

UC Berkeley

IURD Reprint Series

Title

The Multiregional Matrix Growth Operator and the Stable Interregional Age Structure

Permalink

<https://escholarship.org/uc/item/3x50b7g4>

Author

Rogers, Andrei

Publication Date

1966

Peer reviewed

ANDREI ROGERS



The Multiregional Matrix Growth Operator and
the Stable Interregional Age Structure

Center for Planning & Development Research: Institute of Urban & Regional Development
University of California, Berkeley

Reprint No. 21

The Center for Planning and Development Research is a unit within the Institute of Urban and Regional Development. It is devoted to the study of a wide range of activities in environmental design and urban and regional development. Researchers in the Center are engaged in investigating the processes of urban and regional growth, the consequences of alternative patterns of growth, and the scientific bases for development policy and planning.

Toward these ends, current research is directed to improving theories and methods of planning, to building models capable of simulating urban growth processes, and to formulating improved criteria for urban and regional development policies. Fields of interest to the Center include the development of new and established communities, housing and renewal programming, the effects of technological development on cities and urban life, metropolitan planning, the economics of city-building in the United States and in developing nations, and the social planning and human development issues that accompany urbanization.

The Center contains the Architectural Design Laboratory, which is devoted to research in building technology, the behavioral bases for design, and design processes.

THE MULTIREGIONAL MATRIX GROWTH OPERATOR AND THE STABLE INTERREGIONAL AGE STRUCTURE

ANDREI ROGERS*

RESUMEN

Esfuerzos corrientes de predicción de población generalmente adoptan variantes secundarias del método de proyección de cohorte supervivencia. Esta técnica se enfoca sobre una población disgregada en cohortes, un grupo de gente que tiene una o más características comunes en un punto dado en el tiempo, y por la sujeción de cada cohorte a clase específica de tasa de fecundidad, mortalidad y migración neta, genera una distribución de sobrevivientes y descendientes de la población original, a intervalos sucesivos de tiempo.

Aunque el método de cohorte supervivencia se toma sobre un gran número de variaciones, todas ellas son esencialmente tendencias bases, dinámicos modelos espaciales de crecimiento. El elemento temporal es introducido por una estructura recursiva que opera sobre una secuencia de intervalos de unidad de tiempo. La dimensión espacial, cuando es incluida, típicamente es acomodada por réplica del análisis sobre tantas unidades areales que abarque el estudio del área. Realísticamente, sin embargo, tiempo y espacio necesitan ser considerados en conjunto con los modelos proyectados de población. La necesidad por modelos interregionales que sistemáticamente introducen movimientos lugar a lugar y simultáneamente consideran tanto lo espacial como el carácter temporal de los procesos de población interrelacionada están llegando a aumentar aparentemente.

Recientemente varios demógrafos han aprovechado la elegancia conceptual y la simplicidad de cómputo de los métodos de la matriz de análisis de población. Sus modelos, sin embargo, suponen una población "cerrada" que está sujeta solamente a los procesos de fecundidad y mortalidad. Estos, entonces, no son directamente aplicables a sistemas interregionales "abiertos" en los que la migración mas que los nacimientos y las muertes, es frecuentemente una variable y contribuidor importante en los cambios de población. Sin embargo, una extensión natural del modelo de matriz demográfica permite incorporar lugar a lugar la migración y provee un modelo de proyección de población interregional integrado que fácilmente puede ser programado por cualquiera de los modelos actuales de computadores digitales. Tal modelo es esquematizado en este trabajo.

SUMMARY

Current population-forecasting efforts generally adopt minor variants of the cohort-survival projection method. This technique focuses on a population disaggregated into cohorts, a group of people having one or more common characteristics at a point in time and, by subjecting each cohort to class-specific rates of fertility, mortality, and net migration, generates a distribution of survivors and descendants of the original population, at successive intervals of time.

Although cohort-survival methods take on a large number of variations, they all are essentially trend-based, dynamic, aspatial models of growth. The temporal element is introduced by a recursive structure which operates over a sequence of unit time intervals. The spatial dimension, when it is included at all, typically is accommodated by replicating the analysis over as many areal units as comprise the study area. Realistically, however, time and space need to be considered jointly in population-forecasting models. The need for interregional models which systematically introduce place-to-place movements and simultaneously consider the spatial as well as the temporal character of interrelated population processes is becoming increasingly apparent.

Recently several demographers have taken advantage of the conceptual elegance and computational simplicity of matrix methods of population analysis. Their models, however, assume a "closed" population which is subject only to the processes of fertility and mortality. These, therefore, are not directly applicable to interregional "open" systems in which migration is frequently a much more variable and important contributor to population change than births or deaths. However, a natural extension of the demographer's matrix model allows one to incorporate place-to-place migration and provides an integrated interregional population-forecasting model which easily may be programmed for any of the current generation of digital computers. Such a model is outlined in this paper.

INTRODUCTION

Current population-forecasting efforts generally adopt minor variants of the co-

hort-survival projection method. This technique focuses on a population of

* University of California, Berkeley.

"cohorts," a group of persons having one or more characteristics in common at a given time, and advances this population forward through time by subjecting it to the relevant class-specific birth, death, and net migration rates. The model is dynamic but essentially aspatial. Spatial considerations are accommodated by replicating the analysis over as many areal units as comprise the study area. An inter-regional system, therefore, is analyzed one region at a time.

It is becoming increasingly evident that space and time are vital elements that need to be considered jointly in order to comprehend fully the inner workings of interregional systems. In economics, for example, the call by Isard for a redress of the imbalance created by a theoretical literature focusing almost exclusively on one-point economies has triggered concerted efforts toward the development of a general theory of location and space economy.¹ Similarly, in demography the need for theories and methods which simultaneously consider the spatial as well as the temporal character of inter-related population processes is increasingly apparent.

As with many models of economic phenomena, the conditions of birth and sur-

¹ Walter Isard, *Location and Space-Economy* (New York: Technology Press of Massachusetts Institute of Technology and John Wiley & Sons, 1956).

² R. M. Solow, "On the Structure of Linear Models," *Econometrica*, XX (1952), 29-46; G. Debreu and I. N. Herstein, "Non-negative Square Matrices," *Econometrica*, XXI (1953), 597-607; M. A. Woodbury, "Characteristic Roots of Input-Output Matrices," in *Economic Activity Analysis*, ed. O. Morgenstern (New York: John Wiley & Sons, 1954), pp. 365-82.

vivorship of a human population may be conveniently expressed in the form of a set of linear, first order, homogeneous, difference equations with constant coefficients. In economics such models have generated a literature on the stability implications of particular non-negative matrices.² The same interest is beginning to be expressed by mathematical demographers who recently have taken advantage of the conceptual elegance and computational simplicity of matrix representation of population change and movement.³ For example, it is now relatively well established that the dominant characteristic root of a survivorship matrix represents the intrinsic rate of growth of a population at stability and the corresponding characteristic vector its asymptotic age distribution.⁴ These results, however, have been derived only for single-region models. In every instance, the models describe a "closed" population that is subject only to the effects of births and deaths and hence is undisturbed by migration. These models, therefore, are not directly usable in studies of interregional systems of "open" subregions. Their extension to accommodate such systems is the purpose of this paper.

MORTALITY AND FERTILITY

Mathematical demographers have demonstrated that the process of "growing" a population forward through time may be expressed by means of simple matrix multiplication.⁵ Their fundamental model may be summarized by the equation:

$$Sw^{(t)} = w^{(t+1)}, \quad (1)$$

where

$$S_{n \times n} = \begin{pmatrix} 0 & 0 \dots & b_1 & b_2 \dots b_u & 0 \dots 0 \\ {}_2d_3 & 0 & 0 & \dots & 0 \\ 0 & {}_3d_4 & 0 & \dots & 0 \\ 0 & 0 & {}_4d_5 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & {}_{n-1}d_n & 0 \end{pmatrix} \quad w^{(t)} = \begin{pmatrix} w_1^{(t)} \\ w_2^{(t)} \\ \vdots \\ w_n^{(t)} \end{pmatrix}$$

and

$w_r^{(t)}$ = population in the r th age group at time t ;

b_r = birth rate (number of surviving births per person) of the r th childbearing age group;

${}_r d_{r+1}$ = proportion of people in the r th age group who survive to the $r + 1$ st age group after a unit time interval.

This formulation may be extended to an interregional system of m regions simply by attaching a subscript to each population vector w and to every survivorship matrix S , that is,

$$S_i w_i^{(t)} = w_i^{*(t+1)}, \quad (i = 1, \dots, m). \quad (2)$$

At this point an asterisk is introduced to remind us that a component of change unique to interregional population systems, that is, migration, still needs to be incorporated. Since the effects of out-migration may be included in the survivorship proportions, ${}_r d_{r+1}$, only the contribution of in-migration still needs to be expressed. Finally, it should be noted that, in order to maintain the convenience of a single matrix operator, one may adopt the algebra of partitioned matrices and apply equation (2) to all m regions in one step:

$$\begin{pmatrix} S_1 & 0 & 0 & \dots & 0 \\ 0 & S_2 & 0 & & \vdots \\ 0 & 0 & S_3 & & \vdots \\ \vdots & & & \ddots & \\ \vdots & & & & 0 \\ 0 & \dots & \dots & 0 & S_m \end{pmatrix} \begin{pmatrix} w_1^{(t)} \\ w_2^{(t)} \\ \vdots \\ \vdots \\ w_m^{(t)} \end{pmatrix} = \begin{pmatrix} w_1^{*(t+1)} \\ w_2^{*(t+1)} \\ \vdots \\ \vdots \\ w_m^{*(t+1)} \end{pmatrix}. \quad (3)$$

GEOGRAPHICAL MOBILITY

Consider migration as a component of population change in an interregional system. Associated with every region, i , is a population vector w_i . During each unit interval of time, a certain proportion of the i th region's population, in a given age group, r , migrates to region j and enters the $r + 1$ st age group there. Thus for each age group we may construct an interregional transition matrix, ${}_r P_{r+1}$, which describes the number of people, in the r th age group in region i , who during the specified time period move into the $r + 1$ st age group in region j :

$${}_r P_{r+1} = \begin{pmatrix} {}_{r,1} \dot{p}_{r+1,1} & {}_{r,1} \dot{p}_{r+1,2} & \dots & {}_{r,1} \dot{p}_{r+1,m} \\ {}_{r,2} \dot{p}_{r+1,1} & {}_{r,2} \dot{p}_{r+1,2} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ {}_{r,m} \dot{p}_{r+1,1} & \dots & \dots & {}_{r,m} \dot{p}_{r+1,m} \end{pmatrix}, \quad (r = 1, \dots, n), \quad (4)$$

With an estimated set of n transition matrices, one may construct a series of matrix operators, M_{ij} , which, when applied to the age distribution at i , will "mi-

³ Alvaro Lopez, *Problems in Stable Population Theory* (Princeton: Office of Population Research, Princeton University, 1961), and Nathan Keyfitz, "The Population Projection as a Matrix

Operator," *Demography*, I (1964), 56-73.

⁴ Lopez, *op. cit.*, pp. 36-40.

⁵ Keyfitz, *op. cit.*

grate" the requisite number of people from region i to region j and survive them into the next age cohort:

$$M_{ij}w_i^{(t)} = k_{ij}, \quad (5)$$

where

$$M_{ij} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1,ip_{2,j} & 0 & \dots & 0 \\ 0 & 2,ip_{3,j} & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & n-1,ip_{n,j} & 0 \end{pmatrix} \quad k_{ij} = \begin{pmatrix} 0 \\ 2k_{ij} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ nk_{ij} \end{pmatrix}$$

and k_{ij} = the total number of in-migrants, in the r th age group, into region j from region i .

Summing the in-migrants over all origins, i , one finds the total in-migration into region j :

$$rk_{\cdot j} = \sum_i r k_{ij} \quad (r = 1, 2, \dots, n). \quad (6)$$

As in the case of the fertility-mortality process, one may introduce partitioned matrices to express equation (6). For example, for $j = 1$:

$$0M_{21}M_{31} \dots M_{m1} \begin{pmatrix} w_1^{(t)} \\ w_2^{(t)} \\ \cdot \\ \cdot \\ \cdot \\ w_m^{(t)} \end{pmatrix} = k_{\cdot 1}. \quad (7)$$

Thus for $j = 1, 2, \dots, m$ we have:

$$\begin{pmatrix} 0 & M_{21} & M_{31} & \dots & M_{m1} \\ M_{12} & 0 & M_{32} & \dots & M_{m2} \\ M_{13} & M_{23} & 0 & \dots & M_{m3} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ M_{1m} & \dots & \dots & \dots & 0 \end{pmatrix} \begin{pmatrix} w_1^{(t)} \\ w_2^{(t)} \\ \cdot \\ \cdot \\ \cdot \\ w_m^{(t)} \end{pmatrix} = \begin{pmatrix} k_{\cdot 1} \\ k_{\cdot 2} \\ \cdot \\ \cdot \\ \cdot \\ k_{\cdot m} \end{pmatrix}. \quad (8)$$

THE MULTIREGIONAL MATRIX GROWTH OPERATOR

The combined effects of fertility, mortality, and geographical mobility, on an interregional population system, may be expressed by adding the "super" matrix operators of equations (3) and (8).⁶ This defines an over-all "growth" matrix operator G

For the interregional problem, the d_{r+1} of the survivorship matrices, S_i , need to be modified so as to include the effects of out-migration. That is, they now denote the proportion of people in

the r th age group in region i who, after a unit time interval, are survived into the $r + 1$ st age group in the same region.

which, when applied to an interregional population distribution, will carry it forward through time. That is,

$$G = \begin{pmatrix} S_1 & M_{21} & M_{31} & \dots & M_{m1} \\ M_{12} & S_2 & M_{32} & \dots & M_{m2} \\ M_{13} & M_{23} & S_3 & \dots & M_{m3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{1m} & \dots & \dots & \dots & S_m \end{pmatrix} \quad w^{(t)} = \begin{pmatrix} w_1^{(t)} \\ w_2^{(t)} \\ \vdots \\ w_m^{(t)} \end{pmatrix} \quad (9)$$

and

$$Gw^{(t)} = w^{(t+1)}. \quad (10)$$

Each submatrix, S_i , in G now accounts for the effects of fertility and mortality in region i and for the migration out of region i . The matrices M_{ij} account for the migration of people into region i from origins j ($i \neq j$).

EXAMPLE

An example may clarify the above development of the interregional cohort-survival model. Consider the two-region system: California and the rest of the United States. Without loss of generality, assume that the total population of the two-region system is "closed," that is, undisturbed by emigration and immigration. To simplify the problem further, combine male and female cohorts, and consider a total population disaggregated into 9 ten-year age groups. By appropriately adjusting data available from United States census and California Department of Public Health records, we may construct a growth operator for the 1950-60 time interval which, when applied to the 1950 population vector, exactly reproduces the 1960 distribution. Such a matrix operator appears in Figure 1.

ASYMPTOTIC RATE OF GROWTH AND THE STABLE INTERRE- GIONAL AGE STRUCTURE

Equation (10) defines a matrix model of population growth and distribution. With an estimated operator G and an interregional set of age distributions $w^{(t)}$, we can easily compute the vector $w^{(t+1)}$.

Consider now the problem of deriving $w^{(t+n)}$. Clearly,

$$Gw^t = w^{t+1},$$

$$Gw^{t+1} = w^{t+2};$$

therefore, by substitution,

$$G^2w^t = w^{t+2}$$

and, in general,

$$G^n w^{(t)} = w^{(t+n)}. \quad (11)$$

It is of interest to note the gradual process of stabilization of G as it is raised to higher powers. For example, dividing each element of G^{30} by the corresponding element of G^{29} suggests that each member of the former matrix is a multiple of the corresponding member of the latter array. Moreover, one also notices that these multiples appear to be converging toward a common constant. Raising G to higher powers demonstrates that this indeed is the case. For example, dividing the elements of G^{80} by those of G^{79} , we find a matrix whose every member is the constant 1.2238. We conclude, therefore, that for very high powers of G (i.e., at stability):

$$G^s = \lambda G^{s-1}, \quad (12)$$

and since by equation (11)

$$G^s w^{(1)} = Gw^{(s)},$$

by simple substitution we have

$$\lambda w^{(s)} = w^{(s+1)}. \quad (13)$$

Equation (13) establishes that an interregional set of age distributions, if sub-

jected to an unchanging regime of growth G , will ultimately increase at a stable "intrinsic" rate of growth λ , where λ is the asymptotic value approached by the quotients of the elements of successively higher powers of G . Moreover, since λ is a scalar, it is clear that at stability the corresponding elements of successive population vectors w differ only by a factor λ . Thus, at stability, each age group within each region receives a constant share of the total interregional population. This allocation structure we shall define as the *stable interregional age structure* and denote it by the vector v . Therefore, in addition to equation (13), we have the following relationship at stability:

$$v^{t+1} = v^t. \quad (14)$$

Returning to our two-region example, we derive the intrinsic rate of growth $\lambda = 1.2238$, with it solve for the associated characteristic vector w , and then find the stable interregional age structure v . Table 1 traces the historical interregional age structure of our two-region example and compares previous interregional and intercohort allocations with the asymptotic one implied by the G calibrated on the basis of 1950-60 data.

CONCLUSION

This paper has developed an interregional cohort-survival population model in matrix terms. The matrix formulation has allowed us to utilize some well-established results in the theory of non-nega-

Table 1.—PAST AND ASYMPTOTIC INTERREGIONAL AND INTRAREGIONAL AGE STRUCTURES: CALIFORNIA AND THE REST OF THE UNITED STATES

Age group	1950		1960		Stability	
	Interregional age structure	Intraregional age structure	Interregional age structure	Intraregional age structure	Interregional age structure	Intraregional age structure
Total.....	1.0000		1.0000		1.0000	
California						
0-9.....	.0129	.1839	.0187	.2129	.0808	.2775
10-19.....	.0085	.1212	.0140	.1600	.0612	.2102
20-29.....	.0113	.1604	.0112	.1273	.0504	.1731
30-39.....	.0116	.1653	.0129	.1475	.0362	.1244
40-49.....	.0095	.1358	.0114	.1301	.0259	.0889
50-59.....	.0075	.1063	.0085	.0972	.0166	.0570
60-69.....	.0054	.0774	.0061	.0701	.0115	.0396
70-79.....	.0027	.0382	.0036	.0417	.0064	.0220
80+.....	.0008	.0115	.0012	.0132	.0021	.0073
Subtotal	.0702	1.0000	.0876	1.0000	.2911	1.0000
Regional decennial growth rate		1.5326		1.4847		1.2238
Rest of the U.S.						
0-9.....	.1819	.1957	.1989	.2180	.1471	.2075
10-19.....	.1357	.1460	.1532	.1680	.1301	.1836
20-29.....	.1462	.1572	.1097	.1202	.1088	.1534
30-39.....	.1394	.1500	.1233	.1351	.0979	.1382
40-49.....	.1184	.1273	.1140	.1249	.0814	.1148
50-59.....	.0954	.1026	.0921	.1009	.0631	.0891
60-69.....	.0680	.0731	.0686	.0752	.0448	.0631
70-79.....	.0343	.0368	.0412	.0452	.0268	.0378
80+.....	.0105	.0113	.0114	.0125	.0089	.0125
Subtotal	.9298	1.0000	.9124	1.0000	.7089	1.0000
Regional decennial growth rate.....		1.1230		1.1677		1.2238
National decennial growth rate.....	1.1445		1.1900		$\lambda=1.2238$	

tive matrices in order to derive both the intrinsic rate of growth and stable interregional age structure of a population that is subjected to a specified unchanging regime of growth. In this, the interregional model extends the work of stable population theorists.

The expression of an interregional demographic process in matrix terms generates a number of challenging subtopics, which emerge as a direct consequence of the matrix conceptualization. Of these, three have received some attention by the author.

1. Given time series data on an interregional age structure, w (i.e., $w^{(t)}$, $w^{(t+1)}$, ..., $w^{(t+n)}$), estimate a matrix growth operator G which, when applied to $w^{(t)}$, will reproduce as closely as possible the observed time series.⁷

2. Given a matrix growth operator G , calibrated on the basis of an n -year interval, estimate $G^{1/n}$, that is, the growth

⁷ Andrei Rogers, "Matrix Analysis of Interregional Population Growth and Distribution," *Papers and Proceedings of the Regional Science Association*, European Congress, Vienna, 1966 (in press).

operator for an "average" one-year interval.⁸

3. Given an initial interregional population vector w and a desired goal distribution g , determine if the goal is feasible and, if it is, find an intervention vector f which will redirect the system from its original asymptotic distribution toward the desired distribution.⁹

Preliminary explorations in these problem areas have provided encouraging results and point to very fertile directions for future research in matrix analysis of population growth and distribution. As is so frequently true in theory building, what at first appears to be a purely pragmatic conceptualization ultimately leads to insights that are not obtainable by conventional methods of analysis.

⁸ Andrei Rogers, "A Note on the Temporal Decomposition of Interpoint Transition Matrices" (preprint; Center for Planning and Development Research, University of California, Berkeley, 1966).

⁹ Andrei Rogers, "A Markovian Policy Model of Interregional Migration," *Papers and Proceedings of the Regional Science Association*, Philadelphia, 1965 (in press).