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We present a simple formalism describing evolution of a qubit in the process of its measurement in a circuit QED setup. When a phase-sensitive amplifier is used, the evolution depends on only one output quadrature, and the formalism is the same as for a broadband setup. When a phase-preserving amplifier is used, the qubit evolution depends on two output quadratures. In both cases a perfect monitoring of the qubit state and therefore a perfect quantum feedback is possible.

I. INTRODUCTION AND QUALITATIVE DISCUSSION

The goal of this Lecture is to present a physical picture of the process of continuous quantum measurement of a qubit in the circuit quantum electrodynamics (cQED) setup [1–5] (Fig. 1), extending or reformulating the previous theoretical descriptions [6–9]. Understanding of the qubit evolution in the process of measurement is important for developing an intuition, which is useful in many cases, in particular in designing various schemes of the quantum feedback [10–12]. When a quantum measurement is discussed [13], there are usually two different types of questions to answer: we can either focus on obtaining information on the initial state (before measurement) or focus on the quantum state after the measurement (i.e. evolution in the process of measurement). Let us emphasize that we consider the latter problem here and essentially extend the collapse postulate by describing continuous evolution "inside" the collapse timescale.

In the cQED setup (Fig. 1) a qubit interacts with a GHz-range microwave resonator, whose frequency slightly changes depending on whether the qubit is in the state $|0\rangle$ or $|1\rangle$ [1–9]. In turn, this frequency shift affects the phase (and in general amplitude) of a probing microwave, which is transmitted through the resonator (in another setup the microwave is reflected from the resonator, but the difference is not important). The outgoing microwave is amplified, and after that the rf signal is downconverted by mixing it with the original microwave tone, so that the low-frequency (< 100 MHz) output of the IQ mixer provides information on the qubit state. The output noise is mainly determined by the first amplifying stage, the pre-amplifier. With recent development of nearly quantum-limited superconducting parametric amplifiers [5, 14, 15], it is natural to use them as pre-amplifiers [5] instead of cryogenic high-electron-mobility transistors (HEMTs) [1–4], which usually have noise temperature above 3 K.

Continuous quantum measurement in the cQED setup in some sense falls in between a qubit measurement by a quantum point contact (QPC) or a single-electron transistor (SET), the theory of which has been developed over a decade ago [16, 17], and continuous quantum measurement in optics, for which the theory of quantum trajectories has been developed even earlier [18]. Nevertheless, the cQED setup differs from both these cases, and this is probably the reason why there is still a confusion about the proper physical description of the measurement process. The measurement by the QPC or SET is of the broadband type, meaning that the monitored frequency band starts from zero. In contrast, the cQED setup is of the narrowband type: we deal only with a relatively narrow band around the probing microwave frequency ω_m. This necessarily involves two orthogonal quadratures [19]: we work with rf signals of the type $A(t)\cos(\omega_m t) + B(t)\sin(\omega_m t)$, and there are essentially two signals $A(t)$ and $B(t)$ instead of only one in the broadband case. In this sense the cQED setup is similar to the optical (especially cavity QED) setup [18]; however, there is an important difference: in the cQED case the outgoing microwave is amplified (Fig. 1) before being mixed with the original microwave, while there is no amplification stage in the standard optical setup. The operation will obviously depend on whether a phase-sensitive or a phase-preserving amplifier is used, since a phase-preserving amplifier necessarily adds the half quantum of noise into any quadrature [7, 20–22]. Notice that the quantum trajectory theory for the cQED setup was developed in [9]; however, the amplifier stage was essentially missing in the analyzed model.

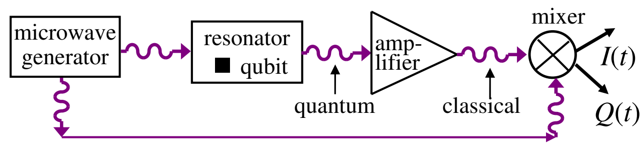

FIG. 1: Schematic of the cQED setup. Microwave field of frequency ω_m is transmitted through (or reflected from) the resonator of frequency ω_r, which slightly changes, $\omega_r \pm \chi$, depending on the qubit state. After amplification the microwave is sent to the IQ mixer, which produces two quadrature signals: $I(t)$ and $Q(t)$. For a phase-preserving amplifier we define $I(t)$ as the quadrature carrying information on the qubit state, while for a phase-sensitive amplifier we define $I(t)$ as corresponding to the amplified quadrature.

In this Lecture we consider the simplest cQED case, assuming dispersive regime [6], exactly resonant microwave frequency, absence of the Rabi drive, and sufficiently wide resonator and amplifier bandwidths for the Markov approximation. Some generalizations are rather straightforward; however, our goal is a simple picture in a simple case.

A description of continuous qubit measurement is essentially a description of the quantum back-action. Following the same quantum Bayesian framework as for the measurement by QPC/SET [16] (see [23] for review), we will discuss two kinds of measurement back-action onto the qubit, which we name here as "spooky" and "realistic". *The "spooky" (or "quantum", "informational", "non-unitary") back-action does not have a physical mechanism* and therefore cannot be described by the Schrödinger equation (in contrast to what people often think, trying to find a mechanism for the quantum collapse); however, *it is a common-sense consequence of acquiring information* on the qubit state in the process of measurement. This is essentially the same back-action which is discussed in the EPR paradox [24] and Bell inequality violation [25]; the only difference is that in our case the information is incomplete and therefore we have to use the quantum Bayes rule [23, 26, 27] instead of the projective collapse rule. In contrast, the "realistic" (or "classical", "unitary") back-action has a physical mechanism: in the cQED case it is a fluctuation of the number of photons in the resonator, which affects the phase of the qubit state. The "realistic" back-action is usually discussed in the standard theories of the cQED measurement [6–8]. Actually, there is a certain spookiness even in the "realistic" back-action (it may be affected by a delayed choice, as discussed in Conclusion); however, we do not want to emphasize it to keep the picture simple. When we measure the z-coordinate of the qubit state on the Bloch sphere (the basis states $|1\rangle$ and $|0\rangle$ correspond to the North and South poles), then the "spooky" back-action changes the z-coordinate and leads to the state evolution along the meridian lines, while the "realistic" back-action leads to the evolution around the z-axis, i.e. along the parallels.

It is important to notice that when the probing microwave leaves the resonator after interaction with the qubit, one quadrature of the microwave carries information about the qubit state, while the orthogonal quadrature carries information on the fluctuating number of photons in the resonator [6–9]. Therefore, if a phase-preserving amplifier is used, then the "spooky" and "realistic" back-actions are fully separated and correspond to two orthogonal quadratures $I(t)$ and $Q(t)$ measured after the mixer (it is trivial to choose the proper linear combinations of the I/Q mixer outputs). The signals $I(t)$ and $Q(t)$ are necessarily noisy, and the measurement back-actions are stochastic; however, there is a correlation (full correlation in the ideal case) between the output noise and the back-action noise in both channels. As a result (derived later), for a quantum-limited phase-preserving amplifier and absence of extra decoherence, the measured quadratures $I(t)$ and $Q(t)$ give us *full information about the back-action*, so that a *random evolution of the qubit wavefunction can be monitored precisely* (a useful analogy is with a Brownian particle under a microscope: we cannot predict its motion, but we can monitor it). This is what is needed, in particular, for arranging a perfect quantum feedback control of the qubit state. It is interesting to notice that for an ensemble-averaged evolution (in which the random but monitorable qubit evolution is replaced by dephasing), exactly one half of the ensemble dephasing Γ comes from the "spooky" back-action, and the other half comes from the "realistic" back-action.

In the case of a phase-sensitive amplifier it is sufficient to measure after the mixer only the quadrature which was amplified; let us still denote it as $I(t)$, though now its phase is determined by the amplifier instead of the microwave-qubit interaction. In this case the "spooky" and "realistic" back-actions are in general mixed (not separated), because there is only one output signal $I(t)$. This situation exactly corresponds to the broadband measurement by the QPC/SET with a correlation between the output and "realistic" back-action noises [23]. The situation simplifies when the amplified quadrature is the one which carries information about the qubit state (z-coordinate). Then in the quantum-limited case the "realistic" back-action is fully absent: we cannot measure the photon number fluctuation and correspondingly it does not fluctuate (in the imperfect case the effect of the remaining "realistic" back-action can be described by an extra dephasing). So we are left with only the "spooky" back-action, and the quantum measurement description coincides with the simpler theory of measurement by a symmetric QPC [23], which does not produce the "realistic" back-action. In contrast, in the case when the photon-number quadrature is amplified, we do not obtain any information on the qubit z-coordinate, and therefore there is no "spooky" back-action, but only the "realistic" one. In a general case, when the amplified quadrature makes an arbitrary angle φ with the qubit-information quadrature, both types of the back-action are present, and their strength depends on φ. It is important to mention that the ensemble dephasing rate Γ does not depend on φ, as required by causality. In particular, in the quantum-limited case the contribution $\Gamma\cos^2\varphi$ comes from the "spooky" back-action, while $\Gamma\sin^2\varphi$ comes from the "realistic" back-action.

Let us emphasize that both the phase-sensitive and phase-preserving amplifiers permit exact monitoring of the qubit state and therefore a perfect quantum feedback. The necessary condition in both cases is that the detection system is quantum-limited.

In the following sections a formal description of the above discussed results is presented. We start with reviewing the Bayesian approach for the broadband qubit measurement, then briefly discuss the difference between phase-preserving and phase-sensitive amplifiers, and then present the formalism of the narrowband continuous measurement of a qubit in the cQED setup. In Conclusion we briefly discuss generalizations of the formalism, quantum feedback, and the causality principle. We note that our approach can be converted into the formal language of the positive-operator-valued-measure(POVM)-type generalized quantum measurement [28] (then separation of

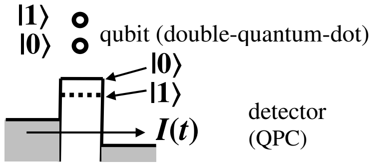

FIG. 2: Schematic of a broadband measurement setup: double-quantum-dot qubit is measured by a QPC (tunnel junction). The output signal $I(t)$ is the QPC current.

the "spooky" and "realistic" back-actions corresponds to the decomposition of the measurement operator into diagonal and unitary parts – see later), and our results for the case of a phase-sensitive amplifier are very similar to the results of Ref. [9].

II. BROADBAND MEASUREMENT

In this section we review the Bayesian formalism [16, 23] for the broadband measurement of a qubit, considering only the simple case without additional evolution, and thus emphasizing the main physical idea of the formalism. We start with the broadband formalism because it is simpler than for the narrowband (cQED) measurement and it can be used as a natural step in understanding the cQED setup.

For definiteness let us assume that the qubit is a double quantum dot populated with one electron (Fig. 2), and the states $|0\rangle$ and $|1\rangle$ correspond to the electron localized in one or the other dot. The qubit is measured by a small-transparency tunnel junction (model of QPC), whose barrier height depends on the electron location, so that the two qubit states correspond to different average currents I_0 and I_1 through the QPC. The voltage across the QPC is sufficiently large to make the detector output classical (Markov approximation), and $|\Delta I| \ll |I_c|$, where $\Delta I = I_1 - I_0$ is the response and $I_c = (I_0 + I_1)/2$ is the mean value; this weak response assumption allows us to consider the QPC current $I(t)$ as a quasicontinuous noisy signal (see [23] for the detailed discussion of required assumptions; the formalism needs only a minor change if $\Delta I \sim I_c$). Then the output signal of the detector is

$$I(t) = I_c + (\Delta I/2)\, z(t) + \xi(t), \;\; S_\xi(\omega) = S, \qquad (1)$$

where $z = \rho_{11} - \rho_{00}$ is the z-component of the Bloch sphere representation of the qubit density matrix $\rho(t)$, and $\xi(t)$ is the white shot noise with spectral density $S = 2eI_c$ (we use the single-sided definition for the spectral density, in which the signal variance ("power") corresponds to $\int_0^\infty S(\omega)\, d\omega/2\pi$; the definition of S is twice smaller in Ref. [7] and 4π times smaller in Refs. [8, 27]). We emphasize that the detector signal $I(t)$ is classical, and the qubit state $\rho(t)$ is practically unentangled from the detector, but obviously depends on $I(t)$.

The detector Hamiltonian and the qubit-detector interaction Hamiltonian are given in Refs. [16, 23], they are not really important for our discussion here. For simplicity let us assume that the qubit Hamiltonian is zero, $H_{qb} = 0$, so that the qubit evolution is due to the measurement only. In this case the qubit evolution during time t happens to be determined only by the time-averaged value of the measured detector output

$$\overline{I}_m(t) = \frac{1}{t}\int_0^t I(t')\, dt', \qquad (2)$$

which would contain full information for a classical measurement.

Because of the correspondence principle, the evolution of the diagonal elements of the qubit density matrix ρ ($\mathrm{Tr}\rho = 1$) should correspond to the classical evolution of probabilities, which are given by the classical Bayes rule. The Bayes rule says that an updated (a posteriori) probability of a system state is proportional to the initial (a priori) probability and the probability (likelihood) of the obtained measurement result assuming this particular state. In our case $\overline{I}_m(t)$ is the measurement result, and its probability for the qubit in the basis state $|j\rangle$ has the Gaussian distribution

$$P_{|j\rangle}(\overline{I}_m) = \frac{1}{\sqrt{2\pi D}} \exp[-(\overline{I}_m - I_j)^2/2D], \;\; D = \frac{S}{2t}, \quad (3)$$

where D is the variance, which decreases with the measurement time t. Therefore the correspondence principle demands the Bayesian evolution

$$\frac{\rho_{11}(t)}{\rho_{00}(t)} = \frac{\rho_{11}(0)}{\rho_{00}(0)} \frac{\exp[-(\overline{I}_m(t) - I_1)^2/2D]}{\exp[-(\overline{I}_m(t) - I_0)^2/2D]}, \qquad (4)$$

which in our terminology is due to the "spooky" back-action; it cannot be described by the Schrödinger equation, but follows from common sense.

If the phase of qubit state is not affected by the measurement process (no "realistic" back-action), then an arbitrary initial wavefunction $|\psi(0)\rangle = \sqrt{\rho_{00}(0)}\,|0\rangle + e^{i\phi}\sqrt{\rho_{11}(0)}\,|1\rangle$ becomes $|\psi(t)\rangle = \sqrt{\rho_{00}(t)}\,|0\rangle + e^{i\phi}\sqrt{\rho_{11}(t)}\,|1\rangle$ with the same phase ϕ; therefore for an arbitrary mixed state we get

$$\rho_{01}(t) = \rho_{01}(0)\, \frac{\sqrt{\rho_{00}(t)\, \rho_{11}(t)}}{\sqrt{\rho_{00}(0)\, \rho_{11}(0)}}\, . \qquad (5)$$

Equations (4) and (5) describe the "spooky" back-action.

Now assume that due to the qubit-detector interaction (e.g. Coulomb interaction), each electron passing through the detector rotates the qubit phase ϕ by a small amount $\Delta\phi$. From the measured result $\overline{I}_m(t)$ we know exactly how many electrons passed through [$n_e = \overline{I}_m(t)t/q$ with q being the electron charge], and can easily introduce the corresponding phase factor into Eq. (5):

$$\rho_{01}(t) = \rho_{01}(0)\, \frac{\sqrt{\rho_{00}(t)\, \rho_{11}(t)}}{\sqrt{\rho_{00}(0)\, \rho_{11}(0)}}\, \exp[iK\overline{I}_m(t)\, t], \qquad (6)$$

where $K=\Delta\phi/q$. The non-stochastic factor $\exp(iKI_c t)$ can be obviously ascribed to the qubit Hamiltonian; however, this is not important here. The factor $\exp[iK\overline{I}_m(t)\,t]$ in Eq. (6) is the effect of the "realistic" back-action. It may or may not be present in a particular physical situation; for example, $K=0$ for measurement by a symmetric QPC, while $K\neq 0$ in an asymmetric QPC or SET case.

Finally, if there is an extra pure dephasing of a qubit with rate γ, then Eq. (6) becomes

$$\rho_{01}(t)=\rho_{01}(0)\,\frac{\sqrt{\rho_{00}(t)\,\rho_{11}(t)}}{\sqrt{\rho_{00}(0)\,\rho_{11}(0)}}\,\exp[iK\overline{I}_m(t)\,t]\,e^{-\gamma t}. \quad (7)$$

Equations (4) and (7) is *the main starting point of the Bayesian formalism* [23]. It is then easy to include non-zero qubit Hamiltonian H_{qb} by differentiating Eqs. (4) and (7) over time (paying attention to whether the Stratonovich or Itô definition of the derivative is used) and adding terms due to H_{qb}. Energy relaxation and other mechanisms of the qubit state evolution can be included in the same way. Actually, there are many ways to derive the Bayesian equations (4) and (7) [16, 17, 23, 29, 30], but we focus here only on their meaning, not on their derivation.

Notice that averaging of Eqs. (4) and (7) over the measurement result $\overline{I}_m$ (i.e. ensemble averaging) with the probability distribution

$$P(\overline{I}_m)=\rho_{00}(0)P_{|0\rangle}(\overline{I}_m)+\rho_{11}(0)P_{|1\rangle}(\overline{I}_m) \quad (8)$$

gives the same evolution as for a pure dephasing: the diagonal matrix elements of ρ do not evolve, while the off-diagonal element ρ_{01} decays as $\rho_{01}(0)\,e^{-\Gamma t}$ with the ensemble dephasing rate [23]

$$\Gamma=\frac{(\Delta I)^2}{4S}+\frac{K^2S}{4}+\gamma, \quad (9)$$

which has clear contributions from the "spooky" back-action, "realistic" back-action, and additional dephasing.

In the case $\gamma=0$ an initially pure qubit state remains pure; in other words we can monitor evolution of a qubit wavefunction. This property can be used as the definition of a quantum-limited detector [16, 23]. The quantum efficiency η can then be naturally defined as $\eta=1-\gamma/\Gamma$. If by some reason the "realistic" back-action is considered as dephasing (i.e. only in the averaged way), then the quantum efficiency can be defined as $\tilde{\eta}=1-\gamma/\Gamma-K^2S/4\Gamma$ (here the definitions of $\tilde{\eta}$ and η are exchanged compared with the definitions in [23]). In other words, $\tilde{\eta}=(\Delta I)^2/4S\Gamma$ is the relative contribution of only the "spooky" back-action in the ensemble dephasing Γ. In particular, this definition is relevant to the peak-to-pedestal ratio of the Rabi spectral peak [31], which is equal to $4\tilde{\eta}$. As an example, if $\gamma=0$ and contributions in Eq. (9) from the "spooky" and "realistic" back-actions are equal to each other (as in the cQED setup with a phase-preserving amplifier), then $\eta=1$ but $\tilde{\eta}=1/2$.

A non-ideal detector ($\eta<1$) can be modeled in two equivalent ways [32]: we either add an extra dephasing γ to the qubit or we add an extra noise to the output of the ideal detector. Only the total dephasing Γ, response ΔI, total output noise S, and correlation factor $K=\delta\langle\phi\rangle/\delta(\overline{I}_m t)$ are the physical (i.e. experimentally measurable) parameters, while distribution of the non-ideality between the extra dephasing and additional output noise is a matter of convenience [here $\phi=\arg(\rho_{01})$, and notation $\langle\phi\rangle$ reminds about averaging over additional classical noise at the output]. We emphasize that the Bayesian formalism deals only with the experimentally measurable parameters ΔI, S, K, Γ, and the output signal $I(t)$.

In the ideal case ($\eta=1$) the evolution equations (4) and (6) can be translated into the language of POVM-type generalized measurement. In this approach the effect of measurement is described as [28]

$$|\psi(t)\rangle=\frac{M_R|\psi(0)\rangle}{||M_R|\psi(0)\rangle||},\;\; \rho(t)=\frac{M_R\rho(0)M_R^\dagger}{\mathrm{Tr}[M_R^\dagger M_R\rho(0)]}, \quad (10)$$

where M_R is the so-called measurement (Kraus) operator, corresponding to the result R. The probability of the result R is $P_R=||M_R|\psi(0)\rangle||^2$ using wavefunctions or $P_R=\mathrm{Tr}[M_R^\dagger M_R\,\rho(0)]$ using density matrices; therefore the POVM elements $M_R^\dagger M_R$ should satisfy the completeness condition $\sum_R M_R^\dagger M_R=\mathbb{1}$. The relation between this approach and the quantum Bayesian approach can be understood via the operator decomposition

$$M_R=U_R\sqrt{M_R^\dagger M_R}, \quad (11)$$

where U_R is unitary and the square root of the positive operator $M_R^\dagger M_R$ is defined in the natural way in the diagonalizing basis. It is easy to see that $\sqrt{M_R^\dagger M_R}$ is essentially the quantum Bayes rule (in the diagonalizing basis); in our terminology it corresponds to the "spooky" back-action, while U_R corresponds to the "realistic" back-action. For the discussed setup the result R is $\overline{I}_m(t)$, the "spooky" back-action $[M_R^\dagger M_R]^{1/2}$ should be determined by the probabilities $P_{|j\rangle}(\overline{I}_m)$ given by Eq. (3), and the "realistic" back-action U_R is given by the phase factor in Eq. (6). Therefore the corresponding measurement operator is

$$M(\overline{I}_m)=\exp(-iK\overline{I}_m t\,\sigma_z/2)\times\left[\sqrt{P_{|0\rangle}(\overline{I}_m)}\,|0\rangle\langle 0|+\sqrt{P_{|1\rangle}(\overline{I}_m)}\,|1\rangle\langle 1|\right], \quad (12)$$

where σ_z is the Pauli matrix.

III. PHASE-PRESERVING VS. PHASE-SENSITIVE AMPLIFIERS

Before discussing microwave amplifiers, let us consider a measurement of an oscillator, for example, a mechanical resonator with frequency ω_r and mass m. This is a very well studied problem [7, 13], so we will only discuss a way to understand the results. A classical resonator position x oscillates as $x_c(t) = A\cos(\omega_r t) + B\sin(\omega_r t)$ (in this section x stands for the usual spatial coordinate, not for the Bloch sphere coordinate). The corresponding quantum state is called the "coherent state" in the optical language; it is represented by the wavefunction $\psi(x,t) = \psi_{gr}[x - x_c(t)]\exp(ip_c x/\hbar)$, where $\psi_{gr}(x)$ is the ground state and $p_c = m\dot{x}_c(t)$ is the classical momentum. So the coherent state is essentially the ground state with oscillating center position. Notice that continuous quantum measurement of a resonator position can be described in the same Bayesian way [33] as in the previous section; for example the "spooky" back-action gives the evolution $\psi(x,t) = \psi(x,0)\exp[-(\overline{I}_m(t) - I(x))^2/4D]/\text{Norm}$, where $I(x)$ is the average detector signal for the resonator position x, and Norm is normalization [see Eqs. (4) and (5)]. The time step t in this case should be chosen much shorter than ω_r^{-1} so that the unitary evolution and evolution due to measurement may be simply added.

Let us consider the following game: Charlie prepares an oscillator in a coherent state with quadratures A and B, gives it to David, and David's goal is to find A as accurately as possible. An optimal strategy is rather obvious: David should make a projective measurement of x at time $t = 2\pi n/\omega_r$ with any integer n (to avoid contribution from the B-term), and the measurement result is the best estimate of A [if the measurement is done at $t = (2\pi n + \pi)/\omega_r$, then the result should be multiplied by -1]. Even though the strategy is optimal, the inaccuracy of David's result is obviously the width (standard deviation) $\sigma_{gr} = \sqrt{\hbar/2m\omega_r}$ of the ground state shape $|\psi_{gr}(x)|^2$; in energy units this inaccuracy corresponds to one half of the energy quantum.

Now assume that David cannot make projective measurements, but only "finite-strength" (i.e. imprecise) measurements. The best accuracy σ_{gr} can still be achieved if the measurement is done in the simple but very clever "quantum non-demolition" (QND) way: many finite-strength measurements are made at times $t = 2\pi n/\omega_r$; this is called "stroboscopic" measurement [13]. Since the oscillator returns to the same state after the period $2\pi/\omega_r$, the unitary evolution is not important, and many finite-strength measurements (described by the Bayesian equation above) are "stacked" to produce a strong, essentially projective measurement. More generally, the necessary condition to have the best accuracy σ_{gr} for A is that the measurement is not sensitive to the quadrature B.

Now assume that David is only allowed to make a continuous measurement with unmodulated weak strength (so that the inaccuracy achieved after ω_r^{-1} is much larger than σ_{gr}). Then the "spooky" back-action gets mixed with the unitary evolution, essentially adding noise into the monitored evolution, so that after a while the resonator state becomes mostly determined by the back-action and almost not dependent on the initial state. As the result, the best accuracy for measurement of A becomes $\sqrt{2}\,\sigma_{gr}$, which in energy units corresponds to two half-quanta [13], that is twice worse than for the projective or stroboscopic measurement. However, the continuous monitoring gives us an information about B in the same way as for A, so the accuracy of B-measurement is also $\sqrt{2}\,\sigma_{gr}$. Therefore, in some sense continuous phase-insensitive measurement brings the same total information as the phase-sensitive (e.g. stroboscopic) measurement; however, in our game only half of this information is useful for the David's goal.

After discussing measurement of the resonator state it is easy to understand the quantum limits for the high-gain microwave amplifiers. Now suppose Charlie prepares a coherent state of a microwave resonator with quadratures A and B, gives it to David to find A, and David uses an amplifier for amplification of the microwave field, which slowly leaks from the resonator until it is empty. There is only classical signal processing after the amplifier, so amplification is essentially the quantum measurement. The results are the same as above [7, 13, 21, 22]: a phase-sensitive amplifier, which amplifies only A-quadrature and "de-amplifies" (attenuates) B-quadrature can measure A with accuracy σ_{gr}, while a phase-preserving amplifier can measure A only with accuracy $\sqrt{2}\,\sigma_{gr}$ (and also measures B with the same accuracy $\sqrt{2}\,\sigma_{gr}$). Technically, the accuracy is limited by the noise at the amplifier output, so this noise should forbid measuring of A with accuracy better than σ_{gr} by a phase-sensitive amplifier, and better than $\sqrt{2}\,\sigma_{gr}$ by a phase-preserving amplifier. Therefore in the quantum-limited case the output noise power of a phase-preserving amplifier (per quadrature) is twice larger than for a phase-sensitive amplifier with the same gain; this is often called an "additional noise", corresponding to one half of the energy quantum (one more half-quantum is present in both cases) [7, 13, 21, 22]. It may be somewhat confusing why this result does not depend on the rate with which the microwave leaks from the resonator. So let us check the scaling: for k times slower leakage (k times larger Q-factor) the microwave amplitude is $\sqrt{k}$ times smaller, but accumulation time is k times longer, therefore the measured signal for the quadrature A is $\sqrt{k}$ times larger, which is the same factor as for the noise accumulation. Therefore, the signal-to-noise ratio which determines the A-accuracy does not depend on the leakage rate (resonator bandwidth).

IV. NARROWBAND (CQED) MEASUREMENT

Using the above discussion of microwave amplifiers, it is easy to extend the Bayesian approach for a broadband quantum measurement to the narrowband cQED setup.

We consider the standard cQED setup [1–9], in which a qubit interacts with a microwave resonator, and assume the dispersive regime with the Hamiltonian

$$H = (\hbar\omega_{qb}/2)\,\sigma_z + \hbar\omega_r a^\dagger a + \hbar\chi a^\dagger a\sigma_z, \qquad (13)$$

where $\omega_{qb} = \omega_{qb,bare} + \chi$ is the Lamb-shifted qubit frequency with no photons in the resonator, $\chi = g^2/(\omega_{qb,bare} - \omega_r)$ is the effective coupling with g being the Jaynes-Cummings coupling, ω_r is the bare resonator frequency, Pauli operator σ_z acts on the qubit state in the energy basis $\{|0\rangle, |1\rangle\}$, and the resonator creation/annihilation operators are $a^\dagger$ and a. Notice that the resonator frequency increases by 2χ when the qubit state changes from $|0\rangle$ to $|1\rangle$; conversely, the qubit frequency increases by 2χ per each additional photon in the resonator. To measure the qubit state, a microwave field with frequency ω_m is either transmitted through or reflected from the resonator, then amplified and sent to the IQ mixer, which measures both quadratures relative to the original microwave tone (Fig. 1). The qubit state affects the resonator frequency and therefore affects the phase (and in general amplitude) of the outgoing microwave.

An elementary Fabry-Pérot analysis gives the classical (complex) microwave field F_r inside the resonator:

$$F_r = \frac{2F_{in}t_{in}/\kappa\tau_{rt}}{1 - 2i(\omega_m - \omega_r)/\kappa}, \tag{14}$$

where F_{in} is the applied incident field, t_{in} is the transmission amplitude of the barrier from the incident side, κ is the resonator bandwidth due to the microwave leakage from the both sides (the Q-factor is ω_r/κ), and the round-trip time is $\tau_{rt} = 2\pi/\omega_r$ for a half-wavelength resonator and $\tau_{rt} = \pi/\omega_r$ for a quarter-wavelength resonator. A similar formula with the same denominator describes a lumped resonator. In presence of the qubit, the resonator frequency ω_r in this formula is substituted by $\omega_r \pm \chi$, depending on the qubit state. Notice that for the quantum measurement analysis *there is no difference between the cases of transmission and reflection* for the same F_r and κ, because the field leaking from the resonator is determined only by F_r and κ. (The reflection case has a technical advantage of dealing with a twice smaller outgoing microwave field for the same measured signal.) However, an important parameter is the collected fraction $\eta_{col} = \kappa_{col}/\kappa$ of the leaking microwave power; we will often assume the ideal case $\eta_{col} = 1$ (for the transmission setup this requires strongly asymmetric coupling, $|t_{in}| \ll |t_{out}|$).

For simplicity we assume the resonant case, $\omega_m = \omega_r$, then the ensemble qubit dephasing due to measurement is [6, 7]

$$\Gamma = 8\chi^2\bar{n}/\kappa, \tag{15}$$

where $\bar{n}$ is the average number of photons in the resonator. It is easy to include Rabi oscillations into the model; however, we do not do it for simplicity and also for more transparent analogy with Sec. II, in which we considered a qubit with zero Hamiltonian, evolving only due to measurement; this case exactly corresponds to the cQED Hamiltonian (13) in the rotating frame.

We will need several assumptions to describe the qubit state evolution in the process of measurement. First, for the validity of the dispersive approximation (13) we need sufficiently large qubit-resonator detuning, $|\omega_{qb} - \omega_r| \gg |g|$, and not too many photons in the resonator, $\bar{n} \ll (\omega_{qb} - \omega_r)^2/g^2$ (we do not consider the recently discovered nonlinear regime [34]). Second, to use the Markov approximation for the evolution we need the so-called "bad cavity" assumption: $\Gamma \ll \kappa \ll \omega_r$ (if the qubit evolves due to Rabi oscillations with frequency Ω_R, we also need $\Omega_R \ll \kappa$). This assumption means that the photons leave the resonator much faster than evolution of the qubit state, and therefore there is practically no entanglement between the qubit and unmeasured microwave field. This assumption also implies that the two resonator states for the qubit states $|0\rangle$ and $|1\rangle$ are almost indistinguishable, $\bar{n}(\chi/\kappa)^2 \ll 1$. Third, we use the "weak response" assumption, which requires a small phase difference between the two resonator states, $|\chi|/\kappa \ll 1$. This means that each outgoing photon carries only a little information about the qubit state. Notice that for $\bar{n} \gtrsim 1$ the previous assumption $\kappa \gg \Gamma$ automatically implies the weak response, and even for $\bar{n} \ll 1$ the weak response assumption is not always needed. Fourth, we will neglect the qubit energy relaxation due to measurement [6, 7], which can be added later.

A coherent state in the resonator with average $\bar{n}$ photons and zero average phase corresponds to the oscillation of the field expectation value $\langle F_r(t)\rangle = 2\sqrt{\bar{n}}\,\sigma_{gr}\cos(\omega_m t)$, where σ_{gr} is the ground state width (rms uncertainty) and we assume $\omega_m = \omega_r$. (Notice that the amplitude σ_{gr} corresponds to 1/4 photon.) Interaction with the qubit slightly changes the phase, $\cos(\omega_m t \mp 2\chi/\kappa)$, depending on the qubit state, so that

$$\langle F_r(t)\rangle = A\cos(\omega_m t) + B\sin(\omega_m t), \;\; A = 2\sqrt{\bar{n}}\,\sigma_{gr},$$
$$B = \pm(4\chi/\kappa)\sqrt{\bar{n}}\,\sigma_{gr} = (4\chi/\kappa)\sqrt{\bar{n}}\,\sigma_{gr}\,z, \tag{16}$$

where z is the qubit Bloch coordinate. Thus the small B-quadrature carries information about the qubit state, while larger A-quadrature may give us information on the fluctuations of the photon number in the resonator. In the optical representation (Fig. 3) with axes A/σ_{gr} and B/σ_{gr}, the two resonator states for the qubit states $|0\rangle$ and $|1\rangle$ are shown as two "error circles" [35] with rms uncertainty 1 along any direction and distance $2\sqrt{\bar{n}}$ between the origin and circle centers (if axes $A/2\sigma_{gr}$ and $B/2\sigma_{gr}$ are used, then the distance is $\sqrt{\bar{n}}$, while the uncertainty is 1/2).

A. Phase-sensitive amplifier

Let us start with the case when a phase-sensitive amplifier is used in the cQED setup. Also, we first assume the most ideal case: the amplifier is quantum-limited, it amplifies the optimal B-quadrature, there is no microwave collection loss ($\kappa_{col}/\kappa = 1$), and there is no extra noise or dephasing. Then, as discussed in the previous section, measuring once the microwave contents of

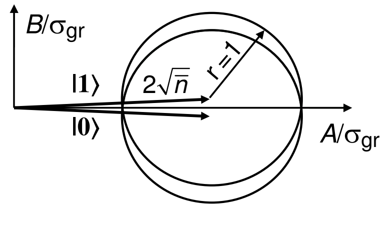

FIG. 3: Phase space representation: for each qubit state the coherent state with $\langle F_r \rangle = A\cos(\omega_r t) + B\sin(\omega_r t)$ in the resonator is shown [35] as an "error circle" with radius 1, shifted by $2\sqrt{\bar{n}}$ from the origin. Axes are normalized by the standard deviation σ_{gr} of the ground state. The B-quadrature carries information about the qubit state, the A-quadrature corresponds to the number of photons in the resonator.

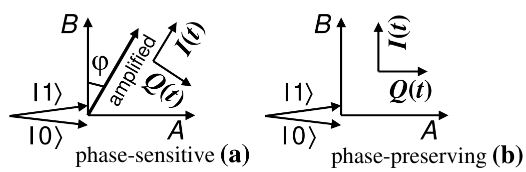

FIG. 4: Relations between relevant quadratures. The quadratures A and B are for the microwave field in the resonator. (a) For a phase-sensitive amplifier the amplified quadrature makes an angle φ with the informational B-quadrature. The corresponding quadrature at the mixer output is defined as $I(t)$ [the output $Q(t)$ is then useless]. (b) For a phase-preserving amplifier we define output quadratures $I(t)$ and $Q(t)$ as corresponding to the resonator quadratures B and A.

the resonator (by fully emptying it) we can measure the B-quadrature with imprecision σ_{gr}. Therefore, in the continuous measurement for time t the B-quadrature is measured with imprecision $\sigma_{gr}/\sqrt{\kappa t}$, which converts into the imprecision $\sqrt{\kappa/t}/(4\chi\sqrt{\bar{n}})$ of the qubit z-coordinate. Following the language of Sec. II, let us discuss the signal and noise at the output of the setup. There are two outputs of the IQ mixer; however, only the amplified quadrature carries an information, so let us denote the corresponding output of the mixer (or their linear combination) as $I(t)$. Then the response $\Delta I = I_1 - I_0$ corresponds to $\Delta z = 2$ and $\Delta B = 8(\chi/\kappa)\sqrt{\bar{n}}\,\sigma_{gr}$. For measurement during time t the above variance $(\kappa/t)/(4\chi\sqrt{\bar{n}})^2$ of the z-coordinate converts into the variance $(\kappa/t)(\Delta I/8\chi\sqrt{\bar{n}})^2$ of the measured output $\bar{I}_m = (1/t)\int_0^t I(t')\,dt'$. Equating it with $D = S/2t$, we find the (single-sided) spectral density of the $I(t)$ noise:

$$S_{min} = (\Delta I_{max})^2 \kappa/(32\chi^2\bar{n}), \tag{17}$$

where we replaced S with S_{min} and ΔI with ΔI_{max} to remind that we consider the quantum-limited case, and the response is maximized by amplifying the optimal quadrature. Notice that since $\Delta I_{max} \propto \chi\sqrt{\bar{n}/\kappa}$, the noise S_{min} does not depend on the qubit or resonator properties; it is essentially the amplified vacuum noise and depends only on the amplifier gain.

Obtaining information on the qubit z-coordinate via the signal $I(t)$ with response ΔI_{max} and noise S_{min}, we necessarily cause the "spooky" back-action described by Eqs. (4) and (5). As discussed in Sec. II, this is the consequence of the corresponding principle or just the common sense. Now averaging the ρ_{01} evolution in Eq. (4) over the measurement result $\bar{I}_m$ with its probability distribution (8) and (3), we see that the "spooky" back-action dephases an ensemble of qubits with the rate [see Eq. (9)] $(\Delta I_{max})^2/4S_{min} = 8\chi^2\bar{n}/\kappa$. This rate coincides with the total ensemble dephasing (15), and therefore the qubit state cannot additionally fluctuate due to any other reason. Thus we derived an important result: *in the ideal case with phase-sensitive amplifier there is only the "spooky" back-action and no "realistic" back-action.* This means that the *number of photons in the resonator does not fluctuate* (otherwise there would be an additional dephasing), which makes sense since we cannot measure the A-quadrature, carrying information on the photon number. Notice that it is also easy to prove this result when $\omega_m \neq \omega_r$. Then from Eq. (14) we obtain that the informational quadrature amplitude is multiplied by the factor $[1+4(\omega_m-\omega_r)^2/\kappa^2]^{-1/2}$ compared with Eq. (16). The response ΔI_{max} is multiplied by same factor, while the noise S_{min} does not change. Therefore, the "spooky" back-action contribution into the ensemble dephasing is multiplied by the factor $[1+4(\omega_m-\omega_r)^2/\kappa^2]^{-1}$, which again coincides with the result [6, 7] for the total ensemble dephasing Γ. This proves the absence of the "realistic" back-action for the non-resonant case $\omega_m \neq \omega_r$ as well.

Now let us consider the case when an ideal phase-sensitive amplifier amplifies the A-quadrature (we again assume $\omega_m = \omega_r$ for simplicity). Then we do not get any information on the qubit z-coordinate, and therefore there is no "spooky" back-action, but there is the "realistic" back-action due to fluctuating number of photons. The description of evolution in this case is essentially the standard description [6, 7]. Let us still denote with $I(t)$ the output signal from the mixer, corresponding to the amplified quadrature. For measurement during time t we measure A-quadrature with imprecision $\sigma_{gr}/\sqrt{\kappa t}$. This is consistent with fluctuation of the number N of emitted photons: $\mathrm{var}(N) = \bar{N}$, $\bar{N} = \bar{n}\kappa t$. The correlation function of photon number in the resonator depends on time as $\exp(-\kappa t/2)$ [6, 7], which means that each extra photon inferred from $I(t)$ fluctuation spends (on average) time $2/\kappa$ in the resonator and therefore changes the qubit phase ϕ by $4\chi/\kappa$ (the correlation time $2/\kappa$ is essentially the lifetime of the field, not power [6, 7]). Then ϕ-variance is $\mathrm{var}(\phi) = (4\chi/\kappa)^2\bar{n}\kappa t$, and the corresponding ensemble dephasing is $\mathrm{var}(\phi)/2t = 8\chi^2\bar{n}/\kappa$. As expected, this reproduces the standard result (15) for the ensemble dephasing, while for individual qubit evolution we have the above discussed correlation: each additional photon inferred from $I(t)$ fluctuation changes ϕ by $4\chi/\kappa$. For the same amplifier gain and noise as for measuring B-quadrature, we get $\delta\sqrt{\bar{n}} = (4\chi/\kappa)\sqrt{\bar{n}}\,(\delta I_m)/\Delta I_{max}$, and therefore the correlation is $K = \delta\langle\phi\rangle/\delta(\overline{I}_m t) = 32(\chi^2/\kappa)\,\bar{n}/\Delta I_{max} = \Delta I_{max}/S_{min}$. It is easy to check that $K^2 S_{min}/4$ [see Eq. (9)] coincides with ensemble dephasing Γ from Eq. (15), as expected for the presence of only "realistic" back-action.

Finally, assume that the phase-sensitive amplifier amplifies the quadrature, which makes angle φ with the optimal B-quadrature and angle $\pi/2-\varphi$ with the A-quadrature. The measured signal $I(t)$ still denotes the output of the IQ mixer, corresponding to the amplified quadrature; now it gives information about both B and A quadratures, with the factors $\cos\varphi$ and $\sin\varphi$, respectively. Combining the "spooky" and "realistic" back-actions, we get the same formulas as for the broadband detection of Sec. II:

$$\frac{\rho_{11}(t)}{\rho_{00}(t)} = \frac{\rho_{11}(0)}{\rho_{00}(0)}\,\exp[\tilde{I}_m(t)\Delta I/D],\;\; D=\frac{S}{2t}, \tag{18}$$

$$\frac{\rho_{01}(t)}{\rho_{01}(0)} = \frac{\sqrt{\rho_{00}(t)\,\rho_{11}(t)}}{\sqrt{\rho_{00}(0)\,\rho_{11}(0)}}\,\exp[iK\tilde{I}_m(t)\,t], \tag{19}$$

$$\Delta I = I_1 - I_0 = \Delta I_{max}\cos\varphi,\;\; K = K_{max}\sin\varphi, \tag{20}$$

$$K_{max} = \Delta I_{max}/S, \tag{21}$$

$$\tilde{I}_m(t) = \frac{1}{t}\int_0^\infty I(t')\,dt' - \frac{I_0+I_1}{2}, \tag{22}$$

$$I(t) = \frac{I_0+I_1}{2} + \frac{\Delta I}{2}\,z(t) + \xi(t),\;\; S_\xi(\omega)=S. \tag{23}$$

Here we introduced $\tilde{I}_m$ by subtracting the constant $(I_0+I_1)/2$ from $\overline{I}_m$ and did a simple algebra to convert Eq. (4) into Eq. (18); the qubit rotating frame corresponds to $\bar{n}$ photons, K_{max} is the above discussed correlation for A-quadrature amplification, ΔI_{max} is the response for B-quadrature amplification, and $S=S_{min}$. Notice that the total ensemble dephasing (9) does not depend on φ:

$$(\Delta I_{max}\cos\varphi)^2/4S_{min} + (K_{max}\sin\varphi)^2 S_{min}/4 = \Gamma. \tag{24}$$

So far we discussed only the ideal case. There are several mechanisms for non-ideality. First, the qubit may have additional environmental dephasing γ_{env}. This will lead to the extra factor $e^{-\gamma_{env}t}$ in Eq. (19) and increase the ensemble dephasing Γ by γ_{env}. Following the definitions in Sec. II, the corresponding quantum efficiency is $\eta_{env} = (1+\gamma_{env}\kappa/8\chi^2\bar{n})^{-1}$. Second, not all microwave power leaking from the resonator may be collected and amplified. This can be characterized by the collection efficiency $\eta_{col}=\kappa_{col}/\kappa$ and multiplies the response ΔI and correlation K by the factor $\sqrt{\eta_{col}}$, while not affecting the output noise S. Third, if the phase-preserving amplifier is not quantum-limited, it introduces additional noise S_{add} compared with the quantum limit S_{min} [given by Eq. (17) when $\eta_{col}=1$]. The corresponding amplifier efficiency is $\eta_{amp} = S_{min}/(S_{min}+S_{add})$. This does not affect ΔI but multiplies K by η_{amp} (because for uncorrelated Gaussian-distributed random numbers x_1 and x_2, the averaging of x_1 for a fixed sum x_1+x_2 gives the correlation $\langle x_1\rangle/(x_1+x_2) = \mathrm{var}(x_1)/[\mathrm{var}(x_1)+\mathrm{var}(x_2)]$).

If all three mechanisms of the non-ideality are present, then the evolution can still be described [32] by Eqs. (18)–(23), but S is now the total (experimental) output noise, ΔI_{max} is the experimental response for $\varphi=0$, so that $S=(\Delta I_{max})^2\kappa/(32\chi^2\bar{n}\,\eta_{col}\,\eta_{amp})$, the correlation $K=\delta\langle\phi\rangle/\delta(\tilde{I}_m t)$ is still given by Eqs. (20)–(21), and the only change is the extra factor in Eq. (19):

$$\frac{\rho_{01}(t)}{\rho_{01}(0)} = \frac{\sqrt{\rho_{00}(t)\,\rho_{11}(t)}}{\sqrt{\rho_{00}(0)\,\rho_{11}(0)}}\,\exp[iK\tilde{I}_m(t)\,t]\,e^{-\gamma t}, \tag{25}$$

$$\gamma = \Gamma - (\Delta I_{max})^2/4S, \tag{26}$$

where the ensemble dephasing is now $\Gamma = 8\chi^2\bar{\eta}/\kappa + \gamma_{env}$. We emphasize that the qubit evolution depends only on the experimentally measurable parameters ΔI_{max}, S, Γ, φ, and the output signal $I(t)$.

The quantum efficiencies can be expressed as

$$\eta = \frac{(\Delta I_{max})^2}{4S\Gamma} = \eta_{amp}\,\eta_{col}\,\eta_{env},\;\; \tilde{\eta} = \eta\cos^2\varphi, \tag{27}$$

where as in Sec. II, η is the relative contribution to Γ from both the "spooky" and "realistic" back-actions, while $\tilde{\eta}$ is the relative contribution from only the "spooky" back-action. The definition of $\tilde{\eta}$ corresponds to replacing the "realistic" back-action factor $\exp(iK\tilde{I}_m t)$ in Eq. (25) with the corresponding ensemble dephasing $\exp(-K^2St/4)$. As mentioned in Sec. II, the peak-to-pedestal ratio of the spectral peak of contiuous Rabi oscillations is $4\tilde{\eta} = 4\eta\cos^2\varphi$.

B. Phase-preserving amplifier

Now assume that a phase-preserving amplifier is used (this includes parametric amplifier, HEMT, etc.). Now both the A-quadrature and B-quadrature of Eq. (16) are amplified independently with the same gain. Correspondingly, both quadratures at the IQ mixer output carry physical information instead of only one quadrature in the case of a phase-sensitive amplifier. Let us denote with $I(t)$ the output of the IQ mixer, corresponding to the B-quadrature; thus $I(t)$ provides information on the qubit z-coordinate. The output signal for the orthogonal quadrature is denoted $Q(t)$; it corresponds to the A-quadrature in the resonator and provides information on the fluctuating number of photons. The main difference from the case of a phase-sensitive amplifier is that now the "spooky" and "realistic" back-actions are related to two different output signals: $I(t)$ and $Q(t)$.

Let us start with the quantum-limited case and assume an amplifier with the same gain as in the phase-sensitive case, so that the $I(t)$-channel response is the same as the optimal phase-sensitive response, $\Delta I = \Delta I_{max}$. The "spooky" back-action is always described by the quantum Bayes formulas (4)–(5), but now the noise S of the output $I(t)$ is twice larger than the value (17) for the phase-sensitive amplifier, $S = 2S_{min}$ (see discussion in Sec. III);

therefore the "spooky" evolution is twice slower than in the phase-sensitive case with $\varphi = 0$. The signal $Q(t)$ has the same noise $S = 2S_{min}$; it is again twice larger than for the phase-sensitive case with $\varphi = \pi/2$; therefore the correlation factor $K = \delta\langle\phi\rangle/\delta[\int_0^t Q(t')\,dt']$ for the "realistic" back-action is twice smaller: $K = K_{max}/2$ (this reduction is similar to the effect of a non-ideal amplifier discussed above). We see that $K = \Delta I/S$, and the ensemble dephasing is at least $(\Delta I)^2/4S + K^2S/4 = (\Delta I)^2/2S = (\Delta I_{max})/4S_{min}$. This again coincides with $\Gamma = 8\chi^2\bar{n}/\kappa$, and therefore there can be no additional evolution of the qubit besides these "spooky" and "realistic" back-actions.

Thus in the ideal case the qubit evolution is

$$\frac{\rho_{11}(t)}{\rho_{00}(t)} = \frac{\rho_{11}(0)}{\rho_{00}(0)}\,\exp[\tilde{I}_m(t)\Delta I/D],\;\; D = \frac{S}{2t}, \qquad (28)$$

$$\frac{\rho_{01}(t)}{\rho_{01}(0)} = \frac{\sqrt{\rho_{00}(t)\,\rho_{11}(t)}}{\sqrt{\rho_{00}(0)\,\rho_{11}(0)}}\,\exp[iK\tilde{Q}_m(t)\,t], \qquad (29)$$

$$\Delta I = I_1 - I_0,\;\; K = \Delta I/S, \qquad (30)$$

$$\tilde{Q}_m(t) = \frac{1}{t}\int_0^\infty Q(t')\,dt' - \langle Q\rangle,\;\; S_Q = S_I = S, \qquad (31)$$

where $\langle Q\rangle$ is the average value of $Q(t)$ (which depends on $\bar{n}$), $\tilde{I}_m(t)$ is defined by Eq. (22), and the channels $I(t)$ and $Q(t)$ both have the same (uncorrelated) noise $S = (\Delta I)^2\kappa/(16\chi^2\bar{n})$. Notice that $(\Delta I)^2/4S = K^2S/4 = 4\chi^2\bar{n}/\kappa$, and therefore *in the phase-preserving case the ensemble dephasing* Γ *contains equal contributions* $\Gamma/2$ *from the "spooky" and "realistic" back-actions.* We emphasize that Eqs. (28)–(29) still *allow us to monitor a qubit wavefunction* if the initial qubit state is pure.

A non-ideal case can be analyzed in the same way as for the phase-sensitive amplifier. An extra dephasing γ_{env} of the qubit is described by $\eta_{env} = (1+\gamma_{env}\kappa/8\chi^2\bar{n})^{-1}$, imperfect collection efficiency is described by $\eta_{col} = \kappa_{col}/\kappa$, and the amplifier efficiency is η_{amp}. We define $\eta_{amp} = S_{ql}/S$ for a phase-preserving amplifier by comparing its output noise S (per quadrature) with the quantum limit for a phase-preserving amplifier: $S_{ql} = 2S_{min}$, so that $\eta_{amp} = 1$ in the quantum-limited case. We also define $\tilde{\eta} = S_{min}/S$ by comparison with a phase-sensitive amplifier having the same gain, so that $\tilde{\eta}_{amp} = \eta_{amp}/2$ and obviously $\tilde{\eta}_{amp} \leq 1/2$. Similarly to the phase-preserving case, incomplete microwave collection multiplies the response ΔI and correlation K by the factor $\sqrt{\eta_{col}}$ but does not change the noise S; the extra noise in the amplifier multiplies K by η_{amp} but does not change ΔI.

The qubit evolution can still be described by Eqs. (28)–(31) with the only change in Eq. (29):

$$\frac{\rho_{01}(t)}{\rho_{01}(0)} = \frac{\sqrt{\rho_{00}(t)\,\rho_{11}(t)}}{\sqrt{\rho_{00}(0)\,\rho_{11}(0)}}\,\exp[iK\tilde{Q}_m(t)\,t]\,e^{-\gamma t}, \qquad (32)$$

$$\gamma = \Gamma - 2(\Delta I)^2/4S, \qquad (33)$$

where now S is the total (experimental) noise per quadrature, ΔI is the experimental response, and $\Gamma = 8\chi^2\bar{\eta}/\kappa + \gamma_{env}$ is the total ensemble dephasing. The qubit evolution is determined by the parameters ΔI, S, Γ, and output signals $I(t)$ and $Q(t)$.

The quantum efficiencies are

$$\eta = \eta_{amp}\eta_{col}\eta_{env} = (\Delta I)^2/2S\Gamma,\;\; \tilde{\eta} = \eta/2. \qquad (34)$$

Here η is the fraction of Γ due to the contribution from both the "spooky" and "realistic" back-actions. The efficiency $\tilde{\eta} = \tilde{\eta}_{amp}\eta_{col}\eta_{env}$ is the fraction from only the "spooky" contribution; it corresponds to replacing the term $\exp(iK\tilde{Q}_m\,t)$ in Eq. (32) with the dephasing term $\exp(-K^2t/4S)$. In particular, the peak-to-pedestal ratio of the Rabi spectral peak for the signal $I(t)$ is $4\tilde{\eta} = 2\eta$.

Let us mention that Eqs. (28)–(30) for the ideal phase-preserving case can also be obtained from Eqs. (18)–(21) for the phase-sensitive case in the following way. Let us think about a phase-preserving amplifier as a phase-sensitive amplifier, in which the angle φ rapidly changes with time, and we have to average over φ. When coefficients $\cos\varphi$ and $\sin\varphi$ in Eq. (20) are substituted into Eqs. (18) and (19), we see a natural formation of the quadratures $\tilde{I}_m$ and $\tilde{Q}_m$ of the phase-preserving setup. Then the exponential factor in Eq. (18) becomes $\exp(\tilde{I}_m\Delta I_{max}/D)$, and the exponential factor in Eq. (19) becomes $\exp(iK_{max}\tilde{Q}_m t)$. Now let us take into account that the average response is $\Delta I = \cos^2\varphi\,\Delta I_{max} = \Delta I_{max}/2$, and the phase-sensitive amplifier noise S splits equally between the $I(t)$ and $Q(t)$ quadratures (the orthogonal, de-amplified quadrature is noiseless). The mutual cancellation of these two factors of 2 leads to the same form of Eq. (28) as in Eq. (18) and the relation $K = \Delta I/S$ in Eq. (30).

One more way to understand the relation between the ideal phase-sensitive and phase-preserving cases is the following. Instead of using a phase-preserving amplifier, let us split the outgoing microwave into two equal parts and use phase-sensitive amplifiers with $\varphi = 0$ and $\varphi = \pi/2$ in the two channels. To keep the same noise S per channel, we increase the gain by the factor $\sqrt{2}$, that also compensates the signal loss at the splitter. Then the channel $\varphi = 0$ produces the "spooky" back-action (28), while the channel $\varphi = \pi/2$ informs us of the "realistic" back-action (29), and the relation (30) between K and ΔI is the same as between K_{max} and ΔI_{max} in (21).

In the ideal case ($\eta = 1$) the qubit evolution description can be translated into the language of the POVM-type measurement. In the same way as in Sec. II, Eqs. (28) and (29) can be converted into the measurement operator

$$M(\overline{I}_m, \tilde{Q}_m) = \exp(-iK\tilde{Q}_m t\,\sigma_z/2) \times \left[\sqrt{P_{|0\rangle}(\overline{I}_m)}\,|0\rangle\langle 0| + \sqrt{P_{|1\rangle}(\overline{I}_m)}\,|1\rangle\langle 1|\right], \qquad (35)$$

where the probabilities $P_{|j\rangle}$ are given by Eq. (3). Similarly, Eqs. (18) and (19) for the case of a phase-sensitive amplifier can be converted into the same measurement operator (35), in which $\tilde{Q}_m$ is replaced with $\tilde{I}_m$.

V. CONCLUSION

We have presented a simple physical picture of the qubit evolution due to its measurement in the circuit QED setup. The "spooky" back-action is universal, it is caused by gradual extraction of information about the qubit state. The "realistic" back-action is due to a specific mechanism: fluctuation of the photon number in the resonator. For a phase-sensitive amplifier the qubit evolution is described by Eqs. (18) and (25); it is determined by the output signal $I(t)$, which corresponds to the amplified quadrature. For a phase-preserving amplifier the evolution is described by Eqs. (28) and (32); it is determined by two output signals: $I(t)$ and $Q(t)$, where $I(t)$ now corresponds to the quadrature, which provides information about the qubit state (B-quadrature in the resonator) and $Q(t)$ corresponds to the orthogonal A-quadrature, which gives us a record of the photon number fluctuations in the resonator.

While the cQED setup significantly differs from both the broadband quantum measurement setup [23] and the standard optical setup [18], we see that the description of the qubit evolution is exactly the same as in both these cases if a phase-sensitive amplifier is used. The description is only slightly different when a phase-preserving amplifier is used: we should assign the "spooky" and "realistic" back-actions to the separate output signals $I(t)$ and $Q(t)$ instead of only one signal. It is also useful to think about the phase-preserving case via the model in which we split the outgoing microwave (the quantum signal) into two equal parts and then use 90-degree-shifted phase-sensitive amplifiers for these two channels.

We intentionally considered only the simplest case, because most of the further steps and generalizations are quite straightforward [23]. In particular, it it very simple to include Rabi oscillations and energy relaxation of the qubit state. For that we have to take time-derivative of the evolution equations and add the terms due to Rabi oscillations and energy relaxation. If the Stratonovich definition of the derivative is used, we get the equations of the Bayesian formalism [23]; if the Itô derivative is used, we get the equations of the quantum trajectory formalism [9, 18]. Generalization to measurement of several entangled qubits is also straightforward [23]. We have considered only the resonant case, $\omega_m = \omega_r$; however, generalization to the case $\omega_m \neq \omega_r$ is quite simple (see [9]): we just need a different definition of the informational B-quadrature and photon-fluctuation A-quadrature. In our formalism we implicitly assumed sufficiently wide bandwidth of the amplifier (much larger than the ensemble dephasing Γ and Rabi frequency Ω_R). If this is not the case, the formalism should change significantly. However, we believe that in most of the practical cases we can take this effect into account by adding a classical narrowband filter to the classical signal at the amplifier output; this will correspond to passing the signals $I(t)$ and $Q(t)$ through the low-pass filters. A much more serious change of the theory is required when the resonator bandwidth κ is comparable to Γ or Ω_R; this still has to be done.

Understanding the difference between the "spooky" and "realistic" back-actions is important for designing the quantum feedback control of the Rabi oscillations [12]. The simplest case is when a phase-sensitive amplifier amplifies the informational B-quadrature. Then there is no "realistic" back-action, and the feedback loop should only modulate the amplitude of the Rabi drive (i.e. the Rabi frequency Ω_R); this case was well studied for the broadband setup [12]. The situation is different for a phase-preserving amplifier. Then we need two feedback channels: the first (usual) channel should modulate the Rabi frequency Ω_R to compensate the "spooky" back-action, while the second channel should compensate the "realistic" back-action by modulating the qubit frequency ω_{qb} or the the frequency of the Rabi drive ω_R. The controller for the second feedback channel is quite simple: it should compensate the contribution $iK\tilde{Q}(t)$ to the qubit phase derivative $\dot{\phi}(t)$ due to the K-term in Eq. (32). Therefore the controller is

$$\Delta(\omega_{qb} - \omega_R) = -K[Q(t) - \langle Q \rangle], \tag{36}$$

i.e. we should directly apply the signal $Q(t)$ to modulate ω_{qb} or ω_R. The second feedback channel essentially eliminates the K-term in Eq. (32) and decreases the ensemble dephasing Γ by $K^2 S/4 = (\Delta I)^2/4S$. Correspondingly, in the absence of the first (main) feedback channel the peak-to-pedestal ratio of the Rabi peak increases from 2 to 4 in the quantum-limited case. The first feedback channel should be the same as for the broadband setup; it depends on the signal $I(t)$ and can be realized using various ideas for the controller ("direct", Bayesian, "simple", etc. [12]). Notice that *without the second channel the feedback performance is determined by the quantum efficiency* $\tilde{\eta}$, *while with the second channel it is determined by* $\tilde{\tilde{\eta}} = \tilde{\eta}/(1-\eta+\tilde{\eta})$ (this is one more combination of the terms in Eq. (9), which can be used for the definition [23] of quantum efficiency). The case of a phase-sensitive amplifier, which amplifies a non-optimal quadrature ($\varphi \neq 0$, $\varphi \neq \pi/2$) is similar to the case of a phase-preserving amplifier, but both feedback channels should start with the same signal $I(t)$. *In both the phase-sensitive and phase-preserving setups a perfect feedback control is possible in the quantum-limited case* $\eta = 1$.

Discussion of the "spooky" and "realistic" back-actions in the cQED setup necessarily raises the question of causality. When the microwave leaves the resonator, it does not yet "know" in which way it will be measured (phase-preserving or phase-sensitive, which angle φ, etc.). Moreover, when a circulator is used for the outgoing microwave, the field in the resonator and the qubit can never "know" in a realistic way which method of measurement is used. Nevertheless, the qubit evolution strongly depends on the measurement method. As we discussed, the "spooky" evolution moves the qubit state along the meridians of the Bloch sphere, the "realistic" back-action moves the state along the parallels, and the measurement method determines whether the qubit experiences the "spooky" or "realistic" back-action (or their combination). In this sense the "realistic" back-action is not fully realistic: it has the physical mechanism, but whether this mechanism works or not is determined in a spooky way. The causality requires that we cannot pass a "useful" information to the qubit by choosing the measurement method. This means that *the ensemble-averaged evolution of the qubit cannot depend on the measurement method (this is the general requirement of causality in quantum mechanics).* It is surely satisfied in our cQED setup.

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