Teachers' and Researchers' Beliefs of Early Algebra Development*

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Abstract

Mathematics teachers and mathematics educational researchers were asked to rank order arithmetic and algebra problems for their predicted problem-solving difficulty for students. It was discovered that these predictions matched closely the view presented implicitly by common mathematics textbooks, but they deviated systematically from actual algebra students' performances in important ways. The Textbook view of early algebra development was contrasted with the Verbal Precedence (VP) model of development. The latter was found to provide a better fit of students' performance data. Implications for student and teacher cognition are discussed in light of these findings.

Introduction

Our current investigation explores the relationship of teachers' predictions of the development of early algebra reasoning, and students' actual performances. *Early algebra* is that area of math learning that builds upon and extends young people's arithmetic thinking to include unknown quantities and general patterns. This research advances our understanding of the development of children's quantitative reasoning.

As students shift from an arithmetic view of problem solving to an algebraic perspective, teachers must also shift their practice. The single greatest influence on teachers' decisions is their perceptions of students' academic abilities (Borko & Shavelson, 1990). Any gains in understanding about teachers' perceptions and misperceptions of students' reasoning abilities strengthens our programs of teacher preparation and instructional practices in the classroom. This work extends models of teacher cognition, and how professional practitioners' knowledge shapes their practice.

This study examines teachers' and researchers' beliefs about the factors that make mathematics problems difficult for early algebra students. The accuracy of teachers' predictions is compared to problem-solving performance data of students' obtained elsewhere (Koedinger, Nathan, & Tabachneck, 1996; Tabachneck, Koedinger, & Nathan, 1994), and discrepancies between teachers' predictions and students' performances are identified, and examined within a model-comparison approach that extends common views of algebraic reasoning. Differences between the models attributed to teachers' predictions and to students' performances appear to be the cause for poor predictions.

Theoretical Framework: Difficulty Factors

The body of work on arithmetic story problem solving of younger children (e.g., Carpenter et al., 1994; De Corte, Greer, & Verschaffel, in press; Shalin & Bee, 1985) provides firm methodological and theoretical bases for this study of high school students' algebraic reasoning and its development and impediments.

Unknown values Problem difficulty is strongly affected by the role (or position) of the unknown quantity within the problem statement. *Result-unknown problems* (P6, Table 1) in which the unknown quantity is the result of the events being described, tend to be significantly easier than *Start-unknown problems* (P3, Table 1) where the unknown value refers to a quantity needed to specify a relationship (Carpenter et al., 1994; Riley & Greeno, 1988).

Riley and Greeno (1988) found that while 1st grade students were 100% correct on result-unknown problems, they were 33% correct on start-unknown problems. Differences like this have also been found at the college level (Koedinger & Tabachneck, 1995). Result-unknown problems lend themselves to direct modeling and arithmetic calculations, and tend to be referred to as "arithmetic problems." Start-unknown problems tend to subvert simple modeling and direct calculation and often require algebraic methods, or sophisticated, modeling (cf. Hall et al., 1989).

Presentation format The format in which a mathematical problem is presented also bears on problem difficulty. Story problems (P1, P4) are in a verbal format and contain contextual information about the problem situation which can be used by the solver as a source of problem elaboration, reframing, and solution constraints (cf. Baranes, Perry, & Stigler, 1989). The observed performance differences between start-unknown and result-unknown problems apply to symbolically presented problems as well as those presented linguistically (Tabachneck et al., 1995). Symbolic arithmetic problems (P6) are typically described as number sentences. There is also a presentation format intermediate to the story problem and symbolic equation format. This *word equation* format (P2, P5) is verbal in

* Some of the data discussed were presented at the American Educational Research Association (AERA) annual meeting, 1996.

presentation and describes the relationship among pure quantities (both known and unknown) with no story context.

Table 1: Sample problems.

P1) When Ted got home from his waiter job, he multiplied his hourly wage by the 6 hours he worked that day. Then he added the $66 he made in tips and found he earned $81.90. How much per hour did Ted make?

P2) Starting with some number, if I multiply it by 6 and then add 66 I get 81.9. What did I start with?

P3) Solve for X: $X * 6 + 66 = 81.90$

P4) When Ted got home from his waiter job, he took the $81.90 he earned that day and subtracted the $66 he received in tips. Then he divided the remaining money by the 6 hours he worked and found his hourly wage. How much per hour does Ted make?

P5) Starting with 81.9, if I subtract 66 and then divide by 6, I get a number. What is it?

P6) Solve for X: $(81.90 - 66) / 6 = X$

Experiment 1: Teachers' Predictions

Subjects and Procedure

Participants were mathematics teachers (n=68) in the southeastern United States from a wide range of settings and socio-economic communities, including predominantly minority-based inner city schools, rural communities, and suburban areas. They were recruited from a teacher workshop during the summer, and included 7th through 12th grades.

The teachers were asked to rank order problems of the six different types shown in Table 1, from easiest to most difficult, based on the criterion, "How hard do I think these problems are for my students?" without discussion[1]. Teachers also provided information about the student grade level that the teacher considered in developing the ranking, and whether they assumed calculator use.

The problems can be mapped on to the factors discussed and presented in Table 2 using the number entries in the table (e.g., P1) as an index to the problems of Table 1. Note that the underlying mathematical relationships are the same across the problems. This underlying structure was not discussed with the participants.

Table 2: Sample problems organized by unknown value and presentation type.

	Verbal problems		
Unknown	Story	Word	Symbol
Result-unk	P4	P5	P6
Start-unk	P1	P2	P3

[1] The authors recommend that the reader take a few minutes to perform this rank ordering task before reading further.

Results and Discussion

The average rank ordering produced by the mathematics teachers in our sample is presented in the first column of Table 3. Teachers' rank ordering of the six problem types was analyzed using a 2-way, repeated-measures ANOVA with position unknown (result v. start) and presentation format (story v. word-equation v. symbol) as within-subjects factors, and difficulty rank (1-6) as the dependent measure.

Unknown values Across all problem types, 84% of the teachers ranked result-unknown (P4,P5,P6) problems (arithmetic) as easier than start-unknown problems. This resulted in a significant main effect for the unknown value factor, $F(1,134)=5.9$, $p<.02$. Arithmetic equations (P6) were favored over arithmetic word-equations (P5), and word equations favored slightly over arithmetic story problems (P4). Six percent viewed start-unknown problems as easier than result-unknown problems, and 10% were inconsistent.

Presentation format The data show 42% of teachers ranking symbol equation (P3,P6) problems (ignoring the unknown value factor) as consistently more difficult than word-equations on average, and 49% ranking equations as consistently more difficult than story problems (P1, P4) on average. Fewer than 30% ranked verbal algebra problems (story and word equation combined) as being as easy to solve as symbolically presented arithmetic problems.

Table 3: Difficulty rank of teachers' and researchers and student performances.

	Teachers (n=68) mean rank	Researchers (n=35) mean rank	Students aggregated (n=76+171) [% correct]	Textbook view
Easy*	P6	P5	P4 [78%]	P6
	P5	P4	P5 [72%]	P5
	P4	P6		P4
Med.*	P3	P3	P1 [59%]	P3
	P1		P6 [55%]	
			P2 [51%]	
Hard*	P2	P2	P3 [31%]	P2
		P1		P1

* Difficulty divisions (Easy, Med, Hard) show significant differences (p<.05) in mean ranking or performance levels.

Start-unknown problems were ranked as most difficult by teachers based on a post hoc comparison, $p=.05$. Within this, verbally presented problems are considered particularly difficult. Over 76% of the teachers ranked story and word-equation start-unknown problems as more difficult than all other problem types. A post hoc comparison among all six problems revealed that teachers (70%) ranked start-unknown (i.e., algebra) word-equation problems as the most difficult problem type given, $p=.05$.

Teachers showed a strong tendency to rank algebra problems as more difficult than matched arithmetic problems, regardless of the presentation format. In comparing formats, teachers tended to rank verbally presented problems (i.e., story and word-equation) as more difficult for students than symbolic equation problems in algebra and arithmetic. There was no significant interaction between presentation format and position unknown.

Textbook view In an attempt to determine the source of teachers' predictions, two prominent mathematics textbook series were analyzed for their treatment of arithmetic and early algebra concepts. The textbook series first presented arithmetic computations in symbol form, followed by the application of these procedures to stories and scenarios. The algebraic formalism was introduced next along with rules of symbol manipulation. Start-unknown problems (and translation methods) were then introduced as applications of the formalism. The chapter organizations followed the general pattern showed in Table 3.

Kendall's rank correlation revealed a highly significant agreement between teachers' rankings (Table 3) and the ordering offered by the mathematics textbook series, $\tau(12)=.867$, $p=.015$. Teachers ranked start-unknown (algebra) problems as significantly more difficult than result-unknown (arithmetic) problems. They also ranked verbally presented problems within each category as more difficult than the corresponding symbol-equation problem. From this, it seems apparent that teachers' decisions regarding problem difficulty are influenced by the order of topics in textbooks. Teachers seem to use the textbook view as a basis for their predictions of problem difficulty for students.

Experiment 2: Researchers' Predictions

Subjects and Procedure

Participants were mathematics educational researchers (n=35) dispersed throughout the US, and were recruited via an Internet discussion group on algebraic thinking and teaching.

Researchers received the same problems and task instructions as the teachers (Experiment 1). The materials were distributed and collected electronically over e-mail.

Results

The mathematics researchers showed strong agreement with teachers and the textbook view. About 66% of the respondents consistently ranked start-unknown problems as harder than result-unknown problems across the 3 presentation forms, while 34% ranked result-unknown problems as more difficult in some but not all cases. About 31% of the respondents consistently ranked equations as easiest. Only 23% of the respondents consistently ranked equations as harder than word and story problems within each of the two levels of the position unknown factor. These data indicate that researchers view story problems as harder than equations for start-unknown (i.e., algebra) problems, and see all forms of start-unknown problems as harder than all result-unknown problems (Table 3).

Student Performance versus Teacher Expectations: A Model Comparison

We discuss the above findings of teachers' and researchers' predictions of problem difficulty in relation to student performance data. Students in two studies (Tabachneck et al., 1995; Koedinger et al., 1996) solved problems based on the six problem types shown in Tables 1 and 2. Koedinger and his colleagues term their investigation a *difficulty factors assessment* (DFA) because it seeks to systematically identify the factors affecting students' problem-solving difficulties. In the two studies, 76 and 117 ninth grade students, respectfully, completed quizzes in their math classes. All students were either currently enrolled in a high school Algebra I class, or had completed it the previous year.

Student performance The initial set of student performance data (n=76) showed highly significant effects of unknown value and presentation format. As predicted by teachers, students scored much lower on start-unknown (algebra) problems than on result-unknown problems, $F(1,75) = 48.9$, $p<.0001$. Contrary to teachers' expectations, however, students' experienced greater difficulties when solving *symbol equation* problems than verbally presented problems, $F(2,75) = 12.6$, $p<.0001$. Also, in contrast to the view proffered by teachers or by textbooks, students' do not find algebra story and word problems to be most difficult. Algebra equation problems were significantly less likely to be correctly solved than either story problems or word-equation problems ($p<.01$ in a post hoc test). Algebra story and algebra word-equation problems were actually found to be equal in difficulty to arithmetic symbol problems, a result predicted by only 4.5% of the teachers in the sample.

A replication of these results was made the following year, and largely parallel the original findings. Students (n=171) showed a greater tendency to accurately solve result-unknown than start-unknown problems, $F(1,170)=138$, $p<.0001$. The effect of presentation format was largely replicated in this study as well, ($F(2,170)=38.4$, $p<.0001$). Symbolic equations were significantly more difficult than either story problems or word equation problems, $p<.01$. However, these data found that result-unknown word-equation problems (P5) were significantly more difficult than result-unknown story problems (P4), $p<.01$.

The pattern of results from both student samples are so similar that the combined results are representative of each group. The relative difficulty ordering of the problems for the combined data sets are shown in Table 3.

Three generalizations can be drawn from the student data: (a) Start-unknown problems are harder for these students than result-unknown ($p < .001$); (b) Symbolic equation problems are harder than both word equation problems and story problems ($p < .001$). The latter two verbal problem types are about equal in difficulty in the original study (n=76), while word equation problems were more difficult for story problems in the replication (n=171); and (c) the relative amount of difficulty attributed to symbolism (b, above) is as large as the amount of difficulty due to start-unknown (algebraic) problems (a, above).

Teachers predicted much of what makes problems difficult for students. A Kendall's Rank Correlation (*Tau*) yields a significant relationship between teachers' ratings and students' performances, $\tau(12)=.61$, $p=.03$ (see Table 3). However, some significant systematic discrepancies with students' performances are also apparent.

Students' solution strategies Tabachneck et al. (1994) observed five major types of solution strategies used by the high school students to solve the 6 classes of problems in Table 2. The first two groups — arithmetic and algebraic methods — are school taught. The other three — diagrams (direct modeling), guess-and-test, and unwinding — are informally adopted and invented strategies. In addition to these, students' may have provided no response, or provided insufficient information to allow coding a strategy. Guess-and-test refers to the class of model-based methods used for iterative analysis or "hand simulations" of the events of the problem (e.g., Hall et al., 1989). Unwinding methods allow the student to work backwards from the givens of the problem and "unwind" or undo the quantitative constraints imposed, in order to isolate the unknown (cf. Kieran & Chalouh, 1990; Polya, 1957).Unwinding often parallels the steps referred to in the scenario of a story problem, or the order of mathematical constraints provided. Unlike an algebraic approach, unwinding makes no use of equations or symbolic place-holders for unknown quantities. Unwinding may be done mentally by the solver (Koedinger & Tabachneck, 1995), or through the solver's written work.

These strategies supported students' problem-solving processes, and appeared to allow students to solve problems which would otherwise be beyond their reach, and avoid school-taught methods that are error-prone.

An analysis of the strategies from original student data (n=76) was conducted. Arithmetic strategies, not surprisingly, were used overwhelmingly in solving result-unknown problems. For start-unknown problems non-standard solution approaches were preferred, and were more successful than standard methods. Story problems tended to elicit the unwinding strategy from students 56% of the time. Story problems seldom elicited the symbol manipulation methods associated with algebra (only 6% of the time). Situation-less word equation problems tended to elicit either a guess-and-test approach (31% of the time) or unwinding (26%). Symbolic equations resulted in no response from these algebra students an alarming 30% of the time, more than twice as often as the other problem types. When students did respond to equations, they tended to stay within the mathematical formalism and apply symbol manipulation methods (26%), or opt for iterative guess-and-test (20%).

The informally acquired unwinding and guess-and-test (GT) methods showed the highest likelihood of success. This is because these methods rely on more familiar and practied representations. (Tabachneck et al., 1994). Both GT and Unwind (UW) rely on a verbally mediated arithmetic. GT makes the problem arithmetic by guessing at a start value and propogating. UW operates in two modes. For story problems, the solver uses the situated nature of the values gleaned from the problem scenario (e.g., maintain the units as one says or writes the values). This minimizes the likelihood of producing absurd values. When used with problems with no context or in the translation form of a story problem, unwinding essentially transforms the algebra problem into a set of arithmetic problems that are then more readily solved. The findings that equate student performance on arithmetic equation problems to that of algebra story and word problems are evidence for this.

Two developmental models Teachers and researchers do not seem to place sufficient weight on the power of students' alternative solution strategies. Rather, students' problem-solving abilities are largely thought of within the textbook view of algebra development (Table 2), where arithmetic symbolic skill is followed by application of that reasoning to verbal problems, and start-unknown problems necessarily rely on translation into symbolic form. Thus, teachers and researchers, examining a problem for its level of relative difficulty, make their decisions based on the question, "how far along the developmental trajectory from symbolic arithmetic to algebra story problem solving has a student progressed?" The analyses of students' problem-solving process data suggests an alternative trajectory for the development of one's early algebra reasoning ability that circumvents some of the difficulties of symbolic algebra.

From this, one can frame students' performances and teachers expectations in the form of two competing models of early algebra development. The textbook and verbal precedence models of students' early algebra development are compared in Figure 1. Each square in the figure depicts the level of problem-solving competency that the student can achieve unaided (cf. Vygotsky, 1978). All sixteen possible levels are shown in Figure 1 with a descriptive label. The number in each level indicates the total number of students from the initial study, n=76) that have progressed to that level but not beyond, as determined by their problem-solving performance. These numbers are used to compare the predictive power of the two models based on student data.

Figure 1 illustrates two different pathways through this space of all competency levels, one which is consistent with the Textbook model (dashed lines) of development and another (heavy lines) which corresponds to our alternative *Verbal Precedence* (VP) model of development. The Verbal Precedence model (left of center) suggested by our research favors the early development of verbal problem-solving abilities over symbolic problem solving (Koedinger & Tabachneck, 1995). The model that follows from the Textbook view (right of center) favors development of arithmetic before algebra, and symbolic problem solving abilities over verbal reasoning.

The VP model predicts that students enter where verbal arithmetic problem solving is a level of competence *by itself*. The Textbook model predicts that verbal arithmetic problem solving is not acquired as an initial competency, but is acquired *after* competency in symbolic arithmetic. Three levels — the Null level at the top of the figure, the All Arithmetic level, and level where all problems in our problem space — are common to both models.

Quantitative model comparison Using the patterns of student performance in the original study (n=76), it is possible to classify each student into a competency level showing their development. Students classified as competent in either none or all of the problem categories fit both models trivially. A student's performance does not fit the model when it demonstrates one competency but lacks another that is presumed to be earlier in the sequence predicted by a particular developmental model. For example, if a student's competency reaches symbolic arithmetic but no further, and the student cannot also solve verbal arithmetic problems, then that student fits the Textbook model but not the VP model. This is because the normative model predicts that competency in symbolic arithmetic problem solving precedes competency in arithmetic verbal problem solving. On the other hand, if a student's competency level reaches verbal arithmetic problem solving but no further, and includes no symbolic arithmetic problem solving, she fits the VP but not the Textbook model.

Alternatively, a student's competence may follow a trajectory different than that predicted by either developmental model. For example, a lone subject is shown to have competency at solving verbal arithmetic and symbolic algebra, but no other areas. Performance of this subject is outside of both models, reducing their predictive power. There are seven such levels (thin solid lines) accounting for 2 of the 76 subjects in the initial study.

As a quantitative measure of the predictive power of each of the two models, the percentage of students who follow each of the proposed trajectories is compared. Of the 76 participants in the original study, 69 (or about 91%) of the students follow the VP model, while 41 (62%) follow the Textbook model. Most of those fitting the Textbook model are the 42 (55%) students in the three levels common to both models (the central column). Only 5 students (7%) uniquely fit the Textbook model, while 27 (36%) uniquely fit the VP model. Two students (3%) remain outside both.

From the original student data, the Verbal Precedence model provides a superior fit over the Textbook model. As a further test, the VP model is also applied to the data from the 171 students in the replication study. Here, 151 students (88%) fit the VP model. In contrast, only 79 students (46%) fit the Textbook model. As before, most of those students who fall along the developmental trajectory of the normative model are the 65 students (38%) who are in the three competency levels common to both models. Only 10 students (6%) uniquely fit the Textbook model, while 86 students (50%) uniquely fit the VP model, and 7 students (4%) remain outside of either model. This is added support that the Verbal Precedence model better reflects students' problem-solving performance than does the Textbook model. Furthermore, the VP accounts for a high percentage of students on an absolute scale, suggesting it captures something basic to students' early algebra development.

Conclusions

Students' problem-solving behaviors differ in systematic ways from those suggested by textbooks, and from the expectations of teachers and mathematics education researchers. Start-unknown (algebra-level) problems, those with an initial unknown quantity, are accurately predicted by teachers to be more difficult overall than arithmetic (result-unknown) problems. However, the impact of verbal presentation formats on students' problem-solving processes is misunderstood. These formats tend to trigger strategies that fall outside of the standard view, and consequently lead to a very different pattern of performance. Students' alternative strategies are general and fairly robust, providing, in many instances, ways for the solver to verify the accuracy of the solution produced. As is evident from the advantages gained from the contextless word-equations, these alternative strategies are not limited to contextually rich problem situations, but extend over a range of verbal presentations.

While many teachers appear to be influenced by the Textbook view in their decisions concerning problem difficulty, the student performance data suggest an alternative development of algebraic reasoning. This places greater emphasis on students' ability to reason about verbal problems. Alternate forms of quantitative reasoning may developmentally precede symbol manipulation ability for many students — as suggested by the VP model of early algebra development. The use of these methods by college students who have been away from algebra for a while has also been observed (Tabachneck et al., 1994), suggesting they are relatively resistant to extinction — unlike algebraic symbol manipulation skills, which were poorly executed by students of Algebra I (Tabachneck et al., 1995).

Implications for Research on Student Cognition

The relative difficulty of start-unknown problems is well-established in the elementary school word problem-solving literature with single operator problems (e.g., DeCorte, Greer, & Verschaffel, in press). The symbol disadvantages apparent in our student data challenges the oft-cited view that story problems are inherently harder than symbolic ones (e.g., Cummins et al., 1988; Mayer, 1982). A number of more recent studies, mostly at the lower grade levels, have shown positive effects of situational context (e.g., Baranes et al., 1989; CTGV, 1993). The relatively high performance on *situation-less* word equations shown here suggests that the general advantage associated with verbal problems involves more than an advantage of context. Students at this stage can solve situation-less word-equations far more readily (by some 17%) than they can reason about symbols. Of course, this finding is certainly developmentally rooted — as symbolic skill develops and students attack harder problems more often, advantages of symbolism are likely to appear.

The student data also show that students can do as well on simple *algebra* problems as they do on *arithmetic* problems. This occurs when the arithmetic problems are presented symbolically while the algebra problems are presented verbally (with or without a context) and thus elicit powerful alternative strategies such as Guess-and-Test and Unwinding.

By building on students' informally acquired knowledge-based strategies, such as Unwinding and Guess-and-test, it may be possible to enhance acquisition, retention, and scope of algebraic reasoning, provide a richer conceptual basis for symbol manipulation, and develop in the student, a stronger sense of agency as a problem solver and mathematician.

Implications for Research on Teacher Cognition

By relying on the Textbook view, teachers tend to make inaccurate predictions of how students will perform and develop. This is of great interest since these beliefs of problem difficulty affect teachers' instructional planning and the development of their assessments (cf. Borko & Shavelson, 1990). The Textbook view is quite popular, and has held the attention of educational researchers, cognitive scientists, and educators for some time (e.g., Mayer, 1982). Beliefs do not easily change. If we want to ultimately bring teachers' views into closer alignment with the empirical findings, it is imperative that teachers are made aware that they hold these views, that the views are explicitly characterized for them, and that their strengths are acknowledged while the limitations are exposed.

Math educators need to be made aware of the power and flexibility of students' alternative reasoning strategies. Findings from this study suggests, however, that any attempts to address this, will need to first address the beliefs held by teachers, principally those in math textbooks.

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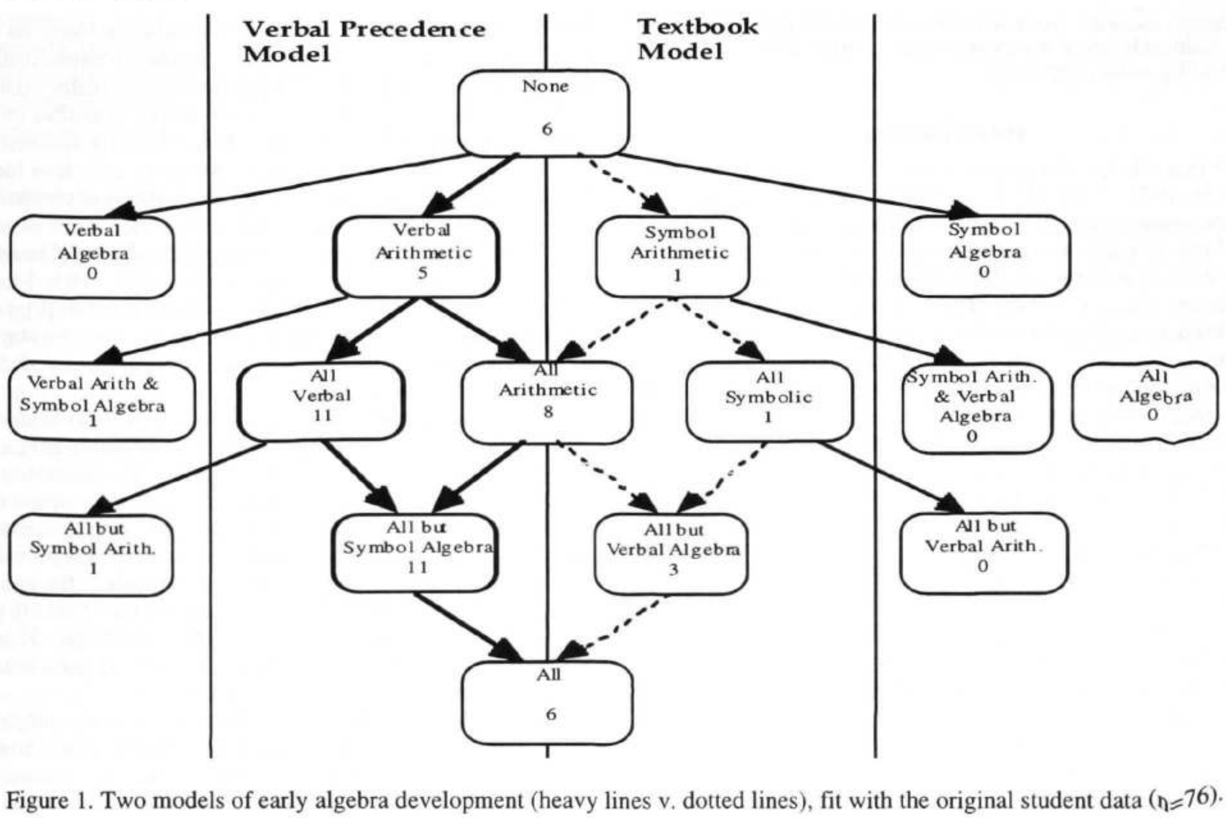

Figure 1. Two models of early algebra development (heavy lines v. dotted lines), fit with the original student data (n=76).