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In a recent article in this Journal, Giaccotto and Ali (5 ) addressed several potential deficiencies in prior studies which investigated heteroskedasticity in the single index market model. Much of the previous work employed standard parametric tests for heteroskedasticity, such as the Goldfeld-Quandt, Bartlett or Glejser tests, which lean heavily on the assumed normality of the disturbances, a potentially untenable assumption in light of the apparent leptokurtotic nature of returns. Moreover, some authors have employed alternative tests, such as the Goldfeld-Quandt peak test, on the ordinary least squares (OLS) residuals from market model regressions under the assumption of the independence of the OLS residuals when, in fact, these residuals are correlated by construction. Finally, the previous literature investigated only a small number of specific forms of heteroskedasticity whereas Giaccotto and Ali (5 ) suggested the virtues of tests which have power against a variety of alternatives.

These considerations led Giaccotto and Ali to consider alternative rank and robust tests which have good power characteristics. Since these test statistics require the use of independent observations, Giaccotto and Ali observed that OLS residuals should not be used in such tests since they are correlated and, hence, dependent. The authors instead employed recursive residuals which were introduced by Brown and Durbin (1 ) and since have been studied extensively in Hedayat and Robson (9 ), Harvey and Phillips (8 ), Harvey and Collier (7 ), Godolphin and DeTullio (6 ), Dufour (2 ), and Garbade (4 ). These recursive residuals are standardized one step ahead prediction errors which by construction are mutually uncorrelated. Giaccotto and Ali concluded that this orthogonality property makes recursive residuals appropriate for the construction of the aforementioned test statistics.

Unfortunately, this unwarranted conclusion involves an elementary flaw in statistical reasoning since uncorrelated random variables are not independent unless their joint distribution is normal. This simple observation renders their claims involving the distribution-free nature of their tests incorrect. In fact, as we shall indicate below, it is not clear whether any of their test statistics possess well-defined distributions in large samples when based on recursive residuals.

In what follows we document the dependence of recursive residuals and examine some of the corresponding implications for the test statistics employed by Giaccotto and Ali. In the next section, we discuss the small and large sample behavior of OLS and recursive residuals and indicate some of the implications of this analysis for various tests for heteroskedasticity. We conclude with a discussion of some of the substantive conclusions reached by Giaccotto and Ali. Detailed analysis of the behavior of OLS and recursive residuals is relegated to an Appendix.

## II. The Behavior of OLS and Recursive Residuals

OLS residuals are not well-suited for hypothesis testing in small samples. These residuals are correlated and, hence, cannot be employed in test statistics which require independence of the basic observations. Moreover, this dependence extends to higher order moments. For example, it is well-known that, even under normality, the average absolute residual tends to be smaller than would be expected from a population of independently distributed variates.

In contradistinction, recursive residuals behave well in finite samples. By construction, they are mutually uncorrelated when the underlying disturbances of the regression model are independent. Note, however, that they are not independent unless the disturbances are normally distributed as well. For example, as shown in the Appendix, if the disturbances are leptokurtotic (or fat-tailed), then squared recursive residuals are positively correlated so that there would tend to be more extremely large or extremely small residuals than would be expected under independence. In consequence, test statistics based on recursive residuals have unknown properties in finite samples when the underlying disturbances are non-normal and it is likely that such questions could only be addressed through Monte-Carlo simulations. It is for this reason that prior work employing recursive residuals for hypothesis testing including for example, the work of Hedayat and Robson (9 ) which developed the peak test employed by the authors, emphasized the crucial nature of the normality assumption.

Paradoxically, the comparative merits of OLS and recursive residuals are reversed in large samples. The OLS residuals converge to the true disturbances in large samples. On the other hand, recursive residuals exhibit no such general tendency. The reason for the convergence of the OLS residuals is simple---each new observation tends to improve the estimate of the slope coefficient and, in consequence, moves each estimated disturbance closer to its true value. By contrast, recursive residuals do not change when new observations are added since the $j^{th}$ recursive residual is a fixed linear combination of the first j observations. Consequently, the dependence among the first several recursive residuals does not vanish and, in fact, does not change as the number of observations grows large. Of course, the dependence of the last several recursive residuals does vanish in large samples and, hence, these residuals may be employed without remorse in large sample tests which require independence. Unfortunately, there is no theoretical guide as to how many residuals constitute "the last several" ones and it is likely that this question can only be answered by Monte Carlo simulations in particular applications.

These simple observations render all of the claims made by Giaccotto and Ali concerning the large sample properties of their test statistics false. The large sample normality of the peak test, Kendall's rank correlation test, Bickel's robust tests, and the rank tests considered by Giaccotto and Ali require the use of independent residuals. Perhaps the large sample properties of these tests are not terribly sensitive to the dependence of the recursive residuals when the underlying disturbances are non-normal but this is surely a matter which should be investigated directly.

It is likely, in fact, that the test statistics in question are somewhat sensitive to this dependence in the market model regressions. On the evidence of Fama (3 ) and many others, stock returns follow distributions which are leptokurtotic relative to the normal. In consequence, there will tend to be more extremely large and extremely small residuals than would be expected under normality and, hence, the recursive residuals would tend to suggest the presence of heteroskedasticity when, in fact, the disturbances were homoskedastic. Since the explicit motivation for the authors' study was that "the incidence of heteroscedasticity detected by these [parametric] tests [employed in previous studies] may be overstated" (p. 1248) due to the fat-tailed nature of returns, it is certainly disturbing that the same type of bias carries over to the allegedly distribution-free tests employed by the authors.

## III. Conclusion

In this paper, we have discussed a number of statistics based on recursive residuals employed by Giaccotto and Ali ( 5 ) to test for heteroskedasticity in the market model. They motivated their use of recursive residuals by suggesting that the orthogonality of recursive residuals make them appropriate for testing for heteroskedasticity when the disturbances are not normally distributed. Unfortunately, the authors failed to note that for non-normal error terms the uncorrelated nature of recursive residuals does not imply their independence. In fact, for the case of leptokurtotic (relative to the normal) disturbances, our analysis indicates a positive correlation between squared recursive residuals, which might cause the test statistics to falsely indicate the presence of heteroskedasticity.

In consequence, it is unclear whether recursive residuals are superior to OLS residuals for the present purposes or whether either set of residuals leads to biased tests in the market model application. The only evidence provided in the paper concerning the relative merits of OLS and recursive residuals is that both sets of residuals yield similar rejection rates of the hypothesis of homoskedasticity with the Kendall rank and Goldfeld-Quandt peak tests, with those based on recursive residuals rejecting only slightly more often. While this suggests that the greater dependence of OLS residuals is perhaps unimportant in testing for heteroskedasticity in the market model, it leaves open the basic question as to whether either set of residuals leads to tests which correctly indicate the presence of heteroskedasticity.

This uncertainty clouds the interpretation of the conclusions reached in the paper. All of the non-normal error distributions considered by Giaccotto and Ali are more leptokurtotic than the normal distribution and, hence, would lead to the problems discussed above. Parenthetically, it is somewhat odd for the authors to speak of heteroskedastic Cauchy disturbances when the Cauchy distribution does not possess finite moments of any order! Moreover, the rank tests which suggest that heteroskedasticity in the market model is related to the squared return on the market suffers from similar difficulties. Our analysis indicates that the degree of dependence among the squared recursive residuals depends on the magnitude of squared variation in the independent variables. Again it is uncertain whether these rank tests are providing us with useful information concerning the stochastic properties of market model regressions or of the sampling properties of residuals.

We do not mean to suggest that heteroskedasticity is not present in the market model or that it is unrelated to the squared return on the market. We are puzzled however, by the lack of concern the authors have for providing an economically motivated rationale to guide this investigation. For example, the analysis in the paper was conducted using data for individual firms over a twelve year sample period. Hence, it is likely that the beta coefficients for many of the firms fluctuated over the sample period. A naive model for individual betas as being the sum of a constant mean plus a random error term would lead to heteroskedasticity being proportional to the squared return on the market. Rather than calling for an ad hoc GLS adjustment, such dependence should suggest the appropriateness of shorter sampling intervals and/or more explicit attention to the economic determinants of fluctuations in beta. If the authors are truly concerned about testing a variety of alternatives to the usual homoskedastic market model, it behooves them to search for economically motivated departures from this formulation and not to restrict their attention to purely statistical alternatives.

## APPENDIX

### A. OLS Residuals

Consider the usual linear regression model:

$$\underline{y} = X\underline{\beta}+\underline{\varepsilon} \tag{1}$$

where $\underline{y}$ is a Tx1 vector of observations on a dependent variable, X is a TXk matrix of observations on k independent variables, $\underline{\beta}$ is a kX1 vector of unknown coefficients and $\underline{\varepsilon}$ is a Tx1 vector of independent and identically distributed unobservable disturbance terms with zero mean vector and scalar covariance matrix, i.e.:

$$E[\underline{\varepsilon}] = \underline{0}$$
$$E[\underline{\varepsilon}\underline{\varepsilon}']=\sigma^2 I_T \tag{2}$$

where $I_T$ is an identity matrix of order T.

The OLS residual vector $\underline{e}$ is defined by:

$$\underline{e} = \underline{y}-X\,\hat{\underline{\beta}} \tag{3}$$

where:

$$\hat{\underline{\beta}} = (X'X)^{-1}X'\underline{y} \tag{4}$$

As is well-known, algebraic manipulation of (3) and (4) yields:

$$\underline{e} = M\,\underline{\varepsilon} \tag{5}$$

where:

$$M = I-X(X'X)^{-1}X'$$
$$MX = 0 \tag{6}$$
$$M'M = M^2 = M$$

Hence the first and second moments of the OLS residuals are given by:

$$E[\underline{e}] = \underline{0}$$
$$E[\underline{e}\,\underline{e}'] = \sigma^2 M \tag{7}$$

In other words, the OLS residuals are correlated and, hence, ought not be treated as independent for the purpose of hypothesis testing.

The large sample analysis of OLS residuals is straightforward. Under weak assumptions, such as:

$$\lim_{T\to\infty} \frac{1}{T} X'X = Q < \infty \tag{8}$$

where Q is a positive definite symmetric matrix, the OLS estimator is consistent:

$$\lim_{T\to\infty} \hat{\underline{\beta}} = \underline{\beta} \tag{9}$$

and, hence, the OLS residual vector converges to the true underlying disturbances:

$$\lim_{T\to\infty} \underline{e} = \underline{\varepsilon} \tag{10}$$

In consequence, if the underlying disturbances are independently distributed over observations, then in large samples the OLS residuals will be independently distributed as well.

B. Recursive Residuals

Let the column vector $\underline{x}_t$ denote the $t^{th}$ row of the Txk matrix X and let the matrix $X_t$ be the first t rows of X. The $t^{th}$ recursive residual is given by:

$$w_t=[y_t-\underline{x}_t' \hat{\underline{\beta}}_{t-1}][1+\underline{x}_t'(X_{t-1}'X_{t-1})^{-1}\underline{x}_t]^{-1/2}$$

where:

$$\hat{\underline{\beta}}_{t-1} = (X_{t-1}X_{t-1})^{-1} \sum_{j=1}^{t-1} \underline{x}_j y_j \tag{12}$$

is the OLS estimator based on the first t-1 observations. In terms of the underlying disturbances, the recursive residual is:

$$\underline{w}_t = [\varepsilon_t-\underline{x}_t(X_{t-1}'X_{t-1})^{-1} \sum_{j=1}^{t-1} \underline{x}_j \; \varepsilon_j][1 + \underline{x}_t'(X_{t-1}X_{t-1})^{-1}\underline{x}_t]^{-1/2} \tag{13}$$

It is well-known that the vector of recursive residuals can be written

$$\begin{aligned} \underline{w} &= H\,\underline{e} \\ &= H\,\underline{y} \\ &= H\,\underline{\varepsilon} \end{aligned} \tag{14}$$

where the (T-k)xT matrix H is obviously formed from the terms involving $\underline{x}_t$ and $X_{t-1}$ in (13) and which satisfies:

$$HX = 0$$

$$HH' = I_{T-k} \qquad (15)$$

$$HMH' = I_{T-K}$$

$$H'H = M$$

where M was defined in (6) above. From these considerations, it follows that:

$$E[\underline{w}] = \underline{0}$$

$$E[\underline{w}\ \underline{w}'] = \sigma^2 I_{T-k} \qquad (16)$$

If, in addition, the disturbances are normally distributed then the recursive residuals are independently distributed as well.

On the other hand, Giaccotto and Ali are expressly concerned about the possibility that the disturbances follow a non-normal leptokurtotic distribution. Hence, Giaccotto and Ali would necessarily presume that the recursive residuals were uncorrelated but not independent. In order to illustrate some of the effects of non-normality, we compute one of the higher order moments---the covariance between squared recursive residuals. Additional higher order moment calculations are available from the authors on request.

Since:

$$w_i^2 = \underline{e}'\underline{h}_i\underline{h}_i'\underline{e} \qquad (17)$$

where $\underline{h}_i$ is a Tx1 vector consisting of the $i^{\underline{th}}$ row of H, then:

$$E[w_i^2\ w_j^2] = E[\underline{\varepsilon}'\underline{h}_i\underline{h}_i'\underline{\varepsilon\varepsilon}'\underline{h}_j\underline{h}_j'\underline{\varepsilon}]$$

$$= E[\text{trace}(\underline{h}_i\underline{h}_i'\underline{\varepsilon\varepsilon}'\underline{h}_j\underline{h}_j\underline{\varepsilon\varepsilon}')]$$

$$= \text{vec}(\underline{h}_i\underline{h}_i')'E[\underline{\varepsilon\varepsilon}'\otimes\underline{\varepsilon\varepsilon}']\text{vec}(\underline{h}_j\underline{h}_j') \qquad (18)$$

where vec( ) denotes the vectorization of a matrix obtained by stacking the matrix column by column and $\otimes$ denotes the usual Kronecker product. By inspection:

$$E[\underline{\varepsilon\varepsilon}'\otimes\underline{\varepsilon\varepsilon}'] = (\mu_4 - 3\sigma^4)\ U + V \qquad (19)$$

where V denotes the expected value of the left-hand side of (19) if $\underline{\varepsilon}$ were normally distributed, U is a diagonal matrix consisting of all zeros except for ones in the $[T(j-1)+j]^{th}$ diagonal position for j = 1, ..., T, $\mu_4$ is the fourth central moment of $\varepsilon$ and $3\sigma^4$ is the fourth central moment of the normal distribution. Hence:

$$E[w_i^2 w_j^2] = (\mu_4 - 3\sigma^4)[\text{vec}(\underline{h}_i\underline{h}_i')' U \text{ vec}(\underline{h}_j\underline{h}_j')]$$
$$+\text{vec}(\underline{h}_i\underline{h}_i')' V \text{ vec}(\underline{h}_j\underline{h}_j')$$
$$= (\mu_4 - 3\sigma^4)[\text{vec}(\underline{h}_i\underline{h}_i')' U \text{ vec}(\underline{h}_j\underline{h}_j')] + \sigma^4$$
$$= (\mu_4 - 3\sigma^4) \sum_{k=1}^{T} h_{ik}^2 h_{jk}^2 + \sigma^4$$

since, under normality, $w_i^2$ and $w_j^2$ are independent. In consequence, if the distribution of the underlying disturbances is leptokurtotic, the covariance between $w_i^2$ and $w_j^2$ is positive, so that we would expect more extremely large values of w and extremely small values of w then would occur if the recursive residuals were actually independent.

The large sample analysis of recursive residuals is even more complicated. This is the obvious result of the fact the the $j^{\underline{th}}$ recursive residual is a fixed linear combination of the first j observations which does not change as the number of observations grows. In contradistinction, the $j^{\underline{th}}$ OLS residual changes as the number of observations increases which accounts for their convergence to the true disturbances documented in (10). Hence, the dependence of the first T recursive residuals does not vanish for any finite T, although the dependence of the $i^{\underline{th}}$ and $j^{\underline{th}}$ recursive residuals does tend to diminish for values of i and j near T as T grows large. The following proposition establishes sufficient conditions for a subset of the recursive residuals to be independent in large samples.

Proposition: Suppose that (8) holds so that the OLS estimator $\hat{\underline{\beta}}$ is consistent. Then, if the underlying disturbances $\underline{\varepsilon}$ are independently distributed, the last L recursive residuals will be independently distributed in large samples if:

$$L/T \to 0 \text{ as } T \to \infty$$

Proof: The result follows directly from the consistency of the OLS estimator (a) and the fact that the last (countably infinite) L recursive residuals will be arbitrarily close to the OLS residuals and, hence, to the true disturbances. Q.E.D.

Note that this, in turn, requires that the first T-L recursive residuals be ignored as T gets large in order to ensure that the remaining L recursive residuals are independent for the purposes of test statistic construction. Again, the magnitude of T-L might be small relative to T in many situations but this question should be addressed directly on a case by case basis.

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