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Differential Operators for Quaternionic Modular Forms

A Dissertation submitted in partial satisfaction of the requirements

for the degree Doctor of Philosophy

in

Mathematics

by

Bryan Hu

Committee in charge:
Professor Aaron Pollack, Chair
Professor Alina Bucur
Professor Kenneth Intriligator
Professor Kiran Kedlaya
Professor Cristian Popescu

2026

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University of California San Diego

2026

DEDICATION

To Emily Zhu

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ABSTRACT OF THE DISSERTATION

Differential Operators for Quaternionic Modular Forms

by

Bryan Hu

Doctor of Philosophy in Mathematics

University of California San Diego, 2026

Professor Aaron Pollack , Chair

In this dissertation, we construct explicit differential operators realizing the restriction of quaternionic discrete series representations of the exceptional group G_2 to an embedded group of type $\mathrm{SU}(2,1)$. These operators are analogous to the classical Maass-Shimura operators and Rankin-Cohen brackets for holomorphic modular forms. If f_1 and f_2 are holomorphic modular forms of weights n_1 and n_2, the Rankin-Cohen bracket $[f_1, f_2]_r$ is a holomorphic modular form of weight n_1+n_2+2r. Furthermore, the Fourier coefficients of $[f_1, f_2]_r$ are rational polynomials in the Fourier coefficients of f_1 and f_2. These two properties have important applications to the study of special values of L-functions. Our differential operators achieve the same for quaternionic modular forms. Starting with a quaternionic modular form F on G_2 of weight n, we construct operators $\mathcal{D}^n_{r,m}$ for any admissible r and m so that $D^n_{r,m}F|_{\mathrm{SU}(2,1)}$ is a quaternionic modular form on $\mathrm{SU}(2,1)$ of weight $(n+\frac{r}{2}, m)$ and show that the Fourier coefficients of $D^n_{r,m}F|_{\mathrm{SU}(2,1)}$ are algebraic linear combinations of the Fourier coefficients of F.

Chapter 1

INTRODUCTION

We begin by recalling some classical constructions. Suppose that $f_1(z)$ and $f_2(z)$ are holomorphic modular forms of weights n_1 and n_2. Then, for any nonnegative integer r, one can produce a modular form of weight $n_1 + n_2 + 2r$ via the r-th Rankin-Cohen bracket of f_1 and f_2. This is the function defined by

$$[f_1, f_2]_r := \frac{1}{(2\pi i)^r} \sum_{a+b=r} (-1)^b \binom{n_1 + r - 1}{a} \binom{n_2 + r - 1}{b} \frac{d^a f_1}{dz} \frac{d^b f_2}{dz}.$$

Rankin-Cohen brackets have applications to the study of special values of L-functions as well as representation theory. The purpose of this thesis is to define differential operators for certain quaternionic exceptional groups that may be viewed as an analogue of Rankin-Cohen brackets for quaternionic modular forms.

To motivate our result, it will first help to place Rankin-Cohen brackets in the context of automorphic representation theory. First, let us replace the $\frac{d}{dz}$ in the definition of Rankin-Cohen brackets with Maass-Shimura differential operators. Applied to the study of special values of L-functions by Shimura in [Shi76], the Maass-Shimura differential operator $\delta_n^{(r)}$ takes (nearly holomorphic) modular forms of weight n to nearly holomorphic modular forms of weight $n + 2r$, and is defined by the formula

$$\delta_n^{(r)} = \left(\frac{1}{2\pi i}\right)^r \left(\frac{n + 2r - 2}{2iy} + \frac{d}{dz}\right) \left(\frac{n + 2r - 4}{2iy} + \frac{d}{dz}\right) \cdots \left(\frac{n}{2iy} + \frac{d}{dz}\right).$$

It is well known that Maass-Shimura operators can take the place of the usual differential in the formula for Rankin-Cohen brackets:

$$[f_1, f_2]_r := \frac{1}{(2\pi i)^r} \sum_{a+b=r} (-1)^b \binom{n_1 + r - 1}{a} \binom{n_2 + r - 1}{b} \left(\delta_{n_1}^{(a)} f_1\right) \left(\delta_{n_2}^{(b)} f_2\right).$$

A proof may be found in [Lan08].

Now, suppose for simplicity that f_1 and f_2 are Hecke eigenforms. In the context of automorphic representation theory, these generate irreducible automorphic representations Π_{n_1} and Π_{n_2} whose archimedean components are holomorphic discrete series representations. In more detail, let $\mathbb{C}(m)$ denote the one dimensional $\mathrm{SO}(2)$ representation for which the matrix $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$ acts as $e^{im\theta}$. Then for $j = 1, 2$, the archimedean component of Π_{n_j} is the discrete series representation π_{n_j} of $\mathrm{SL}_2(\mathbb{R})$ with K-type decomposition

$$\mathbb{C}(n_j) \oplus \mathbb{C}(n_j + 2) \oplus \mathbb{C}(n_j + 4) \oplus \cdots$$

and f_j corresponds to a vector v_j in the lowest K-type $\mathbb{C}(n_j)$.

Let

$$Y = \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad X = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$$

be a standard basis of the complex Lie algebra $\mathfrak{sl}_2$. With suitable normalizations in our correspondence between modular forms and automorphic representations, that f_j is holomorphic corresponds to $Yv_j = 0$, the K-types are H-eigenspaces, and the Maass-Shimura operator is the differential operator X.

Finally, to arrive at a representation theoretic interpretation of the Rankin-Cohen brackets, consider the $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ representation $\pi_{n_1} \boxtimes \pi_{n_2}$ restricted to a diagonally embedded $\mathrm{SL}_2(\mathbb{R})$.

This classical branching problem leads to the decomposition

$$\pi_{n_1} \otimes \pi_{n_2} \cong \bigoplus_{r=0}^{\infty} \pi_{n_1+n_2+2r}.$$

On the right-hand side, $v_1 \otimes v_2$ is a lowest weight vector in $\pi_{n_1+n_2}$; it corresponds to the modular form $f_1 f_2$ of weight $n_1 + n_2$. Furthermore, the Rankin-Cohen bracket $[f_1, f_2]_r$ corresponds exactly to the $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ differential operator that takes $v_1 \otimes v_2$ to a lowest weight vector in $\pi_{n_1+n_2+2r}$.

In this thesis, the exceptional group G_2 takes the place of $\mathrm{SL}_2 \times \mathrm{SL}_2$, and the branching problem we consider is a restriction from G_2 to a group of type $\mathrm{SU}(2,1)$. While G_2 does not possess holomorphic discrete series, it does have a class of representations known as quaternionic discrete series. These representations were first studied by Gross and Wallach in [GW96], and the relevant branching problem is studied by Loke in [Lok99]. Following Gan-Gross-Savin ([GGS02]) and Pollack ([Pol20]), one can study quaternionic modular forms (QMFs), automorphic forms whose archimedean components are quaternionic discrete series. These QMFs have a robust theory of Fourier expansion, and in many cases their Fourier coefficients have established arithmetic properties; for example [Pol24] establishes the arithmeticity of cusp forms on quaternionic exceptional groups.

Suppose that Φ is a QMF on G_2 of weight n. Then $\Phi|_{\mathrm{SU}(2,1)}$ is a QMF on $\mathrm{SU}(2,1)$ of weight n, and our main result is the construction of explicit differential operators that move $\Phi|_{\mathrm{SU}(2,1)}$ to QMFs of any higher weight:

Theorem 1.0.1. *Let $n \geq 1$. For any integers $r \geq 0$ and $m \in \{-r, -r+2, \ldots, r-2, r\}$, there exist differential operators $\mathcal{D}^n_{r,m}$ with the following properties:*

- *If Φ is a QMF on G_2 of weight n, then $\varphi = \mathcal{D}^n_{r,m}\Phi|_{\mathrm{SU}(2,1)}$ is a QMF on $\mathrm{SU}(2,1)$ of weight $(n + \frac{r}{2}, m)$;*
- *the Fourier coefficients of φ are $\overline{\mathbb{Q}}$-linear combinations of the Fourier coefficients of Φ.*

An important property of the Rankin-Cohen brackets in their application to special values of L-functions is the fact that the Fourier coefficients of $[f_1, f_2]_r$ are rational linear combinations of the Fourier coefficients of f_1 and f_2. This is the property underlying Shimura's result on the arithmeticity of the product L-function of modular forms in [Shi76]. We prove that our $\mathcal{D}^n_{r,m}$ also preserves algebraicity of Fourier coefficients. In the future, one should be able to exploit these $\mathcal{D}^n_{r,m}$ to obtain arithmeticity results for the adjoint L-function of $\mathrm{SU}(2,1)$. Indeed, Hundley in [Hun12] constructs an integral representation for the adjoint L-function of $\mathrm{SU}(2,1)$ that uses an Eisenstein series on G_2.

Chapter 2

NOTATION AND PRELIMINARIES

In this section we set up the notation that will be used throughout the thesis. We closely follow the conventions of [Pol21] and [PWZ19].

2.1 The Group G_2 and its Lie algebra

The discussion here is valid over any field of characteristic 0, but we will work over the rational numbers $\mathbb{Q}$. We begin by recalling the Zorn model of the split octonions over $\mathbb{Q}$. Let $V_3/\mathbb{Q}$ be the standard representation of SL_3 and $V_3^\vee$ the dual representation. Let us fix a basis $\{e_1, e_2, e_3\}$ of V_3 with corresponding dual basis $\{e_1^*, e_2^*, e_3^*\}$ and an isomorphism $\wedge^3 V_3 \to \mathbb{Q}$ so that $e_1 \wedge e_2 \wedge e_3 \mapsto 1$. With this identification we also have $\wedge^2 V_3 \cong V_3^\vee$ and $\wedge^2 V_3^\vee \cong V_3$, for example $e_1 \wedge e_2 = e_3^*$ and $e_1^* \wedge e_2^* = e_3$.

We recall the Zorn model of the split octonions. Define

$$\Theta = \left\{ \begin{pmatrix} a & v \\ \phi & d \end{pmatrix} : a, d \in \mathbb{Q}, v \in V, \phi \in V^\vee \right\}.$$

Given an octonion $x = \left(\begin{smallmatrix} a & v \\ \phi & d \end{smallmatrix}\right) \in \Theta$, we can define:

- its conjugate $x^* = \left(\begin{smallmatrix} d & -v \\ -\phi & a \end{smallmatrix}\right)$;
- its norm $N(x) = ad - \phi(v)$;
- its trace $\mathrm{Tr}(x) = a + d$.

Suppose $x = \left(\begin{smallmatrix} a & v \\ \phi & d \end{smallmatrix}\right)$ and $y = \left(\begin{smallmatrix} a' & v' \\ \phi' & d' \end{smallmatrix}\right)$ are two elements of Θ. There is a non-degenerate symmetric bilinear form $(\ ,\ )$ on Θ defined by $(x, y) = N(x+y) - N(x) - N(y)$. We can compute

$$(x, y) = aa' + dd' - \phi(v') - \phi'(v).$$

The multiplication of elements in Θ is described by

$$\begin{pmatrix} a & v \\ \phi & d \end{pmatrix} \cdot \begin{pmatrix} a' & v' \\ \phi' & d' \end{pmatrix} = \begin{pmatrix} aa' + \phi'(v) & av' + d'v - \phi \wedge \phi' \\ a'\phi + d\phi' + v \wedge v' & dd' + \phi(v') \end{pmatrix}.$$

We will sometimes abuse notation and write a for $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) \in \Theta$ and identify elements of V_3 or V_3^* with elements of Θ, for example $e_1 = \left(\begin{smallmatrix} 0 & e_1 \\ 0 & 0 \end{smallmatrix}\right) \in \Theta$.

Define the linear algebraic group G over $\mathbb{Q}$ by

$$G(\mathbb{Q}) = \{g \in \mathrm{GL}(\Theta) : g(xy) = g(x)g(y) \text{ for all } x, y \in \Theta\}.$$

Then G is a split group of type G_2. It is immediate that any $g \in G(\mathbb{Q})$ has the additional properties $g1 = 1$ and $N(gx) = N(x)$ and $\mathrm{Tr}(gx) = \mathrm{Tr}(x)$ for any $x \in \Theta$.

2.2 The Lie algebra of G_2

Let $V_7 \subseteq \Theta$ be the 7-dimensional subspace of trace 0 elements. In fact V_7 is orthogonal to 1 with respect to the bilinear form on Θ. Define for any $x \in \Theta$ its imaginary part $\mathrm{Im}(x) = \frac{1}{2}(x - x^*)$. The decomposition of x into a "real" and "imaginary" part is the same as its orthogonal decomposition, i.e.

$$x = \frac{1}{2}\mathrm{Tr}(x) \cdot 1 + \mathrm{Im}(x)$$

and $\mathrm{Im}(x) \in V_7$. Since G fixes 1, we have a representation $G \to \mathrm{GL}(V_7)$ which in fact factors through $\mathrm{SO}(V_7)$. The Lie algebra of $\mathrm{SO}(V_7)$ can be identified with $\wedge^2 V_7$. In this identification, the

action of $w \wedge x$ on $v \in V_7$ is

$$(w \wedge x) \cdot v = (x, v)w - (w, v)x,$$

the Lie bracket is

$$[w \wedge x, y \wedge z] = (x, y)w \wedge z - (x, z)w \wedge y - (w, y)x \wedge z + (w, z)x \wedge y,$$

and the Killing form is proportional to the pairing

$$(w \wedge x, y \wedge z) = (w, z)(x, y) - (w, y)(x, z).$$

There is an alternating map $\wedge^2 V_7 \mapsto V_7$ given by $w \wedge x \mapsto \mathrm{Im}(wx)$. The kernel of this map, $\mathfrak{g}_0$, is the Lie algebra of G. We view $\mathfrak{g}_0 \subseteq \wedge^2 V_7$ for the purpose of explicit computations.

Abstractly, there is the $\mathbb{Z}/3$-grading $\mathfrak{g}_0 = \mathfrak{sl}_3 \oplus V_3 \oplus V_3^\vee$. We will give an explicit basis for $\mathfrak{g}_0$, as elements of $\wedge^2 V_7$, that corresponds to this $\mathbb{Z}/3$-grading. Let $u_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in V_7$. Define, with indices in $\{1, 2, 3\}$ taken modulo 3:

- $E_{kj} = e_j^* \wedge e_k$
- $v_j = u_0 \wedge e_j + e_{j+1}^* \wedge e_{j+2}^*$
- $\delta_j = u_0 \wedge e_j^* + e_{j+1} \wedge e_{j+2}$.

One can check that v_j and δ_j are in $\mathfrak{g}_0$ for all j, and $E_{jk} \in \mathfrak{g}_0$ if $j \neq k$. Furthermore,

$$\mathfrak{h} = \{\alpha_1 E_{11} + \alpha_2 E_{22} + \alpha_3 E_{33} : \alpha_1 + \alpha_2 + \alpha_3 = 0\}$$

is a Cartan subalgebra for $\mathfrak{g}_0$. The v_j, δ_j, E_{jk} with $j \neq k$ along with $\mathfrak{h}$ span all of $\mathfrak{g}_0$. As suggested by the notation, the E_{jk} correspond to the standard basis for $\mathfrak{sl}_3$ in the $\mathbb{Z}/3$-grading, while $\{v_1, v_2, v_3\}$ is a basis for the V_3 with dual basis $\{\delta_1, \delta_2, \delta_3\}$. The Lie bracket of an element of $\mathfrak{sl}_3$

with a v_j or δ_j is given by the standard or dual representation. That is,

- $[E_{ij}, v_k] = \begin{cases} v_i, & \text{if } j = k \\ 0, & \text{otherwise} \end{cases}$ and
- $[\delta_i, E_{jk}] = \begin{cases} \delta_k, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$

The Lie bracket between other elements is computed as:

- $[\delta_{j-1}, v_j] = 3E_{j,j-1}$
- $[v_{j-1}, \delta_j] = -3E_{j-1,j}$
- $[\delta_{j-1}, \delta_j] = 2v_{j+1}$
- $[v_{j-1}, v_j] = 2\delta_{j+1}$
- $[\delta_j, v_j] = 3E_{jj} - (E_{11} + E_{22} + E_{33})$.

There is a natural choice of Cartan involution on G induced from an involution on Θ. Namely, let $\iota : \Theta \to \Theta$ be given by

$$\iota\left(\begin{pmatrix} a & v \\ \phi & d \end{pmatrix}\right) = \begin{pmatrix} d & -\tilde{\phi} \\ -\tilde{v} & a \end{pmatrix}$$

where $\tilde{e_j} = e_j^*$ and $\tilde{e_j^*} = e_j$, and $\tilde{\cdot}$ is extended linearly to the rest of V_3 and $V_3^\vee$. Corresponding to ι is an involution θ on $\wedge^2 V_7$, given by $\theta(w \wedge x) = \iota(w) \wedge \iota(x)$. One can check that θ preserves $\mathfrak{g}_0$ and is a Cartan involution on $\mathfrak{g}_0$.

Let $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$. We can decompose $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, with $\mathfrak{k}$ the $+1$ eigenspace of θ in $\mathfrak{g}$ and $\mathfrak{p}$ the -1 eigenspace; this is the so-called Cartan decomposition. In this decomposition, the compact part $\mathfrak{k}$ is isomorphic to the direct sum of two copies of $\mathfrak{sl}_2$. We write $\mathfrak{k} \cong \mathfrak{sl}_2^{long} \oplus \mathfrak{sl}_2^{short}$, with the superscripts indicating whether the $\mathfrak{sl}_2$ corresponds to a long or short root. On the group

level, the corresponding maximal compact subgroup K_G of $G(\mathbb{R})$ is isomorphic to $SU(2)^{long} \times SU(2)^{short}/ < (-I,-I) >$. Let V_2^{long} be the standard representation of $SU(2)^{long}$; we take $\{x, y\}$ as a standard basis V_2^{long}. Analogously define V_2^{short} and $\{x_s, y_s\}$. Let $V_G = \mathrm{Sym}^3(V_2^{short})$, a 4-dimensional irreducible representation of $SU(2)^{short}$. We fix symplectic forms on V_2^{long} via $\langle x, y\rangle = 1$ and on V_G via $\langle x_s^3, y_s^3\rangle = 1, \langle x_s^2 y_s, x_s y_s^2\rangle = -1/3$ to identify these representations (and their symmetric powers) with their duals. The noncompact part, $\mathfrak{p}$, has the isomorphism $\mathfrak{p} \cong V_2^{long} \boxtimes V_G$ as a representation of K_G. We will need to record explicit basis elements of $\mathfrak{k}$ and $\mathfrak{p}$. First define, for $j \in \{1, 2, 3\}$ and with indices taken modulo 3,

$$
\begin{aligned}
u_j &= \frac{1}{4}(E_{j+2,j+1} - E_{j+1,j+2} + v_j + \delta_j) \\
r_j &= \frac{1}{4}(3E_{j+2,j+1} - 3E_{j+1,j+2} - v_j - \delta_j) \\
f_j &= E_{j+1,j+2} + E_{j+2,j+1} \\
y_j &= v_j - \delta_j.
\end{aligned}
$$

Note that the u_j and r_j span $\mathfrak{k}$, and the f_j and y_j along with $\mathfrak{h}$ span $\mathfrak{p}$. However, we would like to work with a "good" basis with respect to the representation theory of $\mathfrak{sl}_2$.

For $\mathfrak{k}$, a good basis is as follows:

$$
\begin{aligned}
e_u &= u_1 - iu_3 \\
h_u &= 2iu_2 \\
f_u &= -u_1 - iu_3 \\
e_s &= r_1 - ir_3 \\
h_s &= 2ir_2 \\
f_s &= -r_1 - ir_3
\end{aligned}
$$

In the decomposition $\mathfrak{k} = \mathfrak{sl}_2^{long} \oplus \mathfrak{sl}_2^{short}$, the $\{e_u, h_u, f_u\}$ serves as a standard basis for $\mathfrak{sl}_2^{long}$ and the $\{e_s, h_s, f_s\}$ serves as a standard basis for $\mathfrak{sl}_2^{short}$. We have the bracket relations $[e_*, f_*] = h_*, [h_*, e_*] = 2e_*, [h_*, f_*] = -2f_*$ for $* \in \{s, u\}$. Relative to our standard bases for V_2^{long} and V_2^{short}, these elements act as:

$$
\begin{aligned}
e_u &= x\frac{\partial}{\partial y} \\
h_u &= x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} \\
f_u &= y\frac{\partial}{\partial x} \\
e_s &= x_s\frac{\partial}{\partial y_s} \\
h_s &= x_s\frac{\partial}{\partial x_s} - y_s\frac{\partial}{\partial y_s} \\
f_s &= y_s\frac{\partial}{\partial x_s}
\end{aligned}
$$

For $\mathfrak{p}$, a good basis is as follows:

$$
\begin{aligned}
d_3 &= f_1 - y_1 + i(f_3 + y_3) & h_3 &= 2(E_{33} - E_{11}) + 2if_2 \\
d_1 &= -\frac{2}{3}(E_{11} - 2E_{22} + E_{33}) + \frac{2}{3}iy_2 & h_1 &= \frac{3f_1 + y_1}{3} + i\left(\frac{3f_3 - y_3}{3}\right) \\
d_{-1} &= -\frac{3f_1 + y_1}{3} + i\left(\frac{3f_3 - y_3}{3}\right) & h_{-1} &= -\frac{2}{3}(E_{11} - 2E_{22} + E_{33}) - \frac{2}{3}iy_2 \\
d_{-3} &= 2(E_{33} - E_{11}) - 2if_2 & h_{-3} &= y_1 - f_1 + i(f_3 + y_3).
\end{aligned}
$$

Recall $\mathfrak{p}$ is isomorphic to $V_2^{long} \boxtimes \mathrm{Sym}^3(V_2^{short})$ as a representation of $\mathfrak{k} \cong \mathfrak{sl}_2^{long} \oplus \mathfrak{sl}_2^{short}$. Relative to our standard $\mathfrak{sl}_2$-bases for $\mathfrak{k}$, one can calculate $d_{3-2j} = y \boxtimes x_s^{3-j} y_s^j$ and $h_{3-2j} = x \boxtimes x_s^{3-j} y_s^j$.

2.3 Embedding $\mathrm{SU}(2,1)$ inside G_2

Let $D > 0$ be a square-free integer. Let $v_D = e_2 - De_2^* \in \Theta$. Then $K = \mathrm{Span}_{\mathbb{Q}}\{1, v_D\} \subseteq \Theta$ is a subalgebra isomorphic to $\mathbb{Q}(\sqrt{-D})$, i.e. a embedding from $\mathbb{Q}(\sqrt{-D}) \hookrightarrow \Theta$ is given by $1 \mapsto 1$ and $\sqrt{-D} \mapsto v_D$. We remark that the conjugation on Θ restricts to complex conjugation on $\mathbb{Q}(\sqrt{-D})$.

Let V_D be the orthogonal complement of K in Θ, i.e. $\Theta = V_D \oplus \mathrm{Span}_{\mathbb{Q}}\{1, v_D\}$ is an orthogonal decomposition relative to the bilinear form on Θ. A basis for V_D is given by

$$V_D = \mathrm{Span}_{\mathbb{Q}}\{u_0, e_2 + De_2^*, e_1, e_3^*, e_1^*, e_3\}.$$

One can check directly that $(e_2 - De_2^*)V_D = V_D$. So we may endow V_D with the structure of a 3-dimensional (left) K-vector space, with Hermitian form

$$\mathcal{H}(x,y) = \mathrm{Tr}(xy) - \frac{1}{\sqrt{-D}}\mathrm{Tr}(xv_Dy).$$

Let us check that this is actually a Hermitian form. The difficulty lies in (conjugate) linearity with respect to $v_D \approx \sqrt{-D}$. First

$$\begin{aligned}\mathcal{H}(x, v_Dy) &= \mathrm{Tr}(xv_Dy) - \frac{1}{\sqrt{-D}}\mathrm{Tr}(xv_Dv_Dy)\\ &= \mathrm{Tr}(xv_Dy) - \sqrt{-D}\,\mathrm{Tr}(xy)\\ &= -\sqrt{-D}\cdot\mathcal{H}(x,y)\end{aligned}$$

so it is conjugate linear in the second term. Next in

$$\mathcal{H}(v_Dx, y) = \mathrm{Tr}(v_Dxy) - \frac{1}{\sqrt{-D}}\mathrm{Tr}(v_Dxv_Dy)$$

we must check for example that $\mathrm{Tr}(v_Dxy) = -\mathrm{Tr}(xv_Dy)$. For this, we need the general identity

for $x, z \in \Theta$ that

$$xz + zx = \operatorname{Tr}(x)z + \operatorname{Tr}(z)x - (x, z).$$

Therefore

$$\operatorname{Tr}(v_D xy) = \operatorname{Tr}(xv_D y) = \operatorname{Tr}(x)\operatorname{Tr}(v_D y) + \operatorname{Tr}(v_D)\operatorname{Tr}(xy) - (x, v_D)\operatorname{Tr}(y) - \operatorname{Tr}(xv_D y)$$

but $\operatorname{Tr}(v_D) = 0$, and $\operatorname{Tr}(v_D y) = (v_D, y^*) = 0$ and $(x, v_D) = 0$ when $x, y \in V_D$. It is a similar check that $\operatorname{Tr}(v_D x v_D y) = D\operatorname{Tr}(xy)$. Putting these together yields

$$\begin{aligned}\mathcal{H}(v_D x, y) &= \operatorname{Tr}(v_D xy) - \frac{1}{\sqrt{-D}}\operatorname{Tr}(v_D x v_D y) \\ &= -\operatorname{Tr}(xv_D y) + \sqrt{-D}\operatorname{Tr}(xy) \\ &= \sqrt{-D}\cdot\mathcal{H}(x, y)\end{aligned}$$

as desired.

Let $H \subseteq G$ be the stabilizer of v_D. Then for any $h \in H$ and $x, y \in V_D$, we have $(hx, v_D) = (hx, hv_D) = (x, v_D) = 0$ and also $\mathcal{H}(hx, hy) = \mathcal{H}(x, y)$. Therefore H is a group of type $\mathrm{SU}(2, 1)$. We can calculate that the Lie algebra of H, i.e. the subspace of $\mathfrak{g}_0$ that annihilates v_D, has a basis

$$\{E_{11} - E_{33}, v_2 + D\delta_2, \delta_3 - DE_{12}, E_{23} - Dv_1, E_{13}, E_{21} + Dv_3, \delta_1 + DE_{32}, E_{31}\}.$$

From now on fix $D = 1$. This choice ensures that H is embedded in G in "good position", i.e. the choice we made for the Cartan involution on G restricts to one for H. In fact the "good bases" for $\mathfrak{k}$ and $\mathfrak{p}$ recorded earlier contain good bases for the analogous components in the Cartan decomposition of H. More precisely, one obtains the Cartan decomposition $\mathrm{Lie}(H)\otimes\mathbb{C} = \mathfrak{k}_H \oplus \mathfrak{p}_H$

with

$$\mathfrak{k}_H = \mathrm{Span}_{\mathbb{C}}\{e_u, h_u, f_u, h_s\}$$

$$\mathfrak{p}_H = \mathrm{Span}_{\mathbb{C}}\{h_{-3}, h_3, d_{-3}, d_3\}.$$

The maximal compact of H is $K_H \cong U(2) \cong SU(2)^{long} \times U(1)/ < (-I,-I) >$. Let $V_H \subseteq V_G$ be the subspace $\mathrm{Span}_{\mathbb{C}}\{(y_s^3), (x_s^3)\}$. Then $\mathfrak{p}_H \cong V_2^{long} \boxtimes V_H \cong V_2 \boxtimes (\det^{-3} \oplus \det^3)$ as a representation of K_H.

2.4 Characters and the Heisenberg parabolic

We fix as a standard additive character $\psi : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$ with $\psi(a_\infty) = e^{-2\pi i a_\infty}$ for $a_\infty \in \mathbb{R}$.

Let P_G be the stabilizer in G of E_{13} under the adjoint representation. This P_G is the so-called Heisenberg parabolic subgroup; in its Levi decomposition $P_G = M_G N_G$, one has $M_G \cong \mathrm{GL}_2$ and $Z(N_G) = [N_G, N_G]$. In particular, $\mathrm{Lie}(M_G)$ is spanned by $\{\delta_2, v_2\}$ and $\mathfrak{h}$; an explicit identification of M_G with GL_2 is given in [Pol21] section 2.3. Also $\mathrm{Lie}(N_G)$ is spanned by $\{E_{12}, v_1, \delta_3, E_{23}, E_{13}\}$, with $Z(N_G)$ the one-dimensional space generated by E_{13}.

Then $P_H = H \cap P_G$ is a Heisenberg parabolic (and in this case even a Borel) subgroup of H. We have the Levi decomposition $P_H = M_H N_H$, with $M_H = H \cap M_G$ and $N_H = H \cap N_G$. The Lie algebra of M_H is spanned by $\{E_{11} - E_{33}, v_2 + D\delta_2\}$ and the Lie algebra of N_H is spanned by $\{\delta_3 - DE_{12}, E_{23} - Dv_1, E_{13}\}$. Note that $M_H(\mathbb{R}) = S_H(\mathbb{R}) \times A_H(\mathbb{R}) \cong U(1) \times \mathrm{GL}_1(\mathbb{R})$ in its decomposition into a compact and split part. The $\mathrm{GL}_1(\mathbb{R})$ is generated by $E_{11} - E_{33}$, and the $U(1)$ is generated by $v_2 + \delta_2$.

Define $W_G := N_G^{ab} = N_G/[N_G, N_G]$. Characters χ on $N_G(\mathbb{Q})\backslash N_G(\mathbb{A})$ correspond to elements $\omega \in W_G(\mathbb{Q})$. We now make this explicit.

For any $\mathbb{Q}$-algebra R, we fix a bijection of $W_G(R)$ with the space of binary cubic forms over R via

$$aE_{12} + \frac{b}{3}v_1 + \frac{c}{3}\delta_3 + dE_{23} \in W_G \longleftrightarrow au^3 + bu^2v + cuv^2 + dv^3.$$

Write $\left(a, \frac{b}{3}, \frac{c}{3}, d\right)$ to denote $au^3 + bu^2v + cuv^2 + dv^3$ or its corresponding element of W_G. The adjoint action of M_G on $N_G/[N_G, N_G]$ corresponds to the representation of M_G on the space of binary cubic forms given by

$$(m \cdot f)\left(\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)\right) = \det(m)^2 f\left(m^{-1}\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)\right).$$

There is a symplectic form on W_G given by

$$\left\langle \left(a, \frac{b}{3}, \frac{c}{3}, d\right), \left(a', \frac{b'}{3}, \frac{c'}{3}, d'\right) \right\rangle = ad' - \frac{1}{3}bc' + \frac{1}{3}cb' - da'.$$

This form is preserved, up to similitude, by M_G.

The image of any $X \in N_G$ modulo $[N_G, N_G]$ is some $p_X = \left(a', \frac{b'}{3}, \frac{c'}{3}, d'\right) \in W_G$. Then associated to $\omega = \left(a, \frac{b}{3}, \frac{c}{3}, d\right) \in W_G(\mathbb{Q})$ is the character $\chi_\omega : N_G(\mathbb{Q})\backslash N_G(\mathbb{A}) \to \mathbb{C}^\times$ given by $\chi_\omega(X) = \psi(\langle \omega, p_X \rangle)$. Also, for $\omega \in W_G(\mathbb{R})$, we will slightly abuse notation and write $p_\omega(z)$ for the polynomial $az^3 + bz^2 + cz + d$ and $p_\omega(u, v)$ for the binary cubic form $au^3 + bu^2v + cuv^2 + d$.

Under these identifications, elements μ of $W_H := N_H^{ab}$ correspond to tuples of the form $\mu = (a, -d, -a, d)$, which give rise to characters χ_μ of $N_H(\mathbb{Q})\backslash N_H(\mathbb{A})$ via $\chi_\mu(X) = \psi(\langle \mu, p_X \rangle)$. Write (a, d) for the element $(a, -d, -a, d) \in W_H$ and $p_\mu(z)$ for the polynomial $az^3 - 3dz^2 - 3az + d$. We remark that any such element is "positive" in the sense of [Pol20] as long as it is nonzero. Indeed, one can compute that the discriminant of $p_\mu(z)$ is $108(a^2 + d^2)^2$.

Now let $pr : W_G \to W_H$ be given by

$$\left(a, \frac{b}{3}, \frac{c}{3}, d\right) \mapsto \left(\frac{a - c}{4}, \frac{d - b}{4}\right).$$

This projection is the identity on W_H; the projection onto the orthogonal complement is given by

$$pr^\perp\left(\left(a, \frac{b}{3}, \frac{c}{3}, d\right)\right) = \left(\frac{3a + c}{4}, \frac{3d + b}{12}, \frac{3a + c}{12}, \frac{3d + b}{4}\right).$$

Let $\omega = \left(a, \frac{b}{3}, \frac{c}{3}, d\right) \in W_G(\mathbb{R})$. When we restrict a Fourier coefficient on G_2 to $\mathrm{SU}(2,1)$, the quantities $p_\omega(i)$ will be involved. However we have $p_{pr(\omega)}(i) = p_\omega(i)$ and $p_{pr^\perp(\omega)}(i) = 0$. One way to distinguish different ω at the level of $\mathrm{SU}(2,1)$ is related to the quadratic covariant of binary cubic forms. Namely, given $\omega \in W_G$, define the binary quadratic form

$$q_\omega(u,v) = (b^2 - 3ac)u^2 + (bc - 9ad)uv + (c^2 - 3bd)v^2,$$

and let $q_\omega(z)$ be the associated quadratic polynomial. Then q vanishes on W_H, i.e. $q_{pr(\omega)}(i) = 0$ for any ω. On the other hand,

$$q_{pr^\perp(\omega)}(i) = -\frac{1}{4}(3ai + b + ci + 3d)^2$$

is nonzero as long as $\omega \notin W_H$. These quantities appear in the Fourier coefficients of our differentiated QMFs.

Definition 2.4.1. For $\omega = \left(a, \frac{b}{3}, \frac{c}{3}, d\right) \in W_G(\mathbb{R})$, define the quantities

- $z_\omega := 2p_\omega(i) = 2p_{pr(\omega)}(i) = -2ai - 2b + 2ci + 2d$
- $b_\omega := -4ip'_\omega(i) + 6p_\omega(i) = 6ai + 2b + 2ci + 6d.$

We remark that $b_\omega^2 = -16q_{pr^\perp(\omega)}(i)$ and $b_\omega = 0$ if $\omega \in W_H$.

Chapter 3

QUATERNIONIC MODULAR FORMS

3.1 Quaternionic Discrete Series

For any integer $n \geq 2$, there is a discrete series representation π_n^G of $G(\mathbb{R})$ studied by Gross and Wallach in [GW96], with K_G-type decomposition

$$\pi_n^G|_{K_G} = \bigoplus_{j=0}^{\infty} \mathrm{Sym}^{2n+j}(V_2^{long}) \boxtimes \mathrm{Sym}^j(V_G).$$

Write $\mathbb{V}_n^G = \mathrm{Sym}^{2n}(V_2^{long}) \boxtimes \mathbf{1}$ for the lowest K_G-type. We remark that there is also a limit discrete series at $n = 1$ with the same K_G-type decomposition.

It is quite complicated in general to figure out how $U(\mathfrak{p})$ moves around vectors between K_G-types. All we will need to keep in mind is the following: for $v \in \mathrm{Sym}^{2n}(V_2^{long}) \boxtimes \mathbf{1}$ the lowest K_G-type of π_G^n, the map

$$\mathfrak{p}^{\otimes r} \otimes (\mathrm{Sym}^{2n}(V_2^{long}) \boxtimes \mathbf{1}) \to (\mathrm{Sym}^r(V_2^{long}) \boxtimes \mathrm{Sym}^r(V_G)) \otimes (\mathrm{Sym}^{2n}(V_2^{long}) \boxtimes \mathbf{1})$$
$$\to \mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes \mathrm{Sym}^r(V_G)$$

must just be multiplication on elements. For example, for $x^{2n} \in \mathrm{Sym}^{2n}(V_2^{long})$ and $h_3 d_{-3} \in \mathfrak{p}^{\otimes 2}$, it takes some work to figure out completely the decomposition of

$$h_3 d_{-3} x^{2n} \in \bigoplus_{j=0}^{2} \mathrm{Sym}^{2n+j}(V_2^{long}) \boxtimes \mathrm{Sym}^j(V_G),$$

but on the other hand we can read off that the projection of $h_3 d_{-3} x^{2n}$ to the top piece $\mathrm{Sym}^{2n+2}(V_2^{long}) \boxtimes \mathrm{Sym}^2(V_G)$ is equal to $x^{2n+1}y \boxtimes (y_s^3)(x_s^3)$.

3.1.1 $SU(2,1)$ and Restriction

For any integer $n \geq 1$, $\epsilon \in \{0,1\}$, and integer m with the same parity as ϵ, there is a discrete series representation $\pi^H_{n+\frac{\epsilon}{2},m}$ with K_H-type decomposition

$$\pi^H_{n+\frac{\epsilon}{2},m}|_{K_H} = \bigoplus_{j=0}^{\infty} \mathrm{Sym}^{2n+\epsilon+j}(V_2^{long}) \boxtimes (\mathrm{Sym}^j(V_H) \otimes \det{}^m).$$

Write $\mathbb{V}^H_{n+\frac{\epsilon}{2},m} = \mathrm{Sym}^{2n+\epsilon}(V_2^{long}). \boxtimes \det{}^m$ for the lowest K_H-type. These are the "large" discrete series of $\mathrm{SU}(2,1)$, and are in general neither holomorphic nor antiholomorphic.

Corollary 4.2.2 of [Lok99], describes the restriction of π^G_n to H. In our terms,

$$\pi^G_n|_H = \bigoplus_{r=0}^{\infty} \bigoplus_{\substack{m=-r \\ m\equiv r(2)}}^{r} \pi^H_{n+\frac{r}{2},m}.$$

In particular, there is one copy of $\pi^H_{n+\frac{r}{2},m}$ in $\pi^G_n|_H$ for each $r \geq 0$ and $m \in \{-r, -r+2, \ldots, r-2, r\}$. In terms of the K_G-types of π^G_n, the lowest K_H-type of $\pi^H_{n+\frac{r}{2},m}$ in $\pi^G_n|_H$ is contained in $\$\,\mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes \mathrm{Sym}^r(V_G)$. That is, there is a line $L^{r,m}_n \subseteq \mathrm{Sym}^r(V_G)$ that has h_s-eigenvalue equal to m such that

$$\mathbb{V}^H_{n+\frac{r}{2},m} \cong \mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes L^{r,m}_n.$$

Therefore in the K_G-type decomposition of $\pi^G_n|_H$, each $\mathrm{Sym}^{2n+j}(V_2^{long}) \boxtimes \mathrm{Sym}^j(V_G)$ comes from $\pi^H_{n+\frac{r}{2},m}$ for $r \leq j$. More precisely,

$$\mathrm{Sym}^{2n+j}(V_2^{long}) \boxtimes \mathrm{Sym}^j(V_G) = \bigoplus_{j=0}^{r} \bigoplus_{\substack{-j\leq m\leq j \\ m\equiv r(2)}} \mathfrak{p}_H^{\otimes(r-j)} \cdot \mathbb{V}^H_{n+\frac{j}{2},m}.$$

The point of this discussion is to help us fix a particular basis element $\ell^n_{r,m}$ for each $L^n_{r,m}$. The line

$L^n_{r,m}$ is spanned by an element of the form

$$v^n_{r,m} = \sum_{\substack{0\le a,b,c,d\le r \\ -3a-b+c+3d=m}} C_{a,b,c,d}(y_s^3)^a(x_s y_s^2)^b(x_s^2 y_s)^c(x_s^3)^d.$$

When $(a,b,c,d) = (0,(r-m)/2,(r+m)/2,0)$, we must have $C_{a,b,c,d} \neq 0$. Otherwise, $\mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes \mathbb{C}v^n_{r,m}$ would come from lower weight H-discrete series; it would be in the image of

$$\bigoplus_{j=0}^{r-1} \bigoplus_{\substack{-j\le m\le j \\ m\equiv j(2)}} \mathfrak{p}_H^{\otimes(r-j)} \cdot \mathbb{V}^H_{n+\frac{j}{2},m}.$$

We define $\ell^n_{r,m} = (C_{a,b,c,d})^{-1}v^n_{r,m}$, i.e. by normalizing its $(0,(r-m)/2,(r+m)/2,0)$ coefficient.

For $U(\mathrm{Lie}(H)\otimes\mathbb{C}) \subseteq U(\mathfrak{g})$, we take as a quadratic Casimir element

$$\Omega_H := \frac{1}{6}h_s^2 + \frac{1}{2}h_u^2 + (e_u f_u + f_u e_u) + \frac{1}{16}(h_3 d_{-3} + d_{-3}h_3) - \frac{1}{16}(h_{-3}d_3 + d_3 h_{-3}).$$

Then, one can check (see e.g. *loc. cit.* Theorem 3.3.1 for the infinitesimal character) that Ω_H acts on $\pi^H_{n+\frac{r}{2},m}$ as the scalar $\lambda^n_{r,m} := \frac{m^2}{6} + 2(n+\frac{r}{2})^2 - 2$.

3.2 Quaternionic Modular Forms

3.2.1 G_2

Quaternionic modular forms for G_2 were first studied by Gan-Gross-Savin in [GGS02]. We give a slightly broader definition following Pollack [Pol20].

We first define an operator $\mathcal{D}_n$ on functions $\Phi : G(\mathbb{A}) \to \mathbb{V}_n^\vee$ as follows. Let $\tilde{D}_G$ be the operator defined by

$$\tilde{D}_G\Phi = \sum_{X_i} X_i F \otimes X_i^*$$

where the sum is over a basis $\{X_i\}$ of $\mathfrak{p}$ and $\{X_i^*\}$ is the corresponding dual basis. There is a K_G-equivariant projection $pr_- : \mathbb{V}_n^\vee \otimes \mathfrak{p}^\vee \to \mathrm{Sym}^{2n-1}(V_2^{long})^\vee \boxtimes V_G^\vee$. Finally let $\mathcal{D}_n = pr_- \circ \tilde{D}_G$.

Now we can define QMFs for G:

Definition 3.2.1. Let $n \geq 1$ be an integer and $\Phi : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{V}_n^\vee$. We say that Φ is a modular form on G of weight n if

1. $\Phi(gk) = k^{-1}\Phi(g)$ for all $g \in G(\mathbb{A})$ and $k \in K_G$
2. $\mathcal{D}_n\Phi \equiv 0$
3. Φ is smooth, moderate growth, and $Z(\mathfrak{g})$-finite.

3.2.2 $SU(2,1)$

Quaternionic modular forms for $SU(2,n)$ were first studied by Hilado-McGlade-Yan in [HMY24], following work of Koseki-Oda ([KO95]) on $SU(2,1)$ and Yamashita ([Yam91]) on $SU(2,2)$.

Define $\tilde{D}_H$ analogously to $\tilde{D}_G$, let $pr_-^H : \mathbb{V}^\vee_{n+\frac{\epsilon}{2},m} \otimes \mathfrak{p}_H^\vee \to \mathrm{Sym}^{2n+\epsilon-1}(V_2^{long})^\vee \boxtimes (V_H^\vee \otimes \det^{-m})$, and set $\mathcal{D}^H_{n+\frac{\epsilon}{2},m} = pr_-^H \circ \tilde{D}_H$. Then

Definition 3.2.2. Let $n \geq 1$ be an integer, $\epsilon \in \{0,1\}$, and m an integer with the same parity as ϵ. Let $\varphi : H(\mathbb{Q})\backslash H(\mathbb{A}) \to \mathbb{V}^\vee_{n+\frac{\epsilon}{2},m}$. We say that φ is a modular form on H of weight $(n+\frac{\epsilon}{2}, m)$ if

1. $\varphi(hk) = k^{-1}\varphi(h)$ for all $h \in H(\mathbb{A})$ and $h \in K_H$
2. $\mathcal{D}^H_{n+\frac{\epsilon}{2},m}\varphi \equiv 0$
3. φ is smooth, moderate growth, and $Z(\mathfrak{h})$-finite.

3.3 Fourier Expansion

An explicit Fourier expansion for QMFs on exceptional groups is determined in [Pol20]. Taking the place of the exponential function in the Fourier expansion of holomorphic modular

forms are the classical K-Bessel functions (also known as the Modified Bessel Functions of the Second Kind). For a complex parameter v and real $z > 0$, these may be defined by

$$K_v(z) = \frac{1}{2}\int_0^\infty \alpha^v e^{-z(\alpha+\alpha^{-1})/2}\frac{d\alpha}{\alpha}.$$

One can refer to [Pol20] Section 8.1 or [GR15] Chapter 7 for more details; in any case we only need integer parameters v.

Let Φ be a QMF on G of weight n. For $g = g_f g_\infty \in G(\mathbb{A})$ with $g_f \in G(\mathbb{A}_f)$ and $g_\infty \in G(\mathbb{R})$, the Fourier expansion of Φ refers to an explicit formula for the constant term of Φ along $Z_G = [P_G, P_G]$:

$$\Phi_{Z_G}(g_f g_\infty) = \Phi_0(g_f g_\infty) + \sum_{\substack{\omega \in 2\pi W_G(\mathbb{Q}) \\ \omega > 0}} a_\omega(g_f) W^G_{\omega,n}(g_\infty).$$

In the formula above, the Fourier coefficients $a_\omega : G(\mathbb{A}_f) \to \mathbb{C}$ are locally constant functions. The Whittaker functions $W^G_{\omega,n} : G(\mathbb{R}) \to \mathbb{V}_n^\vee$ have the following formulae on the Levi component $M_G(\mathbb{R})$:

For $\omega \in W_G(\mathbb{R})$, and $m \in M_G(\mathbb{R})$,

$$W^G_{\omega,n}(m) = \sum_{-n \le v \le n} W^{G,v}_{\omega,n}(m)\frac{x^{n+v}}{(n+v)!}\frac{y^{n-v}}{(n-v)!}$$

with

$$W^{G,v}_{\omega,n}(m) = \left(\frac{|j(m,i)p_\omega(z)|}{j(m,i)p_\omega(z)}\right)^v \det(m)^n |\det(m)| K_v(|j(m,i)p_\omega(z)|).$$

Now $W^G_{\omega,n}$ is determined by this formula along with equivariance properties

$$W^G_{\omega,n}(nmk) = e^{-i\langle\omega,n\rangle}k^{-1}W^G_{\omega,n}(m)$$

for $n \in N_H(\mathbb{R}), k \in K_G$.

The Fourier expansion for QMFs on $\mathrm{SU}(2,1)$ is determined in [KO95]. Let φ be a QMF

on H of weight $(n+\frac{\epsilon}{2}, m)$. The Fourier expansion of φ takes the form

$$\varphi_{Z_H}(h_f h_\infty) = \varphi_0(h_f h_\infty) + \sum_{\substack{\mu \in 2\pi W_H(\mathbb{Q}) \\ \mu \neq (0,0)}} a_\mu(h_f) W^H_{\mu, n+\frac{\epsilon}{2}}(h_\infty).$$

for $h_f \in H(\mathbb{A}_f)$ and $h_\infty \in H(\mathbb{R})$. Here, for $\lambda \in A_H(\mathbb{R}) \cong \mathbb{R}^\times$,

$$W^H_{\mu, n+\frac{\epsilon}{2}, m}(\lambda) = \sum_{\substack{-n-\frac{\epsilon}{2} \leq v \leq n+\frac{\epsilon}{2} \\ v \in \mathbb{Z}+\frac{\epsilon}{2}}} W^{H,v}_{\mu, n+\frac{\epsilon}{2}, m}(\lambda) \cdot \frac{x^{n+v}}{(n+v)!} \frac{y^{n-v}}{(n-v)!} \boxtimes w_{-m}$$

with

$$W^{H,v}_{\mu, n+\frac{\epsilon}{2}, m}(\lambda) = \lambda^{2n+\epsilon+2} \left(\frac{|p_\mu(i)|}{p_\mu(i)} \right)^{v+\frac{m}{2}} K_{v+\frac{m}{2}}(|p_\mu(i)|\lambda)$$

The w_{-m} is just meant to record that $W^H_{\mu, n+\frac{\epsilon}{2}}$ is valued in the K_H representation $(\mathrm{Sym}^{2n+\epsilon}(V_2) \boxtimes \det^m)^\vee \cong \mathrm{Sym}^{2n+\epsilon}(V_2) \boxtimes \det^{-m}$. Again $W^H_{\mu, n+\frac{\epsilon}{2}, m}$ is determined by the above formula and its equivariance properties.

Chapter 4

DIFFERENTIAL OPERATORS

4.1 Overview and Strategy

Let $n \geq 1$, and consider the representation $\pi_n^G|_H$. Recall from our discussion in section 3.1.1 that for any $r \geq 0$, and $m \in \{-r, -r+2, \ldots, r-2, r\}$, there is a single $\pi^H_{n+\frac{r}{2},m} \subseteq \pi_n^G|_H$ whose lowest K_H-type is cut out by some $L^n_{r,m}$ with distinguished basis element $\ell^n_{r,m}$. Let $Proj^n_{r,m}$ be the K_H-equivariant projection map

$$Proj^n_{r,m} : \mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes \mathrm{Sym}^r(V_G) \to \mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes L^n_{r,m},$$

unique up to scalar multiple. We pin down $Proj^n_{r,m}$ by enforcing that it is the identity on elements of the form $v \boxtimes \ell^n_{r,m}$. We will also abuse notation and write $Proj^n_{r,m}$ for the projection map $\mathrm{Sym}^r(V_G) \to L^n_{r,m}$. The differential operator of Theorem 1.0.1 will be a suitable normalization of the dual projection composted with $\tilde{D}^r$, that is $(Proj^n_{r,m})^\vee \circ \tilde{D}^r$.

We need to understand the action of this operator on Fourier coefficients. Let $\omega \in 2\pi W_G(\mathbb{Q})$, and recall the definitions of z_ω and b_ω from section 2.4. Let

$$v_\omega := z_\omega(y_s^3) + b_\omega(x_s y_s^2) + \overline{b_\omega}(x_s^2 y_s) + \overline{z_\omega}(x_s)^3 \in V_G = \mathrm{Sym}^3(V_2^{short}),$$

and define

$$Q^n_{r,m}(\omega) := \langle \ell^n_{r,m}, v^r_\omega \rangle$$

where the pairing is taken in $\mathrm{Sym}^r(V_G)$.

Theorem 1.0.1 will follow from the more precise result:

Theorem 4.1.1.

$$((Proj^n_{r,m})^\vee \circ \tilde{D}^r) W^G_{\omega,n}|_H = Q^n_{r,m}(\omega) W^H_{pr(\omega),n+\frac{r}{2}}$$

The path to our proof for Theorem 4.1.1 is to explicitly find a scalar-valued version of our differential operator for the "highest weight" Whittaker functions. Recall

$$W^{G,-n}_{\omega,n} := \langle W^G_{\omega,n}, x^{2n} \rangle.$$

Theorem 4.1.2. *There exists $D^n_{r,m} \in U(\mathfrak{g})$ such that $D^n_{r,m} x^{2n} = x^{2n+r} \boxtimes \ell^n_{r,m}$, and*

$$D^n_{r,m}(W^{G,-n}_{\omega,n})|_H = Q^n_{r,m}(\omega) \cdot W^{H,-(n+\frac{r}{2})}_{\omega,n+\frac{r}{2},m}$$

when both sides are restricted to $A_H(\mathbb{R})^0$.

That Theorem 4.1.1 follows from Theorem 4.1.2, as well as the proof of Theorem 4.1.2, is explained in section 4.5. The remainder of this section is dedicated to defining our $D^n_{r,m}$, and establishing their key properties in Theorems 4.3.3 and 4.4.8.

4.2 Operators for highest weight - definition

We first define the $D^n_{r,m}$. For any $r \geq 0$ and $m \in \{-r, -r+2, \ldots, r-2, r\}$, these are elements of $U(\mathfrak{g})$ that are meant to take $x^{2n} \in \mathbb{V}^G_n \subseteq \pi^G_n$ to a highest weight vector in the lowest-K_H type $\mathbb{V}^H_{n+\frac{r}{2},m} = \mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes \det^m \subseteq \pi^H_{n+\frac{r}{2},m}$ in $\pi^G_n|_H$.

Our differential operators will be defined via recursive equations that involve the following coefficients:

Definition 4.2.1. For any integers $n \geq 1, r \geq 0$ and $m \in \{-r, -r+2, \ldots, r-2, r\}$,

- $A^n_{r,m} := -\frac{1}{3}\frac{(4n+r-m-4)(r-m-2)}{(2n+r-m-4)(2n+r-m-2)}$
- $B^n_{r,m} := -\frac{1}{9}\frac{(4n+r+m-2)(3n+r-2)(n+r-1)(r+m)}{(2n+r+m)(2n+r+m-2)(2n+r-1)(2n+r-2)}$
- $E^n_{r,m} := A^n_{r,-m}$
- $F^n_{r,m} := B^n_{r,-m}$
- $U^n_{r,m} := \frac{1}{2}\frac{(4n+r+m-2)(r+m)}{(2n+r+m-2)}$
- $V^n_{r,m} := \frac{1}{3}\frac{(4n+r-m-2)(3n+r-1)(n+r)(r-m)}{(2n+r-m)(2n+r-m-2)(2n+r-1)}$
- $S^n_{r,m} := U^n_{r,-m}$
- $T^n_{r,m} := V^n_{r,-m}$

Remark 4.2.2. It would be fair to complain that, for example, $A^n_{r,m}$ is not always well defined. However, $A^n_{r,m}$ is only really defined and needed when $-r \leq m \leq r-4$. Any other problems similarly disappear.

Definition 4.2.3. Let $D^n_{0,0} = 1, D^n_{1,1} = h_1, D^n_{1,-1} = h_{-1}$. Define inductively

$$D^n_{r,r} = h_1 D^n_{r-1,r-1} + E^n_{r,r} h_3 D^n_{r-1,r-3}$$

and

$$D^n_{r,m} = h_{-1} D^n_{r-1,m+1} + A^n_{r,m} h_{-3} D^n_{r-1,m+3} + B^n_{r,m} h_{-3} h_3 D^n_{r-2,m}$$

for $m \in \{-r, -r+2, \ldots, r-2\}$.

4.3 Operators for highest weight - algebraic properties

We only really care about what $D^n_{r,m}$ does to the highest weight vector $x^{2n} \in \mathrm{Sym}^{2n}(V_2^{long}) \boxtimes \mathbf{1} \subseteq \pi_n^G$. Let us make this formal.

Definition 4.3.1. Define $\mathfrak{g}_+ = \mathrm{Span}_{\mathbb{C}}\{h_{-3}, h_{-1}, h_1, h_3, e_u\}$. This is a Lie subalgebra of $\mathfrak{g}$. Let $J \subseteq U(\mathfrak{g}_+)$ be the left ideal generated by e_u. Since $[e_u, h_j] = 0$ for $j = \pm 1, \pm 3$, it is actually a two-sided ideal.

Remark 4.3.2. Note that $[\mathfrak{g}_+, \mathfrak{g}_+] = \mathbb{C}e_u$. So by the Poincaré-Birkhoff-Witt theorem, each $D^n_{r,m}$ when taken modulo J is represented by a homogeneous degree r polynomial in the h_i. Furthermore, by our discussion in 3.1.1, no nonzero homogeneous degree r polynomial in the h_i can annihilate x^{2n}. Therefore we can identify $D^n_{r,m}$ with its image in $U(\mathfrak{g}_+)/J$, in other words with a unique polynomial in $\mathbb{Q}[h_{-3}, h_{-1}, h_1, h_3]$.

From the shape of the recurrence formulas, one might guess that there are some symmetries in the formulas defining $D^n_{r,m}$. We will capture some of these with the element $w_s \in K_G$ defined by

$$w_s(x) = x \qquad w_s(y) = y$$
$$w_s(x_s) = iy_s \qquad w_s(y_s) = ix_s.$$

This w_s is a representative for the Weyl group element of G that fixes $\mathfrak{sl}_2^{long}$ and reflects $\mathfrak{sl}_2^{short}$. That is, $\mathrm{Ad}(w_s)e_s = f_s$ and $\mathrm{Ad}(w_s)f_s = e_s$. Therefore, $\mathrm{Ad}(w_s)h_s = -h_s$. We also note the calculation $\mathrm{Ad}(w_s)h_j = (-i)h_{-j}$ for $j = \pm 1, \pm 3$.

For the next theorem statement to make sense, we must note that $Jx^{2n} = 0$, $Ad(w_s)J \subseteq J$, and $[\mathfrak{k}, J] \subseteq J$.

Theorem 4.3.3 (Algebraic Properties of $D^n_{r,m}$)**.**

Let $n \geq 1$. Recall (section 3.1) the (limit of) discrete series representation π_n^G has lowest

K_G-type $\mathbb{V}_n = \mathrm{Sym}^{2n}(V_2^{long}) \boxtimes \mathbf{1}$. Let $x^{2n} \in \mathbb{V}_n$ be a highest weight vector. For any $r \geq 0$ and $m \in \{-r, -r+2, \ldots, r-2, r\}$,

(I) $D^n_{r,m} x^{2n} = x^{2n+r} \boxtimes \ell^n_{r,m}$. *Therefore, it has the following properties:*

- $D^n_{r,m} x^{2n}$ *is an Ω_H-eigenvector with eigenvalue* $\lambda^n_{r,m} = \frac{m^2}{6} + 2(n + \frac{r}{2})^2 - 2$,
- $D^n_{r,m} x^{2n} \in x^{2n+r} \boxtimes L^n_{r,m}$, *where* $\mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes L^n_{r,m} \subseteq \pi^G_n$ *is the lowest K_H-type of the unique* $\pi^H_{n+\frac{r}{2},m} \subseteq \pi^G_n|_H$.

(II) $i^r Ad(w_s) D^n_{r,m} = D^n_{r,-m}$

(III) $D^n_{r,m} = h_1 D^n_{r-1,m-1} + E^n_{r,m} h_3 D^n_{r-1,m-3} + F^n_{r,m} h_3 h_{-3} D^n_{r-2,m}$ *for* $-r < m \leq r$

(IV) $[f_s, D^n_{r,m}] = U^n_{r,m} D^n_{r,m-2} + V^n_{r,m} h_{-3} D^n_{r-1,m+1}$

(V) $[e_s, D^n_{r,m}] = S^n_{r,m} D^n_{r,m+2} + T^n_{r,m} h_3 D^n_{r-1,m-1}$

Proof. Since n is fixed throughout, we will often omit it from the superscripts in our notation. We proceed by induction on r. Assume we have all (I)-(V) for all $r' < r$.

Outline: The proof of (II) for r will require (I) for r. The proof of (III) for r requires (II) for r. The proof of (IV) for r requires (III) for r. The proof of (V) for r requires (II) and (IV) for r.

(I) When $m < r$, we use the defining recurrence

$$D_{r,m} = h_{-1} D_{r-1,m+1} + A_{r,m} h_{-3} D_{r-1,m+3} + B_{r,m} h_3 h_{-3} D_{r-2,m}$$

to compute $\Omega_H D_{r,m} x^{2n}$. Since Ω_H commutes with h_3 and h_{-3}, the only thing we really need to compute is $\Omega_H h_{-1} D_{r-1,m+1} x^{2n}$.

For this, note that

$$\Omega_H h_{-1} - h_{-1} \Omega_H = -[h_{-1}, \Omega_H] = h_{-1} \left(\frac{2}{3} - \frac{1}{3} h_s + h_u \right) + 2 d_{-1} e_u - \frac{2}{3} h_{-3} e_s$$

Each summand can be handled separately:

- By induction, specifically by (I) for $r-1$, $D_{r-1,m+1}x^{2n} \in x^{2n+r-1} \boxtimes L^n_{r-1,m+1}$, so $h_s D_{r-1,m+1}x^{2n} = (m+1)D_{r-1,m+1}x^{2n}$ and $h_u D_{r-1,m+1}x^{2n} = (2n+r-1)D_{r-1,m+1}x2n$. Therefore $h_{-1}\left(\frac{2}{3} - \frac{1}{3}h_s + h_u\right) D_{r-1,m+1}x^{2n} = \left(\frac{2}{3} - \frac{m+1}{3} + 2n + r - 1\right) h_{-1}D_{r-1,m+1}x^{2n}$.
- $e_u D_{r-1,m+1}x^{2n} = [e_u, D_{r-1,m+1}]x^{2n} + D_{r-1,m+1}e_u x^{2n} = 0$.
- $-\frac{2}{3}h_{-3}e_s D_{r-1,m+1}x^{2n} = -\frac{2}{3}h_{-3}\left(S_{r-1,m+1}D_{r-1,m+3} + T_{r-1,m+1}h_3 D_{r-2,m}\right)$ by Statement (V) $r-1$.

Therefore,

$$\Omega_H h_{-1}D_{r-1,m+1}x^{2n} = h_{-1}\Omega_H D_{r-1,m+1}x^{2n} + \left(\frac{2}{3} - \frac{m+1}{3} + 2n + r - 1\right) h_{-1}D_{r-1,m+1}x^{2n} - \frac{2}{3}S_{r-1,m+1}h_{-3}D_{r-1,m+3} - \frac{2}{3}T_{r-1,m+1}h_{-3}h_3 D_{r-2,m}$$

Now observe the identities (where all terms depend on n)

(a) $\lambda_{r,m} = \lambda_{r-1,m+1} + \left(\frac{2}{3} - \frac{m+1}{3} + 2n + r - 1\right)$

(b) $A_{r,m}\lambda_{r,m} = -\frac{2}{3}S_{r-1,m+1} + A_{r,m}\lambda_{r-1,m+3}$

(c) $B_{r,m}\lambda_{r,m} = -\frac{2}{3}T_{r-1,m+1} + B_{r,m}\lambda_{r-2,m}$.

We see that everything combines nicely in

$$\begin{aligned}\Omega_H D_{r,m}x^{2n} &= \Omega_H(h_{-1}D_{r-1,m+1} + A_{r,m}h_{-3}D_{r-1,m+3} + B_{r,m}h_3 h_{-3}D_{r-2,m})x^{2n} \\ &= \Omega_H h_{-1}D_{r-1,m+1}x^{2n} + A_{r,m}h_{-3}\lambda_{r-1,m+3}D_{r-1,m+3}x^{2n} + B_{r,m}h_3 h_{-3}\lambda_{r-2,m}x^{2n}\end{aligned}$$

and conclude that $\Omega_H D_{r,m}x^{2n} = \lambda_{r,m}D_{r,m}x^{2n}$.

For the second part of statement (I), since $[h_u, D_{r,m}]x^{2n} = (2n+r)D_{r,m}x^{2n}$, and $[e_u, D_{r,m}]x^{2n} = 0$, we must have $D_{r,m}x^{2n} \in x^{2n+r} \boxtimes \mathrm{Sym}^r(V_G)$. Combining this data with

its Ω_H-eigenvalue and the calculation $[h_s, D_{r,m}]x^{2n} = mD_{r,m}x^{2n}$ allows us to pin down that $D_{r,m}x^{2n} \in x^{2n+r} \boxtimes L^n_{r,m}$, as desired.

To prove the statement when $m = r$, we can use the recurrence defining $D_{r,r}$ and copy the same method.

(II) We begin by showing $i^r \operatorname{Ad}(w_s)D_{r,m}x^{2n} \in x^{2n+r} \boxtimes L_{r,-m}$. We first check that $\Omega_H i^r \operatorname{Ad}(w_s)D_{r,m}x^{2n} = \lambda^n_{r,-m} i^r \operatorname{Ad}(w_s)D_{r,m}x^{2n}$. Since $\operatorname{Ad}(w_s)\Omega_H = \Omega_H$ and w_s fixes x,

$$\begin{aligned}\Omega_H i^r \operatorname{Ad}(w_s)D_{r,m}x^{2n} &= i^r(\operatorname{Ad}(w_s)\Omega_H)(\operatorname{Ad}(w_s)D_{r,m})x^{2n}\\ &= i^r w_s \Omega_H w_s^{-1} w_s D_{r,m} w_s^{-1} x^{2n}\\ &= i^r w_s \Omega_H D_{r,m} x^{2n}.\end{aligned}$$

Now, by (I) for r, $\Omega_H D_{r,m}x^{2n} = \lambda^n_{r,m} D_{r,m}x^{2n}$. Noting that $\lambda^n_{r,m} = \lambda^n_{r,-m}$,

$$\begin{aligned}\Omega_H i^r \operatorname{Ad}(w_s)D_{r,m}x^{2n} &= i^r w_s \lambda^n_{r,-m} D_{r,m}x^{2n}\\ &= \lambda^n_{r,-m} i^r w_s D_{r,m} w_s^{-1} x^{2n}\\ &= \lambda^n_{r,-m} i^r \operatorname{Ad}(w_s)D_{r,m}x^{2n}\end{aligned}$$

as needed. Next, by (I), $\operatorname{Ad}(w_s)D_{r,m}x^{2n} \in x^{2n+r} \boxtimes w_s L_{r,m} \subset \operatorname{Sym}^{2n+r}(V_2^{long}) \boxtimes \operatorname{Sym}^r(V_G)$. Recall the line $L_{r,m}$ is spanned by $\ell_{r,m} \in \operatorname{Sym}^r(V_G)$, and $\ell_{r,m}$ is homogeneous of degree r in $\mathbb{Q}[h_{-3}, h_{-1}, h_1, h_3]$. Since $w_s h_j = (-i)h_{-j}$ for $j = \pm 1, \pm 3$, we find $w_s L_{r,m}$ is some line with h_s eigenvalue equal to $-m$. Combining this with our computation of $\Omega_H i^r \operatorname{Ad}(w_s)D_{r,m}x^{2n}$, we have enough information to pin down $i^r \operatorname{Ad}(w_s)D_{r,m}x^{2n}$ lies in the desired component, that is $i^r \operatorname{Ad}(w_s)D_{r,m}x^{2n} \in x^{2n+r} \boxtimes L_{r,-m} \subset \pi^H_{n+\frac{r}{2},-m}$. To check that we have the correct normalization, we only need to realize that the coefficient of $h_{-1}^{(r+m)/2} h_1^{(r-m)/2}$ in $w_s \ell_{r,m}$ is equal to $(-i)^r$. So we conclude

$$i^r \operatorname{Ad}(w_s)D_{r,m}x^{2n} = x^{2n+r} \boxtimes \ell_{r,-m}.$$

(III) Here $m > -r$. Using (II),

$$\begin{aligned} D_{r,m} &= i^r Ad(w_s) D_{r,-m} \\ &= i^r Ad(w_s)(h_{-1} D_{r-1,-m+1} + A_{r,-m} h_{-3} D_{r-1,-m+3} + B_{r,-m} h_3 h_{-3} D_{r-2,-m}) \\ &= h_1 D_{r-1,m-1} + E_{r,m} h_3 D_{r-1,m-3} + F_{r,m} h_3 h_{-3} D_{r-2,m}. \end{aligned}$$

(IV) We handle separately the cases $m > -r$ and $m = -r$.

Case 1 of (IV), $m > -r$:

From the formula for $D_{r,m}$ from (III) and inductive hypothesis,

$$\begin{aligned} [f_s, D_{r,m}] &= 2h_{-1} D_{r-1,m-1} + h_1 [f_s, D_{r-1,m-1}] \\ &+ E_{r,m}(3h_1 D_{r-1,m-3} + h_3 [f_s, D_{r-1,m-3}]) + F_{r,m}(3h_{-3} h_1 D_{r-2,m} + h_{-3} h_3 [f_s, D_{r-2,m}]) \\ &= 2h_{-1} D_{r-1,m-1} + h_1 (U_{r-1,m-1} D_{r-1,m-3} + V_{r-1,m-1} h_{-3} D_{r-2,m}) \\ &+ 3E_{r,m} h_1 D_{r-1,m-3} + E_{r,m} h_3 (U_{r-1,m-3} D_{r-1,m-5} + V_{r-1,m-3} h_{-3} D_{r-2,m-2}) \\ &+ 3F_{r,m} h_{-3} h_1 D_{r-2,m} + F_{r,m} h_{-3} h_3 (U_{r-2,m} D_{r-2,m-2} + V_{r-2,m} h_{-3} D_{r-3,m+1}) \end{aligned}$$

We collect everything into the following groups of terms:

(a) $2h_{-1} D_{r-1,m-1} + h_{-3}((V_{r-1,m-1} + 3F_{r,m}) h_1 D_{r-2,m} + F_{r,m} V_{r-2,m} h_3 h_{-3} D_{r-3,m+1})$

(b) $(U_{r-1,m-1} + 3E_{r,m}) h_1 D_{r-1,m-3} + E_{r,m} U_{r-1,m-3} h_3 D_{r-1,m-5}$

(c) $(E_{r,m} V_{r-1,m-3} + F_{r,m} U_{r-2,m}) h_3 h_{-3} D_{r-2,m-2}$

and in each group apply the respective identities:

(a)

$$\begin{aligned} V_{r-1,m-1} + 3F_{r,m} &= V_{r,m} + 2A_{r,m-2} \\ F_{r,m} V_{r-2,m} &= (V_{r,m} + 2A_{r,m-2}) F_{r-1,m+1} \end{aligned}$$

(b)

$$E_{r,m}U_{r-1,m-3} = (U_{r-1,m-1} + 3E_{r,m})E_{r,m-2}$$

(c)

$$E_{r,m}V_{r-1,m-3}+F_{r,m}U_{r-2,m} = (V_{r,m}+2A_{r,m-2})E_{r-1,m+1}+2B_{r,m-2}+(U_{r-1,m-1}+3E_{r,m})F_{r,m-2}$$

Now,

- The second term in (a) can be written as

  $$(V_{r,m} + 2A_{r,m-2})h_{-3}(h_1D_{r-2,m} + F_{r-1,m+1}h_{-3}h_3D_{r-3,m+1})$$

  which, added to the $(V_{r,m} + 2A_{r,m-2})h_{-3}(E_{r-1,m+1}h_3D_{r-2,m-2})$ from (c), becomes

  $$(V_{r,m} + 2A_{r,m-2})h_{-3}(h_1D_{r-2,m} + E_{r-1,m+1}h_3D_{r-2,m-2} + F_{r-1,m+1}h_{-3}h_3D_{r-3,m+1}).$$

  This is exactly equal to

  $$(V_{r,m} + 2A_{r,m-2})h_{-3}D_{r-1,m+1}$$

  by Statement (III) for $r-1$. Add this to the first term in (a) and the $2B_{r,m-2}h_3h_{-3}D_{r-2,m-2}$ from (c) to arrive at

  $$2h_{-1}D_{r-1,m-1} + 2A_{r,m-2}h_{-3}D_{r-1,m+1} + 2B_{r,m-2}h_3h_{-3}D_{r-2,m-2} + V_{r,m}h_{-3}D_{r-1,m+1}$$
  $$= 2D_{r,m-2} + V_{r,m}h_{-3}D_{r-1,m+1}.$$

- Combining (b) with the remaining $(U_{r-1,m-1}+3E_{r,m})F_{r,m-2}h_3h_{-3}D_{r-2,m-2}$ in (c) yields

$$(U_{r-1,m-1}+3E_{r,m})(h_1D_{r-1,m-3}+E_{r,m-2}h_3D_{r-1,m-5}+F_{r,m-2}h_3h_{-3}D_{r-2,m-2}) \\ =(U_{r-1,m-1}+3E_{r,m})D_{r,m-2}.$$

In the last equality we used Statement (III) for r,

To finish, observe that $(U_{r-1,m-1}+3E_{r,m})+2=U_{r,m}$. So

$$2D_{r,m-2}+V_{r,m}h_{-3}D_{r-1,m+1}+(U_{r-1,m-1}+3E_{r,m})D_{r,m-2}=U_{r,m}D_{r,m-2}+V_{r,m}h_{-3}D_{r-1,m+1}$$

as needed.

Case 2 of (IV), $m=-r$:

$$\begin{aligned}[f_s,D_{r,-r}] &= h_{-3}D_{r-1,-r+1}+h_{-1}[f_s,D_{r-1,-r+1}]+A_{r,-r}h_{-3}[f_s,D_{r-1,-r+3}] \\ &= h_{-3}D_{r-1,-r+1}+h_{-1}(V_{r-1,-r+1}h_{-3}D_{r-2,-r+2}) \\ &\quad +A_{r,-r}h_{-3}(U_{r-1,-r+3}D_{r-1,-r+1}+V_{r-1,-r+3}h_{-3}D_{r-2,-r+4})\end{aligned}$$

Which we group into:

(a) $h_{-3}(V_{r-1,-r+1}h_{-1}D_{r-2,-r+2}+A_{r,-r}V_{r-1,-r+3}h_{-3}D_{r-2,-r+4})$

(b) $(1+A_{r,-r}U_{r-1,-r+3})h_{-3}D_{r-1,-r+1}$

In (a), we use the identity

$$A_{r,-r}V_{r-1,-r+3}=V_{r-1,-r+1}A_{r-1,-r+1}$$

to get

$$h_{-3}V_{r-1,-r+1}(h_{-1}D_{r-2,-r+2}+A_{r-1,-r+1}D_{r-2,-r+4})=V_{r-1,-r+1}h_{-3}D_{r-1,-r+1}$$

When we add this to (b), we get

$$(1 + A_{r,-r}U_{r-1,-r+3} + V_{r-1,-r+1})h_{-3}D_{r-1,-r+1} = V_{r,-r}h_{-3}D_{r-1,-r+1}$$

as desired (note $U_{r,-r} = 0$).

(V) We use (II) and (IV) for r. Note that $Ad(w_s)e_s = f_s$. So

$$\begin{aligned}
[e_s, D_{r,m}] &= [Ad(w_s)f_s, i^r Ad(w_s)D_{r,-m}] \\
&= i^r Ad(w_s)[f_s, D_{r,-m}] \\
&= i^r Ad(w_s)(U_{r,-m}D_{r,-m-2} + h_{-3}V_{r,-m}D_{r-1,-m+1}) \\
&= U_{r,-m}D_{r,m+2} + V_{r,-m}h_3 D_{r-1,m-1}
\end{aligned}$$

as desired. □

4.4 Operators for highest weight - analytic properties

The focus of this section is to prove Theorem 4.1.2. This will follow from our more comprehensive Theorem 4.4.8. Throughout this section, we will work with a fixed $\omega = \left(a, \frac{b}{3}, \frac{c}{3}, d\right) \in 2\pi W_G(\mathbb{Q})$.

4.4.1 Coordinates

We will work with coordinates for our Whittaker functions, following the notation and results of [Pol21]. Recall $M_G(\mathbb{R}) \cong GL_2(\mathbb{R})$. Let $B(\mathbb{R})^0$ be the neutral component of the Borel subgroup of $M_G(\mathbb{R})$. Take as coordinates for $B(\mathbb{R})^0$

$$(x, y, t) \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}, \text{ where } x \in \mathbb{R}, y \in \mathbb{R}^{\times}_{>0}, t \in \mathbb{R}^{\times}_{>0}.$$

Also write $z = x + iy$ and $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$.

For a given character $\omega \in 2\pi W_G(\mathbb{Q})$, the weight n Whittaker functions for G and their components $W^{G,v}_{\omega,n}$ are determined by their restriction to $B(\mathbb{R})^0$. This is due to their left N_G-equivariance and right K_G-equivariance properties. On these coordinates,

$$W^{G,v}_{\omega,n}(x,y,t) = t^{2n+2}\left(\frac{|p_\omega(z)|}{p_\omega(z)}\right)^v K_v(|p_\omega(z)|y^{-3/2}t).$$

On the other hand, recall that $M_H(\mathbb{R}) = S_H(\mathbb{R})A_H(\mathbb{R}) \cong U(1) \times \mathbb{R}^\times$ in its decomposition into a compact and split part. We take $t \in \mathbb{R}^\times_{>0}$ as a coordinate for $A_H(\mathbb{R})^0$. Then the Whittaker functions for H are determined by their restriction to $A_H(\mathbb{R})^0$. For $\mu \in 2\pi W_H(\mathbb{Q})$, and weight $(n + \frac{\epsilon}{2}, m)$,

$$W^{H,v}_{\mu,n+\frac{\epsilon}{2},m}(t) = t^{2n+\epsilon+2}\left(\frac{|p_\mu(i)|}{p_\mu(i)}\right)^{v+\frac{m}{2}} K_{v+\frac{m}{2}}(|p_\mu(i)|t).$$

In our embedding of H inside G, the neutral component $A_H(\mathbb{R})^0 = M_H(\mathbb{R}) \cap B(\mathbb{R})^0 = \left\{\left(\begin{smallmatrix} t & 0 \\ 0 & t \end{smallmatrix}\right), t \in \mathbb{R}^\times_{>0}\right\}$. Suppose that $\epsilon = 0$, $m = 0$, and $\mu = pr(\omega)$ or in other words $\chi_\mu = \chi|_{N_H}$; Then we have $W^{H,v}_{\mu,n}(t) = W^{G,v}_{\omega,n}(x,y,t)|_{A_H(\mathbb{R})} = W^{G,v}_{\omega,n}(x,y,t)|_{x=0,y=1}$.

On these coordinates, we have explicit formulas for the actions of $\mathfrak{p}$. First of all, by right K_G-equivariance, for any $m \in M_G(\mathbb{R})$

- $h_u W^{G,v}_{\omega,n}(m) = -2v W^{G,v}_{\omega,n}(m)$ for any $-n \le v \le n$
- $k_s W^{G,v}_{\omega,n}(m) = 0$ for all $k_s \in \{e_s, h_s, f_s\}$ for any $-n \le v \le n$
- $e_u W^{G,-n}_{\omega,n}(m) = 0$

Also, since $W^G_{\omega,n}(nm) = e^{-i\langle\omega,n\rangle}W^{G,v}_{\omega,n}(m)$, it follows that for any $X \equiv (a', \frac{b'}{3}, \frac{c'}{3}, d') \in N_G/[N_G, N_G]$ and any $m \in M_G(\mathbb{R})$,

$$(XW^G_{\omega,n})(m) = -i\langle\omega, m \cdot p_X\rangle W^{G,v}_{\omega,n}(m)$$

Next, following [Pol21], we have the following Iwasawa decompositions (modulo $\mathbb{C}E_{13}$, which acts trivially on anything in $U(\mathfrak{g}) \cdot W^G_{\omega,n}$):

- $h_3 \equiv p_3 + k_3$, where $p_3 = -2(\epsilon_1 + \epsilon_2)$ and $k_3 = -(h_u + h_s)$
- $h_1 \equiv p_1 + k_1$, where $p_1 = 2(v + iu)^2(v - iu)$ and $k_1 = -\frac{4}{3}f_s$
- $h_{-1} \equiv p_{-1} + k_{-1}$, where $p_{-1} = \frac{2}{3}(\epsilon_1 - \epsilon_2 - 2iv_2)$ and $k_{-1} = \frac{1}{3}(3h_u - h_s)$
- $h_{-3} \equiv p_{-3} + k_{-3}$, where $p_{-3} = -2(v + iu)^3$ and $k_{-3} = -4e_u$

One can calculate the effect of these differential operators on the $W^{G,v}_{\omega,n}(x, y, t)$: For any $\phi(x, y, t) : B(\mathbb{R})^0 \to \mathbb{C}$ smooth,

- $(\epsilon_1 + \epsilon_2)\phi = t\partial_t\phi$
- $(\epsilon_1 - \epsilon_2)\phi = 2y\partial_y\phi$
- $v_2\phi = y\partial_x\phi$.

Let us also define the following auxiliary functions on $P_G(\mathbb{R})$:

Definition 4.4.1.

- $Z^\omega_1(nm) := -i\langle\omega, m \cdot p_1\rangle$
- $Z^\omega_{-1}(nm) := -i\langle\omega, m \cdot -\overline{p_1}\rangle$
- $Z^\omega_{-3}(nm) := -i\langle\omega, m \cdot \overline{p_{-3}}\rangle$.
- $Z^\omega_3(nm) := -i\langle\omega, m \cdot -p_{-3}\rangle$.

Note that these Z_j, when restricted to $m = (x, y, t) \in B(\mathbb{R})^0$, do not depend on t.

Lemma 4.4.2. *On $B(\mathbb{R})^0$,*

1. $Z^\omega_{-3}(z) := -2y^{-3/2}p_\omega(z^*)$

2. $Z^\omega_{-1}(z) := \frac{4}{3} i y^{5/2} \partial_{z^*}(p_\omega(z^*) y^{-3})$

3. $Z^\omega_1(z) := \frac{4}{3} i y^{5/2} \partial_z(p_\omega(z) y^{-3})$

4. $Z^\omega_3(z) := 2y^{-3/2} p_\omega(z)$

Proof. This is [Pol21] Lemma 4.5 with some rescaling. □

Next define functions $G^\omega_v : B(\mathbb{R})^0 \to \mathbb{C}$ by $W^{G,v}_{\omega,n}(x, y, t) = t^{2n+2} G^\omega_v(x, y, t)$, i.e.

$$G^\omega_v(x, y, t) = \left(\frac{|p_\omega(z)|}{p_\omega(z)} \right)^v K_v(|p_\omega(z)| y^{-3/2} t)$$

Lemma 4.4.3. *On $B(\mathbb{R})^0$, $p_{-1} = -\frac{8}{3} i y \partial_{\bar{z}}$ and $p_3 = -2t\partial_t$, and*

1. $-\frac{8}{3} i y \partial_{\bar{z}} G^\omega_v = Z^\omega_{-1} t G^\omega_{v-1} + 2v G^\omega_v$

2. $-2t\partial_t G^\omega_v = -Z^\omega_{-3} t G^\omega_{v-1} + 2v G^\omega_v$

3. $-2t\partial_t G^\omega_v = Z^\omega_3 t G^\omega_{v+1} - 2v G^\omega_v$

4. $-Z^\omega_3 t G^\omega_v = Z^\omega_{-3} t G^\omega_{v-2} - 4(v-1) G^\omega_{v-1}$

Proof. Items 1, 2, and 3 are properties of K-Bessel functions; see section 4 of [Pol21], especially Lemma 4.5 and Corollary 4.6. Item 4 follows immediately from 2 and 3. □

Lemma 4.4.4.

1. $p_1 X W^{G,-n}_{\omega,n}(m) = Z^\omega_1(m) t \cdot X W^{G,-n}_{\omega,n}(m)$

2. $p_{-3} X W^{G,-n}_{\omega,n}(m) = -Z^\omega_3(m) t \cdot X W^{G,-n}_{\omega,n}(m)$

for all $X \in U(\mathfrak{g})$ and $m \in B(\mathbb{R})^0$.

Proof. Follows from left N_G-equivariance. □

Finally we will need the relations

Lemma 4.4.5. *On $B(\mathbb{R})^0$,*

1. $p_{-1}(Z^{\omega}_{-3}) = 2Z^{\omega}_{-3} + 4Z^{\omega}_{-1}$

2. $p_{-1}(Z^{\omega}_{-1}) = \frac{2}{3}Z^{\omega}_{-1} + \frac{8}{3}Z^{\omega}_{1}$

3. $p_{-1}(Z^{\omega}_{1}) = -\frac{2}{3}Z^{\omega}_{1} + \frac{4}{3}Z^{\omega}_{3}$

4. $p_{-1}(Z^{\omega}_{3}) = -2Z^{\omega}_{3}$

Proof. Direct computation. For example,

$$\begin{aligned} p_{-1}(Z^{\omega}_{-3}) &= -\frac{8}{3}iy\partial_{\bar{z}}(-2y^{-3/2}p_{\omega}(\bar{z})) \\ &= \frac{16}{3}iy\left(y^{-3/2}\partial_{\bar{z}}p_{\omega}(\bar{z}) + p_{\omega}(\bar{z})\partial_{\bar{z}}(y^{-3/2})\right) \\ &= \frac{16}{3}iy^{-1/2}\partial_{\bar{z}}p_{\omega}(\bar{z}) + 4y^{-3/2}p_{\omega}(\bar{z}) \end{aligned}$$

and on the other hand

$$\begin{aligned} 4Z^{\omega}_{-1} &= \frac{16}{3}iy^{5/2}\partial_{\bar{z}}(p_{\omega}(\bar{z})y^{-3}) \\ &= \frac{16}{3}iy^{5/2}\left(y^{-3}\partial_{\bar{z}}p_{\omega}(\bar{z}) - \frac{3}{2}iy^{-4}p_{\omega}(\bar{z})\right) \\ &= \frac{16}{3}iy^{-1/2}\partial_{\bar{z}}p_{\omega}(\bar{z}) + 8y^{-3/2}p_{\omega}(\bar{z}) \end{aligned}$$

so

$$p_{-1}(Z^{\omega}_{-3}) = 2Z^{\omega}_{-3} + 4Z^{\omega}_{-1}$$

as desired. The other identities can be deduced in the same manner. □

Definition 4.4.6.

Define $P^{n,\omega}_{0,0} = 1, P^{n,\omega}_{1,1} = Z^{\omega}_{1}, P^{n,\omega}_{1,-1} = Z^{\omega}_{-1}$. Then, define $P^{n,\omega}_{r,m}$ by the recurrence formulas.

$$P^{n,\omega}_{r,r} = Z^{\omega}_{1}P^{n,\omega}_{r-1,r-1} + E^{n}_{r,r}Z^{\omega}_{3}P^{n,\omega}_{r-1,r-3}$$

and

$$P^{n,\omega}_{r,m} = Z^{\omega}_{-1}P^{n,\omega}_{r-1,m+1} + A^{n}_{r,m}Z^{\omega}_{-3}P^{n,\omega}_{r-1,m+3} + B^{n}_{r,m}Z^{\omega}_{3}Z^{\omega}_{-3}P^{n,\omega}_{r-2,m}$$

for $m \in \{-r, -r+2, \ldots, r-2\}$.

4.4.2 Main result

In this section we calculate the action of $D_{r,m}$ on

$$W^{G,-n}_{\omega,n} = \langle W^{G}_{\omega,n}, x^{2n} \rangle.$$

Recall we are identifying $D_{r,m}$ with a homogeneous degree r polynomial in $\mathbb{Q}[h_{-3}, h_{-1}, h_1, h_3]$. So we need to make sure this action is well-defined:

Lemma 4.4.7. *Let $F : G(\mathbb{R}) \to \mathbb{V}^{\vee}_{n}$ be a smooth, right K_G-equivariant function. If $Y \in J$, then*

$$Y\langle F, x^{2n} \rangle = 0$$

Proof. We may assume Y is a monomial. By the Poincaré-Birkhoff-Witt theorem, we can write $Y = Y_1 e^{s}_{u}$ with $Y_1 \in \mathfrak{p}^{\otimes r}_{+}$ for some $r \geq 0$ and $s > 0$. Then,

$$Y\langle F, x^{2n} \rangle = -Y_1 \langle F, e^{s}_{u} x^{2n} \rangle = 0.$$

□

We are now ready to state the main result of this section:

Theorem 4.4.8 (Analytic properties of $D^{n}_{r,m}$)**.** *Let $n \geq 1$ be an integer, and $\omega \in W_G(\mathbb{R})$. Then, for any $r \geq 0$ and $m \in \{-r, -r+2, \ldots, r-2, r\}$,*

(I) On the group $B(\mathbb{R})^0$,

$$D^{n}_{r,m}(W^{G,-n}_{\omega,n}) = P^{n,\omega}_{r,m}(z) t^{2n+r+2} G^{\omega}_{-n-(\frac{r-m}{2})}(x, y, t)$$

(II) For $m > -r$, there is the additional recurrence formula

$$P^{n,\omega}_{r,m} = Z^{\omega}_{1} P^{n,\omega}_{r-1,m-1} + E^{n}_{r,m} Z^{\omega}_{3} P^{n,\omega}_{r-1,m-3} + F^{n}_{r,m} Z^{\omega}_{3} Z^{\omega}_{-3} P^{n,\omega}_{r-2,m}$$

(III) Define $S^{n}_{r,m}$ for $r \geq 0$ and $m \in \{-r, -r+2, \ldots, r-2\}$ by $S^{n}_{0,0} = S^{n}_{1,1} = S^{n}_{1,-1} = 0$, and for $r \geq 2$

$$\begin{aligned} S^{n}_{r,m} := \frac{2}{3}(m+1) P^{n,\omega}_{r-1,m+1} + p_{-1}(P^{n,\omega}_{r-1,m+1}) \\ + 2A^{n}_{r,m}(2n+r-m-2) P^{n,\omega}_{r-1,m+3} + 4B^{n}_{r,m}(2n+r-1)(Z_3 P^{n,\omega}_{r-2,m}). \end{aligned}$$

Then, $S^{n}_{r,m} = 0$ for all r and m.

Proof. Since n and ω are fixed throughout, we will often omit them from the notation.

Outline: We proceed by induction on r. We may verify directly that the theorem holds for $r = 0$ and $r = 1$.

Suppose now that $r \geq 2$ and we have Statements (I), (II), and (III) for all $r' < r$.

We first prove (III) for r, which requires (II) and (III) for r'. Next we prove (I) for r, which will require (III) for r as well as the inductive hypothesis. Then we finally prove (II), which will require (I) for r as well as the inductive hypothesis.

(III) We handle separately the three cases $m \in \{-r, -r+2, \ldots, r-6\}$ and $m = r-4$ and $m = r-2$, due to the different recurrence formulas defining $P_{r-1,m+1}$ and $P_{r-1,m+3}$ in these cases.

Case 1 of (III), $m \in \{-r, -r+2, \ldots, r-6\}$:

We will spell out the computations for this case in more detail. The remaining cases utilize the same techniques. The other cases are handled similarly. Set $C_{r,m} = 4B_{r,m}(2n+r-1) -$

$\frac{8}{3}E_{r-1,m+3}$. Then we can write

$$S_{r,m} = \frac{2}{3}(m+1)P_{r-1,m+1} + p_{-1}(P_{r-1,m+1}) \\ + 2A_{r,m}(2n+r-m-2)P_{r-1,m+3} + C_{r,m}(Z_3P_{r-2,m}) \\ + \frac{8}{3}E_{r-1,m+3}(Z_3P_{r-2,m}).$$

In the first two lines above, plug in the recurrence formulas

- $P_{r-1,m+1} = Z_{-1}P_{r-2,m+2} + A_{r-1,m+1}Z_{-3}P_{r-2,m+4} + B_{r-1,m+1}Z_3Z_{-3}P_{r-3,m+1}$
- $P_{r-1,m+3} = Z_{-1}P_{r-2,m+4} + A_{r-1,m+3}Z_{-3}P_{r-2,m+6} + B_{r-1,m+3}Z_3Z_{-3}P_{r-3,m+3}$
- $P_{r-2,m} = Z_{-1}P_{r-3,m+1} + A_{r-2,m}Z_{-3}P_{r-2,m+3} + B_{r-2,m}Z_3Z_{-3}P_{r-4,m}$

at all occurrences; leave the $\frac{8}{3}E_{r-1,m+3}(Z_3P_{r-2,m})$ as is.

For the $p_{-1}(P_{r-1,m+1})$ term, we compute

$$p_{-1}(P_{r-1,m+1}) = p_{-1}(Z_{-1}P_{r-2,m+2} + A_{r-1,m+1}Z_{-3}P_{r-2,m+4} + B_{r-1,m+1}Z_3Z_{-3}P_{r-3,m+1}) \\ = p_{-1}(Z_{-1})P_{r-2,m+2} + Z_{-1}p_{-1}(P_{r-2,m+2}) \\ + A_{r-1,m+1}\left[p_{-1}(Z_{-3})P_{r-2,m+4} + Z_{-3}p_{-1}(P_{r-2,m+4})\right] \\ + B_{r-1,m+1}\left[p_{-1}(Z_{-3}Z_3)P_{r-3,m+1} + Z_{-3}Z_3p_{-1}(P_{r-3,m+1})\right].$$

Applications of Lemma 4.4.5 yields

$$p_{-1}(P_{r-1,m+1}) = \frac{2}{3}Z_{-1}P_{r-2,m+2} + \frac{8}{3}Z_1P_{r-2,m+2} + Z_{-1}p_{-1}(P_{r-2,m+2}) \\ + A_{r-1,m+1}\left[2Z_{-3}P_{r-2,m+4} + 4Z_{-1}P_{r-2,m+4} + Z_{-3}p_{-1}(P_{r-2,m+4})\right] \\ + B_{r-1,m+1}\left[2Z_{-3}Z_3P_{r-3,m+1} + 4Z_{-1}Z_3P_{r-3,m+1} - 2Z_{-3}Z_3P_{r-3,m+1} + Z_{-3}Z_3p_{-1}(P_{r-3,m+1}).\right]$$

When all is said and done, one is left with the sum of the following groups of terms:

(a) $(2A_{r,m}(2n+r-m-2)B_{r-1,m+3}+C_{r,m}A_{r-2,m})Z_3Z_{-3}P_{r-3,m+3}$

(b) $(4A_{r-1,m+1}+2A_{r,m}(2n+r-m-2))Z_{-1}P_{r-2,m+4}$

(c) $2A_{r,m}(2n+r-m-2)A_{r-1,m+3}Z_{-3}P_{r-2,m+6}$

(d) $\frac{2}{3}(m+2)Z_{-1}P_{r-2,m+2}+Z_{-1}p_{-1}(P_{r-2,m+2})+(4B_{r-1,m+1}+C_{r,m})Z_{-1}Z_3P_{r-3,m+1}$

(e)

$$\frac{2}{3}(m+1)B_{r-1,m+1}Z_{-3}Z_3P_{r-3,m+1}+B_{r-1,m+1}Z_{-3}Z_3p_{-1}(P_{r-3,m+1}) + C_{r,m}B_{r-2,m}Z_{-3}Z_3Z_3P_{r-4,m}$$

(f) $\frac{2}{3}(m+4)A_{r-1,m+1}Z_{-3}P_{r-2,m+4}+A_{r-1,m+1}Z_{-3}p_{-1}(P_{r-2,m+4})$

(g) $\frac{8}{3}Z_1P_{r-2,m+2}+\frac{8}{3}E_{r-1,m+3}Z_3P_{r-2,m}$

In groups (a) through (e), we invoke the following readily checked identities involving the coefficients:

(a)

$$2A_{r,m}(2n+r-m-2)B_{r-1,m+3}+C_{r,m}A_{r-2,m}=\frac{8}{3}F_{r-1,m+3}-\frac{8}{3}B_{r-1,m+3} + A_{r-1,m+1}\cdot 4B_{r-1,m+3}(2n+(r-1)-1) + B_{r-1,m+1}\cdot 2A_{r-2,m}(2n+(r-2)-m-2)$$

(b) $4A_{r-1,m+1}+2A_{r,m}(2n+r-m-2)=-\frac{8}{3}+2A_{r-1,m+1}(2n+(r-1)-(m+1)-2)$

(c) $2A_{r,m}(2n+r-m-2)A_{r-1,m+3}=-\frac{8}{3}A_{r-1,m+3}+A_{r-1,m+1}\cdot 2A_{r-1,m+3}(2n+(r-1)-(m+3)-2)$

(d) $4B_{r-1,m+1}+C_{r,m}=4B_{r-1,m+1}(2n+(r-1)-1)$

(e) $C_{r,m}B_{r-2,m} = B_{r-1,m+1} \cdot 4B_{r-2,m}(2n + (r-2) - 1)$

Now,

- The $\frac{8}{3}F_{r-1,m+3}Z_3Z_{-3}P_{r-3,m+3}$ from (a), added to (g), gives

$$\frac{8}{3}Z_1P_{r-2,m+2} + \frac{8}{3}E_{r-1,m+3}Z_3P_{r-2,m} + \frac{8}{3}F_{r-1,m+3}Z_3Z_{-3}P_{r-3,m+3},$$

  which is exactly $\frac{8}{3}P_{r-1,m+3}$ by (II) for $r-1$.

- The $-\frac{8}{3}B_{r-1,m+3}Z_3Z_{-3}P_{r-3,m+3}$ from (a), added to the $-\frac{8}{3}Z_{-1}P_{r-2,m+4}$ from (b), and the $-\frac{8}{3}A_{r-1,m+3}Z_{-3}P_{r-2,m+6}$ from (c), gives $-\frac{8}{3}P_{r-1,m+3}$ by the recurrence formula defining $P_{r-1,m+3}$.

- All the terms in (d), added to the $2A_{r-1,m+1}(2n + (r-1) - (m+1) - 2)Z_{-1}P_{r-2,m+4}$ from (b), combine to give $Z_{-1} \cdot S_{r-1,m+1}$, which vanishes by inductive hypothesis.

- All the terms in (e), added to the $B_{r-1,m+1} \cdot 2A_{r-2,m}(2n + (r-2) - m - 2)Z_3Z_{-3}P_{r-3,m+3}$ from (a), combined to give $B_{r-1,m+1}Z_3Z_{-3} \cdot S_{r-2,m}$,which vanishes by inductive hypothesis.

- All the terms in (f), added to the $A_{r-1,m+1} \cdot 4B_{r-1,m+3}(2n + (r-1) - 1)Z_3Z_{-3}P_{r-3,m+3}$ from (a), added to the $A_{r-1,m+1} \cdot 2A_{r-1,m+3}(2n + (r-1) - (m+3) - 2)Z_{-3}P_{r-2,m+6}$ from (c), combine to give $A_{r-1,m+1}Z_{-3} \cdot S_{r-1,m+3}$, which vanishes by inductive hypothesis.

All terms are accounted for in the list above, and we are left with

$$S_{r,m} = \frac{8}{3}P_{r-1,m+3} - \frac{8}{3}P_{r-1,m+3} = 0$$

as desired.

Case 2 of (III), $m = r-2$:

We wish to show

$$S_{r,r-2} = \frac{2}{3}(r-1)P_{r-1,r-1} + p_{-1}(P_{r-1,r-1}) + 4B_{r,r-2}(2n + r - 1)(Z_3P_{r-2,r-2}) = 0.$$

Set

$$C = 4B_{r,r-2}(2n + r - 1) - 4B_{r-1,r-3}(2n + r - 2),$$

so that

$$S_{r,r-2} = \frac{2}{3}(r-1)P_{r-1,r-1} + p_{-1}(P_{r-1,r-1}) + 4B_{r-1,r-3}(2n + r - 2)Z_3P_{r-2,r-2} + CZ_3P_{r-2,r-2}.$$

We will plug in the recurrence formula of 4.4.6 for $P_{r-1,r-1}$ and $P_{r-2,r-2}$ in everything but the last summand above. For example, using the recurrence formula

$$P_{r-1,r-1} = Z_1P_{r-2,r-2} + E_{r-1,r-1}Z_3P_{r-2,r-4},$$

we compute with Lemma 4.4.5

$$p_{-1}(P_{r-1,r-1}) = -\frac{2}{3}Z_1P_{r-2,r-2} + \frac{4}{3}Z_3P_{r-2,r-2} + Z_1p_{-1}(P_{r-2,r-2}) \\ - 2Z_3E_{r-1,r-1}P_{r-2,r-4} + Z_3E_{r-1,r-1}p_{-1}(P_{r-2,r-4}).$$

In the end, $S_{r,r-2}$ is the sum of the following groups of terms:

(a)

$$\frac{2}{3}(r-2)Z_1P_{r-2,r-2} + Z_1p_{-1}(P_{r-2,r-2}) + 4B_{r-1,r-3}(2n + r - 2)Z_1Z_3P_{r-3,r-3}$$

(b)

$$\frac{2}{3}(r-4)Z_3E_{r-1,r-1}P_{r-2,r-4} + Z_3E_{r-1,r-1}p_{-1}(P_{r-2,r-4}) \\ + (\frac{4}{3} + C)Z_3P_{r-2,r-2} + 4B_{r-1,r-3}(2n + r - 2)E_{r-2,r-2}Z_3Z_3P_{r-3,r-5}$$

Each group vanishes:

(a) This is $Z_1S_{r-1,r-3}$, which vanishes by inductive hypothesis.

(b) We must notice that

$$B_{r-1,r-3}E_{r-2,r-2} = B_{r-1,r-5}E_{r-1,r-1}$$

and

$$\frac{4}{3} + C = E_{r-1,r-1} \cdot 2A_{r-1,r-5}(2n + (r-1) - (r-5) - 2)$$

to see that these terms can be expressed as

$$Z_3 E_{r-1,r-1} \cdot S_{r-1,r-5}$$

which again vanishes by inductive hypothesis.

Therefore $S_{r,r-2} = 0$, as desired.

Case 3 of (III), $m = r - 4$:

Note that

$$2A_{r,r-4}(2n + r - (r-4) - 2) = -\frac{8}{3},$$

so that

$$S_{r,r-4} = \frac{2}{3}(r-3)P_{r-1,r-3} + p_{-1}(P_{r-1,r-3}) - \frac{8}{3}P_{r-1,r-1} + 4B_{r,r-4}(2n + r - 1)Z_3 P_{r-2,r-4}.$$

Set

$$C' = 4B_{r,r-4}(2n + r - 1) - \frac{8}{3}E_{r-1,r-1}$$

so that

$$S_{r,r-4} = \frac{2}{3}(r-3)P_{r-1,r-3} + p_{-1}(P_{r-1,r-3}) - \frac{8}{3}P_{r-1,r-1} + C'Z_3 P_{r-2,r-4} + \frac{8}{3}E_{r-1,r-1}Z_3 P_{r-2,r-4}$$

First, note $A_{r-1,r-3} = 0$, so we can write

$$p_{-1}(P_{r-1,r-3}) = p_{-1}(Z_{-1}P_{r-2,r-2} + B_{r-1,r-3}Z_3Z_{-3}P_{r-3,r-3})$$
$$= \frac{2}{3}Z_{-1}P_{r-2,r-2} + \frac{8}{3}Z_1P_{r-2,r-2} + Z_{-1}p_{-1}(P_{r-2,r-2}) + p_{-1}(B_{r-1,r-3}Z_3Z_{-3}P_{r-3,r-3}). \quad (\clubsuit)$$

The $\frac{8}{3}Z_1P_{r-2,r-2}$ from above, along with the $\frac{8}{3}E_{r-1,r-1}Z_3P_{r-2,r-4}$ and $-\frac{8}{3}P_{r-1,r-1}$ from $S_{r,r-4}$ sum to zero. Indeed, this follows from the definition

$$P_{r-1,r-1} = Z_1P_{r-2,r-2} + E_{r-1,r-1}Z_3P_{r-2}.$$

Now in the remaining terms of $S_{r,r-4}$, namely in

$$\frac{2}{3}(r-3)P_{r-1,r-3} + C'Z_3P_{r-2,r-4},$$

plug in the recurrence formulas defining $P_{r-1,r-3}$ and $P_{r-2,r-4}$. Combined with the terms remaining from our expansion of $p_{-1}(P_{r-1,r-3})$ in ($\clubsuit$), we are left with the following two groups of terms:

(a)
$$\frac{2}{3}(r-2)Z_{-1}P_{r-2,r-2} + Z_{-1}p_{-1}(P_{r-2,r-2}) + (4B_{r-1,r-3} + C')Z_{-1}Z_3P_{r-3,r-3}$$

(b)
$$\frac{2}{3}(r-3)Z_3Z_{-3}B_{r-1,r-3}P_{r-3,r-3} + Z_3Z_{-3}B_{r-1,r-3}p_{-1}(P_{r-3,r-3})$$
$$+ C'B_{r-2,r-4}Z_3Z_{-3}Z_3P_{r-4,r-4}$$

Each group vanishes:

(a) Noting that

$$4B_{r-1,r-3} + C' = 4B_{r-1,r-3}(2n + (r-1) - 1),$$

these terms combine to give $Z_{-1} \cdot S_{r-1,r-3}$ which vanishes by inductive hypothesis.

(b) Noting that

$$C'B_{r-2,r-4} = B_{r-1,r-3} \cdot 4B_{r-2,r-4}(2n + (r-2) - 1),$$

these terms combine to give $B_{r-1,r-3}Z_3Z_{-3} \cdot S_{r-2,r-4}$, which vanishes by inductive hypothesis.

Therefore $S_{r,r-4} = 0$, as desired.

(I) We handle separately the cases $m \in \{-r, -r+2, \ldots, r-2\}$ and $m = r$.

Case 1 of (I), $m \in \{-r, -r+2, \ldots, r-2\}$:

We will spell out the computations for this case in more detail. The remaining cases utilize the same techniques. We compute $D_{r,m}(W_{\omega,n}^{G,-n})$ via its definition

$$D_{r,m} = h_{-1}D_{r-1,m+1} + A_{r,m}h_{-3}D_{r-1,m+3} + B_{r,m}h_{-3}h_3D_{r-2,m}$$

and inductive hypothesis.

Recall that the $P_{r,m}$ are functions on $P_G(\mathbb{R})$. To calculate

$$h_{-1}D_{r-1,m+1}(W_{\omega,n}^{G,-n}) = p_{-1}(D_{r-1,m+1}(W_{\omega,n}^{G,-n})) + k_{-1}(D_{r-1,m+1}(W_{\omega,n}^{G,-n})) \qquad (\dagger)$$

we differentiate the first summand and invoke equivariance in the second. In the first summand of (†), by inductive hypothesis

$$D_{r-1,m+1}(W_{\omega,n}^{G,-n}) = P_{r-1,m+1}t^{2n+(r-1)+2}G_{-n-(\frac{r-m}{2})+1}.$$

We differentiate the right hand side by p_{-1}. By Lemma 4.4.3 (1),

$$\begin{aligned} p_{-1}(P_{r-1,m+1}t^{2n+(r-1)+2}G_{-n-(\frac{r-m}{2})+1}) &= (Z_{-1}P_{r-1,m+1})t^{2n+r+2}G_{-n-(\frac{r-m}{2})} \\ &+ 2(-n-(\frac{r-m}{2})+1)P_{r-1,m+1}t^{2n+(r-1)+2}G_{-n-(\frac{r-m}{2})+1} \\ &+ p_{-1}(P_{r-1,m+1})t^{2n+(r-1)+2}G_{-n-(\frac{r-m}{2})+1}. \end{aligned}$$

For the second summand of (†), recall $k_{-1} = \frac{1}{3}(3h_u - h_s)$ and $D_{r-1,m+1}$ is a homogeneous polynomial in $\mathbb{Q}[h_{-3}, h_{-1}, h_1, h_3]$, all of whose terms have h_u eigenvalue $r-1$ and h_s eigenvalue $m+1$. Therefore

$$k_{-1}D_{r-1,m+1} = [k_{-1}, D_{r-1,m+1}] + D_{r-1,m+1}k_{-1} = ((r-1) - (\frac{m+1}{3}))D_{r-1,m+1}.$$

Furthermore, h_s annihilates $W_{\omega,n}^{-n}$ and h_u acts as $2n$. Putting this together,

$$\begin{aligned} k_{-1}(D_{r-1,m+1}W_{\omega,n}^{-n}) &= [k_{-1}, D_{r-1,m+1}](W_{\omega,n}^{-n}) + D_{r-1,m+1}(k_{-1}W_{\omega,n}^{-n}) \\ &= (2n + (r-1) - (\frac{m+1}{3}))P_{r-1,m+1}t^{2n+(r-1)+2}G_{-n-(\frac{r-m}{2})+1}. \end{aligned}$$

Overall,

$$\begin{aligned} h_{-1}D_{r-1,m+1}(W_{\omega,n}^{-n}) &= (Z_{-1}P_{r-1,m+1})t^{2n+r+2}G_{-n-(\frac{r-m}{2})} \\ &+ 2(-n-(\frac{r-m}{2})+1)P_{r-1,m+1}t^{2n+(r-1)+2}G_{-n-(\frac{r-m}{2})+1} \\ &+ p_{-1}(P_{r-1,m+1})t^{2n+(r-1)+2}G_{-n-(\frac{r-m}{2})+1} \\ &+ (2n + (r-1) - (\frac{m+1}{3}))P_{r-1,m+1}t^{2n+(r-1)+2}G_{-n-(\frac{r-m}{2})+1}. \end{aligned}$$

We follow the same process to compute $h_{-3}D_{r-1,m+3}(W_{\omega,n}^{-n})$ and $h_{-3}h_3D_{r-2,m}(W_{\omega,n}^{-n})$. For $h_{-3}D_{r-1,m+3}(W_{\omega,n}^{-n})$, notice that k_{-3} is a scalar multiple of e_u, which has the properties

$[e_u, D_{r-1,m+3}] = 0$ and $e_u W_{\omega,n}^{-n} = 0$. Therefore, applying Lemma 4.4.4 (2),

$$\begin{aligned} h_{-3}D_{r-1,m+3}(W_{\omega,n}^{G,-n}) &= p_{-3}D_{r-1,m+3}(W_{\omega,n}^{G,-n}) \\ &= -Z_3(P_{r-1,m+3}t^{2n+(r-1)+2}G_{-n-(\frac{r-m}{2})+2}). \end{aligned}$$

Now Lemma 4.4.3 (4) says

$$\begin{aligned} &h_{-3}D_{r-1,m+3}(W_{\omega,n}^{G,-n}) = -Z_3(P_{r-1,m+3}t^{2n+(r-1)+2}G_{-n-(\frac{r-m}{2})+2}) \\ &= (Z_{-3}P_{r-1,m+3})t^{2n+r+2}G_{-n-(\frac{r-m}{2})} - 4(-n-(\frac{r-m}{2})+1)P_{r-1,m+3}t^{2n+(r-1)+2}G_{-n-(\frac{r-m}{2})+1}. \end{aligned}$$

For $h_{-3}h_3D_{r-2,m}(W_{\omega,n}^{-n})$, we first compute the differentiation by h_3. We use Lemma 4.4.3 (3) for the p_3 part and a K_G-computation for the k_3 part to obtain

$$\begin{aligned} h_3D_{r-2,m}(W_{\omega,n}^{G,-n}) &= p_3(P_{r-2,m}t^{2n+(r-2)+2}G_{-n-(\frac{r-m}{2})+1}) + k_3(D_{r-2,m}(W_{\omega,n}^{G,-n})) \\ &= -Z_{-3}P_{r-2,m}t^{2n+(r-1)+2}G_{-n-(\frac{r-m}{2})} - 2(2n+(r-2)+2)P_{r-2,m}t^{2n+(r-2)+2}G_{-n-(\frac{r-m}{2})+1} \\ &\quad + 2(-n-(\frac{r-m}{2})+1)P_{r-2,m}t^{2n+(r-2)+2}G_{-n-(\frac{r-m}{2})+1} \\ &\quad - (2n+(r-2)+m)P_{r-2,m}t^{2n+(r-2)+2}G_{-n-(\frac{r-m}{2})+1}. \end{aligned}$$

Next we again take advantage of k_{-3} annihilating $D_{r-2,m}$ and $W_{\omega,n}^{-n}$, and apply Lemma 4.4.4 (2) to our expansion of $h_3D_{r-2,m}(W_{\omega,n}^{-n})$ to find

$$\begin{aligned} h_{-3}h_3D_{r-2,m}(W_{\omega,n}^{G,-n}) &= p_{-3}h_3D_{r-2,m}(W_{\omega,n}^{G,-n}) \\ &= Z_3Z_{-3}P_{r-2,m}t^{2n+r+2}G_{-n-(\frac{r-m}{2})} \\ &\quad + 4Z_3P_{r-2,m}(2n+r-1)t^{2n+(r-2)+3}G_{-n-(\frac{r-m}{2})+1} \quad (\star) \end{aligned}$$

Collecting terms, we get

$$D_{r,m}(W_{\omega,n}^{G,-n}) = P_{r,m}t^{2n+r+2}G_{-n-\frac{(r-m)}{2}} + (S_{r,m})t^{2n+r+1}G_{-n-(\frac{r-m}{2})+1}$$

where $S_{r,m}$ is

$$\frac{2}{3}(m+1)P_{r-1,m+1}+p_{-1}(P_{r-1,m+1})+4A_{r,m}(n+(\frac{r-m}{2})-1)P_{r-1,m+3}+4B_{r,m}(2n+r-1)(Z_3P_{r-2,m})$$

and vanishes by Statement (III) for r.

Case 2 of (I), $m = r$:

We use the recurrence formula of Theorem 4.3.3 (III) and compute in general

$$D_{r,m}(W_{\omega,n}^{G,-n}) = (h_1D_{r-1,m-1} + E_{r,m}h_3D_{r-1,m-3} + F_{r,m}h_3h_{-3}D_{r-2,m})(W_{\omega,n}^{G,-n}).$$

As in Case 1, we proceed summand by summand. For $h_1D_{r-1,m-1}(W_{\omega,n}^{G,-n})$, notice that $f_sW_{\omega,n}^{G,-n} = 0$. So

$$f_sD_{r-1,m-1}(W_{\omega,n}^{G,-n}) = [f_s, D_{r-1,m-1}](W_{\omega,n}^{G,-n}).$$

This tells us how to handle the $k_1D_{r-1,m-1}$ part, for $k_1 = -\frac{4}{3}f_s$. With Lemma 4.4.4 (1) and the formula for $[f_s, D_{r,m}]$ from Theorem 4.3.3,

$$\begin{aligned} h_1D_{r-1,m-1}(W_{\omega,n}^{G,-n}) &= p_1(D_{r-1,m-1}W_{\omega,n}^{G,-n}) + k_1(D_{r-1,m-1}W_{\omega,n}^{G,-n}) \\ &= Z_1P_{r-1,m-1}t^{2n+r+2}G_{-n-(\frac{r-m}{2})} - \frac{4}{3}[f_s, D_{r-1,m-1}]W_{\omega,n}^{G,-n} \\ = Z_1P_{r-1,m-1}t^{2n+r+2}G_{-n-\frac{r-m}{2}} &- \frac{4}{3}(U_{r-1,m-1}D_{r-1,m-3} + V_{r-1,m-1}h_{-3}D_{r-2,m})W_{\omega,n}^{G,-n}. \end{aligned}$$

To proceed, we compute $h_{-3}D_{r-2,m}$ as before (i.e. noticing that only the p_{-3} differentiation is

relevant and so we only need Lemma 4.4.4 (2) and arrive at

$$h_1 D_{r-1,m-1}(W_{\omega,n}^{G,-n}) = Z_1 P_{r-1,m-1} t^{2n+r+2} G_{-n-\frac{r-m}{2}} - \frac{4}{3} U_{r-1,m-1} P_{r-1,m-3} t^{2n+(r-1)+2} G_{-n-(\frac{r-m}{2})-1} + \frac{4}{3} V_{r-1,m-1} Z_3 P_{r-2,m} t^{2n+(r-1)+2} G_{-n-(\frac{r-m}{2})+1}$$

We again compute $h_3 D_{r-1,m-3} W_{\omega,n}^n$ with Lemma 4.4.3 (2) and a K_G-computation:

$$\begin{aligned} h_3 D_{r-1,m-3}(W_{\omega,n}^{G,-n}) &= p_3(P_{r-1,m-3} t^{2n+(r-1)+2} G_{-n-(\frac{r-m}{2})-1}) + k_3(D_{r-1,m-3}(W_{\omega,n}^{G,-n})) \\ &= (Z_3 P_{r-1,m-3}) t^{2n+r+2} G_{-n-(\frac{r-m}{2})} - 2(2n+(r-1)+2) P_{r-1,m-3} t^{2n+(r-1)+2} G_{-n-(\frac{r-m}{2})-1} \\ &\quad - 2(-n-(\frac{r-m}{2})-1) P_{r-1,m-3} t^{2n+(r-1)+2} G_{-n-(\frac{r-m}{2})-1} \\ &\quad - (2n+(r-1)+(m-3)) P_{r-1,m-3} t^{2n+(r-1)+2} G_{-n-(\frac{r-m}{2})-1}. \end{aligned}$$

Combining the above with the formula for $h_{-3} h_3 D_{r-2,m}(W_{\omega,n}^{G,-n})$ from ($\star$), we get

$$\begin{aligned} &(h_1 D_{r-1,m-1} + E_{r,m} h_3 D_{r-1,m-3} + F_{r,m} h_3 h_{-3} D_{r-2,m})(W_{\omega,n}^{G,-n}) \\ &\quad = (Z_1 P_{r-1,m-1} + E_{r,m} Z_3 P_{r-1,m-3} + F_{r,m} Z_3 Z_{-3} P_{r-2,m}) t^{2n+r+2} G_{-n-(\frac{r-m}{2})} \\ &\quad\quad + (S') P_{r-1,m-3} t^{2n+(r-1)+2} G_{-n-(\frac{r-m}{2})-1} + (S'') Z_3 P_{r-2,m} t^{2n+(r-1)+2} G_{-n-(\frac{r-m}{2})+1}. \end{aligned}$$

A short computation verifies that in fact

$$S' = \left(-\frac{4}{3} U_{r-1}^{m-1} + (-4n-2r-2m+4) E_{r,m} \right) = 0$$

and

$$S'' = \left(\frac{4}{3} V_{r-1}^{m-1} + 4F_{r,m}(2n+r-1) \right) = 0.$$

We are left with only

$$D_{r,m}(W_{\omega,n}^{G,-n}) = (Z_1 P_{r-1,m-1} + E_{r,m} Z_3 P_{r-1,m-3} + F_{r,m} Z_3 Z_{-3} P_{r-2,m}) t^{2n+r+2} G_{-n-(\frac{r-m}{2})}. \quad (\star\star)$$

Setting $m = r$, we recover the defining equation $P_{r,r} = Z_1 P_{r-1,r-1} + E_{r,r} Z_3 P_{r-1,r-3}$ and we are done (note $F_{r,r} = 0$).

(II) The case $m = r$ is the definition of $P_{r,r}$. When $-r < m < r$, we have on one hand from (I) that

$$D_{r,m}(W_{\omega,n}^{G,-n}) = P_{r,m} t^{2n+r+2} G_{-n-(\frac{r-m}{2})}$$

and on the other hand from $(\star\star)$ that

$$D_{r,m}(W_{\omega,n}^{G,-n}) = (Z_1 P_{r-1,m-1} + E_{r,m} Z_3 P_{r-1,m-3} + F_{r,m} Z_3 Z_{-3} P_{r-2,m}) t^{2n+r+2} G_{-n-(\frac{r-m}{2})}$$

so we are done (note that the K-Bessel functions do not vanish).

□

4.5 Final Steps

In this section we finally prove Theorem 4.1.2 and Theorem 4.1.1, and then Theorem 1.0.1. We start with a preliminary lemma:

Lemma 4.5.1. *Let $v \in \mathrm{Sym}^{2n}(V_2^{long}) \boxtimes \mathbf{1}$. If $Y \in \mathfrak{p}^{\otimes r}$, then for any smooth function $F : G(\mathbb{R}) \to \mathbb{V}_n^\vee$,*

$$\langle (\tilde{D}^r) F, Y v \rangle = Y \langle F, v \rangle.$$

Proof. We can check this directly, for example on basis elements of $\mathfrak{p}$ and $\mathbb{V}_n$. □

Lemma 4.5.2. *Let $L \subseteq \mathrm{Sym}^r(V_G)$ be a line. Fix a basis element $\ell \in L$. Let*

$$Pr_L : \mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes \mathrm{Sym}^r(V_G) \to \mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes L$$

be the unique K_H-equivariant projection map that fixes $\mathrm{Sym}^{2n+r} \boxtimes \ell$, and write $Pr_L^\vee$ for the corresponding map on dual spaces.

Let $v \in \mathrm{Sym}^{2n}(V_2^{long}) \boxtimes \mathbf{1}$. If $Y \in \mathfrak{p}^{\otimes r}$, and satisfies $Yv = v' \boxtimes \ell$ with $v' \in \mathrm{Sym}^{2n+r}(V_2^{long})$, then for any smooth function $F : G(\mathbb{R}) \to \mathbb{V}_n^\vee$,

$$\langle (Pr_L^\vee \circ \tilde{D}^r)F, v' \rangle = Y \langle F, v \rangle.$$

Proof. First $\tilde{D}^r F$ is valued in

$$\bigoplus_{j=0}^{r} \left(\mathrm{Sym}^{2n+j}(V_2^{long}) \boxtimes \mathrm{Sym}^j(V_G) \right)^\vee.$$

We can orthogonally decompose the top summand relative to K_H:

$$\mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes \mathrm{Sym}^r(V_G) = \mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes L \oplus \mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes L^\perp.$$

If $Yv = v' \boxtimes \ell$, it follows that $\langle w, Yv \rangle = 0$ for any $w \in \mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes L^\perp$. So

$$\langle \tilde{D}^r F, Yv \rangle = \langle (Pr_L^\vee \circ \tilde{D}^r)F, v' \rangle$$

and we are done via Lemma 4.5.1. □

Proof of Theorem 4.1.2. Note that $P_{r,m}^{\omega,n}$ and $D_{r,m}^n$ are defined by the same recurrence relations, and the line $L_{r,m}^n$ in terms of monomials in $\{(y_s^3), (x_s y_s^2), (x_s^2 y_s), (x_s^3)\}$ can be read off from the recurrence relations defining $D_{r,m}^n$. Furthermore,

- $Z_3^\omega(i) = z_\omega$;
- $Z_1^\omega(i) = -b_\omega/3$;
- $Z_{-1}^\omega(i) = \overline{b_\omega}/3$
- $Z_{-3}^\omega(i) = -\overline{z_\omega}$.

Therefore,

$$P^{\omega,n}_{r,m}(i) = \langle \ell^n_{r,m}, v^r_\omega \rangle = Q^n_{r,m}(\omega)$$

and the theorem follows from Theorem 4.4.8 (I). □

Proof of Theorem 4.1.1. Both sides of Theorem 4.1.1 are quaternionic. So to test equality we can pair with an element of $x^{2n+r} \boxtimes L^n_{r,m}$. Such an element is given by $D^n_{r,m} x^{2n}$, due to Theorem 4.3.3. Then since both sides have the same N_H and K_H equivariance properties, we need only to check equality on $A_H(\mathbb{R})^0$. By Lemma 4.5.2, we calculate

$$\begin{aligned}\langle ((Proj^n_{r,m})^\vee \circ \tilde{D}^r) W^G_{\omega,n}|_H, x^{2n+r} \boxtimes \ell^n_{r,m} \rangle &= \langle ((Proj^n_{r,m})^\vee \circ \tilde{D}^r) W^G_{\omega,n}|_H, D^n_{r,m} x^{2n} \rangle \\ &= D^n_{r,m} \langle W^G_{\omega,n}|_H, x^{2n} \rangle.\end{aligned}$$

That this last quantity is equal to $\langle Q^n_{r,m}(\omega) W^H_{pr(\omega),n+\frac{r}{2}}, x^{2n+r} \boxtimes \ell^n_{r,m} \rangle$ is exactly the content of Theorem 4.1.2. □

Proof of Theorem 1.0.1. Suppose $\omega \in 2\pi W_G(\mathbb{Q})$ has $pr(\omega) = \mu \in 2\pi W_H(\mathbb{Q})$. Then, $Q^n_{r,m}(\omega) \in (2\pi)^r \mathbb{Q}(i)$, and so by Theorem 4.1.1, we can set $\mathcal{D}^n_{r,m} = \pi^{-r}(Proj^n_{r,m})^\vee \circ \tilde{D}^r$ as the desired operator. □

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