On the Gross-Stark and Iwasawa Main Conjectures

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by

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Let $F$ be a totally real number field, $p$ a rational prime, and $\chi$ a finite order totally odd abelian character of $\text{Gal}(\overline{F}/F)$ such that $\chi(p) = 1$ for some $p|p$. Motivated by a conjecture of Stark, Gross conjectured a relation between the derivative of the $p$-adic $L$-function associated to $\chi$ at its exceptional zero and the $p$-adic logarithm of a $p$-unit in the $\chi$ component of $F^\times$. In a recent work, Dasgupta, Darmon, and Pollack have proven this conjecture assuming two conditions: that Leopoldt’s conjecture holds for $F$ and $p$, and that if there is only one prime of $F$ lying above $p$, a certain relation holds between the $L$-invariants of $\chi$ and $\chi^{-1}$. The main result of this work removes both of these conditions, thus giving an unconditional proof of the conjecture. We also describe some applications towards simplifying the Iwasawa Main Conjecture at the weight one and Leopoldt zeroes, as well as some partial results on Gross’s conjecture in the higher rank setting.
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To Roy, Mary, and Anne.
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The prototypical example of an $L$-function is the Riemann zeta function
\[ \zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_p (1 - 1/p^s)^{-1}. \]
The two expressions on the right converge for $\text{Re}(s) > 1$, and have analytic continuation to all of $\mathbb{C} \setminus \{1\}$. A slightly more general notion is the Dedekind zeta function $\zeta_K$ attached to a number field $K$, which has similar analytic properties. The zeroes of these functions are known to be intimately related to the distribution of primes in the underlying number field, but another relationship between these functions and the arithmetic of the underlying field is the following formula originally proved by Dedekind:
\[ \lim_{s \to 0} \frac{\zeta_K(s)}{s^r} = -\frac{h_K R_K}{|\mu(K)|}. \]
Here $r = r_1 + r_2 - 1$ is one less than the total number of real embeddings and pairs of complex embeddings of $K$, $h_K$ is the class number of $K$, $\mu(K)$ is the group of roots of unity of $K$, and $R_K$ is the regulator attached to $K$, which measures the covolume of the image of the units of $K$ under the logarithm maps with respect to all but one archimedean embedding. This was one of the first general instances of a special value formula, relating leading terms of an $L$-function to invariants of the underlying arithmetic object.

After Dedekind, Artin introduced the $L$-function attached to any complex representation of a Galois group of number fields. In a series of papers starting
with [Sta61], Stark studied special values of these $L$-functions, and conjectured a more general formula

$$\lim_{s \to 0} \frac{L(V,s)}{s^{r_V}} = R(V)A(V)$$

where $r_V$ is explicitly given, $R(V)$ is a generalized regulator, and $A(V)$ an algebraic number. Stark’s papers were later clarified and refined by Tate [Tat81]. If the underlying complex representation is one-dimensional, and has order of vanishing one, Stark further conjectures that these leading terms can be used to explicitly construct abelian extensions of the ground field (via so-called Stark units), thus giving a partial solution to Hilbert’s 12th problem.

Around the same time, the arithmetic of $L$-functions was being explored in a different direction, namely the construction of $p$-adic $L$-functions. These are $p$-adic analytic functions which interpolate special values of classical $L$-functions. Most relevant for us is the following

**Theorem** (Deligne-Ribet, Cassou-Nogues, Barsky). Fix a totally real field $F$, a prime number $p$, and embeddings $\mathbb{C} \leftarrow \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. Then for any odd abelian character $\chi : \text{Gal}(\overline{F}/F) \to \overline{\mathbb{Q}}^\times$, there is an analytic function

$$L_p(\chi \omega, s) : \mathbb{Z}_p \to \mathbb{Q}_p(\chi)$$

which satisfies the interpolation property

$$L_p(\chi \omega, -n) = L(\chi \omega^{-n}, -n) \prod_{p|p} (1 - \chi \omega^{-n}(p)Np^n)$$

for all $n \geq 0$.

Let $r$ be the number of primes $p | p$ for which $\chi(p) = 1$, and $K$ the CM field cut out by $\chi$. If $r > 0$, this formula implies $L_p(\chi \omega, 0) = 0$. In this case, guided by Tate’s formulation of Stark’s conjectures, Gross conjectured a formula for the leading term of the $p$-adic $L$-function of $\chi$ at $s = 0$ [Gro81, Conjecture 2.12]:

**Conjecture.** $\lim_{s \to 0} \frac{L_p(\chi \omega, s)}{s^{r}} = R_p(\chi)A(\chi)$. 

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Here $A(\chi)$ is an algebraic number which can be read off from the archimedean $L$-function, and $R_p(\chi)$ is a $p$-adic analogue of Dirichlet’s archimedean regulator: it is the co-volume of a lattice in a $\mathbb{Q}_p$-vector space which is the image of the $\chi$-component of the $p$-units in $K$ under a $p$-adic logarithm (a more precise description will be given in 3.2).

Gross proves this conjecture when $F = \mathbb{Q}$. He also formulates a refinement in the rank one setting, i.e. where there is a unique $p|p$ for which $\chi(p) = 1$. When this place is of degree one over $p$, there is a canonical way of choosing a $p$-unit in the $\chi$-component of $K^\times$. Gross conjectures that certain $n^{th}$ roots of this unit should generate abelian extensions of $F$, which would give a $p$-adic solution to some cases of Hilbert’s 12th problem.

Inspired by Greenberg-Stevens’s work on the Mazur-Tate-Teitelbaum conjecture [GS93], and Wiles’s earlier work on the Iwasawa Main Conjecture [Wil90], Dasgupta-Darmon-Pollack were able to prove the unrefined abelian Gross’s conjecture in rank one under two restrictive hypotheses: that Leopoldt’s Conjecture holds, and a technical condition which essentially says that in certain cases, the leading terms of $L_p(\chi\omega)$ and $L_p(\chi^{-1}\omega)$ shouldn’t cancel (see [DDP11, Theorem 2]).

The main result of this thesis is

**Theorem 1.0.1.** If $r = 1$, then Conjecture 1 holds.

The work is organized as follows.

Chapter 2 covers preliminary material on Deligne-Ribet $p$-adic $L$-functions, Hilbert modular forms and their associated Galois representations, and the Iwasawa Main Conjecture as proved by Wiles.

In Chapter 3, we discuss Stark’s conjecture and Gross’s conjecture for abelian characters in arbitrary rank. Starting with the original formulation we reinterpret Gross’s conjecture in a Galois cohomological setting in all ranks, following
Dasgupta-Darmon-Pollack in rank one. We then show that the Iwasawa Main Conjecture implies an inequality which is required to even get Gross’s conjecture off the ground. Finally, we show in all ranks that the algebraic $L$-invariant vanishes if and only if the analytic $L$-invariant does.

In Chapter 4, we give a complete proof of Gross’s conjecture for rank one zeroes, combining our techniques with Dasgupta-Darmon-Pollack’s. To remove Leopoldt’s conjecture, we construct a certain ordinary family of parallel weight Hilbert modular forms with weight zero specialization equal to the constant form $1$. In order to remove the $L$-invariant condition, we break into cases depending on the exact order of vanishing of the coefficients of certain $\Lambda$-adic forms.

In Chapter 5, we describe a consequence of Gross’s conjecture and its proof for the Iwasawa Main Conjecture at the Leopoldt zeroes. We conclude with the construction of a representation of the Hecke algebra in all ranks which shows that the Hecke algebra “knows” the leading term of $L_{S,p}(\chi, s)$. This would appear to be an essential step in an (as yet non-existent) proof of Gross’s conjecture in higher rank.
CHAPTER 2

Preliminaries

We begin by fixing notations. As in the introduction, $F$ will denote a totally real field, $\overline{F}$ an algebraic closure, $p$ a rational prime number, and $\chi : G_F := \text{Gal}(\overline{F}/F) \to \mathbb{Q}^\times$ a totally odd character. Let $K$ be the CM field cut out by $\chi$. Let $\mathbb{Z}_p[\chi]$ denote the ring obtained by adjoining all values of $\chi$ to $\mathbb{Z}_p$. Let $\omega : \text{Gal}(\mathbb{Q}(\mu_{2p})/\mathbb{Q}) \to (\mathbb{Z}/2p\mathbb{Z})^\times$ denote the Teichmüller character. For any place $p$ of $F$, we let $I_p \subset G_p \subset G_F$ denote a choice of inertia resp. decomposition group at $p$. We implicitly fix embeddings $\mathbb{C}_p \leftrightarrow \mathbb{Q} \leftrightarrow \mathbb{C}$ throughout.

2.1 Deligne-Ribet $p$-adic $L$-functions

Let $S$ be any finite set of primes of $F$ which includes all archimedean primes and all primes dividing the conductor of $\chi$. Associated to $\chi$ and $S$ is a complex analytic function $L_S(\chi,s)$ defined for $\text{Re}(s) > 1$ by

$$L_S(\chi,s) = \sum_{(a,S)=1} \chi(a)N(a)^{-s} = \prod_{p \notin S} (1 - \chi(Frob_p)Np^{-s})^{-1},$$

and which has holomorphic continuation to all of $\mathbb{C}$. By Siegel’s rationality theorem, $L_S(\chi,-n) \in \mathbb{Q}$ for $n \geq 0$. Since we assumed $\chi$ is odd, $L_S(\overline{\chi},1)$ is well-defined and nonzero. Furthermore, by the functional equation and nonvanishing of $L_S(\overline{\chi},1)$, the order of vanishing of $L_S(\chi,s)$ at $s = 0$ is equal to the number of $v \in S$ such that $\chi|_{D_v} = 1$.

Let us now assume the set $S$ contains all places above $p$. Let $F_\infty$ be the cyclotomic $\mathbb{Z}_p$-extension of $F$, and $\Gamma = \text{Gal}(F_\infty/F)$, which is canonically isomorphic
to a subgroup of $1 + 2p\mathbb{Z}_p$. For use later, we fix a topological generator $u$ of $\Gamma$, which gives an isomorphism $\mathbb{Z}_p[\chi][[\Gamma]] \cong \mathbb{Z}_p[\chi][[T]] =: \Lambda$ via $u \mapsto 1 + T$. Let $F_\Lambda = \text{Frac}(\Lambda)$. We will identify $u$ with its image in $1 + 2p\mathbb{Z}_p$.

Following Greenberg, a character of $\text{Gal}(\overline{F}/F)$ is said to be of type $S$, resp. type $W$, if the extension it cuts out is disjoint from $F_\infty$, resp. contained in $F_\infty$. Since $\Gamma$ is pro-cyclic, it is a direct summand of $\text{Gal}(F^{ab}/F)$. Therefore, any character can be decomposed as a product of a type $S$ character and a type $W$ character, which we write as $\chi = \chi_S \chi_W$.

The following theorem is due to Deligne-Ribet [DR80]:

**Theorem.** There is a pseudo-measure $L_{S,\chi}\omega \in \text{Frac}(\mathbb{Z}_p[\chi][[\Gamma]])$ which interpolates classical $L$-values via the formula

$$
\chi^k \varepsilon \psi(\omega^{-k} L_{S,\chi}\omega) = L_S(\psi \chi \omega^{1-k}, 1-k),
$$

where $\psi$ is any finite order character of type $W$ and $k \geq 1$.

We will also use $L_{S,\chi}\omega$ to denote the corresponding element of $F_\Lambda$ via the isomorphism above. The previous formula can then be written

$$
L_{S,\chi}\omega(\zeta u^k - 1) = L_S(\psi \chi \omega^{1-k}, 1-k),
$$

where $\zeta = \psi(u)$. Taking $\zeta = 1$, we get a $p$-adic analytic function

$$
L_{p,S}(\chi,\underline{\omega}), \mathbb{Z}_p \to \overline{\mathbb{Q}_p}
$$

$$
s \mapsto L_{S,\chi\omega}(u^{1-s} - 1).
$$

This is usually referred to as the $p$-adic $L$-function of the even character $\chi\omega$. We see from the above properties that it satisfies

$$
L_{p,S}(\chi, -n) = L_S(\chi^{-n}, -n) \text{ for all } n \geq 0.
$$

It follows that if $L_S(\chi, s)$ vanishes at $s = 0 \in \mathbb{C}$, then $L_{p,S}(\chi, s)$ vanishes at $s = 0 \in \mathbb{Z}_p$. 

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Suppose now that $S$ is minimal. Let $K_{\infty} = KF_{\infty}$ and $M_{\infty}$ the maximal odd unramified $\mathbb{Z}_p$-extension of $K_{\infty}$. The Iwasawa Main Conjecture, as proved in most cases by Wiles in [Wil90], states that the characteristic ideal of $(\text{Gal}(M_{\infty}/K_{\infty}) \otimes \overline{\mathbb{Q}}_p)^{-1}$ as a $\Lambda$-module is generated by $L_{S,\chi}$ (this is not quite true when $\chi$ is a type $W$ twists of $\omega^{-1}$; see Ch. 5).

The basic method of proof was introduced by Ribet [Rib76] and refined by Wiles. One interprets the $L$-value as the constant term of an Eisenstein series, and uses the vanishing of this constant term to construct a congruence with a cusp form. Finally, one uses the irreducible Galois representations attached to cuspforms to construct the unramified Galois extensions.

These same ideas appear in Dasgupta-Darmon-Pollack’s work on Gross’s conjecture, and will be used later on in Chapter 4. However, as we are trying to study actual values of the $L$-function, and not just the order of vanishing, our work requires a careful analysis of the relationships between the nature of the congruences and properties of the Galois representation.

2.2 Hilbert Modular Forms

2.2.1 Classical Forms

In this section and in the rest of the work, we will only consider parallel weight Hilbert modular forms. We include a few more notations. Let $d$ be the different ideal of $F$, $U_F$ denote the units of $O_F$, and $U_F^+$ the totally positive units. Let $CL^+(F)$ denote the group of strict ideal classes of $F$, and let $\mathfrak{c}$ be a representative of some element of $CL^+(F)$, with $\mathfrak{c}^+$ the cone of positive elements. Finally, let $n$ be an integral ideal of $F$ and $\varphi : (O_F/n)^\times \to \overline{\mathbb{Q}}^\times$ a character.

**Definition.** Let $I$ denote the set of embeddings of $F$ into $\mathbb{R}$, and $\mathcal{H}$ denote the complex upper half plane. A complex $\mathfrak{c}$-Hilbert modular form of weight $k$,
level $n$, and character $\varphi$ is a holomorphic function $f$ on $\mathcal{H}^I$, such that for every element of
\[
\Gamma_c(n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(F) | a, d \in \mathcal{O}, b \in c^{-1}d^{-1}, c \in nc\mathcal{O}, ad - bc \in U_F^+ \right\},
\]
we have
\[
(ad - bc)^{k/2}(cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right) = \varphi(a)f(z).
\]

Here we are using the same shorthand as in [Shi78, §1]. That is, the expression
\[
(ad - bc)^{k/2}(cz + d)^{-k}
\]
denotes the complex number obtained as the product over all embeddings $\sigma \in I$ of $\sigma((ad - bc)^{k/2}(cz + d)^{-k})$, taking positive square roots. The expression
\[
f\left(\frac{az + b}{cz + d}\right)
\]
denotes $f$ evaluated at the point of $\mathcal{H}^I$ whose $\sigma$th coordinate is $\frac{\sigma(a)z + \sigma(b)}{\sigma(c)z + \sigma(d)}$.

The modularity condition implies that $f$ has a Fourier expansion
\[
f(z) = a(0) + \sum_{b \in c^+} a(b)q^b
\]
where $q^b = e^{2\pi i \text{Tr}_{F/Q}(b)}$.

The space of such forms is finite dimensional; we denote this space by $M_{k,c,\psi}(n, \mathbb{C})$.

More generally, for any ring $R \subset \mathbb{C}$, let $M_{k,c,\psi}(n, R)$ denote the subset of forms with Fourier coefficients in $R$. Shimura has shown that $M_{k,c,\psi}(n, \mathbb{Q}) \otimes \mathbb{C} = M_{k,c,\psi}(n, \mathbb{C})$; using the fixed embedding $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$, we can define $M_{k,c,\psi}(n, R)$ for any subring of $\mathbb{Q}_p$.

**Definition.** A Hilbert modular form (without the $c$) of weight $k$ and level $n$ is a $|Cl^+(F)|$-tuple of $c$-Hilbert modular forms, where $c$ ranges over a set of representatives of the strict ideal classes.
We will usually write this as \( f = (f_c)_c \). For a ray class character \( \chi \) of conductor dividing \( n \), we will say \( f \) has character \( \chi \) if \( S(a)f = \chi(a) \) for almost all prime ideals \( a \) of \( F \) (see [Shi78, p. 648] for the definition of \( S(a) \)). We denote the space of such forms by \( M_k(n, \chi) \).

Given a Hilbert modular form \( f \) in the latter sense, the normalized Fourier expansion is defined as follows: for each nonzero integral ideal \( a \), there is a unique \( c \) in our choice of strict ideal class representatives for which there is an equality of ideals \( ac = (b) \) for some totally positive \( b \in c \). Then we let

\[
c(a, f) = a_c(b)Nc^{-k/2}.
\]

where \( a_c(b) \) is the coefficient of \( q^b \) in \( f \). For each \( \lambda \in Cl^+(F) \), we also set

\[
c_{\lambda}(0, f) = a_{\epsilon}(0)Nc^{-k/2}.
\]

As the notation suggests, neither of these expressions depend on our choice of \( b \).

For each prime \( \ell \nmid n \), and each prime \( q | n \), there are Hecke operators \( T_\ell \) and \( U_q \) which act on the spaces \( M_k(n, \chi) \). Explicitly, we have

\[
c(m, T_\ell(f)) = \sum_{m+\ell \subseteq a} \chi(a)N(a)\ell^{-1}c(a^{-2}m\ell, f)
\]

\[
c_{\lambda}(0, T_\ell(f)) = \sum_{b | \ell} \chi(b)c_{\lambda}[b][\ell]^{-1}(0, f).
\]

Fix a rational prime \( p \) and suppose \( p | n \) for all \( p | p \). If \( R \) is a complete subring of \( \overline{\mathbb{Q}_p} \), then we say \( f \) is ordinary if \( ef = f \), where \( e := \lim_n \prod_{p \nmid p} U_p^{n!} \).

If \( f \) is an eigenform, i.e. there exist algebraic numbers \( \{a_\ell\} \) such that \( T_\ell f = a_\ell f \) for all \( \ell \), a theorem due to many people including Carayol, Blasius-Rogawski, Wiles, and Taylor says the following:

**Theorem.** For any prime \( q \) of \( \mathbb{Q}(\{a_\ell\}) \), there is a Galois representation

\[
p : G_F \to GL_2(\overline{\mathbb{Q}(\{a_\ell\})_\ell})
\]
unramified outside \( n Nm(\varrho) \), which is uniquely determined by the condition that

\[
\text{tr}(\rho(Frob_\ell)) = a_\ell
\]

for \( \ell \nmid n Nm(\varrho) \). If \( f \) is ordinary at \( p \) and \( \varrho \mid p \), then for any \( p \mid p \), the representation \( \rho|_{G_{F_p}} \) admits a one-dimensional unramified quotient on which \( Frob_p \) acts via multiplication by \( U_p \).

In the next section, we will discuss ordinary \( \Lambda \)-adic forms, but we will not describe the Galois representations attached to \( \Lambda \)-adic eigenforms until we use them in Chapter 4.

### 2.2.2 \( \Lambda \)-adic forms

Let \( m_\Lambda \) be the maximal ideal of \( \Lambda \), \( \Lambda_{(0)} \) the localization of \( \Lambda \) at the prime ideal \((T)\), and \( F_\Lambda \) its field of fractions. For any \( \mathbb{Z}_p[\chi] \)-algebra \( E \), let \( \Lambda_E = E \otimes_{\mathbb{Z}_p[\chi]} \Lambda \).

**Definition.** An ordinary \( \Lambda \)-adic form of character \( \chi \) and level \( n \) is a collection of coefficients

\[
\{c(\lambda, F), \{c(a, F)\} \in \Lambda
\]

where \( \lambda \) runs over \( Cl^+(F) \) and \( a \) runs over the nonzero integral ideals of \( O_F \), such that for almost all pairs \( k \geq 2, \zeta \in \mu_{p^\infty} \), the reduction of this system modulo the ideal \( P_{\zeta,k} = (T + 1 - \zeta u^k) \) gives the normalized Fourier coefficients of an ordinary parallel weight \( k \) Hilbert modular form of level \( \bar{n} := \text{lcm}(p \cdot \text{ord}(\zeta), n) \) and character \( \psi \zeta \chi \omega^{1-k} \).

We use \( \mathcal{M}_\Lambda^{\text{ord}}(n, \chi) \) to denote the space of such forms. By [Wil88, Thm. 1.2.2], this space is finite and torsion-free as a \( \Lambda \)-module. We call the reduction mod \( P_k := P_{1,k} \) the weight \( k \) specialization. For any subalgebra \( \Lambda \subset R \subset F_\Lambda \), we define \( \mathcal{M}_R^{\text{ord}}(n, \chi) = \mathcal{M}_\Lambda^{\text{ord}}(n, \chi) \otimes_\Lambda R \). We also let \( \mathcal{S}_\Lambda^{\text{ord}}(n, \chi), \mathcal{S}_F^{\text{ord}}(n, \chi) \), etc. denote the corresponding spaces of cusp forms.

Let \( \mathbb{T}^{\text{ord}} \) denote the ordinary \( \Lambda \)-adic Hecke algebra of level \( n \) and character
\(\chi\), i.e. the \(\Lambda\)-algebra generated, for \(\ell \nmid np\) and \(q|np\), by the Hecke operators \(T_\ell, U_q\) acting on \(\mathcal{M}_\Lambda^{ord}(n, \chi)\). Formulae for this action in terms of \(q\)-expansions are obtained by interpolating the above formulae at each classical weight; they are given explicitly at the top of page 537 in [Wil88]. The following lemma is probably well known, but as far as we know has not been written down.

**Lemma 2.2.1.** If the weight \(k\) specializations of a collection \(\{c(0, \mathcal{F})\}, \{c(a, \mathcal{F})\} \in \Lambda\) give a classical ordinary form for infinitely many \(k \geq 2\), then they are classical for all but finitely many \(k \geq 2\).

**Proof.** Fix a weight \(k\) and let \(E\) be a finite Galois extension of \(\mathbb{Q}_p[\chi]\) containing all the Hecke eigenvalues appearing in \(M_k^{ord}(n_S, \chi \omega^{1-k})\). Wiles shows in [Wil88, Thm. 1.4.1] that the system of Hecke eigenvalues corresponding to any classical eigenform of weight \(k\) can be realized as a quotient of the \(\Lambda\)-adic Hecke algebra \(\mathcal{T}^{ord}/(1 + T - u^k) \to E\). If \(m\) is the corresponding maximal ideal of \(\mathcal{T}^{ord} \otimes_{\mathbb{Z}_p[\chi]} E\), the space \(\mathcal{M}_\lambda^{ord}(n, \chi)/m.\mathcal{M}_\lambda^{ord}(n, \chi)\) is nonzero by Nakayama’s lemma. Therefore, any classical eigenform can be realized as the weight \(k\) specialization of some \(\Lambda_E\)-adic form (i.e. we don’t need to take a finite extension of \(\Lambda_E\); note that we are not claiming there is an eigenform defined over \(\Lambda_E\)).

We claim that this implies that any ordinary form with coefficients in \(\mathbb{Q}_p[\chi]\) can be realized as the specialization of some \(\Lambda_{\mathbb{Q}_p[\chi]}\)-adic form. Indeed, on the weight \(k\) fiber, we can write the ordinary form as an \(E\)-linear combination of eigenforms. Lifting this linear combination to \(\Lambda_E\) gives a form specializing to the one we need, but a priori only has coefficients in \(\Lambda_E\). However, by averaging over \(\text{Gal}(\Lambda_E/\Lambda_{\mathbb{Q}_p[\chi]})\), and observing that specialization intertwines this action with the action of \(\text{Gal}(E/\mathbb{Q}_p[\chi])\), we get a \(\Lambda_{\mathbb{Q}_p[\chi]}\)-adic form with the desired specialization.

The claim implies that the map

\[
\mathcal{M}_{\Lambda_{\mathbb{Q}_p[\chi]}}^{ord}(n, \chi)/(1 + T - u^k).\mathcal{M}_{\Lambda_{\mathbb{Q}_p[\chi]}}^{ord}(n, \chi) \to M_k(n_S, \mathbb{Q}_p[\chi], \chi \omega^{1-k})
\]
is an isomorphism for all \( k \geq 2 \). In particular,
\[
\text{rank}_{\Lambda_{\mathbb{Q}_p[x]}} \mathcal{M}^{\text{ord}}_{\Lambda_{\mathbb{Q}_p[x]}}(n, \chi) = \dim_{\mathbb{Q}_p[x]}(\mathcal{P}, \mathbb{Q}_p[x]/\chi^{1-k})
\]
for almost all \( k \); call this dimension \( d \). We can choose ideals \( a_1, \ldots, a_d \) of \( \mathcal{O}_F \) so that the map
\[
\pi : \mathcal{M}^{\text{ord}}_{\Lambda_{\mathbb{Q}_p[x]}}(n, \chi) \to (\Lambda_{\mathbb{Q}_p[x]})^d
\]
\[
\mathcal{P} \mapsto (c(a_i, \mathcal{P}))_i
\]
is injective. After inverting a finite set of primes \( \mathcal{S} \) of \( \Lambda_{\mathbb{Q}_p[x]} \), \( \pi \) is an isomorphism. Therefore, for \( P_k \notin \mathcal{S} \), we have
\[
\pi_k : M^{\text{ord}}_k(\mathcal{P}, \mathbb{Q}_p[x], \chi^{1-k}) \cong \mathbb{Q}_p[x]^d
\]
\[
f \mapsto (c(a_i, f))_i
\]
Now suppose we had a collection of coefficients \( \{c(\lambda(0, \mathcal{H}))\}, \{c(a, \mathcal{H})\} \) with infinitely many classical specializations. There is a unique element \( \mathcal{P} \) of \( \mathcal{M}^{\text{ord}}_{\Lambda_{\mathbb{Q}_p[x]}}(n, \chi) \) such that \( c(a_i, \mathcal{P}) = c(a_i, \mathcal{H}) \) for all \( i \). Moreover, at each weight \( k \) with \( P_k \notin \mathcal{S} \), and where \( \mathcal{H} \) is classical, the reduction of \( \mathcal{H} \) must agree with the reduction of \( \mathcal{P} \) by the isomorphism \( \pi_k \). Thus, \( \mathcal{H} \) and \( \mathcal{P} \) must be equal since they agree on a Zariski dense set. This proves the lemma.

\[\square\]

A typical example of ordinary \( \Lambda \)-adic forms of tame level \( n \) and character \( \chi \) are the \( \Lambda \)-adic Eisenstein series \( \mathcal{E}(\eta, \psi) \) attached to a pair of (not necessarily primitive) narrow ray class characters \( \eta, \psi \) such that \( \eta \psi = \chi \), \( \text{cond}(\eta)\text{cond}(\psi) = pn \), \( (p, \text{cond}(\eta)) = 1 \): 
\[
c(\lambda(0, \mathcal{E}(\eta, \psi))) = \delta_\eta2^{-g}\eta^{-1}(c_\lambda)L(\eta, \psi, 1)(N_{\infty} \psi)_{\eta}^{-1},
\]
\[
c(a, \mathcal{E}(\eta, \psi)) = \sum_{\substack{(\tau, p)=1 \atop \tau \mid a}} \eta(\frac{d}{\tau})\psi(\tau)(N\tau^{-1})(1 + T)^{-\log(\tau^{k})/\log(p)}.
\]
Here \( \delta_\eta = 1 \) if \( \text{cond}(\eta) = 1 \), and is zero otherwise.

We denote \( G_\zeta := \mathcal{L}_{(p|p),1} \), so that \( G_\zeta(u^s - 1) = \zeta_{F,p}(1 - s) \). Let

\[
\mathcal{G} := 2^s G_\zeta^{-1} \zeta F,p(1 - s),
\]

so that the constant term of \( \mathcal{G} \) at each infinite cusp is identically 1. It follows from a result of Colmez [Col88] that if Leopoldt’s Conjecture is true for \( (F,p) \) then \( G_{\zeta} \) has a pole of order one at \( T = 0 \). In this case, the form \( \mathcal{G} \), which \textit{a priori} only lies in \( \mathcal{M}^\text{ord}_{\Lambda}(1, \omega^{-1}) \), actually lies in \( \mathcal{M}^\text{ord}_{\Lambda(\omega)}(1, \omega^{-1}) \), with specialization equal to the constant form \( 1 \), i.e. \( c_\lambda(0, \mathcal{G}(0)) = 1 \) and \( c(\alpha, \mathcal{G}(0)) = 0 \) for all \( \lambda, \alpha \). In Chapter 4, we will show that even if Leopoldt fails, there is a suitable cusp form \( \mathcal{J} \) such that \( \mathcal{G} - \mathcal{J} \) has all of these properties.
CHAPTER 3

Gross’s Conjecture

In this chapter, we first describe Stark’s conjecture and Gross’s $p$-adic analogue for totally odd representations. Then assuming our Artin representation is one-dimensional, we give a cohomological reformulation of Gross’s conjecture, and finish by proving a weak form of Gross’s conjecture in all ranks.

To avoid confusion, we remark here that any time we consider the action of a Galois group on a $\mathbb{C}_p$-vector space, it will always be linear.

3.1 Stark’s Conjecture

We mainly follow [Tat81] in our treatment of Stark’s conjecture. Let $F$ be a totally real number field, $K$ a CM extension of $F$, and $\chi$ the character attached to a totally odd Artin representation $V$ of $G := \text{Gal}(K/F)$. Let $S$ be a set of places of $F$ including all infinite places. Suppose $k$ is a field of definition of $V$ which is finite over $\mathbb{Q}$, and that $\chi$ factors through the finite extension $K$. Then the archimedean $L$-function

$$L_S(\chi, s) = \prod_{p \notin S} \det(1 - \text{Frob}_p N_p^{-s} | V^I)^{-1}$$

for $\text{Re}(s) > 1$ has meromorphic continuation to the entire complex plane. By the functional equation for Artin $L$-functions, it is known that

$$r := \text{ord}_{s=0} L_S(\chi, s) = \sum_{p \in S} \dim(V_p^G) - \dim V^G.$$
In order to formulate Stark’s conjecture, we must describe Stark’s regulator \( R(V) \). Let \( S_K \) be the set of places of \( K \) lying above places in \( S \), \( U \) be the \( S_K \)-units in \( K \), and \( X \) be the set of degree zero divisors on \( S_K \). Then in one direction, there is a map

\[
g: U \to X
u \mapsto \sum_{\mathfrak{B} \notin \mathfrak{B}_\infty} f_{\mathfrak{B}} \cdot \text{ord}_{\mathfrak{B}}(u) \cdot \mathfrak{B}
\]

This map is \( \text{Gal}(K/F) \)-equivariant, and so induces a map, which is easily checked to be an isomorphism, on minus parts

\[
g: (U \otimes \mathbb{Q})^- \xrightarrow{\sim} (X \otimes \mathbb{Q})^-.
\]

For Stark’s archimedean conjecture, we need the following Archimedean regulator map:

\[
\lambda_\infty: U \to \mathbb{R} \otimes X
u \mapsto \sum_{\mathfrak{B} \in S} \log \|u\|_{\mathfrak{B}} \cdot \mathfrak{B},
\]

where

\[
\|u\|_{\mathfrak{B}} = \begin{cases} 
    u\overline{u} & \text{if } \mathfrak{B} \text{ is complex;} \\
    |u| & \text{if } \mathfrak{B} \text{ is real;} \\
    N\overline{\mathfrak{B}}^{-\text{ord}_{\mathfrak{B}}(u)} & \text{otherwise}.
\end{cases}
\]

It follows from Dirichlet’s \( S \)-unit theorem that \( \lambda_\infty: \mathbb{R} \otimes U \to \mathbb{R} \otimes X \) is a Galois-equivariant isomorphism. Therefore, \( \lambda_\infty \circ g^{-1} \) induces a Galois-equivariant automorphism of \( \mathbb{C} \otimes X \), and hence a well-defined automorphism

\[
\lambda_\infty \circ g^{-1}: (V \otimes_k (\mathbb{C} \otimes_k X))^G \to (V \otimes_k (\mathbb{C} \otimes_k X))^G.
\]

This last space is of \( r \)-dimensional as a \( \mathbb{C} \)-vector space, and \( R(V) \) is defined as the \( r \times r \) determinant of this automorphism.
Conjecture (Stark). \[ \lim_{s \to 0} \frac{L_S(\chi,s)}{s} = R(V)A(V), \]

where \(A(V)\) is an algebraic number in \(k\) which is equivariant with respect to different embeddings \(k \hookrightarrow \mathbb{C}\). We remark that the right hand side should be decorated with an \(S\) on each term, as \(R(V)\) and \(A(V)\) depend on our choice of \(S\).

In case \(V\) is totally odd, i.e. every complex conjugation acts by the scalar \(-1\), Stark’s conjecture is known. This boils down to the following two relatively elementary facts:

- Stark’s conjecture for one choice of \(S\) implies it for any choice of \(S\).
- If \(V\) is totally odd and \(S\) is taken to be minimal, then \(r = 0, R(V) = 1,\) and \(L_S(\chi,0)\) is algebraic by a theorem of Siegel.

### 3.2 Gross’s \(p\)-adic analogue

We retain all notations from the previous section, and assume that \(V\) is totally odd. We now also assume that \(S\) contains all places above \(p\), as well as the ramified and infinite places.

By the theorem of Deligne-Ribet quoted in Chapter 1, and a variant of Brauer induction due to Serre, we can attach a \(p\)-adic \(L\)-function to \(V\). More precisely, there is a meromorphic function

\[ L_{p,S}(V \otimes \omega, \_): \mathbb{Z}_p \to \mathbb{Q}_p \otimes E \]

which is uniquely determined by its values on classical negative integers:

\[ L_{p,S}(V \otimes \omega, -n) = L_S(V \otimes \omega^n, -n) \quad \text{for all even } n > 0. \]

This equality is known to hold at \(n = 0\) if \(V\) is abelian. In other words, letting \(r_p = \text{ord}_{s=0} L_{p,S}(V \otimes \omega, s)\), we have that for \(V\) abelian,

\[ r > 0 \iff r_p > 0. \]
In chapter 3, assuming $V$ is abelian, we will show that $r_p \geq r$, and that $r = 1$ if and only if $r_p = 1$. More generally, we have the following

**Conjecture 3.2.1** (Gross).

i. $r_p = r$.

ii. $\lim_{s \to 0} \frac{L_p(V \otimes \omega_s)}{\sigma_p} = R_p(V)A(V)$.

Here $A(V)$ is the same algebraic number appearing in Stark’s conjecture, and $R_p(V)$ is a certain $p$-adic regulator. In order to describe it, we need a $p$-adic analogue of $\lambda_\infty$. Define

$$\lambda_p : U^- \otimes \mathbb{Q}_p \to X^- \otimes \mathbb{Q}_p$$

$$u \mapsto \sum_{\mathfrak{B} \in \mathcal{S}_K} \log_p(||u||_{\mathfrak{B}, p}),$$

where

$$||u||_{\mathfrak{B}, p} = \begin{cases} 1 & \text{if } \mathfrak{B} \text{ is complex;} \\ sgn(u) & \text{if } \mathfrak{B} \text{ is real;} \\ N_{\mathfrak{B}/\mathbb{Q}_p}(u) & \text{if } \mathfrak{B} \nmid p; \\ N_{\mathfrak{B}/\mathbb{Q}_p}(u) N_{K_{\mathfrak{B}/\mathbb{Q}_p}}(u) & \text{if } p \mid \mathfrak{B}. \end{cases}$$

Unlike Dirichlet’s map, it is unknown whether this map is injective. As we will see later, an affirmative answer to this question is equivalent to Conjecture 3.2.1.i. In any case, $\lambda_p \circ g^{-1}$ is a well-defined Galois-equivariant endomorphism over $\mathbb{C}_p$, and so defines an endomorphism

$$\lambda_p \circ g^{-1} : (V \otimes_{\mathbb{C}_p} (\mathbb{C}_p \otimes_{\mathbb{Z}} X))^G \to (V \otimes_{\mathbb{C}_p} (\mathbb{C}_p \otimes_{\mathbb{Z}} X))^G.$$ 

This space is $r$-dimensional over $\mathbb{C}_p$, and we define $R_p(V)$ to be the determinant of this endomorphism. Following Greenberg, we will also call this the algebraic $L$-invariant $\mathcal{L}_{alg}$.

For our own convenience, we reformulate Gross’s conjecture into three statements of roughly increasing order of difficulty:
Conjecture 3.2.2 (Gross). i. \( \text{ord}_{s=0} L_{p,S}(V \otimes \omega, s) \geq \text{ord}_{s=0} L_S(V, s) \)

ii. \( L_{p,S}(V \otimes \omega, s) = R_p(V) A(V) s^r + O(s^{r+1}) \)

iii. \( R_p(V) \neq 0 \)

If we let 
\[
\mathcal{L}_{an}(\chi) = r! \frac{L_{S,p}^{(r)}(\chi^\omega, 0)}{L_R(\chi, 0)},
\]
then 3.2.2.ii is equivalent to the equation \( \mathcal{L}_{an} = \mathcal{L}_{alg} \). In 3.4, we will show that Conjecture 3.2.2.i follows from the Iwasawa Main Conjecture, and that \( R_p(V) \neq 0 \) if and only if \( r = r_p \). Chapter 4 is dedicated to the proof of 3.2.2.ii when \( r = 1 \).

### 3.3 Cohomological Reformulation

From now on, we will assume that \( \dim V = 1 \). In this section, we give a Galois cohomological interpretation of \( R_p(V) \) which will be essential for the proof in Chapter 4. Although we will only be proving 3.2.2.ii in the rank one setting, we will give the reformulation for arbitrary rank.

We now assume \( S \) is \( p \)-adically minimal, i.e. consists exactly of the primes above \( p \), the infinite places, and the ramified places. We write \( V = E(\chi) \) where \( E \) is any field containing the values of the character \( \chi : \text{Gal}(K/F) \to \overline{\mathbb{Q}}^\times \), and will often substitute the symbol \( \chi \) for \( V \), as in \( L_{p,S}(\chi, s) = L_{p,S}(V, s) \). In this case, the regulator map admits a more explicit description. First, note that we could have just as well defined \( R_p(\chi) \) to be the determinant of \( g^{-1} \circ \lambda_p \) as an endomorphism of 
\[
(V \otimes_E (\mathbb{C}_p \otimes \mathbb{Z} U))^G = (\mathbb{C}_p \otimes U)^{\chi^{-1}}
\]
where the superscript \( \chi^{-1} \) denotes the component on which Galois acts by \( \chi^{-1} \).

Let \( p_1, \ldots, p_r \) be the primes of \( F \) above \( p \) for which \( \chi(p_i) = 1 \), and \( \mathfrak{B}_i \) some choice of prime in \( K \) above \( p_i \). Let \( \phi_i \) denote the inclusion \( U \hookrightarrow K_{\mathfrak{B}_i}^\times \). Let \( D_{p_i} \subset G_F \) be some choice of decomposition group for \( p_i \), and \( I_{p_i} \) the inertia
subgroup. We choose a basis \( \{ u_i \}_{i=1}^r \) of \((\mathbb{C}_p \otimes U)^{\times -1}\) for which

\[
\text{ord}_{p_i}(u_j) = \delta_{ij},
\]

where \( \delta_{ij} \) is the Kronecker delta. In this basis, the \((i, j)\) coefficient of the endomorphism \( g^{-1} \circ \lambda_p \) is given explicitly by \( \log_p(\text{Norm}_{K_{a_i}}(\phi_i(u_j))) \).

For each \( i \), we have

\[
H^1(F_{p_i}, \mathbb{C}_p(\chi^{-1})) = H^1(F_{p_i}, \mathbb{C}_p) \cong \text{Hom}(G_{p_i}, \mathbb{C}_p).
\]

There are two distinguished elements of this space: \( \kappa_{ur} \), the unramified character which send Frobenius to 1 (additively), and \( \kappa_{cyc} \), which is the restriction of the global character

\[
\text{Gal}(\overline{F}/F) \to \text{Gal}(F_{\infty}/F) \to 1 + p\mathbb{Z}_p \xrightarrow{\log_p} \mathbb{Z}_p \to \mathbb{C}_p.
\]

We let \( W_i \) be the subspace spanned by these two classes, and \( H^1_W(F, \mathbb{C}_p(\chi^{-1})) \) the space of global classes which are unramified away from the primes \( p_i \), and locally at \( p_i \) lie in \( W_i \). Let \( H^1_p(F, \mathbb{C}_p(1)(\chi)) \) the subspace of \( H^1(F, \mathbb{C}_p(1)(\chi)) \) cut out by the following local conditions:

- Arbitrary at the primes \( p_1, \ldots, p_r \)
- Arbitrary at primes \( q \) for which \( \chi|_{G_q} \neq 1 \)
- At primes \( q \) for which \( \chi|_{G_q} = 1 \), lies in the orthogonal complement (under the Tate pairing) of the space spanned by \( \kappa_{ur} \in H^1(G_{q}, \mathbb{C}_p(\chi^{-1})) \).

**Lemma 3.3.1.** \( H^1_p(F, \mathbb{C}_p(1)(\chi)) \cong (\mathbb{C}_p \otimes U)^{\chi^{-1}} \)

**Proof.** By Kummer Theory, we have isomorphisms globally:

\[
H^1(F, \mathbb{C}_p(1)(\chi)) \cong (K^\times \otimes \mathbb{C}_p)^{\chi^{-1}},
\]
and locally at primes \( q \) of \( F \):

\[
H^1(F_q, \mathbb{C}_p(1)(\chi)) \cong (K^\times_\Omega \hat{\otimes} \mathbb{C}_p)^{\chi^{-1}},
\]

where \( \Omega \) is the prime above \( q \) corresponding to the implicit embedding \( \bar{F} \to \bar{F}_q \).

The local conditions say exactly that the restriction of an element of \((K^\times_\Omega \hat{\otimes} \mathbb{C}_p)^{\chi^{-1}}\)

lies in \((U^\times_\Omega \hat{\otimes} \mathbb{C}_p)^{\chi^{-1}}\) at primes away from \( p_1, \ldots, p_r \). This proves the lemma. \( \square \)

**Lemma 3.3.2.** \( \dim_{\mathbb{C}_p} H^1_W(F, \mathbb{C}_p(\chi^{-1})) = r \).

**Proof.** In [DDP11, Lemma 1.5], it is shown that the subspace of \( H^1_W(F, \mathbb{C}_p(\chi^{-1})) \)

which is unramified at all but one \( p_i \) is 1-dimensional. Moreover, the unique (up to a scalar) class in this subspace is ramified at \( p_i \), since e.g. there are no unramified \( \mathbb{Z}_p \)-extensions of \( K \). Therefore, \( H^1_W(F, \mathbb{C}_p(\chi^{-1})) \) is at least \( r \)-dimensional. It is also at most \( r \)-dimensional, since otherwise we could find a nonzero element which is unramified at each \( p_i \). \( \square \)

For each \( j \), let \( \tau_j \in I_{p_j} \) be any element which maps to 1 under \( \kappa_{\text{cyc}} \), and \( \sigma_j \) a lift of Frobenius.

We fix the basis \( \{\kappa_i\}_{i=1}^r \) of \( H^1_W(F, \mathbb{C}_p(\chi^{-1})) \) for which \( \kappa_i(\tau_j) = \delta_{ij} \). Then for \( 1 \leq i, j \leq r \), there exists \( x_{ij} \) such that

\[
\kappa_i|_{G_{p_j}} = \begin{cases} 
\kappa_{\text{cyc}} + x_{ii}\kappa_{ur} & \text{if } i = j; \\
x_{ij}\kappa_{ur} & \text{if } i \neq j.
\end{cases}
\]

**Proposition 3.3.1.** \( \det((x_{ij})) = R_p(V) \).

**Proof.** In fact, we will prove the stronger statement that

\[
x_{ij} = \log_p(\text{Norm}_{K_{\mathbb{Q}_p}}(\phi_i(u_j))).
\]

Fix a choice of \( \kappa_i \in H^1_W(F, \mathbb{C}_p(\chi^{-1})) \) and \( u_j \in H^1_p(F, \mathbb{C}_p(1)(\chi)) \). The global reciprocity law of class field theory applied to this pair implies that

\[
\sum_v \langle \kappa_i|_v, u_j|_v \rangle_v = 0,
\]
where the sum is over all places $v$ in $F$, and $\langle , \rangle_v$ denotes the local Tate pairing at $v$. Away from the primes $p_k$, this pairing is equal to zero. When $v$ is equal to one of the primes $p_k$, and if $u \in F_v^\times \otimes \mathbb{C}_p$, we have the following two identities for the local Tate pairing:

\[
\langle \kappa_{ur} , u|_v \rangle_v = -\text{ord}_v(u)
\]
\[
\langle \kappa_{cy}, u|_v \rangle_v = \log_p(\text{Norm}_{K_{\mathbb{A}_k}}(\phi_k(u)))
\]

Therefore, the following identities hold:

\[
\langle \kappa_{i|p_k,j|p_k}|_{p_k} = \begin{cases} 
-x_{ii} + \log_p(\text{Norm}_{K_{\mathbb{A}_i}}(\phi_i(u_i))) & \text{if } i = j = k; \\
0 & \text{if } i = j \neq k; \\
\log_p(\text{Norm}_{K_{\mathbb{A}_j}}(\phi_j(u_j))) & \text{if } i \neq j, i = k; \\
-x_{ij} & \text{if } i \neq j = k; \\
0 & \text{if } i \neq j \neq k \neq i
\end{cases}
\]

Fixing a choice of $i,j$ and setting the sum over $k$ to zero, we get the desired equality.

\[
3.4 \quad r_p \geq r \text{ and a weak form of Gross's conjecture}
\]

We begin by showing the Iwasawa Main Conjecture implies Conjecture 2.i.

**Lemma 3.4.1.** Let $\chi$ be a finite order character of $F$. If IMC holds for $(\chi_S,p)$ (e.g. if $p > 2$) then

\[
\text{ord}_{s=0}L_{p,S}(\chi \omega, s) \geq \text{ord}_{s=0}L_S(\chi, s).
\]

**Proof.** We may assume $S$ is minimal, i.e. consists only of the primes dividing $\text{cond}(\chi)p\infty$. Then $\text{ord}_{s=0}L_S(\chi, s) = r$. Let $\zeta = \chi_W(u)$. By Lemma 3.3.2, the space of extensions $H^1_{W}(F, E(\chi^{-1}))$ is $r$-dimensional. Over $K_{\infty}$, this gives
an $r$-dimensional unramified extension on which $\text{Gal}(K_{\infty}/F)$ acts by $\chi^{-1}$. The subgroup $\text{Gal}(F_{\chi S,\infty}/F_{\infty}) \cong \text{Gal}(F_{\chi S}/F)$ acts on this extension by $\chi_{\chi S}^{-1}$, and $\text{Gal}(F_{\chi S,\infty}/F_{\chi S}) \cong \text{Gal}(F_{\infty}/F)$ acts by $\chi_{\chi S}^{-1}$. It follows from the Main Conjecture that $r \leq \text{ord}_T \zeta_{\chi S} - 1 \mathcal{L}_{\chi S}$. This implies the lemma. 

The weak form of Gross’s conjecture which we wish to prove is

**Proposition.**

$$r_p > r \iff R_F(V) = 0.$$ 

Equivalently,

$$\mathcal{L}_{an} = 0 \iff \mathcal{L}_{alg} = 0.$$ 

**Proof.** $\Rightarrow$: Let $K_{\infty}$ be the cyclotomic $\mathbb{Z}_p$ extension of $K$, and let $M_{\infty}$ be the maximal unramified $\mathbb{Z}_p$-extension of $K_{\infty}$. Let $X$ denote the maximal subspace of $\text{Gal}(M_{\infty}/K_{\infty}) \otimes \mathbb{Z}_p \mathbb{C}_p$ on which $\text{Gal}(K/F)$ acts by $\chi^{-1}$ and on which $\Gamma = \text{Gal}(K_{\infty}/K)$ acts unipotently. By the Main Conjecture, $\text{dim}_{\mathbb{C}_p} X = r_p$. Therefore, if $\mathcal{L}_{an} = 0$, then $\text{dim}_{\mathbb{C}_p} X > r$.

Let $X_\Gamma$ denote the space of $\Gamma$-coinvariants of $X$. For each $p_i$, let $D_{\infty,i} \subset D_{p_i}$ be the kernel of $\kappa_{\text{cyc}}$, and $I_{\infty,i} = D_{\infty,i} \cap I_{p_i}$. Also, let $D_{\infty,i}^{p-\text{ab}}$ be the $p$-abelianization of $D_{\infty,i}$, and let $I_{\infty,i}$ denote the image of inertia in this group$^1$. Then by Proposition 3.3.1, $\mathcal{L}_{alg} \neq 0$ if and only if the map

$$\prod_{p_i} (D_{\infty,i}/I_{\infty,i} \otimes \mathbb{Z}_p \mathbb{C}_p) \to X_\Gamma$$

is an isomorphism. By Nakayama’s lemma, this is equivalent to the surjectivity of

$$\prod_{p_i} (D_{\infty,i}^{p-\text{ab}} \otimes \mathbb{Z}_p \mathbb{C}_p) \to X.$$ 

However, if in addition $\text{dim}_{\mathbb{C}_p} X > r$, this implies that for some $p_i$, the image of inertia is nontrivial, contradicting the definition of $X$.

$^1$N.B. This is not the $p$-abelianization of $I_{\infty,i}$.
It suffices to show that the map

$$\prod_{p_i}(D_{\infty,i}/I_{\infty,i} \otimes \mathbb{Z}_p \mathbb{C}_p) \to X$$

is injective, since if $\mathcal{L}_{alg} = 0$, the composition of 3.1 with the projection to $X_\Gamma$ is not an isomorphism. In particular, the latter projection is not an isomorphism, so that $X \neq X_\Gamma$.

Let $N_\infty$ be the Galois group of the maximal odd $\mathbb{Z}_p$-extension of $K_\infty$ over $K$ in which we allow ramification at the primes $p_i$, and let

$$Y = (\text{Gal}(N_\infty/K_\infty) \otimes \mathbb{C}_p)(T),$$

where the subscript $(T)$ means localization at the prime ideal $(T)$ of $\Lambda$.

To prove that 3.1 is injective, it is enough to show that

$$\prod_{p_i} \prod_{B \subset \mathcal{O}_{K_\infty}} (D_{2B}^{p-ab} \otimes \mathbb{C}_p)(T)$$

injects into $Y$. To prove this, it is enough to show that the subgroup

$$\prod_{p_i} \prod_{B \subset \mathcal{O}_{K_\infty}} (I_{2B}^{p-ab} \otimes \mathbb{C}_p)(T)$$

injects into $Y$, since by [NSW08, Thm. 11.2.3], both of these groups are free of the same rank as $\Lambda_{(T)}$-modules. Finally, injectivity of this last map can be checked at finite levels of the cyclotomic tower, where it follows from class field theory (see e.g. [KW10, Lemma 5.7 and Corollary 5.8]).
CHAPTER 4

Proof of the Main Theorem

In this chapter, we prove Conjecture 2.ii for the one-dimensional totally odd Artin representation \( V = E(\chi) \), assuming \( r = 1 \). In other words, there is a unique \( p \in S \), which we assume lies above \( p \), such that \( \chi(p) = 1 \).

4.1 A \( \Lambda \)-adic form passing through 1

We revert to letting \( \Lambda \) denote \( \mathbb{Z}_p[[T]] \), and refer to Chapter 2.2 for notation. In this section we prove the following

**Theorem 4.1.1.** There exists an \( F_\Lambda \)-adic cusp form \( J \in S_{\text{ord}}^{\Lambda}(1, \omega^{-1}) \) such that \( \mathcal{G} - J \in M_{\text{ord}}^{\Lambda(0)}(1, \omega^{-1}) \) and \((\mathcal{G} - J)(0)\) is the constant form 1.

4.1.1 Reductions and geometric \( \Lambda \)-adic forms

We begin with some reductions. First, it is enough to find any ordinary \( \Lambda_{E,(0)} \)-adic form of level one with constant weight zero specialization, where \( E \) is a finite extension of \( \mathbb{Q}_p \). Rescaling, we may assume the constant at weight zero is equal to one. Let \( \mathcal{F} \) be such a form, so that

\[
\mathcal{F} \in M_{\text{ord}}^{\Lambda(1)}(1) = \bigoplus_{\chi} M_{\text{ord}}^{\Lambda(1, \chi \omega^{-1})}
\]

where \( \chi \) ranges over all even ray class characters of conductor 1. It causes no harm to assume \( E \) contains sufficiently many roots of unity, so that there is a \( E \)-linear combination of diamond operators projecting onto the \( \omega^{-1} \) component.
above. Specializing at weight 0, this acts by the identity on the constant form, so we may assume $\mathcal{F}$ has character $\omega^{-1}$. We can write $F$ as an $F_{\Lambda_{E}}$-linear combination of a cusp form and Eisenstein series $E(\eta, \eta^{-1}, \omega^{-1})$, where $\eta$ ranges over strict ray class characters of conductor 1. For $\eta \neq 1$, the system of Hecke eigenvalues associated to the Eisenstein series $E(\eta, \eta^{-1}, \omega^{-1})$ at weight zero differs from that of the constant form. Hence, there is a $\Lambda_{E(0)}$-linear combination of Hecke operators which will kill all Eisenstein contributions except $E(1, \omega^{-1})$, and act by the identity on the constant form. Applying this to $F$, we are left with an $F_{\Lambda_{E}}$-linear combination of $E(1, \omega^{-1})$ and a cusp form. Since the constant terms of this form are identically one, it must be equal to $G - J$ for some $F_{\Lambda_{E}}$-cuspform $J$. Finally, we can average over $\text{Gal}(F_{\Lambda_{E}}/F_{\Lambda})$ so that $J$ has coefficients in $F_{\Lambda}$.

Our next reduction requires a new definition. Let $r > 0$ be a natural number, and $\Lambda := \mathbb{Z}_{p}[\frac{T}{p^{r}}]$. For an integral ideal $n$ and odd ray class character $\chi$ of conductor dividing $n$, we define an ordinary $\Lambda$-adic Hilbert modular form to be a collection of coefficients

$$\{c_{\lambda}(0, \mathcal{F})\}, \{c(m, \mathcal{F})\} \in \Lambda$$

such that for infinitely many $k \in p^{r}N$, their image under the specialization $T = u^{k} - 1$ are the coefficients of a classical ordinary Hilbert modular form of parallel weight $k$, level $\text{lcm}(p, n)$, and character $\chi \omega^{1-k}$. We will denote this module by $\mathcal{M}_{\Lambda}^{\text{ord}}(n, \chi)$, for any ring $\Lambda \subset R \subset F_{\Lambda}$, we set $\mathcal{M}_{R}^{\text{ord}}(n, \chi) = \mathcal{M}_{\Lambda}^{\text{ord}}(n, \chi) \otimes_{\Lambda} R$. Using the constant dimensionality of the spaces of ordinary weight $k$ forms, and an argument similar to that used in Lemma 2, one can show that the space of $F_{\Lambda}$-adic forms has the same dimension as the space of $F_{\Lambda}$-adic forms, and hence is identified with $\mathcal{M}_{F_{\Lambda}}^{\text{ord}} \otimes_{F_{\Lambda}} F_{\Lambda}$. Now suppose we can find an $\Lambda_{(\frac{T}{p^{r}})}$-adic form specializing to 1 at weight zero (i.e. modulo $\frac{T}{p^{r}}$). Then writing it as an $F_{\Lambda}$-linear combination of elements of $\mathcal{M}_{\Lambda}^{\text{ord}}$, we can replace the coefficients in $F_{\Lambda}$ with elements of $F_{\Lambda}$ having the same principal part and constant term at weight zero, to arrive at an $F_{\Lambda}$-adic form specializing to 1 at weight zero. Thus, it is enough
to find an ordinary family with coefficients in $F_{\Lambda}$.

To construct a family of level one, we first construct a family of some auxiliary level $q$ using powers of a certain theta series (Lemmas 3 and 4). We will then use the Atkin-Lehner operators $U_q$ and $W_q$ to project the form down to level one. In order to define these operators, we will make use of a geometric description of $\Lambda$-adic forms (see Proposition 2).

Fix $F, p$ as in Section 2 and let $n$ be an integral ideal of $\mathcal{O}_F$. Fix a strict ideal class, and let $c$ be a prime-to-$p$ representative of this class. Let $R$ be a $p$-adically complete DVR. Following [AG05, Definition 3.2], we let $T_{m,n} = \mathfrak{M}(R/p^m, \mu_{p^n}, \Gamma_0(n))$ be the moduli stack over $R/p^m$ whose objects over $S$, for any $R/p^n$-scheme $S$, are given by isomorphism classes of tuples $(A, \iota, \lambda, \phi_n, i_{p^n})$ where

- $A \to S$ is an abelian scheme of relative dimension $g$;
- $\iota : \mathcal{O}_F \hookrightarrow \text{End}_R(A)$ is a ring homomorphism;
- $\lambda : (M_A, M_A^+) \cong (c, c^+)$ is an $\mathcal{O}_F$-linear isomorphism of étale sheaves over $T$ between the module of symmetric $\mathcal{O}_F$-linear homomorphisms from $A$ to its dual $A^\vee$ to the ideal $c$, such that the polarizations $M_A^+$ map to $c^+$;
- $\phi_n \subseteq A$ is an $\mathcal{O}_F$-invariant closed subgroup scheme which is isomorphic to the constant group scheme $(\mathcal{O}_F/n)$ étale locally on $S$;
- $i_{p^n} : \mu_{p^n} \otimes_{\mathbb{Z}} A^{-1} \hookrightarrow A$ is an inclusion of group schemes.

These are referred to as $c$-polarized Hilbert Blumenthal Abelian Varieties (HBAV’s) with level structure. Following [AG05, Definition 11.4], we define a $p$-adic $c$-Hilbert modular form (or $c$-HMF) of level $\Gamma_0(n)$ over $R$ to be an element of

$$V_{\infty,\infty} := \lim_{\leftarrow m} \lim_{\rightarrow n} H^0(T_{m,n}/(R/p^m), \mathcal{O}_{T_{m,n}}).$$
If $\chi : (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times \rightarrow R^\times$ is a finite order character, we will say the form is of (parallel) weight $k \in \mathbb{Z}_p$ and character $\chi$ if for any $\alpha \in (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times$, we have

$$\alpha^*(f) = \chi(\alpha)\text{Nm}(\alpha)^k f,$$

where $\alpha^* f(A, t, \lambda, \phi_n, i_p, \omega) = f(A, t, \lambda, \phi_n, i_p, \omega \circ \alpha^{-1})$, and $\text{Nm} : (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times \rightarrow 1 + 2p\mathbb{Z}_p \rightarrow R^\times$ is induced by the norm map followed by projection onto the 1-units.

Fix an isomorphism $\epsilon : c \otimes \mathbb{Z}_p \cong \mathcal{O}_F \otimes \mathbb{Z}_p$. The $q$-expansion at $\infty$ of $f$ is an element

$$f(q) \in R[[q^b]]_{b \in \epsilon^+ \cup \{0\}},$$

which generalizes the $q$-expansion of classical Hilbert modular forms. We refer to [AG05, Definition 11.6] for the precise definition. In their notation, it is the evaluation of $f$ at the cusp $(c, \mathcal{O}_F, \epsilon, j_\epsilon)$ where $j_\epsilon$ is induced from $\epsilon$ as in [loc. cit., 6.5]. The $q$-expansion principle states that a $p$-adic $c$-HMF of weight $\kappa$ is determined by its $q$-expansion [loc. cit., 11.7].

For every $k \geq 2$, there is a Hecke-equivariant inclusion $M_{k, \epsilon}(\Gamma_0(np), R) \hookrightarrow V_{\infty, \infty}/R$ which preserves $q$-expansions and weights [Kat78, Thm. 1.10.15]. In fact, in the quoted theorem, the space of classical forms on the left hand side of the inclusion is more general than the forms we considered in Section 2: it allows any power of $p$ in the level, and in the complex setting, it consists of those forms invariant under the subgroup of $\Gamma_c(n)$ consisting of matrices of determinant 1 (see [AG05, 6.11]). However, this certainly contains the forms we want to consider, and this is all we will need. We will call a form in the image of the above inclusion classical.

Let $W$ be a finite flat DVR over $\mathbb{Z}_p$. Let $m_\Lambda$ denote the maximal ideal of $\Lambda_W$. We now present two definitions of “$p$-adic” $\Lambda_W$-adic forms and prove that they are the same. We also want similar statements to hold for $\Lambda$-adic forms, and will indicate where changes need to be made.
Recall that $u \in 1 + 2p\mathbb{Z}_p$ is a generator of the image of $\text{Gal}(F_\infty/F)$. Define the map $\phi : (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times \to \Lambda^\times_W$ by the composition

$$(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times \to \text{Gal}(F_\infty/F) \xrightarrow{u^{-1}+T} \Lambda^\times_W$$

We may also consider $\phi$ as a $\Lambda W$-valued character via the inclusion $\Lambda \subset \Lambda$.

**Definition.** A *Wiles* $\Lambda W$-adic $\mathfrak{c}$-Hilbert modular form $\mathcal{F}$, of level $\Gamma_0(n)$, is a multiset of elements $\{c(t, \mathcal{F})\}_{t \in \mathbb{Z}^{+}\cup \{0\}} \subset \Lambda_W$ such that for every $s \in \mathbb{Z}_p$, the sequence of elements of $W$ obtained from the specialization $T \mapsto u^s - 1$ is the $q$-expansion of a $p$-adic $\mathfrak{c}$-Hilbert modular form of level $\Gamma_0(n)$ and weight $s$ over $W$. We use the same definition for $\Lambda_W$-adic $\mathfrak{c}$-Hilbert modular forms, except that we only require the specialization condition to hold for $s \in p^r\mathbb{Z}_p$.

**Remark.** Given a $|\text{Cl}^+(F)|$-tuple of Wiles $\Lambda W$-adic forms, one for each strict ideal class with representative $\mathfrak{c}$, and with infinitely many specializations giving the Fourier coefficients of an ordinary classical Hilbert modular form, we can obtain a $\Lambda$-adic form as in Section 2 as follows. Under the usual normalization for weight $k$ forms, one would set $c(m, \mathcal{F}(u^k-1)) = (N\mathfrak{c})^{-k/2} \cdot a_{\mathfrak{c}}(b)(u^k-1)$, where $b$ is a totally positive generator of $mc$. However, this presents a problem since $(N\mathfrak{c})^{-k/2}$ may not vary $p$-adically continuously with $k$. So instead, we simply set $c(m, \mathcal{F}) = a_{\mathfrak{c}}(b)$, and $c_{\mathfrak{c}}(0, \_\_)$ equal to the constant term. Since this is independent of the choice of $b$ at infinitely many weights, it must be independent $\Lambda$-adically. This modification will not affect what we are ultimately interested in: finding a family whose weight zero specialization is the constant form $1$.

**Definition.** A *Katz* $\Lambda W$-adic $\mathfrak{c}$-Hilbert modular form of level $\Gamma_0(n)$ is an element of the subspace of $V_{\infty,\infty}/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} \Lambda_W := \lim_{\leftarrow} V_{\infty,\infty}/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} \Lambda_W/m^n_{\Lambda}$ satisfying

$$\alpha^*(f) = \phi(\alpha)f.$$  

for every $\alpha \in (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times$. We use the same definition for $\Lambda_W$ mutatis mutandis.

This last property is equivalent to requiring that for every $s \in \mathbb{Z}_p$ (resp. $p^r\mathbb{Z}_p$), reducing the form modulo $(1 + T - u^s)$ yields a $p$-adic $\mathfrak{c}$-Hilbert modular form of
weight $s$ defined over $\Lambda_W/(1 + T - u^s)$. Note that neither of these definitions require any of the specializations to be classical.

By definition, a Katz $\Lambda$-adic $c$-HMF is nothing but a compatible sequence of $p$-adic $c$-HMFs over $\Lambda_W/m^n_\Lambda$ satisfying certain extra conditions. Thus, we may define the $q$-expansion at $\infty$ of a Katz $\Lambda$-adic form to be the inverse limit of these $q$-expansions; it is an element of $\Lambda[[[q^b]]]_{b \in c^+ \cup \{0\}}$.

The following proposition is due to Hida when $F = \mathbb{Q}$ [Hid00, Thm. 3.2.16]. We essentially follow his proof.

**Proposition 4.1.1.** The space of Katz $\Lambda_W$-adic $c$-Hilbert modular forms is identified with the space of Wiles $\Lambda_W$-adic $c$-Hilbert modular forms via $q$-expansion at $\infty$. The same is true for $\Lambda_W$-adic forms.

**Proof.** We first explain the proof for $\Lambda_W$-adic forms. It follows from the definitions that the $q$-expansion of a Katz form is a Wiles $\Lambda_W$-adic form, so we need only show that all Wiles $\Lambda_W$-adic forms arise in this way. Let $\mathcal{F}$ be a Wiles $\Lambda_W$-adic form. We start by reinterpreting $\mathcal{F}$ as a measure $C(\mathbb{Z}_p, \mathbb{Z}_p) \rightarrow V_{\infty, \infty}/W$, defined by sending the function $\binom{x^n}{n}$ to the coefficient of $T^n$ in $\mathcal{F}$ (which is a $p$-adic $c$-HMF by virtue of being a limit of $p$-adic $c$-HMF’s). By the binomial theorem, this measure has the property that for $s \in \mathbb{Z}_p$, the function $x \mapsto u^x$ is sent to $\mathcal{F}(u^s - 1)$, a $p$-adic $c$-HMF of weight $s$.

Taking the completed tensor product with $\Lambda$ of this measure gives a map

$$C(\mathbb{Z}_p, \Lambda) \rightarrow V_{\infty, \infty}/W \hat{\otimes} \Lambda.$$  

The image of the function $x \mapsto (1 + T)^x$ is easily seen to be a Katz $\Lambda_W$-adic form (i.e. obeys the equation (*)), with $q$-expansion equal to $\mathcal{F}$.

Now suppose $\mathcal{F}$ is a Wiles $\Lambda_W$-adic form. Define the submodule $M \subset C(\mathbb{Z}_p, \mathbb{Z}_p)$ by demanding that if $C(\mathbb{Z}_p, \mathbb{Z}_p) \ni f = \sum_{N \geq 0} a_N \binom{x_N}{N}$, then

$$f \in M \iff \frac{a_N}{p^{rN}} \in \mathbb{Z}_p$$

and $\frac{a_N}{p^{rN}} \rightarrow 0$ as $N \rightarrow \infty$. 

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Then we may consider $F$ as a measure $M \rightarrow V_{∞,∞/W}$ which sends $p^r N(x)$ to the coefficient of $\left(\frac{p}{p^r}\right)^N$ in $F$. Then just as before, we take the completed tensor product with $Λ_W$, and consider the image of the function $x \mapsto (1 + T)^x$. This gives the desired Katz $Λ_W$-adic form.

The Katz definition gives us a geometric interpretation of $Λ$-adic forms as follows:

Let $M_c$ denote the functor from the category of $m_Λ$-adically complete $Λ$-algebras to Sets which takes an algebra $R$ to the set of isomorphism classes tuples $(A, ι, λ, φ_n, i_p, i_{p∞})$ as above. Then we can view a $Λ$-adic $c$-HMF as a natural transformation from this functor to the forgetful functor $A^1$, which further satisfies ($*$). For $Λ$-adic $c$-HMFs, the same statement holds if we consider $m_Λ$-adically complete $Λ$-algebras.

4.1.2 Construction of the form

We now construct the level one ordinary family with constant weight zero specialization. We begin by quoting Lemma 1.4.2 of [Wil88], which is attributed to Hida:

**Lemma 4.1.1.** For some prime $q \not| p$, and some $m > 0$, there is a Hilbert modular form $f$ of weight $2^m(p - 1)$ and level $Γ_0(pq)$, with coefficients in $Z_p$, such that $c_λ(0, f) = 1$ for all $λ$, and $f \equiv 1(\text{mod } p)$.

**Remark.** The quoted lemma has a power $p^j$ in the level; however, we can apply the operator $U_p := T_{np}(p) j - 1$ times to decrease the level at $p$ to $Γ_0(p)$ without altering any of the other properties. This lemma is proved using theta
series coming from the extension $F(\mu_p)/F(\mu_p)^+$. An alternative approach is to use lifts of suitable powers of the Hasse invariant (see [AG05, Lemma 11.10]).

Write $f = (f_\ell)_\ell$. Since $f_\ell \equiv 1 (\text{mod } p)$, for any $s \in \mathbb{Z}_p$, we can make sense of $f^s$ as a $p$-adic $c$-Hilbert modular form of level $q$ and weight $2^m(p-1)s$ (here we are using the equivalence of Katz-type and Serre-type $p$-adic Hilbert modular forms in parallel weight; see [AG05, Theorem 11.12]).

Let $e$ be the $p$-adic valuation of $u - 1$. Let

$$r = \begin{cases} e + m + 1 & \text{if } p = 2 \\ e + 1 & \text{if } p > 2 \end{cases}$$

and $\Lambda = \mathbb{Z}_p[[T]]$.

**Lemma 4.1.2.** There is a $\Lambda$-adic $c$-HMF $\mathcal{F}_c$ such that $\mathcal{F}_c(u^s - 1) = f^s_c/2^m(p-1)$ for all $s \in 2^m\mathbb{Z}_p$.

**Proof.** Write $f_c = \sum_{b \in c^+} c_b q^b$. Fix a positive integer $k$, and some $b \in c^+$. Let $\Pi$ be the set of all tuples $\{(k_1, b_1), \ldots, (k_\ell, b_\ell)\}$, where $k_i \in \mathbb{N}_{>0}$ and $b_i \in c^+$, such that $\sum k_i b_i = b$. Note that the cardinality of $\Pi$ is finite and does not depend on $k$. The $q^b$ coefficient of $f^k_c$ is given explicitly by

$$[q^b] f^k_c = \sum_{\Pi} \binom{k}{k_1, \ldots, k_\ell, k - (k_1 + \ldots + k_\ell)} c_{b_1}^{k_1} \cdots c_{b_\ell}^{k_\ell},$$

where we have used that $c_0 = 1$, and we interpret a multinomial coefficient with negative arguments to be zero.

As this sum is finite, it suffices to prove that each element of the above sum is given by evaluating some element of $\Lambda$ at $u^{(p-1)2^mk} - 1$. The multinomial coefficient can be written as $\frac{P(k)}{k_1! \cdots k_\ell!}$ for some polynomial $P$. Since $f_c \equiv 1 (\text{mod } p)$, we have $v_p(c_{b_i}) \geq 1$, so $v_p\left(\frac{c_{b_i}^{k_i}}{k_i!}\right) \geq 0$. Thus, it is enough to show $P(k)$ can be expressed as an element of $\Lambda$, and for this it is enough to show $k$ itself can be. The function $k$
is nothing but the weight divided by $2^m(p - 1)$, so it is given by

$$\frac{\log_p(1 + T)}{(p - 1)2^m \log_p u} \in \Lambda.$$  

This concludes the proof. □

Note that the weight zero specialization of the tuple $\mathcal{F} = (\mathcal{F}_c)_c$ is the constant form 1, and that this form has infinitely many classical specializations. To remove $q$ from the level, we make use of the Hecke operators $W_q$ and $U_q$, interpreting $\mathcal{F}$ as a rule on $c$-polarized HBAV’s for some $c$.

For $C$ a finite subgroup scheme of an HBAV $A$, we let $\pi_C : A \to A/C$. We can geometrically define the operators $U_q$ and $W_q$ on $\Lambda_E$ (or $\Lambda_E$)-adic modular forms of level $\Gamma_0(q)$ in the usual way (see e.g. [Hid06, pg. 320-321]):

$$U_q\mathcal{F}(A, \lambda, i_p, \phi_q) = \frac{1}{N_q} \sum_{C \cap \phi_q = \{0\}} \mathcal{F}(A/C, \pi_{C*}\lambda, \pi_C \circ i_p, \pi_{C*}\phi_q)$$

$$W_q\mathcal{F}(A, \lambda, i_p, \phi_q) = \mathcal{F}(A/\phi_q, \pi_{\phi_q*}\lambda, \pi_{\phi_q} \circ i_p, \pi_{\phi_q*}A[q]),$$

The $F_{\Lambda_E}$-adic form $e\left(\frac{W_q + U_q}{1 + N_q^{-1}}\right)\mathcal{F}$ is ordinary of level one, since its evaluation at any tuple $(A, \lambda, i_p, \phi_q)$ does not depend on level $q$ structure. It has infinitely many classical specializations, and its weight zero specialization is the constant form 1. By the remarks at the beginning of the section, this finishes the proof of Theorem 1. □

4.2 Proof of the Main Theorem

We now give the proof of Conjecture 3.2.2.ii following the basic strategy of [DDP], but also assuming the existence of the form $\mathcal{J}$ in order to remove Leopoldt from their hypotheses. We will assume that there is a unique prime $p$ above $p$ in $F$, as this will highlight how the $L$-invariant hypothesis comes into play. When there is more than one prime above $p$, the arguments we give showing that one can replace $\mathcal{J}$ by $\mathcal{G} - \mathcal{J}$ in the original proof go through unchanged.
We let $E$ be an extension of $\mathbb{Q}_p$ containing the values of all characters of conductor dividing $\text{cond}(\chi)p\infty$, and $E(\chi^{-1})$ the $E[\text{Gal}(\mathcal{F}/F)]$-module which is one-dimensional over $E$ and on which Galois acts by $\chi^{-1}$. For ease of notation, let $\Lambda$ denote $\Lambda_E$ (so $p$ is invertible in $\Lambda$). Let $\pi \in \Lambda$ be the uniformizer at weight one given by $\frac{1}{u \log u}(1 + T - u)$, so that the universal cyclotomic character $\chi_{\text{cyc}}$ can be written

$$\chi_{\text{cyc}} = 1 + \kappa_{\text{cyc}} \pi + O(\pi^2)$$

Finally, let $H^1_p(F, E(\chi^{-1}))$ be the subspace of $H^1(F, E(\chi^{-1}))$ which is unramified at all primes away from $p$, and at $p$, lies in the $E$-linear span of $\kappa_{\text{ur}}$ and $\kappa_{\text{cyc}}$. The space $H^1_p(F, E(\chi^{-1}))$ is one-dimensional; by Proposition 3.3.1, the restriction to $p$ of this global class is given (up to a scalar) by

$$\kappa_{\text{cyc}} - L_{\text{alg}}(\chi)\kappa_{\text{ur}}.$$

In order to prove the equality $L_{\text{alg}} = L_{\text{an}}$, the idea is to use modular forms to explicitly construct a class in $H^1_p(F, E(\chi^{-1}))$ whose restriction to $G_p$ can be shown to be equal (up to a scalar) to $\kappa_{\text{cyc}} - L_{\text{an}}(\chi)\kappa_{\text{ur}}$.

Denote by $\chi_R$ the character of conductor $R$ which has the same primitive as $\chi$. Consider the level $R$ weight one Hilbert modular Eisenstein series $E_1(1, \chi_R)$. We have

$$c_\lambda(0, E_1(1, \chi_R)) = 2^{-g}L_R(\chi, 0) + \delta_{\chi} \chi^{-1}(\lambda)L_R(\chi^{-1}, 0),$$

where $\delta_{\chi} = 1$ if $\text{cond}(\chi) = 1$, and is 0 otherwise. We also have

$$U_p E_1(1, \chi_R) = E_1(1, \chi_R) + E_1(1, \chi_S),$$

which implies that $e = \lim_n (U_p^n)$ acts by the identity on this form. Thus, the ordinary $\Lambda$-adic form $\mathcal{P}^0 := e[(\mathcal{G} - \mathcal{F}) E_1(1, \chi_R)]$ has weight one specialization equal to $E_1(1, \chi_R)$. Moreover, its constant terms satisfy

$$c_\lambda(0, \mathcal{P}^0) = 2^{-g}L_R(\chi, 0) + \delta_{\chi} \chi^{-1}(\lambda)L_R(\chi^{-1}, 0)$$
independent of the weight, since this is clearly true before taking the ordinary pro-
jection, and the only Eisenstein series contributing to these cusps at the classical
higher weight specializations are already ordinary.

Over \( F_{\Lambda} \), we can decompose \( \mathcal{P}^0 \) into a linear combination of a cusp form and
ordinary Eisenstein series. The coefficients \( a(1, \chi) \) and \( a(\chi, 1) \) of the Eisenstein
families \( \mathcal{E}(1, \chi) \) and \( \mathcal{E}(\chi, 1) \) in this decomposition are computed in loc. cit. §2,
using knowledge of the constant terms of \( e[\mathcal{G} E_1(1, \chi R)] \) at all unramified cusps,
not just the infinite cusps. The weight \( k \) specializations of these coefficients are
given respectively by

\[
\begin{align*}
a(1, \chi)(k) &= \frac{L_R(\chi, 0)}{L_{S, p}(\chi \omega, 1 - k)} = \frac{-1}{\mathcal{L}_{an}(\chi, k)} \\
a(\chi, 1)(k) &= \frac{L_R(\chi^{-1}, 0)(Nn)^{k-1}}{L_{S, p}(\chi^{-1} \omega, 1 - k)} = \frac{- (Nn)^{k-1}}{\mathcal{L}_{an}(\chi^{-1}, k)}.
\end{align*}
\]

The Eisenstein series other than \( \mathcal{E}(1, \chi) \) and \( \mathcal{E}(\chi, 1) \) can be killed by an
appropriate application of Hecke operators away from \( p \) without affecting the weight
1 specialization \( E_1(1, \chi R) \), simply by dividing each Hecke operator by its eigen-
value on \( E_1(1, \chi) \). It follows that there is some \( t \), a linear combination of Hecke
operators away from \( p \), acting by the identity on \( E_1(1, \chi) \), such that

\[
\mathcal{F} = t(\mathcal{P}^0 - a(1, \chi)\mathcal{E}(1, \chi) - a(\chi, 1)\mathcal{E}(\chi, 1))
\]
is a cusp form. Note that \( a(1, \chi) \) and \( a(\chi, 1) \) have poles at weight one of order
equal to the order of vanishing of the corresponding \( p \)-adic \( L \)-functions; since
we are in the rank one setting, we know that \( R_p(\chi) \) and \( R_p(\chi^{-1}) \) do not vanish
by [Gro81][Proposition 2.13], and therefore \( \mathcal{L}_{an}(\chi) \) and \( \mathcal{L}_{an}(\chi^{-1}) \) do not vanish
by our Proposition 3.4. From now on, we use the shorthand \( \mathcal{L}_\chi := \mathcal{L}_{an}(\chi) \),
\( \mathcal{L}_{\chi^{-1}} := \mathcal{L}_{an}(\chi^{-1}) \).

We have

\[
\mathcal{L}_\chi = \frac{1}{\pi a(1, \chi)}(0)
\]

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\[ \mathcal{L}_{\chi^{-1}} = \frac{1}{\pi a(1, \chi^{-1})}(0). \]

The maximum order of pole appearing in the coefficients of \( F \) at weight 1 is

- 1 if \( \mathcal{L}_{\chi} + \mathcal{L}_{\chi^{-1}} \neq 0 \)
- 0 if \( \mathcal{L}_{\chi} + \mathcal{L}_{\chi^{-1}} = 0 \)

Suppose we are in the first case. Then we have

\[ F = P_0 - a(1, \chi)E(1, \chi) - a(\chi, 1)E(\chi, 1) \in \pi^{-1}\mathcal{F}_{\text{ord}}(n, \chi). \]

We want to consider the image of this form in

\[ \pi^{-1}\mathcal{F}_{\text{ord}}(n, \chi)/\pi\mathcal{F}_{\text{ord}}(n, \chi) \]

and compute the Hecke action. To do this, we will use the following identities

\[ E(\chi, 1) \equiv E(1, \chi) \equiv E_1(1, \chi_S)(\mod \pi) \]
\[ \mathcal{B}^0 \equiv E_1(1, \chi_R)(\mod \pi) \]

\[ T_\ell E(\chi, 1) = (\chi(\ell) + \chi_{\text{cyc}}(\ell))E(1, \chi) \]
\[ U_p E(\chi, 1) = E(\chi, 1) \]
\[ T_\ell E(1, \chi) = (1 + \chi(\ell))E(1, \chi) \]
\[ U_p E(1, \chi) = E_1(1, \chi_S) \]
\[ T_\ell E_1(1, \chi_R) = (1 + \chi(\ell))E_1(1, \chi_R) \]
\[ U_p E_1(1, \chi_R) = E_1(1, \chi_R) + E_1(1, \chi_S) \]

We compute:

\[ T_\ell \mathcal{F} = (1 + \chi(\ell))\mathcal{B}^0 - (1 + \chi(\ell)\chi_{\text{cyc}}(\ell))a(1, \chi)E(1, \chi) - (\chi_{\text{cyc}}(\ell) + \chi(\ell))a(\chi, 1)E(\chi, 1) \]
\[ = (1 + \chi(\ell))\mathcal{F} - \chi(\ell)\kappa_{\text{cyc}}(\ell)\pi a(1, \chi)E(1, \chi) - \kappa_{\text{cyc}}(\ell)\pi a(\chi, 1)E(\chi, 1) \]
\[ = (1 + \chi(\ell))\mathcal{F} - \chi(\ell)\kappa_{\text{cyc}}(\ell)\pi a(1, \chi) + \kappa_{\text{cyc}}(\ell)\pi a(\chi, 1))E_1(1, \chi_S) \]
\[ = (1 + \chi(\ell))\mathcal{F} + (\chi(\ell)\kappa_{\text{cyc}}(\ell)a(1, \chi) + \kappa_{\text{cyc}}(\ell)a(\chi, 1))(\frac{\pi}{a(1, \chi) + a(\chi, 1)}\mathcal{F}) \]
\[ = \left(1 + \frac{\chi(1)}{a(1, \chi) + a(\chi, 1)}\kappa_{\text{cyc}}(\ell)\right) + \chi(\ell)\left(1 + \frac{\chi(1)}{a(1, \chi) + a(\chi, 1)}\kappa_{\text{cyc}}(\ell)\right)\mathcal{F}; \]
The calculation for the \( U_q \) operators is similar.

Thus we get a map
\[
\mathbb{T}^{\text{ord}} \to \Lambda / \pi^2 \cong E[\pi]/\pi^2
\]

\[
T_\ell \mapsto (1 + \pi \frac{a(\chi, 1)}{a(1, \chi) + a(\chi, 1)} \kappa_{\text{cyc}}(\ell)) + \chi(\ell)(1 + \pi \frac{a(1, \chi)}{a(1, \chi) + a(\chi, 1)} \kappa_{\text{cyc}}(\ell))
\]

\[
U_q \mapsto 1 - \pi \frac{a(\chi, 1)}{a(1, \chi) + a(\chi, 1)} \kappa_{\text{cyc}}(q)
\]

\[
U_p \mapsto 1 + \pi \frac{a(1, \chi)}{a(1, \chi) + a(\chi, 1)} L_{\text{an}}.
\]

Let \( I \) denote the kernel of this map, and \( \mathfrak{m} \) the maximal ideal containing \( I \). Since the image of \( t \) is a unit, this map factors through the cuspidal quotient of \( \mathbb{T}^{\text{ord}} \). Let \( R \) denote the localization at \( \mathfrak{m} \) of this cuspidal quotient, and think of \( I \) and \( \mathfrak{m} \) as being ideals of \( R \). Finally, let \( F_R := R \otimes_\Lambda F_\Lambda \).

There is a Galois representation
\[
\rho : G_F \to GL_2(F_R)
\]

\[
\rho(\sigma) = \left( \begin{array}{cc} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{array} \right).
\]

unramified at all \( \ell \nmid \mathfrak{m} \) such that Trace(\( \text{Frob}_\ell \)) = \( T_\ell \). For some choice of complex conjugation \( c \in G_F \), we may assume \( \rho(c) = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \).

By a standard argument, the \( R \)-submodule of \( F_R \) generated by all \( a_\sigma \) is \( R \), and similarly for all \( d_\sigma \). Moreover,

\[
a_\sigma \equiv 1 + \pi \frac{a(\chi, 1)}{a(1, \chi) + a(\chi, 1)} \kappa_{\text{cyc}}(\sigma) \pmod{I}
\]

\[
d_\sigma \equiv \chi(\sigma)(1 + \pi \frac{a(1, \chi)}{a(1, \chi) + a(\chi, 1)} \kappa_{\text{cyc}}(\sigma)) \pmod{I}.
\]
By a theorem of Wiles, there is a change-of-basis matrix \( \begin{pmatrix} A_p & B_p \\ C_p & D_p \end{pmatrix} \) with the property that

\[
\begin{pmatrix} a \sigma & b \sigma \\ c \sigma & d \sigma \end{pmatrix} \begin{pmatrix} A_p & B_p \\ C_p & D_p \end{pmatrix} = \begin{pmatrix} A_p & B_p \\ C_p & D_p \end{pmatrix} \begin{pmatrix} \chi_{\text{cyc}}\eta_p^{-1}(\sigma) & * \\ 0 & \eta_p(\sigma) \end{pmatrix}
\]

for all \( \sigma \in G_p \). Here \( \eta_p \) is the unramified character \( \text{Frob}_p^k \mapsto U_{\text{cyc}}^k \).

We note that in the basis with complex conjugation diagonalized, \( \rho(G_p) \) is not upper-triangular on any component of \( F_R \). If it were, the function \( c_\sigma \) reduced mod \( m \) would yield a non-trivial element of \( H^1(G_F, E(\chi)) \) which is unramified everywhere (recall that \( p \) is the unique prime above \( p \)). However, as discussed in Chapter 2, there are no unramified elements in this \( H^1 \). Therefore, \( C_p \) is invertible in \( F_R \). Hence, for \( \sigma \in G_p \), we can write

\[
b_\sigma = \frac{A_p}{C_p} [\chi_{\text{cyc}}\eta_p^{-1}(\sigma) - a_\sigma].
\]

If \( B \) is the \( R \)-module generated by \( b_\sigma \) (or equivalently by \( \frac{b_{2\sigma}}{\eta_p} \)) as \( \sigma \) ranges over \( G_F \), then \( B \) is finite over \( R \) by a standard compactness argument, so that \( B/mB \) is nonzero. If \( K : G_F \to B/mB \) is the composition of \( b/d \) with reduction mod \( m \), then \( [K] \in H^1(G_F, B/mB(\chi^{-1})) \) is unramified outside \( p \) and nonzero, since if \( K \) were a coboundary, one can check it would have to be identically zero by considering \( K(c) \). Since there are no everywhere unramified elements of this \( H^1 \), \( [K] \) must be ramified at \( p \); in particular, it is nontrivial at \( p \). This argument, combined with Nakayama’s lemma, shows that \( B \) is in fact generated by \( b_\sigma \) for \( \sigma \in G_p \).

Reducing the above equation modulo \( (I\frac{A_p}{C_p} \cap B) \), we get for \( \sigma \in G_p \),
Since \( \pi \in m \) and \( m^2 \subset I \), the module \( B/(I A_p C_p \cap B) \) is \( m \)-torsion. Furthermore, by our initial assumptions we can choose \( \sigma \) so that the bracketed expression is a unit. It follows that \( B/(I A_p C_p \cap B) \) is cyclic over \( R/m \cong E \), generated by \( \pi A_p C_p \). Since it is \( \pi \)-torsion, we must have \( b(\sigma) = b/d(\sigma) \). Thus, we may view \( b/d(\sigma) \) as a cocycle with coefficients in \( E(\chi^{-1}) \), whose restriction to \( G_p \) is given, after dividing by the unit \( \frac{a(1,\chi)}{a(1,\chi)+a(\chi,1)} \), by \( \kappa_{cyc} - L_{an} \kappa_{ur} \). By Proposition 1, \( L_{alg} = L_{an} \), and the proof is complete. \( \square \)

Now suppose \( L_{\chi} = -L_{\chi^{-1}} \).

In this case, although the second and third term of \( \mathcal{P}^0 - a(1,\chi)\mathcal{E}(1,\chi) - a(\chi,1)\mathcal{E}(\chi,1) \) each have a pole of order 1, the sum does not have a pole. We consider the Hecke action on the image of this form in the \( \Lambda/(\pi) \cong E \)-vector space \( \mathcal{S}^0_{\Lambda} / \mathcal{S}_{\Lambda}^{ord} \). Unlike the previous cases, this form is not an eigenform. Nevertheless, we have

\[
T_{\ell}(\mathcal{P}^0 - a(1,\chi)\mathcal{E}(1,\chi) - a(\chi,1)\mathcal{E}(\chi,1)) = (1 + \chi(\ell))\mathcal{P}^0 - (1 + \chi(\ell))(1 + \pi\kappa_{cyc}(\ell))a(1,\chi)\mathcal{E}(1,\chi) - (1 + \pi\kappa_{cyc}(\ell) + \chi(\ell))a(\chi,1)\mathcal{E}(\chi,1) \\
= (1 + \chi(\ell))[\mathcal{P}^0 - a(1,\chi)\mathcal{E}(1,\chi) - a(\chi,1)\mathcal{E}(\chi,1)] - (\kappa_{cyc}(\ell)\chi(\ell)\pi a(1,\chi) + \kappa_{cyc}(\ell)\pi a(\chi,1)) E_1(1,\chi).
\]
\[
U_q(\mathcal{P}^0 - a(1, \chi)E(1, \chi) - a(\chi, 1)E(1, \chi)) = \\
(\mathcal{P}^0 - a(1, \chi)E(1, \chi) - a(\chi, 1)E(1, \chi)) - \pi \kappa_{\text{cyc}}(q)a(\chi, 1)E_1(1, \chi).
\]

\[
U_p(\mathcal{P}^0 - a(1, \chi)E(1, \chi) - a(\chi, 1)E(1, \chi)) = \\
(\mathcal{P}^0 - a(1, \chi)E(1, \chi) - a(\chi, 1)E(1, \chi)) + E_1(1, \chi).
\]

Thus, although the image of our form is not an eigenvector for the Hecke operators, it is a generalized eigenvector for the \(E_1(1, \chi)\)-system of eigenvalues; the Hecke stable subspace it generates is two-dimensional over \(E\), with a basis given by the image of \((\mathcal{P}^0 - a(1, \chi)E(1, \chi) - a(\chi, 1)E(1, \chi))\) and \(E_1(1, \chi)\). The Hecke action can then be viewed as a homomorphism

\[
T^\text{ord} \rightarrow \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \subset M_2(E).
\]

The image is canonically isomorphic to \(E[\varepsilon]/\varepsilon^2\). Under this identification, the map \(T^\text{ord} \rightarrow E[\varepsilon]/\varepsilon^2\) is given explicitly by

\[
T_\ell \mapsto 1 + \chi(\ell) - \varepsilon \left( \frac{\kappa_{\text{cyc}}(\ell)\chi(\ell)}{\mathcal{L}_\chi} \right) + \frac{\kappa_{\text{cyc}}(\ell)}{\mathcal{L}_\chi^{-1}}
\]

\[
U_q \mapsto 1 - \varepsilon \left( \frac{\kappa_{\text{cyc}}(q)}{\mathcal{L}_\chi^{-1}} \right)
\]

\[
U_p \mapsto 1 + \varepsilon (\delta_{d=1}).
\]

Following the same proof as before, we get a Galois representation \(G_F \rightarrow GL_2(F_R)\) such that

\[
R \ni a_\sigma \equiv 1 - \varepsilon \frac{\kappa_{\text{cyc}}(\sigma)}{\mathcal{L}_\chi}(\text{mod } I)
\]

\[
R \ni d_\sigma \equiv \chi(\sigma)(1 - \varepsilon \frac{\kappa_{\text{cyc}}(\ell)}{\mathcal{L}_\chi})(\text{mod } I).
\]
However, in this case the image of the universal cyclotomic character $\chi_{\text{cyc}}$ in $R/I$ is trivial, as we are working “purely in weight one.” Thus, reducing the equation

$$b_\sigma = \frac{A_p}{C_p} [\chi_{\text{cyc}} \eta_p^{-1} - a_\sigma]$$

modulo $I \frac{A_p}{C_p}$ gives for $\sigma \in G_p$

$$\overline{b/d}(\sigma) = \frac{A_p}{C_p} [\eta_p^{-1} - a_\sigma]$$

$$= \frac{A_p}{C_p} \varepsilon (-\kappa_{ur}(\sigma) + \frac{\kappa_{cyc}(\sigma)}{L_\chi^{-1}}).$$

Just as before, we see that $B/(B \cap I \frac{A_p}{C_p})$ is one-dimensional, generated by $\varepsilon \frac{A_p}{C_p}$, and that the function $\overline{b/d}$ yields a class $[\kappa] \in H^1_p(F, E(\chi^{-1}))$ such that

$$[\kappa]|_{G_p} = -\kappa_{ur} + \frac{1}{L_\chi^{-1}} \kappa_{cyc}$$

$$= -\kappa_{ur} - \frac{1}{L_\chi} \kappa_{cyc}.$$

Multiplying through by $-L_\chi$, the right hand side becomes

$$L_\chi \kappa_{ur} + \kappa_{cyc} = -L_{\text{an}}(\chi) \kappa_{ur} + \kappa_{cyc}.$$

This finishes the proof.
CHAPTER 5

Corollaries and Remarks on Higher Rank

5.1 Application to the Iwasawa Main Conjecture

Theorem 4.1.1 allows us to give a direct construction of the Iwasawa extensions corresponding to the (conjecturally nonexistent) zeroes of the $p$-adic zeta function at $s = 1$ [Wil90]. We remark that C. Khare has given a different simplification of this construction by allowing ramification at an auxiliary prime. A separate proof is needed for these extensions, as the general argument only constructs a space of extensions of rank $\text{ord}_{s=1} \zeta_{F,p}(s)$, but the Main Conjecture predicts that this space has rank $\delta = \text{ord}_{s=1} \zeta_{F,p}(s) + 1$. The proofs given in §§10, 11 of loc. cit. are somewhat indirect, using “patching” arguments similar to what is needed in the weight one case. The proof we give here is relatively straightforward with the help of Theorem 2.

Since the non-constant terms of the form $1$ vanish, we have that for each nonzero integral ideal $\mathfrak{m}$,

$$c(\mathfrak{m}, \mathcal{G} - \mathcal{J}) = c(\mathfrak{m}, 2^n G_{\zeta}^{-1} \mathcal{E}(1, \omega^{-1}) - \mathcal{J}) \in \mathfrak{m}_{(0)}.$$

Thus,

$$c(\mathfrak{m}, \mathcal{E}(1, \omega^{-1}) - 2^{-n} G_{\zeta} \mathcal{J}) \in \mathfrak{m}_{(0)}^{\delta}.$$

Consider the action of the cuspidal Hecke algebra on $2^{-n} G_{\zeta} \mathcal{J}$. We have

$$2^{-n} G_{\zeta} \mathcal{J} \equiv^* \mathcal{E}(1, \omega^{-1}) (\text{mod } \mathfrak{m}_{(0)}^{\delta}).$$
where the asterisk indicates that this holds on $q$-expansions coefficients away from the constant terms. Therefore the Hecke eigenvalues on these two forms agree, so there is a map

$$T^c_{cusp} \to \Lambda / \mathfrak{m}^\delta_{(0)}$$

which is just the $\mathcal{E}(1, \omega^{-1})$ system of Hecke eigenvalues (mod $\mathfrak{m}^\delta_{(0)}$). From here, one can proceed by Ribet’s method to construct the desired extensions.

### 5.2 A homomorphism in higher rank

Although we do not have a complete proof of Gross’s conjecture when $r > 1$, we show in this section that even in higher rank, the ordinary $\Lambda$-adic Hecke algebra “knows” the analytic $L$-invariant. This gives hope that there may be a similar automorphic proof in higher rank.

We fix notations as before, and let $p_1, \ldots, p_r$ denote the primes above $p$ for which $\chi|_{p_i} = 1$. $S$ is the set of primes dividing cond($\chi)p\infty$, $R = S \setminus \{p_1, \ldots, p_r\}$, and

$$\mathcal{T} = E[\{T_\ell\}_{\ell \not\in S}, \{U_q\}_{q \in R}, U_{p_1}, \ldots, U_{p_r}].$$

For simplicity, we will assume there is a prime $\tilde{p}$ above $p$ which is not among the primes $p_i$. We again consider the level $R$ weight one Hilbert modular Eisenstein series $E_1(1, \chi_R)$. This is an eigenform for the Hecke operators $T_\ell$ with eigenvalue $1 + \chi(\ell)$, and for the Hecke operators $U_q$ with eigenvalue 1. To describe what happens at the primes $p_i$, we consider a set of primes $T$ such that $R \subset T \subset S$. Then we have

$$U_{p_i}E_1(1, \chi_T) = \begin{cases} E_1(1, \chi_T) + E_1(1, \chi_{T \cup \{p_i\}}) & \text{if } p_i \not\in T \\ E_1(1, \chi_T) & \text{if } p_i \in T. \end{cases}$$

Therefore, the Hecke module generated by $E_1(1, \chi_R)$ is $2^r$ dimensional, with a basis given by $E(1, \chi_T)$ for each choice of $T$. The Hecke action on this space can
then be described by a homomorphism

\[ T \to E[\epsilon_1, \ldots, \epsilon_r]/(\epsilon_1^2, \ldots, \epsilon_r^2), \]

where

\[ T_\ell \mapsto 1 + \chi(\ell) \]

\[ U_q \mapsto 1 \]

\[ U_{p_i} \mapsto 1 + \epsilon_i. \]

We want to extend this map to the \( \Lambda \)-adic setting. Consider now the ordinary \( \Lambda_{(0)} \)-adic form \( \mathcal{P}^0 := te[(\mathcal{H} - \mathcal{J})E_1(1, \chi_R)] \), where \( t \) is a linear combination of Hecke operators which kills all Eisenstein families other than \( \mathcal{E}(1, \chi) \) (we can use \( U_{\tilde{p}} \) to kill \( \mathcal{E}(\chi, 1) \)). Then

\[ \mathcal{F} = \mathcal{P}^0 - a(1, \chi)\mathcal{E}(1, \chi) \]

is cuspidal, where as before, the weight \( k \) specialization is given by

\[ a(1, \chi)(k) = \frac{L_R(\chi, 0)}{L_S,p(\chi \omega, 1 - k)}. \]

Note that now \( a(1, \chi) \) has a pole of order \( r_p \) at weight one, and so therefore so do the coefficients of \( \mathcal{F} \). We consider the image of \( \mathcal{F} \) in

\[ \pi^{-r_p}M^\text{ord}_\Lambda(n, \chi)/\pi M^\text{ord}_\Lambda(n, \chi), \]

which can be viewed as a faithful \( \Lambda/\pi^{r_p+1} \)-module. Note that \( \mathcal{P}^0 \) lands inside the \( \pi \)-torsion submodule

\[ M^\text{ord}_\Lambda(n, \chi)/\pi M^\text{ord}_\Lambda(n, \chi), \]

i.e. the space of \( p \)-adic ordinary weight one modular forms, and its image is simply \( E_1(1, \chi_R) \). The other term \( a(1, \chi)\mathcal{E}(1, \chi) \) is just an Eisenstein family times a scalar in \( F_\Lambda \).
We are interested in understanding the action of the Hecke algebra on the module generated by $\mathcal{F}$ inside this module. The Hecke operators $T_\ell$ and $U_q$ act in essentially the same way as above:

\[
T_\ell \mapsto 1 + \chi(\ell)\chi_{cyc}(\ell)
\]
\[
U_q \mapsto 1.
\]

It remains to understand the relations among the operators $U_{p_i}$; for this, it is easier to consider the operator $U_{p_i} - 1$. This operator kills $a(1, \chi)\mathcal{F}(1, \chi)$, and sends $E_1(1, \chi_R)$ to $E_1(1, \chi_{R,p_i})$. Explicitly,

\[
(U_{p_i} - 1)\mathcal{F} = E_1(1, \chi_{R,p_i}) \in \mathcal{M}_\Lambda^{\text{ord}}(n, \chi)/\pi\mathcal{M}_\Lambda^{\text{ord}}(n, \chi) \subset \pi^{-r_p}\mathcal{M}_\Lambda^{\text{ord}}(n, \chi)/\pi\mathcal{M}_\Lambda^{\text{ord}}(n, \chi).
\]

Therefore, $U_{p_i} - 1$ squares to zero and is $\pi$-torsion. If we apply each operator $U_{p_i} - 1$ once to $\mathcal{F}$, we get back the form $E_1(1, \chi_S)$. This is actually a $\Lambda/\pi^{r_p+1}$-scalar multiple of $\mathcal{F}$. Namely, it is equal to \(\frac{1}{a(1, \chi)}\mathcal{F}\), since the scalar \(\frac{1}{a(1, \chi)} = \frac{L_{S,p}(\chi_{\omega,s})}{L_{R}(\chi,0)}\) has a zero of order $r_p$ and hence kills the $\pi$-torsion element $\mathcal{F}^0$. However, by definition we have that as elements of $\Lambda/\pi^{r_p+1}$,

\[
\frac{1}{a(1, \chi)} = \mathcal{L}_{an}(\chi)\pi^{r_p}
\]

(strictly speaking, this is only true if $r_p = r$, since otherwise $\mathcal{L}_{an}(\chi)$ vanishes; what this formula and the following actually tell us is that the Hecke algebra knows the leading term of $L_{S,p}$ even if it vanishes to higher-than-expected order).

Putting this all together, we get the following representation of the cuspidal ordinary $\Lambda$-adic Hecke algebra:

\[
\mathbb{T} \rightarrow \Lambda[\epsilon_1, \ldots, \epsilon_r]/(\pi^{r_p+1}, \epsilon_1^2, \ldots, \epsilon_r^2, \pi\epsilon_1, \ldots, \pi\epsilon_r, \prod_{i=1}^{r} \epsilon_i - \mathcal{L}_{an}(\chi)\pi^{r_p}),
\]
where

\[ T_\ell \mapsto 1 + \chi(\ell) \chi_{\text{cyc}}(\ell) \]

\[ U_q \mapsto 1 \]

\[ U_{p_i} \mapsto 1 + \epsilon_i. \]
References


