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Scaling Limit of Anyonic Chains and Quantum Simulation of Conformal Field Theory

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by

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To my family.

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Curriculum Vitæ

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Publications

1. Freedman, M., Shokrian Zini, M. & Wang, Z. "Quantum Computing with Octonions", **Peking Math J** 2, 239–273 (2019).
2. Cui, S.X., Shokrian Zini, M. & Wang, Z. "On generalized symmetries and structure of modular categories", **Sci. China Math.** 62, 417–446 (2019).
3. Shokrian Zini, M., Wang, Z. "Conformal Field Theories as Scaling Limit of Anyonic Chains", **Commun. Math. Phys.** 363, 877–953 (2018).

Abstract

Scaling Limit of Anyonic Chains and Quantum Simulation of Conformal Field Theory

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We provide a mathematical definition of a low energy scaling limit of a sequence of general non-relativistic quantum theories in any dimension, and apply our formalism to anyonic chains. We formulate Conjecture 2.5.3 on conditions when a unitary rational $(1+1)-$d conformal field theory would arise as such a limit and verify the conjecture for the Ising minimal model $M(4,3)$ using $\mathfrak{su}(2)_2$ anyonic chains. Our work is based on the existence of a Fourier transform relation between Temperley-Lieb generators $\{e_i\}$ and some finite stage operators of the Virasoro generators $\{L_m + L_{-m}\}$ and $\{i(L_m - L_{-m})\}$ for unitary minimal models $M(k+2,k+1)$ (proven for $k=2$). An earlier attempt [1], called the Koo-Saleur formula, has slower convergence with characteristics that hinder the convergence of *algebras* of observables, an important contribution of this work. Assuming Conjecture 2.5.3, most of our main results for $M(4,3)$ hold for higher $(k \geq 3)$ unitary minimal models $M(k+2,k+1)$ as well. Our approach is supported by extensive numerical simulation and physical proofs in the physics literature. It is also inspired by an eventual application to an efficient simulation of conformal field theories by quantum computers. We approach the definition of the unitary evolution and correlator simulation problems in the same spirit of topological quantum field theory simulation as established by M. Freedman et. al. [2]. Under certain conditions, we present complexity theoretic hardness results on the simulation problems by using the framework of fermionic quantum computation by Bravyi and Kitaev [3].

Contents

Chapter 1

Introduction

1.1 Thesis outline

In this chapter, we review the background materials for the thesis involving mathematical models of topological and conformal quantum field theory and anyonic chains.

The second chapter is mostly based on our work on "CFT as Scaling Limit of Anyonic Chains" [4]. We start by outlining our results on scaling limit of anyonic chains (2.1). We motivate the discussion behind scaling limit and provide a detailed overview of the main theorems. This includes the ideas behind the proofs and conjectures for future works. With this background, previous works are then discussed (2.2). The next sections (2.3-2.5) are all devoted to proving the results in the outline and eventually (2.6) discussing in more details the conjectures and possible future research directions.

The structure of the third chapter is similar and a continuation of the relevant work in [4]. We introduce the simulation problems briefly (3.1) before reviewing previous works on quantum field theory simulation (3.2). Next, we discuss two simulation problems concerning unitary evolution (3.3) and correlation function (3.4). We motivate our definitions for these problems and outline our partial results. We also include discussions

in each case with regards to why we fail to prove the problems are in **BQP**, and what the alternative approaches could be. The last sections include proving those results (3.5) and showing our definitions are well-defined (3.6).

1.2 Modular Tensor Category (MTC)

Reshetikhin and Turaev introduced a 3-manifold invariant in [5] using modular tensor categories (MTCs), believed to be the mathematical realization of Witten's path integral formulation of TQFT from non-abelian Chern-Simon theories. In this thesis, we will consider MTC as the model of $(2+1)-$d TQFT. An MTC is a category with a long list of extra structures. As the exact details of the definition are not used, we briefly overview the concept while mentioning the connection to physics along the way. We refer to [6] for the details.

A modular tensor category comes with a diagram calculus which allows us to compute morphism composition. The diagram calculus will be introduced in figures below as each structure is defined. In this calculus, morphisms are composed by stacking up their corresponding diagram, and tensor product of morphisms is done by putting their diagrams next to each other (Figure 1.1). The identities that will be introduced later which the so-called structure morphisms will have to satisfy, are the necessary identities making the morphisms invariant with respect to diagram isotopies. Hence, one is able to alter the diagram without changing the morphism as long as the motion is an isotopy (just like in identifying two knot diagrams).

We start with an **abelian semisimple** category $\mathcal{C}$ with $\mathbb{C}-$**linear** Hom spaces and add a **monoidal** structure on $\mathcal{C}$. This is given by a bi-functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ with the following additional data.

- There is an *identity* element *1* (vacuum) with respect to $\otimes$ (corresponding to fu-

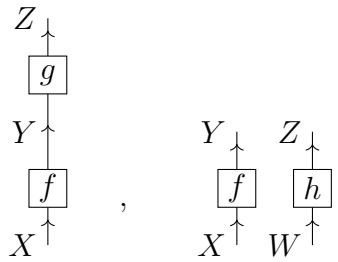

Figure 1.1: $g \circ f : X \to Z, \quad f \otimes h : X \otimes W \to Y \otimes Z$.

sion). More precisely, there are left and right unit constraints $l_X : \mathit{1} \otimes X \xrightarrow{\simeq} X, r_X : X \otimes \mathit{1} \xrightarrow{\simeq} X$.

- There are isomorphism structure maps called associators $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\simeq} X \otimes (Y \otimes Z)$.

These satisfy compatibility axioms expressed in the form of commuting *pentagon* and *triangle* diagrams (Figures 1.2 and 1.3).

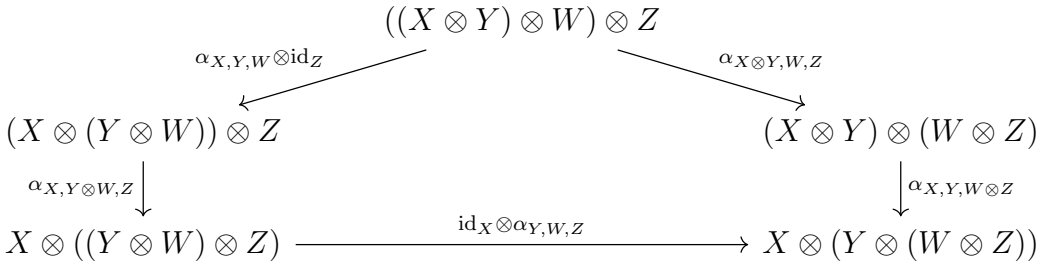

Figure 1.2: Pentagon axiom.

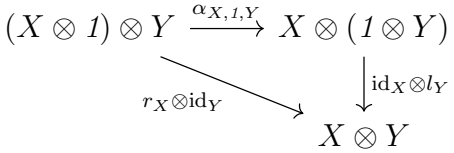

Figure 1.3: Triangle axiom.

Next, assume a **rigidity** structure on $\mathcal{C}$. This corresponds to having a left and right *dual* ${}^*X, X^*$ for each object $X \in \mathrm{Obj}(\mathcal{C})$ with structure maps $\mathrm{ev}_X : X^* \otimes X \to \mathit{1}, \mathrm{coev}_X :$

$\mathit{1} \to X \otimes X^*$ for the right dual and similarly for the left dual $\mathrm{ev}'_X, \mathrm{coev}'_X$. Ultimately, the left and right dual will turn out to be isomorphic and correspond to antiparticles in physics terminology. The structure maps satisfy compatibility axioms with the monoidal structure. One such axiom is the composition below being id_X. Other axioms are derived similarly.

$$X \xrightarrow{l_X^{-1}} \mathit{1} \otimes X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_X} (X \otimes X^*) \otimes X \xrightarrow{\alpha_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X \otimes \mathit{1} \xrightarrow{r_X} X.$$

Figure 1.4: Diagrams for $\mathrm{coev}_X, \mathrm{ev}_X$. By convention, the identity object *1* is represented by "empty" at the bottom of coev_X and top of ev_X.

Let $L(\mathcal{C})$ denote the set of isomorphism class of simple objects. Mathematically, one can view them as irreducible representations of some algebra, and physically, they will represent the quasi-particles (*anyons*) of a $(2+1)-$d TQFT.

Definition 1.2.1 *A fusion category is an abelian semisimple $\mathbb{C}-$linear rigid monoidal category such that 1 is simple and $L(\mathcal{C})$ is finite. We call $|L(\mathcal{C})|$ the rank of $\mathcal{C}$.*

Finiteness of $L(\mathcal{C})$ usually corresponds to some algebra having finitely many irreducible representations. We will see this in the case of *rational* CFTs. A fusion category has *fusion rules* N_{ab}^c. These are nonnegative integers that can be interpreted as the multiplicities of a simple object (anyon) c in the tensor product (fusion) of $a, b \in L(\mathcal{C})$:

$$a \otimes b = \oplus_c N_{ab}^c c.$$

Note $N_{ab}^c = \dim_{\mathbb{C}} \mathrm{Hom}(c, a \otimes b)$ as a $\mathbb{C}-$vector space. A fusion is shown by a *trivalent*

graph (Figure 1.5).

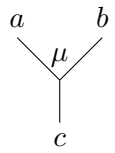

Figure 1.5: Trivalent graph with $\mu \in \mathrm{Hom}(c, a \otimes b)$.

Schur's lemma holds in an MTC, i.e. $\mathrm{Hom}(X, X) \cong \mathbb{C}$ for $X \in L(\mathcal{C})$ (Figure 1.6). This is applied extensively while using diagram calculus; a diagram of any shape sending X to X is always equal to a scalar.

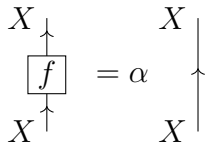

Figure 1.6: Schur's lemma: $f \in \mathrm{Hom}(X, X) \implies f = \alpha\ \mathrm{id}_X$ for $X \in L(\mathcal{C})$.

Next we add a **pivotal** followed by a **spherical** structure. The pivotal structure identifies the *double dual* with the identity functor, allowing us to define the notion of left and right *trace*. The spherical structure is built on top of that to identify the left and right trace.

A pivotal structure is a natural isomorphism $\delta : \mathrm{id}_{\mathcal{C}} \to (\cdot)^{**}$ which is compatible with monoidal and dual structure:

$$\delta_{X \otimes Y} = \delta_X \otimes \delta_Y, \quad \delta_1 = \mathrm{id}_1, \quad \delta_{X^*}^{-1} = \delta_X^*.$$

The notion of spherical structure requires us to define the left and right trace of an endomorphism (Figure 1.7). Let $f \in \mathrm{Hom}(X, X)$ and define the left trace as $\mathrm{Tr}^l(f) = \mathrm{ev}_X \circ (\mathrm{id}_{X^*} \otimes f) \circ (\mathrm{id}_{X^*} \otimes \delta_X^{-1}) \circ \mathrm{coev}_{X^*}$. The right trace is similarly defined. Spherical means $\mathrm{Tr}^l = \mathrm{Tr}^r$, which allows the unambiguous definition of *trace* $\mathrm{Tr} := \mathrm{Tr}^l = \mathrm{Tr}^r$, and

the *quantum dimension* of simple objects $\dim X := \mathrm{Tr}(\mathrm{id}_X)$.

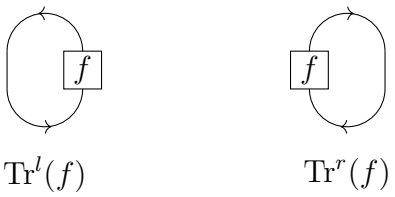

Figure 1.7: Left and right trace for $f : X \to X$.

Next, we introduce **braiding**. This corresponds to moving quasiparticles around each other in a 2−d space. Braiding is a natural isomorphism $c : \otimes \to \otimes \circ \mathit{Swap}$ where *Swap* is the swap functor on $\mathcal{C} \times \mathcal{C}$, in other words $c_{X,Y} : X \otimes Y \xrightarrow{\simeq} Y \otimes X$ (Figure 1.8). Compatibility with all previously introduced structures, notably the monoidal structure, are imposed, including for example the hexagon commutative diagram 1.9.

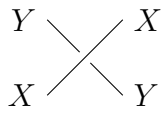

Figure 1.8: $c_{X,Y} : X \otimes Y \xrightarrow{\simeq} Y \otimes X$.

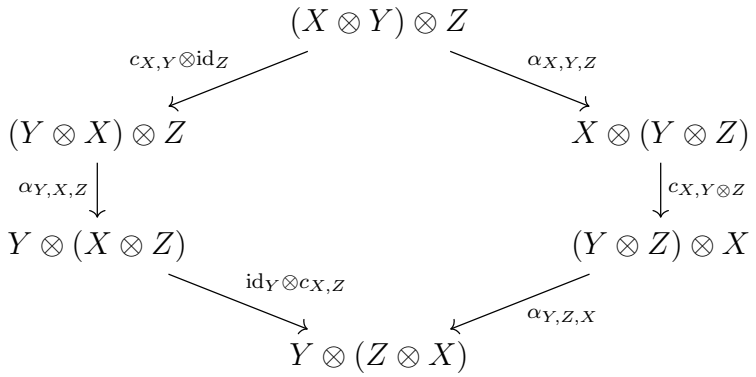

Figure 1.9: Hexagon axiom. The same must hold if we replace $c_{a,b}$ with $c^{-1}_{b,a}$.

Definition 1.2.2 *A spherical braided category is a ribbon category. A pre-modular tensor category is a ribbon fusion category.*

Given a pre-modular tensor category $\mathcal{C}$, one can form the *S-matrix* with entries $s_{X,Y}$ given by (Figure 1.10)

$$s_{XY} = \mathrm{Tr}(c_{Y,X} \circ c_{X,Y}), \quad \forall X, Y \in L(\mathcal{C})$$

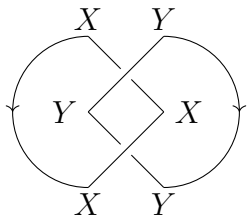

Figure 1.10: $s_{X,Y} = \mathrm{Tr}(c_{Y,X} \circ c_{X,Y})$.

Definition 1.2.3 *If the $S-$matrix is non-degenerate, $\mathcal{C}$ is called a modular tensor category.*

The *T-matrix* is a diagonal matrix given by $T_{XX} = \theta_X$ where $\theta_X \mathrm{id}_X$ is the morphism given by Figure 1.11 (by Schur's lemma).

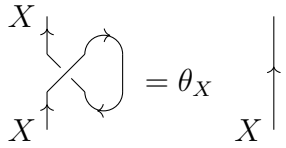

Figure 1.11: Twist θ_X.

These matrices satisfy

$$(ST)^3 = e^{\frac{2\pi i c}{8}} S^2 \quad , \quad S^2 = C.$$

$C = (\delta_{ij^*})_{ij}$ is the charge conjugation matrix and c is the (topological) central charge which is defined modulo 8. Furthermore, the Anderson-Moore-Vafa theorem states that

T is a finite order matrix and the central charge c is rational. This implies $\theta_X = e^{2\pi i h_j}$ for some $h_j \in \mathbb{Q}$. The notation h_j is also encountered later in CFT (conformal weight). The data S, T of an MTC is called its *modular data* as it gives a projective representation of $\mathrm{SL}(2, \mathbb{Z})$.

We will be working with *unitary* MTCs. This means a conjugation $\overline{f}$ of morphisms, with respect to which all structure morphisms introduced so far are *compatible*. Conjugation should be thought of as taking the diagram of f upside-down. Further, we have a positive definite hermitian form on the Hom spaces provided by the trace

$$\langle g, f \rangle = \mathrm{Tr}(\overline{g} \circ f), \qquad \text{for } g : X \to Y, f : X \to Y.$$

Given a unitary fusion category, one can compute the change of basis coefficients in $\mathrm{Hom}(a, (b \otimes c) \otimes d) \to \mathrm{Hom}(a, b \otimes (c \otimes d))$ using the unitary $F-$move in Figure 1.12 (also called the 6−j symbols because of the six parameters $a, b, c, d, e, \widetilde{e}$). If the fusion rules are *multiplicity-free*, meaning $N_{xy}^z \in \{0, 1\}$ for all $x, y, z \in L(\mathcal{C})$, the labels for **intertwiners** like $\nu \in \mathrm{Hom}(e, b \otimes c)$ are discarded.

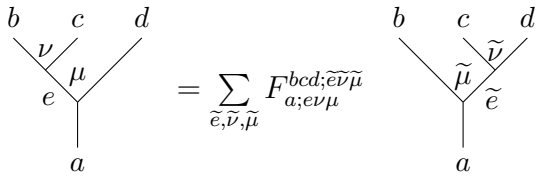

Figure 1.12: The $F-$move.

One can transform a *fusion tree* in Figure 1.13 to any other with any pattern of branching from a to $j_1 \otimes \ldots \otimes j_n$ using repeatedly the $F-$move.

A basis for $\mathrm{Hom}(a, j_1 \otimes \cdots \otimes j_n)$ is given by the *admissible* fusion trees, where admissible means all possible choices of x_i in Figure 1.13 not violating the fusion rules. Computation shows that this is an *orthogonal* basis [7].

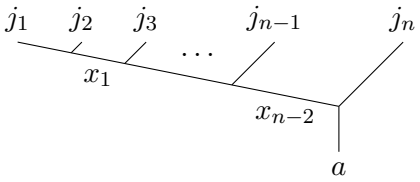

Figure 1.13: Basis for $\mathrm{Hom}(a, j_1 \otimes \cdots \otimes j_n)$ in a multiplicity-free setting.

Remark 1.2.1 *The basis given by the configurations $|x_1 \dots x_{n-2}\rangle$ provides the so-called hidden localities that are used in simulating topological quantum field theory on a quantum computer. The gates in the paradigm of topological quantum computation are the braidings, as unitarity means braiding is also unitary $\overline{c_{X,Y}} = c_{X,Y}^{-1}$.*

1.2.1 Affine Kac-Moody Lie algebra $\mathfrak{su}(2)_k$ MTC

Here, we review the unitary MTC given by irreps of the affine Kac-Moody Lie algebra $\mathfrak{su}(2)_k$ ($\mathfrak{su}(2)$ at level k) for $k \geq 2$. The $k+1$ simple objects are irreducible modules of WZW model $\mathfrak{su}(2)_k$ introduced later in 1.3.7. They are labelled by half-integers $L(\mathcal{C}) = \{0, \frac{1}{2}, \dots, \frac{k}{2}\}$. The fusion rules are the multiplicity-free fusion rules of $\mathfrak{su}(2)$ (Clebsch-Gordan rule) capped at level k:

$$j_1 \otimes j_2 = |j_1 - j_2| \oplus (|j_1 - j_2| + 1) \oplus \dots \oplus \min\{j_1 + j_2, k - j_1 - j_2\}. \tag{1.1}$$

The $S-$matrix is given by

$$S_j^{j'} = \sqrt{\frac{2}{k+2}} \sin\Big(\frac{\pi(2j+1)(2j'+1)}{k+2}\Big),$$

with the central charge $c = \frac{3k}{k+2}$.

We list the important data for $\mathfrak{su}(2)_2$ for future reference. Let us use the labels

$\{1, \sigma, \psi\}$ instead of $\{0, \frac{1}{2}, 1\}$. Fusion rules are as follows

$$\sigma^2 = 1 + \psi, \sigma\psi = \psi\sigma = \sigma, \psi^2 = 1$$

with quantum dimensions $\{1, \sqrt{2}, 1\}$. The twists are $\theta_1 = 1, \theta_\sigma = e^{\frac{3\pi i}{8}}, \theta_\psi = -1$ with central charge $c = \frac{3}{2}$. $S-$matrix is

$$S = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}.$$

Finally, the nontrivial entries of the $F-$move are given by

$$F_\sigma^{\sigma\sigma\sigma} = -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, F_\sigma^{\psi\sigma\psi} = (-1), F_\psi^{\sigma\psi\sigma} = (-1).$$

We refer to [8, section 5.3] for more details.

1.2.2 Temperley-Lieb algebra

The Temperley-Lieb algebra operators e_i are depicted in Figure 1.14. These operators act on a fusion tree in Figure 1.13 when $j_i = j$, by stacking up at x_i-th position (connecting j_{i-1}, j_{i+1}). Note that in $\mathfrak{su}(2)_k$, all anyons are self-dual.

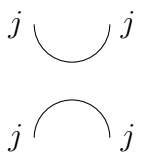

Figure 1.14: TL operator for self-dual spin $j \in L(\mathcal{C})$.

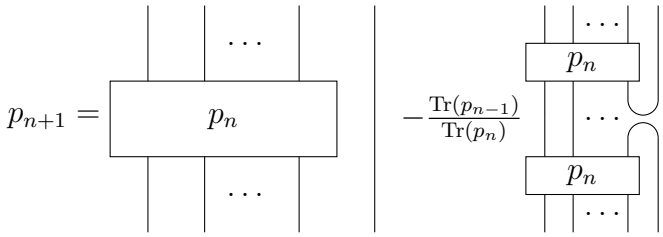

Figure 1.15: Jones-Wenzl projectors recursive relation.

In the case of $\mathfrak{su}(2)_k$, the TL operators satisfy the identities

$$e_i^2 = de_i, \quad e_i e_{i\pm1} e_i = e_i, \quad [e_i, e_j] = 0, \text{ for } |i-j| > 1, \tag{1.2}$$

where $d = \dim j$, and more explicitly

$$e_i \left|x_{i-1} x_i x_{i+1}\right\rangle = \sum_{x_i' \in L(\mathcal{C})} (e[i]^{x_{i+1}}_{x_{i-1}})^{x_i'}_{x_i} \left|x_{i-1} x_i' x_{i+1}\right\rangle, \quad (e[i]^{x_{i+1}}_{x_{i-1}})^{x_i'}_{x_i} = \delta_{x_{i-1},x_{i+1}} \sqrt{\frac{S^0_{x_i} S^0_{x_i'}}{S^0_{x_{i-1}} S^0_{x_{i+1}}}}. \tag{1.3}$$

1.2.3 Jones-Temperley-Lieb JTL_k MTC

There is a closely related MTC to the $\mathfrak{su}(2)_k$ MTC and TL algebra, obtained from the *Jones-Temperley-Lieb* algebra JTL_k. This algebra is itself obtained by taking the quotient of the TL algebra for spin $j = \frac{1}{2}$ by *Jones-Wenzl projectors* $p_{k+1}, k \geq 0$. We only briefly review this MTC to clear any confusion that may arise due to their similarity with $\mathfrak{su}(2)_k$, and the fact that our anyonic chains are based on $\mathfrak{su}(2)_k$ but there are others in the literature ([9]) that are based on JTL_k.

The projectors are defined inductively as in Figure 1.15 with initial values $p_0 = \mathrm{id}_0, p_1 = \mathrm{id}_{\frac{1}{2}}$, the identity morphism of vacuum and $\frac{1}{2}$, respectively. These projectors themselves can form an MTC with the diagram calculus inherited from $\mathfrak{su}(2)_k$.

For k even, JTL_k MTC has simple objects $\{p_0, \ldots, p_k\}$, while for k odd, the even

indices form an MTC. JTL_2 is called the *Ising MTC*, while JTL_3 with objects $\{1, \tau\}$ is called the *Fibonacci* MTC, with τ the Fibonacci anyon. The fusion rules are multiplicity-free and identical to those of $\mathfrak{su}(2)_k$ by identifying $p_i \to \frac{i}{2}$.

As an example, to obtain JTL_2, we quotient TL by

$$p_3 = e_j e_{j+1} + e_{j+1} e_j - \sqrt{2}(e_j + e_{j+1}) + 1.$$

JTL_2 MTC has objects $\{p_0, p_1, p_2\}$ which is also relabelled by $\{1, \sigma, \psi\}$ and is closely related to $\mathfrak{su}(2)_2$ MTC.

Remark 1.2.2 *There is a difference in the* ***Frobenius-Schur*** *indicator of the spin σ [8] which can be observed by comparing θ_σ in both MTCs.*

The fusion rules and quantum dimensions are the same as $\mathfrak{su}(2)_2$. The twist at σ differs:

$$\theta_1 = 1, \quad \theta_\sigma = e^{\frac{\pi i}{8}}, \quad \theta_\psi = -1.$$

Furthermore, the central charge is $c = \frac{1}{2}$ and while the $S-$matrix is the same, the nontrivial fusion entries differ at $F_\sigma^{\sigma\sigma\sigma}$ by a negative sign:

$$F_\sigma^{\sigma\sigma\sigma} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad F_\sigma^{\psi\sigma\psi} = (-1), \quad F_\psi^{\sigma\psi\sigma} = (-1).$$

1.3 Vertex Operator Algebra (VOA)

A VOA is a mathematical axiomatization of the chiral algebra of a CFT. However, the story behind the discovery of this framework has less to do with physics. A series of bold conjectures, named *Monstrous moonshine*, emerged in the late 1970s and early 1980s [10], describing a profound connection between modular forms in number theory

and the Monster group in sporadic group theory.

The field of $SL(2,\mathbb{Z})-$modular invariant functions with complex variable τ in the upper half-plane is generated by the modular function

$$J(q) = q^{-1} + 196884q + 21493760q^2 + \ldots, \quad \text{where } q = e^{2\pi i\tau}. \tag{1.4}$$

The Monstrous moonshine conjectures were based on a relationship behind the sequence of coefficients of J with the dimensions of the irreducible representations of the Monster group (McKay-Thompson and Conway-Norton conjectures).

It was proved in 1980s ([11, 12, 13, 14]) that these numbers are dimensions of representations of the Monster group; but more importantly, it was shown that the (modular invariant) character of a VOA $\mathcal{V}^\natural$, called *Monster* VOA (or the *moonshine module*), coincides with $J(q)$ with some appropriate scaling. Like all VOAs, it is a graded vector space endowed with some algebra structure but its symmetry is provided by the Monster group. We refer to the introductory materials in [15] for more details.

This new mathematical structure $\mathcal{V}^\natural$ was then interpreted as an inherently string-theoretic structure: the "chiral algebra" underlying the $\mathbb{Z}_2-$orbifold *conformal field theory* based on the 24-dimensional Leech lattice [16, 12].

VOA is our preferred framework for (chiral) CFT. We will provide two important examples of VOAs, the unitary minimal models and WZW models $\mathfrak{su}(2)_k$. An important fact common to both of these models ties the discussion of MTC and VOA [17, and references therein]:

The representation category of the VOA forms a unitary MTC.

Remark 1.3.1 *Regarding the connection between $(2+1)-d$ TQFT represented by an MTC and a rational $(1+1)-d$ CFT, we need to mention the FFRS formalism [18, 19,*

20, 21, 22, 23] which is a rigorous construction and classification of all rational $(1+1)-d$ *CFTs on compact surfaces (for all genera). This formalism establishes the connection between* $(2+1)-d$ *TQFT and a rational* $(1+1)-d$ *CFT, but its construction is at a high level and not useful for detailed computational purposes.*

Remark 1.3.2 *VOAs should be thought of as studying CFT infinitesimally at "the level of Lie algebra", where representations of the Virasoro Lie algebra are taken into account. On the other hand, LCN, another framework discussed in section 1.5, provides the Lie group point of view, where representations of the Lie group of orientation preserving diffeomorphisms of the circle* $Diff_+(S^1)$ *are taken into account.*

From a mathematical point of view, VOA is the one-complex-dimensional analogue of *both* Lie algebra and commutative associative algebra [15]; a statement that is made precise using operad language [24]. This remarkable fact shows the richness of the theory. This will be more evident as we will be deriving infinitely many complicated identities involving *non*associative products generated by a single commutativity (or associativity or Jacobi) identity.

The notations and definition for VOA follow closely those of [25] and will use formal calculus as an essential tool. Let $\mathbb{N}_0$ be the set of nonnegative integers. Consider an $\mathbb{N}_0-$graded $\mathbb{C}-$vector space $\mathcal{V} = \oplus_{n=0}^{\infty}\mathcal{V}_n$, where the *weight* spaces $\mathcal{V}_n$ satisfy $\dim \mathcal{V}_n < \infty$, equipped with a linear map called the *vertex operator*, mapping into **formal** Laurent series with coefficients in $\mathrm{End}(\mathcal{V})$

$$Y(\cdot, z) : \mathcal{V} \to \mathrm{End}(\mathcal{V})[[z, z^{-1}]], \quad Y(v, z) = \sum_{n\in\mathbb{Z}} v_{(n)} z^{-n-1},$$

where z is the formal variable and $v_{(n)} \in \mathrm{End}(\mathcal{V})$ are called the *mode* operators of v. The

mode operators satisfy

$$v_{(n)}u = 0, \text{ for all } v, u \in \mathcal{V} \text{ and } n \text{ sufficiently large.}$$

The vertex operator can be looked at as the analogue of the product from the left in an algebra, where $Y(a,z)b$ should be thought of as $a.b$, or complexified as $a^z.b$. This last analogy can be made precise [26], but we will use it as a guide for intuition.

As will be shown later, the vertex operator $Y(a,z)$ implements the state-operator correspondence of CFTs. They are the field operators which insert the state a at a space-time point $z = 0$ with a small neighborhood locally parameterized by z.

As a different notation, which will be motivated later, for a *homogeneous* vector v in some weight space with weight wt v, we can shift the index to obtain

$$Y(v,z) = \sum_{n\in\mathbb{Z}} (v)_n z^{-n-\text{wt } v} = z^{-\text{wt } v} \sum_{n\in\mathbb{Z}} (v)_n z^{-n},$$

where $(v)_n = v_{(n+\text{wt } v-1)}$.

Further, there are two distinguished vectors, the *vacuum* $\Omega \in \mathcal{V}_0$ and the *conformal* or *Virasoro* vector $\omega \in \mathcal{V}_2$.

The vacuum field is the identity $Y(\Omega,z) = \mathbf{1}(= \text{id}_{\mathcal{V}})$. We set a convention throughout this thesis to use $\mathbf{1}$ as the identity operator. The *creation* property holds

$$v_{(-1)}\Omega = v \implies Y(v,z)\Omega = v + \ldots \in \mathcal{V}[[z]].$$

Creation property can be seen as the analog of the unit axiom in the definition of an algebra with unit. It gives the *operator-state* or *field-state correspondence* $\lim_{z\to 0} Y(v,z)\Omega = v$ when we replace the indeterminate z with a complex number. That is why we may use the terms "conformal field" and "conformal vector" interchangeably.

Remark 1.3.3 *Motivated from the analogy of product in an algebra, a generating set $\mathcal{S}$ for the VOA is defined as a set for which $\{\prod_i s^{(i)}_{(n_i)}\Omega | s^{(i)} \in \mathcal{S}, n_i \in \mathbb{Z}\}$ spans $\mathcal{V}$ as a vector space.*

The Virasoro vector ω gives the modes and field

$$\omega_{(n+1)} = (\omega)_n = L_n, \quad Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2},$$

where the L_ns generate the *Virasoro Lie algebra* $\mathcal{L}$. This has relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0} \cdot \mathbf{1}, \ \forall m, n \in \mathbb{Z}, \tag{1.5}$$

where the constant c is called the central charge (also called rank $\mathcal{V}$) and $\mathbf{1}$ is the central element of $\mathcal{L}$ acting as identity on $\mathcal{V}$. It turns out that the Virasoro field is a *descendant* of the vacuum, i.e. obtained by applying Virasoro modes on the vacuum as $\omega = \omega_{(-1)}\Omega = L_{-2}\Omega$.

Remark 1.3.4 *The Virasoro algebra is the central extension of the Witt algebra $\{l_n\}$, the lie algebra of $Diff_+(S^1)$. More explicitly, one has $l_n = -\frac{\partial}{\partial z} z^{-n+1}$. This algebra is related to conformal symmetry (and thus to CFT) as holomorphic maps can be written as exponentials of $\sum_{i=-1}^{\infty} c_i l_i$, while Möbius maps as exponentials of $c_{-1}l_{-1} + c_0 l_0 + c_1 l_1$. The Lie subalgebra $\{L_{-1}, L_0, L_1\}$ generates an action of $SL(2,\mathbb{C})$ on the **formal** variable z by Möbius transformations, where L_{-1} is translation, L_0 is dilation/rotation, and L_1 is inversion. These will be used to define conformal (Möbius) covariance of correlation functions further below, where the **analytic** action of these series is analyzed.*

The grading of $\mathcal{V}$ is the spectral decomposition of L_0, so $L_0 v = nv$ for any homogeneous $v \in \mathcal{V}_n$. A homogeneous vector v is *quasi-primary* (like ω) if $L_1 v = 0$ and it is *primary* (like Ω) if $L_n v = 0, \forall n > 0$.

We also have the *translation covariance* property for the vertex operator

$$\frac{\mathrm{d}}{\mathrm{d}z}Y(v,z) = Y(L_{-1}v,z) = [L_{-1}, Y(v,z)], \tag{1.6}$$

where the left side is the formal derivative of a Laurent series.

Finally, for all $a, b \in \mathcal{V}$, there exists $k \in \mathbb{N}_0$ such that

$$(z_1 - z_2)^k [Y(a,z_1), Y(b,z_2)] = 0 \quad (\textit{locality} \text{ condition}). \tag{1.7}$$

Evidently, we are defining products of vertex operators using formal series, and the relations above should be interpreted in that context. Physically and analytically, we want a framework in which expectation values of $Y(a,z_1)Y(b,z_2)$ (discussed in more details in section 1.3.3) make sense if insertions are time-ordered, i.e. $|z_1| > |z_2|$. This is the case for a VOA with the above axioms, where expectation values of $Y(a,z_1)Y(b,z_2)$ will give a rational function $\frac{h(z_1,z_2)}{(z_1-z_2)^k}$ ($h \in \mathbb{C}[z_1^{\pm}, z_2^{\pm}]$), with singularity at $z_1 = z_2$, that is expanded in its Taylor series in the time-ordered region $|z_1| > |z_2|$.

As a result, the commutator $[Y(a,z_1), Y(b,z_2)]$ is what is called *an expansion of zero*: the difference of the expansion of a rational function in two opposite directions, here $|z_1| > |z_2|$ and $|z_2| > |z_1|$. This is one strength of formal calculus, where such expansion only makes sense in a formal context; by getting rid of the singularity of $z_1 = z_2$ using $(z_1 - z_2)^k$, we get the numerator of the rational function h for both $Y(a,z_1)Y(b,z_2)$ and $Y(b,z_2)Y(a,z_1)$, ensuring the difference is zero.

The above condition is also called *weak* commutativity, a one-complex-dimensional analogue of the commutativity axiom in commutative algebras, analogous to $a^{z_1}.(b^{z_2}.c) = b^{z_2}.(a^{z_1}.c)$, but weak due to the existence of the power $(z_1 - z_2)^k$. This axiom can be

replaced with weak associativity:

$$(z_0+z_2)^l Y(a,z_0+z_2)Y(b,z_2)c = (z_0+z_2)^l Y(Y(a,z_0)b,z_2)c, \tag{1.8}$$

The same discussion above applies, where the regions in which expectation values are defined are $|z_0+z_2|>|z_2|$ and $|z_2|>|z_0|$. Notice by formal calculus convention, the binomial expansion of

$$(x+y)^n = \sum_{k\in\mathbb{N}} \binom{n}{k} x^{n-k}y^k, \quad \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}, \tag{1.9}$$

for $n<0$ is the expansion with nonnegative powers for y, or the Taylor expansion in the region $|x|>|y|$. Note the expansion for $n\geq 0$ is the same for $(x+y)^n$ and $(y+x)^n$ as $\binom{n}{k}=0$ when $k>n$.

Thus the order of addition of formal variables allude to the region where expectation values are analytically defined; the expansion of $Y(a,z_0+z_2)$ assumes $|z_0|>|z_2|$ as it involves negative powers of (z_0+z_2). As a result, we need to take care of the singularity at $z_0=z_2$ to make both sides equal, hence the factor $(z_0+z_2)^l$.

The expansion of the expression $Y(Y(a,z_0)b,z_2)$ is what is called the *operator product expansion* for $Y(a,z_1)Y(b,z_2)$ (for $z_1=z_2+z_0$) in the physics literature. Furthermore, weak associativity is analogous to $a.(b.c)=(a.b).c$ which complexified is $a^{z_0+z_2}.(b^{z_2}.c) = (a^{z_0}.b)^{z_2}.c$. This finishes the description of vertex operator algebra.

Definition 1.3.1 *The tuple $(\mathcal{V},Y,\Omega,\omega)$ with the above properties is called a vertex operator algebra (VOA).*

There exist some immediate implications of the above axioms that we list below. For

any two homogeneous vectors $u, v \in \mathcal{V}$,

$$\text{wt } ((v)_n u) = \text{wt u} - n.$$

Motivated from the above equation and the creation property, modes $(v)_n$ for $n > -\text{wt v}$ are called the annihilation modes while the rest are called the creation modes. This also implies that the vacuum is an example of a primary field as it has energy zero, thus wt $L_n\Omega = -n$. The vacuum also satisfies $L_{-1}\Omega = 0$, a fact that is used later in conformal covariance of correlation functions (Remark 1.3.7).

The locality axiom implies the *Jacobi* or the *Borcherds identity*, the one-complex-dimensional analogue of the Jacobi identity in Lie algebra. To express this identity, we need to introduce the formal delta function, analogous to the Dirac delta function,

$$\delta(z) = \sum_{n\in\mathbb{Z}} z^n. \tag{1.10}$$

The delta function is the most important expansion of zero, that of $(1-z)^{-1} - (-z+1)^{-1}$. It is also the unique function in $\mathbb{C}[[z, z^{-1}]]$ such that $\text{Res}_z f(z)\delta(z) = f(1)$ for all Laurent polynomials $f \in \mathbb{C}[z, z^{-1}]$, where Res_z is the residue giving the coefficient of z^{-1}. More generally,

$$\text{Res}_{z_1} f(z_1) z_2^{-1} \delta(\frac{z_1}{z_2}) = f(z_2). \tag{1.11}$$

Previously in (1.7) and (1.8), it was mentioned how the same rational function is expressed in three different regions. Using the delta function, we can put all those expressions in

one single Jacobi identity as follows

$$z_0^{-1}\delta(\frac{z_1-z_2}{z_0})Y(a,z_1)Y(b,z_2)-z_0^{-1}\delta(\frac{-z_2+z_1}{z_0})Y(b,z_2)Y(a,z_1)= \tag{1.12}$$

$$z_1^{-1}\delta(\frac{z_2+z_0}{z_1})Y(Y(a,z_0)b,z_2).$$

This is analogous to the Lie algebra Jacobi identity $\mathrm{ad}_a(\mathrm{ad}_b c)-\mathrm{ad}_b(\mathrm{ad}_a c)=\mathrm{ad}_{\mathrm{ad}_a b}c$, where ad (the adjoint action) should be replaced by the complexified product, giving $a^{z_1}.(b^{z_2}.c)-b^{z_2}.(a^{z_1}.c)=(a^{z_1-z_2}.b)^{z_2}.c$. This axiom can replace locality in the definition. In fact, any of the associativity or commutativity or Jacobi axioms implies the other two [15].

The above is a generator of infinitely many complicated identities for the nonassociative products $v_{(n)}u$. As an equivalent formulation:

$$\sum_{j=0}^{\infty}\binom{p}{j}[a_{(q+j)}b]_{(p+k-j)}c=\sum_{j=0}^{\infty}(-1)^j\binom{q}{j}a_{(p+q-j)}b_{(k+j)}c \tag{1.13}$$

$$-\sum_{j=0}^{\infty}(-1)^{j+q}\binom{q}{j}b_{(q+k-j)}a_{(p+j)}c,\quad a,b,c\in\mathcal{V},p,q,k\in\mathbb{Z}.$$

1.3.1 VOA modules and intertwiners

The next objects to discuss are the *modules* of a VOA. A module has a structure similar to that of a VOA and some compatibility properties with the VOA.

A module (also called a *sector* or *representation*) (A,Y_A) for a VOA $(\mathcal{V},Y,\Omega,\omega)$, is an $\mathbb{N}_0-$graded vector space A with a linear map

$$Y_A(\cdot,z):\mathcal{V}\to\mathrm{End}(A)[[z,z^{-1}]],\quad Y_A(v,z)=\sum_{n\in\mathbb{Z}}v^A_{(n)}z^{-n-1},$$

where $v^A_{(n)}$ are the mode operators of v. The spectrum of $\omega^A_{(1)} = L_{0,A}$ gives the *weights* or grading of the module

$$A = \bigoplus_{n\in\mathbb{N}_0} A_n.$$

A_0 is called the *top-level* and A_n the $n-$th level of module A. For an *irreducible* module, $L_{0,A}a = h_A a, \forall a \in A_0$ where h_A is called the *conformal* or *highest* weight of A, and

$$L_{0,A}a = (h_A + n)a, \quad \forall a \in A_n.$$

Lastly, there is an analogous notation of $(v)_n$ for a homogeneous vector $v \in \mathcal{V}$,

$$Y_A(v,z) = \sum_{n\in\mathbb{Z}} (v)_{n,A} z^{-n-\text{wt } v},$$

where $(v)_{n,A} = v_{(n+\text{wt } v-1),A}$ and for any two homogeneous vectors $u, v \in \mathcal{V}$,

$$\text{wt } ((v)_{n,A}u) = \text{wt u} - n.$$

The vertex operator and $Y_A(\omega, z)$ satisfy all the axioms of a VOA, except the creativity property which does not make sense in this context unless $A = \mathcal{V}$. Locality holds and as a result, Borcherds identity (1.13) also holds in this case with the obvious necessary changes. The subscript A will be dropped from the mode operators involved as it will be clear from the context.

Finally, the character for a module A is defined as

$$\text{char}(A) = \text{Tr}_A(q^{L_0 - \frac{c}{24}}) = \sum_{n\in\mathbb{N}_0} \dim(A_n)\ q^{n+h-\frac{c}{24}}.$$

Let A, B, C be irreducible modules with corresponding conformal weights h_A, h_B, h_C.

An *intertwiner of type* $\binom{C}{A\ B}$ is a map

$$\mathcal{Y}(\cdot,z) : A \to \mathrm{End}(B,C)[[z,z^{-1}]], \quad \mathcal{Y}(a,z) = \sum_{m\in\mathbb{Z}} a_{(m)} z^{-\tau-m},$$

where $\tau = h_A + h_B - h_C$. It has the following notation for homogeneous $a \in A_k$

$$\mathcal{Y}(a,z) = \sum_{n\in\mathbb{Z}} (a)_n z^{-n-k-\tau}, \tag{1.14}$$

and it satisfies similar axioms as the vertex operator. Intertwiners are part of the fundamental features of a CFT as they describe the fusion rules of the representation category of the VOA.

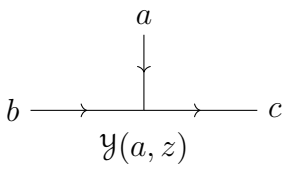

Figure 1.16: Intertwiner $\mathcal{Y}(a,z)$ as a trivalent graph (recall Figure 1.5).

Most important corollary of all the above definitions, is the infinitesimal conformal covariance of the primary fields. Assuming a to be primary, for $n > 0$,

$$[L_n, \mathcal{Y}(a,z)] = (z^{n+1}\partial_z + (\text{wt } a)(n+1)z^n)\mathcal{Y}(a,z). \tag{1.15}$$

1.3.2 Further conditions on VOA and unitarity

There are many properties that the VOAs in our consideration will have, and most of the time we will be assuming such properties. In this writing, the condition C_2-*cofiniteness* is imposed on the VOAs. This is a technical condition that means the space $C_2 = \mathrm{span}\{u_{(-2)}v|\ u,v \in \mathcal{V}\}$ has finite co-dimension $C_{\mathcal{V}} = \dim(\mathcal{V}/C_2) < \infty$. It is

assumed in many settings for its applications, like ensuring the representation category of the VOA is a modular tensor category. However, more relevant to our computational purposes is a result [27, 28] on the growth of the dimension of the weight spaces of an irreducible module A that follows, which states

$$\dim A_n \leq (\dim A_0) \cdot e^{2\pi\sqrt{\frac{c_{\mathcal{V}} n}{6}}}.$$

This at-most-exponential growth is necessary if an approach to simulation requires a truncation of energy up to some N, where one can not afford more than polynomially many qubits to be used to simulate the vector space.

An important class of VOAs consists of the unitary minimal models (UMMs) introduced in section 1.3.5. A UMM $\mathcal{V}$ satisfies many properties such as being *CFT-type*, i.e. $V_0 = \mathbb{C}\Omega$ where only the vacuum has energy zero. Also, $\mathcal{V}$ has finitely many irreducible modules and every $\mathcal{V}-$module is a direct sum of irreducible $\mathcal{V}-$modules (*rationality*). Last but not least, $\mathcal{V}$ is unitary. We define unitarity below and refer to [29, 25] for more details.

In the category of VOAs, morphisms are grade-preserving vector space maps that preserve the *field (product)* and *conformal* structure, i.e. $f : \mathcal{V} \to \mathcal{W}$ satisfies

$$f(Y_{\mathcal{V}}(v,z)u) = Y_{\mathcal{W}}(f(v),z)f(u) \iff f(v_{(n)}u) = f(v)_{(n)}f(u)$$

$$f(\Omega_{\mathcal{V}}) = \Omega_{\mathcal{W}}, \quad f(\omega_{\mathcal{V}}) = \omega_{\mathcal{W}}.$$

For two modules of $\mathcal{V}$, the concept of a $\mathcal{V}-$module map can be naturally defined. Morphisms introduced below to define a unitary VOA (and later full CFT) are morphisms in the category of VOAs.

A unitary VOA has some invariant positive definite hermitian form $(\cdot,\cdot)_{\mathcal{V}} : \mathcal{V}\times\mathcal{V} \to \mathbb{C}$

with $(\Omega, \Omega) = 1$. Further, there is an anti-linear involution (so-called **PCT** operator) $\eta : \mathcal{V} \to \mathcal{V}$ for which $(\eta\cdot, \cdot) = \langle \cdot, \cdot \rangle$ is an *invariant* bilinear form on $\mathcal{V}$ [25, section 5.1]

$$\langle c, Y(a,z)b\rangle = \left\langle Y(e^{zL_1}(-z^{-2})^{L_0}a, z^{-1})c, b\right\rangle, \quad a, b, c \in \mathcal{V}.$$

We will assume anti-linearity in the first and linearity in the second argument of $(\cdot, \cdot)$ when substituting complex numbers instead of z.

It follows that η is antiunitary $(\eta(a), \eta(b)) = (\eta^2(b), a) = (b, a)$. Moreover, for $a, b, c \in \mathcal{V}$ with a homogeneous and quasi-primary (i.e. $e^{zL_1}a = a$), we have

$$(c, Y(a,z)b) = (-1)^{\mathrm{wt}\ a}(Y(\eta(a), \overline{z}^{-1})c, b),$$

in other words $(a)_n^\dagger = (-1)^{\mathrm{wt}\ a}(\eta(a))_{-n}$. A *hermitian field* satisfies $\eta(a) = a$. The conformal field ω is hermitian implying $L_n^\dagger = L_{-n}$.

Remark 1.3.5 *For example,*

$$(L_{-2}\Omega, L_{-2}\Omega) = (\Omega, L_2L_{-2}\Omega) \underbrace{=}_{\Omega \text{ primary so } L_2\Omega=0} (\Omega, [L_2, L_{-2}]\Omega) \tag{1.16}$$

$$\underbrace{=}_{eq.\ (1.5)} (\Omega, (4L_0 + \frac{c}{2}\mathbf{1})\Omega) = \frac{c}{2}.$$

This is a well-known calculation done for anyonic chains to compute the central charge of the scaling limit, where one calculates $(\widetilde{L}_{-2}\widetilde{\Omega}, \widetilde{L}_{-2}\widetilde{\Omega})$ *where* $\widetilde{L}_{-2}, \widetilde{\Omega}$ *are the finite versions of* L_{-2} *and the vacuum.*

One can similarly define *unitary* modules: a positive definite form $(\cdot, \cdot)_A : A \times A \to \mathbb{C}$ with an anti-linear involution $\eta_A : A \to A$ with similar properties as above. Notice the Virasoro field satisfies $L_n^\dagger = L_{-n}$ with respect to $(\cdot, \cdot)_A$ as well.

More generally, for the VOAs in this work, there exists a hermitian conjugate of any intertwiner $\mathcal{Y}$ (of type $\binom{C}{A\ B}$) called $\mathcal{Y}^\dagger$ (of type $\binom{B}{A\ C}$) for which

$$(c, \mathcal{Y}(a,z)b)_C = (-1)^{\mathrm{wt}\ a}(\mathcal{Y}^\dagger(\eta_A(a), \overline{z}^{-1})c, b)_B$$

for $b \in B, c \in C$ and quasi-primary homogeneous $a \in A$.

Using the hermitian form, one can define a norm in the obvious way and get the completion of a graded unitary module (which includes $\mathcal{V}$ itself) denoted by $\overline{A}$.

1.3.3 Correlation functions

In this section, we can define the **correlation function** of intertwiners.

Remark 1.3.6 *$\mathcal{Y}(a,z)$ is not necessarily a linear operator for a general value of $z \in \mathbb{C}$ as it is a Laurent series.*

First, we recall the notion of **configuration space** for $X \subset \mathbb{C}$ as

$$\mathrm{Conf}_n(X) = \{(x_1, \ldots, x_n) \in X \mid x_i \neq x_j\}$$

and let $\mathbb{C}^\times = \mathbb{C} - \{0\}$. Let $z_i \in \mathbb{C}^\times$ and $a^{(i)} \in A_i$ be fields in the irreducible modules A_i. Let $\mathcal{Y}_i$ be of type $\binom{C_i}{A_i\ C_{i-1}}$ with irreducible modules C_i and $u \in C_n, v \in C_0$. The correlation function is defined as

$$(u, \mathcal{Y}_n(a^{(n)}, z_n) \ldots \mathcal{Y}_1(a^{(1)}, z_1)v). \tag{1.17}$$

This is generally a *multi*-valued rational function $\frac{G(z_1,\ldots,z_n)}{\prod_i z_i^{r_i} \prod_{i<j}(z_i - z_j)^{s_{ij}}}$ on $\mathrm{Conf}_n(\mathbb{C}^\times)$ [30], and thus single-valued on the universal covering space $\widetilde{\mathrm{Conf}}_n(\mathbb{C}^\times)$.

There is bound on the order of singularities $s_{ij} \in \mathbb{N}_0$ that depends only on $a^{(i)}$ and

$a^{(j)}$; more precisely, $s_{ij} \leq k_{ij}$ where $(a^{(i)})_n a^{(j)} = 0$ for all $n \geq k_{ij}$. This is the highest mode before which $a^{(i)}$ does not annihilate $a^{(j)}$. For the order r_i of poles at z_i, there is a similar bound depending only on $a^{(i)}$ and v. This order also includes the contribution from $\tau_i = h_{A_i} + h_{C_{i-1}} - h_{C_i}$ corresponding to $\mathcal{Y}_i$. Therefore, r_i can be real numbers as τ_i depends on the conformal weights which are real numbers in general. As a result, we have a *single*-valued rational function if all intertwiners are the vertex operator $Y()$, as in this case $\tau_i = 0$.

In general, the correlator is single-valued when restricting to *time-ordered* insertion points $0 < |z_1| < \ldots < |z_n|$. In fact, in this regime, one can compute the correlation function above by summing up all the terms in the formal calculus sum directly

$$\sum_{k_1,\ldots,k_n} (u, P^{k_n}\mathcal{Y}_n(a^{(n)}, z_n)P^{k_{n-1}}\mathcal{Y}_{n-1}(a^{(n-1)}, z_{n-1})\ldots P^{k_1}\mathcal{Y}_1(a^{(1)}, z_1)v), \tag{1.18}$$

where P^{k_i} is the projection onto $(C_i)_{k_i}$ weight space of the irreducible module C_i. The sum above is equivalent to

$$\sum_{m_1,\ldots,m_n\in\mathbb{Z}} (u, (a^{(n)})_{m_n}(a^{(n-1)})_{m_{n-1}}\ldots(a^{(1)})_{m_1}v)\prod_{i=1}^{n} z_i^{-m_i-\text{wt } a^{(i)}-\tau_i}, \tag{1.19}$$

The reason behind the unambiguity of this sum, where terms can be added in any order, is its absolute convergence. In fact, the sum above is *uniformly* convergent and bounded in any *compact* neighborhood of $(z_1, \ldots, z_n)$ in $\text{Conf}_n(\mathbb{C}^\times)$ (see [30, Thm. 3.5],[31, and references therein]).

Symmetries of a QFT manifest themselves as symmetries of the correlation function. A CFT is expected to have conformal invariant correlation functions. For simplicity, we illustrate this when all intertwiners are the vertex operator.

Primary fields are conformal invariant, while quasi-primary fields are only Möbius

invariant. More precisely, assume $a^{(i)}$s and u, v are primary, i.e. $L_n a^{(i)} = L_n u = L_n v = 0, \forall n > 0$. Let $\mathcal{U} \subset \mathbb{C}$ be an open domain containing all z_i and 0 and $f : \mathcal{U} \to \mathcal{U}$ a biholomorphic map which fixes the origin. According to Remark 1.3.4, one can find a series $g = 0L_{-1} + c_0 L_0 + \ldots$ for which $\exp(g) = f$ where the zero coefficient of L_{-1} ensures there is no translation ($f(0) = 0$). Exponentiation of the infinitesimal conformal covariance relation in (1.15) yields an action e^g on primary fields where

$$e^g Y(a, w) e^{-g} = (\frac{\mathrm{d}f}{\mathrm{d}z}(w))^{\mathrm{wt}\ a} Y(a, f(w)), \quad w \in \mathcal{U}. \tag{1.20}$$

One can view u, v as insertions $Y(u, 0)\Omega, Y(v, 0)\Omega$. As they are primary, $e^g u = f'(0)^{\mathrm{wt}\ u} u$ and $e^g v = f'(0)^{\mathrm{wt}\ v} v$. Applying these equations, we obtain the conformal covariance of the correlator [32]

$$(u, Y(a^{(n)}, z_n) \ldots Y(a^{(1)}, z_1) v) = \tag{1.21}$$

$$\overline{f'(0)}^{\mathrm{wt}\ u} f'(0)^{\mathrm{wt}\ v} \prod_{i=1}^{n} f'(z_i)^{\mathrm{wt}\ a^{(i)}} (u, Y(a^{(n)}, f(z_n)) \ldots Y(a^{(1)}, f(z_1)) v).$$

All the above applies to quasi-primary fields ($L_1 a^{(i)} = L_1 u = L_1 v = 0$), if using a Möbius map coming from $g = 0L_{-1} + c_0 L_0 + c_1 L_1$.

Remark 1.3.7 *If $u = v = \Omega$, the conformal f does not need to fix the origin, since $L_{-1}\Omega = 0$.*

1.3.4 Full CFT

A full CFT on the plane can be constructed from two chiral CFTs, called the *chiral* or *left-moving* $\mathcal{V}_L$ and the *antichiral* or *right-moving* $\mathcal{V}_R$. Notice the category of VOAs is closed under tensor product, i.e. $\mathcal{V}_L \otimes \mathcal{V}_R$ is also a VOA with vacuum $\Omega = \Omega_{\mathcal{V}_L} \otimes \Omega_{\mathcal{V}_R}$

and the conformal vector $\omega = \omega_{\mathcal{V}_L} \otimes \Omega_{\mathcal{V}_R} + \Omega_{\mathcal{V}_L} \otimes \omega_{\mathcal{V}_R}$.

The Hilbert space $\mathcal{H}$ of a full CFT decomposes to a sum of irreducible modules of $\mathcal{V}_L \otimes \mathcal{V}_R$ in the form of

$$\mathcal{H} = \bigoplus_{i,j} Z_{ij} A[i] \otimes B[j]$$

where the sum is over irreps of $\mathcal{V}_L$ and $\mathcal{V}_R$ ($A[i]$ and $B[j]$ respectively) for some integer multiplicities $Z_{ij} \in \mathbb{N}_0$. We shall assume $\mathcal{V}_L = \mathcal{V}_R = \mathcal{V}$.

Remark 1.3.8 *To have a CFT, we need the character of the Hilbert space to be modular invariant. This means $Z = (Z_{ij})_{i,j}$ is modular invariant, commuting with the S, T matrix of the MTC given by the representation category of $\mathcal{V}$.*

A particular class of such full CFTs is the *diagonal* full CFT, which appears in scaling limit of anyonic chains with periodic boundary conditions. Any irreducible module is coupled with its *contragredient* module

$$\mathcal{H} = \bigoplus_{\text{irreducible modules}} A_i \otimes A_i',$$

where A_i' is the contragredient module of $\mathcal{A}_i$. The contragredient module A' is defined as the linear functionals that vanish except on finitely many of the weight spaces, in other words

$$A' = \bigoplus_n A_n',$$

which can be given a $\mathcal{V}-$module structure. In the case of unitary modules, this means another isomorphic copy of the module itself. Thus the $Z-$matrix is identity and obviously modular invariant.

The conformal vector $\omega = \omega_L \otimes \Omega_R + \Omega_L \otimes \omega_R$ has mode operators

$$\mathbb{L}_n = L_n + \overline{L}_n,$$

where the first term is the Virasoro mode for the left-moving part and the second for the right-moving one. The conformal field is defined as

$$\mathbb{Y}(\omega,(z,\overline{z}))=\sum_{n\in\mathbb{Z}}L_n z^{-n-2}+\overline{L}_n\overline{z}^{-n-2}.$$

Primary fields $a\in\mathcal{H}$ are accordingly defined as those satisfying

$$\mathbb{L}_n a=0,\ \forall n>0.$$

Conformal covariance of correlation functions for full CFT has a similar formulation, for which we refer to [29, section 6.1] and references therein for more details.

1.3.5 Unitary Virasoro VOAs

Unitary minimal models (UMMs) $M(k+2,k+1)$ at level $k\geq 2$, form a special class of unitary VOAs with central charge $c<1$. They are completely characterized by their central charge, which form a *discrete series* $c=1-\frac{6}{(k+1)(k+2)}$ for $k\geq 2$.

Following [15, section 6.1] and [33, section 7], we construct the more general class of *Virasoro VOAs*.

Remark 1.3.9 ***Unitary*** *Virasoro VOA is a unitary VOA among Virasoro VOAs $\mathcal{V}_{c,0}$ defined below for $c>0$, and* ***UMM****s specifically refer to the discrete series $c<1$, which are the only unitary Virasoro VOAs when $c<1$.*

To define the *Virasoro VOA*, let us define the Virasoro module

$$\mathcal{L}_{c,0}=U(\mathcal{L})\otimes_{U(\mathcal{L}_{\leq 1})}\mathbb{C}.c\mathbf{1}, \tag{1.22}$$

where $U(\mathcal{L})$ is the universal enveloping algebra of the Virasoro Lie algebra $\mathcal{L}$, and $\mathcal{L}_{\leq 1}=$

$(\oplus_{n\geq -1}\mathbb{C}L_n)\bigoplus \mathbb{C}.c\mathbf{1}$ is the holomorphic part of $\mathcal{L}$ with the central element (note the holomorphic map correspondence described in 1.3.4 for $\mathcal{L}_{\leq 1}$). This has basis

$$L_{-n_l}\dots L_{-n_1}\mathbf{1}, \quad 2\leq n_1\leq \dots \leq n_l. \tag{1.23}$$

This Virasoro module has a VOA structure. We can take $\mathbf{1}$ in $U(\mathcal{L})$ to represent the vacuum state Ω. The grading is given by $(\mathcal{L}_{c,0})_n = \{L_{-n_l}\dots L_{-n_1}\Omega | \sum_{i=1}^{l} n_i = n\}$. The first three weight spaces are $(\mathcal{L}_{c,0})_0 \cong \mathbb{C}\Omega, (\mathcal{L}_{c,0})_1 \cong \emptyset, (\mathcal{L}_{c,0})_2 \cong \mathbb{C}\omega = \mathbb{C}L_{-2}\Omega$.

Remark 1.3.10 *Any VOA with central charge c includes the above states, as the descendants of the vacuum exist in any VOA. This is the motivation behind "minimal" VOA models, meaning they are generated by the vacuum and ω.*

$\mathcal{L}_{c,0}$ is not necessarily an *irreducible* VOA. In fact, there is a maximal proper submodule $\mathcal{M}_{c,0}$, which after taking the quotient of, gives the *(irreducible) Virasoro VOA* $\mathcal{V}_{c,0} = \mathcal{L}_{c,0}/\mathcal{M}_{c,0}$.

$\mathcal{V}_{c,0}$ is **unitary**, rational and C_2-co-finite for the previously mentioned discrete series $M(k+2,k+1)$ when $c<1$. Unitarity for the discrete series and **all** $c\geq 1$ is shown [33] by constructing a hermitian form $(\cdot,\cdot)$, which is defined on $\mathcal{V}_{c,0}\times\mathcal{V}_{c,0}$ and satisfies $L_n^\dagger = L_{-n}$.

Let us describe the UMMs $M(k+2,k+1)$ with $c = 1-\frac{6}{(k+1)(k+2)}$. Their irreducible modules $\mathcal{V}_{c,h}$ are determined by their conformal weights

$$h_{r,s} = \frac{((k+1)r-(k+2)s)^2-1}{4(k+1)(k+2)}, \quad 1\leq r\leq k+1, 1\leq s\leq k.$$

Due to the symmetry $h_{k+2-r,k+1-s} = h_{r,s}$, there are $\frac{k(k+1)}{2}$ many irreducible modules.

Remark 1.3.11 *$\mathcal{V}_{c,h}$ is obtained from a similarly defined $\mathcal{L}_{c,h}$ in (1.22), with a VOA structure in which L_0 acts by a scalar h on the lowest energy state. For $c\geq 1$, the*

irreducible modules $\mathcal{V}_{c,h}$ are classified by a parameter $h \geq 0$. As all conformal weights in $[0, \infty)$ are allowed, the VOA $\mathcal{V}_{c,0}$ is not rational.

1.3.6 Ising CFT

The chiral Ising CFT is the UMM $M(4,3)$. It has with three irreducible modules with conformal weights $h_{1,1} = 0$ (the VOA χ_0 itself), $h_{2,1} = \frac{1}{2}$ (the module $\chi_{\frac{1}{2}}$ corresponding to the free fermionic field ψ), $h_{3,1} = \frac{1}{16}$ (the module $\chi_{\frac{1}{16}}$ corresponding to the spin field σ). With this notation, the MTC given by Ising CFT is the Ising MTC introduced in 1.2.3. Thus, the nontrivial fusion rules are as follows:

$$\chi_{\frac{1}{2}} \otimes \chi_{\frac{1}{2}} = \chi_0,$$

$$\chi_{\frac{1}{16}} \otimes \chi_{\frac{1}{16}} = \chi_{\frac{1}{2}} \oplus \chi_0,$$

$$\chi_{\frac{1}{2}} \otimes \chi_{\frac{1}{16}} = \chi_{\frac{1}{16}}.$$

The *fermionic algebra* is used to generate the Hilbert spaces χ_i. The Hilbert spaces $\chi_0, \chi_{\frac{1}{2}}$ are generated by the fermionic modes $\{\Psi_{n-\frac{1}{2}}\}_{n\in\mathbb{Z}}$ satisfying the anticommutative canonical relations (ACR) $\{\Psi_k, \Psi_{k'}\} = \delta_{k+k',0}$.

The third Hilbert space $\chi_{\frac{1}{16}}$ is generated by $\{\Psi_n\}_{n\in\mathbb{Z}}$ which is another version of the fermionic algebra where the modes are indexed by integers and they satisfy the same properties: $\{\Psi_k, \Psi_{k'}\} = \delta_{k+k',0}$.

The first algebra generates χ_0 and $\chi_{\frac{1}{2}}$ by acting on the vacuum Ω. The vectors

$$\{\Psi_{-k_r} \dots \Psi_{-k_1}\Omega|\ k_1 < \dots < k_r, \quad k_i \in \mathbb{N} - \frac{1}{2}\},$$

with weight $\sum k_i$, give an orthonormal basis for $\chi_0 \oplus \chi_{\frac{1}{2}}$. As a result, the character is

$$q^{-\frac{c}{24}} \prod_{n=1}^{\infty} (1 + q^{n-\frac{1}{2}}). \tag{1.24}$$

As a matter of convenience, the factor $q^{-\frac{c}{24}} = q^{-\frac{1}{48}}$ will sometimes get dropped. Obviously, the part of the series with powers of q in $\mathbb{N} - \frac{1}{2}$ corresponds to $\chi_{\frac{1}{2}}$ and the rest with powers in $\mathbb{N}_0$ corresponds to χ_0.

The second algebra $\{\Psi_n\}_{n\in\mathbb{Z}}$ generates $\chi_{\frac{1}{16}}$ in a similar way, with orthonormal basis

$$\{\Psi_{-k_r} \dots \Psi_{-k_1} |\frac{1}{16}\rangle \mid 0 < k_1 < \dots < k_r, \quad k_i \in \mathbb{N}\}, \tag{1.25}$$

where $|\frac{1}{16}\rangle$ is the vector at the top level satisfying $L_0 |\frac{1}{16}\rangle = \frac{1}{16} |\frac{1}{16}\rangle$. Notice that $|\frac{1}{16}\rangle$ is sent to a scalar multiple of itself by Ψ_0. The character is

$$\mathrm{char}(\chi_{\frac{1}{16}}) = q^{\frac{1}{16}-\frac{1}{48}} \prod_{n=0}^{\infty} (1 + q^n).$$

The hermitian form on this UMM implies the conjugacy relation $\Psi_k = \Psi^{\dagger}_{-k}$. The formulae for L_ns are well-known [34] and will be derived in section 2.4.

Remark 1.3.12 *The conformal vector is simply* $\omega = \frac{1}{2}\Psi_{-\frac{3}{2}}\Psi_{-\frac{1}{2}}\Omega$.

With regards to the intertwiners of the theory, the fusion rules for the Ising model

correspond to three different free fermionic fields

$$\psi^0_{\frac{1}{2}}(z) = \sum_{n\in\mathbb{Z}} z^{-(n-1)}\Psi_{n-\frac{1}{2}} \tag{1.26}$$

$$\psi^0_{\frac{1}{2}}(z)^\dagger = \psi^{\frac{1}{2}}_0(z) = \sum_{n\in\mathbb{Z}} z^{-n}\Psi_{n-\frac{1}{2}} \tag{1.27}$$

$$\psi^{\frac{1}{16}}_{\frac{1}{16}}(z) = \sum_{n\in\mathbb{Z}} z^{-n}\Psi_n, \tag{1.28}$$

where $\psi^j_i : \chi_i \to \chi_j$.

As a final note, the Ising full CFT is

$$\mathcal{H} = \chi_0\overline{\chi}_0 + \chi_{\frac{1}{2}}\overline{\chi}_{\frac{1}{2}} + \chi_{\frac{1}{16}}\overline{\chi}_{\frac{1}{16}},$$

with the corresponding Virasoro operators $\mathbb{L}_n$, derived using the formulae for L_ns. We refer to [33] for more on minimal models, and [33] and particularly the notes [34] for the Ising CFT.

1.3.7 Wess-Zumino-Witten (WZW) models

There are CFTs that have a local symmetry given by the action of a compact Lie group G. To study this infinitesimally, one needs to study VOAs with symmetry from a complex simple Lie algebra $\mathfrak{g}$.

Such a Lie algebra has a *normalized* Killing form $(\cdot,\cdot) : \mathfrak{g}\otimes\mathfrak{g} \to \mathbb{C}$, meaning $(\theta,\theta) = 2$ for θ maximal root of $\mathfrak{g}$. Further, it has an affinization $\hat{\mathfrak{g}} = \mathfrak{g}\otimes\mathbb{C}[t,t^{-1}] \oplus \mathbf{k}\mathbb{C}$ with $\mathbf{k}$ a central element and bracket relations:

$$[a\otimes t^n, b\otimes t^m] = [a,b]\otimes t^{n+m} + \delta_{n+m,0}n(a,b)\mathbf{k} \tag{1.29}$$

We decompose $\hat{\mathfrak{g}}$ into subalgebras:

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_+ \oplus \hat{\mathfrak{g}}_- \oplus \mathfrak{g} \oplus \mathbf{k}\mathbb{C} \ , \ \hat{\mathfrak{g}}_+ = \mathfrak{g} \otimes \mathbb{C}[t]t, \hat{\mathfrak{g}}_- = \mathfrak{g} \otimes \mathbb{C}[t^{-1}]t^{-1}. \tag{1.30}$$

Next, we take the universal enveloping algebra $U(\hat{\mathfrak{g}})$ and quotient by $\{\hat{\mathfrak{g}}_+, \mathbf{k} - k\mathbf{1}\}$ where $\mathbf{1}$ is the identity in $U(\hat{\mathfrak{g}})$ and $k \in \mathbb{N}$. Taking this identity as the vacuum state Ω, and denoting $(a)_n = a \otimes t^n$, this gives a module isomorphic to $U(\hat{\mathfrak{g}}_-)$ with elements

$$(a^{(l)})_{-n_l} \dots (a^{(1)})_{-n_1}\Omega \ , \quad (a^{(i)})_{-n_i} \in \hat{\mathfrak{g}}_-. \tag{1.31}$$

Note that $(a)_n\Omega = 0$ when $n > 0$ (annihilation modes). This module has a maximal proper submodule which taking the quotient of, gives a new $\hat{\mathfrak{g}}-$module.

This new module has a unitary rational C_2-co-finite VOA structure as long as $k \neq -g$, where g is the dual Coxeter number of $\mathfrak{g}$ [35]. This VOA is called the WZW model of $\mathfrak{g}$ at level k, also denoted by $\mathfrak{g}_k$.

The grading of the VOA is given by

$$\mathrm{wt}((a^{(l)})_{-n_l} \dots (a^{(1)})_{-n_1}\Omega) = \sum_{i=1}^{l} n_i.$$

The first two weight spaces are $(\mathfrak{g}_k)_0 \cong \mathbb{C}\Omega$, $(\mathfrak{g}_k)_1 \cong \mathfrak{g}$. The vertex operator on $(\mathfrak{g}_k)_1$ defines the so-called **currents**

$$Y(a, z) = \sum_{n \in \mathbb{Z}} (a)_n z^{-n-1}.$$

Further, unitarity is ensured by using (1.29) with the help of the Chevalley involution $\eta_{\mathfrak{g}}$. For example $((a)_{-n}\Omega, (b)_{-m}\Omega) = (\Omega, (\eta_{\mathfrak{g}}(a))_n(b)_{-m}\Omega)$ and the rest is calculated by using (1.29). The central charge of the theory is given by $\frac{k \dim \mathfrak{g}}{k+g}$.

To construct modules of this VOA, let V be an irreducible representation of $\mathfrak{g}$ with highest weight λ_V. Similar to the definition of $\mathfrak{g}_k$, instead of acting on the vacuum Ω, we let the modes act on the module. This means the elements of irreducible modules of $\mathfrak{g}_k$ are of the form:

$$(a^{(l)})_{-n_l} \dots (a^{(1)})_{-n_1} v \ , \quad (a^{(i)})_{-n_i} \in \hat{\mathfrak{g}}_-, \ v \in V. \tag{1.32}$$

The grading is similar with weighting given by

$$\mathrm{wt}((a^{(l)})_{-n_l} \dots (a^{(1)})_{-n_1} v) = h_V + \sum_{i=1}^{l} n_i.$$

Note the top-level of this module $\mathfrak{g}_{k,V}$ is isomorphic to V. The trivial one-dimensional representation $V = \mathbb{C}\Omega$ gives the VOA $\mathfrak{g}_k$ itself. The conformal weight h_V is given by $\frac{\langle \lambda_V, \lambda_V + \rho \rangle}{k+g}$, where ρ is the Weyl vector of $\mathfrak{g}$. Not all maximal roots give irreducible modules for $\mathfrak{g}_k$; more precisely, only those with integrable weights λ_V satisfying $\lambda_V(\theta) \leq k$ do so [35].

Remark 1.3.13 *Let $\mathfrak{g} = \mathfrak{su}(2)$. Then $M(k+2, k+1)$ can be constructed as* ***cosets*** $\frac{\mathfrak{su}(2)_{k-1} \times \mathfrak{su}(2)_1}{\mathfrak{su}(2)_k}$. *The coset construction is detailed in [15] and is essentially the process of taking the* ***centralizer*** *of the embedding of $\mathfrak{su}(2)_k$ in $\mathcal{V} = \mathfrak{su}(2)_{k-1} \times \mathfrak{su}(2)_1$; i.e. all fields $Y(a,z)$ for $a \in \mathcal{V}$ that commute with all fields $Y(b,z)$ for $b \in \mathfrak{su}(2)_k$.*

Remark 1.3.14 *In $\mathfrak{g}_k$, the conformal vector is given by $\omega = \frac{1}{2(k+g)} \sum_{i=1}^{\dim \mathfrak{g}} (u_i)_{-1} (u_i)_{-1} \Omega$ (Segal-Sugawara construction), where the sum is over an orthonormal basis $\{u_i\}_{i=1}^{\dim \mathfrak{g}}$ with respect to the Killing form of $\mathfrak{g} \cong (\mathfrak{g}_k)_1$, the space of all nontrivial primary fields.*

1.4 Smeared field or Wightman's observable

In addition to VOA, observables coming from Wightman's axioms will also be used. One of the objectives of this work is to obtain the fields in the scaling limit and prove that products of fields are also in the scaling limit, hence obtaining a "scaling limit of algebras". We define the **observables** (or fields) in each framework. As we shall see, observables are related to the fields $Y(a,z)$ that have been used so far. For this section, the definitions and facts follow those of [25].

So far, the observables or fields that are *point-like* have been described; the insertion of the field is exactly at a point. *Smeared* field operators or *Wightman's* observables are insertions of a field where the position of the particle is smeared using a smooth complex-valued smearing function $f \in C^\infty_{\mathbb{C}}(S^1)$. Formally

$$Y(a,f) := \oint Y(a,z) f(z) z^{\text{wt } a} \, \frac{\mathrm{d}z}{2\pi i z} = \sum_{n\in\mathbb{Z}} \hat{f}_n(a)_n, \tag{1.33}$$

where $\hat{f}_n$s are the Fourier coefficients of f. As f is smooth, its Fourier coefficients will be rapidly decreasing:

$$\forall k, \exists N_k \text{ such that } \forall |n| \geq N_k \implies |\hat{f}_n| \leq \frac{1}{n^k}. \tag{1.34}$$

In order to have $Y(a,f)$ defined on $\mathcal{V}$ (before taking its completion), an *energy bound* on the mode operators is needed

$$||(a)_n b|| \leq C_a (|n|+1)^{r_a} ||(L_0 + \mathbf{1})^{s_a} b||, \ \forall b \in \mathcal{V}, \tag{1.35}$$

where the constants $C_a, r_a, s_a > 0$ depend on a, and the norm is given by the unitary

structure. By summing up the above inequalities,

$$||Y(a,f)b|| \leq C_a||f||_{r_a}||(L_0+\mathbf{1})^{s_a}b||, \tag{1.36}$$

where the r_a-*norm* of f is defined as

$$||f||_{r_a} = \sum_n |\hat{f}_n|(|n|+1)^{r_a}.$$

Similarly, we define the $s-$th norm of a state $||b||_s := ||(L_0+\mathbf{1})^s b||$ for any $s \geq 0$. These notations will be useful in the next chapters.

The exact same concepts apply for *smeared intertwiners* $\mathcal{Y}(a,f)$. The domain on which the smeared intertwiners can act are called smooth vectors (or states).

Definition 1.4.1 *For any module A of $\mathcal{V}$, define the set of smooth states $A_{(\infty)} \subset \overline{A}$ as the set of states $a \in \overline{A}$ for which $||a||_s < \infty$ for all $s \geq 0$.*

As they are defined on $\mathcal{V}_{(\infty)}$, one can take their product

$$\prod_{i=1}^{n} Y(a^{(i)}, f_i) : \mathcal{V}_{(\infty)} \to \mathcal{V}_{(\infty)}, \tag{1.37}$$

and define the smeared correlator $(u, \prod_{i=1}^n Y(a^{(i)}, f_i)v)$ on $\mathcal{V}_{(\infty)} \times \mathcal{V}_{(\infty)}$.

Remark 1.4.1 *All intertwiners of unitary Virasoro VOAs, thus including UMMs, and WZW models are energy-bounded [25, see e.g.].*

The conformal smeared field $Y(\omega, f)$ will be denoted by

$$L(f) = \sum_{n\in\mathbb{Z}} \hat{f}_n L_n. \tag{1.38}$$

From the previous remark, for all unitary VOAs, it can be shown that L_n satisfies the energy bound

$$||L_n b|| \leq \sqrt{\frac{c}{2}}(|n|+1)^{\frac{3}{2}}||(L_0+\mathbf{1})b||, \tag{1.39}$$

implying

$$||L(f)b|| \leq \sqrt{\frac{c}{2}}||f||_{\frac{3}{2}}||(L_0+\mathbf{1})b||. \tag{1.40}$$

1.5 Local Conformal Net (LCN)

The next observables are the ones coming from the *chiral* local conformal net (LCN) picture of chiral CFT. In this writing, LCN will always refer to chiral LCN, unless explicitly mentioned otherwise, as in *full* LCN or *boundary* LCN introduced later. We follow the definition of LCN in [25].

Recall a von Neumann algebra is a self-adjoint algebra of bounded operators on a Hilbert space containing identity and closed with respect to weak (equivalently strong) operator topology. The adjoint of O is denoted by $O^\dagger$ as usual. We will axiomatize a family of von Neumann on S^1. Let $\mathcal{I}$ be the family of open, connected, non-empty and non-dense subsets (intervals) of S^1 and $B(\mathcal{H})$ the algebra of bounded linear operators on Hilbert space $\mathcal{H}$.

Definition 1.5.1 *A local Möbius covariant net $\mathcal{A}$ on S^1 is a family of von Neumann algebras $\mathcal{A}(I) \subset B(\mathcal{H})$ on Hilbert space $\mathcal{H}$ satisfying*

- ***Isotony***. $I_1 \subset I_2 \implies \mathcal{A}(I_1) \subset \mathcal{A}(I_2)$.
- ***Locality***. $I_1 \cap I_2 = \emptyset \implies [\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\}$.

- ***Möbius covariance***. *There exists a strongly continuous unitary representation U of $PSL(2,\mathbb{R})$ (Möbius transformations of S^1) on $\mathcal{H}$ such that*

$$U(\gamma)\mathcal{A}(I)U(\gamma)^\dagger = \mathcal{A}(\gamma I),\ \forall\gamma \in PSL(2,\mathbb{R}), \forall I \in \mathcal{I}.$$

- ***Positivity of the energy***. *The generator of the one-parameter rotation subgroup of U (the CFT Hamiltonian L_0) is positive.*

- ***Existence of the vacuum***. *There exists a unit Möbius-invariant vector $\Omega \in \mathcal{H}$ called the **vacuum** which is cyclic for $\bigvee_{I\in\mathcal{I}} \mathcal{A}(I)$, the von Neumann algebra generated by all $\mathcal{A}(I)$s.*

These axioms imply the *Haag duality*,

$$\mathcal{A}(I)' = \mathcal{A}(I'), \quad \forall I \in \mathcal{I}, \tag{1.41}$$

where I' is the interior of $S^1\backslash I$. Next, let $\mathrm{Diff}_+(S^1)$ be the Lie group of orientation-preserving diffeomorphisms of S^1, which contains the Möbius transformations $\mathrm{PSL}(2,\mathbb{R})$ as they restrict to diffeomorphisms on the circle.

Definition 1.5.2 *A **local conformal net** $\mathcal{A}$ is a local Möbius covariant net with the additional conformal covariance property.*

- ***conformal covariance***. *There exists a strongly continuous projective unitary representation U of $Diff_+(S^1)$ on $\mathcal{H}$, extending the $PSL(2,\mathbb{R})$ represenatation, such that $\forall I \in \mathcal{I}$*

$$U(\gamma)\mathcal{A}(I)U(\gamma)^\dagger = \mathcal{A}(\gamma I), \quad \forall\gamma \in Diff_+(S^1), \tag{1.42}$$

$$U(\gamma)OU(\gamma)^\dagger = O, \quad \forall O \in \mathcal{A}(I),\ \forall\gamma \in Diff_+(I'), \tag{1.43}$$

where $Diff_+(I)$ is the group of orientation preserving diffeomorphisms γ of S^1 satisfying $\gamma(z) = z, \forall z \in I'$.

Remark 1.5.1 *It is an ongoing research to identify LCN with VOAs. With regards to well-known CFTs like UMMs or WZW models, their LCN and VOA correspondence has been fully established in [25]. Thus the notion of UMM or WZW CFT is unambiguous whether we are using the mathematical framework of LCN or VOA to describe the CFT. Consequently, all notions (like that of representation category) in one context have analogs in the other. For example, irreducible modules of VOA correspond to irreducible sectors of the local conformal net [36, see e.g. for a definition] and they form the same MTC under similar conditions for the LCN. Hence, we will sometimes use the terminology "irreducible sector" to refer to irreducible modules of the VOA.*

Remark 1.5.2 *Full local conformal net is defined for full CFTs on a two-dimensional Minkowski space, where instead of intervals, one considers open subsets $\mathcal{O} = I \times J \subset \mathcal{M}$ of the Minkowski plane formed by intervals I, J on the light rays $t - x$ and $t + x$. These have associated algebras that are sent to each other by conformal maps on the plane, similar to (1.42). It turns out [36, section 5.1 and references therein] that a full LCN is essentially determined by a left $\mathcal{A}_L(I)$ and right $\mathcal{A}_R(I)$ LCN and a modular invariant Z such that $S_L Z = Z S_R, T_L Z = Z T_R$ for the modular data S, T of the MTC formed by the irreducible sectors of the LCNs $\mathcal{A}_L$ and $\mathcal{A}_R$.*

Remark 1.5.3 (Unitary Virasoro LCN) *For unitary Virasoro VOAs, there is a corresponding unitary Virasoro LCN $\mathcal{A}_{vir}$ where*

$$\mathcal{A}_{vir}(I) = \{e^{iL(f)} |\ f \in C^\infty_{\mathbb{R}}(S^1), supp(f) \subset I\}'' \quad (see\ [37]). \tag{1.44}$$

$\mathcal{A}_{vir}$ is dependent on the choice of the central charge which will be clear from the context.

It is defined as the double-commutant of the algebra generated by unitaries $U(\exp(f)) = e^{iL(f)}$ *associated to real-valued smooth functions with support inside* I*. Note the double-commutant theorem implies that the strong (or weak limit) of the algebra generated by* $e^{iL(f)}$*s is* $\mathcal{A}_{vir}(I)$*. In other words, the von Neumann algebra they generate is* $\mathcal{A}_{vir}(I)$*.*

1.5.1 Boundary net

This section will be important in our characterization of the scaling limit of anyonic chains with boundary conditions in section 2.6.2.

We study boundary CFT on the half-plane $\mathcal{M}_+ = \{(t,x) : x > 0\}$. Note the stress-energy tensor $T = \begin{pmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{pmatrix}$ has to satisfy a boundary condition at $x = 0$. Conservation and vanishing of the trace imply that the left and right moving components $T_L = \frac{1}{2}(T_{00} + T_{01})$ and $T_R = \frac{1}{2}(T_{00} - T_{01})$ are chiral fields and $T_L = T_L(t+x)$, $T_R = T_R(t-x)$. The boundary condition means the absence of energy flow across the boundary,

$$T_{01}(t, x = 0) = 0 \qquad \Leftrightarrow \qquad T_L = T_R \equiv T. \tag{1.45}$$

This means the left and right chiral Virasoro fields are *identified*. The mathematical model must describe what net of algebras the additional non-chiral fields generate.

Remark 1.5.4 *An anyonic chain with boundary conditions exhibits the same behavior in the scaling limit. They contain only one chiral algebra with identification between left and right movers. As a result, the Hilbert space in the scaling limit is a representation that is a direct sum of representations of the chiral algebra, and not a sum of tensor products of representations of two chiral algebras like in a full CFT for periodic chains.*

We now define the boundary net following [36, section 5.2]. This will be the first step

in modeling boundary CFT. Let $\mathcal{K}_+$ be the set of *double cones* $\mathcal{O}$ whose closures are contained in $\mathcal{M}_+$. A double cone is $\mathcal{O} = I \times J$ where $I, J \subset \{x = 0\} \cong \mathbb{R}$ are bounded intervals on t-axis with $I > J$ (meaning $\inf I > \sup J$), and $I \times J$ is defined as the region $\{(t, x) | t + x \in I, t - x \in J\}$. Consider a completely rational local conformal net $\mathcal{A}(I)$ restricted to a net on $\mathbb{R}$ by removing the point ∞. Then the universal cover $\widetilde{\mathrm{PSL}(2, \mathbb{R})}$ acts globally on the universal cover of S^1. The product action on the chiral lines $t+x$ and $t-x$ of $\mathcal{M}$ gives a local action of $\widetilde{\mathrm{PSL}(2, \mathbb{R})} \times \widetilde{\mathrm{PSL}(2, \mathbb{R})}$ on $\mathcal{M}$. To pass to the boundary CFT, we consider the local action of $\mathrm{PSL}(2, \mathbb{R})$ obtained by restricting the local action of $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$ to the diagonal. This action restricts to local actions of $\mathrm{PSL}(2, \mathbb{R})$ on $\mathcal{M}_+$ and its boundary, the time axis.

Definition 1.5.3 (Definition 5.6, [36]) *An assignment $\mathcal{B}_+$ of a von Neumann algebra $\mathcal{B}_+(\mathcal{O})$ on a fixed Hilbert space $H_\mathcal{B}$ to each double cone $\mathcal{O} \in \mathcal{K}_+$ is called a* ***boundary net*** *if it satisfies the following.*

- ***Isotony***. $\mathcal{O}_1 \subset \mathcal{O}_2 \implies \mathcal{B}_+(\mathcal{O}_1) \subset \mathcal{B}_+(\mathcal{O}_2)$.
- ***Locality***. *If $\mathcal{O}_1, \mathcal{O}_2$ are spacelike separated, then $[\mathcal{B}_+(\mathcal{O}_1), \mathcal{B}_+(\mathcal{O}_2)] = 0$.*
- ***Möbius covariance***. *There exists a unitary representation U of $\widetilde{PSL(2, \mathbb{R})}$ on the Hilbert space $H_\mathcal{B}$ such that $U(g)\mathcal{B}_+(\mathcal{O})U(g)^\dagger = \mathcal{B}_+(g\mathcal{O})$ for every $\mathcal{O} \in \mathcal{K}_+$ with $g \in \widetilde{PSL(2, \mathbb{R})}$ having a path of elements $g_s \in \widetilde{PSL(2, \mathbb{R})}$ connecting the identity of $\widetilde{PSL(2, \mathbb{R})}$ and g satisfying $g_s\mathcal{O} \in \mathcal{K}_+$ for all s.*
- ***Positive energy condition***. *The generator of the translation one-parameter subgroup of U (the CFT Hamiltonian) is positive.*
- ***Existence of the vacuum vector***. *There exists a unit vacuum state $\Omega \in H_\mathcal{B}$ such that $\mathbb{C}\Omega$ are the $U-$invariant vectors and cyclicity holds $\overline{\mathcal{B}_+(\mathcal{O})\Omega} = H_\mathcal{B}$ for each $\mathcal{O} \in \mathcal{K}_+$.*

The non-chiral fields of a boundary CFT generate a boundary net of algebras $\mathcal{B}_+(\mathcal{O})$ on $\mathcal{M}_+$ which contains the boundary net generated from a chiral net $\mathcal{A}(I)$. Let us define $\mathcal{A}_+(\mathcal{O}) = \mathcal{A}(I) \vee \mathcal{A}(J)$ for $\mathcal{O} = I \times J$. A boundary net associated to a chiral net defines the model for boundary CFT and we will use both terms boundary net and boundary CFT interchangeably.

Definition 1.5.4 (Definition 5.7, [36]) *A boundary net $\mathcal{B}_+(\mathcal{O})$ associated with $\mathcal{A}(I)$ is a boundary net $\mathcal{B}_+(\mathcal{O})$ satisfying the following.*

- ***Joint irreducibility**. There is a representation π of $\mathcal{A}(I)$ on $H_{\mathcal{B}}$ such that $\pi(\mathcal{A}_+(\mathcal{O})) \subset \mathcal{B}_+(\mathcal{O})$ and $U(g)\pi(\mathcal{A}_+(\mathcal{O}))U(g)^\dagger = \pi(\mathcal{A}_+(g\mathcal{O}))$ for doubles cones $\mathcal{O}, g\mathcal{O} \in \mathcal{K}_+$.*

- *For each double cone $\mathcal{O}$, the von Neumann algebra generated by $\mathcal{B}_+(\mathcal{O})$ and all algebras $\pi(\mathcal{A}(I))$ is $B(\mathcal{H}_{\mathcal{B}})$.*

Remark 1.5.5 *An important example of a boundary net associated to the chiral net $\mathcal{A}_{vir}$ (1.44) for Ising model is $\mathcal{B}_+(\mathcal{O}) = (CAR(I) \otimes CAR(J))^{even}$ [38, Examples in p.20, 47] for $\mathcal{O} = I \times J$, generated as a von Neumann algebra by the monomials of Dirac modes (with even many terms) in $\{\Psi(f)\Psi(g) | supp(f) \subset I, supp(g) \subset J\}$, and acting on the Hilbert space $\mathcal{H}_{\mathcal{B}} = \chi_0 \oplus \chi_{\frac{1}{2}}$.*

Boundary nets can be characterized using the notion of chiral extension of chiral theories. Every maximal local boundary CFT $\mathcal{B}_+$ can be recovered from its "restriction to the boundary". The latter is some (possibly non-local) *chiral extension* $\mathcal{B}$ of a chiral LCN $\mathcal{A}$ defined on the same Hilbert space $\mathcal{H}_{\mathcal{B}}$ as $\mathcal{B}_+$. We refer to [38, 36] for the (re)construction of the boundary CFT from a (non-local) chiral theory and classification of the latter.

1.6 Anyonic Chain (AC)

Although anyonic chains (AC) are closely related to and inspired by spin chains, there are some fundamental differences between them. The most salient difference touches on the trade-off between explicit locality and unitarity in QFTs. Spin chains implement locality explicitly by attaching local state spaces to each site, while the Hilbert spaces of ACs do not have such explicit tensor product decomposition. In general, it is harder to obtain unitary interacting exactly solvable spin chains with CFT scaling limits, while such examples of ACs are ubiquitous [9]. This phenomenon is related to the localization of braid group representations, where finite order unitary $R-$matrices are very rare [39].

This section follows the exposition of anyonic chains in [40, 9]. An anyonic chain is a periodic or open (with boundary condition) chain, along which pairwise interactions occur between quasi-particles (the anyons), e.g. the generalized spin j anyons of $\mathfrak{su}(2)_k$.

Remark 1.6.1 *Our anyonic chains will be based on $\mathfrak{su}(2)_k$ and it should not be confused with anyonic chains based on JTL_k, like the Fibonacci golden chain in [9] (see Remark 1.6.2 for more on this issue).*

The chain is usually presented along a straight path if it is not periodic and as a loop if it is periodic. We will also put the nonperiodic chain along the upper half-circle S^1_+ (Figure 1.17) as this picture will be used in section 2.5.2 to relate the AC to LCN.

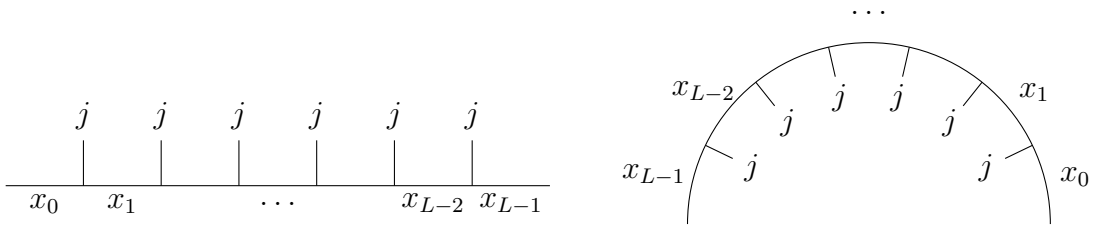

Figure 1.17: Anyonic chain on a straight path and on a half-circle.

A *boundary condition* (a, b) means $x_0 = a$ and $x_{L-1} = b$. The channel between each

two anyons provides the means for fusion. Each *admissible* fusion path has to satisfy the fusion rules of $\mathfrak{su}(2)_k$ in (1.1). All admissible fusion paths form an orthogonal basis of the Hilbert space $\mathrm{Hom}(x_0 \otimes x_{L-1}, j^{\otimes(L-1)})$ where the inner-product comes from the diagram calculus of the unitary MTC $\mathfrak{su}(2)_k$ (section 1.2.1). Generally for all j and k, the resulting Hilbert spaces do not have a tensorial structure, though the case of $k = 2$ does have one.

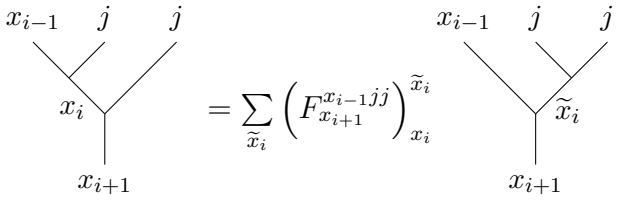

Figure 1.18: The $F-$move applied on the anyonic chain.

We specialize to the case $j = \frac{1}{2}$, and define a Hamiltonian. The motivation of all these settings could be seen as a generalization of the Heisenberg model [40]. In that model, there exists a spin-spin nearest neighbor interaction given by the term

$$\vec{S}_i.\vec{S}_{i+1} = P_i^1 - \frac{3}{4}I_i = -P_i^0 + \frac{1}{4}I_i,$$

where P_i^s is the projection onto the total spin s channel of two spins $\vec{S}_i$ and $\vec{S}_{i+1}$. This leads to the following Hamiltonian

$$H = J\sum_j P_j^0,$$

where J determines if the chain is antiferromagnetic ($J = -1$) or ferromagnetic ($J = 1$). To generalize, we first need to define the projection onto the total spin using the so-called $F-$move in Figure 1.18.

The next step would be to project onto the desired fusion which is 0 (the vacuum)

and go back to the previous basis of fusion path by applying the inverse of the $F-$move:

$$H = -\sum_{i=1}^{L-2} F_i^{-1} P_i^0 F_i,$$

where the antiferromagnetic coupling $J = -1$ has been chosen in order to obtain UMMs in the scaling limit. In the case of spin$-\frac{1}{2}$ chain, letting $d = 2\cos(\frac{\pi}{k+2})$, the quantum dimension of $\frac{1}{2}$,

$$F_i^{-1} P_i^0 F_i = \frac{1}{d} X_i \implies H = -\frac{1}{d} \sum_{i=1}^{L-2} X_i.$$

The operators X_i satisfy the following relations [9, eq. (3)]:

$$X_i^2 = dX_i, \quad X_i X_{i\pm1} X_i = X_i, \quad [X_i, X_j] = 0, \text{ for } |i-j| > 1.$$

These are the same operators e_i of the Temperley-Lieb (TL) algebra (section 1.2.2). Thus,

$$H = -\frac{1}{d} \sum_{i=1}^{L-2} e_i.$$

For the nonperiodic $\mathfrak{su}(2)_2$ AC, there are several possibilities (a, b) for the boundaries as $a, b \in \{0, \frac{1}{2}, 1\}$. For example, the chain $(\frac{1}{2}, \frac{1}{2})$ has odd length $L = 2n+1$ due to the fusion rules and the Hamiltonian is $H = \frac{-1}{\sqrt{2}} \sum_{i=1}^{2n-1} e_i$. However, the periodic chain has always even length $2n$.

Going back to the general case, recall the operator e_i acts nontrivially on the $i-$th particle according to its neighbor particles (1.3). From the MTC point of view, one can think of the (open) AC as a diagram inside $\mathrm{Hom}(x_0 \otimes x_{L-1}, (\frac{1}{2})^{\otimes(L-1)})$ on which e_j with the above entries act, by stacking up above x_j.

Numerical experiments suggest that the scaling limit of the ACs of $\mathfrak{su}(2)_k$ can be described by chiral CFTs data or full CFT (for open boundary condition or periodic

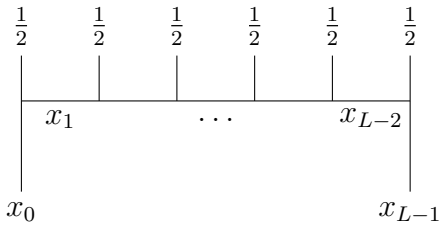

Figure 1.19: AC with $j = \frac{1}{2}$ as an element inside $\mathrm{Hom}(x_0 \otimes x_{L-1}, (\frac{1}{2})^{\otimes(L-1)})$.

chains, respectively). These results are outlined in [40, 9], showing that depending on the boundary condition, we obtain different irreducible modules of UMMs with central charge $c = 1 - \frac{6}{(k+1)(k+2)}$. This happens for the antiferromagnetic chain, and it is expected that one obtains the parafermion CFT with central charge $c = \frac{2(k-1)}{k+2}$ for the ferromagnetic chain.

For periodic chains, exact diagonalization numerically solves the anyonic chain model by finding the excitation spectra [9]. For example, conformal dimensions of the scaling limit CFT are extracted from the energy levels for a length L periodic chain given by

$$E = E_1 L + \frac{2\pi v}{L}(-\frac{c}{12} + h_L + h_R) + O(\frac{1}{L^2}),$$

with h_L, h_R the conformal weights of left and right sector. The scaling limit CFT is stable under symmetry-preserving perturbation; more precisely, the *topological* symmetry that the periodic chain has [40, Fig. 3]. One can imagine a loop inside the chain and repeatedly use the $F-$move until it gets removed. As demonstrated in [9, 40], any perturbation preserving such symmetry will not change the scaling limit .

Remark 1.6.2 *Due to the similarities between JTL_k and $\mathfrak{su}(2)_k$ for the particular case of $k = 2$ (Remark 1.2.2), what would be referred to as the **Ising AC** can be constructed with either of JTL_2 or $\mathfrak{su}(2)_2$ as the scaling limit results would apply to both cases. This is consistent with physics terminology where the Ising chain means a spin$-\frac{1}{2}$ chain $(\mathbb{C}^2)^{\otimes n}$*

with the Hamiltonian

$$H = -\sum_{j=1}^{n} \sigma_j^x - \sum_{j=1}^{n-1} \sigma_j^z \sigma_{j+1}^z$$

given by Pauli operators σ^x, σ^z. As shown in section 2.4, this Hamiltonian is the same as the one constructed earlier. However, the general relation (for $k > 2$) between the two types of JTL_k and $\mathfrak{su}(2)_k$ anyonic chains is likely to be complicated. See [9] for an attempt for $k = 3$, where the golden Fibonacci chain based on JTL_3 is mapped to an $\mathfrak{su}(2)_3$ AC.

As a final note, an important connection between AC model and the Restricted Solid-On-Solid (RSOS) lattice model provides further physical proof that the scaling limits of ACs are CFTs. One can show that the Hamiltonian derived from the logarithmic derivative of the transfer matrix, coincides with the AC Hamiltonian [9]. This lattice model has been studied for a long time and the literature has similar numerical results for this model (see [41], [42], and the references in [9]). While there is no doubt that the two approaches are equivalent in the end, mathematically it seems easier to obtain observable algebras of CFTs as scaling limits in the AC approach as illustrated in section 2.5.

Chapter 2

CFT as Scaling Limit of Anyonic Chains

2.1 Outline of main results

We provide a mathematical definition of a low energy scaling limit of a sequence of quantum theories in any dimension, and apply our formalism to ACs. The formulation is non-relativistic and there will be no treatment of spacetime. Instead, most of our focus will be on the scaling limit of the algebra of observables. Of utmost importance to our applications are the recovery of all algebras of local observables in the scaling limit and the rate of convergence to the scaling limit. Similar ideas for defining related scaling limits for lattice models and spin chains have appeared in the physics and mathematics literature which will be reviewed in 2.2. We show our results for the scaling limits of the Ising ACs and pin the conditions necessary for them to generalize to higher minimal models.

In section 2.3, we define the *scaling limit* of quantum *theories* and address the issues that come up with scaling limit Definition 2.3.2. Quantum theories are Hilbert spaces

$\mathcal{W}_n$ with Hamiltonians H_n and algebras of observables $\mathcal{A}_n$. The scaling limit is a Hilbert space which is a completion of a graded vector space $\mathcal{V}$ with Hamiltonian H. It is built by stacking up the low energy spectrum of $\mathcal{W}_n$s using the embeddings called *connecting maps* $\phi_n^M : \mathcal{W}_n^M \hookrightarrow \mathcal{W}_{n+1}^M$, where M is an energy cut-off. The connecting maps satisfy some compatibility axioms and are assumed to exist only for large enough n given a fixed M. If such embeddings are defined for all M and n, i.e. $\phi_n : \mathcal{W}_n \hookrightarrow \mathcal{W}_{n+1}$, we have a **strong** scaling limit, a scenario conjectured to hold for UMMs and proved for Ising. The grading of $\mathcal{V} = \oplus_{i=0}^{\infty}\mathcal{V}_i$ represents the $\lambda_{(i)}-$energy eigenspaces on which $H|_{\mathcal{V}_i} = \lambda_{(i)}\mathbf{1}$. Next, Definition 2.3.4 defines the scaling limit of observables $O_n \xrightarrow{SL} O$ where O_n's low energy behavior, defined as expectation values $(u_n, O_n v_n)$, converges to that of O, i.e. (u, Ov). Note this applies to the Hamiltonians H_n as well, with scaling limit H. The observables O defined on $\mathcal{V}$ generate the *vector space* $\mathcal{A}$ of observables (Definition 2.3.5). These observables are *almost* linear (but not necessarily linear), where $Ou = \sum_i u_i$ is a formal sum with the grading $u_i \in \mathcal{V}_i$. This reminds one of point-like fields $Y(a,z)u = \sum a_{(n)}uz^{-n-1}$ which expectation values are defined but are not linear operators (Remark 1.3.6). If the sum $\sum_i u_i$ is always finite, then O is linear on $\mathcal{V}$.

In section 2.3.3, we propose a definition of locality with respect to both space and energy, called *space locality* and *energy locality*. Intuitively, energy-local observables are those that do not shift the energy *level* by more than a constant Λ, i.e. they are zero when viewed as a map $\mathcal{V}_i \to \mathcal{V}_j$ for $|i-j| > \Lambda$ (Definition 2.3.9). This definition is **intrinsic** but too flexible and should not be confused with the later definition of locality in the next chapter, which is needed to characterize *local* operators in a quantum computer simulation of CFT. The locality definition for simulation requires a detailed knowledge of the energy (or space) local *degrees of freedom*, which may *not* have a canonical form and is only known in the case of free models like Ising. Still, the concept of energy locality provides the minimal restriction needed for some of our theorems, where we try

to distinguish the smeared Virasoro field $Y(\omega, f)$ in the scaling limit. We explore the energy-local operators in section 2.5 in greater details; Specifically, Theorem 2.5.1 and Theorem 2.5.2, where it is shown that

$$Y(\omega, f) = \sum_{n=-\Lambda}^{\Lambda} \hat{f}_n L_n, \quad \text{with } f = \sum_{n=-\Lambda}^{\Lambda} \hat{f}_n e^{in\theta} \in C^\infty_{\mathbb{R}}(S^1)$$

form *the* energy-local operators as scaling limit of hermitian linear combinations of Temperley-Lieb operators e_j and their commutators $i[e_j, e_{j+1}]$.

Even though there is no treatment of spacetime, we propose a definition of space locality on lattice-based models (Remark 2.3.1 and Definition 2.3.10), which is stronger than the usual notion of locality in quantum computation (QC-locality). This is possible due to the obvious presence of space local degrees of freedom on a lattice model. We point out whenever possible how one could obtain operators of interests such as smeared Virasoro field $Y(\omega, f)$ (or more generally any smeared field $Y(a, f)$) from a sequence of space (or QC-local operators), in Theorem 2.5.2 and Remark 2.5.9.

Previous works have mostly analyzed convergence of Virasoro modes L_m in the scaling limit without taking into account the *algebra* of observables they generate, a crucial characteristic of any quantum theory. There are three important types of observables in the mathematical frameworks of CFT:

(a) (Wightman's) Smeared fields $Y(a, f)$,

(b) (Haag-Kastler or) LCN bounded observables $O \in \mathcal{A}_{lcn}(I)$,

(c) VOA point-like fields $Y(a, z)$.

Given that they are all supposed to describe the same theory, and are also considered to be *physical* (computable), our goal is to show that the scaling limit vector space of observables $\mathcal{A}$ contains all three as a scaling limit **algebra** (*SL-algebra*). Using our

definition of scaling limit of observables, the notion of SL-algebra can be naturally defined as a collection of observables in $\mathcal{A}$ which generate an algebra in $\mathcal{A}$, hence their product has to be a scaling limit of a sequence of observables (Definition 2.3.7). If that sequence is the product of associated sequences $(O_n^{(i)})_n$ to the observables in a generating set $\{O^{(i)}\}$, then it is called strong SL-algebra (Definition 2.3.8). This is the ideal scenario where $O_n^{(i)} \xrightarrow{SL} O^{(i)}$ and scaling limit commutes with taking products of $O_n^{(i)}$s.

However, before we can move forward to show the convergence as an algebra, we need to obtain the Virasoro modes L_m. We further need to analyze the rate of convergence, which would be important to prove convergence of their product. In section 2.4, we obtain the scaling limits of Ising ACs for all boundary conditions. Proving the limits is a computationally involved procedure, where the same technique ([43]) is applied to each case. The proofs for some of the cases can also be found in the physics literature with different or similar approaches [1]. Yet, no mathematically rigorous proof for all Ising ACs using one consistent method and with explicit estimate of the convergence rate for the limits could be found in the literature. Therefore, we prove the Theorem 2.4.1 which is partially recited below.

Theorem 2.4.1 *1- The following* ***strong*** *scaling limits hold,* ***up to some scalings*** *(explained further below) of the Hamiltonians*

(a) $\mathcal{W}_n = (\frac{1}{2}, \frac{1}{2})$, $H_n = -\sum_{j=1}^{2n-1} e_j$. *Then* $(\mathcal{W}_n, H_n) \xrightarrow{SL} (\chi_0 + \chi_{\frac{1}{2}}, L_0)$.

(b) $(\mathcal{W}_n = (0,0)$ *or* $(1,1), H_n) \xrightarrow{SL} (\chi_0, L_0)$.

(c) $(\mathcal{W}_n = (0,1)$ *or* $(1,0), H_n) \xrightarrow{SL} (\chi_{\frac{1}{2}}, L_0)$.

(d) $(\mathcal{W}_n = (\frac{1}{2},1)$ *or* $(\frac{1}{2},0), H_n) \xrightarrow{SL} (\chi_{\frac{1}{16}}, L_0)$.

(e) $\mathcal{W}_n$ *the periodic chain of size* $2n$*, and* $H_n = -\sum_{j=1}^{2n} e_j$*. Then*

$$(\mathcal{W}_n, H_n) \xrightarrow{SL} (\chi_0\overline{\chi}_0 + \chi_{\frac{1}{2}}\overline{\chi}_{\frac{1}{2}} + \chi_{\frac{1}{16}}\overline{\chi}_{\frac{1}{16}}, L_0 + \overline{L}_0)$$

if n *is even.*

Furthermore, restricting energies up to $O(\sqrt[3]{n})$ *gives the rate of convergence* $O(\frac{1}{n})$*.*

2- For the corresponding higher Virasoro generators, with the same rate of convergence as above, given a fixed $m \neq 0$*, we have (**up to some scalings**)*

(a) $O^c_{n,m} = -\sum_{j=1}^{2n-1} \cos(\frac{m(j+\frac{1}{2})\pi}{2n+1}) e_j \xrightarrow{SL} L_m + L_{-m}$,
$O^s_{n,m} = i\sum_{j=1}^{2n-2} \sin(\frac{m(j+1)\pi}{2n+1})[e_j, e_{j+1}] \xrightarrow{SL} i(L_m - L_{-m})$

If $m \leq \sqrt[4]{n}$*, we have a rate of convergence of* $O(\frac{1}{n})$ *for energies up to* $\sqrt[4]{n}$*.*

It turns out one can define operators $\widetilde{L}^c_{\pm m}, \widetilde{L}^s_{\pm m} \xrightarrow{SL} L_{\pm m}$ satisfying the properties mentioned above, being some scalings of $O^c_{n,m}, O^s_{n,m}$,

$$\frac{\widetilde{L}^c_m + \widetilde{L}^c_{-m}}{2} = \alpha^c_n O^c_{n,m} + \beta^c_{n,m}\mathbf{1} \xrightarrow{SL} \frac{L_m + L_{-m}}{2},$$

$$\frac{i(\widetilde{L}^s_m - \widetilde{L}^s_{-m})}{2} = \alpha^s_n O^s_{n,m} + \beta^s_{n,m}\mathbf{1} \xrightarrow{SL} \frac{i(L_m - L_{-m})}{2},$$

where $\alpha^c_n, \alpha^s_n, \beta^c_{n,m}$, and $\beta^s_{n,m}$ are suitable scaling factors. An operator $\widetilde{L}_m$ is desired which has scaling limit L_m so that expressions like $\widetilde{L}(f) := \sum \hat{f}_m \widetilde{L}_m \xrightarrow{SL} L(f) = \sum \hat{f}_m L_m$ can be used where $\hat{f}_m = a_m + ib_m \in \mathbb{C}$ are Fourier coefficients of $f \in C^\infty_{\mathbb{C}}(S^1)$. Let

$$\widetilde{L}_m := \Big(\frac{\widetilde{L}^c_m + \widetilde{L}^s_m}{2} + \frac{\widetilde{L}^c_{-m} - \widetilde{L}^s_{-m}}{2}\Big) \quad \forall m \neq 0, \quad \widetilde{L}_0 = \widetilde{L}^c_0.$$

The above satisfies $\widetilde{L}_m \xrightarrow{SL} L_m$ and inherits the same rate of convergence from $\widetilde{L}^c_m$ and $\widetilde{L}^s_m$ (Remark 2.5.1) and will be used in our next results.

In 2.4.4, we compare our Koo-Saleur formula to the earlier one in the literature [1], and show why ours is a better choice with regards to *algebra* convergence which is addressed below. The important distinction lies in the energy locality of our formula, while the one in [1] mixes high and low energy spaces.

Going back to proving convergence as an SL-algebra, it is not hard to show that the three important types of CFT observables mentioned previously live in $\mathcal{A}$ as a vector space, i.e. all in a single framework. This fact tells us two things we expect. First, they are all physical as they describe some computable convergent sequence. Second, although all three sets are related (see [25]), and each believed to store all the CFT information, they have to be in our set of observables simultaneously.

(a) Smeared fields $Y(a, f)$: In section 2.5.1, using the previous results for Ising, the smeared fields $Y(a, f)$ are recovered as an SL-algebra. The steps to show this are outlined. The VOA is generated by L_ns applied to the primary fields, therefore realizing the L_ns or the smeared conformal field $Y(\omega, f)$ should be the top priority. We illustrate intuitively the idea behind the equations above defining $\widetilde{L}_n$. Consider a nonperiodic Ising chain placed on the upper half-circle S^1_+ (Figure 1.17). We ask for the "finite version" $\widetilde{\omega}$ of ω. Informally, the answer is $\widetilde{\omega} = e$ (the TL operator). For example, for any function f with $f(z) = f(\overline{z})$, with Fourier coefficients of $\sin(n\theta)$s being zero, we have (informally)

$$\int_{S^1_+} f(e^{i\frac{\pi j}{n}})e_j \xrightarrow{SL} Y(\omega, f) = \oint Y(\omega, z)f(z)z^2\frac{\mathrm{d}z}{2\pi i z},$$

where the integral on the left is an integral over a "finite" space, in other words, a summation. Hence, as ω can be regarded either as a state or a field, so does e_j, which can be seen as a vector (with a diagram presentation) or as an operator (stacking up diagrams). Here, Y (the vertex operator) is the analog of the stacking at infinity.

For the opposite situation, i.e. $f(z) = -f(\overline{z})$, where the Fourier coefficients of

$\cos(n\theta)$s are zero, it can be viewed as a *derivative* of the previous case. However, $\frac{\mathrm{d}}{\mathrm{d}z}Y(\omega,z) = Y(L_{-1}\omega,z)$. So it is necessary to find the finite version of $L_{-1}\omega$, which should be the *derivative* (as it is the interpretation of L_{-1}) of e_j. The first candidate that comes to mind is $[e_j, e_{j+1}]$ and we have

$$i\int_{S^1_+} f(e^{i\frac{\pi j}{n}})[e_j,e_{j+1}] \xrightarrow{SL} Y(\omega,f) = \oint Y(\omega,z)f(z)z^2\frac{\mathrm{d}z}{2\pi i z}.$$

For general smooth functions f, a linear combination of e_j and $[e_j, e_{j+1}]$ is required to obtain $Y(\omega, f)$. Once this is achieved, convergence as a **strong** SL-algebra is proved.

Theorem 2.5.6 $\{L(f) \mid f \in C^\infty_{\mathbb{C}}(S^1)\}$ *generates a strong SL-algebra with corresponding sequence* $\widetilde{L}(f)$ *to each* $L(f)$.

As mentioned previously, for a general VOA, one would need to recover the smeared primary fields along with the conformal field. For UMMs, and more generally unitary Virasoro VOAs, the VOA is generated from only the conformal field acting on the vacuum, the only (trivial) primary field $\{L_{-n_k}\dots L_{-n_1}\Omega\} = \mathcal{V}$. Borcherds identity (1.13) allows us to compute a descendant field based on its parents and the conformal field. Using an inductive approach, we obtain all $Y(a,f)$ for all fields a living in an SL-algebra:

Theorem 2.5.8 *The set of operators* $\{Y(a,f)|\ a \in \mathcal{V}, f \in C^\infty_{\mathbb{C}}(S^1)\} \subset \mathcal{A}$ *generate an SL-algebra, with* $Y(a,f)$ *recovered from a QC-local sequence.*

In Remark 2.5.10, we outline the technical reasons why our proof does not recover the algebra as a **strong space**-local SL-algebra and Conjecture 2.5.9 is made along this direction.

The scaling limit Theorem 2.4.1 for Ising and the rate of convergence required for the past theorems on SL-algebras, motivate the formulation of Conjecture 2.5.3 on conditions

when a unitary CFT would arise as such a limit and on the analytical requirements for the above theorems on SL-algebras to hold for higher minimal models.

Conjecture 2.5.3 *For any unitary minimal model $\mathcal{V} = \mathcal{V}_{c,0}$ and a sector $\mathcal{V}_{c,h}$, there is a sequence of (AC or lattice-based) quantum theories $(\mathcal{W}_n, H_n, \mathcal{A}_n)$ with **strong** scaling limit $(\mathcal{V}_{c,h}, L_0)$ such that for each Virasoro generator L_m, there is a sequence $\widetilde{L}_m \in \mathcal{A}_n$ (dependence on n implicit) with the following properties:*

- *$\widetilde{L}_m$ is a **space**-local observable such that $\widetilde{L}_m^\dagger = \widetilde{L}_{-m}$ and $a\widetilde{L}_m + \overline{a}\widetilde{L}_{-m} \in \mathcal{A}_n^H$ for any complex number a, where $\mathcal{A}_n^H$ is the (generating) subset of $\mathcal{A}_n$ consisting of hermitian observables.*
- *$\widetilde{L}_m$ shifts the energy no more than $|m|$.*
- *There exist constants $d_\omega, g_\omega, e_\omega > 0$ such that when $\widetilde{L}_m$ is restricted to energy at most n^{d_ω}, it has an approximation by $L_m|_{n^{d_\omega}}$ with error $O(\frac{1}{n^{g_\omega}})$:*

$$\widetilde{L}_m|_{n^{d_\omega}} = L_m|_{n^{d_\omega}} + O(\frac{1}{n^{g_\omega}}),$$

and the operator norm $||\widetilde{L}_m||$ is bounded by $O(n^{e_\omega})$.

Assuming the above conjecture for UMMs, most theorems in section 2.5 notably Theorem 2.5.8 mentioned above, hold for all UMMs as well (Remark 2.5.6 and 2.5.7 clarify which results generalize).

(b) Local conformal net observables: The algebra of observables in LCN is recovered in section 2.5.2. Finite versions of LCN are defined in the obvious way by considering intervals I in the interval set $\mathcal{I}_+$ of the upper-half circle S^1_+ (Figure 1.17), and algebra of observables on interval I is generated by e_js where $[\frac{j\pi}{2n+1}, \frac{(j+1)\pi}{2n+1}] \subset I$. We denote these by $\mathcal{A}_n(I)$ and obtain a **local** net of **strong SL-algebra** of bounded

observables $\mathcal{A}_b(I)$ in the scaling limit (Definition 2.5.3 and Corollary 2.5.9). We solely focus on characterizing the algebras $\mathcal{A}_b(I)$, however as space is part of the data describing the net, studying the net thoroughly should also allow a treatment of space in the scaling limit process.

We would like to compare this net to that of $\mathcal{A}_{vir}(I)$ (1.44). This may seem to be not a like-to-like comparison at first. Indeed, as explained in section 2.6.3, it is not accurate to call the scaling limit a *chiral* UMM as there is no preferred direction for chirality to emerge. However, it is completely described by chiral data like all other versions of CFT. This comparison helps to understand how the theory is built from chiral data.

Applying Theorem 2.5.6, one gets $e^{i\widetilde{L}(f)} \xrightarrow{SL} e^{iL(f)}$ as a corollary. Let $j(I)$ be I's reflection in the lower half-circle. For $\mathrm{supp}(f) \subset I \cup j(I)$, we know $e^{i\widetilde{L}(f)} \in \mathcal{A}_n(I)$, implying for their scaling limit $e^{iL(f)} \in \mathcal{A}_b(I)$. On the other hand, $\mathcal{A}_{vir}(I)$ is generated as a von Neumann algebra by $\{e^{iL(f)} | \mathrm{supp}(f) \subset I\}$, thus $\mathcal{A}_{vir}(I \cup j(I)) \subset \mathcal{A}_b(I)$. Once an inclusion like the latter is obtained between two nets, where one ($\mathcal{A}_{vir}$) satisfies Haag duality (1.41) and the other ($\mathcal{A}_b$) locality (1.5.1), it is a common trick to form a sequence of inclusions to show their equality. We can do this when $|I \cap \partial S^1_+| = 1$, where for J the complement of I in S^1_+:

$$\mathcal{A}_b(J) \underbrace{\subset}_{\text{locality}} \mathcal{A}_b(I)' \subset \mathcal{A}_{vir}(I \cup j(I))' \underbrace{=}_{\text{Haag duality}} \mathcal{A}_{vir}(J \cup j(J)) \subset \mathcal{A}_b(J).$$

Theorem 2.5.13 *$\mathcal{A}_b(I) = \mathcal{A}_{vir}(I \cup j(I))$ for $I \in \mathcal{I}_+$ with $|I \cap \partial S^1_+| = 1$, where $j(I)$ is I's reflection in the lower half-circle.*

Let us suppose that I does not satisfy $|I \cap \partial S^1_+| = 1$. Then

$$\mathcal{A}_{vir}(I) \vee \mathcal{A}_{vir}(j(I)) \subset \mathcal{A}_b(I) \subset (\mathcal{A}_{vir}(J_1) \vee \mathcal{A}_{vir}(J_2))',$$

where J_1 and J_2 are the two intervals obtained by removing I and $j(I)$ from the circle. The difference between $\mathcal{A}_{vir}(I) \vee \mathcal{A}_{vir}(j(I))$ and $(\mathcal{A}_{vir}(J_1) \vee \mathcal{A}_{vir}(J_2))'$ is given by the *charge transporters*. In particular, we observe that $\mathcal{A}_b(I)$ includes the algebra $(CAR(I) \otimes CAR(j(I)))^{even}$ (Remark 1.5.5). This is a case of charge transporters for $\chi_{\frac{1}{2}}$. In section 2.6.2, we speculate about the boundary CFT (Definition 1.5.4) that corresponds to $\mathcal{A}_b(I)$. More generally, if the scaling limit contains irreducible sectors χ_i, it is conjectured that charge transporters $\rho_i\overline{\rho}_i$ corresponding to χ_i exist in $\mathcal{A}_b(I)$. Even for the Ising model, the exact description of $\mathcal{A}_b(I)$ is not clear yet, showing much is yet to be done.

To generalize Theorem 2.5.13 to higher minimal models, we need a space-local relation between Temperley-Lieb generators and finite versions of Virasoro generators. In Conjecture 2.6.4, it is claimed that the trivial generalization of $O^c_{n,m}, O^s_{n,m}$ in Theorem 2.4.1 should give operators $\widetilde{L}_{\pm m}$ satisfying the properties in Conjecture 2.5.3. We will discuss in 2.6.3, using numerical arguments from [1], why this generalization to $\mathfrak{su}(2)_k$ ACs may not satisfy all properties in Conjecture 2.5.3. We will also speculate on the possible fixes, including adding higher commutators like $[e_j, [e_{j+1}, e_{j+2}]]$ to deal with the finite-size effects and changing the framework from $\mathfrak{su}(2)_k$ to the Jones-Temperley-Lieb JTL_k MTC to ensure space-locality.

(c) Point-like fields $Y(a, z)$: It is not hard to recover the point-like fields simply as an almost linear operator.

Theorem 2.5.14 *$Y(a, z) \in \mathcal{A}$ as an almost linear operator.*

However, we are unable to show that they form an algebra in the scaling limit. Intuitively, one would expect "$e_\theta \xrightarrow{SL} Y(\omega, e^{i\theta})$" where e_θ is the TL operator acting nontrivially on the part of the chain closest to angle θ. This means one can obtain the *point*-like field as an *ultra* space-local operator (acting on constantly many adjacent anyons). It is because of the high non-locality of energy (a result of being highly space-local) that the scaling limit

does not hold. The precise statement is as follows, where a scaling of e_j and $i[e_j, e_{j+1}]$ are supposed to produce the even and odd part $(\cos(), \sin())$ of the point-like field.

Theorem 2.5.15 *We do* ***not*** *have*

$$O_n = \alpha_n^c ||\widetilde{v}_x^c||^2 e_x + i\alpha_n^s ||\widetilde{v}_x^s||^2 [e_x, e_{x+1}] + (\beta_n^c.\widetilde{v}_x^c + \beta_n^s.\widetilde{v}_x^s)\mathbf{1} \xrightarrow{SL} Y(\omega, z)z^2,$$

where $z = e^{i\theta}$ *and we pick the unique* $1 \leq x \leq 2n-1$ *such that* $\theta \in [\frac{x\pi}{2n+1}, \frac{(x+1)\pi}{2n+1}]$.

In Remark 2.5.12, it is shown that one could build a sequence of space *ultra*-local operators (acting on $O(\log(n))$ adjacent anyons on the AC) converging to $Y(\omega, e^{i\theta})$. This is done by comparing the number of variables and the number of equations needed to cancel the energy non-locality. Yet, that sequence of operators may not be the one that could yield an SL-algebra.

Finally, in section 2.6, we go through a list of conceptual and technical gaps in our understanding of the scaling limit, providing directions for future works (some of which are mentioned above).

2.2 Previous works

We discuss prior works in the literature on the mathematically rigorous definition of a scaling limit in the *quantum mechanics approach*, and the recovery of algebras of observables. There is a vast literature on the subject of scaling limits in statistical mechanics [41], and substantial progress has been made in the case of Ising model proving the correlation functions in the limit are conformal invariant (see [44, 45] and references therein). Statistical mechanics approach could also provide techniques with which one could compute the conformal weights present at the scaling limit without actually diagonalizing the Hamiltonian [46].

A recent program to construct CFTs from subfactors is in [47, 48, 49], where the inductive limit of Hilbert spaces is clearly discussed based on planar algebras, which have the same Hilbert spaces of states as ACs (spin chains in these papers are better interpreted as generalized spin chains as in [42]). Our work focuses on the quantum mechanics approach to scaling limits of ACs enriching the inductive limits [48] with explicit Hamiltonians and algebras of local observables.

A scaling limit of spin chains close to our Ising AC was analyzed earlier in [1] starting with the idea of how to take the scaling limit of the Hamiltonians of the chains and also obtain the Virasoro modes L_n from Fourier transforms of the TL generators e_i. More recently, in the first paper of the series [50, 51, 52] on the $\mathfrak{gl}(1|1)$ (free) model, the authors proposed a potentially rigorous definition for the scaling limit [50, section 4.3], obtained operators like our $\widetilde{L}_m$s and computed their commutators to check their convergence to the commutators of the Virasoro modes. Such computations are commonly pursued after one obtains some operators $\widetilde{L}_m \xrightarrow{SL} L_m$ and have been done in different models both rigorously and numerically ([53],[54, p.19 and references therein]). We go beyond the convergence of commutators and further pin down the conditions necessary (Conjecture 2.5.3) to prove the same theorems for higher level UMMs.

In the third paper of the series ([52]), the authors gave a rigorous definition of scaling limit while working on the scaling limit of JTL algebra (with $d = 0 \implies e_i^2 = 0$) as it acts on a $\mathfrak{gl}(1|1)$ periodic spin-chain model (the scaling limit is the $c = -2$ Logarithmic CFT—symplectic fermions theory). Even though the context and the type of the model (Logarithmic CFTs) are quite different from ours (unitary CFTs), our definitions closely mirror theirs. But there are some differences due to our different motivation, emphasis and applications.

As defined in [52, Appendix C], our scaling limit is also dictated by the low energy behavior of Hamiltonians, Hilbert spaces, and observables. In [52], the primary focus

is on the *algebraic* scaling limit of TL operators. However, we focus on the analytic side of scaling limits motivated by our goal of simulating CFTs as we need to know how computations in the finite stages converge. Especially, the unitary evolution and correlation functions involve unbounded operators for which we desire a clear description on how they are obtained in the scaling limit. In fact, even when restricted to the bounded observables, not all bounded operators can be obtained through the algebraic approach (for example the unitary operators $e^{iL(f)}$). Related to this, the analytic approach provides a more direct picture on how the LCN emerges (section 2.5.2) since we still keep the TL operators e_i as our operators of interests and (mostly) do not switch to fermionic fields. This enables us to obtain theorems with proofs general enough for higher UMMs assuming Conjecture 2.5.3.

The algebraic approach, and algebraic-numerical techniques [55, 56], have been used to obtain more information about the algebraic structure of the Hilbert space and the algebra of observables in the scaling but to our knowledge, a mathematically rigorous procedure has been applied mainly for free models like $\mathfrak{gl}(1|1)$. We believe our analytical approach should also apply to such free models.

Recently, emergence of conformal symmetry has been numerically investigated using the Koo-Saleur generators (KSGs) [57, 1]. To compare our version of KSGs with those of [57], first recall our notation $\mathbb{L}_n = L_n + \overline{L}_n$. Our counterparts of the KSGs are operators $\widetilde{\mathbb{L}}_n \pm \widetilde{\mathbb{L}}_{-n}$ on the ACs that give us $\mathbb{L}_n \pm \mathbb{L}_{-n}$ in the scaling limit ([4, Appendix]). On the other hand, using a different diagonalization of the Hamiltonian in [57] (same as that in [1]), the authors found their KSG operators, different from ours, in the AC notation to be $\widetilde{H}_n = \frac{-N}{2\pi} \sum_{j=1}^{2N} e^{\frac{2n(j+1)\pi i}{2N}} e_j$, which converge to $L_n + \overline{L}_{-n}$. Taking the sum and difference of $\widetilde{H}_n$ and $\widetilde{H}_{-n}$ respectively, we obtain $\mathbb{L}_n + \mathbb{L}_{-n}$ from the sum and $(L_n - L_{-n}) - (\overline{L}_n - \overline{L}_{-n})$ from the difference, which does not have a counterpart in our version.

The difference stems from different diagonalizations of the same Hamiltonian, which

illustrates the potential importance of *connecting* maps in our definition of scaling limit in the next section. In [1], the diagonalization of the Hamiltonian is accomplished by constructing creation and annihilation operators from the usual Fourier transform of the Majorana operators. While in our version, the Dirac operators are obtained as cos() and sin() transforms for the left and right moving sectors, respectively, which implies that going from one diagonalization to the other requires a mixing of the left and right moving sectors of the full CFT.

It follows that the scaling limit of $\widetilde{H}_n$ from our diagonalization will have an *interchiral* part which mixes left and right moving sectors, thus clearly different from $H_n = L_n + \overline{L}_{-n}$. The method in [57] works well numerically, and for the Neveu-Schwarz sector $\chi_0\overline{\chi}_0 + \chi_{\frac{1}{2}}\overline{\chi}_{\frac{1}{2}}$, the resulting scaling limit (see e.g. [1] for a proof) gives rise to a full CFT isomorphic to ours by a not necessarily local isomorphism that connects the two different sets of creation and annihilation operators.

Finally, while not directly related, the paper [29] serves as a conceptual inspiration for our work and the techniques introduced there address analytic problems of similar nature to ours.

2.3 Scaling limit of quantum theories

It is commonly believed that QFTs are low energy effective theories such as WCS TQFTs are the low energy effective theories for two dimensional fractional quantum Hall liquids. In this section, we define mathematically a low energy limit of a sequence of quantum theories. Our formalism is closely related to the definition of topological phases in [58] and ideas in [50].

We start with the definition of quantum theories by imagining quantum theories that describe a collection of interacting quantum particles. The theories considered have

a discrete energy spectrum in the scaling limit like all CFTs. Notice this is different than the energy spectrum given by the primary fields. In the context of CFTs, there are non-unitary Virasoro representations with continuous spectrum of primary fields, while still having a discrete energy spectrum in each sector. The definition below is for a finite dimensional theory, with the next definitions defining what the infinite dimensional scaling limit is.

Definition 2.3.1 ([4], Def. 2) *A quantum theory is a 3-tuple $(\mathcal{W}, H, \mathcal{A})$ where*

- *$\mathcal{W}$ is the Hilbert space of states,*
- *H is the Hamiltonian and hermitian,*
- *$\mathcal{A}$ is a von Neumann algebra of observables.*

Remark 2.3.1 *We can also add a number of notions to the definition above. For example, the space information of the system can be thought of a graph G, which is usually the $1-$skeleton of a triangulation of the space. In the following text, G is always a chain. There are also different notions of locality based on the basis we choose. As an example, considering the space information given the graph G, the Hamiltonian H is $r-$space-local for some constant $r > 0$ if $H = \sum_{i=1}^{p} H_i$ such that each local hermitian term H_i is trivial outside the ball $B_r(v_i)$ of distance r at some vertex v_i of G. If $p = 1$, then H is $r-$space **ultra**-local. There will be different notions of locality in this chapter, explained later in section 2.3.3.*

2.3.1 Low energy limit of quantum theories

The first part of a limit theory is a Hilbert space and a Hamiltonian, which are constructed from the low energy spectra of a sequence of quantum theories $(\mathcal{W}_n, H_n)$

with strictly increasing dimensions. The scaling limit of observables will be addressed in the next section.

Assume a sequence of quantum theories $(\mathcal{W}_n, H_n)$ with H_n's eigenvalues being ordered as $\lambda_1^{(n)} \leq \ldots \leq \lambda_{d(n)}^{(n)}$, where $d(n) = \dim(\mathcal{W}_n)$. The Hilbert spaces $\mathcal{W}_n$ decompose into the corresponding one-dimensional eigenspaces

$$\mathcal{W}_n = E_{\lambda_1^{(n)}} \oplus \cdots \oplus E_{\lambda_{d(n)}^{(n)}} .$$

Denote by $\mathcal{W}_n^M$ the Hilbert space $\mathcal{W}_n$ restricted to energies at most M, i.e. $\mathcal{W}_n^M = \bigoplus_{\lambda_i^{(n)} \leq M} E_{\lambda_i^{(n)}}$. Assume the following set of properties (**P**)

- $\lambda_i = \lim\limits_{n\to\infty} \lambda_i^{(n)}$ exists for all $i \in \mathbb{N}$ with the convention $\lambda_i^{(n)} = 0$ for $i > d(n)$, and $\lim\limits_{i\to\infty} \lambda_i = \infty$,
- (*connecting maps*) for all $M > \lambda_1$ where $M \neq \lambda_j$ for all j, there exist *connecting* unitary maps $\phi_n^M : \mathcal{W}_n^M \to \mathcal{W}_{n+1}^M$ for all $n > N_M$ for some N_M depending on M,
- (*extension*) ϕ_n^M is an extension of $\phi_n^{M'}$ when $M \geq M'$, i.e. $\phi_n^M|_{\mathcal{W}_n^{M'}} = \phi_n^{M'}$.

The reason for $M \neq \lambda_j$ for all j in the first property is that energies oscillating around their limit points would make the stabilization of the low energy spectrum impossible for a cut-off $M = \lambda_j$. From now on, any cut-off will be implicitly assumed to be not equal to any λ_j. Consider the sequence $(\mathcal{W}_n^M, \phi_n^M)$ with $M > \lambda_1$ and $M \neq \lambda_j$. Note that this sequence eventually stabilizes due to the existence of *unitary* maps for large enough n.

Taking the colimit of the sequence $(\mathcal{W}_n^M, \phi_n^M)$ gives a finite dimensional vector space, called $\mathcal{V}^M$, along with the unitary maps $\rho_n^M : \mathcal{W}_n^M \to \mathcal{V}^M$. It follows that $\mathcal{V}^M$ has a *natural* Hilbert space structure. Further, for all $M, M' \in (\lambda_j, \lambda_{j+1})$, $\mathcal{V}^M = \mathcal{V}^{M'}$ as $\mathcal{W}_n^M = \mathcal{W}_n^{M'}$ for $n > \max(N_M, N_{M'})$ due to the first property. As such the space $\mathcal{V}^M$ for $M \in (\lambda_j, \lambda_{j+1})$ can be conveniently called $\mathcal{V}^{\lambda_j}$. So there are only countably many

different $\mathcal{V}^M$s. Next we add the following property to **P** on the convergence of H_n^M, the restriction of H_n to $\mathcal{W}_n^M$:

- (*convergence*) The push-forward of H_n^M on $\mathcal{V}^M$ given by ρ_n^M converges to some operator H^M:

$$\rho_n^M H_n^M (\rho_n^M)^{-1} \to H^M.$$

Obviously, H^M will be hermitian. Furthermore, the above property is equivalent to the following diagram "commuting up to ϵ_n^M in the norm operator", which goes to zero as $n \to \infty$:

$$\begin{array}{ccc} \mathcal{W}_n^M & \xrightarrow{\rho_n^M} & \mathcal{V}^M \\ {\scriptstyle H_n^M}\downarrow & & \downarrow{\scriptstyle H^M} \\ \mathcal{W}_n^M & \xrightarrow{\rho_n^M} & \mathcal{V}^M \end{array}.$$

The construction of the **scaling limit** $(\mathcal{V}, H)$ of the sequence is not hard from here.

Properties of the colimit imply that the set $\{(\mathcal{V}^M, H^M)\}$ is unique up to unique isomorphism. Using the extension property of the connecting maps, and some formal diagram chasing involving the colimit construction, one can build $\{(\mathcal{V}^M, H^M)\}$ as restrictions of a single Hilbert space and its Hamiltonian $(\mathcal{V}, H)$ [4].

In other words, one can ensure that the embedding $\mathcal{V}^M \to \mathcal{V}$ is by identity and $\mathcal{V}$ is a union (and colimit) of all $\mathcal{V}^M$s, which have a nested structure. The following diagram will also commute:

$$\begin{array}{ccc} \mathcal{V}^M & \longrightarrow & \mathcal{V} \\ {\scriptstyle H^M}\downarrow & & \downarrow{\scriptstyle H} \\ \mathcal{V}^M & \longrightarrow & \mathcal{V} \end{array}.$$

Since two colimits are taken to obtain the scaling limit (similar to the construction in [50, section 4.3]), the above process is called the *double colimit construction*, allowing the following definition

Definition 2.3.2 ([4], Def. 3) *Given a sequence of quantum theories*

$$\{(\mathcal{W}_n, H_n)\}_{n=1}^{\infty}$$

*with connecting maps ϕ_n^M satisfying properties (**P**), the scaling limit $(\mathcal{V}, H)$ is the result of the double colimit construction. This limit will be written as $(\mathcal{W}_n, H_n) \xrightarrow{SL} (\mathcal{V}, H)$.*

Note also the grading

$$\mathcal{V} = \oplus_{i=1}^{\infty} \mathcal{V}_i$$

for $\mathcal{V}$, where $\mathcal{V}_i$ is the $i-$th eigenspace of H by *strictly* increasing order of energy. This means $H|_{\mathcal{V}_i} = \lambda_j \mathbf{1}$ for some $j \geq i$; note that although $\lim \lambda_i = \infty$, λ_is are not strictly increasing.

We emphasize that as long as the connecting maps are specified the scaling limit process is unique up to unique isomorphism due to the nature of colimit. From now on, whenever a sequence of quantum theories is given with a scaling limit, implicitly, there is a given set of connecting maps. We do not discuss the issue of uniqueness any further and for a relevant example, we refer to the previous discussion in section 2.2 on different diagonalization in the case of the Ising full CFT.

Notice that $\mathcal{V}$ is separable but not complete, i.e. not a Hilbert space. The completion of $\mathcal{V}$ will be denoted by $\overline{\mathcal{V}}$. For notational easiness, The scaling limit will be written as $(\mathcal{V}, H)$ with the understanding that one needs to take a completion whenever the context requires so.

We would like to think of the scaling limit as the result of *stacking up* the low energy spectra of H_ns, and the double colimit construction fulfills this expectation. For example, let E_{λ_1} be the eigenspace of the limit Hamiltonian H corresponding to λ_1, and k be the smallest integer such that $\lambda_k > \lambda_1$, i.e. the first larger eigenvalue of H. Choose some M

such that $\lambda_1 < M < \lambda_k$, then the above construction builds a space E_{λ_1} from $\mathcal{W}_n^M$s with large enough n. These are the eigenvectors of H_n which energies converge to λ_1 in the limit. Note that our notation for the grading of $\mathcal{V}$ implies $\mathcal{V}_1 = E_{\lambda_1}$ and $\mathcal{V}_2 = E_{\lambda_k}$.

Although our definition does not assume an embedding of the whole space $\mathcal{W}_n$ into $\mathcal{W}_{n+1}$, we expect this to be the case for all physical models, like in the case of Ising and other free models [52]. Our discussions in section 2.5 will be based on this assumption:

Definition 2.3.3 ([4], Def. 4) *A sequence of quantum theories $\{(\mathcal{W}_n, H_n)\}_{n=1}^{\infty}$ gives a **strong** scaling limit $(\mathcal{V}, H)$ if in addition to properties **P**, for all n and M, the connecting maps ϕ_n^M are the restriction up to energy M of an isometry*

$$\phi_n : \mathcal{W}_n \hookrightarrow \mathcal{W}_{n+1}$$

for large enough n.

Given the above, the colimit of the sequence of embeddings $\mathcal{W}_n \hookrightarrow \mathcal{W}_{n+1}$ gives $\mathcal{V}$.

Usually, the chosen basis for $\mathcal{W}_n$ closely relates to a notion of space, and locality in this space basis is supposed to represent locality in space. Finding the embedding ϕ_n is not trivial based on this basis. In the scaling limit, the **space** embedding is not the "trivial" embedding, in contrast to the thermodynamical limit [59, Appendix A]. Indeed, the **energy** embedding is the trivial one as shown in the definition. As a result of this trivial energy embedding, the space-local operators in $\mathcal{W}_n$ (as defined later in 2.3.10) like e_is in ACs, are generally space non-local when their actions are pushed forward. This will become clearer in next few sections.

2.3.2 Scaling limit of observables

Assume a sequence of quantum theories $\{(\mathcal{W}_n, H_n, \mathcal{A}_n)\}_{n=1}^{\infty}$ with the scaling limit $(\mathcal{V}, H)$. As $\forall O \in \mathcal{A}_n$, $O + O^\dagger$ and $i(O - O^\dagger)$ are hermitian observables, $\mathcal{A}_n$ is generated by an underlying real vector space of hermitian observables called $\mathcal{A}_n^H$, and $H_n \in \mathcal{A}_n^H$. In the examples of ACs, the space $\mathcal{A}_n^H$ is given by hermitian observables generated by e_js, a subset of which $\{e_j, i[e_j, e_{j+1}]\}$ will be our focus. This choice is motivated on one hand from including the local terms of interaction of the system, and on the other hand to recover the Virasoro algebra in the scaling limit (see Theorem 2.5.2 and Remark 2.5.11).

To build the observables of $\mathcal{V}$ from the observables of $\mathcal{W}_n$, the low energy behavior of the observables has to be taken into account.

Definition 2.3.4 ([4], Def. 5) *Let $O_n \in \mathcal{A}_n$ be any sequence of observables. For a given M and $u, v \in \mathcal{V}^M$, denote by $u_n, v_n \in \mathcal{W}_n^M$ the vectors $(\rho_n^M)^{-1}u, (\rho_n^M)^{-1}v$, which are defined for sufficiently large n. The scaling limit of O_n is a partially-defined (defined on a subset of $\mathcal{V} \times \mathcal{V}$) sesquilinear form $O(\cdot, \cdot)$, where $O(u, v)$ is defined as $\lim_{n\to\infty}(u_n, O_n v_n)$ when it exists. We will denote the scaling limit by $O_n \xrightarrow{SL} O$.*

Notice O exactly stores the information in the limit for the expectation values of O_n. If O can be represented by a linear operator as $O(u, v) = (u, Ov)$, then O will also denote the linear operator.

Definition 2.3.5 ([4], Def. 6) *We define the following sets from the set of sesquilinear forms in Definition 2.3.4:*

- $\overline{\mathcal{A}}$*: the set of observables in Definition 2.3.4,*
- $\mathcal{A}$*: the vector space of sesquilinear forms in $\overline{\mathcal{A}}$ defined on $\mathcal{V} \times \mathcal{V}$,*
- $\mathcal{A}^H$*: the real vector space of sesquilinear forms represented by hermitian operators which are scaling limit of hermitian observables in $\mathcal{A}_n^H$ and defined on $\mathcal{V}$.*

Another difficulty with understanding the scaling limit is the difference in the description of observables at finite stages and at the scaling limit. The chosen basis for $\mathcal{W}_n$ is usually closely related to the notion of space and not energy. In the case of ACs, the e_is are space ultra-local operators. On the other hand in CFTs, a "space-based" description of the operators for computational purposes is hard to find. Among the different frameworks, VOA is the only one suitable for actual calculations. Even though the $Y(a,z)$s are thought to be space ultra-local observables in the continuous spacetime, yet their description is a Fourier-like series of mode operators $\sum a_{(n)}z^{-n-1}$. The mode operators are energy shifting operators while their space action is obscure. Therefore, having a general definition of a Fourier transform on the e_is is essential to understand the relation between TL and Virasoro algebra (see Conjecture 2.6.4 for an attempt). Alternatively, one has to find some space description of $Y(a,z)$.

Remark 2.3.2 *$\mathcal{A}$ is closed under the obvious weak limit. In fact, we can consider the semi-norms $||\cdot||_n$ on $\mathcal{A}$ which are defined by $||O||_n = ||P^nOP^n||$. Here, P^n is the restriction up to energy λ_n, and P^nOP^n should be read as the linear operator that the sesquilinear form O gives once restricted to the finite-dimensional space $\mathcal{V}^{\lambda_n}$. Then, one can consider the usual operator norm $||P^nOP^n||$ of this linear map. By some standard analysis argument [4, Remark 3], it can be shown that $\mathcal{A}$ is a Fréchet space with respect to these (separated) countably many semi-norms, thus providing the* ***scaling limit metric*** *d_{SL}.*

Adding to the above remark, consider the case of a strong scaling limit with connecting maps ϕ_n, and assume there also exist embeddings $\tau_n : \mathcal{A}_n \hookrightarrow \mathcal{A}_{n+1}$ compatible with ϕ_n:

$$\phi_n \circ O_n = \tau_n(O_n)|_{\phi_n(\mathcal{W}_n)},\ \forall O_n \in \mathcal{A}_n.$$

Then scaling limit becomes equivalent to convergence in the metric d_{SL}. Let us call

the colimit of the sequence of embeddings $\mathcal{A}_n \hookrightarrow \mathcal{A}_{n+1}$ the *algebraic* scaling limit. The closure with respect to d_{SL} of the algebraic scaling limit is precisely $\mathcal{A}$, which we can call the *analytic* scaling limit. As τ_n is compatible with ϕ_n, by the scaling limit construction, the algebraic scaling limit contains a copy of each O_n defined on $\mathcal{V}$. The embeddings τ_n exist in the case of $\mathfrak{gl}(1|1)$ studied in [52, Theorem 4.4]. For Ising ACs, the algebra $\mathcal{A}_n$ for $\mathcal{W}_n$ is the *even* algebra generated by Dirac operators (basis given by even products $\{\Psi_{i_1} \dots \Psi_{i_{2k}} | \ -n \leq i_j \leq n\}$), and the embeddings τ_n are obvious. We conjecture that such embeddings exist for higher level anyonic chains.

Remark 2.3.3 *One can ask whether $\mathcal{A}$ "generates" $\overline{\mathcal{A}}$? This is true for any model with an algebraic scaling limit. As the algebraic scaling limit contains a copy of any $O_n \in \mathcal{A}_n$ in $\mathcal{A}$, any observable $O_n \xrightarrow{SL} O \in \overline{\mathcal{A}}$ can be seen as an operator obtained as scaling limit of the copies of O_n inside $\mathcal{A}$, implying that $\mathcal{A}$ generates $\overline{\mathcal{A}}$.*

For the set of observables in each framework for unitary CFTs, there is an underlying generating set of hermitian observables. Indeed the *hermitian fields* (more strongly, hermitian quasi-primary fields) generate the VOA [25]. As for LCN, since the algebra corresponding to an interval I is a von Neumann algebra, it is trivially true that it can be generated by hermitian observables. In the general scaling limit, we do not know whether $\mathcal{A}^H$ generates $\mathcal{A}$.

We wish to identify some subsets of $\overline{\mathcal{A}}$ that may be algebras. Since operators may not be linear, it is not clear how one can have an algebraic structure. In general, there might not be a linear operator which gives the sesquilinear form O.

Sometimes these operators can be *almost linear*, as is the case of point-like fields $Y(a, z)$ or any $O \in \mathcal{A}$. They are *almost* linear since one can formally set $Ov = \sum_i v_i$ where v_i is the vector which dual $(-, v_i)$ represents the functional $O(-, v)$ on $\mathcal{V}_i$, i.e. for any $u \in \mathcal{V}_i$, we have $O(u, v) = (u, v_i)$. If the formal sum is always finite, then O is a linear

operator. For this reason, $\mathcal{A}$ is also called the vector space of almost linear observables.

The definition for the product of such operators is exactly in the same spirit of the correlation function $(u, Y(a^{(1)}, z_1) \dots Y(a^{(k)}, z_k)v)$ in (1.18).

Definition 2.3.6 ([4], Def. 7) *Given $O^{(1)}, \dots, O^{(k)} \in \mathcal{A}$, the product $O^{(1)} \dots O^{(k)}$ as a partially-defined sesquilinear form $F(u, v)$ is well-defined at $(u, v) \in \mathcal{V} \times \mathcal{V}$ if the following is **absolutely** convergent:*

$$\sum_{m_1,\dots,m_k} (u, P^{m_1} O^{(1)} \dots P^{m_k} O^{(k)} v),$$

where P^{m_i} projects onto $\mathcal{V}_{m_i}$.

One basic obstacle to get an algebraic structure is when observables O_n have a *significant mix* of the low and high energy states (energy non-local). For example, define the two sequences below where $v_n^{(i)} \in E_{\lambda_i^{(n)}}$ are pull-back of some $v^{(i)} \in E_{\lambda_i}$:

- $O_{n,1} = v_n^{(1)} (v_n^{d(n)})^\dagger + v_n^{d(n)} (v_n^{(1)})^\dagger$,
- $O_{n,2} = 0$,

where recall $d(n) = \dim \mathcal{W}_n$. Both sequences converge to the zero operator $O = 0$, while being quite different. The significant (non-decaying) mix of low-high energy states in the $O_{n,1}$s manifests itself not in the expectation values of the observables at low energies, but the higher powers of the observables. Indeed, looking at the expectation values of powers, while $O_{n,2}^k \xrightarrow{SL} O^k = 0$ for any k, one has $O_{n,1}^2 \xrightarrow{SL} v^{(1)}(v^{(1)})^\dagger \neq 0$. Next example shows that the rate of decay of this low-high energy mix is important:

- $O_{n,1} = v_n^{(1)} (v_n^{(1)})^\dagger + 2^{d(n)} v_n^{d(n)} (v_n^{d(n)})^\dagger$,
- $O_{n,2} = v_n^{(1)} (v_n^{(1)})^\dagger + \sum_{i=1}^{d(n)} \frac{1}{i^2} \Big(v_n^{(i)} (v_n^{(1)})^\dagger + v_n^{(1)} (v_n^{(i)})^\dagger \Big)$.

It is not hard to check that

$$O^k_{n,1} \xrightarrow{SL} O^k_1, \quad \text{where} \quad O_1 = v^{(1)}(v^{(1)})^\dagger,$$

and

$$O^k_{n,2} \xrightarrow{SL} O^k_2, \quad \text{where } O_2 = v^{(1)}(v^{(1)})^\dagger + \sum_{i=1}^{\infty} \frac{1}{i^2}\Big(v^{(i)}(v^{(1)})^\dagger + v^{(1)}(v^{(i)})^\dagger\Big).$$

The first sequence has a significant high-high energy mix while the second has a decaying low-high energy mix. One can check that $||O_{n,1}O_{n,2}v^{(1)}_n|| \not\to ||O_1O_2v^{(1)}||$, i.e.

$$(v^{(1)}_n, O_{n,2}O_{n,1}O_{n,1}O_{n,2}v^{(1)}_n) \not\to (v^{(1)}, O_2O_1O_1O_2v^{(1)}).$$

So $O_{n,2}O_{n,1}O_{n,1}O_{n,2}$ does not have $O_2O_1O_1O_2$ as a scaling limit. The reason behind this is an *imbalance* between the low-high energy mix decay rate and the rate of high-high energy mix. We note that it is possible to have a collection of observables with high-high energy mix, which is even increasing, and yet have an algebra, as will be shown in the case of Virasoro operators $\widetilde{L}_n \xrightarrow{SL} L_n$. Generally speaking, energy non-locality should be avoided in order to get an algebra. For the discussion of algebras in scaling limit, a natural definition is

Definition 2.3.7 ([4], Def. 8) *Given a set of almost linear observables $\{O^{(i)}\}_{i\in I}$, and the algebra of operators generated by this set. If this algebra is inside $\mathcal{A}$, it is called a scaling limit algebra (SL-algebra).*

Some of these SL-algebras are special, in the sense that each observable has a *nice* associated sequence:

Definition 2.3.8 ([4], Def. 9) *Given an SL-algebra as in Definition 2.3.7, let each $O^{(i)}$*

have an associated sequence $O_n^{(i)}$ *such that for any* $i_1, i_2, \ldots, i_k$,

$$\lim_{n\to\infty}(u, O_n^{(i_1)}\cdots O_n^{(i_k)}v) = (u, O^{(i_1)}\cdots O^{(i_k)}v), \quad \forall u, v \in \mathcal{V}.$$

Then the algebra generated by $\{O^{(i)}\}_{i\in I}$ *is called a* ***strong*** *SL-algebra.*

For example, $\{L(f)|f \in C^\infty_{\mathbb{C}}(S^1)\}$ gives a strong SL-algebra in the case of Ising. The above definition assumes a strong property which is sometimes not easy to show; in section 2.5, we can only show that $\{Y(a,f)|f \in C^\infty_{\mathbb{C}}(S^1)\}$ is an SL-algebra.

2.3.3 Locality in scaling limit

We review the conventions/terminology around *local* observables. In LCN (or more generally for QFTs in Haag Kastler's framework), the adjective local for a local net $\mathcal{A}$ of von Neumann algebras refers to the locality axiom: If I_1 and I_2 are spacelike separated, then observables in $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$ commute. Further, elements inside the *local observables* algebra $\mathcal{A}(I)$ are also called local observables [60, 25].

For VOA or more generally in the Wightman's framework for QFT, observables are described as *(primary) fields*, or *distribution of operators* Φ, or *limits of observables localized at a point* x [61]. In addition, there are *local* smeared fields $\Phi(f)$ with functions f having support in some region O [60, II.4.1] (if f is a so-called test function, then $\Phi(f)$ is "almost local"). We also have a similar locality axiom: Let Φ_1 and Φ_2 be two observables and functions f_1 and f_2 be spacelike separated in their supports, then $[\Phi_1(f_1), \Phi_2(f_2)] = 0$.

The conclusion is that "local" is used in all frameworks to describe the observables in sets satisfying some locality axiom. Our definition of locality (in space and in energy) turns out to be more restrictive.

Energy-local observables

Finding the "energy basis" requires an understanding of the energy local degrees of freedom (EL-DOFs), which comes from an exact diagonalization of the Hamiltonian. Even numerical exact diagonalization is very limited for interacting models. For the Ising AC, exact diagonalization analytically gives us the creation and annihilation operators, which are the EL-DOFs. This, in turn, provides us the energy basis, which allows us to construct the scaling limit at each energy eigenspace. For all the models with known CFT limits, only free theories have mathematical descriptions of their EL-DOFs so far (see [43, 52, 53] for some recent examples). Without a clear knowledge of EL-DOFs, it is hard to make an accurate definition of energy-local operators. Thus our definition later (Definition 2.3.9) will have to be flexible to certainly include those operators that are actually energy-local. Of course, the downside is that our definition also allows operators that should not be called energy-local.

Another goal of the following definition is to find out constraints on observables in the scaling limit that will force them to be of a specific type. Locality is one of these fundamental constraints.

We propose a definition of energy-local operators without using any explicit knowledge of the EL-DOFs. Therefore, it is likely not the most refined definition and should not be used to define local operators in a quantum simulation problem. Still, our *intrinsic* notion of energy locality together with space locality, put enough constraints on operators so that they are easier to work with (Theorem 2.5.1).

All smeared operators $Y(a, f)$ where f has finite Fourier series do not shift the energy of any eigenvector by more than a constant. This is a motivation for the definition of *energy locality*.

Definition 2.3.9 ([4], Def. 10) *Given a scaling limit as in Definition 2.3.2, the se-*

quence $(O_n)_n$ *is* $\Lambda-$*energy-local for* $\Lambda \in \mathbb{N}$*, if for any* n, M *and for all* $u \in \mathcal{V}_i, v \in \mathcal{V}_j$ *with* $|i-j| > \Lambda$*,*

$$((\rho_n^M)^{-1}(u), O_n(\rho_n^M)^{-1}(v)) = 0,$$

where $(\rho_n^M)^{-1}(u) := 0$ *if* $u \notin \rho_n^M(\mathcal{W}_n^M)$*. An observable* $O \in \mathcal{A}$ *which is the scaling limit of such a sequence is called a* $\Lambda-$*energy-local observable.*

Any $\Lambda-$energy-local is a linear operator, as the formal sum Ov is a finite sum with no more than 2Λ terms. Moreover,

Theorem 2.3.1 ([4], Thm. 2.1) $\Lambda-$*energy-local observables for all* $\Lambda \in \mathbb{N}$ *form a strong SL-algebra.*

Proof: Consider Λ_i-energy-local observables $O^{(i)}, 1 \leq i \leq k$ and corresponding sequences $(O_n^{(i)})_n$. Note $(u, O^{(k)} \dots O^{(1)} v)$ is well-defined; if $v \in \mathcal{V}_t$ for some t, then every multiplication by some $O^{(i)}$ makes a vector in a space enlarged by adding/subtracting the energy level by Λ_i. This means taking projections onto $\mathcal{S} = \bigoplus_{i=t-\sum \Lambda_i}^{t+\sum \Lambda_i} \mathcal{V}_i$ called $P_{\mathcal{S}}$, all operators $O^{(i)}$ in the product can be replaced with the linear operator $P_{\mathcal{S}} O^{(i)} P_{\mathcal{S}}$ without changing the result.

Similarly for the expectation values $((\rho_n^M)^{-1}(u), O_n^{(k)} \dots O_n^{(1)}(\rho_n^M)^{-1}(v))$, everything is also happening in a finite dimensional Hilbert space. Let $\lambda_{(i)}$ be the energy corresponding to $\mathcal{V}_i$. The limit can be taken with restriction to $\mathcal{W}_n^M \backslash \mathcal{W}_n^{M'}$, with $\lambda_{(t+\sum \Lambda_j)} < M < \lambda_{(1+t+\sum \Lambda_j)}$ and $\lambda_{(t-1-\sum \Lambda_j)} < M' < \lambda_{(t-\sum \Lambda_j)}$, which is a finite dimensional Hilbert space stabilizing for large enough n and becoming isometric to $\mathcal{S}$. This means for large enough n, we might as well assume that all operators $O_n^{(i)}$ are acting on $\mathcal{S}$, by using the connecting maps followed by the projection $P_{\mathcal{S}}$ like the previous case. In this setting, we have a sequence of operators weakly convergent, but all acting on a finite dimensional Hilbert space. This implies norm convergence and the convergence of their product as a $(\sum \Lambda_j)-$energy-local operator. ■

This is our first example of an algebra which is preserved under the scaling limit. We have required a **constant** Λ to define energy locality. One might think of the possibility to enlarge the set of all $\Lambda-$energy-local observables to include those operators that are scaling limits of $\Lambda(n)-$energy-local observables where $\Lambda(n)$ is a function of n.

The motivation for this modification comes from the smeared operators $Y(a, f)$ where f has *infinite* Fourier series. Any product of these operators is defined on the VOA (1.37), so it is possible that they form a strong SL-algebra. They are not energy-local by themselves, but it is important to recall that the higher shift of energies happens with a magnitude $|\hat{f}_n|$ which is rapidly decaying, faster than inverse of any polynomial (1.34).

Another motivation is from quantum computation, where a local operator is defined to be a sum of operators, each acting on no more than $O(\log(n))$ particles for a system with n particles. This is a discrete way of characterizing locality which is equivalent to defining locality as an action that has exponential decay away from a specific particle. A similar story could apply to energy locality.

Space-local observables

To have a notion of space, some notion of adjacency for particles in $\mathcal{W}_n$ is required.

Definition 2.3.10 ([4], Def. 11) *The $r-$space-local operators in ACs are a sequence of operators $O_n \in \mathcal{A}_n$ that are the sum of $r-$space ultra-local operators. An $r-$space ultra-local operator acts on r many of adjacent particles.*

Remark 2.3.4 *The TL operator e_i is $3-$space ultra-local. Notice the difference between space-locality in our sense and locality in quantum computation (QC-locality). A sequence of observables like $O_n = e_1 e_{\lfloor \frac{n}{2} \rfloor} \in \mathcal{A}_n$ is considered to be QC-local, while it is not space-local. On the other hand, space-locality clearly implies QC-locality.*

More generally, we note that locality in quantum computation applies on sequences O_n where O_n is the sum of operators acting nontrivially on $O(\log(n))$ many particles. Thus we have the following definition as well:

Definition 2.3.11 *A sequence of space ultra-local operators is a sequence of $O(\log(n))$-space ultra-local operators. O_n is a space-local sequence if it is the sum of $O(\log(n))$-space ultra-local operators.*

The picture we hope to obtain for $Y(a, f)$ in finite settings is that of a quantum system with a large number of equidistant particles and some *space ultra-local* operator $\widetilde{a}$ (the *finite* version of a), applied with weight f on each particle and constantly many of its close neighbors. Informally,

$$\sum_j f(e^{i\frac{2\pi j}{n}})\widetilde{a}_j \xrightarrow{SL} Y(a, f).$$

This will be explored further in section 2.5.

2.4 Scaling limit of Ising anyonic chains

The main theorem of the section will be written in its entirety as a reference. We will give an outline of the proof in the most important cases and discuss differences of our formulae with those found in the literature. We refer to [4, appendix] for the details. Recall (a, b) is used to denote the Hilbert space given by the anyonic chain with the two ends of the chain being a and b.

Theorem 2.4.1 ([4], Thm. 3.1) *1- The following **strong** scaling limits hold, **up to some scalings** of the Hamiltonians*

(a) $\mathcal{W}_n = (\frac{1}{2}, \frac{1}{2})$, $H_n = -\sum_{j=1}^{2n-1} e_j$. *Then* $(\mathcal{W}_n, H_n) \xrightarrow{SL} (\chi_0 + \chi_{\frac{1}{2}}, L_0)$.

(b) $\mathcal{W}_n = (0,0)$ *or* $(1,1)$, $H_n = -\sum_{j=2}^{2n-2} e_j$. *Then* $(\mathcal{W}_n, H_n) \xrightarrow{SL} (\chi_0, L_0)$.

(c) $\mathcal{W}_n = (0,1)$ *or* $(1,0)$, $H_n = -\sum_{j=2}^{2n-2} e_j$. *Then* $(\mathcal{W}_n, H_n) \xrightarrow{SL} (\chi_{\frac{1}{2}}, L_0)$.

(d) $\mathcal{W}_n = (\frac{1}{2},1)$ *or* $(\frac{1}{2},0)$, $H_n = -\sum_{j=1}^{2n-2} e_j$. *Then* $(\mathcal{W}_n, H_n) \xrightarrow{SL} (\chi_{\frac{1}{16}}, L_0)$.

(e) Let $\mathcal{W}_n$ *be the periodic chain of size* $2n$, *and* $H_n = -\sum_{j=1}^{2n} e_j$. *Then*

$$(\mathcal{W}_n, H_n) \xrightarrow{SL} (\chi_0\overline{\chi}_0 + \chi_{\frac{1}{2}}\overline{\chi}_{\frac{1}{2}} + \chi_{\frac{1}{16}}\overline{\chi}_{\frac{1}{16}}, L_0 + \overline{L}_0)$$

if n *is even.*

Furthermore, restricting energies up to $O(\sqrt[3]{n})$ *gives the rate of convergence* $O(\frac{1}{n})$.

2- For the corresponding higher Virasoro generators action, given a fixed $m \neq 0$, *we have (up to some scalings)*

(a) $-\sum_{j=1}^{2n-1} \cos(\frac{m(j+\frac{1}{2})\pi}{2n+1}) e_j \xrightarrow{SL} L_m + L_{-m}$,
$i\sum_{j=1}^{2n-2} \sin(\frac{m(j+1)\pi}{2n+1})[e_j, e_{j+1}] \xrightarrow{SL} i(L_m - L_{-m})$

(b) $-\sum_{j=2}^{2n-2} \cos(\frac{m(j+\frac{1}{2})\pi}{2n-1}) e_j \xrightarrow{SL} L_m + L_{-m}$,
$i\sum_{j=2}^{2n-3} \sin(\frac{m(j+1)\pi}{2n-1})[e_j, e_{j+1}] \xrightarrow{SL} i(L_m - L_{-m})$

(c) $-\sum_{j=2}^{2n-2} \cos(\frac{m(j+\frac{1}{2})\pi}{2n-1}) e_j \xrightarrow{SL} L_m + L_{-m}$,
$i\sum_{j=2}^{2n-3} \sin(\frac{m(j+1)\pi}{2n-1})[e_j, e_{j+1}] \xrightarrow{SL} i(L_m - L_{-m})$

(d) $-\sum_{j=1}^{2n-2} \cos(\frac{m(j+\frac{1}{2})\pi}{2n}) e_j \xrightarrow{SL} L_m + L_{-m}$,
$i\sum_{j=1}^{2n-3} \sin(\frac{m(j+1)\pi}{2n})[e_j, e_{j+1}] \xrightarrow{SL} i(L_m - L_{-m})$

(e) $-\sum_{j=1}^{2n} \cos(\frac{2m(j+\frac{1}{2})\pi}{2n}) e_j \xrightarrow{SL} \mathbb{L}_m + \mathbb{L}_{-m}$
$i\sum_{j=1}^{2n} \sin(\frac{2m(j+1)\pi}{2n})[e_j, e_{j+1}] \xrightarrow{SL} i(\mathbb{L}_m - \mathbb{L}_{-m})$

If $m \leq \sqrt[4]{n}$, *we have a rate of convergence of* $O(\frac{1}{n})$ *for energies up to* $\sqrt[4]{n}$.

We will give an outline of the proof by going through the first case of the boundary conditions 1(a) and 2 (a), and briefly mention the full periodic case. We will also give a comparison of our formulae to those in [1] and how ours are better suited to obtain convergence of algebras. We refer to [4, Appendix] for more details.

2.4.1 Boundary condition $(\frac{1}{2}, \frac{1}{2})$

We will follow closely the method used in [43] to obtain all the Virasoro modes. It is therefore necessary to review the general procedure described for the Hamiltonian diagonalization of 1(a) in [43].

Proving $H_n \xrightarrow{SL} L_0$

Consider the operator $-\sum_{j=1}^{2n-1} t_j e_j$, which due to the identities

$$e_{2j} = \frac{1}{\sqrt{2}}(1 + \sigma_j^z \sigma_{j+1}^z), \quad e_{2j-1} = \frac{1}{\sqrt{2}}(1 + \sigma_j^x), \quad \text{where } \sigma^x, \sigma^z, \sigma^y \text{ are Pauli operators,}$$

after a suitable scaling becomes

$$H = -\sum_{j=1}^{n} t_{2j-1}\sigma_j^x - \sum_{j=1}^{n-1} t_{2j}\sigma_j^z \sigma_{j+1}^z,$$

where the coefficients t_j are fixed. Written this way, we see the famous $\mathbb{Z}_2$ symmetry by the spin-flip operator

$$(-1)^F := \prod_{j=1}^{2n} \sigma_j^x.$$

As detailed in [43], to diagonalize this Hamiltonian, the **Majorana** operators should be defined as

$$\psi_{2j-1} = \Big(\prod_{k=1}^{j-1} \sigma_k^x\Big)\sigma_j^z, \qquad \psi_{2j} = i\Big(\prod_{k=1}^{j} \sigma_k^x\Big)\sigma_j^z$$

which satisfy the ACR (Anticommutative Canonical Relations):

$$\{\psi_a, \psi_b\} = 2\delta_{ab}, \quad \forall a, b = 1, \ldots, 2n.$$

It is a well-known fact that these operators and their monomials are linearly independent and this representation of the Clifford algebra is faithful. Using

$$e_a = \frac{1}{\sqrt{2}}(1 + i\psi_a\psi_{a+1}),$$

we rewrite the Hamiltonian

$$H = i\sum_{a=1}^{2n-1} t_a\psi_{a+1}\psi_a.$$

Next, the raising (creation) and lowering (annihilation) **Dirac** operators are introduced, satisfying

$$[H, \Psi] = 2\epsilon\Psi.$$

For any operator linear in the Majorana operators, the commutator with H is also linear in the Majorana operators. Thus, let us choose the following form for Ψ

$$\Psi = \sum_b i^b \mu_b \psi_b,$$

where μ_b are numbers that will turn out to be real. The i^b's factor will ensure that the matrix in (2.1) is hermitian and not skew-hermitian, making the computations easier. Computing μ'_as,

$$\Psi' = [H, \Psi] = \sum_a i^a \mu'_a \psi_a,$$

is same as the following matrix equation

$$\begin{pmatrix} \mu_1' \\ \mu_2' \\ \vdots \\ \\ \mu_{2n}' \end{pmatrix} = 2 \begin{pmatrix} 0 & 1 & 0 & \dots & \\ 1 & 0 & 1 & & \\ 0 & 1 & 0 & & \\ \vdots & & & & 1 \\ & & & 1 & 0 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \\ \mu_{2n} \end{pmatrix}. \tag{2.1}$$

This hermitian matrix has determinant $(-1)^n \prod_{j=1}^{n} t_{2j-1}^2$. The eigenvectors of this matrix give the Dirac operators and each corresponding eigenvalue is the energy that is raised or lowered. Specializing the values of t_js will give the different boundary conditions. $(\frac{1}{2}, \frac{1}{2})$ corresponds to $t_j = 1$ for all j, thus our matrix is (2.1) assuming $t_j = 1$. For other boundary conditions $(0,0), (1,0), (0,1), (1,1)$: $t_1 = t_{2n-1} = 0$, and $(\frac{1}{2}, 0), (0, \frac{1}{2})$: $t_{2n-1} = 0$.

Notation. For $n \in \mathbb{N}$, set $[n] := \{1, \dots, n\}$. We can subtract sets $[2n] - [n] = \{n+1, \dots, 2n\}$. Similarly define $[-n] := \{-1, \dots, -n\}$.

The Dirac operators Ψ_k for $k \in [2n]$, are given by the eigenvectors $\mu_{a,k} = \sin(\frac{ak\pi}{2n+1})$ with corresponding energy $\epsilon_k = 4\cos(\frac{k\pi}{2n+1})$, satisfying

$$[H, \Psi_{\pm k}] = 2\epsilon_{\pm k}\Psi_{\pm k}, \ \{\Psi_{\pm k}, \Psi_{\pm k'}\} = 0, \ \{\Psi_{\pm k}, \Psi_{\mp k'}\} = N_k \delta_{k,k'} \mathbf{1}, \tag{2.2}$$

where $\Psi_{-k} := \Psi_{2n+1-k}$, and $N_k = 2\sum_a |\mu_{a,k}|^2$. The relations are obtained using the identities

$$\{\Psi, \chi\} = \sum_{a,b} i^{a+b} \mu_a \nu_b \{\psi_a, \psi_b\} = 2 \sum_a (-1)^a \mu_a \nu_a,$$

for any two linear Majorana forms $\Psi = \sum_b i^b \mu_b \psi_b, \chi = \sum_b i^b \nu_b \psi_b$. The hermitian matrix has orthogonal eigenvectors, and for any eigenvector $(\mu_{a,k})_a$ giving eigenvalue ϵ_k, there is

a corresponding eigenvector $((-1)^{a+1}\mu_{a,k})_a$ giving eigenvalue $\epsilon_{-k} := -\epsilon_k$. Thus we obtain $\Psi_k^\dagger = \Psi_{-k}$ and the equations (2.2).

We will always work with the normalization of Ψ_k by $\sqrt{N_k}$, hence $\{\Psi_{\pm k}, \Psi_{\mp k'}\} = \delta_{k,k'}\mathbf{1}$. The Dirac operators Ψ_k for $k \in [n]$ are called the *raising or creation* operators and the Dirac operators Ψ_k for $k \in [2n]-[n]$ are the *lowering or annihilation* operators. There will be a renumbering of the operators indices which will make the creation operators have negative index while the annihilation operators will have positive index.

Ψ_ks satisfy the ACR while the dimension of $\mathcal{W}_n$ (the Hilbert space) is 2^n. This implies the existence of an orthonormal basis of $\mathcal{W}_n$ given by

$$\prod_{i\in S} \Psi_i \Omega_n, \ \forall S \subset [n],$$

all of which turn out to be eigenvectors of H, where Ω_n is the *vacuum* or *ground state* annihilated by the annihilation operators. This is due to the energy symmetry of H and well-known properties of the representations of the algebra generated by the Ψ_ks [43]. We recall some easy-to-prove facts on the representations of Dirac operators.

Notation. Denote by $\mathcal{F}_n$ the algebra generated by the Ψ_ks and $\mathcal{F}_n^+$ the sub-algebra generated by the creation operators. Similarly define $\mathcal{F}_n^-$. We will use S as any subset of the indices of creation operators.

Fact 2.4.1 *Let $\mathcal{W}$ be a representation of $\mathcal{F}_n$ which is a Hilbert space with $\dim \mathcal{W} = 2^s$ where $s \geq n$ and $\Psi_k^\dagger = \Psi_{-k}$ with respect to the inner product of $\mathcal{W}$. Consider the image $\mathcal{W}_0$ of the product of all annihilation operators. Taking a unit "vacuum" vector $v \in \mathcal{W}_0$, by definition of $\mathcal{W}_0$ and ACR relations (in particular $\Psi_k^2 = 0$), we get $\mathcal{F}_n^-(v) = \{0\}$. Further, the space $\mathcal{W}_v = \mathcal{F}_n^+(v)$ generated by the creation operators acting on v has dimension 2^n with an orthonormal basis $\{\prod_{i\in S} \Psi_i v| \ \forall S\}$.*

Fact 2.4.2 *With the same settings of Fact 2.4.1, let D be a matrix satisfying $[D, \Psi_k] = 0$ for all $\Psi_k \in \mathcal{F}_n$. Then D preserves $\mathcal{W}_0$ and it is uniquely determined based on how it acts on $\mathcal{W}_0$. In particular, if there is a decomposition of $\mathcal{W}$ into 2^{s-n} irreducible representations where D preserves the corresponding "vacuums", then D acts as a scalar on each one of them. This will be the case here.*

Fact 2.4.3 *In addition to the spin-flip symmetry $(-1)^F$, the matrix H has charge conjugation symmetry (also called energy symmetry) provided by $C = \prod_i \sigma_i^z \prod_i (\sigma_i^x)^i$ satisfying $CH = -HC$. This implies each energy has one corresponding opposite energy. This is a necessary property which helps us to show that some nonzero scalar shift breaking this symmetry for H can not happen.*

From Fact 2.4.1, $(\mathcal{W}_n)_0$ is one dimensional from which a unit vector Ω_n is chosen. Define

$$H' := \sum_{k\in[n]} \epsilon_k(\Psi_{+k}\Psi_{-k} - \Psi_{-k}\Psi_{+k}).$$

H's eigenvectors are $\{\prod_{i\in S} \Psi_i\Omega_n |\ \forall S\}$, each with the corresponding eigenvalue $\sum_{i\in S} \epsilon_i - \sum_{j\notin S} \epsilon_j$. So H' has $C-$symmetry. Further $[H', \Psi_k] = 2\epsilon_k\Psi_k$ and so, for $D = H - H' \implies [D, \Psi_k] = 0$. As $(\mathcal{W}_n)_0$ is one dimensional, $D = \alpha\mathbf{1}$. But H' shifted by any α does not satisfy the energy symmetry. Therefore, $\alpha = 0$ and $H' = H$. Taking the shift $H \to H + \sum \epsilon_k$ and using $\{\Psi_{+k}, \Psi_{-k}\} = \mathbf{1}$,

$$H = \sum_{k\in[n]} 2\epsilon_k\Psi_{+k}\Psi_{-k}.$$

The final change to H is $H \to \frac{2n+1}{8\pi}H$ and the desired Hamiltonian $\widetilde{L}_0^c$ is given by:

$$\widetilde{L}_0^c = \frac{2n+1}{\pi} \sum_{k\in[n]} \cos(\frac{k\pi}{2n+1})\Psi_{+k}\Psi_{-k}. \tag{2.3}$$

Defining the scaling limit requires defining the connecting maps. Before doing so, a renumbering $k \to k - \frac{1}{2} - n$ is performed to get the creation operators indices as $\{-\frac{1}{2}, \ldots, -(n-\frac{1}{2})\}$.

Notation. $[(n-\frac{1}{2})] := \{\frac{1}{2}, \ldots, (n-\frac{1}{2})\}$ and $[-(n-\frac{1}{2})] := \{-\frac{1}{2}, \ldots, -(n-\frac{1}{2})\}$.

This will change the coefficients from $\cos(\frac{k\pi}{2n+1}) = -\sin(\frac{(k-\frac{1}{2}-n)\pi}{2n+1})$ to $\sin(\frac{-k\pi}{2n+1})$, giving

$$\widetilde{L}_0^c = \frac{2n+1}{\pi} \sum_{k \in [(n-\frac{1}{2})]} \sin(\frac{k\pi}{2n+1}) \Psi_{-k} \Psi_k.$$

Next, define

$$\phi_n : \mathcal{W}_n \hookrightarrow \mathcal{W}_{n+1}, \quad \text{where } \forall S : \phi_n(\prod_{i \in S} \Psi_i \Omega_n) = \prod_{i \in S} \Psi_i \Omega_{n+1}.$$

This is consistent with the trivial embedding $\tau_n : \mathcal{F}_n \hookrightarrow \mathcal{F}_{n+1}$ where $\tau_n(\Psi_i) = \Psi_i$, giving us in the limit the algebra of Dirac fermion operators $\mathcal{F}$. We can show that there is a **strong** scaling limit (Definition 2.3.3), where the scaling limit space $\mathcal{V}$ can be constructed as the algebraic colimit of the sequence coming with the natural embedding maps $\rho_n : \mathcal{W}_n \hookrightarrow \mathcal{V}$. The connecting maps will turn out to be the restriction of ρ_n to energy M as it is required in Definition 2.3.3. The natural orthonormal spanning set is

$$\{\prod_{i \in S} \Psi_i \Omega | \ \forall S \subset \mathbb{Z}_{<0} + \frac{1}{2}\}$$

for $\mathcal{V}$, where $\Omega = \rho_n(\Omega_n)$ is the *vacuum* vector. We need to make sure that this is consistent with the definition of scaling limit obtained through the double colimit construction in Definition 2.3.2.

Restricting to energy at most M, one has to check that $\phi_n^M : \mathcal{W}_n^M \to \mathcal{W}_{n+1}^M$ is unitary

for large enough n. Any eigenvector $\prod_{-k\in S}\Psi_{-k}\Omega_n$ with energy

$$\frac{(2n+1)}{\pi}(\sum_{-k\in S}\sin(\frac{k\pi}{2n+1}))<M$$

has the energy $\frac{(2n+3)}{\pi}(\sum_{-k\in S}\sin(\frac{k\pi}{2n+3}))$ given by H_{n+1} also smaller than M for large enough n. Indeed, by using the Taylor expansion we obtain

$$\frac{(2n+1)}{\pi}(\sum_{-k\in S}\sin(\frac{k\pi}{2n+1}))=\sum_{-k\in S}k-\sum_{-k\in S}\frac{k^3\pi^2}{6(2n+1)^2}+\ldots=\sum_{-k\in S}k+O(\frac{1}{n}). \quad (2.4)$$

Further, by taking the limit, the above shows the energy of $\prod_{-k\in S}\Psi_{-k}\Omega$ is $\sum_{-k\in S}k$. Hence, $\widetilde{L}_0^c\xrightarrow{SL}L_0$, where

$$L_0=\sum_{k\in\mathbb{N}-\frac{1}{2}}k\Psi_{-k}\Psi_k.$$

This gives the character

$$\prod_{k=1}^{\infty}(1+q^{k-\frac{1}{2}})$$

which agrees with the character of $\chi_0+\chi_{\frac{1}{2}}$ (1.24).

Rate of convergence

Consider the natural action of $\widetilde{L}_0^c$ on $\mathcal{V}$ obtained through the embedding $\mathcal{F}_n\hookrightarrow\mathcal{F}$. We could alternatively take the "less" natural action by extending $\widetilde{L}_0^c$ by zero on the orthogonal complement of $\mathcal{W}_n$ in $\mathcal{V}$ and this will not impact the convergence rate. Either way, the restriction of $\widetilde{L}_0^c$ and L_0 to subspace with energy at most $\sqrt[3]{n}$, denoted by $\widetilde{L}_0^c|_{\sqrt[3]{n}}$ and $L_0|_{\sqrt[3]{n}}$, can be compared. To finish the proof of 1(a), one needs to show

$$\widetilde{L}_0^c|_{\sqrt[3]{n}}=L_0|_{\sqrt[3]{n}}+O(\frac{1}{n}).$$

This is a stronger result than restriction to some finite energy M. This equation demands M to be changing according to n and yet have a convergence. We shall show that $\widetilde{L}_0^c|_{\sqrt[3]{n}}$ gives the same energy as $L_0|_{\sqrt[3]{n}}$ on its eigenvectors up to an error of $O(\frac{1}{n})$. This would imply that the difference has *operator norm* at most $O(\frac{1}{n})$ as $\widetilde{L}_0^c|_{\sqrt[3]{n}}$ and $L_0|_{\sqrt[3]{n}}$ share the same eigenvectors in $\mathcal{V}$. We estimate the difference

$$|\frac{(2n+1)}{\pi}(\sum_{-k\in S}\sin(\frac{k\pi}{2n+1}))-\sum_{-k\in S}k|,$$

assuming $\sum_{-k\in S}k<\sqrt[3]{n}$. According to (2.4),

$$=|-\sum_{-k\in S}\frac{k^3\pi^2}{6(2n+1)^2}+\text{h.o.t}|\leq|\sum_{-k\in S}\frac{k^3\pi^2}{6(2n+1)^2}|+|\text{h.o.t}|.$$

In general, if the sum $\sum_k x_k = t$ of nonnegative numbers x_k is a fixed value t, then $\sum_k x_k^j \leq t^j$ with equality if and only if one of the numbers is t and the others are zero. This implies that the maximum above happens when $S=\{-\lfloor\sqrt[3]{n}\rfloor+\frac{1}{2}\}$. The h.o.t is at most $O(\frac{1}{n^2})$ and the first term is $O(\frac{1}{n})$. This finishes the proof of 1(a).

2.4.2 The higher Virasoro modes L_ms

Changing the coefficients t_j to a cos() and sin() transform of the e_js is necessary to obtain the higher Virasoro modes L_ms. We will prove the case 2(a) $(\chi_0+\chi_{\frac{1}{2}})$ in Theorem 2.4.1. The proof for the convergence rate is similar to above and we refer to [4] for more details.

The $\cos()$ **transform and** $\widetilde{L}^c_m + \widetilde{L}^c_{-m}$

The operator $L_m + L_{-m}$ that we want to obtain in the scaling limit is given by (see e.g. [33])

$$\sum_{k\geq\frac{m+1}{2},k\in\mathbb{Z}+\frac{1}{2}}(k-\frac{m}{2})\Psi_{-k+m}\Psi_k+\sum_{k\geq\frac{-m+1}{2},k\in\mathbb{Z}+\frac{1}{2}}(k+\frac{m}{2})\Psi_{-k-m}\Psi_k. \tag{2.5}$$

To understand what the observable $O = i\sum t_j(m)\psi_{j+1}\psi_j$ will be in terms of Ψ_k's, one has to use the matrix equation (2.1). Similar to how we built H' previously, we build an observable O' using Ψ_k's which has scaling limit $L_m + L_{-m}$, and satisfies $[O - O', \Psi_k] = 0$. Then, going through the usual arguments like the previous case where we showed $H' = H$, after some suitable scaling, one has $O' = O \xrightarrow{SL} L_m + L_{-m}$.

Notice that $[L_m + L_{-m}, \Psi_k]$ is the sum of exactly two Dirac operators with indices differing by m from k. $t_j(m)$ should be such that the same result for $[O, \Psi_k]$ happens, with coefficients going to $k \pm \frac{m}{2}$ in the scaling limit. Using the indices *before* the renumbering ($k \in [2n]$), a natural candidate for the coefficients would be $\cos(\frac{(k\mp\frac{m}{2})\pi}{2n+1})$. Hence, computing $[O, \Psi_k]$ using (2.1), the following must hold

$$t_j(m)\mu_{k,j+1} + t_{j-1}(m)\mu_{k,j-1} = \tag{2.6}$$

$$\cos\Big(\frac{(k+\frac{m}{2})\pi}{2n+1}\Big)\mu_{k+m,j} + \cos\Big(\frac{(k-\frac{m}{2})\pi}{2n+1}\Big)\mu_{k-m,j}.$$

From simple trigonometric identities, the right side is equal to

$$\cos\Big(\frac{(k+\frac{m}{2})\pi}{2n+1}\Big)\sin\Big(\frac{(k+m)j\pi}{2n+1}\Big) + \cos\Big(\frac{(k-\frac{m}{2})\pi}{2n+1}\Big)\sin\Big(\frac{(k-m)j\pi}{2n+1}\Big) \tag{2.7}$$

$$= \cos(\frac{m(j+\frac{1}{2})\pi}{2n+1})\sin\Big(\frac{k(j+1)\pi}{2n+1}\Big) + \cos(\frac{m(j-\frac{1}{2})\pi}{2n+1})\sin\Big(\frac{k(j-1)\pi}{2n+1}\Big),$$

which is in fact

$$\cos(\frac{m(j+\frac{1}{2})\pi}{2n+1})\mu_{k,j+1}+\cos(\frac{m(j-\frac{1}{2})\pi}{2n+1})\mu_{k,j-1}.$$

Thus $t_j(m)$ are *forced* to be $\cos(\frac{m(j+\frac{1}{2})\pi}{2n+1})$.

Remark 2.4.1 *The coefficients* $\cos(\frac{mj\pi}{2n})$*, as used in the conjecture [1, (7.5)], do not satisfy the identity (2.6). This is the important difference of our Koo-Saleur formula with the ones in [1, 57]. The implications will be discussed more in the future.*

We change O to $O/2$ to cancel the factor two present in the identity for the matrix (2.1). Although the identities above determine what O' should be, what happens at the *boundaries* when $k+m>2n$ or $k-m<1$ must be examined more carefully. In these cases, one has to consider $\sin(\frac{(k+m)j\pi}{2n+1}) = -\sin(\frac{(2(2n+1)-k-m)j\pi}{2n+1}) = -\mu_{2(2n+1)-(k+m),j}$ if $k+m>2n$ and $\sin(\frac{(k-m)j\pi}{2n+1}) = -\sin(\frac{(m-k)j\pi}{2n+1}) = -\mu_{m-k,j}$ if $k-m<1$. Therefore, O' is defined as

$$\Big(\sum_{k+m\leq 2n}\cos\Big(\frac{(k+\frac{m}{2})\pi}{2n+1}\Big)\Psi_{k+m}\Psi_k^\dagger-\sum_{k+m>2n}\cos\Big(\frac{(k+\frac{m}{2})\pi}{2n+1}\Big)\Psi_{2(2n+1)-k-m}\Psi_k^\dagger\Big) \tag{2.8}$$

$$+\Big(\sum_{k-m\geq 1}\cos\Big(\frac{(k-\frac{m}{2})\pi}{2n+1}\Big)\Psi_{k-m}\Psi_k^\dagger-\sum_{k-m<1}\cos\Big(\frac{(k-\frac{m}{2})\pi}{2n+1}\Big)\Psi_{m-k}\Psi_k^\dagger\Big).$$

Finally we define the finite version of the Virasoro modes for the cos() transform as:

$$\widetilde{L}_m^c+\widetilde{L}_{-m}^c=\frac{2n+1}{2\pi}O', \tag{2.9}$$

where $\widetilde{L}_m^c$ is the first and $\widetilde{L}_{-m}^c$ is the second parenthesis in (2.8). Using the Taylor expansion, we can prove $\widetilde{L}_{\pm m}^c \xrightarrow{SL} L_{\pm m}$, and through a similar argument to $H=H'$, we can prove $O=O'$. We refer to [4] for further details.

The sin() **transform and** $i(\widetilde{L}^s_m - \widetilde{L}^s_{-m})$

To recover $i(L_m - L_{-m})$, the exact same procedure applies, this time to

$$O = -i\sum_{j=1}^{2n-2} t_j(n)[e_j, e_{j+1}] \propto i\sum_j t_j(n)\psi_j\psi_{j+2}.$$

The corresponding matrix for $[O, \Psi]$ where $\Psi = \sum i^b\mu_b\psi_b$ can be found as follows:

$$[O, \Psi] = \Psi' = i(\sum_b i^b\mu'_b\psi_b)$$

where we have

$$\mu'_b = 2(t_b(m)\mu_{b+2} - t_{b-2}(m)\mu_{b-2})$$

except at the boundaries where the formula will be different. In the case of the nonperiodic chain $(\frac{1}{2}, \frac{1}{2})$, this can be turned into the corresponding matrix

$$\begin{pmatrix} 0 & 0 & t_1(m) & & & 0 \\ 0 & 0 & 0 & t_2(m) & & \\ -t_1(m) & 0 & 0 & & \ddots & \\ & -t_2(m) & & & & \\ & & & & & \\ & & \ddots & 0 & 0 & t_{2n-2}(m) \\ & & & 0 & 0 & 0 \\ 0 & & & -t_{2n-2}(m) & 0 & 0 \end{pmatrix}. \tag{2.10}$$

Similar to the previous procedure, we need the $t_j(m)$ to satisfy the following identity

$$t_j(m)\mu_{k,j+2} - t_{j-2}(m)\mu_{k,j-2} =$$

$$\sin\Big(\frac{(2k+m)\pi}{2n+1}\Big)\sin\Big(\frac{(k+m)j\pi}{2n+1}\Big)-\sin\Big(\frac{(2k-m)\pi}{2n+1}\Big)\sin\Big(\frac{(k-m)j\pi}{2n+1}\Big).$$

The above equals

$$\sin\Big(\frac{m(j+1)\pi}{2n+1}\Big)\sin\Big(\frac{k(j+2)\pi}{2n+1}\Big)-\sin\Big(\frac{m(j-1)\pi}{2n+1}\Big)\sin\Big(\frac{k(j-2)\pi}{2n+1}\Big), \tag{2.11}$$

which suggests the right candidate for $t_j(m)$ is $\sin\Big(\frac{m(j+1)\pi}{2n+1}\Big)$. On the other hand, the candidate for our observable is $-iO' =$

$$\Big(\sum_{k+m\leq 2n}\sin\Big(\frac{(2k+m)\pi}{2n+1}\Big)\Psi_{k+m}\Psi_k^\dagger-\sum_{k+m>2n}\sin\Big(\frac{(2k+m)\pi}{2n+1}\Big)\Psi_{2(2n+1)-k-m}\Psi_k^\dagger\Big) \tag{2.12}$$

$$-\Big(\sum_{k-m\geq 1}\sin\Big(\frac{(2k-m)\pi}{2n+1}\Big)\Psi_{k-m}\Psi_k^\dagger-\sum_{k-m<1}\sin\Big(\frac{(2k-m)\pi}{2n+1}\Big)\Psi_{m-k}\Psi_k^\dagger\Big).$$

Let us define

$$i(\widetilde{L}^s_m-\widetilde{L}^s_{-m})=\frac{2n+1}{2\pi}O', \tag{2.13}$$

where $\widetilde{L}^s_m$ is the first and $\widetilde{L}^s_{-m}$ corresponds to the second parenthesis in (2.12). Using the Taylor expansion, we conclude $\widetilde{L}^s_{\pm m}\xrightarrow{SL}L_{\pm m}$. Finally, after some suitable scaling, we find $O'=O\xrightarrow{SL}i(L_m-L_{-m})$.

2.4.3 Periodic case

If n is even, the scaling limit is the diagonal full CFT $\chi_0\overline{\chi}_0 + \chi_{\frac{1}{2}}\overline{\chi}_{\frac{1}{2}} + \chi_{\frac{1}{16}}\overline{\chi}_{\frac{1}{16}}$ and if n is odd, it is $\chi_0\overline{\chi}_{\frac{1}{2}} + \chi_{\frac{1}{2}}\overline{\chi}_0 + \chi_{\frac{1}{16}}\overline{\chi}_{\frac{1}{16}}$. Similar to how (2.1) was derived, the matrix is

$$\begin{pmatrix} \mu_1' \\ \mu_2' \\ \vdots \\ \\ \mu_{2n}' \end{pmatrix} = 2 \begin{pmatrix} 0 & t_1 & 0 & \dots & (-1)^{F+(n+1)}t_{2n} \\ t_1 & 0 & t_2 & & \\ 0 & t_2 & 0 & & \\ \vdots & & & & t_{2n-1} \\ (-1)^{F+(n+1)}t_{2n} & & & t_{2n-1} & 0 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \\ \mu_{2n} \end{pmatrix}, \tag{2.14}$$

where by $(-1)^F$ in the entries, the **sign** of the operator $(-1)^F$ when restricted to ± 1 sector is considered. We specialize to $t_j = 1$ for all j. There are four cases based on the parity of n and the ± 1 sector. For n even, the scaling limit yields $\chi_0\overline{\chi}_0 + \chi_{\frac{1}{2}}\overline{\chi}_{\frac{1}{2}}$ for the $+1$ sector and $\chi_{\frac{1}{16}}\overline{\chi}_{\frac{1}{16}}$ for the -1 sector. The analysis is a more involved version of previous nonperiodic cases [4].

On the other hand, if n is odd, the $+1$ sector gives $\chi_{\frac{1}{16}}\overline{\chi}_{\frac{1}{16}}$ and the -1 sector gives $\chi_0\overline{\chi}_{\frac{1}{2}} + \chi_{\frac{1}{2}}\overline{\chi}_0$. The reason the -1 sector is not diagonal is the fact that odd (even) number of left-moving operators have to act with even (odd) number of right-moving operators to take the vacuum living in the $+1$ sector to -1 sector, where the lowest energy is 1. This is not a full CFT as the character is not modular invariant.

2.4.4 Comparison of Koo-Saleur formulae

The conjecture [1, (7.5)] asserts that $l_m + l_{-m} \xrightarrow{SL} L_m + L_{-m}$ where $l_m + l_{-m}$ is some scaling of $\sum_j \cos(\frac{mj\pi}{2n})e_j$. The difference with Conjecture 2.6.4 is the factor j instead of $(j+\frac{1}{2})$ and the denominator $2n$ instead of $2n+1$. Recall that the point $\frac{(j+\frac{1}{2})\pi}{2n+1}$ is interpreted as the "center" of the action of e_j in the half-circle in Figure 1.17. Also, the identity (2.6)

forced the coefficient $\cos(\frac{m(j+\frac{1}{2})\pi}{2n+1})$, therefore any other coefficient (including $\cos(\frac{mj\pi}{2n})$) should have some *undesirable effect*, although these effects could vanish in the scaling limit. We can compute the coefficient of a term $\Psi_x\Psi_{-y}$ in $O'_s = i\sum_{j=1}^{2n-1}\cos(\frac{mj\pi}{2n})\psi_{j+1}\psi_j$ for any $1 \leq x, y \leq n$:

$$\frac{1}{N_xN_y}\sum_{j=1}^{2n-1}\cos(\frac{mj\pi}{2n})\Big(\sin(\frac{jx\pi}{2n+1})\sin(\frac{(j+1)y\pi}{2n+1}) + \sin(\frac{(j+1)x\pi}{2n+1})\sin(\frac{jy\pi}{2n+1})\Big).$$

This is coefficient is zero if and only if $m + x + y \not\equiv 0 \pmod 2$. As an example, for $m = 9, x = 14, n = 52, y = 49 = n - 3$, the above gives approximately $-0.0256625 \neq 0$. Of course, these terms should vanish at the scaling limit, along with all terms with nonzero coefficient giving energy shift other than m. A numerical simulation shows that happening but with a slower rate (as in [1, table 19]). Hence, the rate of convergence can be a reason to consider the operators in Conjecture 2.6.4 for obtaining the higher Virasoro modes.

More important is the mixing of low-high energy that suggests the conjecture [1, (7.5)] does not provide the right candidates if a strong SL-algebra is desired. These terms could make even the convergence of simple products such as the convergence of commutators to the commutators of scaling limit impossible. Similar results hold for $i(l_m - l_{-m})$, which is some scaling of $i\sum_j \sin(\frac{mj\pi}{2n})[e_j, e_{j+1}]$.

2.4.5 Notations for future sections

We set some notations for studying scaling limit of algebras. For the Hamiltonians, we recall the notation $\widetilde{L}_0^c$ in (2.3) as a *scaling* of H_n which has scaling limit L_0. The notations and scalings for the case 1(a) in Theorem 2.4.1, i.e. $\chi_0 + \chi_{\frac{1}{2}}$, are

$$\widetilde{L}_0^c = \alpha_n^c H_n + \beta_{n,0}^c \mathbf{1} \xrightarrow{SL} L_0,$$

where $\alpha_n^c = \frac{(2n+1)\sqrt{2}}{8\pi}$ and $\beta_{n,0}^c \in \mathbb{R}$. For the higher Virasoro generators in Theorem 2.4.1, let $O_{n,ms}^c, O_{n,m}^s$ denote the first and second operator (superscript c, s refer to $\cos, \sin$). Then similar notations are defined:

$$\frac{\widetilde{L}_m^c + \widetilde{L}_{-m}^c}{2} = \alpha_n^c O_{n,m}^c + \beta_{n,m}^c \mathbf{1} \xrightarrow{SL} \frac{L_m + L_{-m}}{2},$$

$$\frac{i(\widetilde{L}_m^s - \widetilde{L}_{-m}^s)}{2} = \alpha_n^s O_{n,m}^s + \beta_{n,m}^s \mathbf{1} \xrightarrow{SL} \frac{i(L_m - L_{-m})}{2}.$$

Here, $\alpha_n^s = \frac{(n+\frac{1}{2})(\sqrt{2})^2}{8\pi} = \alpha_n^c$ and $\beta_{n,m}^c, \beta_{n,m}^s \in \mathbb{R}$. Similarly for the full CFT, $\widetilde{\mathbb{L}}_m^c + \widetilde{\mathbb{L}}_{-m}^c$ and $i(\widetilde{\mathbb{L}}_m^s - \widetilde{\mathbb{L}}_{-m}^s)$ can be defined using some scaling of:

$$-\sum_{j=1}^{2n} \cos(\frac{m(j+\frac{1}{2})2\pi}{2n}) e_j, \tag{2.15}$$

$$i\sum_{j=1}^{2n} \sin(\frac{m(j+1)2\pi}{2n}) [e_j, e_{j+1}] \tag{2.16}$$

Recall the splitting $\widetilde{L}_{\pm m}^c$ and $\widetilde{L}_{\pm m}^s$ is such that each have scaling limit $L_{\pm m}$. In our notation $\widetilde{L}_m^c$ and $\widetilde{L}_m^s$, there is no explicit mention of n, though they depend on the size of the chain that should be clear from the context.

One can easily recover the scaling factors by following the proof in 2.4.2. We will only need the rate of growth of these scaling factors which will be at most $O(n^2)$ and α_n^s, α_n^c do not depend on m while $\beta_{n,m}^c$ and $\beta_{n,m}^c$ do.

2.5 Scaling limit algebras in $\overline{\mathcal{A}}$

Our goal is to obtain the observables of each of these three types and prove they form an SL-algebra:

(a) smeared fields $Y(a, f)$,

(b) LCN observables $O \in \mathcal{A}(I)$,

(c) point-like fields $Y(a, z)$.

It is not hard to show that they live in $\mathcal{A}$ as a vector space, i.e. all in a single framework. This means they are all physical as they describe some computable convergent sequence. Further, although each of the three sets are believed to store all the information of the CFT, they have to be in our set of observables simultaneously.

We will first obtain (a), and use it to recover the observables (b), and lastly, some comments will be made on (c). Only the case $\mathcal{V} = \chi_0 + \chi_{\frac{1}{2}}$ will be analyzed, and all theorems can be similarly stated for other boundary conditions.

2.5.1 Wightman's observables

We will try to identify when sequence of space-local hermitian observables linearly generated by $e_j, i[e_j, e_{j+1}]$ are energy-local.

Theorem 2.5.1 ([4], Thm. 4.1) *$(O_n)_n$ is a $\Lambda-$energy-local observable made from a linear combination of e_j and $[e_j, e_{j+1}]$s and the identity*

$$\eta_n \mathbf{1} + \sum_j t_j^{(n)} e_j \quad \& \quad \eta_n \mathbf{1} + i \sum_j t_j^{(n)} [e_j, e_{j+1}]$$

if and only if it is of the form

$$O_n = \gamma_n \mathbf{1} + a_0^{(n)} \widetilde{L}_0 + \sum_{m=1}^{\Lambda} \left(a_m^{(n)} \widetilde{L}_m^c + i b_m^{(n)} \widetilde{L}_m^s \right) + \sum_{m=1}^{\Lambda} \left(a_m^{(n)} \widetilde{L}_{-m}^c - i b_m^{(n)} \widetilde{L}_{-m}^s \right),$$

where $a_m^{(n)}, b_m^{(n)} \in \mathbb{R}$.

Proof: [Proof sketch] A trigonometric interpolation of the $t_j^{(n)}$s with $\cos(\frac{m(j+\frac{1}{2})\pi}{2n+1})$ or $\sin(\frac{m(j+1)\pi}{2n+1})$ can be performed. For the observable $O_n = \eta_n \mathbf{1} + \sum_{j=1}^{2n-1} t_j^{(n)} e_j$, such an

interpolation gives

$$t_j^{(n)} = \alpha_n^c \sum_{m=0}^{2n-2} a_m^{(n)} \cos(\frac{m(j+\frac{1}{2})\pi}{2n+1})$$

$$\implies O_n = \gamma_n \mathbf{1} + a_0^{(n)} \widetilde{L}_0^c + \sum_{m=1}^{2n-2} a_m^{(n)} \frac{\widetilde{L}_m^c + \widetilde{L}_{-m}^c}{2},$$

where γ_n is some multiple of identity. Next, suppose O_n does not shift the energy level in $\mathcal{V}$ more than some given Λ. Then since $\widetilde{L}_m^c$ shifts the energy by at most $|m|$ (see (2.8)), we can inductively prove that the higher modes coefficients $a_{2n-2}^{(n)}, a_{2n-1}^{(n)}, \ldots$ are all zero up to $a_\Lambda^{(n)}$. A similar argument applies to $\eta_n \mathbf{1} + i \sum_j t_j^{(n)} [e_j, e_{j+1}]$. Adding up cases $(\widetilde{L}_m^c + \widetilde{L}_{-m}^c)$ and $i(\widetilde{L}_m^s - \widetilde{L}_{-m}^s)$ proves the statement. ■

Remark 2.5.1 *Dealing with $a_m \widetilde{L}_m^c + i b_m \widetilde{L}_m^s$ every time can be cumbersome. An operator $\widetilde{L}_m$ is desired which has scaling limit L_m so that expressions like $\sum \hat{f}_m \widetilde{L}_m \xrightarrow{SL} \sum \hat{f}_m L_m$ can be used where $\hat{f}_m = a_m + i b_m \in \mathbb{C}$. The choice below resolves this issue*

$$\widetilde{L}_m := \Big(\frac{\widetilde{L}_m^c + \widetilde{L}_m^s}{2} + \frac{\widetilde{L}_{-m}^c - \widetilde{L}_{-m}^s}{2} \Big) \quad \forall m \neq 0, \quad \widetilde{L}_0 = \widetilde{L}_0^c.$$

The above is a definition for an operator satisfying $\widetilde{L}_m \xrightarrow{SL} L_m$ and inheriting the same rate of convergence from $\widetilde{L}_m^c$ and $\widetilde{L}_m^s$. Indeed, by Theorem 2.4.1, restricting $\frac{\widetilde{L}_{-m}^c - \widetilde{L}_{-m}^s}{2}$ to subspace with energy at most $\sqrt[4]{n}$, will give an operator with a norm at most $O(\frac{1}{n})$ which becomes part of the error of the approximation of L_m by $\widetilde{L}_m$. The rest of the operator acting on energy higher than $\sqrt[4]{n}$ will join that of $\frac{\widetilde{L}_m^c + \widetilde{L}_m^s}{2}$.

Convention. Given a scalar $E \geq 0$, let P^E denote the projection to subspace with energy at most E. Then define $O|_E = O P^E$ and $O|_{>E} := O(\mathbf{1} - P^E)$.

Notation. From now on, n will be used for the sequence index which will be related

to the size of the chain ($2n+1$ for boundary condition $(\frac{1}{2},\frac{1}{2})$). For example

$$\widetilde{L}_m = L_m|_{\sqrt[4]{n}} + O(\frac{1}{m}) + R_{m,n},$$

where $R_{m,n} = \widetilde{L}_m|_{>\sqrt[4]{n}}$. We can now state our first result for the scaling limit of observables.

Theorem 2.5.2 ([4], Thm. 4.2) *The set of energy-local observables which have an associated hermitian energy-local sequence in the real vector space spanned by $e_j, i[e_j, e_{j+1}]$ and the identity, form the real vector space*

$$\{L(f) + \gamma\mathbf{1} \mid f \in C^\infty_{\mathbb{R}}(S^1) \text{ has finite Fourier series, } \gamma \in \mathbb{R}\}.$$

Proof: Assume a sequence of $\Lambda-$energy-local operators

$$O_n = \gamma_n\mathbf{1} + \sum_{j=-\Lambda}^{\Lambda} \hat{f}_j^n \widetilde{L}_j,$$

where $\hat{f}^n_{-j} = \overline{\hat{f}^n_j}$ and $O_n \xrightarrow{SL} O$. To show that $O = L(f) + \gamma\mathbf{1}$ for some function f with finite Fourier series, restrict O_n to some energy $M > 2\Lambda$,

$$O_n|_M = \gamma_n\mathbf{1} + \sum_{j=-\Lambda}^{\Lambda} \hat{f}_j^n \widetilde{L}_j|_M.$$

According to the properties of $\widetilde{L}_j$s, for large enough n,

$$O_n|_M = \gamma_n\mathbf{1} + \sum_{j=-\Lambda}^{\Lambda} \hat{f}_j^n L_j|_M + \hat{f}_j^n O(\frac{1}{n}).$$

Since $O_n|_M$ has a limit in the operator norm to $O|_M$, $\hat{f}^n_j$s must have a limit. To prove

that, we compute

$$(L_{-\Lambda}|_M\Omega, O_n|_M\Omega) = f_{-\Lambda}^n \ ||L_{-\Lambda}\Omega|| + (L_{-\Lambda}\Omega, (\sum_j \hat{f}_j^n O(\frac{1}{n}))\Omega) \to (L_{-\Lambda}|_M\Omega, O\Omega),$$

where $|_M$ is dropped as it is no longer needed. Notice all the errors $O(\frac{1}{n})$ corresponding to $\widetilde{L}_j$ give at most $|j|$ energy shift. This means the only nonzero term is

$$f_{-\Lambda}^n \ ||L_{-\Lambda}\Omega|| + (L_{-\Lambda}\Omega, (f_\Lambda^n O(\frac{1}{n}) + f_{-\Lambda}^n O(\frac{1}{n}))\Omega).$$

$f_{-\Lambda}^n \ ||L_{-\Lambda}\Omega||$ can be exactly computed and is of order $f_{-\Lambda}^n \Lambda^{\frac{3}{2}}$. The rest can have norm at most $O(\frac{1}{n})|f_{-\Lambda}^n|$ as $f_\Lambda^n = \overline{f_{-\Lambda}^n}$. It follows that $f_{-\Lambda}^n$ must have a limit, say $f_{-\Lambda}$.

Next step is to subtract $f_\Lambda^n \widetilde{L}_\Lambda + \overline{f_\Lambda^n}\widetilde{L}_{-\Lambda}$ from O_n and repeat the procedure. For the special case of $j = 0$, $\gamma_n \mathbf{1} + f_0^n \widetilde{L}_0$ can be seen to give the same conclusion. Denoting $\lim_{n\to\infty} \hat{f}_j^n = \hat{f}_j, \lim_{n\to\infty} \gamma_n = \gamma$, we have $O = \gamma\mathbf{1} + \sum_{j=-\Lambda}^{\Lambda} \hat{f}_j L_j$. ■

Remark 2.5.2 *By Theorem 2.3.1, we have a strong SL-algebra. Note the operators* $O_n = \gamma_n \mathbf{1} + \sum_{j=-\Lambda}^{\Lambda} \hat{f}_j^n \widetilde{L}_j$ *used for obtaining* $L(f)$ *are also space-local.*

For UMMs, higher level ACs are conjectured to give the same results as in Theorem 2.4.1, implying the above theorem for UMMs. A relaxed version of 2.4.1 for UMMs would still ensure the next results in this section hold for all UMMs:

Conjecture 2.5.3 *For any UMM VOA* $\mathcal{V} = \mathcal{V}_{c,0}$ *and a sector* $\mathcal{V}_{c,h}$*, there is a sequence of (AC or lattice-based) quantum theories with* ***strong*** *scaling limit* $(\mathcal{V}_{c,h}, L_0)$ *such that for each* L_m*, we have a sequence* $\widetilde{L}_m \in \mathcal{A}_n$ *with the following properties:*

- *It is a space-local observable with hermitian operators* $a\widetilde{L}_m + \overline{a}\widetilde{L}_{-m} \in \mathcal{A}_n^H, \forall a \in \mathbb{C}$.
- *It shifts the energy no more than* $|m|$.

- *Restricted to energy at most n^{d_ω} it has an $O(\frac{1}{n^{g_\omega}})$ approximation by $L_m|_{n^{d_\omega}}$:*

$$\widetilde{L}_m = L_m|_{n^{d_\omega}} + O(\frac{1}{n^{g_\omega}}) + R_{m,n},$$

  where d_ω, g_ω are positive constants and $R_{m,n} = \widetilde{L}_m|_{>n^{d_\omega}}$.

- *$||\widetilde{L}_m||$ is bounded by $O(n^{e_\omega})$ for some constant e_ω.*

Remark 2.5.3 *Two important observations about the first and third item above:*

- *For space-locality to be meaningful, we need an AC or lattice-based model as Definition 2.3.10 of space-locality applies to these contexts.*

- *Note that $R_{m,n} = \widetilde{L}_m(\mathbf{1} - P^{n^{d_\omega}})$ and thus the norm of the remainder is also bounded by $O(n^{e_\omega})$.*

Remark 2.5.4 *The second and third item above have a meaning after the push-forward of the map $\widetilde{L}_m$ acting on $\mathcal{V}_{c,h}$ is assumed. This is done by the embedding $\rho_n : \mathcal{W}_n \hookrightarrow \mathcal{V}_{c,h}$ from the strong scaling limit; the map $\rho_n \widetilde{L}_m(\rho_n)^{-1}$ acts on the copy of $\mathcal{W}_n$ inside $\mathcal{V}_{c,h}$ and is extended by zero on the orthogonal complement.* ***This push-forward will be implicitly assumed whenever it is necessary****. Note that in an algebraic scaling limit (like for the Ising model), there is a copy of each $O \in \mathcal{A}_n$ inside $\mathcal{A}$, and it is equal to our push-forward copy restricted to $\rho_n(\mathcal{W}_n)$, which is what will be important in our analysis.*

Remark 2.5.5 *The last item applies to the Ising chain as $\widetilde{L}^c_m$ and $\widetilde{L}^s_m$ are a sum of $2n$ terms of e_js which have constant norm. Taking the norm of the scaling factors α^c_n and $\beta^c_{n,m}, \beta^s_{n,m}$ into account, one has $||\widetilde{L}_m|| \leq O(n^2)$.*

Remark 2.5.6 *Assuming the above conjecture, except for Theorem 2.5.1, 2.5.2 and 2.5.15, all other theorems in sections 2.5.1 and 2.5.3 hold for all UMMs. In fact, even Theorem 2.5.2 would be true for all UMMs if the statement is changed to: the energy-local scaling limit* ***contains*** $\{L(f)|\ f \in C^\infty_{\mathbb{R}}(S^1)$ *has finite Fourier series*$\}$*. For all theorems in section 2.5.2, the stronger Conjecture 2.6.4 on how to recover the higher Virasoro modes for UMMs has to be assumed. Therefore, even though theorems below will be proved using the Ising AC, by replacing some of the powers by appropriate constants (d_ω, etc), the results hold for UMMs assuming the above conjecture.*

Remark 2.5.7 *It is conjectured that all* ***nice*** *VOAs as described in section 1.3.2 (unitary, C_2-co-finite, rational, etc) satisfy energy boundedness [25, Conjecture 8.18]. A generalization of the Conjecture 2.5.3 to all CFTs which satisfy energy boundedness is possible. Sequences in the same fashion of the Virasoro modes have to exist for all elements inside a minimal quasi-primary hermitian field generator set of the VOA. Then, all theorems in sections 2.5.1 and 2.5.3, with the exception of Theorem 2.5.1, 2.5.2 and 2.5.15, can be recovered. In UMMs, the generator is only ω and in WZW models, the currents corresponding to the Lie algebra $\mathfrak{g}$ (see [62] for a numerical demonstration on obtaining the $\mathfrak{g}_k$ currents in the scaling limit and [53] for $W-$algebra currents).*

Notation. Set $L(f)_{\leq m} = \sum_{|j|\leq m} \hat{f}_j L_j$ and similarly for $\widetilde{L}(f)$. Define $L(f)_{>m} = L(f) - L(f)_{\leq m}$ and similarly $\widetilde{L}(f)_{>m}$. Also set

$$||f||_s^{\leq E} = \sum_{|i|\leq E} |\hat{f}_i|(|i|+1)^s,$$

and

$$|f|^{\leq m} := \sum_{|i|\leq m} |\hat{f}_i|, \quad |f| := \sum_i |\hat{f}_i|.$$

We wish to show that the choice of the "natural" sequence corresponding to $L(f)$ gives a strong SL-algebra. Some lemmas are needed.

Lemma 2.5.4 ([4], Lem. 4.4) *We have*

$$\widetilde{L}(f) = \sum_{j=-\infty}^{\infty} \hat{f}_j \widetilde{L}_j \in \mathcal{A}_n^H, \quad \textit{for all } f \in C_{\mathbb{R}}^{\infty}(S^1)$$

Proof: Note that $\hat{f}_j$s are rapidly decreasing. Also, from Remark 2.5.5,

$$||\widetilde{L}_j|| \leq O(n^2). \tag{2.17}$$

The estimation does not depend on j. This gives an absolute convergence to an operator with norm bounded by $|f|O(n^2)$. On the other hand, for each j, we have $\hat{f}_j\widetilde{L}_j + \hat{f}_{-j}\widetilde{L}_{-j} \in \mathcal{A}_n^H$ implying $\widetilde{L}(f) \in \mathcal{A}_n^H$. ■

Lemma 2.5.5 ([4], Lem. 4. 5) $\widetilde{L}(f) \xrightarrow{SL} L(f)$.

Proof: The result and techniques shown here on the convergence behavior of $\widetilde{L}(f)$ will be useful in the next theorems. Take any $k \in \mathbb{N}$ and note that $\exists N_{f,(10k)^3+1} \in \mathbb{N}$: $|\hat{f}_j| < \frac{1}{j^{(10k)^3+1}}, \forall j > N_{f,(10k)^3+1}$. The coefficient 10 is just for convenience and any high enough coefficient will be sufficient. For n large enough such that $\sqrt[10k]{n} > N_{f,(10k)^3+1}$,

$$||\widetilde{L}(f)_{>\sqrt[10k]{n}}|| = ||\sum_{|j|>\sqrt[10k]{n}} \hat{f}_j\widetilde{L}_j|| \leq O(n^2) \sum_{|j|>\sqrt[10k]{n}} |\hat{f}_j| \tag{2.18}$$

$$\leq O(n^2) \int_{\sqrt[10k]{n}}^{\infty} \frac{1}{x^{(10k)^3+1}} \mathrm{d}x < O(n^2) \frac{(10k)^3+1}{n^{(10k)^2}} = O(n^{-(10k)^2+2}),$$

where (2.17) is used in the first inequality. The next step is an estimate for $L(f)_{>\sqrt[10k]{n}}$

via energy bounds (1.39):

$$||f||^{>\sqrt[10k]{n}}_{\frac{3}{2}} < 2\sum_{j>\sqrt[10k]{n}}^{\infty}\frac{(j+1)^{\frac{3}{2}}}{j^{(10k)^3+1}} < \int_{\sqrt[10k]{n}}^{\infty}\frac{1}{x^{(10k)^3-10k+1}} = O(n^{-(10k)^2+1}),$$

therefore

$$||L(f)_{>\sqrt[10k]{n}}v|| < O(n^{-(10k)^2+1})||(L_0+\mathbf{1})v||. \tag{2.19}$$

Next, given a vector $v \in \mathcal{V}$ and the embedding $\mathcal{W}_n \hookrightarrow \mathcal{V}$,

$$(\widetilde{L}(f)-L(f))v = (\widetilde{L}(f)_{\leq\sqrt[10k]{n}} - L(f)_{\leq\sqrt[10k]{n}})v + (\widetilde{L}(f)_{>\sqrt[10k]{n}} - L(f)_{>\sqrt[10k]{n}})v$$

The two estimations above imply that the second part vanishes. For the first part,

$$\widetilde{L}(f)_{\leq\sqrt[10k]{n}} = L(f)_{\leq\sqrt[10k]{n}}|_{\sqrt[4]{n}} + O\Big(\frac{|f|_{j\leq\sqrt[10k]{n}}}{n}\Big) + R(f)_{k,n}, \tag{2.20}$$

where $R(f)_{k,n} = \widetilde{L}(f)_{\leq\sqrt[10k]{n}}|_{>\sqrt[4]{n}}$. Since $v \in \mathcal{V}^M$ for some large enough M, for large enough n with $\sqrt[4]{n} > M$, $R(f)_{k,n}v = 0$ and $L(f)_{\leq\sqrt[10k]{n}}|_{\sqrt[4]{n}}v = L(f)_{\leq\sqrt[10k]{n}}v$. This implies $||(\widetilde{L}(f)-L(f))v|| \to 0$, which is stronger than $\widetilde{L}(f) \xrightarrow{SL} L(f)$. ■

Remark 2.5.8 *Note that we proved for every $v \in \mathcal{V}$, for large enough n, there exists $v_n = \rho_n^{-1}(v) \in \mathcal{W}_n$, such that $\rho_n(\widetilde{L}(f)v_n) \to L(f)v$. This stronger result will be useful in the next section. Furthermore, even though we assumed $f \in C^\infty_{\mathbb{R}}(S^1)$, it is clear that one can generalize to $C^\infty_{\mathbb{C}}(S^1)$.*

Theorem 2.5.6 ([4], Thm. 4.6) *$\{L(f) \mid f \in C^\infty_{\mathbb{C}}(S^1)\}$ generates a strong SL-algebra with corresponding sequence $\widetilde{L}(f)$ to each $L(f)$.*

Proof: [Proof sketch] Let $\Omega_n = (\rho_n)^{-1}\Omega$. We need to show

$$(\Omega_n, \prod_{j=1}^{k} \widetilde{L}(f^{(j)})\Omega_n) \to (\Omega, \prod_{j=1}^{k} L(f^{(j)})\Omega).$$

Proving the above for any two vectors $u, v \in \mathcal{V}$ instead of Ω will be similar.

We can prove it by using triangle inequality after estimating the *intermediate* terms

$$|(\Omega, \prod_{j=1}^{t-1} L(f^{(j)})(L(f^{(t)}) - \widetilde{L}(f^{(t)})) \prod_{j=t+1}^{k} \widetilde{L}(f^{(j)})\Omega)|, \quad 1 \leq t \leq k,$$

where the embedding ρ_n is used implicitly. For each $1 \leq j \leq t$, write $\widetilde{L}(f^{(j)}) = \widetilde{L}(f^{(j)})_{\leq \sqrt[10k]{n}} + \widetilde{L}(f^{(j)})_{> \sqrt[10k]{n}}$. Denote $y_t = \prod_{j=t+1}^{k} \widetilde{L}(f^{(j)})\Omega$ and let $y_t = y_t^1 + y_t^2$, where

$$y_t^1 = \prod_{j=t+1}^{k} \widetilde{L}(f^{(j)})_{\leq \sqrt[10k]{n}}\, \Omega, \quad y_t^2 := y_t - y_t^1.$$

Note y_t^1 lives inside $\mathcal{V}^{(k-t)\sqrt[10k]{n}} \subset \mathcal{V}^{k\sqrt[10k]{n}}$ of vectors with energies at most $k\sqrt[10k]{n}$. This decomposition of y_t is made in order to separate the lower energy component of the vector with high norm from the rest (y_t^2). Using equations (2.18) and (2.20), we get

$$||\widetilde{L}(f^{(j)})_{\leq \sqrt[10k]{n}}|| \leq O(\sqrt[10k]{n}).$$

Recall $||\widetilde{L}(f^{(j)})_{> \sqrt[10k]{n}}|| \leq O(n^{-(10k)^2+2})$ and $||\widetilde{L}(f^{(j)})|| \leq O(n^2)$. Thus, we can establish the following upper bounds for $||y_t^1||$ and $||y_t^2||$:

$$||y_t^1|| \leq O((\sqrt[10k]{n})^{k-t}) < O(\sqrt[10]{n}), \tag{2.21}$$

$$||y_t^2|| < (2^{k-t}-1)O(n^{-(10k)^2+2})O((n^2)^{k-t})) \leq O(n^{-(10k)^2+2k+2}), \tag{2.22}$$

which imply

$$||y_t|| < 2||y_t^1|| < O(\sqrt[10]{n}). \tag{2.23}$$

Let $x_t := \Omega^\dagger \prod_{j=1}^{t-1} L(f^{(j)}), p := \max_{t=1,\ldots,k} ||x_t||$, and $q := \max_{t=1,\ldots,k} ||(L_0+\mathbf{1})x_t^\dagger||$. It can be shown that $p \leq r||(L_0^k+\mathbf{1})\Omega||$, and that the parameters r, p, q do not depend on n ([63, Lemma 3.2.1]). Our task is to approximate

$$|x_t(\widetilde{L}(f^{(t)}) - L(f^{(t)}))(y_t^1 + y_t^2)|.$$

by using the same decomposition of $(\widetilde{L}(f^{(t)}) - L(f^{(t)}))$ in Lemma 2.5.5:

$$(\widetilde{L}(f^{(t)})_{\leq \sqrt[10k]{n}} - L(f^{(t)})_{\leq \sqrt[10k]{n}}) + (\widetilde{L}(f^{(t)})_{> \sqrt[10k]{n}} - L(f^{(t)})_{> \sqrt[10k]{n}}).$$

For the second part, using the estimates (2.21-2.23) for $||y_t||$, (2.18) for $||\widetilde{L}(f^{(t)})_{> \sqrt[10k]{n}}||$, and finally (2.19), we can show $|x_t(\widetilde{L}(f^{(t)})_{> \sqrt[10k]{n}} - L(f^{(t)})_{> \sqrt[10k]{n}})y_t| \xrightarrow{n\to\infty} 0$. For the first part, using the approximation of $\widetilde{L}$s for energies up to $k\sqrt[10k]{n}$,

$$\widetilde{L}(f^{(t)})_{\leq \sqrt[10k]{n}} - L(f^{(t)})_{\leq \sqrt[10k]{n}} = -L(f^{(t)})_{\leq \sqrt[10k]{n}}\Big|_{>k\sqrt[10k]{n}} + O\Big(\frac{|f^{(t)}|_{\leq \sqrt[10k]{n}}}{n}\Big) + R(f^{(t)})_{k,n}. \tag{2.24}$$

Each of the three terms above leads to a vanishing term. For example, the second term, which is the only term where our approximation gets somewhat tight, when acting on y_t, the estimation (2.21) gives a bound of $O(\frac{\sqrt[10]{n}}{n})$ on the result which goes to zero. The relatively small bound on $||y_t^2||$ shows other terms vanish as well. ■

$\mathcal{A}^H$ contains more than just the strong SL-algebra above when $f \in C^\infty_{\mathbb{R}}(S^1)$:

Theorem 2.5.7 ([4], Thm. 4.7) *We have* $\{L(f) \mid ||f||_{\frac{3}{2}} < \infty\} \subset \mathcal{A}^H$ *which contains* $\{L(f) | f \in C^\infty_{\mathbb{R}}(S^1)\}$ *as a maximal strong SL-algebra.*

Proof: [Proof sketch] For the maximality part, one can estimate the norm of $L_0^k L(f)\Omega$ for any $k \in \mathbb{N}$, and conclude that the Fourier series of f must be rapidly decreasing, and $L(f)$ hermitian, hence $f \in C^\infty_{\mathbb{R}}(S^1)$. To show $L(f) \in \mathcal{A}^H$ for $||f||_{\frac{3}{2}} < \infty$, take $O_n = \widetilde{L}(f)_{\leq \log(n)} \xrightarrow{SL} L(f)$. ■

We try to generalize Theorem 2.5.6 to all fields. We recall three facts ([25])

- In a UMM, the descendants of ω span the VOA.
- Due to the Virasoro algebra identities, all descendants of ω can be obtained only by applying operators $L_n (n \geq -2)$.
- In a UMM, all fields are energy bounded:

$$||Y(a, f)v|| \leq C_a ||f||_{r_a} ||(L_0 + \mathbf{1})^{s_a} v||.$$

Theorem 2.5.8 ([4], Thm. 4.8) $\{Y(a, f)|\ a \in \mathcal{V}, f \in C^\infty_{\mathbb{C}}(S^1)\} \subset \mathcal{A}$ *generates an SL-algebra.*

Remark 2.5.9 *It can be shown that* $Y(a, f)$ *can be obtained by sequences that are local but in the quantum computation sense of locality (QC-locality) (see Remark 2.3.4).*

Proof: [Proof sketch] The proof works by induction using the Borcherds identity as a recursive identity to obtain the descendant fields from the Virasoro field. Choose a basis with descendants of ω. For the field $L_{i_r} \dots L_{i_1}\omega = a$ with $i_j \geq -2$, we want to obtain operators $(\widetilde{a_E})_m$ for energy E and mode m with the hypotheses below for the induction on r. The hypotheses are relaxed version of the ones in Conjecture 2.5.3:

There exists d_a such that for all $E \leq n^{d_a}$ there are operators $(\widetilde{a_E})_m \in \mathcal{A}_n$ where

- $\exists v_a < d_a$ such that for any m and $n^{v_a} \geq |m|$, $(\widetilde{a_E})_m|_E$ provides an energy shift at most $K_a(E + |m|)$ for some constant $K_a \geq 1$. This is a more relaxed version of $\widetilde{L}_m$ shifting energy by at most $|m|$. It is also the obstacle to get a *strong* SL-algebra.
- $(\widetilde{a_E})_m$ has norm at most n^{e_a} where e_a only depends on a.
- There exists a constant $g_a > 0$ such that

$$(\widetilde{a_E})_m = (a)_m|_E + O(\frac{1}{n^{g_a}}) + R^{a,m}_{E,n}, \text{ for } n^{v_a} \geq |m| \tag{2.25}$$

  where $(a)_m|_E$ is the restriction of $(a)_m$ in the VOA to energy at most E. $O(\frac{1}{n^{g_a}})$ should be regarded as the error in the approximation of $(\widetilde{a_E})_m|_E$ by $(a)_m|_E$, and it has norm at most $O(\frac{1}{n^{g_a}})$. Finally, the last term is $R^{a,m}_{E,n} = (\widetilde{a_E})_m(\mathbf{1} - P^E)$.

Notice the last hypothesis implies the same for restriction of energy to any $E' \leq E$ since projection to energy E' has norm at most 1 and the rest will mix with $R^{a,m}_{E,n}$. Further, the base of induction ω is essentially done. For $E \leq n^{\frac{1}{4}}$ and any $\sqrt[4]{n} \geq |m|$, $(\widetilde{\omega_E})_m = \widetilde{L}_m$ provides an energy shift of at most $|m|$ for any mode m, in other words, at most $1 \times (E + |m|)$.

The induction works by assuming hypotheses have been shown to hold for the field b, and then prove the same for $a = L_{-2}b$; the Borcherds identity shows that this is the hardest case and $L_k b$ for $k \geq -1$ are easier. In Borcherds identity (1.13), putting $p = 0$ and $q = -1$, and some index shifting gives

$$(a)_m = (L_{-2}b)_m = \sum_{j=0}^{\infty}(L_{-2-j}(b)_{m+j+2} + (b)_{m-j+1}L_{j-1}),$$

which is an infinite sum. Note that had we chosen $a = L_k b$ for $k \geq -1$, the Borcherds

identity would have given a finite sum, which can be handled similarly to what comes next.

When restricted to energy $E \leq n^{d_a}$, where d_a will be determined, the summation above will be finite. Indeed, as shown in [25],

$$(a)_m|_E = \sum_{j=0}^{E-\max\{-1,m+2\}} (L_{-2-j}(b)_{m+j+2}P^E + (b)_{m-j+1}L_{j-1}P^E).$$

Putting redundant projections in the middle of the operators leads to $(a)_m|_E =$

$$\sum_{j=0}^{E-\max\{-1,m+2\}} (L_{-2-j}|_{E+K_b(E+|m+j+2|)}(b)_{m+j+2}|_E + (b)_{m-j+1}|_{E+|j-1|}L_{j-1}|_E). \tag{2.26}$$

This will be important as the last induction hypothesis for ω and b will be applied separately. Based on the above identities, our choice for $(\widetilde{a_E})_m$ will be

$$(\widetilde{a_E})_m = \sum_{j=0}^{E-\max\{-1,m+2\}} (\widetilde{L}_{-2-j}(\widetilde{b_E})_{m+j+2} + (\widetilde{b_{E+|j-1|}})_{m-j+1}\widetilde{L}_{j-1}) \tag{2.27}$$

where we recall that $(\widetilde{\omega_E})_j = \widetilde{L}_j$ for all E. Each hypothesis can be checked in a straightforward way by simply applying the hypothesis for b and ω.

It remains to show that $\{Y(a,f)|a \in \mathcal{V}, f \in C^\infty_{\mathbb{C}}(S^1)\}$ generates an SL-algebra. Using the properties in the induction hypotheses, the proof is a more involved version of Theorem 2.5.6. To get the product $\prod_{j=1}^k Y(a^{(j)}, f^{(j)})$ in the scaling limit, the operators

$$\widetilde{Y}_{E_j}(a^{(j)}, f^{(j)}) = \sum_m \hat{f}^{(j)}_m (\widetilde{a^{(j)}_{E_j}})_m \in \mathcal{A}_n$$

have to be chosen where E_js need to be determined carefully by taking into account the constants in the energy bound inequalities for all a_js, and also all other constants, notably

d_{a_j}s and g_{a_j}s, so that we can use the approximation provided by the last hypothesis. The choice of E_js will not be universal and depends on the product which is why there is no single associated sequence and we can not claim a *strong* SL-algebra. ■

Remark 2.5.10 *As mentioned during the proof, the reason we could not obtain a strong SL-algebra generating set is the dependence of the energy shift on the energy itself. If somehow all vectors were obtained by only applying $L_r, r \geq -1$ (which is unlikely) or if we knew that the base of induction $\widetilde{L}_m$ shifts the energy **exactly** by m (not true even in the case of Ising), this issue would not be present. Still, we believe this to be a technicality that can be resolved.*

Remark 2.5.11 *One would wish to get the hermitian fields, which give self-adjoint $Y(a, f)$s, as a scaling limit of hermitian observables generated by the e_is. This was not the case in the construction above. Still, as [25] demonstrates, quasi-primary hermitian fields generate (see Remark 1.3.3) any unitary VOA. For UMMs, that generating set is $\{\omega\}$, for which there is a corresponding hermitian sequence from $\mathcal{A}_n^H$.*

2.5.2 Local conformal net observables

Recall that for UMMs, the observables algebra on an interval I is given by $\{e^{iL(f)} |\ f \in C^\infty_{\mathbb{R}}(S^1)$ and $\text{supp}(f) \subset I\}''$ (1.44). From results of the previous section, the following is immediate

Corollary 2.5.9 ([4], Cor. 4.9) *The sequence of observables below give a strong SL-algebra:*

$$e^{i\widetilde{L}(f)} \xrightarrow{SL} e^{iL(f)}, \forall f \in C^\infty_{\mathbb{R}}(S^1).$$

Proof: [Proof sketch] The convergence is a direct application of the Trotter-Kato approximation theorem (see e.g. [64, p. 141]) on the sequence $i\widetilde{L}(f)$ and its scaling

limit $iL(f)$ (see also Remark 2.5.8). The scaling limit is a strong SL-algebra due to the uniform boundedness of the operators involved, all being unitary. ■

Corollary 2.5.10 ([4], Cor. 4.10) $\{e^{iL(f)}\}'' \subset \mathcal{A}$ *as a strong SL-algebra.*

Proof: [Proof sketch] We will be using Kaplansky's density theorem multiple times. This theorem allows us to obtain self-adjoint bounded observables as the scaling limit of uniformly norm bounded self-adjoint observables. Restricting to self-adjoint operators is without loss of generality, as it is a fact that observable algebras are generated by self-adjoint operators.

Let $O \in \{e^{iL(f)}\}''$ be a bounded self-adjoint operator which we aim to show that it is in the scaling limit. As $O \in \{e^{iL(f)}\}''$, consider a sequence of self-adjoint operators $O^{(i)}$ in the algebra generated by $e^{iL(f)}$s with a strong limit to O. Each $O^{(i)}$ has a corresponding sequence of self-adjoint $(O_n^{(i)})_n$ with scaling limit $O^{(i)}$ which can be thought of replacing any $e^{iL(f)}$ in $O^{(i)}$'s expression by $e^{i\widetilde{L}(f)}$. By this construction, this sequence $(O_n^{(i)})_n$ is uniformly norm bounded. To make it bounded by $||O^{(i)}||$, we apply Kaplansky's density theorem. Thus, new sequences $(O_n^{(i)})_n$ are found, norm bounded by $||O^{(i)}||$. From these sequences, by a diagonal selection, we would like to get a uniformly bounded sequence O_n with scaling limit O which would give a strong SL-algebra as in the previous theorem.

As long as $O^{(i)}$s are uniformly norm bounded, that sequence can easily be built. This includes the case where O is in the **norm-operator** closure of the algebra generated by $\{e^{iL(f)}\}$, implying $||O^{(i)}|| \to ||O||$. So the C^*-algebra generated by $\{e^{iL(f)}\}$ can be recovered. If O is not in the C^*-algebra, applying again Kaplansky's density theorem on the sequence $O^{(i)}$, gives a new uniformly norm bounded (by $||O||$) sequence $(O^{(i)})_i$ in the C^*-algebra. Having previously proven that $O^{(i)}$ can be obtained by a uniformly norm bounded (by $||O^{(i)}||$) sequence $(O_n^{(i)})_n$, this gives the same settings as the easy case at the beginning of this paragraph. ■

Hence, all observables in LCN form a strong SL-algebra. The next question is whether there exists some definition of the finite-size local observables net $\mathcal{A}_n(I)$ and how the *bounded* scaling limit would compare to the LCN, called $\mathcal{A}_{vir}(I)$. As we shall see, the anyons must be on the upper half-circle as in Figure 1.17.

Definition 2.5.1 ([4], Def. 12) *Consider the upper half-circle S^1_+ with its two points on the boundary. The set of intervals $\mathcal{I}_+$ are the connected sets in one of the following forms:*

- *Open intervals I inside S^1_+ for which $\partial I \cap \partial S^1_+ = \emptyset$,*
- *Closed-open intervals I where $|I \cap \partial S^1_+| = |\partial I \cap \partial S^1_+| = 1$,*
- S^1_+.

On these sets, the following nets of observables are defined

Definition 2.5.2 ([4], Def. 13) *Given $I \in \mathcal{I}_+$, $\mathcal{A}_n(I)$ is generated by the identity and the e_js where $[\frac{j\pi}{2n+1}, \frac{(j+1)\pi}{2n+1}] \in I$.*

The definition implies locality, i.e. $[\mathcal{A}_n(I_1), \mathcal{A}_n(I_2)] = \{0\}$ for $I_1 \cap I_2 = \emptyset$, and isotony, i.e. $I_1 \subset I_2 \implies \mathcal{A}_n(I_1) \subset \mathcal{A}_n(I_2)$.

Definition 2.5.3 ([4], Def. 14) *Consider the set of self-adjoint bounded linear operators O in the scaling limit of the algebra of observables $\mathcal{A}_n(I)$ such that there exists a self-adjoint sequence $O_n \in \mathcal{A}_n(I)$ with bounded norm and*

$$\rho_n(O_n(\rho_n)^{-1}(u)) \to Ou, \quad \forall u \in \mathcal{V},$$

*where $(\rho_n)^{-1}(u) = 0$ if $u \notin \rho_n(\mathcal{W}_n)$. This means there is sequence with "**strong** SL convergence" to O or the strong-operator convergence in $\mathcal{V}$. Define $\mathcal{A}_b(I)$ as the von Neumann algebra generated by this set.*

Locality is the reason behind the above definition. Consider two sequences of operators $x_n \xrightarrow{SL} x$ and $y_n \xrightarrow{SL} y$ which are self-adjoint and commuting. In order to ensure $[x, y] = 0$, it can be easily observed that the weak-limit offered by scaling limit is not enough and we need at least a strong type of that limit (which is the above definition). However, that may not be enough as $x_n y_n \xi \to xy\xi$ for $\xi \in \mathcal{V}$ does not necessarily hold. Let us rewrite:

$$(x_n y_n - xy)\xi = x_n(y_n - y)\xi + (x_n - x)y\xi.$$

The first and second term are not guaranteed to go to zero unless x_ns are uniformly bounded and $x_n \to x$ in the strong-operator topology (of $\overline{\mathcal{V}}$ as $y\xi \in \overline{\mathcal{V}}$). It turns out that the strong SL convergence (which is strong-operator convergence in $\mathcal{V}$) **and** norm boundedness are equivalent to convergence in the strong-operator topology in $\overline{\mathcal{V}}$. One direction is clear and the other is the application of Kaplansky's density theorem to get such a sequence with norms uniformly bounded. The definition above imposes these properties and as a result, $\mathcal{A}_b(I)$ satisfies locality and isotony. Further, similar to the procedure carried out in Corollary 2.5.10, there is a sequence associated to any of its elements which are norm bounded and converge strongly to that element. Therefore, it is a strong SL-algebra.

How does this "net" compare to $\mathcal{A}_{vir}(I)$? Denote by $j(I)$ the reflection of the interval I with respect to the $x-$axis where $j : z \to \bar{z}$.

Theorem 2.5.11 ([4], Thm. 4.11) *Given a function* $f = \sum \hat{f}_m e^{im\theta} \in C^\infty_{\mathbb{C}}(S^1)$ *with* $supp(f) \subset I \cup j(I)$, *and* $\hat{f}_m = a_m + ib_m$, *define* $\widetilde{e}(f) =$

$$\alpha_n^c \sum_{j=1}^{2n-1} f_c\Big(e^{i\frac{\pi(j+\frac{1}{2})}{2n+1}}\Big)e_j + i\alpha_n^s \sum_{j=1}^{2n-2} f_s\Big(e^{i\frac{\pi(j+1)}{2n+1}}\Big)[e_j, e_{j+1}] + (\sum_{m=-\infty}^{\infty} a_m\beta^c_{n,m} + b_m\beta^s_{n,m})\mathbf{1}, \tag{2.28}$$

which is inside $\mathcal{A}_n(I)$ *(for large enough* n*), and*

$$f_c(z) = \frac{f(z) + f(j(z))}{2} \in C^\infty_{\mathbb{C}}(S^1_+), \quad \forall z \in S^1_+,$$

$$f_s(z) = \frac{f(z) - f(j(z))}{2} \in C^\infty_{\mathbb{C}}(S^1_+), \quad \forall z \in S^1_+.$$

we have $\widetilde{e}(f) \xrightarrow{SL} L(f)$*.*

Proof: [Proof sketch] By direct calculation, $\widetilde{e}(f) = \widetilde{L}(f)$ and the rest is Theorem 2.5.6. ■

Taking real-valued functions $f \in C^\infty_{\mathbb{R}}(S^1)$,

Corollary 2.5.12 ([4], Cor. 4.12) $\{e^{iL(f)}\}'' \subset \mathcal{A}_b(I)$ *for* $supp(f) \subset I \cup j(I)$*.*

This hints to the relation between $\mathcal{A}_b(I)$ and $\mathcal{A}_{vir}$. Assume I touches the boundary of S^1_+. Then, $I \cup j(I)$ is some connected interval in the circle and so $\{e^{iL(f)} | \text{supp}(f) \subset I \cup j(I)\}'' = \mathcal{A}_{vir}(I \cup j(I))$. By the corollary above,

$$\mathcal{A}_{vir}(I \cup j(I)) \subset \mathcal{A}_b(I).$$

On the other hand, due to Haag duality for the conformal net $\mathcal{A}_{vir}$ and locality for $\mathcal{A}_b$, for J the complement of I in S^1_+,

$$\mathcal{A}_b(J) \subset \mathcal{A}_b(I)' \subset \mathcal{A}_{vir}(I \cup j(I))' = \mathcal{A}_{vir}(J \cup j(J)) \subset \mathcal{A}_b(J).$$

Therefore, one recovers exactly the LCN by taking the bounded scaling limit.

Theorem 2.5.13 ([4], Thm. 4.13) $\mathcal{A}_b(I) = \mathcal{A}_{vir}(I \cup j(I))$ *for* $I \in \mathcal{I}_+$ *with* $|I \cap \partial S^1_+| = 1$*.*

The above is true for all UMMs assuming Conjecture 2.5.3.

2.5.3 Vertex operators $Y(a,z)$

In Theorem 2.5.8, $(a)_m$ was found to be in the scaling limit using QC-local operators. We know $Y(a,z)$ is the weak limit of a sequence $Y(a,f)$ where f shrinks to the δ Dirac function, implying $Y(a,z) \in \mathcal{A}$ (Remark 2.3.2). Alternatively, one can directly construct a sequence to show this.

Theorem 2.5.14 ([4], Thm. 4.14) *$Y(a,z) \in \mathcal{A}$ as an almost linear operator.*

Proof: [Proof sketch] Choose $O_n = \sum_{|m|<\frac{g_a \log(n)}{2\log(|z|)}} (\widetilde{a_{\log(n)}})_m z^{-m-\mathrm{wt}\ a}$. Using the properties of $(\widetilde{a_{\log(n)}})_m$ as outlined in Theorem 2.5.8 one can prove $O_n \xrightarrow{SL} Y(a,z)$. ■

In 2.1, we discussed the thinking behind the construction of finite stages of the smeared Virasoro field

$$\int f(e^{i\frac{\pi j}{n}})e_j \xrightarrow{SL} Y(\omega,f) = \oint Y(\omega,z)f(z)z^2\frac{\mathrm{d}z}{2\pi i z}.$$

Informally, let us take this smooth function f to converge to the δ Dirac function at some point corresponding to angle θ. One would expect

$$e_x \xrightarrow{SL} Y(\omega,e^{i\theta}).$$

where x, implicitly depending on n, is chosen such that $\theta \in [\frac{x\pi}{2n+1}, \frac{(x+1)\pi}{2n+1}]$. Of course, from our Koo-Saleur formulae, with e_js, the cos() part of $Y(\omega,e^{i\theta})$ appears in the scaling limit. For the imaginary part, the bracket $[e_x, e_{x+1}]$ must be used.

Finding some space ultra-local operator in $\mathcal{A}_n$ giving us the field operator in the scaling limit would be a "proof" that the field operator $Y(w,z)$ should not only be thought of the analog of a space-local observable in the continuum, but of a space ultra-local observable. Unfortunately, the *easy* guess does not work, as shown below.

Notations. Define β_n^c and β_n^s inside $\mathbb{R}^{\mathbb{N}_0}$ with entries $\beta_{n,m}^c$ and $\beta_{n,m}^s$ for all $m \geq 0$, respectively. Define the vectors $\widetilde{v}_x^c = (\cos(\frac{m(x+\frac{1}{2})\pi}{2n+1}))_{0\leq m\leq 2n}$ and $\widetilde{v}_x^s = (\sin(\frac{m(x+1)\pi}{2n+1}))_{1\leq m\leq 2n}$, and extend them by zeros so they also become elements of $\mathbb{R}^{\mathbb{N}_0}$.

Theorem 2.5.15 ([4], Thm 4.15) *We do* ***not*** *have*

$$O_n = \alpha_n^c ||\widetilde{v}_x^c||^2 e_x + i\alpha_n^s ||\widetilde{v}_x^s||^2 [e_x, e_{x+1}] + (\beta_n^c.\widetilde{v}_x^c + \beta_n^s.\widetilde{v}_x^s)\mathbf{1} \xrightarrow{SL} Y(\omega, z)z^2,$$

where $z = e^{i\theta}$ *and we pick the unique* $1 \leq x \leq 2n-1$ *such that* $\theta \in [\frac{x\pi}{2n+1}, \frac{(x+1)\pi}{2n+1}]$.

Notice x can only exist for large enough n if θ is close to the boundaries.

Proof: [Proof sketch] The choice for the coefficients involved in O_n becomes clear once we rewrite O_n as

$$O_n = \sum_{m=-2n}^{2n} \Big(\cos(\frac{m(x+\frac{1}{2})\pi}{2n+1}) + i\sin(\frac{m(x+1)\pi}{2n+1}) \Big) \widetilde{L}_m.$$

This is the "ideal" candidate expression that one would expect to converge to $Y(\omega, z)z^2$ in the scaling limit. It is clear that any finite sum up to some $|m| \leq M$ for O_n goes to $\sum_{|m|\leq M} e^{im\theta} L_m$. However, restricting to some finite energy M from right **and** left $P^M O_n P^M$ does not necessarily give $\sum_{|m|\leq M} e^{im\theta} L_m$.

What makes this convergence impossible are the finite-size effects which implies that $\widetilde{L}_m$ with *high* $|m|$, more precisely $2n+1-2M \leq |m| \leq 2n$, have low-low energy mixing terms. Therefore after the restriction to energy up to M, we have (unwanted) terms other than those from $\widetilde{L}_m$ with $|m| \leq M$ in our summation. Their coefficients can be computed from formulae in equations (2.8) and (2.9) and are divergent, hence the scaling limit does not hold. ■

Remark 2.5.12 *As all problems emerge from the* $|m|$ *being close to* $2n$*, the observable*

O_n as follows has the desired scaling limit

$$\sum_{m=-2n+\lfloor\log(n)\rfloor}^{2n-\lfloor\log(n)\rfloor}\Big(\cos(\frac{m(x+\frac{1}{2})\pi}{2n+1})+i\sin(\frac{m(x+1)\pi}{2n+1})\Big)\widetilde{L}_m \xrightarrow{SL}\sum e^{im\theta}L_m.$$

In fact, any function $f(n)$ instead of $\log(n)$, with sublinear growth (and $f(n)\xrightarrow{n\to\infty}\infty$) would work. Unfortunately, this does not translate to the nice ultra-local expression in terms of just e_x and $[e_x, e_{x+1}]$ we desired for O_n.

A space ultra-local sequence for $Y(\omega, z)$

One can construct a space ultra-local sequence in terms of e_x and $O(\log(n))$ many of its neighbor TL operators with the desired scaling limit (Definition 2.3.11). O_n will be a linear combination of:

$$\sum_{y=-\lfloor 8\log(n)\rfloor}^{\lfloor 8\log(n)\rfloor} t_y e_{x+y},\quad i\sum_{y=-\lfloor 8\log(n)\rfloor}^{\lfloor 8\log(n)\rfloor} t_y[e_{x+y}, e_{x+y+1}] \text{ and } \mathbf{1}. \tag{2.29}$$

The idea is to select the t_y's such that O_n becomes of the form:

$$\sum_{m=-\lfloor\log(n)\rfloor}^{\lfloor\log(n)\rfloor}\Big(\cos(\frac{m(x+\frac{1}{2})\pi}{2n+1})+i\sin(\frac{m(x+1)\pi}{2n+1})\Big)\widetilde{L}_m+\sum_{\substack{m=-2n+\lfloor\log(n)\rfloor,\\ m\notin[-\lfloor\log(n)\rfloor,\lfloor\log(n)\rfloor]}}^{2n-\lfloor\log(n)\rfloor}\widetilde{c}_m\widetilde{L}_m, \tag{2.30}$$

where we need specific coefficients for $m\in[-\lfloor\log(n)\rfloor, \lfloor\log(n)\rfloor]$, more precisely

$$\Big(\cos(\frac{m(x+\frac{1}{2})\pi}{2n+1})+i\sin(\frac{m(x+1)\pi}{2n+1})\Big),$$

and zero for $m<-2n+\lfloor\log(n)\rfloor$ or $m>2n-\lfloor\log(n)\rfloor$. This means that there are in total $4\lfloor\log(n)\rfloor$ many equations for each of the cos() and sin() part, thus $8\lfloor\log(n)\rfloor$ in total. On the other hand, there are $2\lfloor 8\log(n)\rfloor$ variables t_ys. As the matrix involved in these

linear equations is the one used in Theorem 2.5.1 for the trigonometric interpolation, this system has infinitely many solutions. Next, it is not hard to see that observables like (2.30) provide the desired scaling limit. Indeed, the finite-size effect are no longer present as for $|m|$ close to $2n$, the coefficients of $\widetilde{L}_m$ are zero. Further, restricting to energy up to M, $P^M O_n P^M$ for large enough n equals:

$$P^M\Big(\sum_{m=-\lfloor\log(n)\rfloor}^{\lfloor\log(n)\rfloor}\Big(\cos(\frac{m(x+\frac{1}{2})\pi}{2n+1})+i\sin(\frac{m(x+1)\pi}{2n+1})\Big)\widetilde{L}_m\Big)P^M \tag{2.31}$$

which converges to $P^M Y(\omega, e^{i\theta})e^{2i\theta}P^M$.

It is still unknown if this sequence can give rise to a strong SL-algebra. The question likely boils down to the following analysis problem: Is there a sequence of functions that have a support of width $O(\log(n))$, with their Fourier series having specific coefficients at both ends $m < -2n + \lfloor\log(n)\rfloor$ and $m > 2n - \lfloor\log(n)\rfloor$, and around the zero mode $m \in [-\lfloor\log(n)\rfloor, \lfloor\log(n)\rfloor]$, and also having coefficients $\widetilde{c}_m$ at all other modes with fast enough decay rate (with respect to n).

2.6 Conjectures and future directions

In this section, we provide a list of problems that need to be addressed for a clearer picture of the structures in the scaling limit.

2.6.1 SL-algebras in the scaling limit

The smeared fields $Y(a, f)$ were obtained as QC-local operators.

Conjecture 2.6.1 ([4], Conjecture 5.2) *There is a **spanning** set $\mathcal{S} = \{a\}_{a\in\mathcal{V}}$ of the VOA such that for any $a \in \mathcal{S}$, the smeared field $Y(a, f)$ is a scaling limit of a space-local sequence.*

One obstacle to space locality is the *absence* of commutators in the Borcherds identity. Otherwise, the terms involving product of far apart e_js would disappear. For example, if $Y(a,f)$ can be expressed in terms of commutators of $Y(\omega,f)$ with the Virasoro generators, the above conjecture can be proved.

Closely related to conformal invariance, is the scaling limit of the product of unitaries $e^{i\widetilde{L}(f)}$ and the smeared field operators $\widetilde{L}(f)$.

Conjecture 2.6.2 ([4], Conjecture 5.3) *Prove that the algebras in Corollary 2.5.10 and Theorem 2.5.6 together generate a (strong) SL-algebra.*

As a remark on the emergence of conformal invariance in the scaling limit, notice that due to Corollary 2.5.9 , for $e^{iL(g)} \in \mathcal{A}_{vir}(I)$, we have

$$e^{i\widetilde{L}(f)}e^{i\widetilde{L}(g)}e^{-i\widetilde{L}(f)} \xrightarrow{SL} e^{iL(f)}e^{iL(g)}e^{-iL(f)}.$$

Further, the scaling limit above is itself in $\mathcal{A}_{vir}(\exp(f)(I))$. This is the conformal covariance axiom (1.42) for the LCN:

$$U(\gamma)\mathcal{A}_{vir}(I)U(\gamma)^\dagger = \mathcal{A}_{vir}(\gamma(I)), \quad \gamma \in \mathrm{Diff}_+(S^1).$$

Similar to how $\mathcal{A}(I)$ is generated by $e^{iL(f)}$s, it is easy to show that $\mathcal{A}_n(I)$ can also be generated by $e^{i\widetilde{L}(f)}$s. Due to Corollary 2.5.9 and Theorem 2.5.13, this implies that, loosely speaking, the two sets $e^{i\widetilde{L}(f)}\mathcal{A}_n(I)e^{-i\widetilde{L}(f)}$ and $\mathcal{A}_n(\exp(f)(I))$ become the same in the scaling limit (at least for I and $\exp(f)(I)$ satisfying the condition in Theorem 2.5.13). Therefore, conformal invariance emerges in the scaling limit. This may not be satisfying as it is not clear what the group of operators $e^{i\widetilde{L}(f)}$ really is, and whether it is "the natural finite" version of $\mathrm{Diff}_+(S^1)$ that should recover its action in the scaling limit (see [49] for a different candidate, the Thompson's group).

We could not get all types of observables as an SL-algebra. Specifically, we believe

Conjecture 2.6.3 ([4], Conjecture 5.4) *Field operators $Y(a,z)$ form a strong SL-algebra, with space ultra-local observables associated sequence to each field.*

The techniques and analysis used in [29], involving the convergence of the truncated versions of the field operators to the correlation function, may be found to be useful.

It is known that the product of QC ultra-local hermitian operators can be simulated efficiently [65, Claim 6.2] on a quantum computer. If one can show an efficient convergence of algebras of space ultra-local (thus QC ultra-local) hermitian observables to hermitian point-like fields, an efficient quantum computer simulation of the correlation functions becomes feasible.

2.6.2 LCN and boundary CFT in the scaling limit

Theorem 2.5.13 only shows the scaling limit of bounded nets for intervals I touching the boundary of the half-circle. Let us suppose that I does not have this property. Then it gives an observable algebra between:

$$\mathcal{A}_{vir}(I) \vee \mathcal{A}_{vir}(j(I)) \subset \mathcal{A}_b(I) \subset (\mathcal{A}_{vir}(J_1) \vee \mathcal{A}_{vir}(J_2))'$$

where J_1 and J_2 are the two intervals obtained by removing I and $j(I)$ from the circle, implying $J_1 \cup J_2 \cup I \cup j(I) = S^1 - \{\text{four points}\}$.

It is a fact in local conformal net that the difference between $\mathcal{A}_{vir}(I) \vee \mathcal{A}_{vir}(j(I))$ and $(\mathcal{A}_{vir}(J_1) \vee \mathcal{A}_{vir}(J_2))'$ is given by the charge transporters. In the case of Ising, this means the smeared fermionic fields $\Psi(f)\Psi(g)$ with $\mathrm{supp}(f) \subset I$, $\mathrm{supp}(g) \subset j(I)$, and the smeared fields $a(f)a(g)$. Here, $a : \chi_0 \to \chi_{\frac{1}{16}}$ is the intertwiner given by $\mathcal{Y}(\sigma, z)$ where $\sigma = |\frac{1}{16}\rangle \in \chi_{\frac{1}{16}}$ is the spin state (1.25) and we use the smeared version of this intertwiner

sending the vacuum sector to the spin sector. As the Ising anyonic chain with boundary condition $(\frac{1}{2},\frac{1}{2})$ gives $\chi_0+\chi_{\frac{1}{2}}$, we can only expect for charge transporters of the type $\Psi(f)\Psi(g)$. Indeed, it can be easily shown that $\Psi(f)\Psi(g)\in\mathcal{A}_b(I)$. More precisely, the subalgebra of the Dirac algebra $CAR(I)\otimes CAR(j(I))$ generated by basis elements with even many Dirac operators is inside $\mathcal{A}_b(I)$, where $CAR(I)=\{\Psi(f)|\text{supp}(f)\subset I\}$. Note the finite algebra $\mathcal{A}_n$ is the even subalgebra of the Majorana operators ψ_j which is isomorphic to the even subalgebra of $\mathcal{F}_n=\{\Psi_{-n+\frac{1}{2}},\dots,\Psi_{n-\frac{1}{2}}\}$.

On the other hand, we recall the boundary CFT (BCFT) in Remark 1.5.5 given by $\mathcal{B}_+(\mathcal{O})=(CAR(I)\otimes CAR(J))^{even}$ for $\mathcal{O}=I\times J$. Our results on the scaling limit of the Ising anyonic chain with boundary condition $(\frac{1}{2},\frac{1}{2})$, strongly suggest that the scaling limit theory is related to the restriction of the mentioned BCFT to spacetime regions "$I\times j(I)$".

The story for higher level anyonic chains should be similar: If an anyonic chain with boundary condition (x_0,x_{L-1}) gives the Hilbert space $\sum_i\chi_i$, the corresponding net to this is conjectured to include all charge transporters for the irreducible modules, denoted usually by $\rho_i\bar{\rho}_i$ where ρ_i is the irreducible sector corresponding to χ_i. This is also what happens in the case of TQFT, where string operators on the lattice converge to charge transporters in the limit [66].

The framework of nets of observables could also allow a precise treatment of spacetime in the scaling limit process, as the local finite nets $\mathcal{A}_n(I)$, encode spacetime information by how they are relative to each other.

2.6.3 Higher minimal models

Due to the numerical results on higher level ACs, it was conjectured in 2.5.3 that the theorems in section 2.5 are true for higher level UMMs. Here, we propose exact identities

giving us the Virasoro algebra.

Conjecture 2.6.4 *[4, Conjecture 5.5] The Hilbert space and the Virasoro algebra action of every chiral UMM with central charge $c = 1 - \frac{6}{(k+1)(k+2)}$ can be obtained as a scaling limit of some $\mathfrak{su}(2)_k$ AC. More precisely, from the operators*

$$O^c_{n,m} = -\sum_{j=1}^{2n-1} \cos(\frac{m(j+\frac{1}{2})\pi}{2n+1})e_j, \quad O^s_{n,m} = i\sum_{j=1}^{2n-2} \sin(\frac{m(j+1)\pi}{2n+1})[e_j, e_{j+1}],$$

we get operators $\widetilde{L}^c_{\pm m}, \widetilde{L}^s_{\pm m} \xrightarrow{SL} L_{\pm m}$ where

$$\frac{\widetilde{L}^c_m + \widetilde{L}^c_{-m}}{2} = \alpha^c_n O^c_{n,m} + \beta^c_{n,m}\mathbf{1} \xrightarrow{SL} \frac{L_m + L_{-m}}{2},$$

$$\frac{i(\widetilde{L}^s_m - \widetilde{L}^s_{-m})}{2} = \alpha^s_n O^c_{n,m} + \beta^s_{n,m}\mathbf{1} \xrightarrow{SL} \frac{i(L_m - L_{-m})}{2},$$

satisfying the properties in Conjecture 2.5.3 for $\alpha^c_n, \alpha^s_n, \beta^c_{n,m}$, and $\beta^s_{n,m}$ being suitable scaling factors.

Remark 2.6.1 *The size of the chain was assumed to be $2n-1$. In general, this depends on the boundary condition which needs to be adjusted accordingly.*

A correction should be made to the above conjecture. It is not accurate to call the scaling limit a *chiral* UMM, though it is *completely described by chiral data*. It is clear that there does not exist a preferred direction for chirality to emerge; indeed, chirality only emerges in the boundary of a system and the anyonic chain as defined is not a boundary component of some $(2+1)-$d QFT. Furthermore, the Hamiltonian is obviously symmetric with respect to flipping the chain.

The situation is likely similar to boundary CFTs in statistical mechanics (RSOS) models [1, section 7] as the transfer matrix-Hamiltonian correspondence demonstrates.

As discussed in the previous section, the anyonic chain scaling limit as an algebraic QFT is likely related to what is called in the literature a "non-diagonal" BCFT [38, section 7]. The precise correspondence is a topic of current research.

Yet, the most important criticism towards the above conjecture is about the identities for the higher Virasoro modes.

Following a numerical argument made in [1, eq. (3.31-3.43)], using the known expression of the free energy of the vertex model (or the continuous Potts model) in the thermodynamic limit [41], one can see whether the above identities deliver at least the correct central charge. Estimating $(\Omega_n, [\widetilde{L}_2, \widetilde{L}_2]\Omega_n)$, assuming the conjecture, should give $(\Omega, [L_2, L_{-2}]\Omega) = \frac{c}{2}$ in the limit (Remark 1.3.5). The above identities work exactly for the case of Ising, but give only an approximation of the central charge for higher levels. We still believe that a space-local Koo-Saleur formula should exist that gives a strong SL-algebra in the scaling limit. It is a subject of current research on how to include higher commutators of e_js in the identities above (while preserving the space-locality of the formulae) in order to correct the error.

One option is to use the closely related Jones-Temperley-Lieb JTL_k MTC instead of $\mathfrak{su}(2)_k$. As introduced in section 1.2.3, JTL_k is obtained by taking the quotient of the TL algebra by Jones-Wenzl projectors $p_{k+1}, k \geq 0$. The expression $p_{k+1} = 0$ imposes a recursive relation on commutators. Thus higher commutators can always be written in terms of lower (up to $[e_i, [\ldots, [e_{i+k-1}, e_{i+k}]\ldots]]$) commutators ensuring space-locality.

Chapter 3

Efficient Quantum Simulation of CFT

3.1 Outline of main results

The original motivation behind the work in the previous chapter was an efficient quantum simulation of CFT. In each simulation problem, there will be **local** observables for which their expectation values is to be efficiently computed in polynomial time with respect to the inputs.

To define local observables, consider the problem of simulation of unitary evolution [67] of a quantum many-body system. There, the ($k-$)local Hamiltonians are a sum of *polynomially* many ($k-$)ultra-local operators. Quantum computation gives a precise definition of locality. A fundamental aspect of this definition is the explicit or implicit assumption of a **sequence** of operators O_n, which are the sum of *ultra*-local operators acting on at most $O(\log(n))$ many particles. So one can distinguish between local and non-local (sequence of) operators.

Definition 3.1.1 *Given $(\mathbb{C}^d)^{\otimes n}$, the tensor product of n qudits $\mathbb{C}^d$, for an ultra-local*

sequence of operators $(O_n)_n$, O_n acts nontrivially on at most $O(\log(n))$ many qudits. A local operator is a sum of ultra-local operators.

Locality is a salient feature for any physical QFT. Since TQFTs are low energy effective theories, their locality is not intrinsic and usually hidden. For example, in the Witten-Chern-Simons (WCS) modeling of the fractional quantum Hall liquids, the WCS theory is an effective description for the emergent anyons, it follows that locality of WCS TQFTs should be derived from that of the underlying electron systems. However, the simulation of TQFTs in [68] uses a hidden locality given by pairs of pants decomposition of the space surfaces ($|x_1 \dots x_{n-2}\rangle$ in Figure 1.13). Similarly, there are no intrinsic (infinite) local degrees of freedom for a CFT to define locality.

We wish to declare a subset of observables in CFTs to be ultra-local. Depending on what problem one wants to solve, we need to efficiently simulate the local observables the subset generates. The argument behind the declaration of those observables as ultra-local is motivated from discussions in the previous chapter, on space and energy locality (see e.g. 2.5.3).

Theorems in the previous chapter on convergence of expectation values could also be helpful. Yet there is no guarantee that using those approximations is the right path for the problems ahead. In fact, we will also point out the disadvantages of the anyonic chain approach for each simulation problem.

In this chapter, we will explore the possible definitions for the two problems that CFT simulation includes. Before discussing the problems, we make a review on related works on QFT simulation. Next, we try to define each simulation problem, and show our results after making the attempts on the definitions.

In the first part, we start with defining the unitary evolution simulation problem.

Definition 3.3.1 *(CFT UNITARY EVOLUTION)* ***Inputs:*** *Given k and real-valued*

functions $f^{(j)}$ with n_j many Fourier coefficients for $1 \leq j \leq k$, each coefficient $(\hat{f^{(j)}})_m$ computable in polynomial time $p(m)$ with respect to m using a classical Turing machine.

Output: *An approximation of the following up to an error ϵ, with complexity measured with respect to $\{\frac{1}{\epsilon}, n_1, \ldots, n_k, k\} \cup \{(\hat{f^{(j)}})_m\}_{j,m}$*

$$|(\Omega, \prod_{j=1}^{k} e^{iL(f^{(j)})}\Omega)|. \tag{3.1}$$

Using Fock space construction of CFTs along with the trick developed in [3] to ultra-localize the Dirac fermionic operators, it is shown in 3.5 how to get an approximation of this expectation value. We can only go so far with this approach and prove the rate of convergence is efficient under certain conditions.

Theorem 3.5.3 *Consider the same setting as above with the condition that*

$$\sum_{j=1}^{k} ||(f^{(j)})'||_{r_\omega+2}$$

*is of order $O(\log(n_1, \ldots, n_k, k))$. Then CFT UNITARY EVOLUTION is in **BQP**.*

This technical condition is due to our inability to accurately estimate some norm of $(\mathbf{1} - P^N)\prod_{j=1}^{k} e^{iL(f^{(j)})}\Omega$, which is the higher energy part of the state (see Remark 3.5.2). We instead opt for an estimate on some norm of the whole state $\prod_{j=1}^{k} e^{iL(f^{(j)})}\Omega$, and yet obtain the nontrivial result above. Lastly, other alternative approaches will be discussed.

Afterwards, we move to the more complicated correlation function simulation problem. There are two possible formulations, each corresponding to choosing the **space** or **energy basis** as the computational basis, based on local degrees of freedom of space or energy.

The energy basis formulation corresponds to the case of smeared correlators, where space position is given by smearing functions $f(e^{i\theta}) = e^{ik\theta}$, which represent the Fourier

transform that takes the space basis to the energy basis. In this scenario, the states undergoing some unitary operations U are obtained by creation modes acting on the vacuum $\mathcal{Y}_n(a^{(n)}, e^{-ik_n\theta})\dots\mathcal{Y}_1(a^{(1)}, e^{-ik_1\theta})\Omega = \prod_{i=1}^{n}(a^{(i)})_{-k_i}\Omega$. We will argue that U is a product of unitary *gates* from a set $U_{\mathcal{V}} = \bigcup_{\text{hermitian primary fields}} U_a$, where U_a is formed by the following gates

- $e^{i\beta(a)_n^\dagger(a)_n}$,
- $e^{i(\alpha(a)_n^\dagger(a)_n(a)_m^\dagger(a)_m+\overline{\alpha}(a)_m^\dagger(a)_m(a)_n^\dagger(a)_n)}$,
- $e^{i(\alpha(a)_n^\dagger(a)_m+\overline{\alpha}(a)_m^\dagger(a)_n)}$,
- $e^{i(\alpha(a)_n(a)_m+\overline{\alpha}(a)_m^\dagger(a)_n^\dagger)}$,

for all $\beta \in \mathbb{R}, \alpha \in \mathbb{C}$.

Definition 3.4.1 (CFT Correlation Function Simulation Problem)

Inputs: *Given error rate $\epsilon > 0$, hermitian primary fields $\{b^{(j)}\}_{j=1}^{l}, \{a^{(i)}\}_{i=1}^{n}$, with corresponding modes m_j, k_i and intertwiners $\mathcal{Y}_{\widetilde{j}}, \mathcal{Y}_i$ of type $\binom{\widetilde{C}_{j-1}}{B_j\ \widetilde{C}_j}, \binom{C_{i-1}}{A_i\ C_i}$ where $B_j, \widetilde{C}_j, A_i, C_i$ are irreps of $\mathcal{V}$ satisfying $\widetilde{C}_0 = C_0 = \mathcal{V}$ and $\widetilde{C}_l = C_n$, and finally a description of a unitary U as a product of N unitaries from the set $U_{\mathcal{V}}$ with all chosen modes used to describe the N unitaries forming a set M_U.*

Output: *An approximation of the normalized correlation function below with error rate ϵ, with complexity measured with respect to $\{m_l, \dots, m_1, k_n, \dots, k_1, \frac{1}{\epsilon}, N\} \cup M_U$ where $m_j, k_i \geq 0$.*

$$\frac{|\Big(\mathcal{Y}_{\widetilde{l}}(b^{(l)}, e^{-im_l\theta})\dots\mathcal{Y}_{\widetilde{1}}(b^{(1)}, e^{-im_1\theta})\Omega, U\mathcal{Y}_n(a^{(n)}, e^{-ik_n\theta})\dots\mathcal{Y}_1(a^{(1)}, e^{-ik_1\theta})\Omega\Big)|}{||\mathcal{Y}_{\widetilde{l}}(b^{(l)}, e^{-im_l\theta})\dots\mathcal{Y}_{\widetilde{1}}(b^{(1)}, e^{-im_1\theta})\Omega||.||\mathcal{Y}_n(a^{(n)}, e^{-ik_n\theta})\dots\mathcal{Y}_1(a^{(1)}, e^{-ik_1\theta})\Omega||}. \tag{3.2}$$

Conjecture 3.4.1 *The CFT Correlation Function Simulation Problem is in **BQP** and generically **BQP**-complete.*

It is a corollary of fermionic quantum computation in [3] that for the case of Ising CFT, the problem is **BQP**-hard.

For the point-like version, the correlator is the insertion of point-like fields on the unit circle (space at present time), i.e. $|z_i| = |w_i| = 1$:

$$|\Big(\mathcal{Y}_{\tilde{l}}(b^{(l)}, w_l)\dots\mathcal{Y}_{\tilde{1}}(b^{(1)}, w_1)\Omega, U(\gamma)\mathcal{Y}_n(a^{(n)}, z_n)\dots\mathcal{Y}_1(a^{(1)}, z_1)\Omega\Big)|. \tag{3.3}$$

Surprisingly, a substantial amount of analysis is necessary (section 3.6) to show the above is well-defined. The correlator should be interpreted as a probability density function, but it is not rigorously known how one should integrate it in order to have a proper normalization like in the smeared case. This is crucial for understanding what the quantum computer should approximate. Finally, we discuss some ideas on what the right approach could be for the above problems.

3.2 Previous works

The circuit model of quantum computing is based on quantum mechanics. From early research on computational complexity of QFTs [2], it was stated explicitly as a conjecture that QFTs would not provide extra computational power beyond quantum mechanics (*extended quantum polynomial Church thesis*), as suggested by the efficient simulation of $(2+1)-$d TQFTs [68]. An important difference between TQFTs and CFTs is that while TQFTs are realized as gapped quantum systems, CFTs represent universality classes of gapless (*massless*) critical phases. Our program seems to be a first attempt towards a quantum simulation of gapless QFTs mathematically.

More recently, there has been effort towards simulating scattering amplitudes of massive QFTs, although some are made with support from numerical and physical arguments.

For a general survey we refer to [69]. Below we discuss the important results.

The first recent work [70] in this domain was a quantum estimate of relativistic scattering probabilities in massive scalar ϕ^4 quantum field theories. It has been shown to be in **BQP** for four and fewer dimensions. This is among the simplest interacting theories in fewer than four dimension (where it is believed to be free). This provides an exponential speedup over previous classical algorithms. The algorithm uses an approximation of the field by discretization of space via a lattice, and discretization of the field value at each lattice site. Creating the initial state for the simulation is done by a modified version of adiabatic state preparation for preparing non-eigenstates such as wavepackets. Given a list of momenta of incoming particles, the quantum algorithm outputs the probability distribution of outgoing particles up to a precision of ϵ efficiently. The algorithm works by [70, Appendix]

- creating the *free* vacuum adiabatically,
- exciting the wavepackets afterwards using evolution of some Hamiltonian H_ψ,
- turning on the interaction term adiabatically and simulating the Hamiltonian time evolution,
- and finally adiabatically turning off the interactiong term and measuring the occupation numbers of momentum modes.

This work would support the extended Church thesis if the convergence of lattice models to the continuum is addressed mathematically. When the coupling constant is sufficiently small, QFT scattering amplitudes are computed using perturbation theory. Perturbative analysis helps in the weak coupling regime to find the convergence rate. If the coupling constant is strong, perturbation theory no longer applies and non-perturbative methods are used to determine the rate of convergence.

More recently, in [71], it was shown that the above problem for $2-$d spacetime is in fact **BQP**-*complete* even for the weakly coupled regime. To implement a universal gate set, an external field $J(t, x)$ is chosen so that the scaling limit is a collection of double-well potentials. The $|0\rangle$ and $|1\rangle$ states of a qubit can be represented by choosing the particle occupation (in the ground state) of the left well for $|0\rangle$ and the right well for $|1\rangle$. Varying the source term J as a function of time moves the potential wells. Moving the left and right wells of a single qubit closer together implements single-qubit gates (like Pauli gates) through tunneling between the wells. Moving the wells of neighboring qubits closer together implements inter-particle interactions which are generically entangling 2-qubit gates. There is still a question regarding the mathematical rigor of the methods, particularly the validity of the perturbative analysis, as also mentioned by the authors themselves in [71, section 2.4].

There is also a parallel work [72] by the same authors for the massive $(1 + 1)-$d *fermionic* QFT called Gross-Neveu model, a theory with quartic interactions. Unlike free bosonic operators in the previous settings which commute and have easy-to-realize spatial localization, the localization of the free anticommuting fermionic operators is tricky and uses the technique developed by Bravyi and Kitaev [3], which we shall also use in section 3.5. The algorithm follows the same line of thought, where a free theory is excited and interaction term is adiabatically turned on and off and measurement is eventually made.

Note how the use of free field theory in both settings mirrors our use of Fock space in 3.5 to construct (interacting) CFTs as subtheories. It is difficult to realize interacting theories *directly* (but see 3.4.3 in this regard).

3.3 Unitary evolution simulation problem

Following functorial $(n+1)-$d QFT, unitary evolution of a system given by a manifold X of dimension n, is described by unitary maps assigned by the functor to elements inside the automorphism group of X in $(n+1)-$cobordism category $\mathrm{Aut}_{(n+1)-\mathrm{cob}}(X)$. In $(2+1)-$d TQFT, it is well-known that the mapping class group of a surface X is isomorphic to $\mathrm{Aut}_{(2+1)-\mathrm{cob}}(X)$; the proof is by simply using the cobordism cylinder X_ϕ construction for any diffeomorphism $\phi \in \mathrm{Diff}(X)$. Therefore, the unitary evolution is a representation of the mapping class group which is generated by braids and Dehn twists. Those are operators for which one can have an ultra-local expression. It is this point of view that is adopted when simulating the unitary evolution of TQFT and showing its **BQP**-completeness in [2], and we seek a similar characterization for unitary evolution in CFT.

In functorial CFT [73], the unitary evolution is guided by unitary maps called $U(\gamma)$. Therefore, our goal is to simulate $|(\Omega, U(\gamma)\Omega)|$, with $U(-)$ a positive-energy projective unitary representation of $\mathrm{Diff}_+(S^1)$ and γ a diffeomorphism in this Lie group. It is well-known ([74]) that the representation U corresponds to a unitary positive energy representation of the Virasoro algebra. Following a result of [74], there exist $f^{(j)}$s in $C^\infty_{\mathbb{R}}(S^1)$ such that $\gamma = \exp(f^{(1)}) \circ \cdots \circ \exp(f^{(k)})$, and further $U(\exp(f)) = e^{iL(f)}, \forall f \in C^\infty_{\mathbb{R}}(S^1)$. Accordingly, the simulation target becomes

$$|(\Omega, \prod_{j=1}^{k} e^{iL(f^{(j)})}\Omega)|. \tag{3.4}$$

Therefore, it makes sense to understand the *complexity* of an evolution infinitesimally, in terms of its decomposition to a product of $e^{iL(f)}$s, similar to how local unitary operations can be defined as (a product of) e^{iH} for local Hamiltonians H.

Theoretically, given any accuracy, (3.4) is computable as long as functions $f^{(j)}$s are computable. However, we can not assume *any* computable function in our definition of the problem. This is similar to how the problem of unitary evolution of quantum many-body systems only assumes local Hamiltonians that are a sum of ultra-local interaction terms. Thus, the set of *local* operators should be specified. Let $\mathcal{F}$ be the set of local operators among $e^{iL(f)}$s that can be considered in (3.4) as part of the inputs of the simulation problem.

Ideally, this set should be similar to its analog in other settings, where it is generated by an *ultra*-local generating set. It is believed that the ultra-local terms are *not* among $e^{iL(f)}$s; this argument is made in the next part on correlation functions (3.4.2) and Remark 3.4.1. Our investigation in this part is on the operators $e^{iL(f)}$s that should be among the *local* operators.

For example, the operator $e^{if_0L_0}$ which is the evolution by the CFT Hamiltonian corresponding to the constant function $f \equiv f_0$ is certainly one of the operators one would want in $\mathcal{F}$. More generally, $e^{i(\hat{f}_m L_m + \overline{\hat{f}_m} L_{-m})}$ corresponding to the real-valued function $f = \hat{f}_m e^{im\theta} + \overline{\hat{f}_m} e^{-im\theta}$ must be in $\mathcal{F}$ as well. Therefore, it is reasonable to ask a finite combination of these to be simulated efficiently. This means $e^{iL(f)} \in \mathcal{F}$ for f having finite Fourier series.

We can wonder which functions with infinite Fourier series can be considered for the simulation problem. An analogy in quantum computation, would be to think of a hermitian matrix H that may be non-local, but its non-local effect, by the non-local interaction terms, is exponentially small. This translates to a unitary operator which is non-local but has an action exponentially close to a local operator around some "centers" of action. In general, recall the Fourier coefficients $\hat{f}_n$ are rapidly decaying

$$\forall k, \exists N_k \text{ such that } \forall |n| \geq N_k \implies |\hat{f}_n| \leq \frac{1}{n^k}$$

but what the rate of this decay (or equivalently, the growth rate of N_k) should be is unclear. Perhaps, an exponentially decaying $\hat{f}_n$ (corresponding to analytic functions) or a polynomial growth for N_k is the answer. Computing the dependence of the rate of convergence of the scaling limit in Corollary 2.5.10 on N_k could help answer this question.

So far, we can safely assume that the set $\mathcal{F}$ has all operators corresponding to functions with finite Fourier series, hence, the following definition for the unitary evolution simulation problem.

Definition 3.3.1 *[**CFT UNITARY EVOLUTION**] **Inputs:** Given k and real-valued functions $f^{(j)}$ with n_j many Fourier coefficients for $1 \leq j \leq k$, each coefficient $(\hat{f^{(j)}})_m$ computable in polynomial time $p(m)$ with respect to m using a classical Turing machine.*

***Output:** An approximation of the following up to an error ϵ, with complexity measured with respect to $\{\frac{1}{\epsilon}, n_1, \ldots, n_k, k\} \cup \{(\hat{f^{(j)}})_m\}_{j,m}$*

$$|(\Omega, \prod_{j=1}^{k} e^{iL(f^{(j)})}\Omega)|. \tag{3.5}$$

The conjecture in the same spirit of TQFT is

Conjecture 3.3.1 ([4], Conjecture 5.6) *CFT UNITARY EVOLUTION is in **BQP**. Generically, the problem is **BQP**-complete.*

In TQFT on the sphere, unitary evolution on the vacuum is trivial, as it is a one dimensional space. In CFT, due to the existence of descendants provided by L_{-m} for $m > 1$, we have a nontrivial problem in an infinite dimensional Hilbert space, the unitary Virasoro VOA $\mathcal{V}_{c,0}$.

Remark 3.3.1 *One can generalize the above to other sectors $\mathcal{V}_{c,h}$ or even replace Ω with the primary fields other than the vacuum in the VOA. Thus, the generalization of*

Definition 3.3.1 includes the expectation value for any primary state ξ instead of Ω. We will address this in section 3.5, using the Fock space construction of VOAs.

The AC approach provides some motivation for which operators have to be in $\mathcal{F}$. Further, it gives a discretized picture of what unitary evolution looks like in a CFT: Take many particles on a chain where it is allowed to have fusion between nearby particles with certain penalties for the undesired fusion. The value of the penalties is what gives the function f. If one lets the system evolve in this setting, the unitary evolution guided by those constraints is $e^{i\widetilde{L}(f)}$. If f is the constant function, then the CFT Hamiltonian is recovered.

But the AC approach may not be the right one. To approximate (3.4) using ACs, we first need a proof that AC gives the VOA in the scaling limit. Furthermore, a bigger obstacle could be proving the **BQP**-*completeness*, as the expression for approximating the operators $e^{iL(f)}$ are exponentials of weighted sum of **all** e_is. It is hard to build specific unitary operators, like a nice universal gate set, without a more detailed understanding of $e^{i\widetilde{L}(f)}$s.

Also note that L_ns are considered to be local terms, but $L_n = Y(\omega, f)$ is given by the smearing (position) function $f(e^{i\theta}) = e^{in\theta}$. Fourier transforms send locality in spacetime to that of energy-momentum. If one considers $Y(\omega, z)$ as (ultra-)local in space basis, then any of its Fourier modes, like L_n, should be viewed as (ultra-)local in an energy basis (based on energy local degrees of freedom). We speculate that it is unlikely that the problem defined above has an algorithm based on a computational basis that uses a discretization of space(time), like AC (or other statistical 2d lattice models).

Finally, one needs to prepare the vacuum which is also a nontrivial problem (see [75] for the case of Ising). By using a basis based on energy, this issue is non-existent, as demonstrated later in section 3.5 using the Fock space construction of VOAs.

3.3.1 Alternative approach

The first alternative approach is to use a combinatorial realization of the action of Virasoro generators called path representations, originally developed to compute the character of VOAs involving counting the dimension of each weight space. Path representations of the states of the Hilbert space and how the Virasoro algebra acts on these paths can be analyzed. We refer to [76] for nonunitary minimal models $M(2,q)$ (with q odd) where the action of every Virasoro generator is obtained, and [77] for unitary minimal models, where actions of higher Virasoro generators is not known yet.

Another approach would be to derive an exact expression for $|(\Omega, U(\gamma)\Omega)|$, which would be easier to think about from a computational perspective. Exact expression can be obtained for all free models [78, see Theorem 6.2.3 and section 7], but they are not suitable for computation, and no such *closed formula* is known for higher (interactive) minimal models.

3.3.2 Fock space construction of CFTs

A more promising approach is investigated in section 3.5, with results under certain conditions (Theorem 3.5.3), using large enough tensor power of the *Fock space*. This contains many interacting models as subtheories, including all $\mathfrak{su}(2)_k$ WZW models and all minimal models (see [79, section 4] for a list). The Fock space is a free theory given by $(\chi_0 + \chi_{\frac{1}{2}})(\overline{\chi}_0 + \overline{\chi}_{\frac{1}{2}})$ and can be modelled using an ultra-local realization of the Dirac operators $\Psi_k, \overline{\Psi}_k$s ([3]).

Let us briefly discuss the idea behind this ultra-localization following [72, section 3.2]. The naive localization attempt of Dirac operators corresponds to the mapping $\Psi_{-k_1} \dots \Psi_{-k_l}\Omega \to |0 \dots 1 \dots 1 \dots 0\rangle$ where $0 < k_1 < \dots < k_l < n$ and the one states are used for positions $k_i + \frac{1}{2}$ only. We call the occupation number for $k-$th position o_k, which

is 1 if $\Psi_{-k+\frac{1}{2}}$ is among the acting operators on the vacuum and 0 otherwise. To know the action of any operator Ψ_{-m}, we generally need to know the parity of total occupation number $t_m = \sum_{i \leq m+\frac{1}{2}} o_i$, as the Dirac operators anticommute to put Ψ_{-m} in its place in the ordered sequence $k_1 < \ldots < k_l$. Thus, the operator is not ultra-local as its action in the worst case is dependent on all qubits. Bravyi and Kitaev [3] use the following trick to compute the parity of t_m in $O(\log(n))$ with a clever ordering of the mode indices. First, represent any mode index $k_i + \frac{1}{2} \in \{1, \ldots, n\}$ by its bit string of length $s = \lceil \log(n) \rceil$. Then consider the following partial ordering: $x = x_s \ldots x_1 \preceq y = y_s \ldots y_1$ if for some r, $x_i = y_i$ for $i > r$ and $y_{r-1} = y_{r-2} = \ldots = y_1 = 1$. Now let $n_j = \sum_{r \preceq j+\frac{1}{2}} o_r$. Any total occupation number $t_m = \sum_{r \leq m+\frac{1}{2}} o_j$ can be computed from the n_j quantities in $O(\log(n))$ time and changing the occupation number of any mode o_j requires updating only $O(\log(n))$ of the n_j quantities.

Using the above localization, it can be shown that a local expression for the Virasoro generators of any unitary subtheory can be derived using the Dirac operators. By some energy truncation and taking the scaling limit, one should compute the rate of convergence and show that it is efficient. This could provide a faster convergence than AC; indeed, CFT has quantized energy but continuous spacetime. Using the Fock space means using the energy local degrees of freedom as the local basis for quantum computation, instead of the space local degrees of freedom as in AC.

3.4 Correlation function simulation problem

The second problem is the simulation of the correlation functions of CFT. Similar to TQFT ([2, 80]), we have insertions of n fields (like a fusion tree in a Temperley-Lieb diagram) followed by a unitary operator (braiding) and the probability of getting back to some state (inner product with another fusion tree). Loosely speaking, for any QFT,

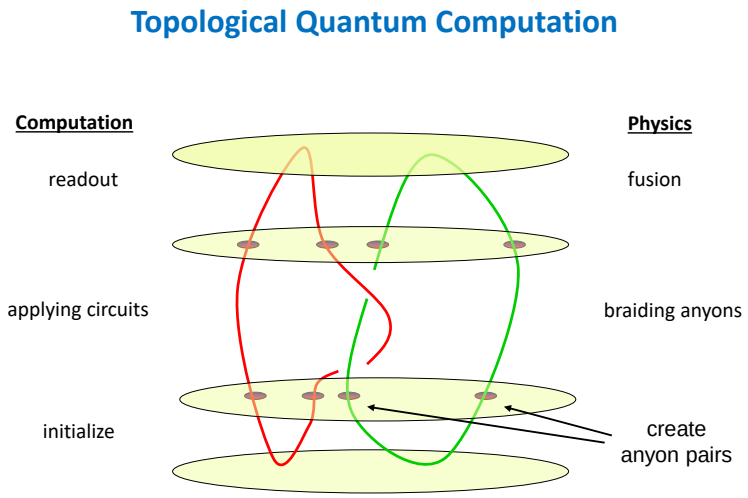

Figure 3.1: Time flows from bottom to top ([58, Figure 3]).

the following two seem to be our target

$$|(\prod_{j\in S'}\rho_j(w_j)\Omega, U\prod_{i\in S}\phi_i(z_i)\Omega)|, \tag{3.6}$$

$$|(\prod_{j\in S'}\rho_j(g_j)\Omega, U\prod_{i\in S}\phi_i(f_i)\Omega)|, \tag{3.7}$$

with U some local unitary operator, ϕ_i, ρ_j some fields from set of sectors S, S', and insertion points or smearing functions z_i, w_j (f_i, g_j).

As quantum computation is usually performed in the Schrödinger picture, we pick our insertion points or smearing functions so that they do not have any explicit parameter of time. In the case of $(1+1)-$d CFT, this means insertion points are on the unit circle S^1 (space at present time in the radial setting) and smearing functions are smooth complex-valued functions on the unit circle, i.e. inside $C^\infty_{\mathbb{C}}(S^1)$. It is only through U that time

evolution occurs. In the next parts, we will address the following questions:

- Ideally, both quantities above are probability amplitudes between 0 and 1, but they are not so by default. What is the correct normalization?
- What is the nature of the computational basis in each version?
- What are the set of ultra-local unitary gates used to construct the unitary U?

These issues are discussed for both versions of the correlator for chiral CFT, but all the definitions and ideas can be readily generalized to full CFT.

3.4.1 Simulation of the point-like correlation function

Throughout this and the next part, we will assume a nice VOA: unitary, CFT-type, rational (including C_2-co-finite), will all intertwiners being energy-bounded. Given our knowledge so far, our simulation target seem be the following

$$|\Big(\mathcal{Y}_{\widetilde{l}}(b^{(l)},w_l)\dots\mathcal{Y}_{\widetilde{1}}(b^{(1)},w_1)\Omega,U(\gamma)\mathcal{Y}_n(a^{(n)},z_n)\dots\mathcal{Y}_1(a^{(1)},z_1)\Omega\Big)|, \tag{3.8}$$

where $|w_j|=|z_i|=1$, $a^{(i)}\in A_i, b^{(j)}\in B_j$ are hermitian primary fields in the irreducible modules, and $\mathcal{Y}_i,\mathcal{Y}_{\widetilde{j}}$ are of type $\binom{C_i}{A_i\ C_{i-1}},\binom{\widetilde{C}_j}{B_j\ \widetilde{C}_{j-1}}$ with irreducible modules $C_i,\widetilde{C}_j$. To have a nonzero result, we need $\widetilde{C}_0=C_0=\mathcal{V}$ and $\widetilde{C}_l=C_n$.

Even if intuitively the VOA mathematical framework of CFT should easily allow the computation of such correlation, it involves some analysis (carried out in section 3.6) to show that (3.8) is well-defined. Note that correlators are generally well-defined only when insertions are time-ordered, i.e. $|z_1|<\ldots<|z_n|$, and similarly for $|w_i|$s. Also, observe that singularities in (3.8) need to be avoided. For example $z_i\neq z_j$. This means $(z_i)_i\in \mathrm{Conf}_n(S^1)$ and similarly $(w_j)_j\in\mathrm{Conf}_l(S^1)$, where we recall the configuration space for $X\subset\mathbb{C}$ is defined as $\mathrm{Conf}_n(X)=\{(x_1,\dots,x_n)\in X\mid x_i\neq x_j\}$.

This correlation function is not a probability amplitude. In QFT, this is interpreted as an evaluation of some probability density function on the continuous space. Thus one expects the integral of (3.8) over the space of eligible insertions points (non-singularities) with some measure to be finite. However, this is not clear even in the case of a simple Ising fermion correlation function with γ the identity diffeomorphism, where $\int\int |\frac{1}{w_1-z_1}|^2 = \infty$. Generally speaking, in QFT, the rigorous mathematical formulation of the probability density function is not known, even though it is obvious it involves the term (3.8). Without this, it is not possible to define this version of the simulation problem. Of course, once the density function is known, it is a valid mathematical question whether sampling such density function or computing it directly can be simulated efficiently using a quantum computer.

Assuming a rigorous formulation of the problem, one can speculate on the nature of the computational basis of the quantum algorithm. Since the ultra-localization is happening with respect to *continuous* space, with uncountably many points, the computational basis likely *approximates* $\mathcal{Y}_n(a^{(n)}, z_n) \dots \mathcal{Y}_1(a^{(1)}, z_1)\Omega$, with the help of some lattice and scaling limit.

Approximating CFTs with finite anyonic systems can be done through scaling limit. However, as demonstrated in the previous chapter, the space embedding in scaling limit is non-local. Furthermore, it is unclear how to approximate the intertwiners on the anyonic chains, as our success has been limited to the trivial intertwiner, i.e. the vertex operator $Y() = \mathcal{Y}\binom{\mathcal{V}}{\mathcal{V}\ \mathcal{V}}$. Even in this case, there is much to be done to ensure a scaling limit algebra convergence (see section 2.5.3).

In (3.8), the unitary is set as $U(\gamma)$. It is shown later in Lemma 3.6.14 by using conformal covariance, that fields inserted at z_i essentially move to $\gamma(z_i)$, making our

target

$$|\Big(\mathcal{Y}_{1^\dagger}(\eta_1(a^{(1)}),\gamma(z_1))\dots\mathcal{Y}_{n^\dagger}(\eta_n(a^{(n)}),\gamma(z_n))\mathcal{Y}_{\tilde{l}}(b^{(l)},w_l)\dots\mathcal{Y}_{\tilde{1}}(b^{(1)},w_1)\Omega,U(\gamma)\Omega\Big)|. \quad (3.9)$$

Thus, avoiding the singularities also requires $\gamma(z_i)\neq w_j$. Note $\eta_i(a^{(i)})=a^{(i)}$, as the primary fields are chosen to be hermitian. As alluded to previously, it is believed that $U(\gamma)$ are local but not ultra-local unitary operators, implying the possibility of having unitaries other than $U(\gamma)$ in (3.8). This is discussed in more details in the next part.

3.4.2 Simulation of the smeared correlation function

A smearing function is interpreted as the probability density function of the position of the particle. As the formal definition below suggests:

$$\mathcal{Y}(a,f):=\oint \mathcal{Y}(a,z)f(z)z^{\text{wt }a+\tau}\,\frac{\mathrm{d}z}{2\pi i z}=\sum_{n\in\mathbb{Z}}\hat{f}_n(a)_n,$$

the definition of the smeared correlator should be the probability *distribution* that comes from the weighted integration of the probability *density* functions (point-like insertion) by the smearing functions over S^1. This interpretation is formalized in [31, Proposition 3.12] and Theorem 3.6.12.

Based on (3.7), our simulation target seems to be the following

$$|\Big(\mathcal{Y}_{\tilde{l}}(b^{(l)},g_l)\dots\mathcal{Y}_{\tilde{1}}(b^{(1)},g_1)\Omega,U(\gamma)\mathcal{Y}_n(a^{(n)},f_n)\dots\mathcal{Y}_1(a^{(1)},f_1)\Omega\Big)|, \quad (3.10)$$

with $f_i,g_j\in C^\infty_{\mathbb{C}}(S^1)$. For the point-like version, the correlation function had to be shown that it is well-defined. Here, as all intertwiners are energy-bounded, both vectors involved in the inner product are smooth vectors inside $\mathcal{V}_{(\infty)}$, and this is a usual inner product of

the Hilbert space $\overline{\mathcal{V}}$. This also allows us to perform the normalization in order to have an actual probability amplitude

$$\frac{|\Big(\mathcal{Y}_{\tilde{l}}(b^{(l)},g_l)\dots\mathcal{Y}_{\tilde{1}}(b^{(1)},g_1)\Omega, U(\gamma)\mathcal{Y}_n(a^{(n)},f_n)\dots\mathcal{Y}_1(a^{(1)},f_1)\Omega\Big)|}{||\mathcal{Y}_{\tilde{l}}(b^{(l)},g_l)\dots\mathcal{Y}_{\tilde{1}}(b^{(1)},g_1)\Omega||.||\mathcal{Y}_n(a^{(n)},f_n)\dots\mathcal{Y}_1(a^{(1)},f_1)\Omega||}. \tag{3.11}$$

In the point-like discussion, we observed that the computational basis should *approximate* the continuous space basis. This makes the analysis far more difficult and is not the best approach given that CFTs naturally arise as the low *energy* behavior of increasingly large size quantum theories. Further, unlike space, energy is quantized. As quantum computation occurs in finite discrete spaces, a computational basis that relates to energy is more desirable, which is the case for the smeared version. The functions f_i, g_j are generated by Fourier modes $e^{in\theta}$, and one can think of $\mathcal{Y}(a, e^{-in\theta})\Omega = (a)_{-n}\Omega$ as the ultra-local insertion of a primary field in the energy basis, where the smearing function f, i.e. the position probability density function, is $e^{-in\theta}$. Thus the possible values for f_i, g_j are the Fourier modes $e^{ik\theta}$. For Ising, this means our computational basis are excitations of the vacuum with Dirac Ψ or spin σ operators depending on the sector, e.g. $\Psi_{-k_n}\dots\Psi_{-k_1}\Omega$.

With a judicious choice of the basis, one should be able to prepare these states quickly, perhaps in constant time, similar to the case of TQFT [7, Chapter 7.2-7.3]. As an example, using Fock space construction of a CFT, the operators Ψ_{-k} can be ultra-localized [3] and the computational basis $\{\Psi_{-k_n}\dots\Psi_{-k_1}\Omega\}$ can be constructed efficiently.

In terms of the unitary evolution, one can rewrite (3.10) as

$$|\Big(\mathcal{Y}_{1^\dagger}(\eta_1(a^{(1)}),\overline{\beta_{d_{a^{(1)}}}(\gamma)(f_1)})\dots\mathcal{Y}_{n^\dagger}(\eta_n(a^{(n)}),\overline{\beta_{d_{a^{(n)}}}(\gamma)(f_n)})\mathcal{Y}_{\tilde{l}}(b^{(l)},g_l)\dots\mathcal{Y}_{\tilde{1}}(b^{(1)},g_1)\Omega, U(\gamma)\Omega\Big)|. \tag{3.12}$$

The adjoint action $\beta_{d_a}(\gamma)$ is defined in (3.86) and this rewriting follows from the conformal covariance of the smeared field. The unitaries $U(\gamma)$ are all symmetries of the theory and are products of exponentiation of a smeared *quasi*-primary field $L(f)$, not a primary field. This leads us to speculate on whether other physical unitaries could replace U in (3.7), which generate $e^{iL(f)}$s.

As mentioned in [3], in a fermionic system with computational basis $\{\Psi_{-k_n} \dots \Psi_{-k_1}\Omega\}$, the physical unitary operators involve:

- $e^{i\beta\Psi_{-j}\Psi_j}$ (action by an external potential),
- $e^{i\beta\Psi_{-j}\Psi_j\Psi_{-k}\Psi_k}$ (two-particle's interaction),
- $e^{i(\alpha\Psi_{-j}\Psi_k+\overline{\alpha}\Psi_{-k}\Psi_j)}$ (tunneling),
- $e^{i(\alpha\Psi_k\Psi_j+\overline{\alpha}\Psi_{-j}\Psi_{-k})}$ (interaction with a superconductor),

where $\beta \in \mathbb{R}, \alpha \in \mathbb{C}$. These operators generate a universal gate set. As a result, if one considers such operators as physical in Ising CFT, the simulation of Ising CFT correlation function is **BQP**-hard.

Inspired from this, the following physical unitaries may be what the *ultra*-local unitary gates should be. They are the exponentiation of the ultra-local field insertions of some hermitian primary field a, i.e. $\mathcal{Y}(a, e^{in\theta}) = (a)_n$, and their adjoint:

$$\{e^{i\beta(a)_n^\dagger(a)_n}, e^{i(\alpha(a)_n^\dagger(a)_n(a)_m^\dagger(a)_m+\overline{\alpha}(a)_m^\dagger(a)_m(a)_n^\dagger(a)_n)}, e^{i(\alpha(a)_n^\dagger(a)_m+\overline{\alpha}(a)_m^\dagger(a)_n)}, e^{i(\alpha(a)_n(a)_m+\overline{\alpha}(a)_m^\dagger(a)_n^\dagger)}\}. \tag{3.13}$$

The intertwiner $\mathcal{Y}$ is of type $\binom{C}{A\ B}$ given the sector B on which the unitary is supposed to act on. We define $U_{\mathcal{V}} = \bigcup_{\text{hermitian primary fields}} U_a$, where U_a is the set given above for all $\beta \in \mathbb{R}, \alpha \in \mathbb{C}$.

Remark 3.4.1 *It is important to note that* $U(\gamma) = \prod e^{iL(f_j)}$ *simulation would likely require the ability to simulate the unitaries above; the Virasoro field* $L(f)$ *in all UMMs or WZW models is generated by some other primary fields (we refer to Remarks 1.3.12 and 1.3.14 for the formulae). As a result of the Borcherds identity, the formula for* $L_n + L_{-n}$ *is an infinite hermitian sum of terms like* $(a)^{\dagger}_{n+m}(a)_m$ *for all* $m \in \mathbb{Z}$*, as we saw in the case of Ising when writing* $L_n + L_{-n}$ *as a sum of* $\Psi_{-k-n}\Psi_k$ *and their adjoints in eq. (2.5). Even though this sum is infinite, when restricted to any energy cut-off* E*, it only involves poly*(E) *many terms. Thus, it makes sense to set the unitary gates as proposed above, and try to solve the simulation problem with these unitaries. Given that, some analysis similar to section 3.5 for unitary evolution (a generalization of the Trotter-Kato approach), should follow to approximate* $e^{iL(f)}$*s.*

Definition 3.4.1 (CFT Correlation Function Simulation Problem)

Inputs*: Given error rate* $\epsilon > 0$*, hermitian primary fields* $\{b^{(j)}\}_{j=1}^{l}, \{a^{(i)}\}_{i=1}^{n}$*, with modes* m_j, k_i *and intertwiners* $\mathcal{Y}_{\widetilde{j}}, \mathcal{Y}_i$ *of type* $\binom{\widetilde{C}_j}{B_j\ \widetilde{C}_{j-1}}, \binom{C_i}{A_i\ C_{i-1}}$ *where* $B_j, \widetilde{C}_j, A_i, C_i$ *are irreps of* $\mathcal{V}$ *satisfying* $\widetilde{C}_0 = C_0 = \mathcal{V}$ *and* $\widetilde{C}_l = C_n$*, and finally a unitary* U *description as a product of* N *unitaries* $U_{\mathcal{V}}$ *with all chosen modes used to describe* U *forming a set* M_U*.*

Output*: An approximation of the normalized correlation function below with error rate* ϵ *with complexity measured with respect to* $\{m_l, \ldots, m_1, k_n, \ldots, k_1, \frac{1}{\epsilon}, N\} \cup M_U$ *where* $k_i, m_j \geq 0$*.*

$$\frac{|\Big(\mathcal{Y}_{\widetilde{l}}(b^{(l)}, e^{-im_l\theta}) \ldots \mathcal{Y}_{\widetilde{1}}(b^{(1)}, e^{-im_1\theta})\Omega, U\mathcal{Y}_n(a^{(n)}, e^{-ik_n\theta}) \ldots \mathcal{Y}_1(a^{(1)}, e^{-ik_1\theta})\Omega\Big)|}{||\mathcal{Y}_{\widetilde{l}}(b^{(l)}, e^{-im_l\theta}) \ldots \mathcal{Y}_{\widetilde{1}}(b^{(1)}, e^{-im_1\theta})\Omega||.||\mathcal{Y}_n(a^{(n)}, e^{-ik_n\theta}) \ldots \mathcal{Y}_1(a^{(1)}, e^{-ik_1\theta})\Omega||}. \tag{3.14}$$

Conjecture 3.4.1 *The CFT Correlation Function Simulation Problem is in* ***BQP*** *and generically* ***BQP****-complete.*

Note we did not manage to simulate the spin field σ and its modes for Ising CFT. As

mentioned previously, it was shown in [3] to be **BQP**-complete if one only uses the fermionic field Ψ. Thus the problem above for Ising is known to be **BQP**-hard.

3.4.3 Alternative approach

CFT in nature arises from finite systems in some limit, and as such, it is always the case that CFT in the continuum is the *approximation* of physical reality and not itself. The mathematical frameworks for this continuum provide many tools to study the large size finite models in their low energy spectrum, similar to how calculus can be thought of as a powerful tool to study discrete functions defined at many points. Therefore, one may ask whether these facts need to be accounted for in the simulation problem. If so, the strictly *physical* version of the problem is likely easier as there is no need for a potentially difficult convergence analysis. Of course, mathematical curiosity requires that it be formulated also in the continuum, and the convergence rate of discrete to continuum rigorously be studied.

However, we need a *canonical* definition of *discrete* CFT as there is usually a wide range of lattice models with the same CFT in the scaling limit, otherwise there is no natural discrete version of the CFT simulation problem. One can speculate that this discretization is provided by some $q-$deformation of the symmetry algebra, the Virasoro algebra. There are many attempts in this area [12, 81, 82], but none provide the framework we want to.

This suggests a new approach to the whole CFT and the simulation problem, where one needs to formulate the $q-$deformation of the Virasoro algebra and study its representation theory as the discrete CFT. This should be followed by a study of the limit (and convergence rate) when $q \to 1$.

3.5 Unitary evolution simulation using Fock space construction

Define the Fock space as $\mathcal{H} = (\chi_0 + \chi_{\frac{1}{2}})(\overline{\chi}_0 + \overline{\chi}_{\frac{1}{2}})$ with Hamiltonian

$$\mathbb{L}_0 = L_0 + \overline{L}_0 = \sum_{k \in \mathbb{N} - \frac{1}{2}} k \Psi_{-k} \Psi_k + \sum_{k \in \mathbb{N} - \frac{1}{2}} k \overline{\Psi}_{-k} \overline{\Psi}_k$$

and character $\prod_{k \in \mathbb{N} - \frac{1}{2}} (1 + q^k)^2$. The $\Psi_k, \overline{\Psi}_k$s are the Dirac operators of the left and right moving sectors. They act as $\Psi_k \otimes \mathbf{1}$ and $\mathbf{1} \otimes \Psi_k$, respectively. For this reason, they are relabelled by (commuting operators) Ψ_k^L, Ψ_k^R.

3.5.1 Construction of the restrictions of Virasoro modes

The first steps of our approach starts by building this Fock space as a strong scaling limit of finite spaces. Then, restricting to a desired subtheory gives restrictions of the Virasoro field to finite subspaces. We can divide this construction into the following steps:

(a) First, consider the tensor product $\mathcal{H}_n = \mathcal{H}_n^L \otimes \mathcal{H}_n^R$, where each $\mathcal{H}_n^{L,R}$ with dimension 2^n is generated by the action of n Dirac operators on their *vacuums* $\Omega_n^{L,R}$. The n creation Dirac operators are $\Psi_{-k}^{L,R}$ with $k \in S = \{\frac{1}{2}, \frac{3}{2}, \ldots, n - \frac{1}{2}\}$ and their adjoints are the annihilation operators, $\Psi_k^{L,R}$ acting on $\mathcal{H}_n^{L,R}$, respectively.

(b) Each $\mathcal{H}_n^{L,R}$ is given a special basis ([3]) using qubits $(\mathbb{C}^2)^{\otimes n}$, so that the Dirac operators are all *ultra*-local operators acting nontrivially on $O(\log(n))$ many qubits. In particular, any bilinear expression in terms of Dirac operators is a local operator. If hermitian and acting on a finite dimensional Hilbert space, its evolution can be simulated efficiently on a quantum computer. This will be the case for the

approximation of the hermitian operator $L(f)$. Moreover, this construction leaves the vacuum $\Omega_n = \Omega_n^L \otimes \Omega_n^R$ as a basis element $|00\ldots0\rangle \otimes |00\ldots0\rangle$, making vacuum preparation trivial.

(c) Define the Hamiltonian $H_n = \sum_{k\in S} k\Psi^L_{-k}\Psi^L_k + \sum_{k\in S} k\Psi^R_{-k}\Psi^R_k$. We have a **strong** scaling limit to $(\mathcal{H}, \mathbb{L}_0)$ in which the vacuum Ω_n is sent to the vacuum $\Omega \in \mathcal{H}$. More precisely, $H_n = \mathbb{L}_0 P^n|_{\mathcal{H}_n}$ where P^n is the projection of $\mathcal{H}$ onto $\mathcal{H}_n$ and thus H_n can be extended to $\mathcal{H}$ (by zero) using $\mathbb{L}_0 P^n$. Note that $P^n\mathbb{L}_0 P^n = \mathbb{L}_0 P^n$, as $\mathbb{L}_0$ does not increase the energy.

(d) The above generalizes. For *any* operator O formed by *bilinear* terms of Dirac operators each of which decreases the energy (like $\mathbb{L}_m$ with $m > 0$), the operators $P^nOP^n = OP^n$ and $P^nO^\dagger P^n = P^nO^\dagger$ (like $\mathbb{L}_{-m}$ with $m > 0$) are both *local* operators acting on $\mathcal{H}_n$ with scaling limit O and $O^\dagger$. Indeed, OP^n ($P^nO^\dagger$) is exactly given by the bilinear terms of O ($O^\dagger$) that do not include any Dirac operator with mode higher than n or lower than $-n$.

(e) By abuse of notation, we will use the same notation P^n for the projection onto $\mathcal{H}_{n,p}$ which is the subspace of $\mathcal{H}^{\otimes p}$ with elements of total energy smaller than n. This is *strictly* contained in $\mathcal{H}_n^{\otimes p}$; e.g. for $n = 3, p = 2$, we have $\Psi^L_{-\frac{5}{2}}\Omega_3 \otimes \Psi^L_{-\frac{3}{2}}\Omega_3 \in \mathcal{H}_3^{\otimes 2}$ but it is not in $\mathcal{H}_{3,2}$ since its energy is higher than 3.

(f) The facts mentioned in step (d) hold for tensor powers of $\mathcal{H}$. Indeed, this applies to Virasoro modes of any subtheory (as mentioned below) of $\mathcal{H}^{\otimes p}$.

(g) Many chiral CFTs (including minimal models) are unitary subVOAs of some $\mathcal{H}^{\otimes p}$; see [79, p.42-43]. The tensor power p depends on the subVOA $\mathcal{V}$, thus is a constant in the simulation problem. $\mathcal{V}$ inherits the grading of $\mathcal{H}^{\otimes p}$ and its conformal vector

is the projection of that of $\mathcal{H}^{\otimes p}$ to $\mathcal{V}$ ([25, proposition 5.29] and [79]). The conformal vector $\omega \in \mathcal{V}_2$ is inside the weight 2 subspace of $\mathcal{H}^{\otimes p}$, which has an explicit orthonormal basis formed by all creation Dirac operators applied on the vacuum $\Omega^{\otimes p}$ $\Omega^{\otimes p}$ (which is denoted by Ω by abuse of notation):

$$\{v_i\} = \{\Psi_{-k}^{(L,R),j}\Psi_{-k'}^{(L,R),j'}\Omega \ | k' = \frac{1}{2}, k = \frac{3}{2}, 1 \leq j, j' \leq p\} \tag{3.15}$$

with weight $-\frac{1}{2}, -\frac{3}{2}$, coming from the $j, j'-$th copy in the tensor power and belonging to either of the (L, R) sectors. The basis has $(2p)^2$ elements v_i. Given the VOA, for some computable constants μ_i in constant time, we have

$$\omega = \sum_i \mu_i v_i. \tag{3.16}$$

(h) The restriction of the vertex operator $Y(-, z)$ of $\mathcal{H}$ to $\mathcal{V}$ is the vertex operator of the latter, as $\mathcal{V}$ is a subVOA of $\mathcal{H}$. Taking this on both sides of the above equation, we obtain the Virasoro field for $\mathcal{V}$. On the other side, for each v_i, $Y(v_i, z)$ at each mode is a bilinear expression of Dirac operators. More precisely, each v_i defined in 3.15 is either a tensor product of $\Psi_{-\frac{1}{2}}\Omega$ and $\Psi_{-\frac{3}{2}}\Omega$ (when $j \neq j'$ or Ψs are not of the same chirality L or R), or is of the form $\Psi_{-\frac{3}{2}}\Psi_{-\frac{1}{2}}\Omega$ (when $j = j'$ and both Ψs are of the same chirality L or R). The vertex operator corresponding to these are respectively linear (fermionic field $\Psi_{-\frac{1}{2}}\Omega$) and linear (derivative of the fermionic field $\Psi_{-\frac{3}{2}}\Omega$), and bilinear (the Ising model Virasoro field $\frac{1}{2}\Psi_{-\frac{3}{2}}\Psi_{-\frac{1}{2}}\Omega$). Thus the Virasoro mode L_m of $\mathcal{V}$ will also be bilinear in Dirac operators. Since $\mathcal{V}$ is a *unitary* subVOA, its hermitian form is also a restriction of that of $\mathcal{H}^{\otimes p}$, and we are completely familiar with the hermitian form of the latter. Therefore, due to steps (d-f), for $m \geq 0$, The *hermitian* operator $P^n L_{-m} + L_m P^n$ acts on $\mathcal{H}_{n,p}$

as a local operator. Notice, there is no error $O(\frac{1}{n})$ like in the previous chapter approximations, as we are directly taking the restriction of L_m to some subspace.

Remark 3.5.1 *Unless $\mathcal{V} = \mathcal{H}^{\otimes p}$, the subspace $\mathcal{H}_{n,p}$ is strictly larger than $\mathcal{V}^n$, the subspace of energy at most n, described by the VOA Hamiltonian L_0, and does not give $\mathcal{V}$ in the scaling limit. This does not matter for simulation purposes, and is modelled by what is usually called a junk J^n computational subspace $\mathcal{V}^n \oplus J^n = \mathcal{H}_{n,p}$. The finite local version of the Virasoro modes $L_m P^n, P^n L_{-m}$ for $m > 0$ are converging to L_m, L_{-m} where their action is extended to $\mathcal{H}^{\otimes p}$. Furthermore, $L_m P^n, P^n L_{-m}$ act on the whole $\mathcal{H}^{\otimes p}$ by definition, though they preserve $\mathcal{H}_{n,p}$. This allows us to compare them directly to $L_{\pm m}$ without any artificial push-forward, as was the case in the previous chapter using some scaling limit construction. Finally, note the vacuum for $\mathcal{V}$ is also Ω, which means no preparation for the vacuum is required for simulation as mentioned in (b). These last two points are significant advantages that this approach provides over that of AC.*

3.5.2 Convergence of unitary evolution on Fock space

With the previous settings $\mathcal{V} \hookrightarrow \mathcal{H}^{\otimes p}$, and Ω being the vacuum, let us consider a finite sum of the Virasoro modes $L(f) = \sum_{|m|\leq N} \hat{f}_m L_m$ where $f(e^{i\theta}) = \sum_{|m|\leq N} \hat{f}_m e^{im\theta} \in C^\infty_{\mathbb{R}}(S^1)$ has a finite Fourier series.

First, we try a very simple case of Conjecture 3.3.1. We will generalize our method and mention the obstacles to proving the full version of the conjecture.

Theorem 3.5.1 *Given $f \in C^\infty_{\mathbb{R}}(S^1)$ with finitely many Fourier coefficients, a quantum computer can approximate the following up to an error ϵ in polynomially many steps in $\frac{1}{\epsilon}$*

$$|(\Omega, e^{iL(f)}\Omega)|. \tag{3.17}$$

In the above, the complexity of the algorithm is only measured with respect to the error. We do *not* claim the above can not be **exactly** computed (see [83] for the similar problem in TQFT which is **#P**-hard). But as far as we are aware of the literature, there is a lack of an exactly computable description of this simplest case of unitary evolution.

We set $\hat{f}_m = 0$ for $|m| > N$. Define the unitary $e^{iL^n(f)}$ given by the hermitian operator

$$L^n(f) = \sum_{m<0} \hat{f}_m P^n L_m + \sum_{m\geq 0} \hat{f}_m L_m P^n \tag{3.18}$$

acting on the finite dimensional Hilbert space $\mathcal{H}_{n,p}$. From the construction steps outlined in 3.5.1, $e^{iL^n(f)}$ can be simulated efficiently on a quantum computer with $2np$ many qubits, where p depends on $\mathcal{V}$. Hence, it suffices to prove that for n larger than some constant C,

$$||(e^{iL^n(f)} - e^{iL(f)})\Omega|| < \frac{1}{n}, \tag{3.19}$$

and take $n = \max(\lceil C \rceil, \lceil \frac{1}{\epsilon} \rceil + 1)$ to finish the proof.

Our arguments will be based on estimations found in [84] regarding the action of $e^{iL(f)}$. Below, we introduce and recall some notations and the relevant inequalities.

Notations. Let $A = L_0 + \mathbf{1}$ and $\mathcal{V}_{(s)}$ for $s \geq 0$ be all $\xi \in \overline{\mathcal{V}}$ such that the $s-th$ *norm* is finite (not to be confused with $\mathcal{V}_s$, the energy s eigenspace):

$$||\xi||_s := ||A^s \xi|| = ||(L_0 + \mathbf{1})^s \xi|| < \infty \tag{3.20}$$

Notice $||\cdot||_0 = ||\cdot||$ the norm on $\mathcal{V}$. Let $\mathcal{V}_{(\infty)} = \cap_s \mathcal{V}_{(s)}$ to be the set of *smooth* vectors for A . As $\mathcal{V}_{(s)} \subset \mathcal{V}_{(s')}$ for $s > s'$, clearly $\mathcal{V} \subset \mathcal{V}_{(\infty)}$. $||.||_s$ gives two continuous semi-norms on

skew-adjoint operators X (which will be assumed to be some $iL(f)$)

$$|X|_{s+1} = \sup_{\xi \in \mathcal{V}_\infty} \frac{||X\xi||_s}{||\xi||_{s+1}}, \tag{3.21}$$

$$|X|_{A,s+1} = \sup_{\xi \in \mathcal{V}_\infty} \frac{||[A,X]\xi||_s}{||\xi||_{s+1}}. \tag{3.22}$$

Finally, recall the $r-$th norm for $r \geq 0$,

$$||f||_r := \sum_m |\hat{f}_m|(|m|+1)^r. \tag{3.23}$$

There are two obvious observations that will be used implicitly in our estimates

$$||f'||_r = \sum_m |m\hat{f}_m|(|m|+1)^r \leq ||f||_{r+1} \ , \ ||(L_0 + \mathbf{1})\xi||_s = ||\xi||_{s+1}. \tag{3.24}$$

Let us prove the above seminorms are finite for smeared Virasoro fields. As the VOA is unitary, the representation of Virasoro algebra gives an energy bounded conformal field ([25, 84]), i.e.

$$||L(f)\xi|| \leq C_\omega ||f||_{r_\omega} ||\xi||_1, \tag{3.25}$$

for some $C_\omega, r_\omega > 0$ depending on $\mathcal{V}$. This follows from the sum of

$$||L_m\xi|| \leq C_\omega (|m|+1)^{r_\omega} ||\xi||_1, \quad \forall m. \tag{3.26}$$

Further, $[A,X] = iL(f')$. Iteratively applying this relation leads to (see also Lemma

3.6.13)

$$||X\xi||_s \leq M_s||f||_{r_\omega+s}||\xi||_{s+1} \text{ (where } M_s = 2^s C_\omega) \implies |X|_{s+1} \leq M_s||f||_{r_\omega+s} \tag{3.27}$$

$$||[A,X]\xi||_s = ||L(f')\xi||_s \leq M_s||f'||_{r_\omega+s}||\xi||_{s+1} \implies |X|_{A,s+1} \leq M_s||f'||_{r_\omega+s} \tag{3.28}$$

The above also shows why $L(f)$ is a map from $\mathcal{V}_{(\infty)} \to \mathcal{V}_{(\infty)}$. A stronger statement is true for the exponentiation where we have $e^X : \mathcal{V}_{(s)} \to \mathcal{V}_{(s)}$ as a unitary operator. Moreover the following estimate holds ([84, proof of Proposition 2.1]):

$$\forall \xi \in \mathcal{V}_{(s)} \ : \ ||e^X\xi||_s \leq e^{2s|X|_{A,s}}||\xi||_s \tag{3.29}$$

Going back to the problem, let $Y = iL^n(f)$. To estimate $||(e^X - e^Y)\xi||$, we will proceed as in the proof of [84, Corollary 2.3]. Take $\xi \in \mathcal{V}_{(\infty)}$ and define $F(t) = e^{-tY}e^{tX}\xi$. $F(t)$ is differentiable and $\dot{F}(t) = e^{-tY}(X-Y)e^{tX}\xi$, implying

$$||(e^X - e^Y)\xi|| =$$

$$||e^Y \int_0^1 \dot{F}(t)\mathrm{d}t|| \leq \int_0^1 ||e^{(1-t)Y}(X-Y)e^{tX}\xi||\mathrm{d}t = \int_0^1 ||(X-Y)e^{tX}\xi||\mathrm{d}t, \tag{3.30}$$

where the second equality holds as $e^{(1-t)Y}$ is a unitary. Let $\xi_t = e^{tX}\xi$. Using the definition of Y in (3.18), the goal is to have an estimate on

$$||(X-Y)\xi_t|| = ||(\sum_{m<0} \hat{f}_m(\mathbf{1}-P^n)L_m + \sum_{m\geq 0} \hat{f}_m L_m(\mathbf{1}-P^n))\xi_t||. \tag{3.31}$$

Recall that for $|m| > N$, the Fourier coefficients are zero. From Remark 3.5.1, we know

$\mathcal{V}^n \subset \mathcal{H}_{n,p}$. Therefore, for $-N \leq m < 0 : L_m : \mathcal{V}^{n+m} \to \mathcal{V}^n \subset \mathcal{H}_{n,p}$, which implies $(\mathbf{1} - P^n)L_m = (\mathbf{1} - P^n)L_m(\mathbf{1} - P^{n+m}) = (\mathbf{1} - P^n)L_m(\mathbf{1} - P^{n-N})$. If $n < N$, then $P^{n-N} = \mathbf{0}$. By choosing $n > N$,

$$||(X - Y)\xi_t|| = ||(\sum_{m<0} \hat{f}_m(\mathbf{1} - P^n)L_m(\mathbf{1} - P^{n-N}) + \sum_{m\geq 0} \hat{f}_m L_m(\mathbf{1} - P^n))\xi_t|| \leq \quad (3.32)$$

$$\leq C_\omega ||f||_{r_\omega} ||(\mathbf{1} - P^{n-N})\xi_t||_1.$$

The last inequality is derived using the triangle inequality followed by

$$||(\mathbf{1}-P^n)L_m(\mathbf{1}-P^{n-N})\xi_t|| \leq ||L_m(\mathbf{1}-P^{n-N})\xi_t|| \ , \ ||L_m(\mathbf{1}-P^n)\xi_t|| \leq ||L_m(\mathbf{1}-P^{n-N})\xi_t||,$$

and applying (3.26). We wish to show that (3.32) is smaller than $\frac{1}{n}$ when $\xi = \Omega$ and n is larger than some constant. We need to use

$$||\xi_t||_3 = ||e^{tX}\xi||_3 \leq e^{6|tX|_{A,3}}||\xi||_3 \leq e^{6|X|_{A,3}}||\xi||_3 \leq e^{6M_2||f'||_{r_\omega+2}}||\xi||_3 \quad (3.33)$$

which follows from (3.28, 3.29). Let $\xi_t = \sum_l (\xi_t)_l$ with $(\xi_t)_l \in \mathcal{V}_l$. Recall

$$||\xi_t||_s = ||(L_0 + \mathbf{1})^s \xi_t|| = (\sum_{l=0}^{\infty} (l+1)^{2s} ||(\xi_t)_l||^2)^{\frac{1}{2}}. \quad (3.34)$$

Let us suppose that (3.30) is $> \frac{1}{n}$. We find a constant upper bound for n:

$$\frac{1}{n} < ||(e^X - e^Y)\xi|| = \int_0^1 ||(X-Y)e^{tX}\xi||\mathrm{d}t \implies \tag{3.35}$$

$$\frac{1}{n} < C_\omega ||f||_{r_\omega} \int_0^1 ||(\mathbf{1} - P^{n-N})\xi_t||_1 \mathrm{d}t = C_\omega ||f||_{r_\omega} \int_0^1 (\sum_{l>n-N} (l+1)^2 ||(\xi_t)_l||^2)^{\frac{1}{2}} \mathrm{d}t \implies \tag{3.36}$$

$$\frac{(n-N)^2}{n} < C_\omega ||f||_{r_\omega} \int_0^1 (\sum_{l>n-N} (l+1)^6 ||(\xi_t)_l||^2)^{\frac{1}{2}} \mathrm{d}t < C_\omega ||f||_{r_\omega} \int_0^1 ||\xi_t||_3 \mathrm{d}t \tag{3.37}$$

where we multiplied both sides by $(n-N)^2$ in the last line. By (3.33), this implies

$$\frac{(n-N)^2}{n} < C_\omega ||f||_{r_\omega} e^{6M_2 ||f'||_{r_\omega+2}} ||\xi||_3. \tag{3.38}$$

Finally, specializing to the case of Theorem 3.5.1, where $\xi = \Omega \implies ||\xi||_3 = 1$, and the norms depending on f and N are also constant (with regards to how complexity of the algorithm is measured). Clearly for n higher than some constant the above inequality does not hold, meaning (3.30) is smaller than $\frac{1}{n}$. This proves Theorem 3.5.1.

The proof actually provided an efficient estimate of the **state** $e^{iL(f)}\xi$ for any homogeneous ξ with given energy E. This energy can be a parameter of the algorithm complexity, as $||\xi||_3$ is a polynomial in E in (3.38).

If N and f are accounted for in the complexity of the algorithm, the bound on n will be polynomial *if* $||f'||_{r_\omega+2}$ is of log order $O(\log(N))$. The following is the best we can prove for this simulation problem on $e^{iL(f)}\xi$.

Theorem 3.5.2 ***Inputs:*** *Given function f with N many Fourier coefficients, with each coefficient $\hat{f}_m$ computable in polynomial time $p(m)$ with respect to m using a classical*

Turing machine, and a unit homogeneous vector ξ with energy E prepared in advance. Also given the promise that $||f'||_{r_\omega+2}$ is of order $O(\log(N))$.

Output: *The state $e^{iL(f)}\xi$ up to an error ϵ. The complexity of the algorithm is measured with respect to $\{\frac{1}{\epsilon}, N, E, \hat{f}_m\}$.*

*The above problem is in **BQP**.*

The above is a nontrivial result. Regarding the norm condition, it includes cases where the Fourier coefficients $\hat{f}_m$ are of order $\frac{\log(N)}{(N+1)^{r_\omega+4}}$.

Next case is the product of unitary operators $e^{iL(f^{(j)})}$ (Conjecture 3.3.1). Following the previous theorem, our best result on the full version of Conjecture 3.3.1 is derived:

Theorem 3.5.3 *Consider the same settings as in Conjecture 3.3.1, with the exception of a prepared homogeneous state ξ of energy E instead of Ω, and the promise that*

$$\sum_{j=1}^{k} ||(f^{(j)})'||_{r_\omega+2} \tag{3.39}$$

is of order $O(\log(n_1, \ldots, n_k, k))$. Then one can efficiently approximate the state

$$\prod_{j=1}^{k} e^{iL(f^{(j)})}\xi \tag{3.40}$$

up to error ϵ in polynomially many steps in $\{\frac{1}{\epsilon}, n_1, \ldots, n_k, k, E\} \cup \{(\hat{f^{(j)}})_m\}_{j,m}$ on a quantum computer.

Proof: Let $X_j = iL(f^{(j)})$ and $Y_j = iL^n(f^{(j)})$. Using the intermediate sums technique

$$\prod_{j=1}^{k} e^{Y_j} - \prod_{j=1}^{k} e^{X_j} = \sum_{b=1}^{k} (\prod_{j=1}^{b-1} e^{Y_j}(e^{Y_b} - e^{X_b}) \prod_{j=b+1}^{k} e^{X_j}), \tag{3.41}$$

followed by the triangle inequality to sum up estimates on

$$||\prod_{j=1}^{b-1} e^{Y_j}(e^{X_b} - e^{Y_b}) \prod_{j=b+1}^{k} e^{X_j}\xi|| \tag{3.42}$$

will give the error rate of the estimate of (3.40). If each of the above is smaller than $\frac{1}{n}$, the error is $\frac{k}{n}$ and choosing a higher n like nk, the desired rate of error is achieved. For each of the intermediate terms in (3.42),

$$||\prod_{j=1}^{b-1} e^{Y_j}(e^{X_b} - e^{Y_b}) \prod_{j=b+1}^{k} e^{X_j}\xi|| = ||(e^{X_b} - e^{Y_b}) \prod_{j=b+1}^{k} e^{X_j}\xi||, \text{ as } e^{Y_j} \text{ is unitary.} \tag{3.43}$$

After repeating the previous argument in Theorem 3.5.1, replacing ξ with $\prod_{j=b+1}^{k} e^{X_j}\xi$, N with n_b and f with $f^{(b)}$ in (3.38),

$$\frac{(n-n_b)^2}{n} < C_\omega ||f^{(b)}||_{r_\xi} e^{6M_2||(f^{(b)})'||_{r_\omega+2}||} \prod_{j=b+1}^{k} e^{X_j}\xi||_3. \tag{3.44}$$

According to (3.29, 3.28),

$$|| \prod_{j=b+1}^{k} e^{X_j}\xi||_3 \leq e^{\sum_{j=b+1}^{k} 6|X_j|_{A,3}}||\xi||_3 = e^{6M_2 \sum_{j=b+1}^{k} ||(f^{(j)})'||_{r_\omega+2}}||\xi||_3. \tag{3.45}$$

As a result

$$\frac{(n-n_b)^2}{n} < C_\omega ||f^{(b)}||_{r_\omega} e^{6M_2 \sum_{j=b+1}^{k} ||(f^{(j)})'||_{r_\omega+2}}||\xi||_3. \tag{3.46}$$

By making sure that none of the k inequalities above for $b = 1, \ldots, k$ hold, the theorem follows. The promise in the statement is needed to be able to have a polynomial bound on n. ■

Remark 3.5.2 *In order to discard the promise, the first possible approach is to find a better estimate for $||\xi_t||_s$ than what (3.29) provides, which is not exponential in $||f'||_{r_\omega+s-1}$. Below, an example is provided that shows the tightness of this bound. There is a need to estimate accurately $||(\mathbf{1}-P^{n-N})\xi_t||_s$, instead of replacing it with $||\xi_t||_s$ as is done in (3.37). One can hope that $||(\xi_t)_l||_s$ has a rate of growth similar to the Taylor series term $\frac{||f'||^l_{r_\omega+s-1}}{l!}$ of $e^{||f'||_{r_\omega+s-1}}$, in which case $||(\mathbf{1}-P^{n-N})\xi_t||_s$ is polynomially bounded.*

Remark 3.5.3 *This method has potential and provided some progress into the simulation problem. The power of the Fock space to yield accurate restriction of the Virasoro field on finite spaces is clearly more than that of ACs. The analysis of the convergence rate becomes doable as well. Further, as mentioned in Remark 3.5.1, the issue of preparation of any primary state in the vacuum sector is completely dealt with, as Ω is simply $|0\dots0\rangle$. We leave it to the future to apply this method on functions with infinite Fourier series, and to find out what possible additional promise is needed in that case.*

We prove the claim made in Remark 3.5.2 regarding the growth rate of $||\xi_t||_s$. Notice that for any $\gamma(e^{i\theta}) = e^{i\rho(\theta)} \in \mathrm{Diff}_+(S^1)$, where ρ is a $2\pi-$periodic diffeomorphism of $\mathbb{R}$, and $h \in C^\infty_{\mathbb{R}}(S^1)$, we have ([85, (15)])

$$U(\gamma)L(h)U(\gamma)^\dagger = L(\gamma_*(h)) + r(c,h,\gamma)\mathbf{1}, \tag{3.47}$$

where $r(c,h,\gamma)$ is some constant, computed in [86, (3.27)]. The adjoint action γ_* of $\gamma \in \mathrm{Diff}_+(S^1)$ is defined as

$$\gamma_*(h)(z) = \frac{1}{(\rho^{-1})'(\theta)} h(\gamma^{-1}(z)), \quad z = e^{i\theta} \tag{3.48}$$

The function h is chosen to be $h \equiv 1$, giving $L(h) = L_0$. This is required to take a power

of $(L_0 + \mathbf{1})$ acting upon ξ_t:

$$||\xi_t||_s = ||(L_0 + \mathbf{1})^s U(\gamma^{-1})\Omega|| = ||(L_0 + \mathbf{1})^s U(\gamma)^\dagger \Omega|| = \tag{3.49}$$

$$||U(\gamma^{-1})(L(\gamma_*(h)) + (r(c,h,\gamma)+1)\mathbf{1})^s\Omega|| = ||(L(\gamma_*(h)) + (r(c,h,\gamma)+1)\mathbf{1})^s\Omega||.$$

Therefore, our example would be to choose γ such that the above is exponential. Let $\gamma = \exp(\cos(N\theta))$, hence $U(\gamma) = e^{iL(f)}$ with $f(e^{i\theta}) = \cos(N\theta) \implies L(f) = L_N + L_{-N}$. From standard Lie group theory $\gamma_* = \exp(\mathbf{ad}_f)$. The map $\mathbf{ad}$ is defined as

$$\mathbf{ad}_f(g) = [f,g] = f'g - g'f, \text{ where } f' := \frac{\mathrm{d}}{\mathrm{d}\theta} f(e^{i\theta}). \tag{3.50}$$

We will show a lower bound for (3.49) that is exponential in N. We compute

$$\mathbf{ad}_f(h) = f' = -N\sin(N\theta) \tag{3.51}$$

$$\mathbf{ad}_f(\sin(N\theta)) = -N\sin^2(N\theta) - N\cos^2(N\theta) = -N = -Nh.$$

As a result, $\mathbf{ad}_f : \text{span}\{h, \sin(N\theta)\} \to \text{span}\{h, \sin(N\theta)\}$. This implies the exponential of the adjoint map, restricted to the subspace generated by the above two functions, is precisely equal to its Taylor series:

$$\exp(\mathbf{ad}_f)h = \sum_{k=0}^{\infty} \frac{(\mathbf{ad}_f)^k}{k!} h = (\sum_{k=0}^{\infty} \frac{N^{2k}}{(2k)!})h - (\sum_{k=0}^{\infty} \frac{N^{2k+1}}{(2k+1)!})\sin(N\theta). \tag{3.52}$$

We will choose $s = 1$ and compute $||\xi_{t=1}||_1$ in this setting:

$$||\Big(L\big((\sum_{k=0}^{\infty} \frac{N^{2k}}{(2k)!})h - (\sum_{k=0}^{\infty} \frac{N^{2k+1}}{(2k+1)!})\sin(N\theta)\big) + (r(c,h,\gamma)+1)\mathbf{1}\Big)\Omega|| \tag{3.53}$$

As the operator corresponding to the smearing field above is a sum of L_0, L_N, and L_{-N}, the result is a sum of two vectors of weight 0 and N as we are acting on Ω. The two vectors are orthogonal to each other, and in order to obtain a lower bound on the norm above, we compute the norm of the weight N vector:

$$||(\sum_{k=0}^{\infty} \frac{N^{2k+1}}{(2k+1)!})L_{-N}\Omega|| = (\sum_{k=0}^{\infty} \frac{N^{2k+1}}{(2k+1)!})\sqrt{\frac{cN(N^2-1)}{12}} = (\frac{e^N - e^{-N}}{2})\sqrt{\frac{cN(N^2-1)}{12}}. \tag{3.54}$$

This is clearly exponential in N, thus exponential in $||f'||_{r_\omega+s-1} = N(N+1)^{r_\omega+s-1}$ (we chose $s = 1$, but this estimate works for any $||\xi_1||_s$).

3.6 Point-like correlation function on the unit circle

To compute correlation functions, ideally, we would want the product of point-like field insertions to give a well-defined vector. This is not the case in general, hence why the observables are called *almost* linear and only their expectation values are well-defined. Nevertheless, if the field insertions are inside the unit circle (meaning insertions are done in the *past* in the radial setting), and they are energy-bounded, their product is a vector.

All theorems proven below are formulated using the vertex operator $Y(a, z)$ for energy-bounded primary field a. The hermitian assumption on a is not needed and is only assumed in the quantum simulation problem. Moreover, one can also add the quasi-primary Virasoro field ω to this collection, due to its conformal covariance property similar to that of primary fields.

We will always implicitly adjust the weighting of the vertex operator by $Y(a, z)z^{\mathrm{wt}\ a} = \sum(a)_m z^{-m}$, so that the modes $(a)_m$ with coefficient z^{-m} change weights by $-m$. This does not impact any of the convergence results as it is a simple global multiplication.

Energy-boundedness implies:

$$||(a)_m\xi|| \leq C_a(|m|+1)^{r_a}||(L_0+\mathbf{1})\xi||, \forall \xi \in \mathcal{V}_{(\infty)},$$

where if $\xi \in \mathcal{V}_k$ is homogeneous, then

$$||(a)_m\xi|| \leq C_a(|m|+1)^{r_a}(k+1)||\xi||.$$

All theorems generalize for intertwiners $\mathcal{Y}(a,z)$. We will make some remarks on the possible subtleties of this generalization in the end.

As the product order can be important, we set a convention for the order of any product as follows: $\prod_{i=k}^{l} A_i = A_l \dots A_k$, where the bottom index k of $\prod$ appears first in the product. We also recall the definition of configuration space for $X \subset \mathbb{C}$ as $\mathrm{Conf}_n(X) = \{(x_1,\dots,x_n) \in X \mid x_i \neq x_j\}$ and denote the closed unit disk by D_1 and its interior by $\mathring{D}_1$ with the zeros removed giving $D_1^\times, \mathring{D}_1^\times$. This is important as the insertion points will always be nonzero.

The final result in Theorem 3.6.16 is to show that the correlation of point-like insertions on the unit circle, i.e. (3.8), rewritten here for the vertex operator $\mathcal{Y} = Y$

$$(Y(b^{(l)},w_l)\dots Y(b^{(1)},w_1)\Omega, U(\gamma)Y(a^{(n)},z_n)\dots Y(a^{(1)},z_1)\Omega), \tag{3.55}$$

is well-defined on $\mathrm{Conf}_l(S^1) \times_\gamma \mathrm{Conf}_n(S^1)$. $\times_\gamma$ means the cartesian product with the additional condition $\gamma(z_i) \neq w_j$. We prove this by demonstrating that (3.55) is equal to (3.9) (up to some scalar), rewritten here for the vertex operator $\mathcal{Y} = Y$

$$(Y(\eta(a^{(1)}),\gamma(z_1))\dots Y(\eta(a^{(n)}),\gamma(z_n))Y(b^{(l)},w_l)\dots Y(b^{(1)},w_1)\Omega, U(\gamma)\Omega). \tag{3.56}$$

The proof involves using estimates and analytic continuation to find the definition of correlators on $\mathrm{Conf}_l(S^1) \times_\gamma \mathrm{Conf}_n(S^1)$ that are closer and closer in form to our target correlator in (3.55).

- An important Lemma 3.6.1, which shows if insertions are done in the past, i.e. $|z_1| < \ldots < |z_n| < 1$, then $\prod_{i=1}^n Y(a^{(i)}, z_i)\Omega \in \mathcal{V}_{(\infty)}$ is an actual state (note that $Y(a^{(i)}, z_i)$ are not linear operators on $\mathcal{V}_{(\infty)}$).

- This implies Corollary 3.6.8, where by some analytic continuation argument,

$$(\Omega, U(\gamma) \prod_{i=1}^n Y(a^{(i)}, z_i)\Omega)$$

  is an analytic function defined on $\mathrm{Conf}_n(D_1^\times)$.

- In Corollary 3.6.10, the first nontrivial result on the domain of (3.55) is obtained:

$$(\prod_{i=1}^l Y(b^{(i)}, w_i)\Omega, U(\gamma) \prod_{i=1}^n Y(a^{(i)}, z_i)\Omega)$$

  is an analytic function defined on the region $\mathrm{Conf}_l(\mathring{D}_1^\times) \times \mathrm{Conf}_n(\mathring{D}_1^\times)$. Fixing either of z_i or w_i allows analytic extension to $\mathrm{Conf}_l(D_1^\times)$ or $\mathrm{Conf}_n(D_1^\times)$.

- To extend the domain to the boundary of the above configuration spaces, we will need to relate the correlator to their smeared versions. By doing so, one can use the known conformal covariance of smeared correlators, and send the smeared functions to delta functions, in order to recover conformal covariance for point-like insertions on the unit circle (Theorem 3.6.12 and Lemma 3.6.14):

$$(v, U(\gamma) \prod_{i=1}^n Y(a^{(i)}, z_i)\Omega) = \prod_{i=1}^n \rho'(\phi_i)^{d_{a^{(i)}}} (v, \prod_{i=1}^n Y(a^{(i)}, \gamma(z_i))U(\gamma)\Omega),$$

where $\gamma(e^{i\theta}) = e^{i\rho(\theta)}, d_a = \text{wt } a$ and $v = \prod_{i=1}^{l} Y(b^{(i)}, w_i)\Omega$ for $|w_i| < 1$.

- From one of our first results, the RHS of the previous equation is known to analytically extend to $|w_i| = 1$. By taking the adjoint of the fields $Y(a^{(i)}, \gamma(z_i))$ to the other side of the inner product, the main Theorem 3.6.16 follows. It shows (3.55) is well-defined on $\text{Conf}_l(S^1) \times_\gamma \text{Conf}_n(S^1)$ and equals (3.56) up to $(-1)^{\sum d_{a^{(i)}}} \prod_{i=1}^{n} \rho'(\phi_i)^{d_{a^{(i)}}}$.

We start the procedure outlined above by showing that past insertions form a smooth vector.

Lemma 3.6.1 *For insertion points $|z_1| < \ldots < |z_n| < 1$*

$$\prod_{i=1}^{n} Y(a^{(i)}, z_i)\Omega = v = \sum_{k=0}^{\infty} v_k, \tag{3.57}$$

where v_k with energy k has norm smaller than $B(|\frac{z_1}{z_2}|, \ldots, |\frac{z_{n-1}}{z_n}|)z_n^k$, for some function B depending on $a^{(i)}$. As $|z_n| < 1$, it follows $v \in \mathcal{V}_{(\infty)} \subset \overline{\mathcal{V}}$.

Proof: Fixing k, we have to estimate

$$\sum_{k_1,\ldots,k_{n-1}} ||P^k Y(a^{(n)}, z_n) P^{k_{n-1}} Y(a^{(n-1)}, z_{n-1}) \ldots P^{k_1} Y(a^{(1)}, z_1)\Omega||, \tag{3.58}$$

where P^m projects onto $\mathcal{V}_m$. Computing each term

$$|z_n|^{m_n} \ldots |z_1|^{m_1} ||(a^{(n)})_{-m_n} \ldots (a^{(1)})_{-m_1}\Omega||, \tag{3.59}$$

where $\sum_{i=1}^{t} m_i = k_t \geq 0, \forall t \leq n-1$ and $\sum_{i=1}^{n} m_i = k$. Notice $m_n \leq k$ as taking the

adjoint of $P^k(a^{(n)})_{-m_n}$, if $m_n > k$, gives zero. Next, using energy boundedness repeatedly,

$$||(a^{(n)})_{-m_n} \dots (a^{(1)})_{-m_1}\Omega|| \leq \tag{3.60}$$

$$(\prod_{i=1}^{n} C_{a^{(i)}}(|m_i|+1)^{r_{a^{(i)}}})(m_1 + \ldots + m_{n-1} + 1)^{s_{a^{(n)}}} \dots (m_1+1)^{s_{a^{(2)}}}$$

Let $C_{a^{(1)},\dots,a^{(n)}} := \prod_{i=1}^{n} C_{a^{(i)}}$. Replace m_n with $m_n + k$, implying $\sum_{i=1}^{n} m_i = 0$ and $m_n \leq 0$, and replace $m_i = k_i - k_{i-1}$ for $1 \leq i \leq n$ with $k_0 = k_n = 0$. Rewriting the inequality

$$|z_n|^k \prod_{i=1}^{n} |z_i|^{m_i} ||(a^{(n)})_{-m_n-k} \dots (a^{(1)})_{-m_1}\Omega|| \leq$$

$$C_{a^{(1)},\dots,a^{(n)}} |z_n|^k \prod_{i=1}^{n-1} (k_i+1)^{s_{a^{(i+1)}}} |\frac{z_i}{z_{i+1}}|^{k_i} . \prod_{i=1}^{n} (|k_i - k_{i-1}|+1)^{r_{a^{(i)}}} . \tag{3.61}$$

The factor z_n^k has been obtained. The rest needs to be shown to be smaller than some function $B()$ of $|\frac{z_i}{z_{i+1}}|$. The terms $\prod_{i=1}^{n}(|k_i - k_{i-1}|+1)^{r_{a^{(i)}}}$ can be replaced by the following rough estimate

$$|k_i - k_{i-1}| + 1 \leq k_i + k_{i-1} + 1 \leq (k_i+1)(k_{i-1}+1). \tag{3.62}$$

Rewriting accordingly for the whole sum,

$$\sum_{k_1,\dots,k_{n-1} \geq 0} \prod_{i=1}^{n-1} (k_i+1)^{s_{a^{(i+1)}} + r_{a^{(i)}} + r_{a^{(i+1)}}} |\frac{z_i}{z_{i+1}}|^{k_i} \tag{3.63}$$

Notice that $|\frac{z_i}{z_{i+1}}| < 1$, implying

$$\sum_{k_1,\ldots,k_{n-1}\geq 0} \prod_{i=1}^{n-1} |\frac{z_i}{z_{i+1}}|^{k_i} = \frac{1}{\prod_{i=1}^{n-1}(1-|\frac{z_i}{z_{i+1}}|)}. \tag{3.64}$$

The factors $(k_i+1)^{s_{a^{(i+1)}}+r_{a^{(i)}}+r_{a^{(i+1)}}}$ in (3.63) can be obtained by multiplying the above by enough powers of $|\frac{z_i}{z_{i+1}}|$ and taking a derivative with respect to it $s_{a^{(i+1)}}+r_{a^{(i)}}+r_{a^{(i+1)}}$ times. The above function has its Taylor expansion also converging to its derivative (term by term differentiation), implying uniform convergence to a function $B(|\frac{z_1}{z_2}|,\ldots,|\frac{z_{n-1}}{z_n}|)$. ■

Corollary 3.6.2 *For $u \in \mathcal{V}_{(\infty)}$ the correlation function $(u, \prod Y(a^{(i)}, z_i)\Omega)$ is absolutely convergent where $|z_1| < \ldots < |z_n| \leq 1$. More precisely, the sum*

$$\sum_{k,k_1,\ldots,k_n} |(P^k u, P^k Y(a^{(n)}, z_n)\ldots P^{k_1} Y(a^{(1)}, z_1)\Omega)| \tag{3.65}$$

is uniformly convergent in any compact neighborhood.

Proof: By the previous lemma, for $u = \sum_k c_k u_k$ where u_k a unit vector with energy k, the above is bounded by

$$B(|\frac{z_1}{z_2}|,\ldots,|\frac{z_{n-1}}{z_n}|)\sum_k |c_k||z_n|^k. \tag{3.66}$$

As $u \in \mathcal{V}_{(\infty)}$, c_k is decreasing faster than any polynomial. Hence, even for $|z_n| = 1$, the sum is uniformly convergent in any compact neighborhood. ■

Remark 3.6.1 *Due to the above results, we can unambiguously write the correlation*

function as

$$\sum_{k,k_1,\ldots,k_n} (u, P^k Y(a^{(n)}, z_n) \ldots P^{k_1} Y(a^{(1)}, z_1)\Omega), \tag{3.67}$$

or any other sum alike, where projections are done only in $1 \leq j \leq n$ *fixed places such as*

$$\sum_{k,k_{i_1},\ldots,k_{i_j}} (P^k u, \ldots P^{k_{i_j}} Y(a^{(i_j)}, z_{i_j}) \ldots P^{k_{i_1}} Y(a^{(i_1)}, z_{i_1}) \ldots Y(a^{(1)}, z_1)\Omega). \tag{3.68}$$

Remark 3.6.2 *It is easy to see that* Ω *can be replaced by any state in* $\mathcal{V}$*, as that would only introduce finitely many poles at* z_1*.*

From the previous corollary, as $U(\gamma) : \mathcal{V}_{(\infty)} \to \mathcal{V}_{(\infty)}$,

Corollary 3.6.3 *For* $\gamma \in Diff_+(S^1)$*, the correlation function* $(U(\gamma)\Omega, \prod Y(a^{(i)}, z_i)\Omega)$ *is absolutely convergent for* $|z_1| < \ldots < |z_n| \leq 1$*.*

The next step is to extend the domain of the correlation function to the closed unit disk in the configuration space, i.e. $\mathrm{Conf}_n(D_1^\times) = \{(z_1, \ldots, z_n) \mid z_i \in D_1,\ z_i \neq z_j\}$. Let us define $F_k(z_1, \ldots, z_n) := (u_k, \prod_{i=1}^n Y(a^{(i)}, z_i)\Omega)$. It is well-known (following Huang's results in [30]) that F_k is an analytic function defined on the configuration space $\mathrm{Conf}_n(\mathbb{C}^\times)$. More precisely, F_k is a rational function of the form

$$F_k = \frac{G_k(z_1, \ldots, z_n)}{\prod_{i<j}(z_i - z_j)^{s_{ij}}} \tag{3.69}$$

where G_k is a **polynomial**, and the order of the poles for $(z_i - z_j)$ is **at most** s_{ij}, where $(a^{(i)})_m a^{(j)} = 0$ for all $m \geq s_{ij}$. Note the absence of $z_i = 0$ poles is due to the action on the vacuum Ω [87, Remark 3.5.2]. Replacing Ω by a higher energy state would produce

poles at z_i, order of which is dependent on $a^{(i)}$ and the state [15, section 3]. Next, define

$$F(z_n, \ldots, z_1) := (u, \prod_{i=1}^{n} Y(a^{(i)}, z_i)\Omega) = \sum_k c_k F_k, \tag{3.70}$$

which has been shown so far to be analytic for $|z_1| < \ldots < |z_n| \leq 1$.

We shall bound the coefficients of the polynomial G_k, so that $G := \sum c_k G_k$ is uniformly convergent on compact neighborhoods in $\mathrm{Conf}_n(D_1^\times)$. As a result, $F = G\prod_{i<j}(z_i - z_j)^{-s_{ij}}$ is also analytically extended to that space. Most importantly, this will allow us to define F on n distinct points with $|z_i| = 1$, getting us closer to the formulation in (3.8).

Lemma 3.6.4 *The order of z_i for any i in G_k is at most $k + \sum_j s_{ij}$.*

Proof: This fact comes from the same proof that G_k is a polynomial [15, Proposition 3.2.7]. By weak commutativity, the following holds in *formal* calculus

$$\prod_{i<j}(z_i - z_j)^{s_{ij}}(u_k, \prod_{i=1}^{n} Y(a^{(i)}, z_i)\Omega) = G_k = \prod_{i<j}(z_i - z_j)^{s_{ij}}(u_k, \prod_{i=1}^{n} Y(a^{(\sigma(i))}, z_{\sigma(i)})\Omega), \tag{3.71}$$

where σ is any permutation. Consider z_i for some fixed i and σ such that $\sigma(1) = i$. This shows that z_i has no poles (as pointed out previously) since $z_i^{-m}(a^{(i)})_m\Omega = 0, \forall m > 0$. Taking $\sigma(n) = i$, as u_k has weight k, shows z_i has maximum positive order k. Finally, the product $\prod(z_i - z_j)^{s_{ij}}$ gives a contribution of $\sum_j s_{ij}$ to the maximum order of z_i. ■

Let $G_k = \sum_{j_1,\ldots,j_n \geq 0} d_{j_1,\ldots,j_n,k} \prod_i z_i^{j_i}$.

Lemma 3.6.5 *There exist constants $Q_{a^{(1)},\ldots,a^{(n)}}, R_{a^{(1)},\ldots,a^{(n)}} > 0$ depending on $a^{(i)}$ and n such that*

$$d_{j_1,\ldots,j_n,k} < Q_{a^{(1)},\ldots,a^{(n)}} k^{R_{a^{(1)},\ldots,a^{(n)}}},$$

for large enough k.

Proof: Consider a substitution $|z_1| < \ldots < |z_n|$ of complex numbers for the left equality in (3.71). Then one can use the expansion below of the correlator

$$\sum_{j_1,\ldots,j_n \geq 0} d_{j_1,\ldots,j_n,k} \prod_i z_i^{j_i} = \prod_{i<j} (z_i - z_j)^{s_{ij}} (u_k, \prod_{i=1}^{n} Y(a^{(i)}, z_i)\Omega) = \tag{3.72}$$

$$\prod_{i<j} (z_i - z_j)^{s_{ij}} \Big(\sum_{\sum m_i = k} \prod_i z_i^{m_i} (u_k, \prod_{i=1}^{n} (a^{(i)})_{-m_i} \Omega) \Big).$$

To compute the coefficient of $\prod_i z_i^{j_i}$, we first decide which powers are selected from the product $(z_i - z_j)^{s_{ij}}$. Let $S = \max_i \sum_j s_{ij}$. There are at most $(\max(s_{ij}+1))^{\binom{n}{2}} < (S+1)^{\binom{n}{2}}$ choices that can be made from the product $\prod_{i<j}(z_i - z_j)^{s_{ij}}$. Afterwards, there is a unique choice of m_i in the expansion of the correlation function that would give $\prod_i z_i^{j_i}$.

There are also bounds on these choices. For z_i, by the previous lemma $j_i \leq k + S$, and no positive power more than S can emerge from $\prod_{j \neq i}(z_i - z_j)^{s_{ij}}$. Therefore the choice for the m_i is bounded by $-S \leq j_i - S \leq m_i \leq j_i \leq k + S$. The minimum for m_i happens when the maximum power ($\leq S$) for z_i is selected from $\prod_{j \neq i}(z_i - z_j)^{s_{ij}}$. The maximum for m_i happens when no z_i is selected from the product $(z_i - z_j)^{s_{ij}}$.

Let us now compute the coefficient corresponding to each choice. The binomial coefficient for a power of z_i in $\prod_{j \neq i}(z_i - z_j)^{s_{ij}}$ can be no more than $\binom{S}{\frac{S}{2}}^{n-1} < 2^{(S-1)(n-1)}$.

As m_i is bounded between $-S$ and $k+S$, it follows $|m_i| \leq k+S$. Let $R = \max(s_{a^{(i)}} + r_{a^{(i)}})$. Using the following loose estimates

$$(m_1 + \ldots + m_t + 1)^{s_{a_t}} \leq (|m_1| + \ldots + |m_t| + 1)^{s_{a_t}} < (nk + nS + 1)^R \tag{3.73}$$

$$(|m_i| + 1)^{r_{a^{(i)}}} < (k + S + 1)^R, \tag{3.74}$$

we obtain $|(u_k, \prod_{i=1}^{n}(a^{(i)})_{-m_i}\Omega)| \leq$

$$C_{a^{(1)},\ldots,a^{(n)}}(m_1+\ldots+m_{n-1}+1)^{s_{a^{(n)}}}\ldots(m_1+1)^{s_{a^{(2)}}}\prod_i(|m_i|+1)^{r_{a^{(i)}}} < \tag{3.75}$$

$$C_{a^{(1)},\ldots,a^{(n)}}(nk+nS+1)^{Rn}(k+S+1)^{Rn} < C_{a^{(1)},\ldots,a^{(n)}}(nk+nS+1)^{2Rn}.$$

It follows

$$|d_{j_1,\ldots,j_n,k}| < C_{a^{(1)},\ldots,a^{(n)}}(S+1)^{\binom{n}{2}}2^{(S-1)(n-1)n}(nk+nS+1)^{2Rn}, \tag{3.76}$$

where the first term is the maximum number of possible choices to get $\prod_i z_i^{j_i}$, the second comes from the bound on the coefficient of z_i in $\prod_{j\neq i}(z_i-z_j)^{s_{ij}}$ for each $i=1,\ldots,n$, and the third term is the bound calculated for the unique choice of m_i. The above is clearly smaller than $Q_{a^{(1)},\ldots,a^{(n)}}k^{R_{a^{(1)},\ldots,a^{(n)}}}$, for large enough k, for a suitable choice of constants $Q_{a^{(1)},\ldots,a^{(n)}}, R_{a^{(1)},\ldots,a^{(n)}}$. ■

Corollary 3.6.6 *There exist constants $Q'_{a^{(1)},\ldots,a^{(n)}}, R'_{a^{(1)},\ldots,a^{(n)}} > 0$ such that*

$$\forall(z_1,\ldots,z_n)\in D_1^n \ : \ |G_k(z_1,\ldots,z_n)| \leq Q'_{a^{(1)},\ldots,a^{(n)}}k^{R'_{a^{(1)},\ldots,a^{(n)}}},$$

for large enough k.

Proof: There are at most $(k+S+1)^n$ many coefficients $|d_{j_1,\ldots,j_n,k}|$ since $0\leq j_i\leq k+S$. When $|z_i|\leq 1$,

$$|G_k(z_1,\ldots,z_n)| \leq \sum_{j_1,\ldots,j_N}|d_{j_1,\ldots,j_n,k}| < (k+S+1)^n Q_{a^{(1)},\ldots,a^{(n)}}k^{R_{a^{(1)},\ldots,a^{(n)}}}. \tag{3.77}$$

A suitable choice for $Q'_{a^{(1)},\ldots,a^{(n)}}, R'_{a^{(1)},\ldots,a^{(n)}}$ clearly exists to satisfy the statement. ■

Corollary 3.6.7 *$F = \sum c_k F_k$ is an analytic function defined on $Conf_n(D_1^\times)$ with order of singularity at $z_i = z_j$ at most s_{ij}. In particular, the same holds for*

$$(U(\gamma)\Omega, \prod_{i=1}^{n} Y(a^{(i)}, z_i)\Omega).$$

Proof: As $F_k = \prod_{i<j}(z_i - z_j)^{s_{ij}} G_k$ and this product is independent of k, bounding $|F_k|$ only requires bounding $|G_k|$. By the previous corollary, this is bounded uniformly by $Q'_{a^{(1)},\ldots,a^{(n)}} k^{R'_{a^{(1)},\ldots,a^{(n)}}}$ for large enough k. As c_k decays faster than any polynomial in k, uniform convergence on any local compact neighborhood in $\mathrm{Conf}_n(D_1^\times)$ is ensured, and F is analytic on $\mathrm{Conf}_n(D_1^\times)$. For the statement on the order of singularities, note $G = \sum_k c_k G_k = \prod_{i<j}(z_i - z_j)^{s_{ij}} F$ is defined on the entire D_1^n. ■

Corollaries below show that one can take adjoints of operators $Y(a^{(i)}, z_i)$ or $U(\gamma)$ in the inner product when evaluating the correlation function at all points in the domain.

Corollary 3.6.8 *$(\Omega, U(\gamma)\prod_{i=1}^{n} Y(a^{(i)}, z_i)\Omega)$ is an analytic function on $Conf_n(D_1^\times)$ with order of singularity at $z_i = z_j$ at most s_{ij}. In fact*

$$(\Omega, U(\gamma)\prod_{i=1}^{n} Y(a^{(i)}, z_i)\Omega) = (U(\gamma^{-1})\Omega, \prod_{i=1}^{n} Y(a^{(i)}, z_i)\Omega),$$

on $Conf_n(D_1^\times)$.

Proof: Note that $U(\gamma)^\dagger = U(\gamma^{-1})$. Even though this corollary seems obvious, note that one can use the definition of the correlator above as a sum of inner products (see (3.67)) only on $|z_1| < \ldots < |z_n| \leq 1$ where absolute convergence holds. When $|z_i| = |z_j|$, evaluation is done by taking the analytic extension, thus taking a limit. We note that

$$(\Omega, P^0 U(\gamma) P^k Y(a^{(n)}, z_n) P^{k_{n-1}} \ldots P^{k_1} Y(a^{(1)}, z_1)\Omega) =$$

$$(P^k U(\gamma^{-1})\Omega, P^k Y(a^{(n)}, z_n) P^{k_{n-1}} \dots P^{k_1} Y(a^{(1)}, z_1)\Omega),$$

which summing over $k, k_1, \dots, k_{n-1}$, implies

$$(\Omega, U(\gamma) \prod_{i=1}^{n} Y(a^{(i)}, z_i)\Omega) = (U(\gamma^{-1})\Omega, \prod_{i=1}^{n} Y(a^{(i)}, z_i)\Omega).$$

As $|z_1| < \ldots < |z_n| \leq 1$ and its permutations form a dense subset in $\mathrm{Conf}_n(D_1^\times)$, the rest follows from the previous corollary. ■

One can also take adjoints of the fields. Recall the anti-linear involution $\eta : \mathcal{V} \to \mathcal{V}$ for which

$$(x, \prod_{i=1}^{n} Y(a^{(i)}, z_i)y) = (-1)^{\sum \mathrm{wt}\ a^{(i)}} (\prod_{i=n}^{1} Y(\eta(a^{(i)}), \overline{z_i}^{-1})x, y) \tag{3.78}$$

for all $x, y \in \mathcal{V}$. We would like to show the same holds when $x \in \mathcal{V}_{(\infty)}$ and $(z_i)_i \in \mathrm{Conf}_n(D_1^\times)$.

Corollary 3.6.9 *$(\prod_{i=n}^{1} Y(\eta(a^{(i)}), \overline{z_i}^{-1})U(\gamma)\Omega, \Omega)$ is an analytic function on $Conf_n(D_1^\times)$ with order of singularity at $z_i = z_j$ at most s_{ij}. In fact*

$$(-1)^{\sum wt\ a^{(i)}} (\prod_{i=n}^{1} Y(\eta(a^{(i)}), \overline{z_i}^{-1})U(\gamma)\Omega, \Omega) = (U(\gamma)\Omega, \prod_{i=1}^{n} Y(a^{(i)}, z_i)\Omega),$$

on $Conf_n(D_1^\times)$. More generally, one could take adjoints of only some of the insertions giving

$$(-1)^{\sum_{i=n}^{j} wt\ a^{(i)}} (\prod_{i=n}^{j} Y(\eta(a^{(i)}), \overline{z_i}^{-1})U(\gamma)\Omega, \prod_{i=1}^{j-1} Y(a^{(i)}, z_i)\Omega) = (U(\gamma)\Omega, \prod_{i=1}^{n} Y(a^{(i)}, z_i)\Omega),$$

on $Conf_n(D_1^\times)$.

Proof: First, the equality of the inner products involved in the expansion (3.67) in

the region $|z_1| < \ldots < |z_n| \leq 1$ is shown:

$$(-1)^{\sum_{i=n}^{1} \text{wt } a^{(i)}}(Y(\eta(a^{(1)}), \overline{z_1}^{-1})P^{k_1} \ldots P^{k_{n-1}}Y(\eta(a^{(n)}), \overline{z_n}^{-1})P^k U(\gamma)\Omega, \Omega) =$$

$$(-1)^{\sum_{i=n}^{2} \text{wt } a^{(i)}}(P^{k_1} \ldots P^{k_{n-1}}Y(\eta(a^{(n)}), \overline{z_n}^{-1})P^k U(\gamma)\Omega, Y(a^{(1)}, z_1)\Omega) =$$

$$(-1)^{\sum_{i=n}^{2} \text{wt } a^{(i)}}(Y(\eta(a^{(2)}), \overline{z_2}^{-1}) \ldots P^{k_{n-1}}Y(\eta(a^{(n)}), \overline{z_n}^{-1})P^k U(\gamma)\Omega, P^{k_1}Y(a^{(1)}, z_1)\Omega) =$$

$$\ldots = (P^k U(\gamma)\Omega, Y(a^{(n)}, z_n)P^{k_{n-1}} \ldots P^{k_1}Y(a^{(1)}, z_1)\Omega)$$

where each equality is implied from (3.78), and taking P^{k_i} to the other component. Using the analytic extension as proven in Corollary 3.6.7 for $(U(\gamma)\Omega, \prod_i Y(a^{(i)}, z_i)\Omega)$ implies the equality on the entire $\text{Conf}_n(D_1^\times)$. The last statement has a similar proof. ■

The combination of above corollaries gets us closer in proving the well-definedness of (3.8). Let $\mathring{D}_1$ be the interior of D_1.

Corollary 3.6.10 $(\prod_{i=1}^{l} Y(b^{(i)}, w_i)\Omega, U(\gamma)\prod_{i=1}^{n} Y(a^{(i)}, z_i)\Omega)$ *is an analytic function defined on the region* $Conf_l(\mathring{D}_1^\times) \times Conf_n(\mathring{D}_1^\times)$.

Fixing either of the two points in $(z_i)_i \in Conf_n(\mathring{D}_1^\times)$ *or* $(w_j)_j \in Conf_l(\mathring{D}_1^\times)$, *the function can be extended analytically to* $Conf_l(D_1^\times)$ *or* $Conf_n(D_1^\times)$.

The same holds for all other formulations of the inner product (which do not involve any interchanging of $U(\gamma)$ *with the fields), like taking adjoints of the fields or* $U(\gamma)$.

Proof: By lemma 3.6.1, the correlator is analytic in the region $\text{Conf}_l(\mathring{D}_1^\times) \times \text{Conf}_n(\mathring{D}_1^\times)$, and vectors on both components of the inner product are inside $\mathcal{V}_{(\infty)}$. More explicitly, for the expansion

$$\sum_{k,k',k_i,k_i'} |(P^k \prod_{i=1}^{l} P^{k_i}Y(b^{(i)}, w_i)\Omega, P^k U(\gamma)P^{k'} \prod_{i=1}^{n} Y(a^{(i)}, z_i)P^{k_i'}\Omega)| \leq$$

$$\sum_{k,k',k_i,k_i'} ||P^k \prod_{i=1}^{l} P^{k_i} Y(b^{(i)}, w_i)\Omega||.||P^k U(\gamma) P^{k'} \prod_{i=1}^{n} Y(a^{(i)}, z_i) P^{k_i'}\Omega|| \leq$$

$$\sum_{k,k',k_i,k_i'} ||P^k \prod_{i=1}^{l} P^{k_i} Y(b^{(i)}, w_i)\Omega||.||P^{k'} \prod_{i=1}^{n} Y(a^{(i)}, z_i) P^{k_i'}\Omega||,$$

where the last inequality is due to $U(\gamma)$ being unitary. According to Lemma 3.6.1, both terms above are exponentially decreasing with order w_l^k and $z_n^{k'}$. Hence, the sum is absolutely convergent on $\mathrm{Conf}_l(\mathring{D}_1^{\times}) \times \mathrm{Conf}_n(\mathring{D}_1^{\times})$, and uniformly convergent on any compact neighborhood.

If the z_is are fixed in $\mathrm{Conf}_n(\mathring{D}_1^{\times})$, then $U(\gamma)\prod_{i=1}^{n} Y(a^{(i)}, z_i)\Omega \in \mathcal{V}_{(\infty)}$, and by Corollary 3.6.7, we have an analytic extension to $\mathrm{Conf}_l(D_1^{\times})$.

The same can be done when w_is are fixed, where

$$(\prod_{i=1}^{l} Y(b^{(i)}, w_i)\Omega, U(\gamma)\prod_{i=1}^{n} Y(a^{(i)}, z_i)\Omega) = (U(\gamma^{-1})\prod_{i=1}^{l} Y(b^{(i)}, w_i)\Omega, \prod_{i=1}^{n} Y(a^{(i)}, z_i)\Omega)$$

as both components are vectors in $\mathcal{V}_{(\infty)}$. Similarly, using Corollary 3.6.7, we can extend analytically to $\mathrm{Conf}_n(D_1^{\times})$. Note this also shows one example for the last statement of the corollary and other formulations of the inner product can be similarly shown to deliver the same correlator. ■

Eventually, the definition of the correlator should be extended to include $\mathrm{Conf}_l(S^1) \times_\gamma \mathrm{Conf}_n(S^1)$ where $\times_\gamma$ means $\gamma(z_i) \neq w_j$, as alluded to in (3.9). For this purpose, we shall study the relationship between the smeared and the point-like correlation functions on the unit circle, from which conformal covariance of point-like correlation functions on S^1 follows. Our work so far and the Proposition 3.12 in [31] provides the basis for this discussion.

Lemma 3.6.11 *For $u \in \mathcal{V}_{(\infty)}$ we have*

$$(u, \prod_{i=1}^{n} Y(a^{(i)}, f_i)\Omega) = \frac{1}{(2\pi)^n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} (u, \prod_{j=1}^{n} Y(a^{(j)}, e^{i\theta_j})\Omega) \prod_j f_j(e^{i\theta_j}) \prod_i d\theta_i, \quad (3.79)$$

where $f_i \in C^{\infty}_{\mathbb{C}}(S^1)$ with disjoint support.

Remark 3.6.3 *Note the disjoint support of f_i is needed to not hit singularities of the correlation function at $z_i = z_j$. Further, as $z_j = e^{i\theta_j}$ have the same norm, the value of $(u_k, \prod_{j=1}^{n} Y(a^{(j)}, e^{i\theta_j})\Omega)$ is determined by the analytic extension of $(u_k, \prod_{i=1}^{n} Y(a^{(i)}, z_i)\Omega)$ to $Conf_n(D_1^{\times})$, which is a continuous extension to the boundary on which the point $(z_j)_j = (e^{i\theta_j})_j$ lives. As mentioned previously, one can **not** use the expansion:*

$$(u_k, \prod_{i=1}^{n} Y(a^{(i)}, z_i)\Omega) = \sum_{\sum m_i = k} \prod_i z_i^{m_i} (u_k, \prod_{i=1}^{n} (a^{(i)})_{-m_i}\Omega),$$

as this expansion does not absolutely converge when z_is have equal norms, thus is not well-defined. The value of F is obtained by taking a limit from the interior of the disk:

$$\lim_{z_j \to e^{i\theta_j}} F = \lim_{z_j \to e^{i\theta_j}} \sum_k c_k F_k = \sum_k c_k \lim_{z_j \to e^{i\theta_j}} F_k,$$

where swapping the limit and the infinite sum is allowed since the convergence was proven to be uniform in any compact neighborhood on $Conf_n(D_1^{\times})$.

Proof: We first show these two identities:

$$\lim_{K \to \infty} (\sum_{i=1}^{K} c_k u_k, \prod_{i=1}^{n} Y(a^{(i)}, f_i)\Omega) = (u, \prod_{i=1}^{n} Y(a^{(i)}, f_i)\Omega) \quad (3.80)$$

$$\lim_{K\to\infty}\sum_{k=1}^{K}\frac{1}{(2\pi)^n}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}c_kF_k(e^{i\theta_1},\ldots,e^{i\theta_n})\prod_j f_j(e^{i\theta_j})\prod_i \mathrm{d}\theta_i = \tag{3.81}$$

$$\frac{1}{(2\pi)^n}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}F(e^{i\theta_1},\ldots,e^{i\theta_n})\prod_j f_j(e^{i\theta_j})\prod_i \mathrm{d}\theta_i.$$

The first identity is obvious due to the continuity of the inner product on $\mathcal{V}_{(\infty)}\times\mathcal{V}_{(\infty)}$ and $\sum_{i=1}^{K}c_ku_k\to u\in\mathcal{V}_{(\infty)}$. The second is due to the uniform convergence of $\sum c_kF_k$ to F on compact neighborhoods given by the union of supports of f_is. This allows interchanging the limit and integral. The statement immediately follows as the LHS of the above two identities are equal, as shown in [31, Proposition 3.12]. ■

Remark 3.6.4 *One can replace the vacuum by any finite energy vector v_k, and it follows by the exact same argument after taking adjoints that*

$$(\prod_{i=1}^{n}Y(a^{(i)},f_i)u,v_k)=\frac{1}{(2\pi)^n}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\Big(\prod_{j=1}^{n}Y(a^{(j)},e^{i\theta_j})u,v_k\Big)\prod_j f_j(e^{i\theta_j})\prod_i d\theta_i, \tag{3.82}$$

Ultimately, we will need a stronger version of the above lemma where the vacuum is replaced by any vector coming from past insertions, i.e. $v=\prod_{i=1}^{l}Y(b^{(i)},w_i)\Omega$ for some $|w_1|<\ldots<|w_l|<1$. Stating the desired theorem explicitly, given a number of past insertions $Y(b^{(i)},w_i)$ in addition to our present insertions $Y(a^{(j)},e^{i\theta_j})$ (on the unit circle), we want to be still able to integrate over the present insertions. Recall from Lemma 3.6.1 that $v=\sum_k v_k\in\mathcal{V}_{(\infty)}$.

Theorem 3.6.12 *For $u\in\mathcal{V}_{(\infty)}$ and v as defined above,*

$$(u,\prod_{i=1}^{n}Y(a^{(i)},f_i)v)=\frac{1}{(2\pi)^n}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}(u,\prod_{j=1}^{n}Y(a^{(j)},e^{i\theta_j})v)\prod_j f_j(e^{i\theta_j})\prod_i d\theta_i, \tag{3.83}$$

where $f_i\in C^\infty_{\mathbb{C}}(S^1)$ with disjoint support.

We shall generalize the proof of [31, Proposition 3.12]. A few energy-bound estimates are needed which are recited from [31].

Lemma 3.6.13 (Proposition 3.9, [31]) *Recall the property of an energy-bounded field a, where*

$$||Y(a,f)v|| \leq C_a||f||_{r_a}||v||_{s_a}, \forall v \in \mathcal{V}_{(\infty)}, \forall f \in C^\infty_{\mathbb{C}}(S^1),$$

for some $C_a, r_a, s_a > 0$, with $r, s-$th norms defined in (3.24). We have

$$||Y(a,f)v||_p \leq 2^p C_a||f||_{r_a+p}||v||_{s_a+p}, \forall v \in \mathcal{V}_{(\infty)}, p \in \mathbb{N}.$$

Proof: This can be proved using the infinitesimal rotational covariance:

$$[L_0, Y(a,f)] = iY(a,f').$$

We will do induction on p. For $p = 0$, it is the assumption. Assume

$$||Y(a,f)v||_p \leq 2^p C_a||f||_{r_a+p}||v||_{s_a+p}, \forall v \in \mathcal{V}_{(\infty)}.$$

Then

$$||Y(a,f)v||_{p+1} = ||(L_0 + \mathbf{1})Y(a,f)v||_p = ||Y(a,f)(L_0+\mathbf{1})v + iY(a,f')v||_p \leq$$

$$||Y(a,f)(L_0+\mathbf{1})v||_p + ||Y(a,f')v||_p \leq$$

$$2^p C_a||f||_{r_a+p}||(L_0+\mathbf{1})v||_{s_a+p} + 2^p C_a||f'||_{r_a+p}||v||_{s_a+p} =$$

$$2^p C_a(||f||_{r_a+p}||v||_{s_a+p+1} + ||f'||_{r_a+p}||v||_{s_a+p}) \leq 2^{p+1} C_a||f||_{r_a+p+1}||v||_{s_a+p+1}$$

where the first inequality is using the induction hypothesis and the last is using the

observations we made in (3.24). ■

Remark 3.6.5 *For $f = e^{in\theta}$ in the lemma above,*

$$||(a)_n v||_p \leq 2^p C_a(|n|+1)^{r_a+p}||v||_{s_a+p}, \forall v \in \mathcal{V}_{(\infty)}. \tag{3.84}$$

This implies that for $(u, \prod_{i=1}^n Y(a^{(i)}, f_i)v)$ for $u, v \in \mathcal{V}_{(\infty)}$ with $||u|| = 1$, the expansion of the inner product like in (3.67) is absolutely convergent:

$$\sum_{m_1,\ldots,m_n} |(u, \prod_{i=1}^n (a^{(i)})_{m_i} v)| \prod_{i=1}^n |(\hat{f}_i)_{m_i}| \leq \sum_{m_1,\ldots,m_n} ||\prod_{i=1}^n (a^{(i)})_{m_i} v||. \prod_{i=1}^n |(\hat{f}_i)_{m_i}| \leq$$

$$M_{a^{(1)},\ldots,a^{(n)}} \sum_{m_1,\ldots,m_n} \prod (|m_i|+1)^{r_{a^{(i)}}+\ldots+r_{a^{(n)}}} ||v||_{\sum_{j=1}^n s_{a^{(j)}}} \prod_{i=1}^n |(\hat{f}_i)_{m_i}|,$$

where the last inequality is obtained by iteratively applying (3.84), and $M_{a^{(1)},\ldots,a^{(n)}}$ is a constant that is a product of the powers of 2 and the $C_{a^{(i)}}$s. It follows that

$$\leq M_{a^{(1)},\ldots,a^{(n)}} ||v||_{\sum_i s_{a^{(i)}}} \prod_{i=1}^n (\sum_{m_i} (|m_i|+1)^{r_{a^{(i)}}+\ldots+r_{a^{(n)}}} |(\hat{f}_i)_{m_i}|)$$

$$\leq M_{a^{(1)},\ldots,a^{(n)}} ||v||_{\sum_i s_{a^{(i)}}} \prod_{i=1}^n ||f_i||_{r_{a^{(i)}}+\ldots+r_{a^{(n)}}} < \infty.$$

Remark 3.6.6 *Let $u = \sum c_k u_k \in \mathcal{V}_{(\infty)}$, and $v = \sum v_k \in \mathcal{V}_{(\infty)}$ a result of past insertions at $w_1, \ldots, w_l$, where u_k, v_k have energy k and $||u_k|| = 1$. From our previous computation,*

$$\sum_{m_1,\ldots,m_n} |c_k||(u_k, \prod_{i=1}^n (a^{(i)})_{m_i} v_{k'})| \prod_{i=1}^n |(\hat{f}_i)_{m_i}| \leq$$

$$|c_k| M_{a^{(1)},\ldots,a^{(n)}} ||v_{k'}||_{\sum_i s_{a^{(i)}}} \prod_{i=1}^n ||f_i||_{r_{a^{(i)}}+\ldots+r_{a^{(n)}}}.$$

which sum over k, k' is finite:

$$M_{a^{(1)},\ldots,a^{(n)}} \prod_{i=1}^{n} ||f_i||_{r_{a^{(i)}}+\ldots+r_{a^{(n)}}} \sum_{k,k'} |c_k| \; ||v_{k'}||_{\sum_i s_{a^{(i)}}} =$$

$$M_{a^{(1)},\ldots,a^{(n)}} \prod_{i=1}^{n} ||f_i||_{r_{a^{(i)}}+\ldots+r_{a^{(n)}}} \sum_{k} |c_k| \sum_{k'} ||v_{k'}||_{\sum s_{a^{(i)}}},$$

but $\sum_k |c_k| < \infty$ and $\sum_{k'} ||v_{k'}||_{\sum s_{a^{(i)}}} < B((|\frac{w_i}{w_{i+1}}|)_{i=1}^{l-1}) \sum_{k'} (k'+1)^{\sum_i s_{a^{(i)}}} |w_l|^{k'} < \infty$, due to Lemma 3.6.1.

Proof: [Proof of Theorem 3.6.12] Choose $|w_l| < r_1 < \ldots < r_n < 1$, such that $\frac{|w_l|}{r_1} < c$ for some fixed $|w_l| < c < 1$, i.e. r_1 not arbitrarily close to $|w_l|$. Let $v'_{k'} = \frac{v_{k'}}{w_l^{k'}}$. By the previous remark,

$$|c_k|.|(u_k, P^{k_n} Y(a^{(n)}, f_n) P^{k_{n-1}} \ldots P^{k_1} Y(a^{(1)}, f_1) v'_{k'}).(\frac{w_l}{r_1})^{k'} \prod_{i=1}^{n-1} (\frac{r_i}{r_{i+1}})^{k_i} r_n^k|$$

is bounded by a constant (as $|w_l| < r_1 < \ldots < r_n < 1$ and $\frac{|w_l|}{r_1} < c$) multiplied by

$$|c_k|.|(u_k, P^{k_n} Y(a^{(n)}, f_n) P^{k_{n-1}} \ldots P^{k_1} Y(a^{(1)}, f_1) v'_{k'})| c^{k'},$$

the sum of which over $k, k', k_1, \ldots, k_{n-1}$ is finite. Indeed, by a similar argument in the previous remark, it is less than

$$M_{a^{(1)},\ldots,a^{(n)}} \prod_{i=1}^{n} ||f_i||_{r_{a^{(i)}}+\ldots+r_{a^{(n)}}} \sum_{k} |c_k| \sum_{k'} ||v'_{k'}||_{\sum_i s_{a^{(i)}}} c^{k'}$$

and $\sum_{k'} ||v'_{k'}||_{\sum_i s_{a^{(i)}}} c^{k'} < B((|\frac{w_i}{w_{i+1}}|)_{i=1}^{l-1}) \sum_{k'} (k'+1)^{\sum_i s_{a^{(i)}}} c^{k'}$ is still finite as $c < 1$. It is crucial that v is not any state in $\mathcal{V}_{(\infty)}$, but a state with energy components $v_{k'}$ with exponential norm decay (a corollary of being a result of past insertions). Otherwise, the

summation over k' might have been divergent due to presence of $r_1^{-k'}$. Thus as long as the conditions $|w_l| < r_1 < \ldots < r_n < 1$ and $\frac{|w_l|}{r_1} < c$ are satisfied, one can apply dominated convergence theorem to interchange the limit and the sum in the equation below

$$(u, \prod_{i=1}^{n} Y(a^{(i)}, f_i)v) =$$

$$\sum_{k',k,k_1,\ldots,k_n} \lim_{r_1,\ldots,r_n \to 1} c_k(u_k, P^{k_n} Y(a^{(n)}, f_n) P^{k_{n-1}} \ldots P^{k_1} Y(a^{(1)}, f_1) v'_{k'}).(\frac{w_l}{r_1})^{k'} \prod_{i=1}^{n-1} (\frac{r_i}{r_{i+1}})^{k_i} r_n^k$$

$$= \lim_{r_1,\ldots,r_n \to 1} \sum_{k',k,k_1,\ldots,k_n} c_k(u_k, P^{k_n} Y(a^{(n)}, f_n) P^{k_{n-1}} \ldots P^{k_1} Y(a^{(1)}, f_1) v'_{k'}).(\frac{w_l}{r_1})^{k'} \prod_{i=1}^{n-1} (\frac{r_i}{r_{i+1}})^{k_i} r_n^k$$

$$= \lim_{r_1,\ldots,r_n \to 1} \sum_{k',k,k_1,\ldots,k_n} \frac{1}{(2\pi)^n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} c_k(u_k, \prod_{j=1}^{n} P^{k_j} Y(a^{(j)}, e^{i\theta_j}) v'_{k'}) \prod_{j=1}^{n} f_j(e^{i\theta_j})$$

$$\prod_{i=1}^{n} \mathrm{d}\theta_i (\frac{w_l}{r_1})^{k'} \prod_{i=1}^{n-1} (\frac{r_i}{r_{i+1}})^{k_i} r_n^k$$

where the last equality is simply the integral giving the Fourier coefficients of f_is at $k_{i-1} - k_i$ for $i > 1$ and $k' - k_1$ for f_1. The next step is to use the r_i to change the insertion points to:

$$= \lim_{r_1,\ldots,r_n \to 1} \sum_{k',k,k_1,\ldots,k_n} \frac{1}{(2\pi)^n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} c_k(u_k, \prod_{j=1}^{n} P^{k_j} Y(a^{(j)}, r_j e^{i\theta_j}) v_{k'}) \prod_{j=1}^{n} f_j(e^{i\theta_j}) \prod_{i=1}^{n} \mathrm{d}\theta_i.$$

Next, we exchange the sum and integral. This is possible because of our first lemma on insertions in the past and the uniform convergence of the expansion of inner product in a locally compact neighborhood (see Remark 3.6.1). Note the assumption that $v = \prod_{i=1}^{l} Y(b^{(i)}, w_i)\Omega$ is an insertion at points $|w_1| < \ldots < |w_l| < r_1$ is being used here where $c_k\Big(u_k, \prod_{j=1}^{n} P^{k_j} Y(a^{(j)}, r_j e^{i\theta_j}) v_{k'}\Big)$ is similar to (3.68), which includes all the fields $a^{(i)}$ and $b^{(i)}$, but projections are not done after the past insertions $b^{(i)}$, except for $P^{k'}$ after the

last past insertion $b^{(l)}$. It follows

$$= \lim_{r_1,\dots,r_n\to 1} \frac{1}{(2\pi)^n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} (u, \prod_{j=1}^{n} Y(a^{(j)}, r_j e^{i\theta_j}) v) \prod_{j=1}^{n} f_j(e^{i\theta_j}) \prod_{i=1}^{n} \mathrm{d}\theta_i.$$

The last step is to exchange the limit and the integrals, which is possible due to the analyticity of $(u, \prod_{i=1}^{n} Y(a^{(i)}, z_i) \prod_{i=1}^{l} Y(b^{(i)}, w_i)\Omega)$ in $\mathrm{Conf}_{n+l}(D_1^{\times})$. ■

Set $d_a = \mathrm{wt}\, a$ and let v be an insertion in the past like in the previous setting.

Lemma 3.6.14 *For $\gamma(e^{i\theta}) = e^{i\rho(\theta)}$ with ρ a $2\pi-$periodic diffeomorphism of $\mathbb{R}$, i.e. $\rho(\theta + 2\pi) = \rho(\theta) + 2\pi$, we have*

$$(v, U(\gamma) \prod_{i=1}^{n} Y(a^{(i)}, z_i)\Omega) = \prod_{i=1}^{n} \rho'(\phi_i)^{d_{a^{(i)}}} (v, \prod_{i=1}^{n} Y(a^{(i)}, \gamma(z_i)) U(\gamma)\Omega), \tag{3.85}$$

where $z_j = e^{i\phi_j}$ are distinct points on S^1.

The lemma is essentially conformal covariance, but on the unit circle for *point-like* insertions. Recall the smeared version of conformal covariance for primary fields, where

$$U(\gamma) Y(a, f) U(\gamma)^{\dagger} = Y(a, \beta_{d_a}(\gamma)(f)) \tag{3.86}$$

as operators on smooth vectors, which implies

$$(v, U(\gamma) \prod_{i=1}^{n} Y(a^{(i)}, f_i)\Omega) = (v, \prod_{i=1}^{n} Y(a^{(i)}, \beta_{d_{a^{(i)}}}(\gamma)(f_i)) U(\gamma)\Omega). \tag{3.87}$$

Here, $\beta_{d_a}(\gamma)(f)(z = e^{i\theta}) = \left(\frac{1}{(\rho^{-1})'(\theta)}\right)^{d_a - 1} f(\gamma^{-1}(z))$, a generalization of the adjoint action γ_* defined in (3.47). (3.86) can be derived very similarly to [88, eq. (39)] and (3.47);

from the infinitesimal conformal covariance

$$[L_n, Y(a,z)] = (z^{n+1}\partial_z + (\text{wt } a)(n+1)z^n)Y(a,z) \tag{3.88}$$

one obtains the smeared infinitesimal conformal covariance

$$[iL(f), Y(a,g)] = Y(a, (\text{wt a})f'g - g'f), \tag{3.89}$$

which can be exponentiated. Then using the fact below gives the desired equation (3.86)

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp(\beta_{d_a}(\gamma_t)(g))|_{t=0} = (\text{wt a})f'g - g'f, \quad \text{where } \gamma_t = \exp(tL(f)).$$

Proof: [Proof of Lemma 3.6.14] Let $U(\gamma)^\dagger v = u_v, U(\gamma)\Omega = u_\Omega$ with $u_v, u_\Omega \in \mathcal{V}_{(\infty)}$. We first rewrite the RHS of (3.87) as

$$(-1)^{\sum d_{a^{(i)}}}(\prod_{i=n}^{1} Y(\eta(a^{(i)}), \overline{\beta_{d_{a^{(i)}}}(\gamma)(f_i)})v, u_\Omega),$$

where $\eta : \mathcal{V} \to \mathcal{V}$ gives the adjoint field. By the previous lemma, the LHS and RHS of (3.87) are themselves equal to

$$(u_v, \prod_{i=1}^{n} Y(a^{(i)}, f_i)\Omega) = \frac{1}{(2\pi)^n}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}(u_v, \prod_{j=1}^{n} Y(a^{(j)}, e^{i\theta_j})\Omega)\prod_{j=1}^{n} f_j(e^{i\theta_j})\prod_{i=1}^{n}\mathrm{d}\theta_i, \tag{3.90}$$

$$(-1)^{\sum d_{a^{(i)}}}(\prod_{i=n}^{1} Y(\eta(a^{(i)}), \overline{\beta_{d_{a^{(i)}}}(\gamma)(f_i)})v, u_\Omega) = \tag{3.91}$$

$$\frac{1}{(2\pi)^n}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}(-1)^{\sum d_{a^{(i)}}}(\prod_{j=n}^{1} Y(\eta(a^{(j)}), e^{i\theta_j})v, u_\Omega)\prod_{j=1}^{n}\beta_{d_{a^{(j)}}}(\gamma)(f_j)(e^{i\theta_j})\prod_{i=1}^{n}\mathrm{d}\theta_i.$$

Thus RHS of both equations above are equal. The terms inside the integrals have to be shown to be related according to the statement.

We shall take the limit as f_i converges to a delta function at $z_j = e^{i\phi_j}$ to recover the statement. Consider f_{δ_j}, parametrized by $\delta_j > 0$, as a smooth function on the circle, nonzero symmetrically around $e^{i\phi_j}$ with the measure of the support interval equal to δ_j, and with integral on the circle equal to 2π. As δ_j goes to zero, since $\phi_k \neq \phi_j$, the functions will have disjoint support which allows the use of the previous lemma. Hence,

$$(u_v, \prod_{i=1}^{n} Y(a^{(i)}, f_{\delta_i})\Omega) - (u_v, \prod_{j=1}^{n} Y(a^{(j)}, e^{i\phi_j})\Omega) = \tag{3.92}$$

$$\frac{1}{(2\pi)^n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \Big((u_v, \prod_{j=1}^{n} Y(a^{(j)}, e^{i\theta_j})\Omega) - (u_v, \prod_{j=1}^{n} Y(a^{(j)}, e^{i\phi_j})\Omega) \Big) \prod_{j=1}^{n} f_{\delta_j}(e^{i\theta_j}) \prod_{i=1}^{n} \mathrm{d}\theta_i.$$

$(u_v, \prod_{j=1}^{n} Y(a^{(j)}, e^{i\theta_j})\Omega)$ is uniformly continuous on any compact subset in $\mathrm{Conf}_n(D_1^{\times})$. As the integral can be taken not on the whole circle but only on δ_i compact neighborhood of ϕ_i, the norm of the difference inside the integrand gets smaller than any $\epsilon > 0$. Therefore as $\delta_i \to 0$

$$(u_v, \prod_{j=1}^{n} Y(a^{(j)}, e^{i\phi_j})\Omega) - \epsilon \leq (3.92) \leq (u_v, \prod_{j=1}^{n} Y(a^{(j)}, e^{i\phi_j})\Omega) + \epsilon \implies \tag{3.93}$$

$$(u_v, \prod_{i=1}^{n} Y(a^{(i)}, f_{\delta_i})\Omega) \to (u_v, \prod_{j=1}^{n} Y(a^{(j)}, e^{i\phi_j})\Omega).$$

Let us now evaluate the same limit for (3.91). For a function f with integral 2π, the function $\beta_{d_a}(\gamma)(f)$, by change of variable and periodicity of ρ, has integral

$$\int_{-\pi}^{\pi} \beta_a(\gamma)(f)\mathrm{d}\theta = \int_{-\pi}^{\pi} \Big(\frac{1}{(\rho^{-1})'(\theta)} \Big)^{d_a - 1} f(\gamma^{-1}(e^{i\theta}))\mathrm{d}\theta = \tag{3.94}$$

$$\int_{-\pi}^{\pi}\Big(\frac{1}{(\rho^{-1})'(\rho(\theta))}\Big)^{d_a-1}f(\gamma^{-1}(e^{i\rho(\theta)}))\mathrm{d}\rho(\theta)=\int_{-\pi}^{\pi}\rho'(\theta)^{d_a}f(e^{i\theta})\mathrm{d}\theta.$$

If f is converging to a delta function at $e^{i\phi}$, then the above will converge to $2\pi\rho'(\phi)^{d_a}$. Taking (3.91), we have

$$(\prod_{i=n}^{1}Y(\eta(a^{(i)}),\beta_{d_{a^{(i)}}}(\gamma)(f_{\delta_i}))v,u_\Omega)-\frac{1}{(2\pi)^n}(\prod_{j=n}^{1}Y(\eta(a^{(j)}),\gamma(e^{i\phi_j}))v,u_\Omega)\times$$
$$\prod_{i=1}^{n}\int_{-\pi}^{\pi}\beta_{d_{a^{(i)}}}(\gamma)(f_{\delta_i})\mathrm{d}\theta_i=$$
$$\frac{1}{(2\pi)^n}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\Big((\prod_{j=n}^{1}Y(\eta(a^{(j)}),e^{i\theta_j})v,u_\Omega)-(\prod_{j=n}^{1}Y(\eta(a^{(j)}),\gamma(e^{i\phi_j}))v,u_\Omega)\Big)\times$$
$$\prod_{j=1}^{n}\beta_{d_{a^{(j)}}}(\gamma)(f_{\delta_j})(e^{i\theta_j})\prod_{i=1}^{n}\mathrm{d}\theta_i$$

Since γ is smooth, as $\delta_i\to 0$, the support of $\beta_{d_{a^{(j)}}}(\gamma)(f_{\delta_j})(e^{i\theta_j})$ which depends on the support of $f_{\delta_j}(\gamma^{-1}(z_j))$ goes to zero as well and the functions will have disjoint support. Hence the integral can be taken on arbitrarily small neighborhoods of $\gamma(e^{i\phi_j})$s. With an argument similar to (3.92), given that the integrals $\int_{-\pi}^{\pi}\beta_{d_{a^{(i)}}}(\gamma)(f_{\delta_i})\mathrm{d}\theta_i\to\rho'(\phi_i)^{d_{a^{(i)}}}$, it follows

$$(u_v,\prod_{j=1}^{n}Y(a^{(j)},e^{i\phi_j})\Omega)=(-1)^{\sum d_{a^{(i)}}}\prod_{i=1}^{n}\rho'(\phi_i)^{d_{a^{(i)}}}(\prod_{j=n}^{1}Y(\eta(a^{(j)}),\gamma(e^{i\phi_j}))v,u_\Omega)\implies \tag{3.95}$$

$$(v,U(\gamma)\prod_{j=1}^{n}Y(a^{(j)},e^{i\phi_j})\Omega)=\prod_{i=1}^{n}\rho'(\phi_i)^{d_{a^{(i)}}}(v,\prod_{j=1}^{n}Y(a^{(j)},\gamma(e^{i\phi_j}))U(\gamma)\Omega),$$

where the last implication follows from the last statement in Corollary 3.6.9. ■

Corollary 3.6.15 *$(\prod_{i=1}^{l}Y(b^{(i)},w_i)\Omega,U(\gamma)\prod_{i=1}^{n}Y(a^{(i)},z_i)\Omega)$ on $Conf_l(\mathring{D}_1^{\times})\times Conf_n(S^1)$,*

i.e. $|w_i| < 1$ *and* $z_j = e^{i\phi_j}$, *equals*

$$(-1)^{\sum d_{a^{(i)}}} \prod_{i=1}^{n} \rho'(\phi_i)^{d_{a^{(i)}}} (\prod_{i=n}^{1} Y(\eta(a^{(i)}), \gamma(z_i)) \prod_{i=1}^{l} Y(b^{(i)}, w_i)\Omega, U(\gamma)\Omega).$$

Proof: This is a direct application of the previous lemma followed by taking adjoints of $Y(a^{(i)}, \gamma(z_i))$ to the other side as shown in the last statement of Corollary 3.6.9. ■

By Corollary 3.6.7 $(\prod_{i=n}^{1} Y(\eta(a^{(i)}), z_i') \prod_{i=1}^{l} Y(b^{(i)}, w_i)\Omega, U(\gamma)\Omega)$ is an analytic function on $\mathrm{Conf}_{n+l}(D_1^{\times})$, implying the main theorem of this section:

Theorem 3.6.16 *The correlation function* $(\prod_{i=1}^{l} Y(b^{(i)}, w_i)\Omega, U(\gamma) \prod_{i=1}^{n} Y(a^{(i)}, z_i)\Omega)$ *is well-defined on* $Conf_l(S^1) \times_\gamma Conf_n(S^1)$ *and equals*

$$(-1)^{\sum d_{a^{(i)}}} \prod_{i=1}^{n} \rho'(\phi_i)^{d_{a^{(i)}}} (\prod_{i=n}^{1} Y(\eta(a^{(i)}), \gamma(z_i)) \prod_{i=1}^{l} Y(b^{(i)}, w_i)\Omega, U(\gamma)\Omega).$$

3.6.1 General case for intertwiners

Below we explain the necessary changes required for the generalization of the results in the previous part to intertwiners. Each result which proof needs some additional remarks is mentioned below.

- Throughout the previous part, there was a convention that $Y(a, z)z^{\mathrm{wt}\, a}$ be used instead of $Y(a, z)$. This had no impact in the convergence result, as it was a product by a constant *integer* power of z. More importantly, this also had no impact on the question of existence of analytic extensions. For intertwiners $z^{\mathrm{wt}\, a}$ is replaced by $z^{\mathrm{wt}\, a+\Delta_{\mathcal{Y}}}$, with $\Delta_{\mathcal{Y}} = (h_A + h_B) - h_C$ for conformal weights h of the irreps involved in the intertwiner type $\mathcal{Y}\binom{C}{A\ B}$. This number can be fractional. When doing analytic extension, this can no longer be ignored, as is explained below.

- The function $F_k = (u_k, \prod_{i=1}^{n} \mathcal{Y}_i(a^{(i)}, z_i)\Omega)$ is generally a *multi*-valued rational func-

tion on $\mathrm{Conf}_n(\mathbb{C}^\times)$, thus single-valued on the universal covering space $\widetilde{\mathrm{Conf}_n(\mathbb{C}^\times)}$. The important equation $F_k = \frac{G_k}{\prod_{i<j}(z_i-z_j)^{s_{ij}}}$ still holds with the caveat that G is a polynomial if multiplied by suitable fractional powers of z_is.

- Lemma 3.6.4 is proven similarly, with an important change in (3.71), where right equality needs to be changed to

$$\prod_{i<j}(z_i - z_j)^{s_{ij}} \sum_{\bar{1},\ldots,\bar{n}} B_{\sigma,\bar{1},\ldots,\bar{n}}(u_k, \prod_{i=1}^{n} \mathcal{Y}_{\bar{i}}(a^{(\sigma(i))}, z_{\sigma(i)})\Omega),$$

  where braiding coefficients $B_{\sigma,\bar{1},\ldots,\bar{n}}$ appear to allow the permutation of intertwiners. This is the same as performing the necessary braidings on a fusion tree to implement a given permutation σ on the anyons. Here, summation is over $\bar{i}$, which represents all intertwiners of type $\binom{C}{A_i\ B}$ for all irreps B, C. Of course, only admissible types matter according to fusion rules. Recall that the equations (3.71) are in formal calculus, and that is enough to bound the degrees of z_i in G_k. The bound $k+\sum_j s_{ij}$ in Lemma 3.6.4 will only change by some constants, all depending on Δ_y of the intertwiners involved in the equation above.

- Regarding Corollary 3.6.7 and all similar results showing certain correlators are analytic or equal on some configuration space, in this generalized setting, they now imply that the correlators are multi-valued analytic functions or equal as multi-valued analytic functions.

- Proposition 3.12 in [31] is the basis of the results involving the relationship between smeared and point-like correlators on the unit circle. This proposition is formulated for general intertwiners. As such, all arguments leading to Theorem 3.6.12 easily generalize. Then, Lemma 3.6.14 and the main Theorem 3.6.16 follow similarly.

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