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Summary

This report describes the use of cubic spline fitting to compute, for the median plane z = 0, the vertical magnetic-field component, B(u, w, 0) and its first, second, and third partial derivatives from a set of measurements of B at mesh points (u_i, w_j). The arguments (u, w) may be either rectilinear (x, y) or polar (θ, R).

The results of the fitting are used for a third-order approximation of the field components B_u, B_w, and B_z at any point (u, w, z) within the domain of measurement for (u, w) and for small z. These field components are used in the equations of motion for tracking charged particles through the magnetic field. In particular, the application of this process to the magnetic field of the Bevatron (Berkeley) is discussed.

Introduction

Median field measurements of the vertical component, B(u, w, 0) of a magnetic field are usually available only on a relatively coarse and often nonuniform mesh. Some interpolation and approximation of differentiation must be applied to these measurements to compute B and its partial derivatives for use in a third-order approximation of the field components B_u, B_w, and B_z at specified points (u, w, z).[1] These field components appear in the equations of motion of a charged particle through the magnetic field.

Cubic spline fitting provides for interpolation and approximate differentiation up to third order for a function of one variable when the function values are known only for discrete (not necessarily uniformly spaced) values of the argument.[2] Although the median plane vertical component of the field is a function of two variables, separation may be possible:

$$B(u, w, 0) = g(u) \cdot f(w)$$

or, in turn each of the arguments may be held fixed and B, with the one fixed argument, treated as a function of the nonfixed argument:

$$B(u, w_j, 0) = B^{(j)}(u)$$
$$B(u_i, w, 0) = B^{(i)}(w) .$$

The fitting can be applied to g and f or to $B^{(j)}$ and $B^{(i)}$ to compute values for each of these functions and its first three derivatives with respect to its argument.

Cubic Spline Fitting

Properties

Let h(u) be any function, h, of any variable, u, whose values, h_i, are known only for a distinct and increasing set of points:

$$u_i, \ i = 1, I \text{ with } I \geq 3$$

and whose terminal first derivatives

$$h'_1 \equiv h'(u_1) \text{ and } h'_I \equiv h'(u_I)$$

are known (or can be stipulated). The cubic spline fit, s(u) for h, has the following properties:
1. The function s(u) is defined for the interval $[u_1, u_I]$.
2. On any subinterval $[u_i, u_i+1]$, s is a cubic in u.
3. Known values for h are fitted exactly --

$$s_i \equiv s(u_i) = h_i, \ s'_1 \equiv s'(u_1) = h'_1 \text{ and } s'_I \equiv s'(u_I) = h'_I .$$

4. On the whole interval $[u_1, u_I]$, s has continuous first and second derivatives.
5. The third derivative is piecewise continuous (from 2, s''' is constant on each subinterval (u_i, u_i+1); i = 1, I - 1).

Construction

From Properties 2 and 4 above, we deduce

$$d_i s'_{i-1} + 2(d_i + d_{i-1})s'_i + d_{i-1} s'_{i-1}$$

$$= 3 \left\{ d_i(s_i - s_{i-1})/d_{i-1} + d_{i-1}(s_{i+1} - s_i)/d_i \right\}$$

for i = 2, I - 1, where $d_i = u_{i+1} - u_i$. In accordance with Property 3, we substitute the known values of h_i, h'_1, and h'_I for the s_i, s'_1, and s'_I a system of linear equations to solve for s'_i, i = 2, I - 1. This system is tridiagonal with diagonal dominance; hence, it is always determinate.[3]

*This work was done under the auspices of the U. S. Atomic Energy Commission.

The values of s, s', s'', and s''' are readily determined for any point u in any subinterval $[u_i, u_{i+1}]$ by (u_i, s_i, s'_i) and $(u_{i+1}, s_{i+1}, s'_{i+1})$. The values, s'_I and S''_I can be determined from $(u_{I-1}, s_{I-1}, s'_{I-1})$ and (u_I, s_I, s'_I). We can find an interpolative value of h and estimated values for h', h'', and h''' at any $u \epsilon [u_1, u_I]$ by assuming that these are equal respectively to s, s', s'', and s''' at u.

Fitting Field Measurements

Polar Mesh

Where the median-plane vertical component of the field is expressible as the product of a function of radius alone and a function of azimuth alone (this is the case in each quarter section of the Bevatron with $1.°8 \leq \theta \leq 88.°2$), we have

$$B(\theta, R, 0) = g(\theta) \cdot f(R).$$

With a set of measurements $f_i \equiv f(R_i)$; i = 2, I - 1, we assume that for some sufficiently small R_1 and sufficiently large R_I that f_1, f'_1, f_I, and f'_I are all zero. We can then fit a cubic spline through the points (R_i, f_i): i = 1, I. Values for f, f', f'', and f''' can then be estimated for any R in $[R_1, R_I]$.

With a set of measurements $g_j \equiv g(\theta_j)$; j = 1, J with J > 3, θ measured in radians for later convenience, we must make a careful estimate $g'_1 \equiv g'(\theta_1)$ and $g'_J \equiv g'(\theta_J)$. Then, we can construct a cubic spline fit for g, and values for g, g', g'', and g''' can be computed for any θ in $[\theta_1, \theta_J]$.

Values for B and its partial derivatives up to third order for any point, (θ, R), in $[\theta_1, \theta_J] \times [R_1, R_I]$ can be readily computed from:

$B = g f$,

$\partial B/\partial\theta = g'f$, $\quad \partial B/\partial R = g f'$,

$\partial^2B/\partial\theta^2 = g''f$, $\quad \partial^2B/\partial\theta\partial R = g'f'$, $\quad \partial^2B/\partial R^2 = g f''$,

$\partial^3B/\partial\theta^3 = g'''f$, $\quad \partial^3B/\partial\theta^2\partial R = g''f'$,

$\partial^3B/\partial\theta\partial R^2 = g'f''$, $\quad \partial^3B/\partial R^3 = g f'''$.

Rectilinear Mesh

Separation of the rectilinear variables, (x, y), is not often practical (usually not possible). When the vertical component B has been measured on a complete rectangular grid (x_i, y_j); i = 1, I ≥ 3; j = 1, J ≥ 3, we can, for each fixed j, use the values

$$B_i^{(j)} \equiv B^{(j)}(x_i) \equiv B(x_i, y_i)$$

and carefully estimate the terminal derivatives with respect to x,

$$B_1^{(j)'} \equiv B^{(j)'}(x_1)$$

and

$$B_I^{(j)'} \equiv B^{(j)'}(x_I),$$

then construct the cubic spline fit of $B^{(j)}$, obtaining $B_i^{(j)'} \equiv B^{(j)'}(x_i)$ for i = 2, I - 1.

Similarly, for each fixed x_i, we use the values

$$B_j^{(i)} \equiv B^{(i)}(y_j) \equiv B(x_i, y_j),$$

to estimate terminal first derivatives with respect to y and construct the cubic spline fit of $B^{(i)}$, determining the remaining first derivative values at the y_j.

At the grid points (x_i, y_j); i = 1, I and j = 1, J, we can now set

$$\partial B/\partial x = B_i^{(j)'} \text{ and } \partial B/\partial y = B_j^{(i)'}.$$

Interpolation and higher-order differentiation is discussed later.

Sometimes (as was the case in the Bevatron) the measurement grid is not rectangular. For one of the rectilinear variables, say y, for some set of y_j; j = 1, J ≥ 3, we may have a set of measurements $B(x_{ij}, y_j)$; ij = 1, IJ ≥ 3. For each fixed j, we use the values

$$B_{ij}^{(j)} \equiv B_i^{(j)}(x_{ij}) \equiv B(x_{ij}, y_j),$$

estimate terminal derivatives, and construct a cubic spline fit for each $B^{(j)}$. We select some set of x, x_i; i = 1, I ≥ 3 with

$$x_1 \geq \max_j x_{1j}$$

and

$$x_I \leq \min_j x_{Ij}.$$

We then interpolate on the cubic spline to obtain values for $B^{(j)}(x_i)$, then set

$$B(x_i, y_j) = B^{(j)}(x_i),$$

which gives us values for B on the rectangular mesh (x_i, y_j). We can now proceed as outlined in the previous paragraphs.

Now, from the cubic spline fits of $B^{(j)}(x)$, $B^{(i)}(y)$ on the complete rectangular mesh, we compute at each grid point (x_i, y_j) values for $B_j^{(i)''}$, $B_j^{(i)'''}$, $B_i^{(j)''}$, and $B_i^{(j)'''}$ and set

$$\partial^2B/\partial x^2 = B_j^{(i)''} \qquad \partial^3B/\partial x^3 = B_j^{(i)'''}$$

$$\partial^2B/\partial y^2 = B_i^{(j)''} \qquad \partial^3B/\partial y^3 = B_j^{(i)'''}.$$

The mixed third partial derivatives, $\partial^3B/\partial x\partial y^2$ and $\partial^3B/\partial x^2\partial y$ cannot be obtained directly from the spline fit. The assumption that $\partial^3B/\partial x\partial y^2$ is constant over the subintervals, (x_i, x_{i+1}); i = 1, I - 1 for each fixed j; j = 1, J gives

$$\frac{\partial^3B}{\partial x dy^2}(x_i, y_j)$$

$$= \left[\frac{\partial^2B}{\partial y^2}(x_{i+1}, y_j) - \frac{\partial^2B}{\partial y^2}(x_i, y_j)\right] / [x_{i+1} - x_i],$$

and we may set

$$\frac{\partial^3 B}{\partial x \partial y^2}(x_I, y_j) = \frac{\partial^3 B}{\partial x \partial y^2}(x_{I-1}, y_j) .$$

Similarly, for j = 1, J - 1, with each fixed i; i = 1, I

$$\frac{\partial^3 B}{\partial x^2 \partial y}(x_i, y_j)$$

$$= \left[\frac{\partial^2 B}{\partial x^2}(x_i, y_{j+1}) - \frac{\partial^2 B}{\partial x^2}(x_i, y_j)\right] / y_{j+1} - y_j ,$$

and for J

$$\frac{\partial^3 B}{\partial x^2 \partial y}(x_i, y_J) = \frac{\partial^3 B}{\partial x^2 \partial y}(x_i, y_{J-1}) .$$

There are several ways to compute $\partial^2 B/\partial x \partial y$ at grid points. One is to expand $\partial B/\partial y$ from (x_i, y_j) to (x_{i+1}, y_j) for each j and for i = 1, I - 1 and solve for $\partial^2 B/\partial x \partial y\ (x_i, y_j)$:

$$\frac{\partial^2 B}{\partial x \partial y}(x_i, y_j)$$

$$= \left[\frac{\partial B}{\partial y}(x_{i+1}, y_j) - \frac{\partial B}{\partial y}(x_i, y_j)\right] / h_i$$

$$- \left[h_i \frac{\partial^3 B}{\partial x^2 dy}(x_i, y_j)\right] / 2$$

where $h_i = x_{i+1} - x_i$. Then set

$$\frac{\partial^2 B}{\partial x \partial y}(x_I, y_j) = \frac{\partial^2 B}{\partial x \partial y}(x_{I-1}, y_j)$$

$$+ h_{I-1} \frac{\partial^3 B}{\partial x^2 \partial y}(x_{I+1}, y_j) .$$

We now have for every grid point; (x_i, y_j); i = 1, I and j = 1, J, values for the median-plane vertical field component B and all its partial derivatives up to third order.

For any nongrid point (x, y) in $[x_1, x_I] \times [y_1, y_J]$ there is some i, such that $x_i < x < x_{i+1}$ and some j such that $y_j < y < y_{j+1}$. We can write expansions for B and its first and second derivatives from (x_i, y_j) to (x, y) terminating with the third derivatives at (x_i, y_j). Third derivatives at (x, y) may simply be set equal to their counterparts at (x_i, y_j).

Field Components Off Median Plane

The nonvertical components B_u and B_w are zero in the median field. From the scalar potential, it can be shown that they are odd functions of z.[2] On the other hand, the vertical component is an even function of z, assuming the value B in the median plane (z = 0). With small z, we may use a third-degree approximation:

$$B_u(u, w, z) = A_u z + C_u z^3$$

$$B_w(u, w, z) = A_w z + C_w z^3$$

$$B_z(u, w, z) = B(u, w, 0) + C_z z^2 ,$$

where the coefficients A_u, C_u, A_w, C_w, and C_z are functions of (u, w) which are defined below for polar variables (θ, R) and rectilinear variables (x, y), respectively.

Polar Variables

Expressing the scalar potential in polar coordinates and differentiating to obtain field components yields:[1]

$$A_\theta(\theta, R) = \frac{1}{R} \frac{\partial B}{\partial \theta}$$

$$C_\theta(\theta, R) = -\frac{1}{6}\left\{\frac{1}{R^3}\frac{\partial^3 B}{\partial \theta^3} + \frac{1}{R^2}\frac{\partial^2 B}{\partial \theta \partial R} + \frac{1}{R}\frac{\partial^3 B}{\partial \theta^2 \partial R}\right\}$$

$$A_R(\theta, R) = \frac{\partial B}{\partial R}$$

$$C_R(\theta, R) = -\frac{1}{6}\left\{\frac{\partial^3 B^3}{\partial R^3} + \frac{1}{R}\cdot\frac{\partial^2 B}{\partial R^2} + \frac{1}{R^2}\left(\frac{\partial^3 B}{\partial \theta^2 \partial R} - \frac{\partial B}{\partial R}\right) - \frac{2}{R^3}\frac{\partial^2 B}{\partial R^2}\right\}$$

$$C(\theta, R) = \frac{1}{2}\left\{\frac{1}{R^2}\frac{\partial^2 B}{\partial \theta^2} + \frac{1}{R}\frac{\partial B}{\partial R} + \frac{\partial^2 B}{\partial R^2}\right\} ,$$

where all terms on the right-hand side are evaluated at (θ, R, 0).

The field components are computed by using:

$$B_\theta(\theta, R, z) = A_\theta(\theta, R) z + C_\theta(\theta, R) z^3$$

$$B_R(\theta, R, z) = A_R(\theta, R) z + C_R(\theta, R) z^3$$

$$B_z(\theta, R, z) = B(\theta, R, 0) + C(\theta, R) z^2 .$$

Rectilinear Variables

From the scalar potential in rectilinear coordinates, we obtain:[1]

$$A_x(x, y) = \frac{\partial B}{\partial x}$$

$$C_x(x, y) = -\frac{1}{6}\left\{\frac{\partial^3 B}{\partial x^3} + \frac{\partial^3 B}{\partial x \partial y^2}\right\}$$

$$A_y(x, y) = \frac{\partial B}{\partial y}$$

$$C_y(x, y) = -\frac{1}{6}\left\{\frac{\partial^3 B}{\partial x^2 \partial y} + \frac{\partial^3 B}{\partial y^3}\right\}$$

$$C(x, y) = -\frac{1}{2}\left\{\frac{\partial^2 B}{\partial x^2} + \frac{\partial^2 B}{\partial y^2}\right\},$$

with terms on the right evaluated at (x, y, 0).

Then the field components are computed by using:

$$B_x(x, y, z) = A_x(x, y)\, z + C_x(x, y)\, z^3$$

$$B_y(x, y, z) = A_y(x, y)\, z + C_y(x, y)\, z^3$$

$$B_z(x, y, z) = B(x, y, 0) + C\, z^2 .$$

Computer Codes

Computer codes SPYGTH, SPBVFR, SPYBYT, SPXTBY, and SPYTBX have been written in FORTRAN 66 for the CDC 6600 which use an existing (Berkeley) library subroutine, SPLYND, to construct spline fits of various Bevatron measured field data. The computer code, BEVORB, tracks particles through the Bevatron magnetic field as expressed by the spline fits. This code has subroutines for interpolation of the spline-fit results. The equations of motion are expressed with x or θ as independent variables (valid when other momentum components are small compared with p_x or p_θ). A Runge-Kutta process of fourth order with input integration steps is used.

Descriptions, listings, and card-input decks are available from the author. It should be realized that these codes were written explicitly for the Bevatron. However, our experience has been that they can be readily modified for other magnetic field configurations.

Conclusion

Computer results of tracking particles through the magnetic field of the Bevatron have been in consistently good agreement with actual results in the accelerator. Since the code, BEVORB, obtains all of its field component information by the methods described in this article, we feel we have strong empirical evidence of the validity of this application of cubic spline fitting.

The cubic-spline-fitting curve is, in general, less likely to have extreme local curvature which may appear when high-degree polynomial fitting is used. Any fitting method involving least squares may introduce considerable distortion of derivative estimates. Local fitting (such as cubic fitting on each set of four successive points) does not preserve continuity of derivatives nor involve any global properties of the data. The cubic spline fit is consistent with the third-order approximation of the field components. If indicated by the data, other third-order splines, such as the hyperbolic spline[4] or damped cubic spline[5] could be used.

In our experience on the Bevatron field, the cubic spline fit makes available in useful form the information contained in measurements with very little distortion.

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