigma)$ is independent of the choices of $\tilde\phi_j$, $j=0,1$. It is also independent of the choice of the spin embedding $X_1\hookrightarrow S^4$, as any two such embeddings are isotopic. Hence, $\widetilde{KhR}_2^+(W,\Sigma)$ is well-defined.

The functoriality of $\widetilde{KhR}_2^+(W,\Sigma)$ is proved similarly. The identity morphism induces the identity map by Theorem 4.6.2. If $(W_j,\Sigma_j)\colon (S_j,L_j)\to(S_{j+1},L_{j+1})$, $j=0,1$, are two composable morphisms in $\mathbf{Links}_1^{spin}$, then we can write $W_j=X_{j+1}\backslash int(X_j)$, $j=0,1$, for some spin 4-dimensional 1-handlebodies X_0,X_1,X_2. Choose a spin embedding $X_2\hookrightarrow S^4$ and parametrizations $\tilde{\phi}_j\colon -(S^4\backslash int(X_j))\xrightarrow{\cong} D_{std,j}$, so that $\phi_j\colon S_j\xrightarrow{\cong} S_{std,j}$ is L_j-admissible, $j=0,1,2$. We obtain concrete models for the abstract induced maps $\widetilde{KhR}_2^+(W_j,\Sigma_j)$, $j=0,1$, as well as $\widetilde{KhR}_2^+(W_1\circ W_0,\Sigma_1\circ\Sigma_0)$. A similar argument as before using Theorem 4.6.2 shows that compositions are functorial on these concrete models, hence on the abstract induced maps themselves.

The lax monoidality of $\widetilde{KhR}_2^+$ is clear from the construction of the classical Rozansky–Willis homology in Section 4.2.1. □

4.6.2 Decomposition into elementary morphisms

If Theorem 4.6.2 holds for embeddings $D_{std,j+1}\hookrightarrow int(D_{std,j})$ and composable link cobordisms in $D_{std,j}\backslash int(D_{std,j+1})$, $j=0,1$, then it holds for the composition as well.

Therefore, it suffices to decompose any morphism into a composition of some elementary morphisms, and check Theorem 4.6.2 for these elementary ones. We first state such a decomposition result for abstract morphisms, namely morphisms in $\mathbf{Links}_1$.

Proposition 4.6.3. *Every morphism in $\mathbf{Links}_1$ is a composition of some elementary morphisms of the following forms. See Figure 4.21.*

(i) Product morphisms: abstract $(I\times S_{std},\Sigma)\colon (S_{std},L_0)\to(S_{std},L_1)$ for some $S_{std}=\sqcup_{i=1}^k\#^{m_i}(S^1\times S^2)$.

(ii) Ball creations: abstract $((I\times S_{std})\sqcup B^4, I\times L)\colon (S_{std},L)\to(S_{std}\sqcup S^3,L)$ for some $S_{std}=\sqcup_{i=1}^k\#^{m_i}(S^1\times S^2)$.

(iii) Connected sums: abstract $(W_\#,I\times L)\colon (S_{std},L)\to(S_{std,\#},L)$ for some $S_{std}=\sqcup_{i=1}^k\#^{m_i}(S^1\times S^2)$, $k\ge 2$, where $S_{std,\#}=(\sqcup_{i=1}^{k-2}\#^{m_i}(S^1\times S^2))\sqcup(\#^{m_{k-1}+m_k}(S^1\times S^2))$, $W_\#$ is a (standard) 1-handle attachment between the $(k-1)$-th and k-th component of S_{std} attached near $\infty\in\#^{m_i}(S^1\times S^2)$, $i=k-1,k$, and $I\times L$ is the trace of L in $W_\#$.

(iv) 1-handle attachments: abstract $(W_+,I\times L)\colon (S_{std},L)\to(S_{std,+},L)$ for some $S_{std}=\sqcup_{i=1}^k\#^{m_i}(S^1\times S^2)$, $k\ge 1$, where $S_{std,+}=(\sqcup_{i=1}^{k-1}\#^{m_i}(S^1\times S^2))\sqcup(\#^{m_k+1}(S^1\times S^2))$, W_+ is a (standard) 1-handle attachment on the k-th component of S_{std} that misses L, and $I\times L$ is the trace of L in W_+.

(v) Canceling 2-handle attachments: abstract $(W_-,I\times L)\colon (S_{std},L)\to(S_{std,-},L)$ for some $S_{std}=\sqcup_{i=1}^k\#^{m_i}(S^1\times S^2)$, $k\ge 1$, $m_k\ge 1$, where $S_{std,-}=(\sqcup_{i=1}^{k-1}\#^{m_i}(S^1\times S^2))\sqcup$

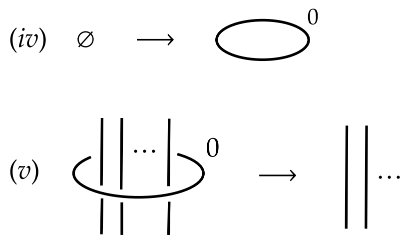


Figure 4.21: Forward: Type (iv) and (v) morphisms in Proposition 4.6.3. Backward: Type (iv) and (v) morphisms in Corollary 4.6.4.

> $(\#^{m_k-1}(S^1 \times S^2))$, *$W_-$ is a (standard) 2-handle attachment on the k-th component of S_{std} that misses L and cancels the last $S^1 \times S^2$ connected summand, and $I \times L$ is the trace of L in W_-.*

Moreover, in each concrete model $(W, \Sigma)\colon (S_{std,0}, L_0) \to (S_{std,1}, L_1)$ described above, L_0, L_1 can be assumed to be admissible.

The usage of "standard" in Proposition 4.6.3 is in a sense similar to that in Proposition 4.5.6. We will not be pedantic about this distinction below.

Proof. We claim that it suffices to perform the decomposition of a morphism $(W, \Sigma)\colon (S_0, L_0) \to (S_1, L_1)$ on the 4-manifold level. Indeed, if such a 4-manifold-level decomposition is given, by general position, further decomposing and making the handles thin in the cocore direction, we may assume that

- The boundary of each layer intersects Σ transversely.
- Each 1-handle is disjoint from Σ.
- Each 2-handle intersects Σ in a disjoint union of interior cocores. In particular, the attaching region of each 2-handle is disjoint from Σ.

By further splitting off product pieces, we can assume that 2-handles have short cores, thus by general position,

- Each 2-handle is disjoint from Σ.

Admissibility of boundary links can be achieved by choosing nice parametrizations. The claim follows.

Now we forget about L_0, L_1, Σ and exhibit a 4-manifold-level decomposition for W. Write $W = X_1 \backslash int(X_0)$ for 4-dimensional 1-handlebodies X_0, X_1. Without loss of generality, say W, hence X_1, is nonempty and connected. By the ball creation or the connected sum operation, we may assume that X_0 is nonempty and connected as well.

Fix a decomposition of X_0, X_1 into handlebodies each with a single 0-handle. Assume the 0-handle of X_0 is contained in that of X_1. Now $\pi_1(X_1)$ is a free group with generators given by the 1-handles of X_1. For each 1-handle of X_1, attach a 1-handle to X_0 inside $int(X_1)$ that represents the corresponding generator in X_1 (up to isotopy, there is no choice for this attachment). Next, each original 1-handle in X_0 can slide over these new 1-handles ambiently in X_1 so that it represents the trivial element in $\pi_1(X_1)$. Again, there is only one possible configuration, hence we see after sliding, each original 1-handle can be canceled by an ambient canceling 2-handle, making the rest of the X_0 complement a product. $\square$

Corollary 4.6.4. *Any $(i\colon D_{std,1} \hookrightarrow D_{std,0}, \Sigma^t \subset W^t)$ in the statement of Theorem 4.6.2 can be decomposed into a composition of elementary ones of the following forms.*

(i) Cobordisms in twisted products: $i\colon D_{std,1} \xrightarrow[\cong]{\tilde{\phi}} D_{std,0} \hookrightarrow D_{std,0}$ where $\tilde{\phi}$ is an orientation-preserving diffeomorphism and the second map is a collar-thickening; Σ^t is any cobordism between admissible links.

(ii) Ball annihilations: $D_{std,1} = D_{std,0} \# B^4$, i is the 4-handle attachment that caps off the last S^3 boundary component; Σ^t is the trace in W^t of an admissible link $L \subset S_{std,1}$ missing the last component.

(iii) Separating 3-handle attachments: $D_{std,1} = \#_{i=1}^{k-2} \natural^{m_i}(D^2 \times S^2) \# \natural^{m_{k-1}+m_k}(D^2 \times S^2)$, $D_{std,0} = \#_{i=1}^{k} \natural^{m_i}(D^2 \times S^2)$, $k \geq 2$, i is a standard 3-handle attachment onto the last summand of $S_{std,1}$ that separates the first m_{k-1} and the last m_k 2-handles; Σ^t is the trace in W^t of an admissible link $L \subset S_{std,1}$ that misses the 3-handle attaching region.

(iv) Canceling 3-handle attachments: $D_{std,1} = \#_{i=1}^{k-1} \natural^{m_i}(D^2 \times S^2) \# \natural^{m_k+1}(D^2 \times S^2)$, $D_{std,0} = \#_{i=1}^{k} \natural^{m_i}(D^2 \times S^2)$, $k \geq 1$, i is a standard 3-handle attachment that cancels the last 2-handle; Σ^t is the trace in W^t of an admissible link $L \subset S_{std,1}$ that misses the 3-handle attaching region.

(v) Local 0-framed 2-handle attachments: $D_{std,1} = \#_{i=1}^{k-1} \natural^{m_i}(D^2 \times S^2) \# \natural^{m_k-1}(D^2 \times S^2)$, $D_{std,0} = \#_{i=1}^{k} \natural^{m_i}(D^2 \times S^2)$, $k \geq 1$, $m_k \geq 1$, i is a standard local 0-framed 2-handle attachment in the last boundary component; Σ^t is the trace in W^t of an admissible link $L \subset S_{std,1}$ that misses the 2-handle attaching region.

Proof. Turn the moves in Proposition 4.6.3 upside down and use type (i) morphisms to absorb parametrization changes. □

4.6.3 Cobordisms in twisted products

We prove Theorem 4.6.2 for type (i) morphisms in Corollary 4.6.4.

We first decompose such a morphism into a composition of the following more elementary ones.

(a) A collar-thickening: $i\colon D_{std} \hookrightarrow D_{std}$ is a collar-thickening; Σ^t is any cobordism between admissible links.

(b) A reparametrization: $i\colon D_{std} \xrightarrow[\cong]{\tilde{\phi}} D_{std} \hookrightarrow D_{std}$ where $\tilde{\phi} \in \mathrm{Diff}^+(D_{std})$ with $\phi = \tilde{\phi}|_{S_{std}}$ L-admissible for an admissible link $L \subset S_{std}$, and the second map is a collar-thickening; Σ^t is the product cobordism from L to $\phi(L)$;

(c) A name change: $i\colon D_{std,1} \xrightarrow[\cong]{\tilde{\phi}} D_{std,0} \hookrightarrow D_{std,0}$ where $\tilde{\phi}$ is a standard orientation-preserving diffeomorphism that exchanges some i-th and $(i+1)$-th connected summands of $D_{std,1}$, and the second map is a collar thickening; Σ^t is the product cobordism from some admissible link L to $\phi(L)$.

Theorem 4.6.2 for each type of morphisms above is now a consequence of our previous work.

(a): The statement follows from Theorem 4.4.3.

(b): The statement follows from Theorem 4.5.3.

(c): The statement follows from an argument similar to that of Section 4.5.6.

4.6.4 Ball annihilations

We prove Theorem 4.6.2 for type (ii) morphisms in Corollary 4.6.4.

In the sequence of isomorphisms (SZ1)–(SZ5) leading to (SZ), the effect of the extra 4-handle $i\colon D_{std,1} \hookrightarrow D_{std,0}$ comes in nowhere. Therefore, terms in each row (SZ1)–(SZ5) for $(D_{std,1}, L)$ and $(D_{std,0}, L)$ are isomorphic via the obvious isomorphisms. We conclude that $\mathcal{S}_0^{2,\mathrm{mul}}(W^t;\Sigma^t)$ is equal to $\widetilde{KhR}_2^+(\Sigma) \otimes i_*$ in terms of row (SZ5), where $\widetilde{KhR}_2^+(\Sigma)$ is induced by the canonical isomorphism $\widetilde{CKhR}_2^+(\emptyset) \xrightarrow{\cong} R$, $\emptyset$ being the empty link in the last S^3 factor.

4.6.5 Separating 3-handles

We prove Theorem 4.6.2 for type (iii) morphisms in Corollary 4.6.4.

The argument is as in Section 4.6.4, as the separating 3-handle missing L intertwines with the isomorphisms (SZ1)–(SZ5) in the evident way. We conclude that $\mathcal{S}_0^{2,\text{mul}}(W^t;\Sigma^t)$ is equal to $\widetilde{KhR}_2^+(\Sigma)\otimes i_*$ in terms of row (SZ5), where $\widetilde{KhR}_2^+(\Sigma)$ is induced by the canonical chain homotopy equivalence $\widetilde{CKhR}_2^+(L_{k-1}\sqcup L_k)\xrightarrow{\cong}\widetilde{CKhR}_2^+(L_{k-1})\otimes\widetilde{CKhR}_2^+(L_k)$. Here L_i denote the part of $L\subset S_{std,0}$ in the i-th boundary component.

4.6.6 Canceling 3-handles

We prove Theorem 4.6.2 for type (iv) morphisms in Corollary 4.6.4.

This morphism can be visualized locally as the reverse of (iv) in Figure 4.21. In each colimit summand in the term corresponding to $\mathcal{S}_0^{2,\text{mul}}(D_{std,1};L)$ in each of the rows (SZ1)–(SZ4), a tensorial factor corresponding to belts coming from the last 2-handle can be split off. The canceling 3-handle evaluates the terms in these tensorial factors to scalars by sending x to 1 and 1 to 0. We conclude that $\mathcal{S}_0^{2,\text{mul}}(W^t;\Sigma^t)$ is equal to $\widetilde{KhR}_2^+(\Sigma)\otimes i_*$ in terms of row (SZ5), where $\widetilde{KhR}_2^+(\Sigma)$ is induced by the chain isomorphism that forgets the empty projector $P_{0,0}^\vee$ coming from the last surgery region in $S_{std,1}$.

4.6.7 Local 2-handles

We prove Theorem 4.6.2 for type (v) morphisms in Corollary 4.6.4.

This morphism can be visualized locally as the reverse of (v) in Figure 4.21. In terms of rows (SZ1) or (SZ2), $\mathcal{S}_0^{2,\text{mul}}(W^t;\Sigma^t)$ is equal to the inclusion as the term with $(n_+)_m=(n_-)_m=r_m=0$, where $m=\sum_{i=1}^k m_i$. In terms of rows (SZ3) or (SZ4), $\mathcal{S}_0^{2,\text{mul}}(W^t;\Sigma^t)$ is equal to the map to this $(n_+)_m=(n_-)_m=r_m=0$ term induced by the unit map $1_{\ell_m}\to P_{\ell_m,0}^\vee$ at the last surgery region in $S_{std,0}$. We conclude that $\mathcal{S}_0^{2,\text{mul}}(W^t;\Sigma^t)$ is equal to $\widetilde{KhR}_2^+(\Sigma)\otimes i_*$ in terms of row (SZ5), where $\widetilde{KhR}_2^+(\Sigma)$ is induced by the unit map $1_{\ell_m}\to P_{\ell_m,0}^\vee$. This finishes the proof of Theorem 4.6.2.

4.7 Remove the spin assumption

In this section, we remove the spin assumption in Theorem 4.6.1 and promote it to Theorem 4.3.2. To this end, we review the Gluck twist operation in Section 4.7.1 and define induced maps on Khovanov skein lasagna modules by Gluck twists in Section 4.7.2. This allows us to define $\widetilde{KhR}_2^+$ on objects of $\mathbf{Links}_1$ in Section 4.7.3 and on morphisms of $\mathbf{Links}_1$ in Section 4.7.4, strengthening results in Section 4.5 and Section 4.6, respectively.

4.7.1 The Gluck twist operation

Let X be a compact oriented 4-manifold and $S \subset int(X)$ be an embedded unoriented 2-sphere with trivial normal bundle. The closed tubular neighborhood $\overline{\nu(S)}$ of S is diffeomorphic to $D^2 \times S^2$. The boundary $S^1 \times S^2$ admits a nonspin diffeomorphism τ given by a Dehn twist along the S^2-factor, or more explicitly $(\theta, x) \mapsto (\theta, \mathrm{rot}_\theta(x))$ where $\mathrm{rot}_\theta \colon S^2 \to S^2$ is the rotation-by-θ map along some fixed axis. The *Gluck twist* of X along S, denoted X_S, is the 4-manifold obtained by cutting out $\nu(S)$ and regluing it back by a τ-twist. We make some elementary observations:

(1) Since the natural inclusion $O(2) \times O(3) \subset \mathrm{Diff}(S^1 \times S^2)$ is a homotopy equivalence [Hat81], $\pi_1(\mathrm{Diff}^+(D^2 \times S^2)) \xrightarrow{\partial} \pi_1(\mathrm{Diff}^+(S^1 \times S^2))$ is surjective, so the manifold X_S is well-defined up to a canonical diffeomorphism (up to isotopy, omitted below).

(2) If S is isotopic to S', then X_S is diffeomorphic to $X_{S'}$ via a diffeomorphism determined by an isotopy from S to S'.

(3) If $S \subset X$ is unknotted, then X_S is diffeomorphic to X via some diffeomorphism determined by a bounding 3-ball B.

(4) The manifold X_S contains another copy of S, and the iterated Gluck twist $(X_S)_S$ is canonically diffeomorphic to X.

(5) If $S = S_0 \cup \cdots \cup S_k$ is disjoint union of unoriented 2-spheres in $int(X)$ with trivial normal bundles, then the iterated Gluck twist $(\cdots(X_{S_0})_{S_1}\cdots)_{S_k}$ is independent of the ordering of $S_0, \cdots, S_k$ up to canonical diffeomorphisms. Write X_S or $X_{S_0,\cdots,S_k}$ for this iterated Gluck twist.

(6) Let $S_0, S_1 \subset int(X)$ be disjoint embedded unoriented 2-spheres with trivial normal bundles, and γ be a path in $int(X)$ equipped with a nonvanishing normal vector field, connecting S_0 and S_1 with interior disjoint from $S_0 \cup S_1$, whose tangent and normal vectors at the endpoints are transverse to S_0, S_1. Let $S_0 \# S_1 = S_0 \#_\gamma S_1$ be the connected sum of S_0 and S_1 along γ. Then $X_{S_0 \# S_1}$ is canonically diffeomorphic to X_{S_0,S_1} via some diffeomorphism determined by S_0, S_1, γ, as can be seen from relative Kirby diagrams of the twisted $\nu(S_0 \cup \gamma \cup S_1)$'s rel the common boundary, as shown in Figure 4.22. If one switches the roles of S_0, S_1, then the diffeomorphism changes by a barbell diffeomorphism implanted from $\nu(S_0 \cup \gamma \cup S_1)$.

(7) Let X, S be as above and $L \subset \partial X$ be an oriented link. We assign two distinguished isomorphisms

$$H_2(\tau_S)_\pm \colon H_2^L(X) \xrightarrow{\cong} H_2^L(X_S) \tag{4.16}$$

as follows. Let $\Sigma \subset X$ be an oriented surface bounding L that intersects S transversely. Take $\nu(S)$ small so that $\Sigma \cap \nu(S)$ is a disjoint union of cocore disks. Normally frame Σ near $\Sigma \cap \nu(S)$ and give $\Sigma \cap \partial\nu(S)$ the induced framing. Take $\Sigma' \looparrowright X_S$ to be the immersed

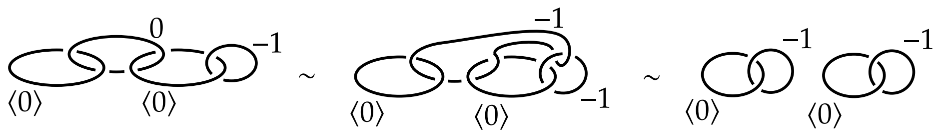


Figure 4.22: Kirby moves exhibiting a diffeomorphism from $(\nu(S_0\cup\gamma\cup S_1))_{S_0\# S_1}$ to $(\nu(S_0\cup\gamma\cup S_1))_{S_0,S_1}$ rel boundary. Each move slides the 2-handle on the left across one other handle.

surface given by Σ outside $\nu(S)$, and $\#U$ embedded disks capping $\tau(U)$ off inside the twisted $\nu(S)$, each having self-intersection number ±1. We demand $H_2(\tau_S)_\pm([\Sigma])=[\Sigma']$. One can check that this is well-defined. If S is unknotted, then $H_2(\tau_S)_\pm=\mathrm{id}$, where the codomain and the domain are identified via the canonical diffeomorphism $X_S\cong X$ determined by a given bounding 3-ball of S. Similarly, under the canonical isomorphism $(X_S)_S\cong X$, we have $H_2(\tau_S)_\mp\circ H_2(\tau_S)_\pm=\mathrm{id}_{H_2^L(X)}$ and $H_2(\tau_S)_\pm^2\colon H_2^L(X)\xrightarrow{\cong}H_2^L(X)$ is given by $\alpha\mapsto\alpha\pm(\alpha\cdot[S])[S]$, where we give S an arbitrary orientation to regard $[S]$ as a class in $H_2(X)$.

4.7.2 Lasagna induced maps by Gluck twists

Theorem 4.7.1. *Let X be a compact oriented 4-manifold, $L\subset\partial X$ be a framed oriented link, and $S\subset int(X)$ be an embedded unoriented 2-sphere with trivial normal bundle. There are two natural maps*

$$\tau_{X,L,S,\pm}\colon \mathcal{S}_0^{2,\mathrm{mul}}(X;L)\to\mathcal{S}_0^{2,\mathrm{mul}}(X_S;L) \tag{4.17}$$

of R-modules. Moreover,

(1) For any $\alpha\in H_2^L(X)$, $\tau_{X,L,S,\pm}$ restricts to a map $\mathcal{S}_0^{2,\mathrm{mul}}(X;L;\alpha)\to\mathcal{S}_0^{2,\mathrm{mul}}(X_S;L;H_2(\tau_S)_\pm(\alpha))$ homogeneous with bidegree shift $(\mp(\alpha\cdot[S])^2/2,\pm(\alpha\cdot[S])^2/2)$. Here $H_2(\tau_S)_\pm$ is the isomorphism (4.16).

(2) If S is unknotted, then $\tau_{X,L,S,\pm}=\mathrm{id}$, where the codomain and the domain of $\tau_{X,L,S,\pm}$ are identified via the canonical diffeomorphism $X_S\cong X$ determined by a given bounding 3-ball of S.

(3) Under the canonical diffeomorphism $(X_S)_S\cong X$, $\tau_{X_S,L,S,\mp}\circ\tau_{X,L,S,\pm}=\mathrm{id}$. In particular, $\tau_{X,L,S,\pm}$ is an isomorphism.

(4) If $S=S_0\cup\cdots\cup S_k\subset int(X)$ is a disjoint union of unoriented 2-spheres, then the induced map $\tau_{X_{S_0,\cdots,S_{k-1}},L,S_k,\pm}\circ\cdots\circ\tau_{X,L,S_0,\pm}\colon\mathcal{S}_0^{2,\mathrm{mul}}(X;L)\to\mathcal{S}_0^{2,\mathrm{mul}}(X_{S_0,\cdots,S_k};L)$ is independent of the ordering of $S_0,\cdots,S_k$ up to canonical diffeomorphisms. Write for short $\tau_{X,L,S,\pm}$ for this composition.

As we will only be using $\tau_{X,L,S,+}$ in the sequel, write for short $\tau_{X,L,S} = \tau_{X,L,S,+}$.

The proof of Theorem 4.7.1 takes up most of Section 4.7.2.

A precursor of the fact that the Khovanov skein lasagna module is invariant under Gluck twists was obtained in Section 3.5.10 in Chapter 3, where it was shown that the Khovanov skein lasagna module (over $\mathbb{Q}$) does not detect potential exotic 4-spheres obtained from Gluck twists.

Corollary 4.7.2. *Let X, L, S be as in Theorem 4.7.1. If S is null-homologous, then $\mathcal{S}_0^{2,\mathrm{mul}}(X;L)$ and $\mathcal{S}_0^{2,\mathrm{mul}}(X_S;L)$ are isomorphic as graded R-modules.* $\square$

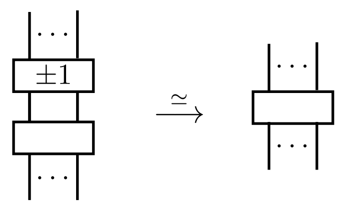
Corollary 4.7.3. *Let X, L, S be as in Theorem 4.7.1. For any $\alpha \in H_2^L(X)$, we have an isomorphism*
$$\mathcal{S}_0^{2,\mathrm{mul}}(X;L;\alpha + (\alpha\cdot[S])[S]) \cong \mathcal{S}_0^{2,\mathrm{mul}}(X;L;\alpha)$$
with bidegree shift $((\alpha\cdot[S])^2, -(\alpha\cdot[S])^2)$.

Proof. $(\tau_{X_S,L,S}\circ\tau_{X,L,S})^{-1}$ gives such an isomorphism. $\square$

We take a slight detour before giving the construction of (4.17).

A belt link in $S^1\times S^2$, in the sense of Section 4.5.4, is a framed oriented link that is isotopic to a union of some even number of S^1 fibers with standard framing and various orientations. We defined, for a standard belt link U as shown on the left of Figure 4.13, a distinguished class $1 \in \mathcal{S}_0^{2,\mathrm{mul}}(D^2\times S^2;U)$ as the class $1\otimes 1 \in \widetilde{KhR}_2^+(U)\otimes\mathcal{S}_0^{2,\mathrm{mul}}(D^2\times S^2)$ under the isomorphism (SZ). It has homological degree 0, quantum degree $-\#U$, and skein degree α_U. Alternatively, it is the class represented by the standard union of cocores that cap off U as a lasagna filling without input balls. We claim that in fact every collection of disks capping off U with framing and orientation represents the class 1, and consequently, as every belt link is isotopic to a standard one, there is a well-defined element $1\in\mathcal{S}_0^{2,\mathrm{mul}}(D^2\times S^2;U)$ for every belt link U which is independent of the parametrization of the pair $(D^2\times S^2, U)$. When $U=\emptyset$ there is nothing to show. When $U\neq\emptyset$, to see the claim, use Gabai's 4-dimensional lightbulb theorem [Gab20, Theorem 10.1] to isotope one component C_1 of the collection of disks to standard position. The complement of $\nu(C_1)$ is diffeomorphic to a 4-ball, in which the other $\#U-1$ components of U form an unlink on the boundary, capped off by other disks in the collection. These $\#U-1$ disks evaluate to the standard element $1\otimes\cdots\otimes 1$ on the boundary (this can be seen by passing to Lee homology). Hence, one can replace these $\#U-1$ disk components by standard cocores without changing the evaluation, and the claim follows.

A *twisted belt link* is a framed oriented link in $S^1\times S^2$ that is $\tau(U)$ for some belt link $U\subset S^1\times S^2$. A *standard positive/negative twisted belt link* is a twisted belt link that takes a standard form as shown in Figure 4.23, which in particular is admissible. For a standard positive/negative twisted belt link T, define a distinguished class $1_\pm \in \mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2; T)$ as the class $1_\pm \otimes 1 \in \widetilde{KhR}_2^+(T) \otimes \mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2)$ under the isomorphism (SZ). Here, if U denotes the standard belt link with strands having orientations matching those of T, then we have a grading-preserving chain homotopy equivalence $\widetilde{CKhR}_2^+(T) \xrightarrow{\simeq} \widetilde{CKhR}_2^+(U)$ given by termwise simplifying Reidemeister I, II maps plus some higher differentials, well-defined up to sign, and $1_\pm \in \widetilde{KhR}_2^{+,0,-\#T}(T)$ is defined as the element corresponding to $1 \in \widetilde{KhR}_2^{+,0,-\#U}(U)$ under the induced isomorphism on homology. This chain homotopy equivalence is induced by the following chain homotopy equivalence that "absorbs" the twist into the Rozansky projector (see [Wil21, Lemma 2.26])



(4.18)

To fix the sign, see Appendix C.4.4. The class $1_\pm \in \mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2; T)$ has homological degree $-\alpha_T^2/2$, quantum degree $\alpha_T^2/2 - \#T$, and skein degree α_T.

Alternatively, if T is a standard positive belt link, write $n = \#T$, and say T has n_+ (resp. n_-) strands oriented upward (resp. downward). Then up to sign, $1_+ \in \widetilde{KhR}_2^+(T)$ is the image of 1 under $\mathbb{Z} \cong \widetilde{KhR}_2^{+,0,-n}(T(n,n)_{n_+,n_-}; \mathbb{Z}) \underset{\cong}{\xrightarrow{\iota}} \widetilde{KhR}_2^{+,0,-n}(T; \mathbb{Z}) \to \widetilde{KhR}_2^{+,0,-n}(T)$ where $T(n,n)_{n_+,n_-}$ is the torus link $T(n,n)$ in S^3 with orientation on n_- of the strands reversed, ι is induced by the unit map creating the Rozansky projector, which is an isomorphism by the proofs in [Man+23], and the first isomorphism is due to [Sto09, Theorem 3], suitably renormalized. To see this alternative description, it suffices to show that $1 \in \widetilde{KhR}_2^+(U; \mathbb{Z}) \cong \widetilde{KhR}_2^+(T; \mathbb{Z})$ is primitive, which follows from the fact that the Rozansky projector is idempotent up to homotopy. By an abuse of notation, below we also write frequently $1 \in \widetilde{KhR}_2^+(T(n,n)_{n_+,n_-})$ for the image of $1 \in \mathbb{Z} \cong \widetilde{KhR}_2^{+,0,-n}(T(n,n)_{n_+,n_-}; \mathbb{Z})$ in $\widetilde{KhR}_2^{+,0,-n}(T(n,n)_{n_+,n_-})$, and $1 \in KhR_2^+(T(n,n)_{n_+,n_-})$ for its image under the renormalization. The sign of $1 \in \widetilde{KhR}_2^+(T(n,n)_{n_+,n_-})$ is fixed by demanding it to map to $1_+ \in \widetilde{KhR}_2^+(T)$ under the unit map. By this alternative description, the element $1_+ \in \mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2; T)$ is represented by the standard lasagna filling $(I \times T(n,n)_{n_+,n_-}, 1)$ of $(D^2 \times S^2, T)$ with one input ball being a shrunk 0-handle, input link $T(n,n)_{n_+,n_-}$ with label $1 \in KhR_2^+(T(n,n)_{n_+,n_-}) \cong (t^{-1}q)^{(n_+-n_-)^2/2}\widetilde{KhR}_2^+(T(n,n)_{n_+,n_-})$, and a product skein contained in a collar neighborhood of the boundary of the 0-handle.

Lemma 4.7.4. *(1) For a standard positive/negative twisted belt link $T \subset S^1 \times S^2$, the class $1_\pm \in \mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2; T)$ is independent of the parametrization of the pair $(D^2 \times$*

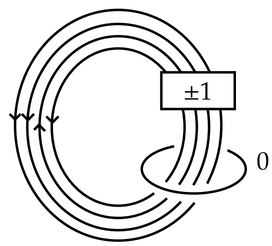


Figure 4.23: Diagram of a standard positive/negative twisted belt link in $S^1 \times S^2$.

$S^2, T)$. In particular, for any twisted belt link T, there are two distinguished classes $1_\pm \in \mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2; T)$.

(2) If $\Sigma\colon T \to T'$ is an annular cobordism annihilating two components of a twisted belt link $T \subset S^1 \times S^2$, where the annihilating annulus component is ∂-parallel, then $\mathcal{S}_0^{2,\mathrm{mul}}(I \times S^1 \times S^2; \Sigma)\colon \mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2; T) \to \mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2; T')$ maps $1_\pm$ to 0.

(3) Let $\Sigma\colon T \to T'$ be the cobordism in (2) with an extra dot on the annihilating annulus component, then $\mathcal{S}_0^{2,\mathrm{mul}}(I \times S^1 \times S^2; \Sigma)\colon \mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2; T) \to \mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2; T')$ maps $1_\pm$ to $1_\pm$.

(4) Let $T_0 \subset S^1 \times S^2$ be a standard negative twisted belt link. Let $T_1 \subset S^1 \times S^2$ (resp. $U \subset S^1 \times S^2$) be the standard positive twisted belt link (resp. standard belt link) with the same number of components (with orientations) as T_0. The gluing map $\mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2; T_0) \otimes \mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2; T_1) \to \mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2; U)$ as shown in Figure 4.24 maps $1_- \otimes 1_+$ to 1.

(5) Let $U_0, U_1 \subset S^1 \times S^2$ be belt links and $T_0, T_1 \subset S^1 \times S^2$ be twisted belt links, so that T_i has the same number of components (with orientations) as U_i, $i = 0, 1$, as shown in Figure 4.25. Suppose U_0 and U_1 together have n components, of which n_+ are oriented upward, and n_- downward. The image of $1 \otimes 1 \otimes 1 \in \mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2; U_0) \otimes \mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2; U_1) \otimes \widetilde{KhR}_2^+(T(n,n)_{n_+,n_-})$ in $\mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2; T_0) \otimes \mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2; T_1)$ under the gluing map shown in Figure 4.25 is equal to $1_+ \otimes \mathrm{id}_\alpha(1_+)$ plus terms with lower lasagna quantum gradings, for some $\alpha \in H_2(D^2 \times S^2)$.

Proof. (1) Let $T, T' \subset S^1 \times S^2$ be standard twisted belt links, both of which are positive/negative, and $\tilde{\phi}\colon D^2 \times S^2 \to D^2 \times S^2$ be an orientation-preserving diffeomorphism mapping T to T' with framing and orientation. We need to show that $\mathcal{S}_0^{2,\mathrm{mul}}(\tilde{\phi})\colon \mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2; T) \to \mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2; T')$ sends $1_\pm$ to $1_\pm$.

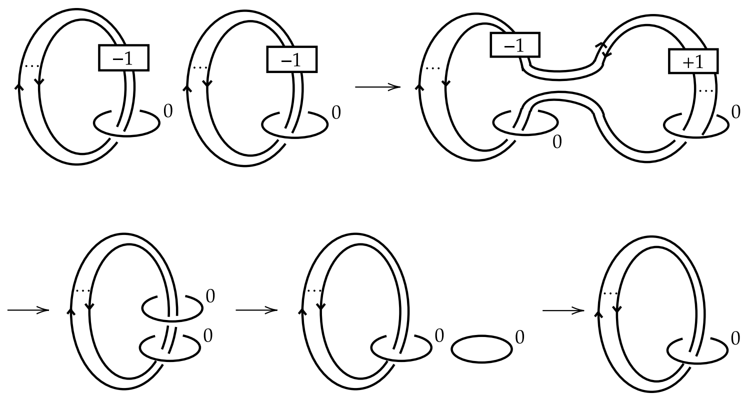


Figure 4.24: A gluing map $(D^2 \times S^2 \sqcup D^2 \times S^2; T_0 \sqcup T_1) \to (D^2 \times S^2; U)$.

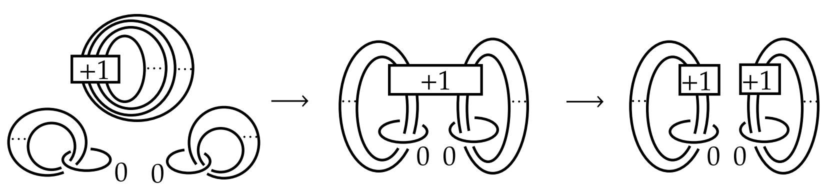


Figure 4.25: A gluing map $(D^2 \times S^2 \sqcup D^2 \times S^2 \sqcup B^4; U_0 \sqcup U_1 \sqcup T(n,n)_{n_+,n_-}) \to (D^2 \times S^2 \natural D^2 \times S^2; T_0 \cup T_1)$. The second map slides the strands on the left over the 2-handle on the right.

We decompose $\tilde{\phi}\colon D^2 \times S^2 \to D^2 \times S^2$ into the composition of the following four diffeomorphisms:

(i) A diffeomorphism $\tilde{\phi}_1$ of $D^2 \times S^2$ with $\phi_{1,*} = \phi_*$ on $H_*(S^1 \times S^2)$ (here and below, dropping the tilde indicates restricting to the boundary) that sends T to a standard positive/negative twisted belt link T_1, which is either

   (a) the identity diffeomorphism; or

   (b) a diffeomorphism that sends the 0-handle (resp. 2-handle) of $D^2 \times S^2$ to itself, preserving the core and the cocore of the 2-handle but reversing each of their orientations.

(ii) A diffeomorphism $\tilde{\phi}_2$ that is the identity on the 2-handle of $D^2 \times S^2$ as well as on a shrunk copy of the 0-handle, and the trace of a braiding away from the 2-handle attaching region that permutes strands of T_1 in a collar neighborhood of the boundary of the 0-handle, that sends T_1 to T'.

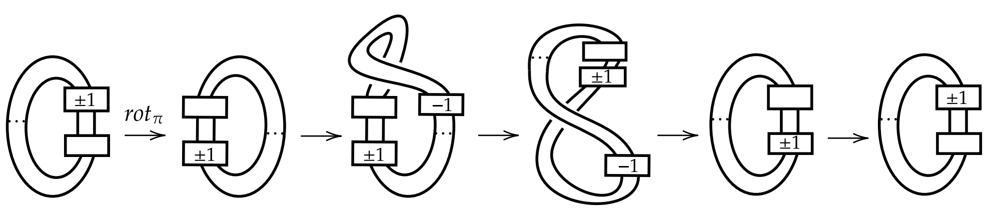
(iii) A diffeomorphism $\tilde{\phi}_3$ that is the identity on a shrunk copy of $D^2 \times S^2$, and the trace of an isotopy of $S^1 \times S^2$ in a collar neighborhood of $\partial(D^2 \times S^2)$ which preserves T' as a set, so that $\phi_3 \circ \phi_2 \circ \phi_1 = \phi$.

(iv) A diffeomorphism $\tilde{\phi}_4$ that is rel boundary.

Here, if $T \neq \emptyset$, changing the braiding in (ii) by a pure braid if necessary, one can always find $\tilde{\phi}_3$ in (iii) as claimed, thanks to Waldhausen's classical work [Wal68] applied to the Haken manifold $S^1 \times S^2 \backslash T'$. If $T = \emptyset$, the existence of $\tilde{\phi}_3$ is trivial.

We show that each $\tilde{\phi}_i$ sends $1_\pm$ to $1_\pm$.

If $\tilde{\phi}_1 \neq \mathrm{id}$, then we may choose it so that its effect on T is the composition of the inverse of a standard inversion as in Proposition 4.5.6(v), and the trace of some isotopy in S_{std} that consists of only overpass/underpass moves and isotopies via admissible links, in the sense of Proposition 4.4.4(i)(v). We note that in the proofs of Theorem 4.4.3 and Theorem 4.5.3, taking associated graded spaces and introducing shifting isomorphism were only necessary for the handleslide move (Proposition 4.4.4(vi)), the barbell move and the trace of an isotopy involving handleslides (special cases of Proposition 4.5.6(i)(ii)). Since $\tilde{\phi}_1^{-1}$ admits a decomposition in which none of these special cases arise, the proofs of these theorems show that $\mathcal{S}_0^{2,\mathrm{mul}}(\tilde{\phi}_1)$ is exactly equal to $\widetilde{KhR}_2^+(\phi_1^{-1}) \otimes \mathrm{id}\colon \widetilde{KhR}_2^+(T) \otimes \mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2) \to \widetilde{KhR}_2^+(T_1) \otimes \mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2)$ under the isomorphism (SZ), where $\widetilde{KhR}_2^+(\phi_1^{-1})$ is the composite map



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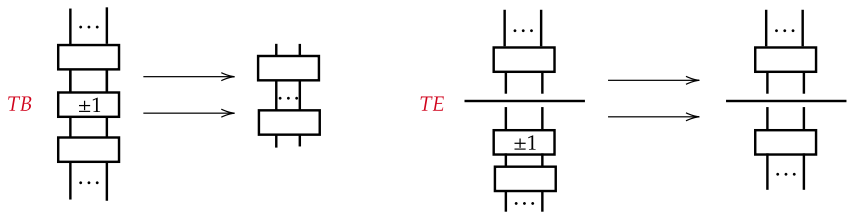
To check that $\widetilde{KhR}_2^+(\phi_1^{-1})$ sends $1_\pm$ to $1_\pm$, by exploiting strategies similar to those in Section 4.4, one reduces to show the commutativity of regions TB, TE shown as follows:



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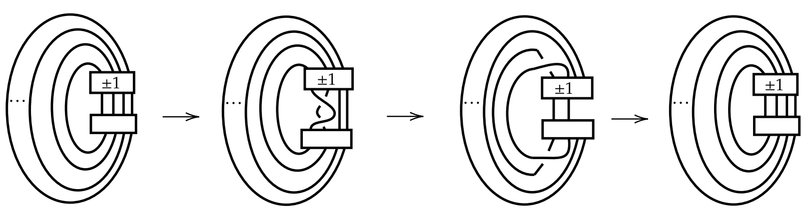
Here, in each region, the second map is given by absorbing the ± 1 twist into the second projector (see (4.18)), and the first map is given by pushing the ± 1 twist up and absorbing it into the first projector.

The commutativity of these regions is proved similarly as regions B, E in Section 4.4. For region TB, the composition of the inverse of the first map with the second map is given termwise by rotating the middle crossingless unlink by 2π, hence is the identity chain map up to sign. To fix the sign, see Appendix C.4.4. This proves the commutativity of region TB. The proof of region TE exactly follows that of region E. This proves that $\mathcal{S}_0^{2,\mathrm{mul}}(\tilde{\phi}_1)$ sends $1_\pm$ to $1_\pm$.

Analogously, $\mathcal{S}_0^{2,\mathrm{mul}}(\tilde{\phi}_2)$ is exactly equal to $\widetilde{KhR}_2^+(\phi_2^{-1})\otimes\mathrm{id}$, where $\widetilde{KhR}_2^+(\phi_2^{-1})\colon \widetilde{KhR}_2^+(T_1) \to \widetilde{KhR}_2^+(T')$ is the map that braids the strands according to $\tilde{\phi}_2$, shown as a composition of maps of the form (the braiding can happen between any two adjacent strands, either positively or negatively; only a special case is depicted)



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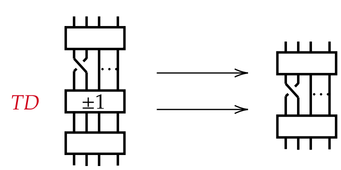
To check that $\widetilde{KhR}_2^+(\phi_2^{-1})$ maps $1_\pm$ to $1_\pm$, in addition to the commutativity of region TB above, one also needs the commutativity of region TD (and its mirrored version):



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In order to apply the strategy for region D, we need to show that the chain map that pushes the ± 1 twist up past the crossing (called X) is nondecreasing in the homological degree contributed by the distinguished crossing X. This can be realized by choosing the twist-pushing map to be the composition of the creation of a canceling pair of twists above X (a ± 1 twist above a ∓ 1 twist) and a rotation map that cancels the ∓ 1 twist above and the ± 1 twist below X, carefully chosen as the composition of two "φ" maps in [CY25, Lemma 4.5]. This proves that $\mathcal{S}_0^{2,\mathrm{mul}}(\tilde{\phi}_2)$ sends $1_\pm$ to $1_\pm$.

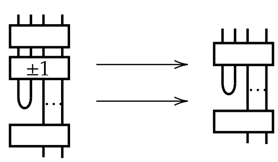
The diffeomorphism $\tilde{\phi}_3$ induces the identity map, since any lasagna filling of $(D^2 \times S^2, T')$ can be isotoped to be $I \times T'$ in a collar neighborhood of the boundary.

Finally, by Gabai's 4-dimensional lightbulb theorem (or Proposition 4.5.7), $\tilde{\phi}_4$ is isotopic rel boundary to a diffeomorphism supported on a local 4-ball, hence also induces the identity map.

(2) Pick T, T' to be standard positive or negative twisted belt links and Σ to be standard. Then, by the same argument used for $\tilde{\phi}_1, \tilde{\phi}_2$ in (1) above, the induced map takes the form $\widetilde{KhR}_2^+(\Sigma) \otimes \mathrm{id}$ under the isomorphism (SZ). This claim now follows from the fact that $\widetilde{KhR}_2^{+,0,-\#T'-2}(T') = 0$ by [Man+23; Sto09].

(3) As in (2), pick T, T', Σ to be standard. One shows that $\widetilde{KhR}_2^+(\Sigma)$ maps $1_\pm \in \widetilde{KhR}_2^+(T)$ to $1_\pm \in \widetilde{KhR}_2^+(T')$ by using the commutativity of region TB above, as well as the commutativity of region TC:

TC



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which is proved in the same way as for region C.

(4) Let $n = \#T_i$ with n_+ (resp. n_-) strands oriented upward (resp. downward). By the description preceding Lemma 4.7.4, $1_+ \in \mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2; T_0)$ is the image of $1 \in KhR_2^+(T(n,n)_{n_+,n_-}) \cong \mathcal{S}_0^{2,\mathrm{mul}}(B^4; T(n,n)_{n_+,n_-})$ under the natural 2-handle attachment map. Thus, the image of $1_- \otimes 1_+$ under the stated gluing map is also the image of $1_- \otimes 1 \in \mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2; T_0) \otimes \mathcal{S}_0^{2,\mathrm{mul}}(B^4; T(n,n)_{n_+,n_-})$ under the gluing map given by n saddles followed by an isotopy. The claim that the image equals 1 follows from the commutativity of region TX:

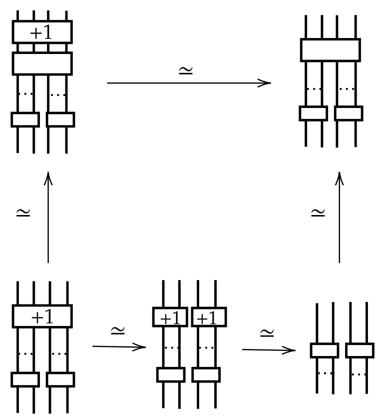
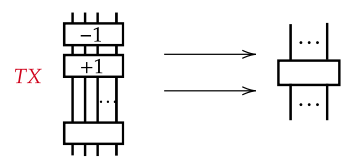


Figure 4.26: Check compatibility of various elements 1 under connected sums.



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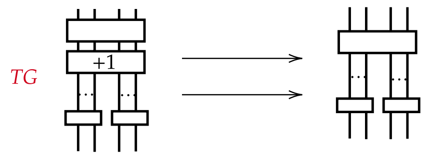
where the first map is the composition of two absorptions of twists into the projector, and the second map is the composition of Reidemeister-induced maps that cancel the two twists. As in the proof of other similar regions, the composition of the inverse of the first map with the second map is termwise the identity chain map up to sign as the composition of the cobordisms is isotopic to identity. To fix the sign, see Appendix C.4.4.

(5) Consider the diagram of isomorphisms on $\widetilde{KhR}_2^+$ in Figure 4.26, where the first map on the second row is obtained by termwise sliding the strands connected to the bottom-left projector across the empty region in the bottom-right projector, thereby undoing the $+1$ linking between the two groups of strands, plus some higher differentials. We first reduce to checking the commutativity of Figure 4.26.

The image of $1 \otimes 1 \otimes 1 \in \mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2; U_0) \otimes \mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2; U_1) \otimes \widetilde{KhR}_2^+(T(n,n)_{n_+,n_-})$ in the middle term in Figure 4.25 is $v \otimes 1 \in \widetilde{KhR}_2^+(L) \otimes \mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2 \natural D^2 \times S^2) \cong \mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2 \natural D^2 \times S^2; L)$ where L is the admissible link shown on the boundary, and v is the element in the (closure of the) bottom left term in Figure 4.26 whose image in the (closure of the) bottom right term under the composite isomorphism that goes through the first row is equal to $1 \otimes 1$. The image of $v \otimes 1$ in the last term in Figure 4.25, by the description of handleslide maps in Section 4.4.7 and the definition of the element 1's, is equal to $f(v) \otimes (1 \otimes \mathrm{id}_\alpha(1)) \in \widetilde{KhR}_2^+(T_0 \cup T_1) \otimes \mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2 \natural D^2 \times S^2)$ plus terms with lower lasagna quantum gradings, where f is the first map on the second row of Figure 4.26. To check $f(v) = 1_+ \otimes 1_+$, it is thus sufficient to check the commutativity of Figure 4.26.

In turn, the commutativity of Figure 4.26 is reduced to the commutativity of region TG:



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where the first map is the twist absorption into the projector on the top, and the second map is obtained by sliding the strands connected to the bottom-left projector across the bottom-right one and then performing two twist absorptions into the bottom projectors. The commutativity of region TG is checked by proceeding termwise as before. $\square$

We also prove the following technical lemma as a consequence of Lemma 4.7.4(5). This is not needed for the proof of Theorem 4.7.1.

Lemma 4.7.5. *Write $D_{std} := \#_{i=1}^k \natural^{m_i}(D^2 \times S^2)$ and $S_{std} = \partial D_{std}$. Let X be an abstract D_{std} together with an orientation-preserving identification $\partial X = S_{std}$. Let $S_0, S_1 \subset int(X)$ be unoriented 2-spheres, $S_0 \# S_1$ be the connected sum of them along a given path γ, and $L \subset S_{std}$ be an admissible link. Suppose that*

- *$\psi \colon X_{S_0,S_1} \xrightarrow{\cong} D_{std}$ and $\psi' \colon X_{S_0\#S_1} \xrightarrow{\cong} D_{std}$ are orientation-preserving diffeomorphisms rel boundary;*
- *The complement of $\nu(S_0 \cup \gamma \cup S_1)$ in X is a 4-dimensional relative 1-handlebody complement W^t with $\partial_- W^t = -S_{std}$ and $\partial_+ W^t = -\partial\nu(S_0 \cup \gamma \cup S_1)$.*

Then, the composition

$$\mathcal{S}_0^{2,\mathrm{mul}}(D_{std}; L) \xrightarrow[\cong]{\mathcal{S}_0^{2,\mathrm{mul}}(\psi^{-1})} \mathcal{S}_0^{2,\mathrm{mul}}(X_{S_0,S_1}; L) \xrightarrow[\cong]{\tau^{-1}_{X,L,S_0\cup S_1}} \mathcal{S}_0^{2,\mathrm{mul}}(X; L)$$
$$\xrightarrow[\cong]{\tau_{X,L,S_0\#S_1}} \mathcal{S}_0^{2,\mathrm{mul}}(X_{S_0\#S_1}; L) \xrightarrow[\cong]{\mathcal{S}_0^{2,\mathrm{mul}}(\psi')} \mathcal{S}_0^{2,\mathrm{mul}}(D_{std}; L)$$

induces a map on the 0-th associated graded spaces with respect to the lasagna quantum grading. Under the isomorphism (SZ), this induced map is equal to $\mathrm{id} \otimes gr_0(\mathrm{id}_\alpha)$ for some $\alpha \in H_2(D_{std})$.

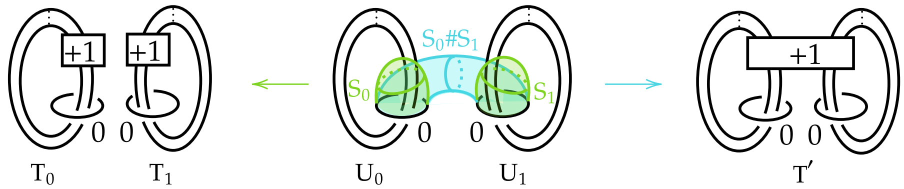


Figure 4.27: Middle: The neighborhood $X_1 = \nu(S_0 \cup \gamma \cup S_1) \cong D_{std,1}$, with the belt link $U_0 \cup U_1$ shown on the boundary, the spheres S_0, S_1 shown in green, and the sphere $S_0\#S_1$ shown in blue. The parts of spheres shown are slightly pushed into the interior of the 0-handle. Left: $(X_1)_{S_0,S_1}$ reparametrized as $D_{std,1}$, with $U_0 \cup U_1$ shown as the twisted belt link $T_0 \cup T_1$ on the boundary. Right: $(X_1')_{S_0\#S_1}$ reparametrized as $D_{std,1}$, with $U_0 \cup U_1$ shown as the admissible link T' on the boundary.

Proof. Orient S_0, S_1 so that the connected sum operation respects the orientations. Let $X_1 = \nu(S_0 \cup \gamma \cup S_1)$, which is naturally identified with $D_{std,1} := \natural^2(D^2 \times S^2)$ using the orientations of S_0, S_1.

A generic lasagna filling (Σ_+^t, v) representing some element $x \in \mathcal{S}_0^{2,\mathrm{mul}}(X; L)$ can be assumed to have input balls away from X_1 and skein Σ_+^t intersecting X_1 in a disjoint union of cocores transverse to $S_0 \cup S_1$. The boundary of these cocores is a belt link $U_0 \cup U_1$ in $\partial X_1 \cong S_{std}$, as shown in the middle of Figure 4.27. Let (Σ^t, v) denote the part of (Σ_+^t, v) outside X_1, which can be thought of as a lasagna filling of $(W^t, -(U_0 \cup U_1) \sqcup L)$. The spheres $S_0, S_1, S_0\#S_1$ are also shown in the middle of Figure 4.27. The Gluck twist $(X_1)_{S_0,S_1}$ can be naturally reidentified with $D_{std,1}$ via a diffeomorphism sending $U_0 \cup U_1$ to the twisted belt link $T_0 \cup T_1$ shown on the left of Figure 4.27. More explicitly, the Gluck twist on S_j is performed along the copy of S_j on $\partial\nu(S_j)$ closest to ∂X_1 by a counterclockwise full rotation around the center of the part of S_j shown in Figure 4.27 (the north pole) and the center of the 2-core part of S_j (the south pole), $j = 0, 1$, while the identification $(X_1)_{S_0,S_1} \cong D_{std,1}$ pushes the effect of these Gluck twists towards the boundary. By construction, $\tau_{X,L,S_0 \cup S_1}(x)$ is the image of $1_+ \otimes 1_+ \in \mathcal{S}_0^{2,\mathrm{mul}}(D_{std,1}; T_0 \cup T_1)$ under gluing the lasagna filling (Σ^t, v).

On the other hand, let X_1' denote a slightly shrunken copy of X_1. The Gluck twist $(X_1')_{S_0\#S_1}$ can be naturally reidentified with $D_{std,1}$ via a diffeomorphism sending $U_0 \cup U_1$ to the admissible link T' shown on the right of Figure 4.27, by a description analogous to the previous case. The boundary parametrizations of $(X_1)_{S_0,S_1}$ and $(X_1')_{S_0\#S_1}$ differ by $\phi = \tau_{S_0\#S_1} \circ (\tau_{S_0} \circ \tau_{S_1})^{-1} \in \mathrm{Diff}^{spin}(\partial D_{std,1})$, where here we abuse the notation and use S_0, S_1 to denote the two core spheres of $\partial D_{std,1}$, and τ_S to denote the Dehn twist along the sphere S. The mapping class of ϕ is trivial, hence the parametrization $(X_1')_{S_0\#S_1} \cong D_{std,1}$ extends by a levelwise diffeomorphism to a parametrization $(X_1)_{S_0\#S_1} \cong D_{std,1}$ whose boundary parametrization agrees with that of $(X_1)_{S_0,S_1} \cong D_{std,1}$. This levelwise diffeomorphism can be chosen to isotope the link T' to $T_0 \cup T_1$ by sliding the strands on the left over the second 2-handle. Let X_1'' be a slightly shrunken copy of X_1', and push $S_0 \# S_1$ slightly outside X_1'', within X_1'. To calculate $\tau_{X,L,S_0\#S_1}(x)$, we start with $1 \in \mathcal{S}_0^{2,\mathrm{mul}}(X_1''; U_0 \cup U_1)$. Add in an input ball in $int(X_1') \backslash X_1''$ with an input link $T(n,n)_{n_+,n_-}$ carrying the label $1 \in KhR_2(T(n,n)_{n_+,n_-})$, as shown on the left of Figure 4.25, where n_+ and n_- denote the numbers of positively and negatively oriented strands of $U_0 \cup U_1$, respectively, and $n = n_+ + n_-$. Then evaluate to $((X_1')_{S_0\#S_1}, T')$, implementing the effect of the lasagna Gluck twist along $S_0 \# S_1$ within X_1'. Next, evaluate to $((X_1)_{S_0\#S_1}, T_0 \cup T_1)$ by gluing in the trace of the isotopy $T' \sim T_0 \cup T_1$. Finally, glue in (Σ^t, v) to obtain $\tau_{X,L,S_0\#S_1}(x) \in \mathcal{S}_0^{2,\mathrm{mul}}(X_{S_0\#S_1}; L)$. By Lemma 4.7.4(5), $\tau_{X,L,S_0\#S_1}(x)$ is thus equal to the image of $1_+ \otimes \mathrm{id}_\alpha(1_+) + \cdots \in \mathcal{S}_0^{2,\mathrm{mul}}(D_{std,1}; T_0 \cup T_1)$ under gluing in the lasagna filling (Σ^t, v), for some $\alpha \in H_2(D^2 \times S^2)$, where $\cdots$ are terms with negative lasagna quantum degrees.

Now, after changing ψ' by a diffeomorphism rel boundary if necessary (which does not affect the statement thanks to Proposition 4.5.4), we may assume ψ and ψ' agree outside X_1, and differ by the composite diffeomorphism $(X_1)_{S_0,S_1} \cong D_{std,1} \cong (X_1)_{S_0\#S_1}$ inside X_1. As such, the element $\mathcal{S}_0^{2,\mathrm{mul}}(\psi)(\tau_{X,L,S_0\cup S_1}(x))$ (resp. $\mathcal{S}_0^{2,\mathrm{mul}}(\psi')(\tau_{X,L,S_0\#S_1}(x))$) is equal to the image of $z_1 := 1_+ \otimes 1_+ \in \mathcal{S}_0^{2,\mathrm{mul}}(D_{std,1}; T_0 \cup T_1)$ (resp. $z_2 := 1_+ \otimes \mathrm{id}_\alpha(1_+) + \cdots \in \mathcal{S}_0^{2,\mathrm{mul}}(D_{std,1}; T_0 \cup T_1)$) under gluing $\psi_*(\Sigma^t, v)$, the pushforward of (Σ^t, v) under ψ. We drop ψ from the notation near $D_{std,1}$ in what follows.

By changing $\psi_*(\Sigma^t, v)$ in its equivalence class, we may assume that it contains a single input ball, which is near $\partial D_{std,1}$. Further decompose $\psi_*(\Sigma^t, v)$ into

(i) A lasagna filling in a collar neighborhood $I \times \partial D_{std,1}$ of $\partial D_{std,1}$ rel $-(T_0 \cup T_1) \sqcup (T_0 \cup T_1 \sqcup L_0)$ with an input link L_0 and skein $I \times (T_0 \cup T_1) \cup [1/2, 1] \times L_0$. Here, when regarded as a link in $\partial D_{std,1}$, L_0 is admissible and distant from $T_0 \cup T_1$.

(ii) A link cobordism in (a shrunk copy of) $\psi(W^t)$ from $T_0 \cup T_1 \sqcup L_0$ to L.

Since gluing (i) amounts to putting some label v in the new tensorial factor $\widetilde{KhR}_2^+(L_0)$ of $\widetilde{KhR}_2^+(T_0 \cup T_1 \sqcup L_0)$ (which does not contribute to lasagna quantum degrees), the images of z_1, z_2 under (i) also differ by a shifting isomorphism plus terms with negative lasagna quantum degrees. Finally, in view of Theorem 4.6.2, their images under the whole gluing of (Σ^t, v) differ by a shifting isomorphism plus terms with negative lasagna quantum degrees, as desired. $\square$

We now give the proof of Theorem 4.7.1.

Proof of Theorem 4.7.1. The map $\tau_{X,L,S,\pm}$ in (4.17) is constructed as follows.

Let (Σ, v) be a lasagna filling of (X, L) representing a given element $x \in \mathcal{S}_0^{2,\mathrm{mul}}(X; L)$. By general position, we may assume that the input balls of Σ are disjoint from S, and that Σ intersects S transversely at some finitely many points. Let $\overline{\nu(S)} \cong D^2 \times S^2$ be a closed tubular neighborhood of S, so that Σ intersects $D^2 \times S^2$ in some finite number of cocore disks, with some boundary $U \subset S^1 \times S^2$.

If U has an odd number of components, then Theorem 4.2.2 implies that (Σ, v) defines the zero class when restricted to a lasagna filling of $(D^2 \times S^2, U)$, hence it also defines the zero class in $\mathcal{S}_0^{2,\mathrm{mul}}(X; L)$.

Suppose now U has an even number of components. Then U is a belt link. Write $(X, \Sigma) = (X \backslash \nu(S), \Sigma \backslash \nu(S)) \cup (D^2 \times S^2, \Sigma \cap D^2 \times S^2)$, we see that $x = [(\Sigma, v)]$ is the image of 1 under the map

$$\mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2; U) \to \mathcal{S}_0^{2,\mathrm{mul}}(X; L)$$

that glues in the lasagna filling $(\Sigma \backslash \nu(S), v)$ of $(X \backslash \nu(S), L \cup (-U)) \cong (X_S \backslash \nu(S), L \cup (-\tau(U)))$.

The Gluck twist X_S is obtained from X by cutting out $D^2 \times S^2$ and regluing it back via the Dehn twist τ. The belt link U, on the boundary of this glued-in $D^2 \times S^2$, is thus the twisted belt link $\tau(U)$. We define $\tau_{X,L,S,\pm}(x)$ to be the image of $1_\pm$ under the gluing map

$$\mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2; \tau(U)) \to \mathcal{S}_0^{2,\mathrm{mul}}(X_S; L)$$

that glues in the lasagna filling $(\Sigma \backslash \nu(S), v)$ of $(X \backslash \nu(S), L \cup (-U))$.

We check that $\tau_{X,L,S,\pm}$ is well-defined.

First, by Lemma 4.7.4(1), $\tau_{X,L,S,\pm}([(\Sigma, v)])$ is independent of the parametrization $\overline{\nu(S)} \cong D^2 \times S^2$.

Next, we check that $\tau_{X,L,S,\pm}([(\Sigma, v)])$ is independent of (Σ, v). It is clearly linear in the label v. If (Σ', v') is another lasagna filling whose input balls contains those of Σ and are disjoint from S, the two gluing maps leading to $\tau_{X,L,S,\pm}(x)$ are equal. Hence, it remains to check that $\tau_{X,L,S,\pm}([(\Sigma, v)])$ is invariant under isotoping the skein Σ rel boundary.

By general position, every isotopy of the skein Σ rel boundary is decomposed into some finger/Whitney moves that create/annihilate transverse intersections $\Sigma \cap S$ in pairs, and some isotopies that do not create/annihilate intersections with S. We need to show that $\tau_{X,L,S,\pm}([(\Sigma, v)])$ is invariant under a finger move from Σ to some Σ' across S.

By a Bar-Natan neck-cutting, we may assume that the component of Σ that undergoes the finger move is a local 2-sphere S_0, which can be either undotted or dotted. When S_0 is undotted, $\tau_{X,L,S,\pm}([(\Sigma, v)])$ is zero since S_0 evaluates to zero. For (Σ', v), since a collar neighborhood of the boundary of $\nu(S)$ outside $\nu(S)$ contains Σ' as an undotted annulus annihilation map, we conclude from Lemma 4.7.4(2) that $\tau_{X,L,S,\pm}([(\Sigma', v)])$ is also zero. Similarly, when S_0 is dotted, we find using Lemma 4.7.4(3) that $\tau_{X,L,S,\pm}([(\Sigma, v)])$ and $\tau_{X,L,S,\pm}([(\Sigma', v)])$ are equal. This shows that $\tau_{X,L,S,\pm}$ is well-defined.

It is clear from the definition that $\tau_{X,L,S,\pm}$ preserves the homological and the quantum degrees, and covers $H_2(\tau_S)_\pm$ in the skein degree.

We prove the addenda (1) to (4). (4) is trivial. (1) follows by comparing the degree of $1 \in \mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2; U)$ with those of $1_\pm \in \mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2; \tau(U))$.

(2) By general position, we may isotope any skein in X rel L to be disjoint from a given 3-ball bounding S. The statement follows.

(3) Push $S \subset X_S$ to be disjoint from $\nu(S) \subset X_S$, and let S' denote this pushoff copy. Find a copy of $D^2 \times S^2$ in $int(X)$ containing $\nu(S) \cup \nu(S')$ as the tubular neighborhood of two core spheres. Any given lasagna filling (Σ, v) of (X, L) can be assumed to have input balls disjoint from $D^2 \times S^2$ and Σ intersecting $D^2 \times S^2$ in some number of cocores. The statement follows from Lemma 4.7.4(4) applied to the gluing of $((D^2 \times S^2)\backslash(\nu(S) \cup \nu(S')), \Sigma\backslash(\nu(S) \cup \nu(S')))$ onto $(\nu(S) \sqcup \nu(S'), \tau(\partial\nu(S) \cap \Sigma) \sqcup \tau(\partial\nu(S') \cap \Sigma))$. $\square$

4.7.3 $\widetilde{KhR}_2^+$ on objects

As before, let $D_{std} = \#_{i=1}^k \natural^{m_i}(D^2 \times S^2)$ and $S_{std} = \partial D_{std} = \sqcup_{i=1}^k \#^{m_i}(S^1 \times S^2)$. We prove the following theorem.

Theorem 4.7.6. *Theorem 4.5.1 is still true when S is only assumed to be an abstract S_{std}.*[3]

In other words, if $L \subset S_{std}$ is admissible and $\phi \in \mathrm{Diff}^+(S_{std})$ is any L-admissible orientation-preserving diffeomorphism, we wish to assign an isomorphism $\widetilde{KhR}_2^+(\phi)\colon \widetilde{KhR}_2^+(\phi(L)) \xrightarrow{\cong} \widetilde{KhR}_2^+(L)$ functorially in ϕ. This is provided by the following theorem.

Theorem 4.7.7. *Let $L \subset S_{std}$ be an admissible link and $\phi \in \mathrm{Diff}^+(S_{std})$ be an orientation-preserving L-admissible diffeomorphism. Let $S \subset int(D_{std})$ be a ∂-parallel finite union of disjoint embedded unoriented 2-spheres, and $\tilde{\phi}\colon D_{std} \xrightarrow{\cong} (D_{std})_S$ be a diffeomorphism that extends ϕ. Then the composition*

$$\mathcal{S}_0^{2,\mathrm{mul}}(D_{std}; \phi(L)) \xrightarrow{\tau_{D_{std},\phi(L),S}} \mathcal{S}_0^{2,\mathrm{mul}}((D_{std})_S; \phi(L)) \xrightarrow{\mathcal{S}_0^{2,\mathrm{mul}}(\tilde{\phi}^{-1})} \mathcal{S}_0^{2,\mathrm{mul}}(D_{std}; L)$$

[3]We warn the reader that the notation S is used both for a 3-manifold that is an abstract S_{std}, and a union of disjoint embedded 2-spheres in a 4-manifold. In the rest of this section, S will only appear for the second purpose.

induces a map on the 0-th associated graded spaces with respect to the lasagna quantum grading. Under the isomorphism (SZ), this induced map is uniquely of the form

$$\widetilde{KhR}_2^+(\phi)\otimes gr_0(\mathrm{id}_\alpha\circ\phi_*^{-1})\colon \widetilde{KhR}_2^+(\phi(L))\otimes gr_0\mathcal{S}_0^{2,\mathrm{mul}}(D_{std})\to \widetilde{KhR}_2^+(L)\otimes gr_0\mathcal{S}_0^{2,\mathrm{mul}}(D_{std}) \tag{4.19}$$

for some isomorphism $\widetilde{KhR}_2^+(\phi)$ independent of S and $\tilde\phi$ induced by a chain map, and some $\alpha\in H_2(D_{std})$. Here, ϕ_* in (4.19) is defined as the composition $\mathcal{S}_0^{2,\mathrm{mul}}(D_{std})\cong\mathcal{S}_0^{2,\mathrm{mul}}(I\times S_{std})\xrightarrow[\cong]{\mathcal{S}_0^{2,\mathrm{mul}}(I\times\phi)}\mathcal{S}_0^{2,\mathrm{mul}}(I\times S_{std})\cong\mathcal{S}_0^{2,\mathrm{mul}}(D_{std})$.

The existence of an extension $\tilde\phi$ for some S as in Theorem 4.7.7 can be seen as follows. The set of spin structures on S_{std} is affine over $H^1(S_{std};\mathbb{Z}/2)$. There is a distinguished spin structure, denoted $\mathfrak{s}_0$, which is the restriction of the unique spin structure on D_{std}. Write $\phi_*(\mathfrak{s}_0)=\mathfrak{s}_0+\beta$ for some $\beta\in H^1(S_{std};\mathbb{Z}/2)\cong H_2(S_{std};\mathbb{Z}/2)\cong H_2(D_{std};\mathbb{Z}/2)$, represented by some $S=\sqcup_{i=1}^m S_i\subset int(D_{std})$, a ∂-parallel union of disjoint embedded unoriented 2-spheres. The restriction of the spin structure on $(D_{std})_S$ on S_{std} is $\mathfrak{s}_0+[S]=\phi_*(\mathfrak{s}_0)$, and the existence of $\tilde\phi$ follows from the surjectivity of $\mathrm{Diff}^+(D_{std})\to\mathrm{Diff}^{spin}(S_{std})$.

Before deducing Theorem 4.7.6 from Theorem 4.7.7, we make the following topological observations.

Lemma 4.7.8. *If $S_0,S_1\subset S_{std}$ are two collections of disjoint embedded unoriented 2-spheres, then S_0 is related to some other collection S_0' that is disjoint from S_1 via a sequence of (isotopies and) inverses of the connected sum operation.*

Proof. Put S_0 and S_1 into transverse position. Surger S_0 along innermost disks on S_1 bounded by $S_0\cap S_1$. □

Lemma 4.7.9. *If $S_0,S_1\subset S_{std}$ are two collections of disjoint embedded unoriented 2-spheres with $[S_0]=[S_1]\in H_2(S_{std};\mathbb{Z}/2)$, then S_0 is related to S_1 via a sequence of (isotopies and) connected sums, separating 2-sphere creations, and their inverses.*

Proof. Without loss of generality, say S_{std} is connected. Take connected sums to make both S_0 and S_1 connected. If $[S_0]=[S_1]=0$, then S_0,S_1 are both separating, hence they are related by the moves. If $[S_0]=[S_1]\neq 0$, apply Lemma 4.7.8 to make S_0 disjoint from S_1. Since S_1 is nonseparating, we may take connected sums outside S_1 to make S_0 connected again. Now form a connected sum $S:=S_0\#S_1$, which may be assumed to be disjoint from $S_0\cup S_1$. Since $[S]=[S_0]+[S_1]=0$, S is separating. Therefore, S_0 is related to S_1 via one separating sphere creation and one connected sum: $S_0\sim S_0\cup S\sim S_1$. □

Lemma 4.7.10. *Let $S_0,S_1\subset S_{std}=\partial D_{std}$ be two disjoint 2-spheres, $\gamma\subset S_{std}$ be an arc connecting them, and $D_{std}\subset S^4$ be the standard embedding with complement a 4-dimensional*

1-handlebody. Then the complement of a neighborhood of $S_0 \cup \gamma \cup S_1$ in S^4 is a 4-dimensional 1-handlebody.

Proof. Since S_0, S_1 lie on the same connected component of S_{std}, by capping off extra components if necessary, we may assume $D_{std} = \natural^m (D^2 \times S^2)$. It suffices to show that S_0, S_1 bound disjoint 3-balls B_0, B_1 in S^4 whose interiors are disjoint from γ. We divide into four cases.

Case 1: S_0, S_1 are both separating.

Write $D_{std} = (D^2 \times S^2) \natural \cdots \natural (D^2 \times S^2)$ and $S_{std} = (S^1 \times S^2) \# \cdots \# (S^1 \times S^2)$. By a spin parametrization change of S_{std} (which extends to D_{std}), we may assume that S_j is a copy of the a_j-th connected sum 2-sphere in S_{std}, $j = 0, 1$, $0 \le a_0 \le a_1 \le m - 1$ (when $a_j = 0$, we interpret this as saying that S_j bounds a ball in S_{std}). We may choose B_j to be a copy of the a_j-th boundary connected sum 3-ball in D_{std}, $j = 0, 1$ (when $a_j = 0$, we interpret this as saying that B_j is a ∂-parallel 3-ball).

Case 2: S_0 is separating, S_1 is nonseparating.

As in Case 1, S_0 bounds a 3-ball B_0 in D_{std}. By applying a spin parametrization change of S_{std} to make S_1 standard, we see that S_1 is the belt sphere of the cocore of a 1-handle in the 4-dimensional 1-handlebody $S^4 \backslash D_{std}$. Hence we may choose B_1 to be this cocore 3-ball.

Case 3: $S_0 \cup S_1$ is nonseparating.

By a spin parametrization change of S_{std}, we see that S_0, S_1 are belt spheres of the cocores of two different 1-handles in $S^4 \backslash D_{std}$. Hence we may choose B_0, B_1 to be these cocore 3-balls.

Case 4: S_0, S_1 are nonseparating, but $S_0 \cup S_1$ is separating.

If S_0 and S_1 are isotopic, they bound two parallel cocores of a 1-handle in $S^4 \backslash D_{std}$. If they are nonisotopic, by a spin parametrization change of $S_{std} = (S^1 \times S^2) \# \cdots \# (S^1 \times S^2)$, we may assume that they are embedded in the i-th connected summand in S_{std} as two nonisotopic S^2 factors, for some $1 < i < m$. It is clear that they bound disjoint 3-balls B_0, B_1 in the i-th boundary connected summand of $S^4 \backslash D_{std} = (S^1 \times B^3) \natural \cdots \natural (S^1 \times B^3)$. □

Proof of Theorem 4.7.6 assuming Theorem 4.7.7. This is similar to deducing Theorem 4.5.1 from Theorem 4.5.3. The only difference is that the lasagna level functoriality itself is more involved, which we now address.

Suppose $\phi_j \in \mathrm{Diff}^+(S_{std})$ with lifts $\tilde{\phi}_j \colon D_{std} \xrightarrow{\cong} (D_{std})_{S_j}$, $j = 0, 1$, so that ϕ_0 is L-admissible and ϕ_1 is $\phi_0(L)$-admissible. Let $N := (-1, 0] \times S_{std}$ be a collar neighborhood of the boundary of D_{std}. After some isotopies, we may assume that

- $S_j \subset \{-(j+1)/3\} \times S_{std} \subset N$, $j = 0, 1$;
- $\tilde{\phi}_j = \mathrm{id} \times \phi_j$ on N, $j = 0, 1$.

The composition map $\phi := \phi_1 \circ \phi_0$ extends to a map $\tilde{\phi}\colon D_{std} \to (D_{std})_S$, where $S = \tilde{\phi}_1(S_0) \cup S_1$. More precisely, we take $\tilde{\phi}$ as the composition

$$D_{std} \xrightarrow{\tilde{\phi}_0} (D_{std})_{S_0} \xrightarrow{(\tilde{\phi}_1)_{S_0}} ((D_{std})_{S_1})_{\tilde{\phi}_1(S_0)} \cong (D_{std})_S,$$

where the middle map is the natural map induced by $\tilde{\phi}_1$.

Consider the following diagram of isomorphisms:

$$\begin{tikzcd}
\mathcal{S}_0^{2,\mathrm{mul}}(D_{std}; \phi(L)) \arrow[r, "\tau_{D_{std},\phi(L),S}"] \arrow[d, "\tau_{D_{std},\phi(L),S_1}"'] & \mathcal{S}_0^{2,\mathrm{mul}}((D_{std})_S; \phi(L)) \arrow[r, "\mathcal{S}_0^{2,\mathrm{mul}}(\tilde{\phi}^{-1})"] \arrow[dr, "\mathcal{S}_0^{2,\mathrm{mul}}((\tilde{\phi}_1)_{S_0}^{-1})"] & \mathcal{S}_0^{2,\mathrm{mul}}(D_{std}; L) \\
\mathcal{S}_0^{2,\mathrm{mul}}((D_{std})_{S_1}; \phi(L)) \arrow[ur, "\tau_{(D_{std})_{S_1},\phi(L),\tilde{\phi}_1(S_0)}"] \arrow[r, "\mathcal{S}_0^{2,\mathrm{mul}}(\tilde{\phi}_1^{-1})"'] & \mathcal{S}_0^{2,\mathrm{mul}}(D_{std}; \phi_0(L)) \arrow[r, "\tau_{D_{std},\phi_0(L),S_0}"'] & \mathcal{S}_0^{2,\mathrm{mul}}((D_{std})_{S_0}; \phi_0(L)). \arrow[u, "\mathcal{S}_0^{2,\mathrm{mul}}(\tilde{\phi}_0^{-1})"']
\end{tikzcd} \tag{4.20}$$

The upper left triangle is commutative by Theorem 4.7.1(4). The lower quadrilateral is commutative by naturality of the lasagna Gluck twist construction. The upper right triangle is trivially commutative. Note that the first row of (4.20) is a lasagna defining map for $\widetilde{KhR}_2^+(\phi)\colon \widetilde{KhR}_2^+(\phi(L)) \to \widetilde{KhR}_2^+(L)$ in the sense of Theorem 4.7.7, even though S is not necessarily ∂-parallel. This is because one may apply Lemma 4.7.8 to change $\tilde{\phi}_1(S_0)$ by some inverse connected sum operations within $\{-1/3\} \times S_{std}$ to make S ∂-parallel, while Lemma 4.7.10 and Lemma 4.7.5 guarantee that the induced map of the first row of (4.20), in terms of (4.19), is unchanged except possibly by altering the shift $\alpha \in H_2(D_{std})$. Theorem 4.7.6 now follows from Theorem 4.7.7 by an argument analogous to that in the proof of Theorem 4.5.1. In particular, $\widetilde{KhR}_2^+$ is defined on objects of $\mathbf{Links}_1$ as a "cross-section" of $\prod_{\rho \in P(S,L)} \widetilde{KhR}_2^+(\rho(L))$, or alternatively the limit of $\widetilde{KhR}_2^+(\bullet(L))$ over $P(S, L)$, as in the proof of Theorem 4.5.1, where $P(S, L)$ is the set or indiscrete category of L-admissible parametrizations of S. $\square$

Proof of Theorem 4.7.7. The independence of the statement on the choice of $\tilde{\phi}$ follows from Proposition 4.5.4.

We prove its independence on S, a ∂-parallel finite union of disjoint unoriented 2-spheres. Suppose S' is another such union of 2-spheres, then $[S] = [S'] = \phi_*(\mathfrak{s}_0) - \mathfrak{s}_0 \in H_2(D_{std}; \mathbb{Z}/2) \cong H_2(S_{std}; \mathbb{Z}/2)$. Note that every separating 2-sphere in S_{std} is unknotted in D_{std}, thus by Lemma 4.7.9, S and S' are related by a sequence of

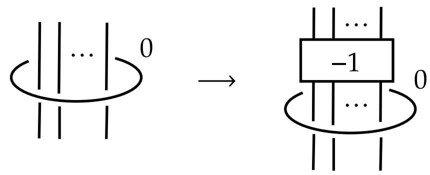


Figure 4.28: The Dehn twist on S_{std} along a core 2-sphere.

(a) unknotted 2-sphere creations,

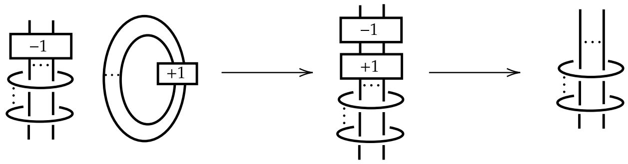
(b) ∂-parallel connected sums,

or their inverses, through ∂-parallel union of 2-spheres. This sequence of moves determines a canonical identification $(D_{std})_S = (D_{std})_{S'}$ rel boundary. Theorem 4.7.1(2) implies the independence of S under type (a) moves, while Lemma 4.7.10 and Lemma 4.7.5 imply the independence of S under type (b) moves.

In view of the commutative diagram (4.20), now it suffices to decompose ϕ into elementary diffeomorphisms and prove the theorem for each elementary one. The mapping class group of S_{std} is generated by the spin mapping class group $\pi_0(\mathrm{Diff}^{spin}(S_{std}))$ together with Dehn twists along each of the $\sum_{i=1}^k m_i$ core S^2 factors, each of which can be taken to have a standard form, visualized as in Figure 4.28 in the presence of an admissible link.

When $\phi \in \mathrm{Diff}^{spin}(S_{std})$, choose $S = \emptyset$. The statement follows from the spin case, Theorem 4.5.3.

When ϕ is the negative Dehn twist along the j-th S^2 factor in S_{std}, let $S \subset int(D_{std})$ be a slight pushin of the Dehn twist 2-sphere in S_{std} and $\tilde{\phi}\colon D_{std} \to (D_{std})_S$ be the natural diffeomorphism extending ϕ, given by pushing the effect of the twist towards the boundary. Take a standard lasagna filling $(I\times\phi(L)\cup(n_+,n_-)\text{ cores}, v)$, $v \in \widetilde{KhR}_2^+(\phi(L)\cup(n_+,n_-)\text{ belts})$ representing some element $x \in \mathcal{S}_0^{2,\mathrm{mul}}(D_{std};\phi(L))$, $n_\pm \in \mathbb{Z}^{\sum_{i=1}^k m_i}$. By construction of the Gluck twist map (4.17) in the proof of Theorem 4.7.1 and the alternative description of the element 1_+ preceding Lemma 4.7.4, the image of x under the composition map $\mathcal{S}_0^{2,\mathrm{mul}}(\tilde{\phi}^{-1})\circ \tau_{D_{std},\phi(L),S}$ is represented by the standard lasagna filling $(I\times L\cup(n_+,n_-)\text{ cores}, w)$, where $w \in \widetilde{KhR}_2^+(L\cup(n_+,n_-)\text{ belts})$ is the image of $v\otimes 1 \in \widetilde{KhR}_2^+(\phi(L)\cup(n_+,n_-)\text{ belts})\otimes \widetilde{KhR}_2^+(T(\ell,\ell)_{\ell_+,\ell_-})$ under the composition (shown near the j-th surgery region)



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where ℓ_+, ℓ_- denote the number of times L intersects the j-th surgery region positively, negatively, respectively, and $\ell = \ell_+ + \ell_-$. Since this map is distant from the surgery region, locality implies that the statement holds for $S, \tilde{\phi}$ with $\alpha = 0$ and $\widetilde{KhR}_2^+(\phi)$ given by saddling in $1 \in \widetilde{KhR}_2^+(T(\ell,\ell)_{\ell_+,\ell_-})$ right above the j-th projector, composed with Reidemeister-induced maps that cancel the ± 1 twists. This can be realized on the chain level as saddling in a cocycle in $\widetilde{CKhR}_2^+(T(\ell,\ell)_{\ell_+,\ell_-})$ representing the element 1. Alternatively, one can check that $\widetilde{KhR}_2^+(\phi)$ is induced by the chain homotopy equivalence that absorbs the -1 twist into the projector. $\square$

4.7.4 $\widetilde{KhR}_2^+$ on morphisms

We are finally able to prove Theorem 4.3.2.

Proof of Theorem 4.3.2. Let $(W, \Sigma)\colon (S_0, L_0) \to (S_1, L_1)$ be a morphism in $\mathbf{Links}_1$, where $W = X_1 \backslash int(X_0)$ for 4-dimensional 1-handlebodies X_0, X_1. As in the proof of Theorem 4.6.1, choose an orientation-preserving embedding $i\colon X_1 \hookrightarrow S^4$ and parametrizations $\tilde{\phi}_j\colon -(S^4 \backslash int(X_j)) \xrightarrow{\cong} D_{std,j} = \#_{i=1}^{k(j)} \natural^{m_i^{(j)}} (D^2 \times S^2)$ making $\phi_j := \tilde{\phi}_j|_{S_j}$ L_j-admissible, $j = 0, 1$. Then the map $\widetilde{KhR}_2^+(\tilde{\phi}_0(W), \tilde{\phi}_0(\Sigma))\colon \widetilde{KhR}_2^+(\phi_1(L_1)) \to \widetilde{KhR}_2^+(\phi_0(L_0))$ determines a map $\widetilde{KhR}_2^+(W, \Sigma)\colon \widetilde{KhR}_2^+(S_1, L_1) \to \widetilde{KhR}_2^+(S_0, L_0)$. We have to check that $\widetilde{KhR}_2^+(W, \Sigma)$ is independent of the embedding $i\colon X_1 \hookrightarrow S^4$. The functoriality will be automatic, as we may pick any spin structure on the relevant 4-dimensional 1-handlebodies and repeat the arguments in the proof of Theorem 4.6.1. Furthermore, it suffices to check the independence on i when (W, Σ) is an elementary morphism as described in Proposition 4.6.3.

Suppose $i'\colon X_1 \hookrightarrow S^4$ is another choice of embedding, and $\tilde{\phi}_j'\colon -(S^4 \backslash int(X_j)) \xrightarrow{\cong} D_{std,j}$ are parametrizations, $j = 0, 1$. Choose a diffeomorphism

$$\psi_1\colon -(S^4 \backslash int(i'(X_1))) \xrightarrow{\cong} (-(S^4 \backslash int(i(X_1))))_S$$

rel boundary for some ∂-parallel union of 2-spheres S, and let

$$\psi_0\colon -(S^4 \backslash int(i'(X_0))) \xrightarrow{\cong} (-(S^4 \backslash int(i(X_0))))_S$$

be the extension of ψ_1 rel W.

We have the following commutative diagram

$$\begin{tikzcd}
\mathcal{S}_0^{2,\mathrm{mul}}(D_{std,1};\phi_1(L_1)) \arrow[r, "\mathcal{S}_0^{2,\mathrm{mul}}(\tilde{\phi}_0(i(W^t));\tilde{\phi}_0(i(\Sigma^t)))"] & \mathcal{S}_0^{2,\mathrm{mul}}(D_{std,0};\phi_0(L_0)) \\
\mathcal{S}_0^{2,\mathrm{mul}}(-(S^4\backslash int(i(X_1)));L_1) \arrow[r, "\mathcal{S}_0^{2,\mathrm{mul}}(i(W^t);i(\Sigma^t))"] \arrow[u, "\mathcal{S}_0^{2,\mathrm{mul}}(\tilde{\phi}_1)", "\cong"'] \arrow[d, "\tau_{-(S^4\backslash int(i(X_1))),L_1,S}"', "\cong"] & \mathcal{S}_0^{2,\mathrm{mul}}(-(S^4\backslash int(i(X_0)));L_0) \arrow[u, "\cong", "\mathcal{S}_0^{2,\mathrm{mul}}(\tilde{\phi}_0)"'] \arrow[d, "\cong"', "\tau_{-(S^4\backslash int(i(X_0))),L_0,S}"] \\
\mathcal{S}_0^{2,\mathrm{mul}}((-(S^4\backslash int(i(X_1))))_S;L_1) \arrow[r, "\mathcal{S}_0^{2,\mathrm{mul}}(i(W^t);i(\Sigma^t))"] & \mathcal{S}_0^{2,\mathrm{mul}}((-(S^4\backslash int(i(X_0))))_S;L_0) \\
\mathcal{S}_0^{2,\mathrm{mul}}(-(S^4\backslash int(i'(X_1)));L_1) \arrow[r, "\mathcal{S}_0^{2,\mathrm{mul}}(i'(W^t);i'(\Sigma^t))"] \arrow[u, "\mathcal{S}_0^{2,\mathrm{mul}}(\psi_1)", "\cong"'] \arrow[d, "\mathcal{S}_0^{2,\mathrm{mul}}(\tilde{\phi}_1')"', "\cong"] & \mathcal{S}_0^{2,\mathrm{mul}}(-(S^4\backslash int(i'(X_0)));L_0) \arrow[u, "\cong", "\mathcal{S}_0^{2,\mathrm{mul}}(\psi_0)"'] \arrow[d, "\cong"', "\mathcal{S}_0^{2,\mathrm{mul}}(\tilde{\phi}_0')"] \\
\mathcal{S}_0^{2,\mathrm{mul}}(D_{std,1};\phi_1'(L_1)) \arrow[r, "\mathcal{S}_0^{2,\mathrm{mul}}(\tilde{\phi}_0'(i'(W^t));\tilde{\phi}_0'(i'(\Sigma^t)))"] & \mathcal{S}_0^{2,\mathrm{mul}}(D_{std,0};\phi_0'(L_0)).
\end{tikzcd}$$

Here, the composition down the first column gives the lasagna defining map for $\widetilde{KhR}_2^+(\phi_1\circ\phi_1'^{-1})\colon \widetilde{KhR}_2^+(\phi_1(L_1))\to\widetilde{KhR}_2^+(\phi_1'(L_1))$ in the sense of Theorem 4.7.7, because one can commute the first two isomorphisms.

Similarly, we claim that the composition down the second column serves as lasagna defining map for $\widetilde{KhR}_2^+(\phi_0\circ\phi_0'^{-1})$, which will finish the proof that $\widetilde{KhR}_2^+(W,\Sigma)$ is well-defined. Since we have assumed (W,Σ) to be an elementary morphism as described in Proposition 4.6.3, we check for each case.

(i)(v): $S\subset\partial_+W$ is parallel to ∂_-W.

(ii): Up to capping off some unknots in W (which does not affect the composite map by Theorem 4.7.1(2)), S is parallel to ∂_-W.

(iii)(iv): One can surger as in Lemma 4.7.8 to make $S\subset\partial_+W$ disjoint from the belt sphere of the 1-handle (which does not affect the induced map on $\widetilde{KhR}_2^+$ by Lemma 4.7.10 and Lemma 4.7.5), so that it is parallel to ∂_-W.

The proof is complete. $\square$

Appendix A

Lee canonical generators

We explain our choice of canonical generators $x_{\mathfrak{o}} \in KhR_{Lee}(L)$ in $\mathfrak{gl}_2$ Lee homology in Section 2.6.1 and prove Proposition 2.6.1. To streamline the procedure, we define these canonical generators using the formalism in [MWW24]; see especially their Section 3.1 and Section 3.2. Note that [MWW24] uses a different renormalization convention from ours; see their Remark 2.3. We will also remark on a description of the canonical generators $x_{\mathfrak{o}}$ up to sign in terms of Rasmussen's generators [Ras10] in the usual Lee homology.

As homologically $\mathbb{Z}$-graded vector spaces, the $\mathfrak{gl}_2$ Lee homology canonically and functorially decomposes into (twisted) $\mathfrak{gl}_1$ homologies [MWW24, Equation (1)]:

$$KhR_{Lee}(L) \cong \bigoplus_{\mathfrak{o}} t^{-2\ell k(L_+(\mathfrak{o}),L_-(\mathfrak{o}))}(KhR_1^{(2)}(L_+(\mathfrak{o});\mathbb{Q}) \otimes KhR_1^{(-2)}(L_-(\mathfrak{o});\mathbb{Q})). \tag{A.1}$$

Here, $\mathfrak{o}$ runs over orientations of L as an unoriented link, and $L_+(\mathfrak{o})$ (resp. $L_-(\mathfrak{o})$) is the sublink of L consisting of components whose orientations agree (resp. disagree) with $\mathfrak{o}$. The functor KhR_1 is the Khovanov–Rozansky $\mathfrak{gl}_1$ homology. It is determined by canonical graded isomorphisms $KhR_1(L) \cong \mathbb{Z}$ for each framed oriented link $L \subset S^3$ where $\mathbb{Z}$ is concentrated in (homological, quantum) bidegree $(0,0)$, such that the vacuum state in $KhR_1(\emptyset)$ corresponds to $1 \in \mathbb{Z}$; under these identifications, all Reidemeister I-induced framing-change maps induce $\mathrm{id}_{\mathbb{Z}}\colon \mathbb{Z} \to \mathbb{Z}$, and $KhR_1(\Sigma) = \mathrm{id}_{\mathbb{Z}}\colon \mathbb{Z} \to \mathbb{Z}$ for each framed oriented cobordism $\Sigma\colon L_0 \to L_1$. Alternatively, KhR_1 is the universal trivial link homology theory H presented in Section 2.5 in Chapter 2, concentrated in bidegree $(0,0)$. The superscript $(\cdot)^{(d)}$, $d \in \mathbb{Q}^\times$ indicates a d-twist on morphisms according to the Euler characteristic; explicitly, one may renormalize so that there are canonical graded isomorphisms

$$KhR_1^{(d)}(L;\mathbb{Q}) \cong \mathbb{Q}$$

for any framed oriented link $L \subset S^3$ where $\mathbb{Q}$ is concentrated in bidegree $(0,0)$, under which the vacuum in $KhR_1^{(d)}(\emptyset;\mathbb{Q})$ corresponds to $1 \in \mathbb{Q}$; under these identifications the Reidemeister I-induced framing-change maps each induce $\mathrm{id}_{\mathbb{Q}}\colon \mathbb{Q} \to \mathbb{Q}$, and

$$KhR_1^{(d)}(\Sigma;\mathbb{Q}) = d^{n(\Sigma)}\mathrm{id}_{\mathbb{Q}}\colon \mathbb{Q} \to \mathbb{Q}$$

for any framed oriented link cobordism $\Sigma\colon L_0 \to L_1$, where

$$n(\Sigma) = \frac{-\chi(\Sigma) + \#\partial_+\Sigma - \#\partial_-\Sigma}{2} \in \mathbb{Z}$$

is a normalized Euler characteristic as defined in (2.14), which is additive under gluing cobordisms.

In the formulation of [MWW24], the orientation $\mathfrak{o}$ in (A.1) corresponds to a coloring of components of L by $\{\pm 1\}$, so that $L_\pm(\mathfrak{o})$ is colored by ± 1. The colors $1, -1$ are also regarded as colors by idempotents $\mathbf{a} = (1+X)/2, \mathbf{b} = (1-X)/2$ in $V_{Lee} = \mathbb{Q}[X]/(X^2-1)$, respectively.

Note that one does not recover the quantum $\mathbb{Z}/4$-grading and the quantum filtration on $KhR_{Lee}(L)$ from (A.1).

Definition A.0.1. The canonical Lee generator $x_\mathfrak{o} \in KhR_{Lee}(L)$, under the isomorphism (A.1), is the element

$$1 \otimes 1 \in KhR_1^{(2)}(L_+(\mathfrak{o}); \mathbb{Q}) \otimes KhR_1^{(-2)}(L_-(\mathfrak{o}); \mathbb{Q})$$

with homological degree shifted by $-2\ell k(L_+(\mathfrak{o}), L_-(\mathfrak{o}))$, in the $\mathfrak{o}$-th direct summand of the right-hand side of (A.1).

Remark A.0.2. There is a more natural renormalization for $KhR_1^{(d)}$, but it requires choosing a square root $d^{1/2}$ of d, and extending the coefficient field to $\mathbb{Q}(d^{1/2})$ if necessary. Namely, one may renormalize so that $KhR_1^{(d)}(L; \mathbb{Q}(d^{1/2})) \cong \mathbb{Q}(d^{1/2})$ and $KhR_1^{(d)}(\Sigma; \mathbb{Q}(d^{1/2})) = d^{-\chi(\Sigma)/2}\mathrm{id}_{\mathbb{Q}(d^{1/2})}$. We will not use this renormalization as this would require that we work over $\mathbb{Q}(\sqrt{2}, i)$.

The isomorphism (A.1) is functorial in L in the sense that if $S\colon L_0 \to L_1$ is a framed oriented cobordism, then $KhR_{Lee}(S)\colon KhR_{Lee}(L_0) \to KhR_{Lee}(L_1)$ is determined in the canonical bases by the matrix entries

$$KhR_{Lee}(S)_{x_{\mathfrak{o}_0}, x_{\mathfrak{o}_1}} = \sum_{\substack{\mathfrak{O}|_{L_0} = \mathfrak{o}_0 \\ \mathfrak{O}|_{L_1} = \mathfrak{o}_1}} 2^{n(S_+(\mathfrak{O}))}(-2)^{n(S_-(\mathfrak{O}))} = 2^{n(S)} \sum_{\substack{\mathfrak{O}|_{L_0} = \mathfrak{o}_0 \\ \mathfrak{O}|_{L_1} = \mathfrak{o}_1}} (-1)^{n(S_-(\mathfrak{O}))}, \tag{A.2}$$

where $\mathfrak{o}_i$ runs over orientations of L_i, $i = 0, 1$, $\mathfrak{O}$ runs over orientations of S compatible with $\mathfrak{o}_0, \mathfrak{o}_1$, and $S_\pm(\mathfrak{O})$ denotes the union of components of S whose orientations agree/differ with that of $\mathfrak{O}$.

Remark A.0.3. More classically, if D is a diagram of an oriented link L and $\mathfrak{o}$ is an orientation of L, Rasmussen [Ras10] defined a cocycle $\mathfrak{s}_\mathfrak{o}(D) \in CKh_{Lee}(D)$, whose class $[\mathfrak{s}_\mathfrak{o}(D)] \in Kh_{Lee}(L)$ in the Lee homology of L is independent of the choice of D up to a scalar of the form $\pm 2^k$, $k \in \mathbb{Z}$. Better, Rasmussen [Ras05] defined a renormalized cocycle $\widetilde{\mathfrak{s}_{\mathfrak{o}}}(D) := 2^{(w(D)-r(D))/2}\mathfrak{s}_{\mathfrak{o}}(D) \in CKh_{Lee}(D) \otimes \mathbb{Q}(\sqrt{2})$, where $w(D)$ is the writhe of D, and $r(D)$ is the number of Seifert circles in the oriented resolution of D. Its class $[\widetilde{\mathfrak{s}_{\mathfrak{o}}}] = [\widetilde{\mathfrak{s}_{\mathfrak{o}}}(D)] \in Kh_{Lee}(L) \otimes \mathbb{Q}(\sqrt{2})$ is well-defined up to sign, and behaves nicely under cobordisms as described in [Ras05, Proposition 3.2][1], consistent with the renormalization in Remark A.0.2. We have used these renormalized generators in Section 1.4 in Chapter 1. One can check, analogously to the proof of item (4) below, that the canonical generator $x_{\mathfrak{o}} \in KhR_{Lee}(L)$ for a framed oriented link L in Definition A.0.1 corresponds, up to sign, to the element $2^{\ell k(L_+(\mathfrak{o}), L_-(\mathfrak{o})) - \#L/2}[\widetilde{\mathfrak{s}_{-\mathfrak{o}}}] \in Kh_{Lee}(-L)$ under the canonical (up to sign) isomorphism $KhR_{Lee}(L) \cong Kh_{Lee}(-L)$ (cf. (2.11); be aware that this isomorphism may involve conjugations on the Frobenius algebra V_{Lee}, see [Bel+23]), with changes in gradings and filtration suppressed.

Proof of Proposition 2.6.1. (3) follows from the definition. (5) is a reformulation of (A.2).

To prove the other items, one needs to understand the isomorphism (A.1) more explicitly, as described in [MWW24]. We sketch the arguments below. With appropriate (intricate) sign fixes, one may also prove these in terms of Rasmussen's generators.

(1) The canonical isomorphism $KhR_{Lee}(\emptyset) \cong KhR_1^{(2)}(\emptyset; \mathbb{Q}) \otimes KhR_1^{(-2)}(\emptyset; \mathbb{Q})$ is determined by demanding that the vacuum corresponds to the vacuum. The canonical generator $x_\emptyset \in KhR_{Lee}(\emptyset)$ thus corresponds to the vacuum $1 \in \mathbb{Q}$.

(2) For $(U, \mathfrak{o})$ the 0-framed oriented unknot, let D be the spanning disk of U in S^3, with interior pushed slightly into $int(B^4)$. Then the canonical isomorphism $KhR_{Lee}(U) \cong q^{-1}V_{Lee}$ of homologically graded, quantum filtered vector spaces is determined by identifying the class in $KhR_{Lee}(U)$ born from the $\mathbf{a}, \mathbf{b}$-colored disk D with $\mathbf{a}, \mathbf{b}$ (with filtration degree shifts), respectively. On the other hand, the class represented by the $\mathbf{a}$-colored (resp. $\mathbf{b}$-colored) disk corresponds to $1 \otimes 1$ in the $\mathfrak{o}$-summand (resp. $\bar{\mathfrak{o}}$-summand) on the right-hand side of (A.1). This proves the statements for the 0-framed unknot. For a general n-framed unknot $(U^n, \mathfrak{o})$, the statements follow from the fact that Reidemeister-I-induced framing-change maps induce a canonical isomorphism $KhR_{Lee}(U^n) \cong q^{-n}KhR_{Lee}(U)$, and that (A.1) intertwines with the Reidemeister-I-induced maps on the two sides.

(4) For $L = \emptyset$, the conclusion follows from (1) (note that $0 \in \mathbb{Q}$ is homogeneous of arbitrary degree). For L the n-framed unknot, the conclusion follows from (2).

Next, we check the statement for L the positive (resp. negative) Hopf link where both components have zero framings, and $\mathfrak{o}$ an orientation on L making it a negative (resp. positive) Hopf link. The corresponding canonical generator $x_{\mathfrak{o}}$ is denoted h_+ (resp. h_-) in [MWW24], which involves a scalar choice. Note that $x_{\mathfrak{o}} = h_+$ determines $x_{\bar{\mathfrak{o}}}$ by symmetry, and h_+ determines h_- by (A.2). Moreover, $q_{\mathbb{Z}/4}(x_{\mathfrak{o}} \pm x_{\bar{\mathfrak{o}}})$ is independent of the scalar choice of

[1]The equation in [Ras05, Proposition 3.2] missed a $\frac{1}{2}$ multiplicative factor on the exponent on the right-hand side.

h_+. Thus, for convenience, we may pick the diagram of L consisting of two counterclockwise oriented crossingless circles, with exactly 2 positive (resp. negative) crossings between them, and assign any $x_{\mathfrak{o}}$ for the purpose of proving (4). The advantage of this choice is that there is a planar isotopy of the diagram exchanging the two components of L, sending $\mathfrak{o}$ to $\bar{\mathfrak{o}}$. With this choice of diagram, one may write down explicit representatives for $x_{\mathfrak{o}}$ and $x_{\bar{\mathfrak{o}}}$ analogously to [MWW24, Example 3.7] and verify the claims about quantum $\mathbb{Z}/4$ degrees.

If $L = L_1 \sqcup L_2$ is a split link, $\mathfrak{o}_1, \mathfrak{o}_2$ are orientations on L_1, L_2, and $\mathfrak{o}$ is the corresponding orientation on L, then

$$x_{\mathfrak{o}} \pm x_{\bar{\mathfrak{o}}} = x_{\mathfrak{o}_1} \otimes x_{\mathfrak{o}_2} \pm x_{\overline{\mathfrak{o}_1}} \otimes x_{\overline{\mathfrak{o}_2}} = \tfrac{1}{2}((x_{\mathfrak{o}_1} + x_{\overline{\mathfrak{o}_1}}) \otimes (x_{\mathfrak{o}_2} \pm x_{\overline{\mathfrak{o}_2}}) + (x_{\mathfrak{o}_1} - x_{\overline{\mathfrak{o}_1}}) \otimes (x_{\mathfrak{o}_2} \mp x_{\overline{\mathfrak{o}_2}})).$$

Hence if $(L_1, \mathfrak{o}_1)$ and $(L_2, \mathfrak{o}_2)$ both satisfy the claims on quantum $\mathbb{Z}/4$ gradings, so does $(L, \mathfrak{o})$.

Finally, for any pair $(L, \mathfrak{o})$ where L is a framed oriented link, and $\mathfrak{o}$ is an orientation of L as an unframed link, there exists a cobordism $S\colon L \to L'$ so that

- L' is a disjoint union of some unknots with some framings, and some 0-framed positive and negative Hopf links;
- there exists a unique orientation $\mathfrak{O}$ of S compatible with $\mathfrak{o}$ on L, whose restriction on L', denoted $\mathfrak{o}'$, does not agree with the orientation of any Hopf link disjoint summands or their opposites.

In particular, $(L', \mathfrak{o}')$ satisfies the claims on quantum $\mathbb{Z}/4$ degrees thanks to the previous arguments. On the other hand, $KhR_{Lee}(S)$ is homogeneous of quantum $\mathbb{Z}/4$ degree $-\chi(S)$, so

$$\begin{aligned} q_{\mathbb{Z}/4}(x_{\mathfrak{o}} \pm x_{\bar{\mathfrak{o}}}) &= q_{\mathbb{Z}/4}(KhR_{Lee}(S)(x_{\mathfrak{o}} \pm x_{\bar{\mathfrak{o}}})) + \chi(S) \\ &= q_{\mathbb{Z}/4}(2^{n(S)}(-1)^{n(S_-(\mathfrak{O}))}(x_{\mathfrak{o}'} \pm (-1)^{n(S)} x_{\overline{\mathfrak{o}'}})) + \chi(S) \\ &= q_{\mathbb{Z}/4}(x_{\mathfrak{o}'} \pm x_{\overline{\mathfrak{o}'}}) + 2n(S) + \chi(S) \\ &= -w(L) - \#L - 1 \pm 1, \end{aligned}$$

as desired. □

Appendix B

Proof of comparison lemma

In this appendix, we prove Lemma 3.4.9, which was used in Section 3.4.4 in Chapter 3 in the proof of the diagrammatic nonvanishing result for Khovanov skein lasagna modules of 2-handlebodies with sufficiently small framings.

We first rewrite Lemma 3.4.9 in terms of the classical Khovanov homology. For a framed link L, let $Q(L)(t,q)$ denote the Poincaré polynomial of $Kh(L)$, and $\tilde{Q}(L)(t,q) := (tq^3)^{-\frac{1}{2}w(L)}Q(L)$. Then by (2.11), $P(L)(t,q) = q^{-w(L)}Q(-L)(t,q^{-1})$ and $\tilde{P}(L)(t,q) = \tilde{Q}(-L)(t,q^{-1})$. Thus Lemma 3.4.9 is equivalent to the following Khovanov version. For $p,w \in \mathbb{Z}^m$, let P,W be diagonal matrices with entries given by p,w.

Lemma B.0.1 (Comparison lemma).

(1) For $n \in \mathbb{Z}^m_{\geq 0}$, $p,w \in \mathbb{Z}^m$, $p \geq w$, we have

$$\tilde{Q}(K^p(n)) \leq \sum_{\substack{0 \leq n' \leq n \\ 2|n-n'}} R_{n,n',p-w}\tilde{Q}(K^w(n'))$$

for some polynomials $R_{n,n',p-w} \in \mathbb{Z}[t^{\pm 1/2}, q^{\pm 1/2}]$ independent of K with

$$R_{n,n',p-w}(t,q) = (tq)^{-\frac{1}{2}n'^T(P-W)n'}q^{-\langle p-w,n'\rangle} \sum_{\substack{n' \leq n'' \leq n \\ 2|n-n''}} \dim\left(\frac{n+n''}{2}, \frac{n-n''}{2}\right) q^{|n'|-|n''|} + \cdots$$

where $\cdots$ are terms with lower t-degrees.

(2) For $n \in \mathbb{Z}^m_{\geq 0}$, $p,w \in \mathbb{Z}^m$, $j \in \{1,\cdots,m\}$, with $p - e_j \geq w$, we have

$$(tq)^{n_j/2}q^{-1}\tilde{Q}(K^{p-e_j/n_j}(n)) \leq \sum_{\substack{0 \leq n' \leq n \\ 2|n-n' \\ n'_j = n_j}} R_{n,n',p-w}\tilde{Q}(K^w(n')).$$

Lemma B.0.1 can be reduced to an elementary piece. Suppose $J \cup L \subset S^3$ is a framed oriented link, where J is a knot and L is a link. Let J^1 be J with framing one higher. For $n \geq 0$, let $J(n) \cup L$ denote the link obtained from $J \cup L$ by replacing J by an n-cable of itself, and $J^1(n) \cup L$ similarly. The link $J^{1-1/n}(n) \cup L$ is obtained from $J^1(n) \cup L$ by inserting a $(-1/n)$-twist along cables around J (with writhe also dropped by n).

Lemma B.0.2 (Elementary comparison lemma).

(1) In the notations above,

$$\tilde{Q}(J^1(n) \cup L) \leq \sum_{\substack{0 \leq n' \leq n \\ 2 | n-n'}} R_{n,n'} \tilde{Q}(J(n') \cup L) \tag{B.1}$$

for some polynomials $R_{n,n'} \in \mathbb{Z}[t^{\pm 1/2}, q^{\pm 1/2}]$ independent of $J \cup L$ with

$$R_{n,n'}(t,q) = (tq)^{-\frac{n'^2}{2}} q^{-n'} \sum_{r=0}^{\frac{n-n'}{2}} \dim \left(\frac{n+n'}{2} + r, \frac{n-n'}{2} - r \right) q^{-2r} + \cdots$$

where $\cdots$ are terms with lower t-degrees.

(2) In the notations above,

$$(tq)^{-n/2} q^{-1} \tilde{Q}(J^{1-1/n}(n) \cup L) \leq (tq)^{-\frac{n^2}{2}} q^{-n} \tilde{Q}(J(n) \cup L). \tag{B.2}$$

It is easy to check that Lemma B.0.1 is obtained by iteratively applying Lemma B.0.2 to components of K until the framing drops to w. Therefore we only prove Lemma B.0.2.

Before starting the proof, we introduce an auxiliary family of links defined by Stošić [Sto07; Sto09] and re-exploited by Tagami [Tag13] (our notation aligns with Section 1.5 in Chapter 1, which is different from Stošić's or Tagami's).

For $n, m \geq 0$, $0 \leq i \leq n-1$, let $D^i_{n,m}$ denote the braid closure of $(\sigma_1 \cdots \sigma_{n-1})^m \sigma_1 \cdots \sigma_i \in B_n$, regarded as an annular link. Thus $D^0_{n,m+1} = D^{n-1}_{n,m}$, $D^0_{n,n}$ is the annular torus link $T(n,n)$, and $D^i_{n,m} = \emptyset$ if $n = 0$. For $i > 0$, in the standard diagram of $D^i_{n,m}$ defined by the braid word, resolving the last letter σ_i leads to the 0-resolution which is exactly $D^{i-1}_{n,m}$, and some 1-resolution, denoted $E^{i-1}_{n,m}$.

Lemma B.0.3. *The annular link $E^{i-1}_{n,m}$ is isotopic to some $D^{i_0}_{n_0,m_0}$ or its flip (i.e. the braid closure of $\sigma_{n_0-i_0} \cdots \sigma_{n_0-1}(\sigma_1 \cdots \sigma_{n_0-1})^{m_0}$, still denoted $D^{i_0}_{n_0,m_0}$ by an abuse of notation), possibly disjoint union with a local unknot. Moreover,*

(1) $0 \leq n_0 < n$, $2 | n - n_0$.

(2) If $m < n$, then we may assume $m_0 < n_0$ or $n_0 = 0$.

(3) $E^{n-2}_{n,n-1} \simeq D^{n-3}_{n-2,n-3} \sqcup U$ and $E^i_{n,n-1} \simeq D^i_{n-2,n-3}$ if $i < n-2$. Here, we define $D^{-1}_{0,-1} = D^0_{0,0}$.

(4) If $m < n$, then the number of negative crossings in the 1-resolution of $D^i_{n,m}$ oriented via $D^{i_0}_{n_0,m_0}(\sqcup U)$, denoted $\omega^i_{n,m}$, is less than or equal to $\frac{n^2-n_0^2}{2} - 1$, with equality if and only if $m = n-1$.

Sketch proof. The fact that $E^{i-1}_{n,m}$ is isotopic to some $D^{i_0}_{n_0,m_0}$ or $D^{i_0}_{n_0,m_0} \sqcup U$ is checked carefully in [Tag13, Figure 15˜20], from which (1) and (2) are also clear. (3) is exactly a half of Lemma 1.5.1.

For (4), when $m = n-1$ there are exactly $2n-3 = \frac{n^2-(n-2)^2}{2} - 1$ negative crossings as claimed in the paragraph following Lemma 1.5.1.

When $m < n-1$, we count the contribution as the turnback arc in Figure 15 to Figure 18 of [Tag13] travels along the longitude. Note that the whole isotopy $E^{i-1}_{n,m} \simeq D^{i_0}_{n_0,m_0}(\sqcup U)$ is decomposed into $\frac{n-n_0}{2}$ full longitude traversals of the turnback arc and an extra final move according to the four types listed in Figure 15 to Figure 18 of [Tag13].

Interpret n as the number of through strands in the bottom box and m as the number of full $\sigma_1 \cdots \sigma_{n-1}$ rotations. Then each full longitude travel decreases n by 2, decreases m by 0 or 2 (other than the last longitude travel of Type 4, which will be discussed below). The negative crossing contribution of this full longitude travel is $m+n-2+\epsilon$ if m is decreased by 2 and $m+\epsilon$ if m is not decreased, where $\epsilon = 0$ or 1 is the contribution outside the bottom box. The last longitude travel has $\epsilon = 0$. The negative crossing contribution from the final move is 0 for type $1, 2$, and at most $n-1$ for type $3, 4$ (for these two types, the last longitude travel decreases m by either 0 or 1, contributing m negative crossings). In total, the number of negative crossings is at most

$$\sum_{r=0}^{(n-n_0)/2-1} ((m-2r)+n-2r-1) \leq \sum_{r=0}^{(n-n_0)/2-1} (2n-4r-3) = \frac{n^2-n_0^2}{2} - \frac{n-n_0}{2} \leq \frac{n^2-n_0^2}{2} - 1.$$

If the equality holds, we must have $m = n-2$ and $n_0 = n-2$. This implies $i = n-1$ or $n-2$. One can check $E^{n-2}_{n,n-2} \simeq D^{n-3}_{n-2,n-4} \simeq E^{n-3}_{n,n-2}$, and the 1-resolutions of $D^{n-1}_{n,n-2}$, $D^{n-2}_{n,n-2}$ have $2n-4, 2n-5$ crossings, respectively, which are strictly less than $\frac{n^2-(n-2)^2}{2} - 1$. $\square$

Proof of Lemma B.0.2. Let $D^i_{n,m}(J) \cup L \subset S^3$ be the link obtained by replacing J by its $D^i_{n,m}$ satellite. For $i > 0$, the resolution of $D^i_{n,m}$ at the last letter σ_i induces a resolution of $D^i_{n,m}(J) \cup L$ into $D^{i-1}_{n,m}(J) \cup L$ and $E^{i-1}_{n,m}(J) \cup L$, the latter of which is isotopic to some $D^{i_0}_{n_0,m_0}(J) \cup L$ possibly disjoint union with an unknot.

Fix a framed link diagram of $J \cup L$. Then $D^i_{n,m}(J) \cup L$ have (almost) standard diagrams by taking the blackboard n-cable of J and inserting the standard diagrams of $D^i_{n,m}$ locally.

Then, the number of negative crossings of $D^i_{n,m}(J) \cup L$ is

$$n^2 \cdot n_-(J) + n \cdot n_-(J,L) + n_-(L)$$

where $n_\pm(J), n_\pm(L), n_\pm(J,L)$ are the numbers of positive/negative crossings of J, of L, and between J, L, respectively. Since the orientations of $\frac{n-n_0}{2}$ of the strands are reversed in the 1-resolution of $D^i_{n,m}(J) \cup L$, the number of negative crossings in the 1-resolution is

$$\left((\frac{n+n_0}{2})^2 + (\frac{n-n_0}{2})^2\right) n_-(J) + 2 \cdot \frac{n+n_0}{2} \cdot \frac{n-n_0}{2} \cdot n_+(J) + \omega^i_{n,m}$$
$$+ \frac{n+n_0}{2} n_-(J,L) + \frac{n-n_0}{2} n_+(J,L) + n_-(L),$$

The difference between the numbers of negative crossings of the 1-resolution of $D^i_{n,m}(J) \cup L$ and $D^i_{n,m}(J) \cup L$ itself is

$$\omega^i_{n,m}(J,L) := \frac{n^2 - n_0^2}{2} w(J) + (n - n_0)\ell k(J,L) + \omega^i_{n,m}. \tag{B.3}$$

The skein exact triangle on Khovanov homology corresponding to the resolution is

$$\begin{tikzcd} t^{\omega^i_{n,m}(J,L)+1} q^{3\omega^i_{n,m}(J,L)+2} Kh(E^{i-1}_{n,m}(J) \cup L) \arrow[rr] & & Kh(D^i_{n,m}(J) \cup L) \arrow[dl] \\ & q^1 Kh(D^{i-1}_{n,m}(J) \cup L), \arrow[ul, "{[1]}"] & \end{tikzcd} \tag{B.4}$$

which implies that

$$Q(D^i_{n,m}(J) \cup L) \le qQ(D^{i-1}_{n,m}(J) \cup L) + t^{\omega^i_{n,m}(J,L)+1} q^{3\omega^i_{n,m}(J,L)+2} Q(E^{i-1}_{n,m}(J) \cup L). \tag{B.5}$$

Start with $J^1(n) \cup L = D^{n-1}_{n,n-1}(J) \cup L$ and apply (B.5) or the identity $D^0_{n,m+1} = D^{n-1}_{n,m}$ iteratively. By Lemma B.0.3, each $E^{i-1}_{n,m}$ is some simpler $D^{i_0}_{n_0,m_0}$ (possibly with an unknot component which contributes a multiplicative factor of $q + q^{-1}$ to Q), and thus we must end with an inequality

$$Q(J^1(n) \cup L) \le \sum_{\substack{0 \le n' \le n \\ 2 | n-n'}} S_{n,n',J,L} Q(J(n') \cup L) \tag{B.6}$$

for some polynomials $S_{n,n',J,L}$. These polynomials are determined as follows. Define a graph with vertex set $(\{(n,m,i) : 0 \le i < n,\ 0 \le m < n\} \cup \{(0,0,0)\})/\sim$ where the equivalence relation identifies $(n,m,0)$ with $(n,m-1,n-1)$. For $i > 0$, construct a directed edge $(n,m,i) \to (n,m,i-1)$ with weight q, and a directed edge $(n,m,i) \to (n_0,m_0,i_0)$ with weight $t^{\omega^i_{n,m}+1} q^{3\omega^i_{n,m}+2}$ if $E^{i-1}_{n,m} \simeq D^{i_0}_{n_0,m_0}$ and weight $t^{\omega^i_{n,m}+1} q^{3\omega^i_{n,m}+2}(q + q^{-1})$ if $E^{i-1}_{n,m} \simeq D^{i_0}_{n_0,m_0} \sqcup U$

(here, we demand $(n_0, m_0, i_0) = (0, 0, 0)$ if $D^{i_0}_{n_0,m_0} = \emptyset$ or U). The weight of a directed path from (n, m, i) to (n', m', i') is the product of its edge weights, with an extra multiplicative factor $(tq^3)^{\frac{n^2-n'^2}{2}w(J)+(n-n')\ell k(J,L)}$ to account for the renormalization factors in (B.3). Then $S_{n,n',J,L}$ is the sum of the path weights over directed paths from $(n, n, 0)$ to $(n', 0, 0)$.

By Lemma B.0.3(4), the t-degree of every path $(n, n, 0) \to (n', 0, 0)$ is at most $\frac{n^2-n'^2}{2}w(J) + (n - n')\ell k(J, L) + \frac{n^2-n'^2}{2}$, with equality achieved if and only if every edge on it of the form $(n, m, i) \to (n_0, m_0, i_0)$ that decreases n has $m = n - 1$. We can also enumerate all paths that achieve equality thanks to Lemma B.0.3(3). A few path-counting combinatorics then yield (cf. Figure 1.5)

$$S_{n,n',J,L}(t, q) = (tq^3)^{\frac{n^2-n'^2}{2}w(J)+(n-n')\ell k(J,L)} t^{\frac{n^2-n'^2}{2}} q^{\frac{3n^2-n'^2}{2}-n'} \sum_{r=0}^{\frac{n-n'}{2}} \dim\left(\frac{n+n'}{2} + r, \frac{n-n'}{2} - r\right) q^{-2r} + \cdots$$

where $\cdots$ are terms with lower t-degrees. Since $w(J^1(n) \cup L) = n^2(w(J) + 1) + 2n\ell k(J, L) + w(L)$ and $w(J(n') \cup L) = n'^2 w(J) + 2n'\ell k(J, L) + w(L)$, (B.6) renormalizes to yield (B.1). The proof of (B.2) is analogous but much easier, and is indeed a sub-counting of the path-counting for (B.1) (there is a unique path to be counted). □

Appendix C

Sign fixes

In this appendix, we prove Theorem C.2.1, the precise version of Theorem 4.1.6, and explain how to use the $\mathfrak{gl}_2$ webs and foams formalism to fix various sign ambiguities appearing in Chapter 4.

In Section C.1, we recall the topological setup of $\mathfrak{gl}_2$ webs in S^3 and $\mathfrak{gl}_2$ foams between them, and introduce singular $\mathfrak{gl}_2$ foams that are of interest to us. In Section C.2, we show that the closed Lee foam evaluation in $\mathbb{R}^4$ agrees with the abstract Lee foam evaluation, assign maps on $\mathfrak{gl}_2$ homology induced by singular foams, and deduce Theorem C.2.1. In Section C.3, we sketch the definition of $\mathfrak{gl}_2$ Rozansky projectors. Throughout, if not otherwise indicated, we work over $\mathbb{Z}$, except finally in Section C.4 where we work over R as in the setup of Chapter 4 and address all sign issues in the proof of Theorem 4.1.2 in Chapter 4.

C.1 $\mathfrak{gl}_2$ webs and singular $\mathfrak{gl}_2$ foams

We set up the notion of (embedded) $\mathfrak{gl}_2$ webs and foams relevant to us. See [QW24] for a more general topological setup.

For us, a $\mathfrak{gl}_2$ *web* in $\mathbb{R}^3$ (resp. S^3) is an embedded trivalent graph $W \subset \mathbb{R}^3$ (resp. S^3) together with the following data:

(1) A label 1 or 2 on each edge;

(2) An orientation on each edge;

(3) An oriented ribbon R of W, i.e. a smoothly embedded oriented surface $R \subset \mathbb{R}^3$ (resp. S^3) that has W as its core.

The data are subject to the following constraints[1]:

(1) At each vertex, two of the adjacent edges are labeled 1 and one is labeled 2;

(2) At each vertex, the two 1-labeled edges induce the same orientation on the vertex, which is opposite to the orientation induced by the 2-labeled edge.

Each vertex of a $\mathfrak{gl}_2$ web W is assigned the orientation induced from the 2-labeled edge adjacent to it. Moreover, the orientation of the ribbon R at a vertex coupled with the vertex orientation induces a cyclic ordering of the three adjacent edges: for a positive vertex, we take the cyclic ordering determined by the orientation of the ribbon, and for a negative one the opposite.

A *(regular)* $\mathfrak{gl}_2$ *foam* between $\mathfrak{gl}_2$ webs W_0, W_1 in $\mathbb{R}^3$ (resp. S^3) is an embedded singular surface $F \subset I \times \mathbb{R}^3$ (resp. $I \times S^3$) cobounding $\{0\} \times W_0$ and $\{1\} \times W_1$, whose interior points have local neighborhood diffeomorphic to either $\mathbb{R}^2 \subset \mathbb{R}^4$ or $Y \times \mathbb{R} \subset \mathbb{R}^4$, where $Y \subset \mathbb{R}^2 \subset \mathbb{R}^3$ is the neighborhood of a trivalent vertex of an embedded graph in $\mathbb{R}^2$ (a "Y shape"), together with the following additional data:

(1) A label 1 or 2 on each face;

(2) An orientation on each face;

(3) An oriented ribbon R of F, i.e. a smoothly properly embedded oriented 3-manifold $R \subset I \times \mathbb{R}^3$ (resp. $I \times S^3$) that has F as its core.

Points on F whose neighborhoods are of the form $Y \times \mathbb{R}$ form a 1-manifold in $I \times \mathbb{R}^3$ (resp. $I \times S^3$) cobounding vertices of W_0 and vertices of W_1, each component of which is called a *seam* of F. Each component of the exterior of seams in F is a *face* of F. The additional data of F are subject to the following constraints:

(1) Around each seam, two of the adjacent faces are labeled 1 and one is labeled 2;

(2) Around each seam, the two 1-labeled faces induces the same orientation on the seam, which is opposite to the orientation induced by the 2-labeled face;

(3) The labels and orientations on faces are compatible with the labels and orientations on edges when restricted to the boundary (here, as usual, for W_0 this means its edge orientations are given by $-\partial F$);

[1] [QW24] further requires the tangent vectors of all three edges at a trivalent vertex to point in the same direction, as this would cut down the number of generic Reidemeister-type moves and movie moves (a similar condition is posed on $\mathfrak{gl}_2$ foams in $I \times \mathbb{R}^3$ or $I \times S^3$). We ignore this difference.

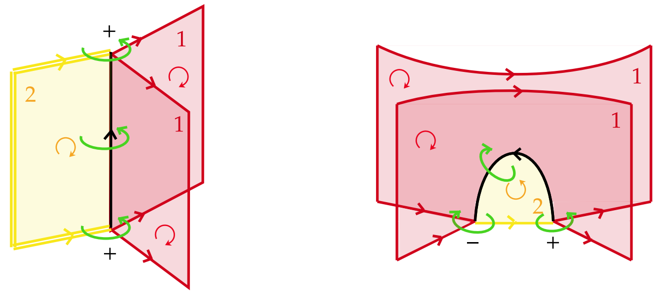


Figure C.1: Examples of $\mathfrak{gl}_2$ foams between $\mathfrak{gl}_2$ webs that are contained in $I \times \mathbb{R}^2 \times \{0\} \subset I \times \mathbb{R}^3$, shown locally, with ribbons given by thickenings of the pictures in $I \times \mathbb{R}^2 \times \{0\}$ with the induced orientations from $I \times \mathbb{R}^2 \times \{0\}$. The orientations of the vertices, edges, seams, and faces, as well as cyclic orientations around vertices and seams are indicated.

(4) The ribbon restricts to the ribbons of the boundary webs, with compatible orientations.

Each seam of a $\mathfrak{gl}_2$ foam F is assigned the orientation induced from the 1-labeled faces adjacent to it. Moreover, the orientation of the ribbon R at a seam coupled with the seam orientation induces a cyclic ordering of the three adjacent faces by the right-hand rule. As a consequence, both the orientations on seams and the cyclic orientations around them are compatible with those for vertices of the boundary webs.

See Figure C.1 for two examples of $\mathfrak{gl}_2$ foams contained in $I \times \mathbb{R}^2 \times \{0\} \subset I \times \mathbb{R}^3$ shown locally, with orientation data indicated in the picture.

Below, when we speak about $\mathfrak{gl}_2$ webs and foams without specifications, we always mean $\mathfrak{gl}_2$ webs in $\mathbb{R}^3$ or S^3 and $\mathfrak{gl}_2$ foams in $I \times \mathbb{R}^3$ or $I \times S^3$.

A *dotted* $\mathfrak{gl}_2$ foam is a $\mathfrak{gl}_2$ foam with finitely many distinguished points (called *dots*) on the interior of its 1-labeled faces.

For our purposes, we also wish to allow more general $\mathfrak{gl}_2$ foams.

Definition C.1.1. A *singular* $\mathfrak{gl}_2$ *foam* is a dotted $\mathfrak{gl}_2$ foam, but with finitely many singularities away from the seams and the dots of the following types allowed:

(1) Transverse double points between 1- and 2-labeled faces or between 2-labeled faces. The ribbon is also immersed near the transverse double points, but is embedded when restricted to a neighborhood in each sheet.

(2) Framing points, which are $\mathbb{Z}$-labeled points on the interior of faces of the foam away from double points. In a closed neighborhood of such a k-labeled point, the foam is

regular of the form $D^2 = D^2 \times \{0\} \subset D^2 \times D^2$, but the ribbon is only immersed of the form $D^2 \times (-\epsilon, \epsilon) \to D^2 \times D^2$, $(z,t) \mapsto (z, tz^k)$, where we denote points in $D^2 \subset \mathbb{C}$ using complex numbers.

Remark C.1.2. (1) After taking the Khovanov–Rozansky $\mathfrak{gl}_2$ functor (see Section C.2), the framing points in a singular $\mathfrak{gl}_2$ foam play the same role as framing-changing input balls in [MWW24, Definition 2.5] when $N = 2$ (note that they use a different renormalization convention).

(2) We could also have allowed transverse double points between 1-labeled edges, and insist that they act as immersion point input balls (as considered in the Lee case in [MWW24, Example 3.7]) on the $\mathfrak{gl}_2$ homology. Theorem 4.1.6 will still be valid once we fix cocycles in the Khovanov–Rozansky $\mathfrak{gl}_2$ chain complexes of the positive/negative Hopf links (see the construction in Section C.2). This can be used to give a shortcut proving the main theorem of [Car+25]. We remark that no choice of cocycles will result in cobordism maps that are invariant under finger/Whitney moves between 1-labeled faces.

We think of $1, 2$-labeled edges and faces as having thickness $1, 2$, respectively. The *writhe* of a $\mathfrak{gl}_2$ web W, denoted $w(W)$, is the linking number between W and a push-off of itself in the normal direction of the ribbon. Analogously, by interpreting framing points as introducing local twistings, one could also define the *self-intersection number* of a singular $\mathfrak{gl}_2$ foam $F\colon W_0 \to W_1$; more explicitly, it is $[F] \cdot [F] := 2i(F) + i_1(F) + 4i_2(F)$, where $i(F)$ is twice the sum of the signed intersection number between 1- and 2-labeled faces plus four times the sum of the signed intersection number between 2-labeled faces, and $i_k(F)$ is the sum of labels on the framing points on k-labeled faces, $k = 1, 2$. Thus, $[F] \cdot [F] = w(W_1) - w(W_0)$. In particular, the writhe of a $\mathfrak{gl}_2$ web in S^3 is the self-intersection number of any singular $\mathfrak{gl}_2$ foam in B^4 that bounds it (such a singular foam always exists). A regular $\mathfrak{gl}_2$ foam has self-intersection number 0, and the writhe of a $\mathfrak{gl}_2$ web in S^3 is a complete obstruction to having a regular $\mathfrak{gl}_2$ foam in B^4 bounding it.

C.2 Functoriality of singular $\mathfrak{gl}_2$ foams

For $M = \mathbb{R}^3, S^3$, let $\mathbf{Links}_M$ denote the category of admissible framed oriented links in M and framed oriented link cobordisms between them up to isotopy rel boundary, where admissibility of the link is in the same sense as in Section 4.2.1, namely that the link is contained in $\mathbb{R}^3$ and the projection onto $\mathbb{R}^2 \times \{0\}$ is generic. The link diagram of an admissible framed link comes with $\mathbb{Z}$-labeled framing points away from crossings, which may move freely along the link components, combine or split in a weight-preserving way, and 0-labeled framing points may be created or annihilated at will[2]. Reidemeister I moves come at a cost of ± 1-labeled framing points.

[2]Strictly speaking, a generic projection of the ribbon would only equip the diagram with $\pm$-half-framing points, which create or cancel in pairs only when we isotope the ribbon. Since framing points only affect the

Similarly, let $\mathbf{Webs}_M$ (resp. $\mathbf{Webs}_M^{\text{sing}}$) denote the category of admissible $\mathfrak{gl}_2$ webs in M and dotted (resp. singular) $\mathfrak{gl}_2$ foams between them up to isotopy rel boundary (resp. isotopy rel boundary, weight-preserving collision/separation of labeled framing points on the same face, and creation/annihilation of 0-labeled framing points). Here, in addition to the requirements of generality as in the case of links, the admissibility of $\mathfrak{gl}_2$ webs further requires that the projection to $\mathbb{R}^2 \times \{0\}$ is orientation-preserving on the ribbon at each trivalent vertex. Framing points in a web diagram are not allowed to move across trivalent vertices. We have the following diagram of functors

$$\begin{tikzcd} \mathbf{Links}_{\mathbb{R}^3} \arrow[r] \arrow[d, hook] & \mathbf{Links}_{S^3} \arrow[d, hook] \\ \mathbf{Webs}_{\mathbb{R}^3} \arrow[r] \arrow[d, hook] & \mathbf{Webs}_{S^3} \arrow[d, hook] \\ \mathbf{Webs}_{\mathbb{R}^3}^{\text{sing}} \arrow[r] & \mathbf{Webs}_{S^3}^{\text{sing}}. \end{tikzcd} \tag{C.1}$$

Here, all vertical arrows are inclusions of categories. All horizontal arrows are full, but not faithful. All functors except the vertical ones from the first row to the second are bijective on objects.

The Khovanov–Rozansky $\mathfrak{gl}_2$ homology was first defined for links in $\mathbb{R}^3$ and S^3 and link cobordisms by Khovanov [Kho00] (referred to as Khovanov homology, where the renormalization convention is different from ours), although its functoriality turns out to be more difficult. The functoriality for links in $\mathbb{R}^3$, up to sign, was proved by Jacobsson [Jac04]. Many sign fixes appeared in the literature, but the one that first introduced webs and foams in the language we will be using is due to Blanchet [Bla10]; thus, we obtain a functor

$$CKhR_2\colon \mathbf{Links}_{\mathbb{R}^3} \to \mathbf{K}^b(\mathbb{Z})^{\mathbb{Z}}, \tag{C.2}$$

where the target is the bounded homotopy category of cochain complexes of quantum $\mathbb{Z}$-graded abelian groups, with quantum grading shifts allowed for morphisms. The functoriality for links in S^3 requires an additional check for a global elementary movie move called the sweep-around move, and was obtained only recently by Morrison–Walker–Wedrich [MWW22]. This means (C.2) descends to a functor

$$CKhR_2\colon \mathbf{Links}_{S^3} \to \mathbf{K}^b(\mathbb{Z})^{\mathbb{Z}}.$$

The Khovanov–Rozansky $\mathfrak{gl}_2$ homology was also extended to $\mathfrak{gl}_2$ webs in $\mathbb{R}^3$ or S^3 as the $N=2$ special case of Wu [Wu14] and studied by many other authors. For us, singularities in the diagram of a $\mathfrak{gl}_2$ web, in addition to trivalent vertices, are crossings of various kinds and $\mathbb{Z}$-labeled framing points where the ribbon twists along the strands by multiples of full turns rel the blackboard framing. One resolves the singularities and builds a cube of resolutions

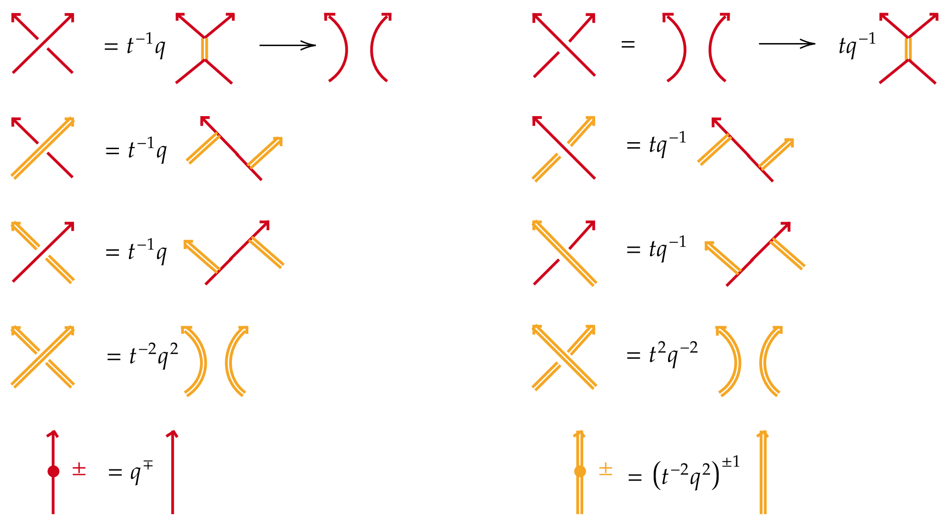


Figure C.2: Resolution of crossings or framing points in a web diagram into complexes of planar webs. Here t and q denote the homological and quantum degree shifts, respectively.

using the rules in Figure C.2. The functoriality of $CKhR_2$ for webs in $\mathbb{R}^3$ was proved recently by Queffelec [Que22]. This means (C.2) extends to a functor

$$CKhR_2 \colon \mathbf{Webs}_{\mathbb{R}^3} \to \mathbf{K}^b(\mathbb{Z})^{\mathbb{Z}}.$$

We warn the reader that our convention differs from that of Queffelec by mirroring the webs and foams (or on the level of diagrams, changing the signs of all crossings and framing points).

One could also define the universal version of Khovanov–Rozansky $\mathfrak{gl}_2$ homology in the various settings above, by replacing the underlying Frobenius algebra $\mathbb{Z}[x]/(x^2)$ by its universal deformation $\mathbb{Z}[E_1, E_2, x]/(x^2 - E_1 x + E_2)$ over $\mathbb{Z}[E_1, E_2]$. The proof of functoriality is similar, and one obtains a functor $CKhR_2^{univ}$ to $\mathbf{K}^b(\mathbb{Z}[E_1, E_2])^{\mathbb{Z}}$, the bounded homotopy category of cochain complexes of quantum $\mathbb{Z}$-graded $\mathbb{Z}[E_1, E_2]$-modules, from $\mathbf{Links}_{\mathbb{R}^3}, \mathbf{Links}_{S^3}$, or $\mathbf{Webs}_{\mathbb{R}^3}$ (see [MWW24, Theorem 2.2] for the functoriality for links in S^3). Here, the free variables E_1, E_2 have q-degrees $2, 4$, respectively. By setting $E_1 = E_2 = 0$, one recovers the functoriality results for the undeformed theories.

homology by global bigrading shifts, we ignore this technical difference. Once we restrict to integral framing points (by imposing this as a part of the admissibility condition), homological shifts are always even.

Theorem C.2.1. *The universal Khovanov–Rozansky $\mathfrak{gl}_2$ homology on objects extends to a functor*

$$CKhR_2^{univ} \colon \mathbf{Webs}_{S^3}^{\mathrm{sing}} \to \mathbf{HCh}^b(\mathbb{Z}[E_1, E_2])^{\mathbb{Z}}. \tag{C.3}$$

Here, $\mathbf{HCh}^b$ denotes the cohomology category of the dg category of bounded chain complexes, i.e. the extension of $\mathbf{K}^b$ where homological degree shifts are allowed for morphisms[3]*. Moreover,*

(1) The bigrading shifts of $CKhR_2^{univ}(F) \colon CKhR_2^{univ}(W_0) \to CKhR_2^{univ}(W_1)$ for a singular foam $F \colon W_0 \to W_1$ is $(t^{-1}q)^{i(F)+2i_2(F)} q^{-\chi(u(F))-i_1(F)+2\#(dots)}$, where $u(F)$ denotes the closure of the union of 1-labeled faces in F.

(2) $CKhR_2^{univ}(F)$ (up to homotopy) is independent of the embedding of the interior of the 2-labeled faces of F. In fact, up to sign, it is determined by the abstract dotted surface $u(F)$. The sign is further determined by the germ of $u(F)$ in F with all data (orientations, ribbon, dots, and framing points) and the mod 4 total Euler characteristic of the 2-labeled faces.

Remark C.2.2. The functor $CKhR_2^{univ}$ in Theorem C.2.1 gives rise to functors from every term in the diagram (C.1). When restricted to the first two rows of (C.1), the image is contained in the subcategory $\mathbf{K}^b(\mathbb{Z}[E_1, E_2])^{\mathbb{Z}}$ where morphisms preserve the homological degree. Thus, Theorem C.2.1 is a simultaneous generalization of all previous functoriality results in the context of $\mathfrak{gl}_2$ homology.

The proof of Theorem C.2.1 is divided into two steps. In Section C.2.1, by proving a 4-dimensional Lee foam evaluation formula, we upgrade Queffelec's functoriality to webs in S^3 and nonsingular foams between them. In Section C.2.3, we further extend the functoriality to singular foams. This allows the construction of (braided) monoidal 2-categories in Section C.2.4. In Section C.2.5 we interpret the moves involving singular foams in terms of sylleptic centers.

C.2.1 Functoriality of regular $\mathfrak{gl}_2$ foams

In this section, we show that Queffelec's functor

$$CKhR_2^{univ} \colon \mathbf{Webs}_{\mathbb{R}^3} \to \mathbf{K}^b(\mathbb{Z}[E_1, E_2])^{\mathbb{Z}} \tag{C.4}$$

descends to a functor

$$CKhR_2^{univ} \colon \mathbf{Webs}_{S^3} \to \mathbf{K}^b(\mathbb{Z}[E_1, E_2])^{\mathbb{Z}}. \tag{C.5}$$

This amounts to showing that the sweep-around movie move [MWW22, (1-1)] induces the identity chain map up to chain homotopy (for us, the strand that sweeps around can be either 1-labeled or 2-labeled).

[3] In fact, morphisms in the image of $CKhR_2^{univ}$ are always of even homological degree.

For a $\mathfrak{gl}_2$ web W (resp. $\mathfrak{gl}_2$ foam F), let $u(W)$ (resp. $u(F)$) denote the closure of the union of 1-labeled edges (resp. faces), thought of as an unoriented link (resp. link cobordism). If $F\colon W_0 \to W_1$ is a $\mathfrak{gl}_2$ foam between admissible $\mathfrak{gl}_2$ webs, then $u(F)\colon u(W_0) \to u(W_1)$ is an orientable unoriented link cobordism between admissible unoriented links, and the Bar-Natan formalism of Khovanov homology gives a chain map $CKhR_2^{univ}(u(F))\colon CKhR_2^{univ}(u(W_0)) \to CKhR_2^{univ}(u(W_1))$, well-defined up to sign and chain homotopy, between the Khovanov chain complexes of links $u(W_0), u(W_1)$, where the bigrading is only well-defined up to an overall even shift. The main result of Beliakova–Hogancamp–Putyra–Wehrli [Bel+23] implies that there are isomorphisms $\iota_\bullet$ canonical up to signs making the following diagram commute up to sign and chain homotopy (as homologically $\mathbb{Z}/2$-graded, $\mathbb{Z}$-relatively graded chain complexes).

$$\begin{tikzcd} CKhR_2^{univ}(W_0) \arrow[r, "CKhR_2^{univ}(F)"] \arrow[d, "\iota_{W_0}", "\cong"'] & CKhR_2^{univ}(W_1) \arrow[d, "\iota_{W_1}", "\cong"'] \\ CKhR_2^{univ}(u(W_0)) \arrow[r, "CKhR_2^{univ}(u(F))"] & CKhR_2^{univ}(u(W_1)). \end{tikzcd} \tag{C.6}$$

Now, let $F\colon W \to W$ be the movie of a sweep-around move. Then $u(F)\colon u(W) \to u(W)$ is either the movie of a sweep-around move (if a 1-labeled edge sweeps around) or the identity movie (if a 2-labeled edge sweeps around) for links and link cobordisms, whose induced chain map $CKhR_2^{univ}(u(F))$ is the identity chain map up to sign and chain homotopy. Indeed, any version of Khovanov homology with analogs of (C.6) commuting up to global sign and homotopy inherit the triviality of the sweep-around move up to sign from $CKhR_2^{univ}$, for which it is proven in [MWW24, Theorem 2.2] following [MWW22]. By the commutativity of (C.6), we deduce that $CKhR_2^{univ}(F)$ is chain homotopic to the identity map up to sign.

To fix the sign, as usual, it suffices to do so on the level of an appropriately defined Lee homology. To this end, we denote by

$$CKhR_{Lee}\colon \mathbf{Webs}_{\mathbb{R}^3} \to \mathbf{K}^b(\mathbb{Q})^{\mathbb{Z}}$$

the result of base-changing $CKhR_2^{univ}$ to $\mathbb{Q}$ by tensoring all complexes with the $\mathbb{Z}[E_1, E_2]$-module $\mathbb{Q}$, on which E_1 acts by 0 and E_2 by -1. Further, we write KhR_{Lee} for the homology of the complexes computed by $CKhR_{Lee}$ and refer to this as *Lee homology*.

Returning to the movie of a sweep-around move $F\colon W \to W$, we now check the induced map $KhR_{Lee}(F)\colon KhR_{Lee}(W) \to KhR_{Lee}(W)$ is the identity map. Since the underlying abstract $\mathfrak{gl}_2$ foam (by which we mean the underlying singular surface together with labels, orientations, and cyclic orderings around seams) of F is the same as that of the identity foam cobordism $W \to W$, this follows from the next theorem using a tracing argument.

Theorem C.2.3 (4-dimensional Lee foam evaluation). *Let $F\colon \emptyset \to \emptyset$ be a closed dotted $\mathfrak{gl}_2$ foam in $\mathbb{R}^4$. Then the following two rational numbers are equal:*

(a) *The* 4*-dimensional Lee evaluation* $\langle F\rangle_{Lee}$ *of* F*, i.e. the eigenvalue of the endomorphism* $KhR_{Lee}(F)$ *of* $KhR_{Lee}(\emptyset) = \mathbb{Q}$ *provided by the functoriality of Lee homology.*

(b) *The Lee evaluation of* F *as an abstract foam, defined as in Blanchet [Bla10, Section 1.5 and 4].*

Corollary C.2.4. *Let* $F\colon \emptyset \to \emptyset$ *be a closed dotted* $\mathfrak{gl}_2$ *foam in* $\mathbb{R}^4$*. The* 4*-dimensional Khovanov–Rozansky* $\mathfrak{gl}_2$ *evaluation of* F *agrees with the* $\mathfrak{gl}_2$ *evaluation of* F *as an abstract foam, defined as in [Bla10].*

Thus, if $u(F)$ contains a component that is not a one-dotted sphere or an undotted torus, then the 4-dimensional evaluation is $\langle F\rangle_{\mathfrak{gl}_2} = 0$. Otherwise, $\langle F\rangle_{\mathfrak{gl}_2} = \pm 2^{\#\{\text{torus components in } u(F)\}}$, with the sign determined by the remaining data of F viewed as an abstract foam. In particular, the sign is positive if F has no 2-labeled faces.

Proof. If the q-degree shift of $KhR_2(F)\colon KhR_2(\emptyset) \to KhR_2(\emptyset)$ is nonzero, then both the concrete and the abstract $\mathfrak{gl}_2$ evaluations of F are zero. If the q-degree shift of $KhR_2(F)$ is zero, then $\langle F\rangle_{\mathfrak{gl}_2} = \langle F\rangle_{Lee}$ is equal to the abstract Lee/$\mathfrak{gl}_2$ evaluation. □

For our purposes, it is convenient to allow *Lee idempotent-colored foams*, namely (undotted) $\mathfrak{gl}_2$ foams whose 1-labeled faces are decorated with colors $+ = (1+x)/2$ or $- = (1-x)/2$ (which are idempotents in the Lee Frobenius algebra $\mathbb{Q}[x]/(x^2-1)$). Every $\mathfrak{gl}_2$ foam can be rewritten as a formal linear combination of idempotent-colored ones, and both evaluations in Theorem C.2.3 are extended to closed idempotent-colored foams by linearity.

We refer the reader to [Que22, (3.1)-(3.19)] for a collection of local skein relations satisfied by the Lee evaluation of abstract $\mathfrak{gl}_2$ foams, some of which will be useful to us. Relations (3.1) and (3.3)-(3.12) therein can be converted into idempotent-colored skein relations in the natural way. We also note one more skein relation that if the two 1-labeled faces around a seam are colored by the same idempotent, then the abstract Lee evaluation is zero.

The local pictures in these skein relations can also be interpreted as foams embedded in B^3, with ribbon given by a thickening of the core surface, oriented as a submanifold of B^3. To evaluate in Lee homology an idempotent-colored foam F in $\mathbb{R}^4$, one can apply these local relations to simplify F. More explicitly, this means that if in some local B^3, F with its ribbon is given by one side of a skein relation, then one can replace the local picture of F by the other side of the skein relation (this is justified by Queffelec's functoriality, since one can rotate the local B^3 to sit in the first three coordinates of $I \times \mathbb{R}^3$ (with orientation) and evaluate).

Proof of Theorem C.2.3. It suffices to prove the case when F is idempotent-colored by $\pm$. If two of the 1-labeled faces around a seam in F are colored by the same idempotent, then both evaluations (a) and (b) are zero. Thus, it remains to prove Theorem C.2.3 for *compatibly* idempotent-colored foams, namely the ones where two 1-labeled faces around each seam have opposite colors $\pm$.

Let F_1, F_2 denote the closures of the union of $1, 2$-labeled faces in F, respectively. As we observed above, one may simplify F by the skein relations [Que22, (3.1)-(3.19)] without affecting the truth of the statement.

Step 1: We may assume that F_1 is connected.

This is because we can find a collection of paths connecting different components of F_1 in the complement of F, and perform the inverses of [Que22, (3.15) or (3.18)].

Step 2: We may assume that F_1 and F_2 are disjoint smoothly embedded closed oriented surfaces, or equivalently, F has empty seams.

The collection of seams is an embedded multicurve γ on the closed surface F_1. The inverses of [Que22, (3.4)] allow us to change γ by oriented band surgeries on F_1. If $\gamma \neq \emptyset$, since F_1 is compatibly colored, we may apply band surgeries to assume γ has a single component, which is necessarily a separating curve on F_1. By further band surgeries, we may assume γ is a contractible curve on F_1. Since γ bounds the surface F_2 in the complement of F_1, the twisting of the germ of F_2 along γ around F_1 is zero, hence we may apply the neck-cutting relation [Que22, (3.2)] and then [Que22, (3.6)] to detach F_2 from F_1.

Step 3: A ribbon structure on F is equivalent to framings on the embedded surfaces $F_1, F_2 \subset \mathbb{R}^4$. Changing these framings does not affect the Lee foam evaluation.

This is because F with two different framings differ in a generic movie presentation levelwise by some framing points, and the assignment of induced maps is insensitive to framing points.

Step 4: We may assume $F = \tau(S^1 \times H)$, where $H \subset B^3$ is the positive Hopf link with one 1-labeled and one 2-labeled component, both 0-framed, and $\tau\colon S^1 \times B^3 \hookrightarrow \mathbb{R}^4$ is the twisted embedding induced by an oddly framed circle in $\mathbb{R}^4$.

Suppose $W_1 \cup W_2 \subset I \times \mathbb{R}^4$ is a paired oriented cobordism between $F_1 \cup F_2 \subset \mathbb{R}^4$ and some $F_1' \cup F_2' \subset \mathbb{R}^4$. Upon changing the framings of F_1, F_2, we may assume $W_1 \cup W_2$ to be a framed cobordism (note that closed oriented 3-manifolds embedded in $\mathbb{R}^5$ have trivial normal bundles). By making the projection $W_1 \cup W_2 \subset I \times \mathbb{R}^4 \to I$ Morse, we see that (F_1, F_2) and (F_1', F_2') (with all components of F_1' colored by the color of F_1) are related by a sequence of skein relations [Que22, (3.2)(3.15)(3.17)] and their inverses. The claim now follows from the result by Sanderson [San87, Example 1.3] that the oriented unframed cobordism group of the pair (F_1, F_2) in $\mathbb{R}^4$ is isomorphic to $\mathbb{Z}/2$, with the underlying unframed pair of $\tau(S^1 \times H)$ representing the nontrivial bordism class.

Step 5: Theorem C.2.3 holds for $F = \tau(S^1 \times H)$.

If $W \subset \mathbb{R}^3$ is an admissible web and T_W is the foam given by the trace of W under a 2π-rotation in $\mathbb{R}^3$, then by choosing the rotation to be along the z-axis, we see the induced map

$$CKhR_2^{univ}(T_W)\colon CKhR_2^{univ}(W) \to CKhR_2^{univ}(W)$$

is chain homotopic to the identity map. Therefore, we may replace the twisted embedding $\tau\colon S^1 \times B^3 \hookrightarrow \mathbb{R}^4$ by the untwisted embedding $i\colon S^1 \times B^3 \hookrightarrow \mathbb{R}^4$ without affecting the Lee foam evaluation. But $i(S^1 \times H)$ is a null-cobordant pair, so the proof is complete. □

C.2.2 A local statement

Before extending the functor (C.5) of the previous section to singular foams, we establish notation for a local version of functoriality in the spirit of Bar-Natan's notion of canopolis [Bar05], see [ETW18, Section 2.2] and [QW21]. We also prove a lemma that will be useful. This section only plays a minor role in proving Theorem C.2.1.

Let S denote an oriented surface, and let ϵ denote a collection of oriented points $p \subset \partial S$, each with a label 1 or 2. We now consider the graded $\mathbb{Z}[E_1, E_2]$-linear additive category $\mathbf{Foams}_{S,\epsilon}$ of $\mathfrak{gl}_2$ foams in the thickened surface. (A version with $\epsilon = \emptyset$ and $E_1 = E_2 = 0$ was defined in [QW21, Definition 3.1].) The category $\mathbf{Foams}_{S,\epsilon}$ has as objects $\mathfrak{gl}_2$ webs $W \subset S$ with $\partial W = \epsilon$, as well as formal grading shifts and direct sums thereof. The morphisms in $\mathbf{Foams}_{S,\epsilon}$ are (matrices of) $\mathbb{Z}[E_1, E_2]$-linear combinations of dotted $\mathfrak{gl}_2$ foams in $I \times S$ rel $I \times \epsilon$ between such webs, up to isotopy rel boundary and appropriate skein relations[4].

For tangled webs in the thickening of S we similarly define the category $\mathbf{Webs}_{I\times S,\epsilon}$ with objects admissible $\mathfrak{gl}_2$ webs $W \subset I \times S$ with $\partial W = \{1/2\} \times \epsilon$, and morphisms dotted $\mathfrak{gl}_2$ foams in $I \times I \times S$ rel $I \times \{1/2\} \times \epsilon$ between such webs; these webs and foams are defined analogously to Section C.1, with webs admissible if their projections onto S are generic. (A version with $\epsilon = \emptyset$ was defined in [QW21, Definition 4.6].)

The local invariant we consider is a functor of the form

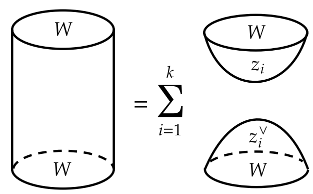
$$\mathbf{Webs}_{I\times S,\epsilon} \xrightarrow{[\![\cdot]\!]} \mathbf{K}^b(\mathbf{Foams}_{S,\epsilon}), \tag{C.7}$$

where $\mathbf{K}^b(\mathbf{Foams}_{S,\epsilon})$ is the bounded homotopy category of cochain complexes in $\mathbf{Foams}_{S,\epsilon}$. The existence of a functor (C.7) that categorifies the evaluation of tangled webs in $\mathfrak{gl}_2$ skein theory was proven for tangles (i.e. purely 1-labeled webs) in [QW21, Theorem 1.1], conjectured in full in [QW21, Conjecture 4.8] and proven in [Que22].

With this notation in place, we describe the central idea of the local lemma we will need. Suppose we have a closed web $W \subset B^3$ within some larger web diagram W', and suppose

[4]Specifically, the relations [Que22, (3.3)-(3.12)] together with equivariant versions of sphere and neck-cutting relations, see e.g. [Bel+23, Definition 2.6]. All these relations arise as local relations from (an equivariant analog of) Blanchet's abstract foam evaluation [Bla10], as explained in detail in [ETW18, Section 2].

that we would like to slide W either under or over some other strand in W'. We would like to prove that such a move induces "the identity map on both $CKhR_2^{univ}(W)$ and the other strand." To interpret this idea properly, we first note that for any planar $\mathfrak{gl}_2$ web $W \in \mathbf{Foams}_{D^2,\emptyset}$, the universal construction implies that we can neck-cut the identity foam morphism id_W and write it as a finite sum



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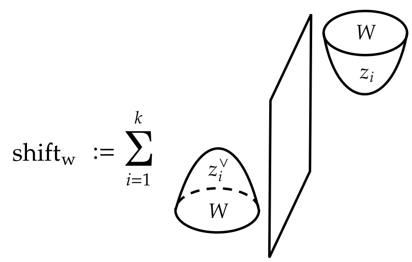
where z_i's are foams representing a basis of $KhR_2^{univ}(W)$, and $z_i^\vee$'s are foams representing a basis of $KhR_2^{univ}(-W)$ dual to the basis representing by the z_i's. If we now let W_L (resp. W_R) denote the (planar) web in $\mathbf{Foams}_{D^2,\epsilon}$ consisting of W sitting to the left (resp. right) of a single strand through the disk D^2 (with endpoint data ϵ), we can define the "identity" map between such webs by the "shift" map $W_L \to W_R$ in $\mathbf{Foams}_{D^2,\epsilon}$ given by



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More generally, if W is any admissible closed web in $B^3 = I \times D^2$, we let $W_L, W_R \in \mathbf{Webs}_{I\times D^2,\epsilon}$ be defined similarly and then define the shift map $[\![W_L]\!] \xrightarrow{\iota_{sh}} [\![W_R]\!]$ by applying the shift described above termwise between the two complexes.

Lemma C.2.5. *Let W be an admissible closed $\mathfrak{gl}_2$ web in $B^3 = I \times D^2$, and let W_L (resp. W_R) be the web in $\mathbf{Webs}_{I\times D^2,\epsilon}$ formed by placing W to the left (resp. to the right) of an arbitrarily oriented and labeled through-strand in $\{1/2\} \times D^2$ (with endpoint data ϵ). Let $W_L \xrightarrow{\eta_{un}} W_R$ (resp. $W_L \xrightarrow{\eta_{ov}} W_R$) denote the foam in $\mathbf{Webs}_{I\times D^2,\epsilon}$ tracing out the isotopy of passing W under (resp. over) the through-strand. Then the three maps $\iota_{sh}, [\![\eta_{un}]\!], [\![\eta_{ov}]\!]$ are equal morphisms in $\mathbf{K}^b(\mathbf{Foams}_{D^2,\epsilon})$.*

Remark C.2.6. One could reprove the sweep-around move required for the $\mathbb{R}^3$ to S^3 functoriality upgrade using the equality of $[\![\eta_{un}]\!]$ and $[\![\eta_{ov}]\!]$ in Lemma C.2.5, similarly as in [CY25, Corollary 5.7]. Nevertheless, we proceeded as in Section C.2.1, hoping that Theorem C.2.3 or its proof might be of independent interest.

Proof. We only prove $[\![\eta_{un}]\!] = \iota_{sh}$, as the other half is analogous. We remark that, in the case when the middle strand is 1-labeled and the web W is a (1-labeled) link, the equality follows from the argument of the sweep-around move in [MWW22, Theorem 1.1]. Following their idea, we proceed as follows:

Special case: W is planar.

The universal $\mathfrak{gl}_2$ homology of W is generated by foams capping it off in B^3. Birthing a foam on the left of the strand and sliding the boundary web under the strand to the right is isotopic to birthing the foam on the right. By Queffelec's functoriality [Que22], this proves the desired equality.

The general case:

The map $W_L \xrightarrow{\eta_{un}} W_R$ can be decomposed into a sequence of elementary moves

$$W_L = W_0 \xrightarrow{\eta_1} W_1 \xrightarrow{\eta_2} \cdots \xrightarrow{\eta_n} W_n = W_R,$$

where each η_i is either a Reidemeister II move, a fork slide, or a Reidemeister III move involving three strands having non-alternating orientations. Letting N denote the number of crossings in W, each complex $[\![W_t]\!]$ can be viewed as a twisted complex with terms indexed by $\delta \in \{0,1\}^N$ via the cube of resolutions for W, and we have termwise maps

$$[\![W_L]\!]_\delta = [\![W_0]\!]_\delta \xrightarrow{[\![\eta_1]\!]_\delta} [\![W_1]\!]_\delta \xrightarrow{[\![\eta_2]\!]_\delta} \cdots \xrightarrow{[\![\eta_n]\!]_\delta} [\![W_n]\!]_\delta = [\![W_R]\!]_\delta$$

induced by the corresponding isotopies on the resolutions. If $W_{t-1} \xrightarrow{\eta_t} W_t$ is a Reidemeister II move or a fork slide, $[\![\eta_t]\!] = \bigoplus_\delta [\![\eta_t]\!]_\delta$ with no cross terms. If η_t is a Reidemeister III move involving only 1-labeled strands (with non-alternating orientations), [MWW22] showed that $[\![\eta_t]\!]$ can be chosen carefully in its chain homotopy class to ensure that the cross terms $[\![\eta_t]\!] - \bigoplus_\delta [\![\eta_t]\!]_\delta$ strictly increase the *internal homological degree*, defined as the homological degree of the index δ. Since the entire composition $[\![W_0]\!] \xrightarrow{[\![\eta_{un}]\!]} [\![W_1]\!]$ preserves the (internal=total) homological degree, the contributions from these cross terms vanish in the full composition $[\![\eta_{un}]\!]$.

It remains to consider $[\![\eta_t]\!]$ for Reidemeister III moves involving 2-labeled strands (with non-alternating orientations). One can check, case by case, using [Que22, Appendix A], that in these cases $[\![\eta_t]\!] = \bigoplus_\delta [\![\eta_t]\!]_\delta$ again with no cross terms, so that $[\![\eta_{un}]\!] = \bigoplus_\delta [\![\eta_{un}]\!]_\delta$, and the desired statement follows from the planar case. We outline an alternative effortless proof. Note that the endomorphism space between the local source and target of a Reidemeister III induced map with 2-labeled strands involved in the quantum grading 0 is isomorphic to $\mathbb{Z}$. Thus the local map must agree with the local termwise Reidemeister maps up to sign; combining this with the rest of the argument thus far, we see $[\![\eta_{un}]\!] = \pm\iota_{sh}$. To fix the sign, we pass to Lee homology. We add in a 1-labeled framed crossingless unknot as needed to force W to have writhe zero, so that there exists a $\mathfrak{gl}_2$ foam that caps off W in B^4, compatibly idempotent-colored in the sense explained in the proof of Theorem C.2.3. Then the functoriality of (C.7) [Que22] shows that birthing W on the left of the middle strand and sliding it to the right induces the same map on Lee homology as birthing W on the right of the strand. Since both maps are nonzero, this proves $[\![\eta_{un}]\!] = \iota_{sh}$. □

Remark C.2.7. In fact, by taking advantage of Bar-Natan's canopolis formalism, one can view $(\cdot)_L$ (resp. $(\cdot)_R$) as a functor $\mathbf{Webs}_{I\times D^2,\emptyset} \to \mathbf{Webs}_{I\times D^2,\epsilon}$ which plugs objects and morphisms into a local thickened disc to the left (resp. to the right) of a through-strand in $\{1/2\} \times D^2$ with endpoint data ϵ. In this language, the functoriality of (C.7) ensures that our maps in Lemma C.2.5 induce natural transformations (indeed, natural isomorphisms) $\iota_{sh}, [\![\eta_{un}]\!], [\![\eta_{ov}]\!]$ between the functors

$$[\![(\cdot)_L]\!], [\![(\cdot)_R]\!] : \mathbf{Webs}_{I\times D^2,\emptyset} \to \mathbf{K}^b(\mathbf{Foams}_{D^2,\epsilon}).$$

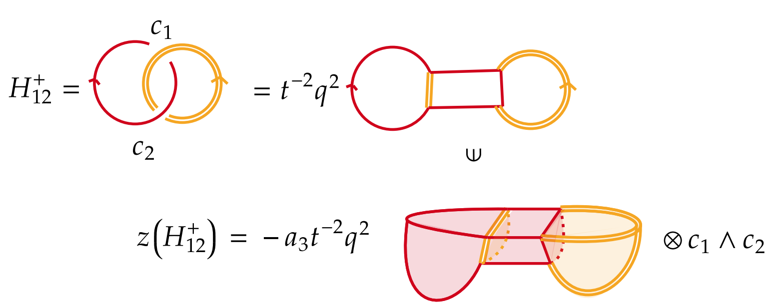
Lemma C.2.5 implies that these natural isomorphisms are in fact equal.

C.2.3 Extension to singular $\mathfrak{gl}_2$ foams

We now extend the functor (C.5) defined in Section C.2.1 to singular foams, obtain the claimed functor (C.3) in Theorem C.2.1, and prove the extra assertions in Theorem C.2.1.

We first describe the link of singularity L for each singularity model in singular $\mathfrak{gl}_2$ foams, fix admissible representatives of them, and fix explicit cocycles $z(L) \in CKhR_2^{univ}(L)$, which will be useful in the construction. Below, we follow the notation in [Que22]. In particular, $a_3, a_4 \in \{\pm 1\}$ are free sign variables for the universal Khovanov–Rozansky $\mathfrak{gl}_2$ homology theory, which affects only the sign assignments to morphisms.

(i) A positive transverse double point between 1- and 2-labeled faces:



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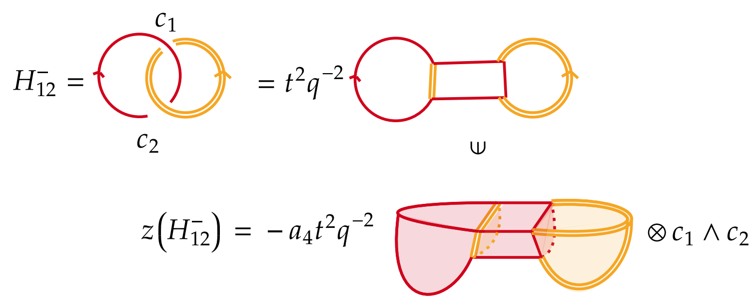
(ii) A negative transverse double point between 1- and 2-labeled faces:



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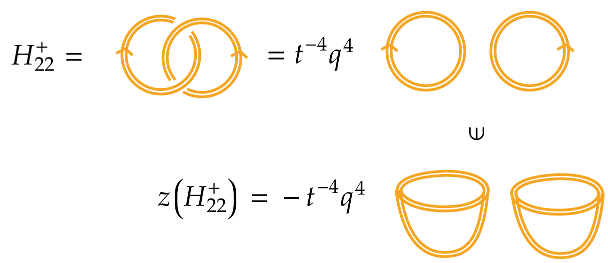
(iii) A positive transverse double point between 2-labeled faces:



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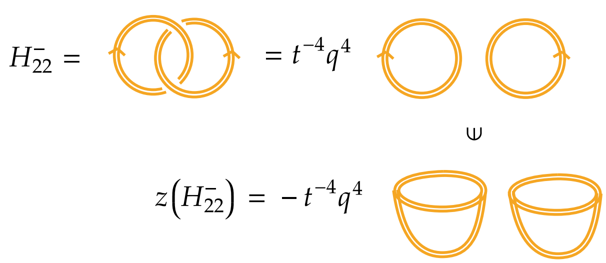
(iv) A negative transverse double point between 2-labeled faces:



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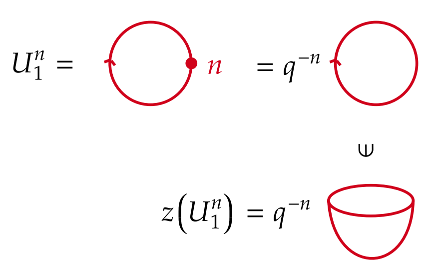
(v) An n-labeled framing point on a 1-labeled face:



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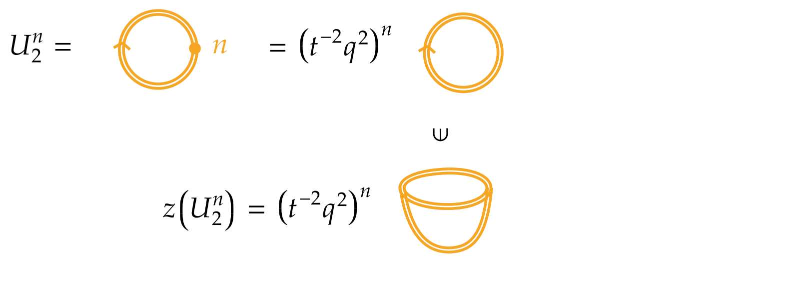
(vi) An n-labeled framing point on a 2-labeled face:



Now, for a singular $\mathfrak{gl}_2$ foam $F \subset I \times S^3$ between admissible $\mathfrak{gl}_2$ webs $W_0, W_1 \subset S^3$, we delete small 4-balls $B_1, \cdots, B_k$ from $I \times S^3$ around singular points of F, and choose disjoint framed paths in the exterior of $F \cup \sqcup_{i=1}^k B_i$ to tube each ∂B_i to $\{0\} \times S^3$ so that the links of singularities land disjointly near $\{0\} \times \infty \in \{0\} \times S^3$ as admissible links of one of the standard models (i)-(vi) above, denoted $L_1, \cdots, L_k$. We have built a nonsingular dotted $\mathfrak{gl}_2$ foam $F^\circ \colon W_0 \sqcup (\sqcup_{i=1}^k L_i) \to W_1$ in $I \times S^3$, which induces a map

$$CKhR_2^{univ}(F^\circ) \colon CKhR_2^{univ}(W_0) \otimes (\otimes_{i=1}^k CKhR_2^{univ}(L_i)) \to CKhR_2^{univ}(W_1).$$

Evaluating at cocycles $z(L_1), \cdots, z(L_k)$ at all but the first tensorial factor in the source, we obtain a map

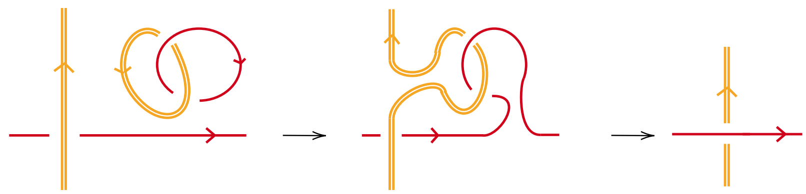
$$CKhR_2^{univ}(F) := CKhR_2^{univ}(F^\circ)(- \otimes (\otimes_{i=1}^k z(L_i))) \colon CKhR_2^{univ}(W_0) \to CKhR_2^{univ}(W_1).$$

By the same argument as in [MWW22, Theorem 5.2], $CKhR_2^{univ}(F)$ up to chain homotopy is independent of the choices of the ordering of singular points and of framed paths, and is functorial under composition of singular foams ([MWW22] only argued this on the level of homology, but the relevant facts, namely the triviality of the sweep-around move and the $\pi_1(SO(3))$-action, both hold on the chain level up to homotopy).

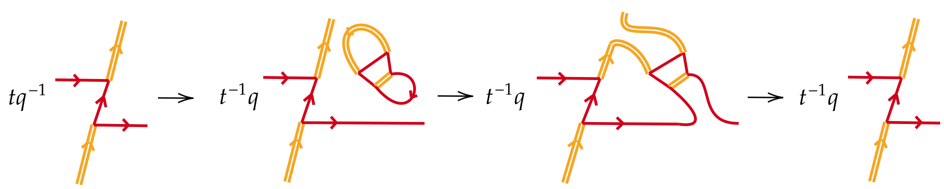
It remains to prove the two extra items in Theorem C.2.1. Since Theorem C.2.1(1) holds for nonsingular dotted $\mathfrak{gl}_2$ foams and respects compositions, it suffices to check it when F has a single singular point of type (i)-(vi) and is a product elsewhere. By our construction, in addition to the contribution from $\chi(u(F))$, the bigrading shift of a singular point of type (i)-(vi) is given by $t^{-2}q^2$, t^2q^{-2}, $t^{-4}q^4$, t^4q^{-4}, q^{-n}, $(t^{-2}q^2)^n$, respectively, each of which is consistent with Theorem C.2.1(1), proving the statement.

Before proving Theorem C.2.1(2), we reinterpret the assignment $CKhR_2^{univ}(F)$ for singular foams F on the level of diagrams. If F has a single type (i) singularity and is a product elsewhere, then up to precomposing and postcomposing with isotopies, we may assume that F takes a standard form, which is a negative-to-positive crossing change between a 1-labeled edge and a 2-labeled edge as shown in Figure C.3.

The induced map $CKhR_2^{univ}(F)$ is represented by the movie which births the element $z(H_{12}^+)$ in the complex for the Hopf link H_{12}^+ near the point at ∞, drags this Hopf link near the relevant crossing (which maintains the element $z(H_{12}^+)$ via Lemma C.2.5), and then performs the saddles and Reidemeister moves shown below.



On the level of the resolution cube, this becomes



,

where the first map is the birth of $z(H_{12}^+)$, the second map is induced by two saddles, and the third map is a Reidemeister II induced map, given explicitly by the downward arrow in [Que22, (A.15)] (note that our diagrams correspond to the mirror of those in [Que22], hence we are using his (A.15) instead of (A.16))[5]. One can check that the composition of the underlying foams is isotopic to the identity foam, and the total shift and coefficient carried by the composition are $t^{-2}q^2$. Therefore, the induced map of a standard type (i) singularity as shown in Figure C.3 is the degree shift by $t^{-2}q^2$.

In an analogous manner, one can compute the induced map of other type singularities explicitly in terms of movies on diagrams. We collect the results as follows.

Lemma C.2.8. *The induced map of a $\mathfrak{gl}_2$ foam with a single singularity, represented as a chosen movie of diagrams, is given on the level of resolutions by the maps determined from Table C.1.* □

We will not use the last two rows of Table C.1, but they are included for readers' convenience. It would be reasonable to impose the additional constraint $a_3 = a_4$ in [Que22] to remove the extra sign twists in the descriptions.

[5]The foam F depicted in [Que22, (A.15)] and some other foams therein have some color inconsistency, which the reader may ignore.

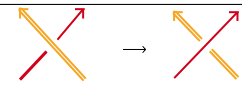
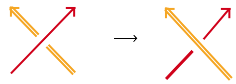
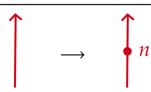
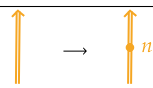
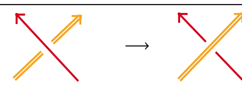
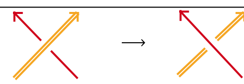
Type of singularity	Diagram	Induced map on resolution
(i)		$t^{-2}q^{2}$
(ii)		$t^{2}q^{-2}$
(iii)		$-t^{-4}q^{4}$
(iv)		$-t^{4}q^{-4}$
(v)		q^{-n}
(vi)		$(t^{-2}q^{2})^{n}$
(i)		$a_3a_4t^{-2}q^{2}$
(ii)		$a_3a_4t^{2}q^{-2}$

Table C.1: Induced maps by a singularity in a singular $\mathfrak{gl}_2$ foam.

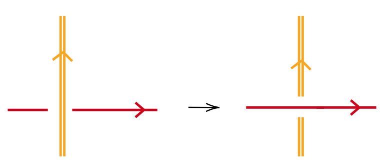


Figure C.3: The movie of a standard type (i) singularity.

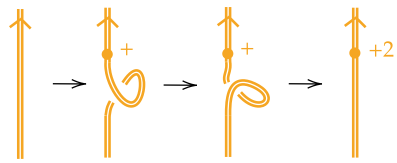


Figure C.4: A movie of a local positive self-intersection of a 2-labeled face

We are now ready to prove Theorem C.2.1(2). We first show the following topological lemma.

Lemma C.2.9. *If two singular $\mathfrak{gl}_2$ foams $F, F' \colon W_0 \to W_1$ in $I \times S^3$ have $u(F) = u(F')$, along which the germs of all data in F and F' agree, then F is related to F' by a sequence of the following moves:*

(1) Creation of a local unknotted 2-labeled framed oriented 2-sphere disjoint from F, or its inverse.

(2) Tubing 2-labeled faces along a framed arc ending on F with interior disjoint from F, or its inverse.

(3) Trading a local $\pm$-self-intersection (see Figure C.4) of a 2-labeled face with a ± 2-labeled framing point, or its inverse.

(4) A finger/Whitney move between 1- and 2-labeled faces.

(5) A finger/Whitney move between 2-labeled faces.

(6) A fork version of the finger/Whitney move through a 2-labeled face, as shown in Figure C.5.

(7) A framing change in the interior of 2-labeled faces.

(8) Collision of framing points on 2-labeled faces in a weight-preserving way, or its inverse.

Proof. Let F_1, F_2 denote the closures of the union of $1, 2$-labeled faces in F, respectively. We divide into two cases.

Case 1: F has empty seams.

Without loss of generality, assume $F_2' \neq \emptyset$. Further assume F_2' is connected by applying some moves (2). The immersed surface $(\{0\} \times F_2) \cup (I \times \partial F_2) \cup (\{1\} \times F_2') \subset \partial(I \times (I \times S^3))$ bounds an immersed 3-manifold $W \looparrowright I \times (I \times S^3)$. By general position (and further modification of W for the third item below if necessary), we may assume:

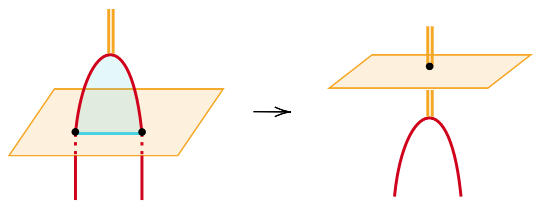


Figure C.5: A time slice of a fork-Whitney move between a 2-labeled face and a neighborhood of a point on a seam. The shaded blue region indicates a Whitney disk. Any compatible orientation is allowed.

- W has only transverse double point singularities along arcs and circles, where arcs may end either on $\partial W \subset \partial(I \times (I \times S^3))$ or on Whitney umbrella singularities in $int(W)$ [Whi44].
- W and $I \times F_1$ are in general position.
- The projection of W to the first I coordinate is Morse without index 3 critical points, and with all critical points away from $I \times F_1$.
- The projection of $W \cap (I \times F_1)$ to the first I coordinate is Morse.
- The projection of the interior of the double point locus of W to the first I coordinate is Morse.
- Local neighborhoods of Whitney umbrella singularities on W are in general position with respect to the Morse function; thus, they appear in the time movie as local $\pm$-self-intersection creations/annihilations.

By going up the first I coordinate, we see that there exists some singular $\mathfrak{gl}_2$ foam $F''\colon W_0 \to W_1$ that differs from F' only by some framing points on 2-labeled faces, so that F and F'' are related by a sequence of moves (1)(2)(3)(4)(5)(7). The self-intersections of F'' and F' are equal, since they are both determined by the common boundary data W_0, W_1; therefore, they are further related by some moves (8), as desired.

Case 2: F has nonempty seams.

Using move (2), we assume F_2 (and F_2') to be connected. If $p \in F_2$ is a self-intersection point, pick a generic path γ on F_2 connecting p to a point on a seam. One can then tube the other sheet of F_2 at p along γ and use move (6) to remove the intersection point p. Thus, we may assume F_2 (and F_2') to have no self-intersections.

Let $s = \partial F_2 = \partial F_2'$, and $\nu(s)$ be a tubular neighborhood of s. The relative homology classes represented by F_2 and F_2' in $I \times S^3 \backslash \nu(s)$ rel the common boundary differ by some meridian spheres of seams of F. If s_0 is a seam of F, using move (3), we may create a local self-intersection on F_2 near s_0. Then we may slide the self-intersection off the seam s_0 as in the previous paragraph, changing the relative homology class of F_2 by a meridian sphere of s_0. Hence, we may arrange so that F_2 and F_2' are homologous rel the common boundary in $I \times S^3 \backslash \nu(s)$.

Since F_2 and F_2' are embedded and homologous rel boundary, there is an embedded 3-manifold $W \subset I \times (I \times S^3)$ cobounding $\{0\} \times F_2$ and $\{1\} \times F_2'$, which agrees with $I \times F_2$ in $I \times \nu(s)$. Now the same Morse theory argument as in Case 1 shows that one may change F_2 to F_2' by a sequence of moves (1)(2)(4)(7)(8). $\square$

Hence, to prove Theorem C.2.1(2), it suffices to show that $CKhR_2^{univ}(F)$ changes sign under moves (1)(2) and is invariant under moves (3)-(8) described in Lemma C.2.9.

Let F and F' be related by one of the moves (1)-(8).

(1)(2): F and F' are related by a local skein relation [Que22, (3.2)] which introduces a sign change.

(7): The induced maps by F and F' are equal by the same proof as in Claim 3 of the proof of Theorem C.2.3.

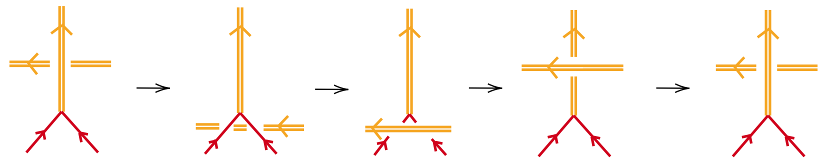
(8): This follows from the description of induced maps for type (vi) singularities in Lemma C.2.8.

(4): If F' is obtained from F by a finger move, then locally we can represent F by the constant movie and F' by a movie that does a crossing change shown in the first row of Table C.1, followed by the inverse change shown in the second row of Table C.1. Lemma C.2.8 implies that F' induces the same map as F.

(5): This is similar to (4), where we use the third and fourth rows of Table C.1 instead of the first two.

(3): For positive local self-intersections, we need to run through the moves in Figure C.4 and show that the induced map of the composition agrees with that of a type (vi) singularity with $n = 2$. By the description of moves [Que22, (A.3)(A.4)] and the third row of Table C.1, the composition induces the degree shift by $t^{-4}q^4$, which agrees with the description in the sixth row of Table C.1 for $n = 2$. The calculation for negative local self-intersections is similar; alternatively, one may decompose a negative local self-intersection annihilation as a positive local self-intersection creation followed by a Whitney move.

(6): If F' is obtained from F by a fork-finger move, then locally we can represent F by the constant movie and F' by the movie



or its reverse. The reverse of every move in the movie induces the inverse map of the corresponding forward move. Hence it suffices to check that the forward movie induces the identity map, or more conveniently, that the composition of the first two maps is equal to the composition of the inverses of the last two maps. This follows from moves [Que22, (A.57)(A.58)] and the second and fourth rows of Table C.1.

The proof of Theorem C.2.1 is complete. □

Before moving on, we note that the proof of Theorem C.2.1 can be used to provide a singular version of the functor (C.7) with domain $\mathbf{Webs}^{\text{sing}}_{I\times S,\epsilon}$, namely:

$$\mathbf{Webs}^{\text{sing}}_{I\times S,\epsilon} \xrightarrow{[\![\cdot]\!]} \mathbf{HCh}^b(\mathbf{Foams}_{S,\epsilon}). \tag{C.8}$$

C.2.4 Monoidal 2-categories and braidings

In Section C.2.2, we regarded $\mathbf{Foams}_{S,\epsilon}$, $\mathbf{Webs}_{I\times S,\epsilon}$ as 1-categories, with boundary conditions fixed. In the special case $S = I^2$ one may instead leave the boundary data free (with certain restrictions) and proceed as follows.

(1) We let $\mathbf{Foams}_{I^2,-}$ denote the monoidal 2-category defined as follows. Objects are finite subsets $\sigma \subset (0,1)$ of $\{1,2\}$-labeled signed points. The morphism category from σ_0 to σ_1 is precisely $\mathbf{Foams}_{I^2,\epsilon(\sigma_0,\sigma_1)}$, where $\epsilon(\sigma_0,\sigma_1) = (-\sigma_0 \times \{0\}) \sqcup (\sigma_1 \times \{1\})$. Thus 1-morphisms are (direct sums of grading shifts of) planar webs from σ_0 to σ_1, and 2-morphisms are foams between them up to skein relations. The monoidal structure is given by placing webs and foams side-by-side and rescaling. Every 1-morphism in the endomorphism category of the distinguished object $\emptyset$, i.e. every closed web, is isomorphic to a direct sum of grading shifts of $\mathrm{id}_\emptyset$, i.e. the empty web. Since $\mathrm{End}(\mathrm{id}_\emptyset)$ is the ground ring $\mathbb{Z}[E_1,E_2]$, the representable functor $\mathrm{Hom}(\mathrm{id}_\emptyset,-)$ witnesses an equivalence with the category of finitely generated graded free $\mathbb{Z}[E_1,E_2]$-modules which is braided (in fact, symmetric) monoidal.

(2) We let $\mathbf{Webs}^{\text{sing}}_{I\times I^2,-}$ denote the braided monoidal 2-category with the same objects as $\mathbf{Foams}_{I^2,-}$, whose morphism category from σ_0 to σ_1 is $\mathbf{Webs}^{\text{sing}}_{I\times I^2,\epsilon(\sigma_0,\sigma_1)}$. The braiding 1-morphisms use standard crossings in $I\times I^2$.

(3) We let $\mathbf{HCh}^b(\mathbf{Foams}_{I^2,-})$ denote the monoidal 2-category whose objects are the same as in $\mathbf{Foams}_{I^2,-}$, 1-morphisms are chain complexes over $\mathbf{Foams}_{I^2,-}$, and 2-morphisms are equivalence classes of homogeneous closed morphisms between such chain complexes, with components given by 2-morphisms in $\mathbf{Foams}_{I^2,-}$, considered up to homogeneous exact morphisms. It is expected, but not yet proven, that this monoidal 2-category admits a braiding[6], such that the functors (C.8) assemble into a braided monoidal 2-functor

$$\mathbf{Webs}^{\mathrm{sing}}_{I\times I^2,-} \xrightarrow{[\![\cdot]\!]} \mathbf{HCh}^b(\mathbf{Foams}_{I^2,-}).$$

Since endomorphisms of $\emptyset$ in $\mathbf{Webs}^{\mathrm{sing}}_{I\times I^2,-}$ can be identified with $\mathbf{Webs}^{\mathrm{sing}}_{\mathbb{R}^3}$, composing with a representable functor recovers the singular version

$$CKhR_2^{univ}\colon \mathbf{Webs}^{\mathrm{sing}}_{\mathbb{R}^3} \to \mathbf{HCh}^b(\mathbb{Z}[E_1, E_2])^{\mathbb{Z}}$$

of Queffelec's functor (C.4).

(4) As in [MWW22, Section 6], one can construct a braided monoidal 2-category without further higher-algebraic complications by a mixture between (2) and (3): objects and 1-morphisms are as in $\mathbf{Webs}^{\mathrm{sing}}_{I\times I^2,-}$, but spaces of 2-morphisms are computed inside $\mathbf{HCh}^b(\mathbf{Foams}_{I^2,-})$ after applying the functor (C.7). The axioms of a braided monoidal 2-category then follow from the functoriality of (C.7). We invite the reader to keep this braided 2-category in mind for the following discussion of sylleptic centers.

Remark C.2.10. Note that all (higher) categories here decompose by $\mathbb{Z}$-valued *weight* computed on objects as signed sum of all labels.

C.2.5 Graded sylleptic considerations

Recall that in a braided monoidal 2-category, every pair of objects A, B admits a braiding 1-morphism

$$A \boxtimes B \xrightarrow{R_{A,B}} B \boxtimes A$$

which is invertible up to 2-isomorphisms and equipped with higher coherence 2-morphisms that express categorified analogs of the naturality and hexagon axioms of braided monoidal 1-categories. A braided 1-category is symmetric if the double braiding of any two objects is the identity. For braided 2-categories the situation is more subtle. A coherent trivialization of the double braiding:

$$\nu_{A,B}\colon (A \boxtimes B \xrightarrow{R_{A,B}} B \boxtimes A \xrightarrow{R_{B,A}} A \boxtimes B) \xrightarrow{\cong} (A \boxtimes B \xrightarrow{\mathrm{id}} A \boxtimes B)$$

[6]In light of the homological algebra involved, it may be more natural to model this braided monoidal 2-category as a truncation of an $\mathbb{E}_2$-monoidal $(\infty, 2)$-category, i.e. as a $\mathfrak{gl}_2$ version of [Liu+24; SW24]. We refer to these articles for an in-depth discussion of the necessary higher algebra, which then accommodates all higher movie moves.

is called a *syllepsis*. Coherence of the trivializations requires, amongst others, naturality in both arguments. Note that a syllepsis can also be considered as a coherent identification

$$(A \boxtimes B \xrightarrow{R_{A,B}} B \boxtimes A) \xrightarrow{\cong} (A \boxtimes B \xrightarrow{R_{B,A}^{-1}} B \boxtimes A)$$

between positive and negative (inverse) braiding 1-morphisms.

Given a braided monoidal 2-category, one can consider its *sylleptic center* [Cra98, Section 5.1][7], which consists of objects A equipped with a coherent trivialization of the double braiding with any other object B. True to the naming, the sylleptic center is then naturally a sylleptic monoidal 2-category itself [Cra98, Theorem 5.1].

For the putative braided 2-category $\mathbf{HCh}^b(\mathbf{Foams}_{I^2,-})$ from Section C.2.4 we expect that all purely 2-labeled objects can be interpreted as objects in a $\mathbb{Z}$-*crossed* analog of the sylleptic center. Note that we have a natural $\mathbb{Z}$-action on 1-morphisms (i.e. complexes of planar webs) with the generator $1 \in \mathbb{Z}$ acting by the grading shift autoequivalence t^2q^{-2}. We observe that Section C.2.3 (with any choice of signs a_3 and a_4) provides for every purely 2-labeled object A and every other object B a coherent identification

$$(A \boxtimes B \xrightarrow{R_{A,B}} B \boxtimes A) \xrightarrow{\cong} (t^2q^{-2})^{|A|\cdot|B|/2}(A \boxtimes B \xrightarrow{R_{B,A}^{-1}} B \boxtimes A)$$

of the braiding of A and B with a grading shift of the inverse braiding, see Table C.1.

The ($\mathbb{Z}$-crossed) sylleptic center is the natural home of objects whose identity 1-morphisms admit a coherent system of *unbelting* 2-isomorphisms. The bottom projector $P^{\vee}_{\sigma,0}$ to be constructed in Section C.3 can be interpreted as projection onto the full sub-2-category on purely 2-labeled objects and thus, possibly, into the $\mathbb{Z}$-crossed sylleptic center.

C.3 $\mathfrak{gl}_2$ Rozansky projectors

In this section, we follow a recipe of Hogancamp in [Hog20, Section 5.2] to construct Rozansky projectors in the universal $\mathfrak{gl}_2$ webs and foams setting. We also give sketch proofs of properties of the Rozansky projectors stated in Proposition 4.2.1 in this setup. We follow the terminology in [Hog20], with the caveat that we are applying the dual construction of [Hog20] (see the comment at the beginning of Section 1.2 in [Hog20]).

Let σ be an object in $\mathbf{Foams}_{I^2,-}$, namely a finite $\{1,2\}$-labeled signed subset of $(0,1)$. Let $u(\sigma) \subset \sigma$ denote the subset of 1-labeled points, with signs forgotten, and let $\ell := \#u(\sigma)$. We build a Rozansky projector $P^{\vee}_{\sigma,0} \in \mathbf{K}^+(\mathbf{Foams}_{I^2,-}(\sigma,\sigma))$ which, upon forgetting the thick edges, orientations, and setting $E_1 = E_2 = 0$, recovers the Rozansky projector $P^{\vee}_{\ell,0}$ that appeared in Section 4.2.2. Here, $\mathbf{K}^+(\mathbf{Foams}_{I^2,-}(\sigma,\sigma))$ denotes the bounded below homotopy category of cochain complexes in $\mathbf{Foams}_{I^2,-}(\sigma,\sigma) = \mathbf{Foams}_{I^2,\epsilon(\sigma,\sigma)}$.

[7] Named 2-*center* there.

Let $\mathcal{B}_{u(\sigma)}$ denote the finite set of crossingless matchings of $u(\sigma)\times\{1\}\subset I^2$ up to isotopy rel boundary[8]. For each $\delta\in\mathcal{B}_{u(\sigma)}$, we pick a web $W_\delta\in\mathbf{Foams}_{I^2,-}(\tau,\sigma)$ for some (necessarily 2-labeled) object τ so that forgetting the 2-labeled edges and orientations gives $u(W_\delta)=\delta$. We also pick a 2-morphism $\eta_\delta\colon 1_\sigma\to q^{\ell/2}W_\delta\otimes W_\delta^t$ which recovers the cobordism $1_\ell\to q^{\ell/2}\delta\otimes\delta^t$ given by $\ell/2$ saddles upon forgetting 2-labeled faces and orientations. Here, 1_σ is the identity 1-morphism at σ, and $(\cdot)^t$ denotes vertical reflection composed with orientation reversal.

Let $C:=q^{\ell/2}(\oplus_{\delta\in\mathcal{B}_{u(\sigma)}}W_\delta\otimes W_\delta^t)$ be an object in the monoidal category $\mathcal{A}=\mathcal{A}_\sigma:=\mathbf{Foams}_{I^2,-}(\sigma,\sigma)$, equipped with the unit map

$$\eta_C=\oplus_{\delta\in\mathcal{B}_{u(\sigma)}}\eta_\delta\colon 1_\sigma\to C.$$

Then, an object in $\mathcal{A}$ is (left, or equivalently right) C-injective in the dual sense of [Hog20, Definition 2.5] if and only if it factors through a purely 2-labeled object.

Let $P_C^\vee\in\mathbf{K}^+(\mathcal{A})$ be defined by applying the cobar construction dual to [Hog20, (3.1)] to the unital object C in $\mathcal{A}$, which comes with a unit map $\iota_C\colon 1_\sigma\to P_C^\vee$. By Hogancamp [Hog20, Theorem 3.12], $(P_C^\vee,\iota_C)$ is an idempotent algebra in $\mathbf{K}^+(\mathcal{A})$ characterized up to homotopy by

(a) $P_C^\vee$ is a complex of C-injective objects;

(b) $\mathrm{id}_C\otimes\iota_C\colon C\to C\otimes P_C^\vee$ and/or $\iota_C\otimes\mathrm{id}_C\colon C\to P_C^\vee\otimes C$ is a homotopy equivalence.

Moreover, if $P_C^\vee, P_C'^\vee$ are two unital idempotent algebras in $\mathbf{K}^+(\mathcal{A})$ satisfying (a) and (b), then there is a unique homotopy equivalence $P_C^\vee\simeq P_C'^\vee$ up to homotopy that interwines with the unit maps up to homotopy.

We may further deloop all loops formed by 1-labeled edges in the components of $P_C^\vee$ to obtain a preferred model of the Rozansky projector, denoted $(P_{\sigma,0}^\vee,\iota_\sigma)$. Properties (1) and (2) in Proposition 4.2.1 in their webs and foams versions thus follow from the construction. When σ is purely 2-labeled, we may choose $C=1_\sigma$, hence (after a further homotopy) $P_{\sigma,0}^\vee=1_\sigma$ with the identity unit map. This shows the analog corresponding to Proposition 4.2.1(3). The analog of Proposition 4.2.1(4) follows from [Hog20, Remark 3.4].

Let σ^- denote the orientation-reversal of the flip of σ. The π-rotation of $P_{\sigma,0}^\vee$ is a complex in $\mathbf{K}^+(\mathcal{A}_{\sigma^-})$ satisfying the characterizing properties of $P_{\sigma^-,0}^\vee$. Hence, there is a canonical homotopy equivalence satisfying the analog of Proposition 4.2.1(8).

To prove the analogs of Proposition 4.2.1(5)(6)(7) in our setup, we observe the following property of the Rozansky projectors. Let $\mathbf{Foams}_{I^2,-}^{(1.5)}$ denote the collection of full subcategories of the hom-categories in $\mathbf{Foams}_{I^2,-}$ on all 1-morphisms that factor through

[8] Note that for ℓ odd, $\mathcal{B}_{u(\sigma)}=\emptyset$, and the following construction will produce $P_{\sigma,0}^\vee=0$.

purely 2-labeled objects. We think of $\mathbf{Foams}^{(1.5)}_{I^2,-}$ as an ideal of $\mathbf{Foams}_{I^2,-}$ with respect to the horizontal composition. Thus, by construction, $P^{\vee}_{\sigma,0} \in \mathbf{K}^+(\mathcal{A}^{(1.5)}) \subset \mathbf{K}^+(\mathcal{A})$, where $\mathcal{A}^{(1.5)} = \mathcal{A}^{(1.5)}_{\sigma} := \mathbf{Foams}^{(1.5)}_{I^2,-}(\sigma,\sigma)$. By [Hog20, Theorem 3.12], if $X \in \mathbf{K}^+(\mathbf{Foams}^{(1.5)}_{I^2,-}(\sigma',\sigma))$ for some σ', then $\iota_\sigma \otimes \mathrm{id}_X \colon X \xrightarrow{\simeq} P^{\vee}_{\sigma,0} \otimes X$ is a homotopy equivalence. Similarly, if $X \in \mathbf{K}^+(\mathbf{Foams}^{(1.5)}_{I^2,-}(\sigma,\sigma'))$ for some σ', then $\iota_\sigma \otimes \mathrm{id}_X \colon X \xrightarrow{\simeq} P^{\vee}_{\sigma,0} \otimes X$ is a homotopy equivalence. Morally, one should think of $P^{\vee}_{\sigma,0}$ as projecting $\mathbf{K}^+(\mathcal{A})$ onto $\mathbf{K}^+(\mathcal{A}^{(1.5)})$. The analog of Proposition 4.2.1(5) and the first part of (7) directly follow from these properties. The second part of (7) follows by bending up the lower right half of the diagrams. Finally, (6) follows by bending down the two sides of the over/understrand, since the complex in the source is then termwise, hence overall, homotopy equivalent to a complex in some $\mathbf{K}^+(\mathbf{Foams}^{(1.5)}_{I^2,-}(\sigma',\sigma))$.

C.4 The sign fixes

In this section we use the webs and foams formalism to resolve all sign ambiguities present in the proof of Theorem 4.1.2 throughout Chapter 4. Throughout this section we consider webs and foams versions of various earlier diagrams. In particular, in any previous diagram involving $P^{\vee}_{\ell,0}$, we note that the orientations of the ℓ strands determine a (purely 1-labeled) sign sequence $\sigma \in \mathbf{Foams}_{I^2,-}$, and we replace such $P^{\vee}_{\ell,0}$ (boxes in the diagrams) with $P^{\vee}_{\sigma,0}$ from Section C.3.

C.4.1 Sliding belts down

In Section 4.2.3, when deriving the isomorphism (SZ) by breaking it into a sequence of isomorphisms, the isomorphism on row (SZ4) by "sliding off" the belts was only well-defined up to sign. In the webs and foams formalism, the "sliding-off" maps are termwise given by singular foams that drag the belts off, hitting the 2-labeled strands in the middle level transversely—these can be interpreted as components of the syllepsis in the sense of Section C.2.5. By Theorem C.2.1, these termwise maps fit into a "sliding-off" isomorphism supplying (SZ4).

C.4.2 Regions B to H and R_1

We fix the sign in the proof of the commutativity of the lower triangle in region R_1 in Section 4.5.9. It suffices to fix the sign termwise. On each term of the twisted complex, the two composite cobordisms agree up to re-embedding the interior of 2-labeled faces, hence the commutativity follows from Theorem C.2.1.

We fix the sign in the proof of the commutativity of region B in Section 4.4.4, and the fixes for regions C, D, E, F, G, H are analogous. This is a consequence of Lemma C.4.1 below.

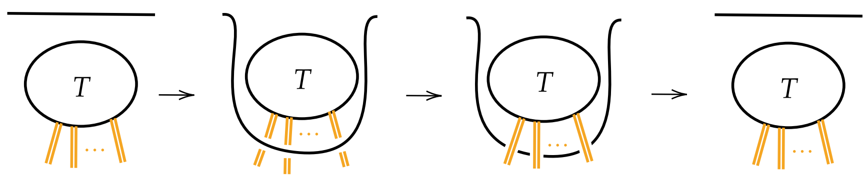


Figure C.6: The movie of a "sweep-around-across" singular foam, where T is an arbitrary tangled web, the black strand has an arbitrary label, and all strands shown have arbitrary orientations.

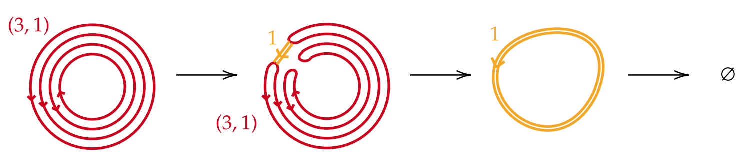
Lemma C.4.1 (Enhanced sweep-around move). *The singular $\mathfrak{gl}_2$ foam given by the movie in Figure C.6 induces the identity chain map up to homotopy on the universal $\mathfrak{gl}_2$ tangle invariant.*

Proof. Say there are k incoming 2-labeled strands connected to T, and hence k outgoing ones. When $k = 0$, the statement follows from applying Lemma C.2.5 twice. In general, use k saddles to pair up the incoming and outgoing strands, exploiting functoriality and reduce to the case $k = 0$. □

C.4.3 The barbell move

We fix the sign $c = 1$ near the end of the proof of Lemma 4.5.8.

In the webs and foams formalism, when fixing the constant c, instead of capping the (m_+, m_-) (resp. (n_+, n_-)) strands off by dotted annuli in each row of Figure 4.16, we need to perform a combination of dotted annular caps and zips, illustrated on $(+,+,+,-)$ (write for short, ambiguously, still as $(3,1)$) strands as



,

where the first map is a zip and a saddle, the second map is two dots, a cap, and a zip-cap, and the third map is a cap. Here, cyclic orderings on seams and placements of dots before zip-caps are fixed once and for all, for each sequence of $+, -$ of even length. As this capping procedure is more complicated than before, we present the relevant diagram chasing

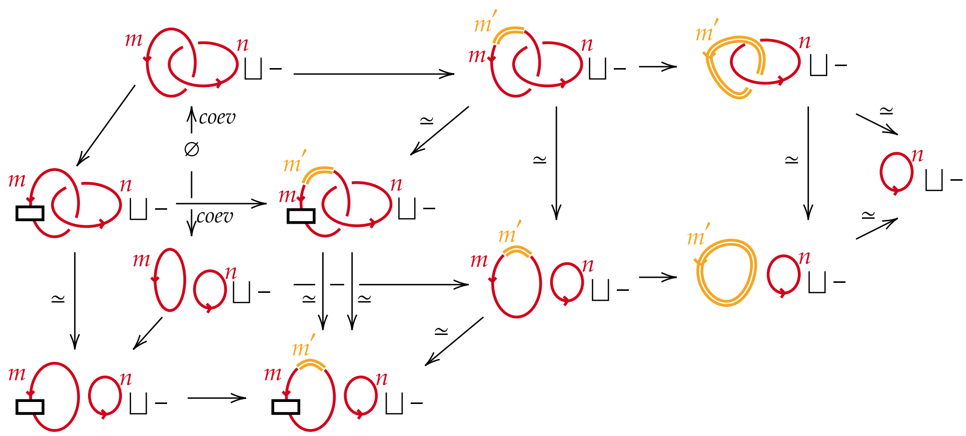


Figure C.7: Diagram chase for fixing $c = 1$ in the proof of Lemma 4.5.8.

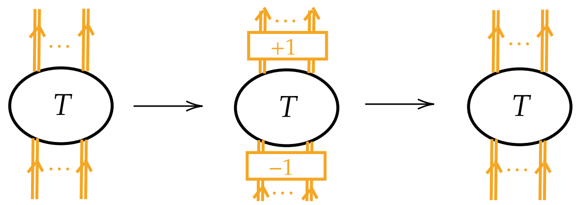
in Figure C.7. Here, only a half of each term is drawn, where the other half is understood as obtained by switching m and n, m' and n', and reversing the orientations on the m, m' strands. All squares and the triangle commute by locality and Theorem C.2.1. The two downward maps to the front middle term in the bottom are equal by Lemma C.4.1. We need to show that the composite maps from $\emptyset$ to the rightmost term are equal in KhR_2, under either the upward or the downward coevaluation map composed with any other path of maps or their inverses (if invertible). For this purpose, we stay in the terms without projectors in Figure C.7, from which the equality of the two compositions follows from Theorem C.2.1.

C.4.4 The Gluck twist

We fix the sign ambiguities in Section 4.7.2 that arose when defining various element 1's in the homology of twisted belt links, as well as when showing some compatibilities of these element 1's in Lemma 4.7.4.

Let $T \subset S^1 \times S^2$ be a standard positive/negative twisted belt link and $U \subset S^1 \times S^2$ be the corresponding standard belt link obtained by untwisting. In the webs and foams formalism, the isomorphism $\widetilde{KhR}_2^+(T) \cong \widetilde{KhR}_2^+(U)$ is obtained termwise by pushing the ± 1 twist above the Rozansky projector region to the vertical 2-labeled edges in the middle of the Rozansky projector region by simplifying Reidemeister I,II moves and fork twist moves (in the sense of [Que22]), post-composed with the singular $\mathfrak{gl}_2$ foam that undoes the ± 1 twist on the 2-labeled strands by crossing changes together with a ∓ 1 framing change on each strand. This sign fix consequently fixes the signs of $1_\pm \in \widetilde{KhR}_2^+(T)$, $1_\pm \in \mathcal{S}_0^{2,\mathrm{mul}}(D^2 \times S^2; T)$, and $1 \in \widetilde{KhR}_2^+(T(n,n)_{n_+,n_-})$.

It remains to fix the termwise signs when proving the commutativities of regions TB, TC, TD, TE, TG, TX. For regions TB, TC, TD, TE, it suffices to check that the composition



induces the identity map on homology (rather than its negative), where T is any tangle, the first map is a twist around T, the second map is an unlinking of the ±1 twists together with framing changes on 2-labeled edges. It suffices to check this after we close up the 2-labeled strands. In the closed up diagram, the second map can be replaced by the map that cancels the twists in the outer part without changing its induced map, thanks to Theorem C.2.1. The composition is now equal to the rotation by 2π around the core of the solid torus that the closed up diagrams live in. Since this rotation extends to a 2π rotation in S^3, it induces the identity map. By a re-embedding of 2-labeled faces, the same argument applies to fix the sign for region TG. The sign fix for region TX is easier and we omit the proof.

Appendix D

Elementary moves on Rozansky–Willis homology

In Chapter 4, we showed that the (bounded above, renormalized) Rozansky–Willis chain complex $\widetilde{CKhR}_2^-$ defines a lasagna $(3,1,1)$-link homology theory. On the level of objects, the functor

$$\widetilde{CKhR}_2^- : \mathbf{Links}_1 \to \mathbf{K}^{-,1/2}(\mathbb{Z})^{\mathbb{Z}}$$

is understood as follows. If $(S,L) \in \mathbf{Links}_1$, and $\phi\colon S \to S_{std}$ is an L-admissible parametrization, then the classical Rozansky–Willis chain complex $\widetilde{CKhR}_2^-(\phi(L))$ for the admissible link $\phi(L)$ is a model for $\widetilde{CKhR}_2^-(L)$, and any two such models are canonically identified.

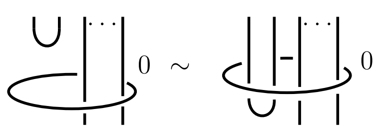
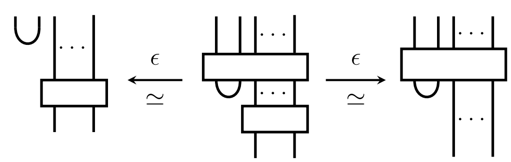
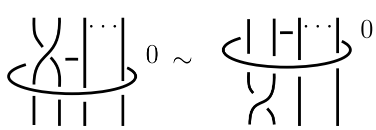
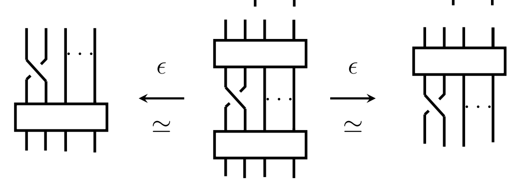
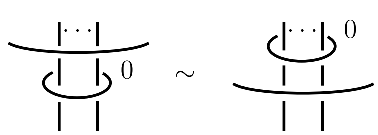
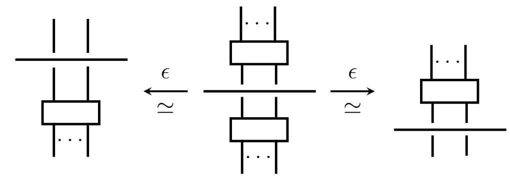
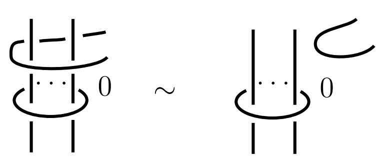
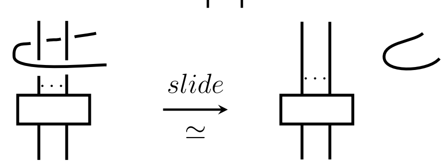
Thus, for a morphism $(W,L)\colon (S_0,L_0) \to (S_1,L_1)$, to understand its induced map, it suffices to assume $S_0 = S_{std,0}$, $S_1 = S_{std,1}$ are admissibly parametrized with respect to L_0 and L_1, and describe $\widetilde{CKhR}_2^-(W,L)$ as a chain map $\widetilde{CKhR}_2^-(L_0) \to \widetilde{CKhR}_2^-(L_1)$ between the classical Rozansky–Willis chain complexes for the admissible links L_0, L_1. In Section 4.4–Section 4.7 in Chapter 4, we described a complete list of elementary moves that any such (W,Σ) (with admissibly parametrized domain/target) decomposes into, in a dual setup. In this appendix, we collect the necessary elementary moves and their induced chain maps on $\widetilde{CKhR}_2^-$ in Table D.1. The description for chain maps is obtained by a dualization of the computations (or their inverse or mirrored versions) in Chapter 4.

In the diagrams representing Bar-Natan chain complexes in Table D.1, the boxes denote the dual of the Rozansky projectors introduced in Sections 4.2.1 (or Section C.3 in the proper $\mathfrak{gl}_2$ webs and foams setup), and are denoted $P_{\ell,0}$, where ℓ is the number of strands passing through the projector. These projectors enjoy properties dual to those described in Section 4.2.2 and the main text of Chapter 4; in particular, the following maps are well-defined up to chain homotopy:

- Counit maps, denoted $\epsilon = \epsilon_\ell \colon P_{\ell,0} \to 1_\ell$, that are dual to the unit maps $\iota_\ell \colon 1_\ell \to P^\vee_{\ell,0}$.
- Slide-off maps that slide a winding strand off a projector, denoted *slide*, that are dual to the homotopy inverses of the mirrored versions of (4.10).
- Twist absorption maps that absorb a (positive/negative) full twist into a projector, denoted *absorb*, that are dual to the homotopy inverses of (4.18).

Finally, unlike in Chapter 4, for the description presented in Table D.1, we do not keep track of the orderings of the connected components of a standard manifold $S_{std} = \sqcup_{i=1}^k \#^{m_i}(S^1 \times S^2)$ in which admissible links live, saving us some obvious elementary moves. This is justified by the symmetric monoidality of $\widetilde{CKhR}_2^-$ under disjoint unions. We omit isotopies via admissible links, which induce isotopies of the corresponding diagrams that give rise to chain complexes via Bar-Natan's construction. We also omit versions of moves E7 and E8 in which the pushing is performed on two adjacent strands that are not the leftmost ones, an upside-down version of move E7, and mirrored versions of moves E8 and E9.

Table D.1: Elementary moves and their induced maps on $\widetilde{CKhR}_2^-$.

Move	**Name**	**Where**	**Local picture**	**Chain map**
E1–E3	Reidemeister I–III moves	Sec. 4, (ii)	Classical	See e.g. [MWW22, Section 3.3–3.5]
E4–E6	birth, saddle, death	Sec. 4, (ii)	Classical	Structural maps of the Frobenius algebra $\mathbb{Z}[x]/(x^2)$
E7	finger move	Sec. 4, (iii)		
E8	crossing move	Sec. 4, (iv)		
E9	passing move	Sec. 4, (v)		
E10	handleslide	Sec. 4, (vi)		

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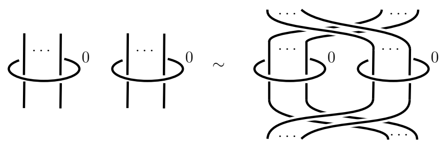
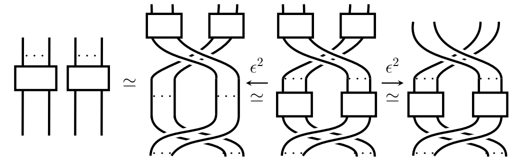
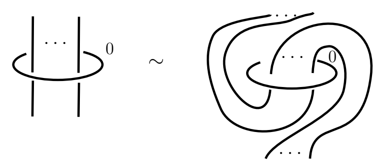
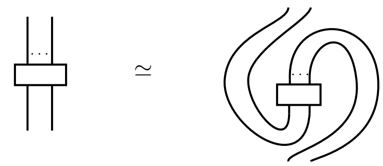
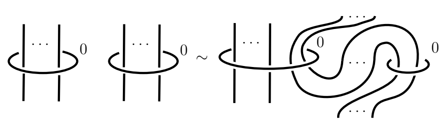
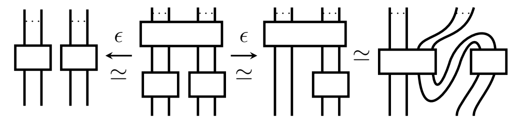
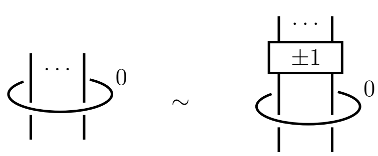
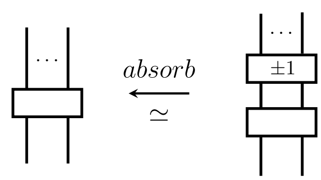
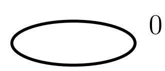
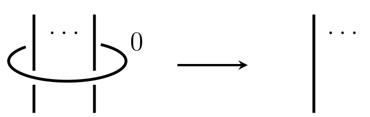
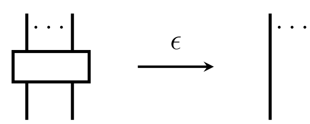
Move	**Name**	**Where**	**Local picture**	**Chain map / induced map**
E11	summand exchange	Sec. 5, (iv)		
E12	inversion	Sec. 5, (v)		
E13	2-handle slide	Sec. 5, (vi)		
E14	Dehn twist	Sec. 7		
E15	ball creation	Sec. 6, (ii)	$(\emptyset, \emptyset) \;\to\; (S^3, \emptyset)$	$\mathbb{Z} \xrightarrow{=} \mathbb{Z}$
E16	connected sum	Sec. 6, (iii)	$(\#^{m_1}(S^1 \times S^2) \sqcup \#^{m_2}(S^1 \times S^2), L_1 \sqcup L_2) \to (\#^{m_1+m_2}(S^1 \times S^2), L_1 \sqcup L_2)$	$\widetilde{CKhR}_2^-(L_1) \otimes \widetilde{CKhR}_2^-(L_2) \xrightarrow{=} \widetilde{CKhR}_2^-(L_1 \sqcup L_2)$
E17	1-handle attachment	Sec. 6, (iv)	$\emptyset \longrightarrow$ 	$\mathbb{Z} \xrightarrow{\cong} P_{0,0}$
E18	canceling 2-handle attachment	Sec. 6, (v)		

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