Title
Mathematics MOVES Me—Digital Solutions for Co-ordinating Enactive and Symbolic Resources: The Case of Positive and Negative Integer Arithmetic

Authors
Anton, Jacqueline
Cosentino, Giulia
Gelsomini, Mirko
et al.

Publication Date
2024-09-29

DOI
10.1007/s40751-024-00158-5

Digital Experiences in Mathematics Education
https://doi.org/10.1007/s40751-024-00158-5

Mathematics MOVES Me—Digital Solutions for Co-ordinating Enactive and Symbolic Resources: The Case of Positive and Negative Integer Arithmetic

Jacqueline Anton[1,2] · Giulia Cosentino[3] · Mirko Gelsomini[3,4] · Kshitij Sharma[3] · Michail N. Giannakos[3] · Dor Abrahamson[1,5]

Accepted: 9 September 2024

Abstract

We present an innovative educational design for basic arithmetic that responds to students' documented difficulties with adding and subtracting single-digit positive and negative numbers. The design utilizes MOVES, a technological architecture that combines floor- and wall-projected interactive interfaces. Students enact arithmetic operations, e.g., "3 – (-2)" by walking along a projected body-scale number line, while their actions are captured and analyzed to provide in-the-moment feedback on elements of their solution procedure. Next, a screen-based avatar is introduced who mimics their whole-body movements. Finally, analogous problems are presented on a tablet that utilizes tangible interaction, where seated students walk the avatar, now as an action-figure, along a standard-sized number line. Our theoretical framework, design conjecture, product evaluation, and data analysis all pertain to fostering conceptual understanding through co-ordinating full-body egocentric experiences on a body-scale number line with the allocentric experience of "puppeting" the avatar along the desk-scale number line. We introduce the educational design and report on its pilot trials. Based on these preliminary results, we speculate on the nature and type of supports that students require to co-ordinate these perspectives and discuss implications for future iterations of the design.

Keywords Embodiment · Integer operations · Multisensory interaction · Number line · Perspective

---

✉ Jacqueline Anton
jacqueline_anton@berkeley.edu

[1] University of California, Berkeley, USA

[2] San Francisco State University, San Francisco, USA

[3] Norwegian University of Science and Technology, Trondheim, Norway

[4] Scuola Universitaria Professionale Della Svizzera Italiana, Manno, Switzerland

[5] Korea University, Seoul, South Korea

Introduction

The purpose of this snapshot is to describe an innovative educational design (MOVES–NL) which utilizes the number line (NL) as a semiotic resource for students to learn how to add and subtract positive and negative integers. In this design, students experiment with interactive floor and wall projections of NLs generated by the MOVES multisensory technological system, which we describe below.

Relevance for Mathematics Teaching and Learning

Foundational mathematics skills are important predictors of secondary and post-secondary success for students (Varma & Schwartz, 2011). In early mathematics classrooms, students often struggle when they encounter negative integers (Bossé, 2016; Hawthorne et al., 2022). Negative integers, unlike positive integers, are not easily modelled by teachers. For instance, if students learn to count, add, and subtract by manipulating concrete “things” (i.e., fingers, blocks, and toys), then negative numbers pose a substantial challenge because it is unclear how to generate and display negative numbers as things that are available for inspection and enumeration. Still, an understanding of negative integers can plausibly draw on an understanding of positive integers. Indeed, teachers should support students in expanding their understanding of number, such as assimilating the notion of negative numbers onto the notion of positive numbers, by providing resources designed explicitly to forge conceptual continuity (Varma & Schwartz, 2011). Specifically, the NL is a beneficial pedagogical resource for learning negative-number arithmetic because it affords conceptual opportunities to assimilate negative numbers as spatially deployed variant elaborations of positive-number ontology (Bossé, 2016). The present design attempts to augment the NL’s affordance for learning positive-and-negative integer arithmetic by installing prior designs in interactive digital media that offer supports for perceptual co-ordination across scale (body-scale versus desk-scale) while supplementing the activities with conceptually oriented feedback regimens.

Perspectival Co-ordination

The MOVES–NL design creates conditions for students to experience integer arithmetic on the NL through both an egocentric and an allocentric perspective. Previous studies that incorporated activities of enacting arithmetic along a body-scale NL (e.g., Nurnberger-Haag, 2018) found these experiences to be more beneficial than others (i.e., chip models). However, even after 5 weeks of training on the body-scale NL, students still struggled to solve later addition and subtraction problems when given a traditional paper-and-pencil task, especially with problems involving double-digit integers. We propose that the issue may have not been a shortcoming in the fundamental rationale of body-scale arithmetic enactment but, rather, that the study participants had little to no opportunities to co-ordinate their body-scale *egocentric* (first-person) experience of walking along a floor-based NL with the *allocentric* (third-person) perspective of looking at a NL from the "outside," which is typically required in classrooms (see Papert, 2004, on "paper math").

Indeed, in an early iteration of this design (see Anton & Abrahamson, under review), students struggled to solve problems on the desk-scale NL, even after solving identical problems on the walking NL moments before. The three interactions described, below, demonstrate how the design's new technological affordances enhance our goal of supporting students in grounding their fluency with the traditional NL in their enacted experience on the floor NL. Building on Benally et al. (2022), we frame this design effort as supporting students in first achieving *perspectival mutuality* (i.e., using an alternate perspective to inform their own) and, eventually, achieving *perspectival synergy* (i.e., blending two perspectives into a greater structure). This perspectival synergy would subsist of a linear, spatial–numerical, mental NL (Mock et al., 2019) that grounds negative-integer arithmetic in concrete action (Varma & Schwartz, 2011).

Embodied Multisensory Interactions for Learning

Finding new methods to improve and facilitate learning is a key objective in both child–computer interaction and learning-technology research (Giannakos et al., 2020). Several advances in sensing technologies (e.g., size, affordability, ease of use, and supporting frameworks) allow human–computer interaction researchers to "sense" and "respond" to the users' presence, as well as their gestures, affective states, motions, and manipulations, while simultaneously orchestrating the interactions in numerous ways. In addition, sensing technologies have been particularly beneficial for enabling children's play and learning as naturalistic and meaningful (Sharma & Giannakos, 2021).

In particular, multisensory technologies focus on supporting learners' needs and developmental progression, with early investigations identifying the benefits of multisensory interaction to support academic learning (Zou et al., 2017). Multisensory technologies are digitally connected, controllable, and interactive, providing children with more affordances and enabling ludic-cum-educational experiences in an organic and embodied manner, all of which are important for children's learning and development (Hourcade, 2015; Malinverni et al., 2019). For example, Kosmas et al. (2019) investigated how to employ a motion-based, embodied, learning game to improve students' memory performance when learning a second language.

Technologically enabled embodied interaction activities have proven their potential to enhance children's learning (Lee-Cultura et al., 2020). In addition, Cosentino et al., (2023a, 2023b) investigated the benefits, challenges, and trade-offs between different interaction modalities (e.g., full body interactions) in the context of educational multisensory environments for children. The interaction modalities presented in the next sections (walking NL, walking NL with mirrored avatar, and small NL with figurine), when orchestrated together, we submit, are especially relevant for learning mathematical content. Taken as a whole, the proposed activity rationale reflects an enactivist perspective (Varela, Thompson & Rosch, 1992/2000), which acknowledges the sensorimotor foundations of learning. Traditional classroom practices that require students to sit still and learn without tapping into the potential afforded by their full-body perceptuomotor systems do not accord with this enactivist epistemological conjecture. Symbolic notation typical of mathematical practice must be *grounded* in perceptuomotor action, in order for students to make meaning of traditional educational tasks (Glenberg, 2010). In line with the enactivist motto that "cognitive structures emerge from recurrent sensorimotor patterns that enable action to be perceptually guided" (Varela et al., 1992/2000, p. 173), our design seeks to ground integer arithmetic in movement patterns.

The Design

The MOVES–NL utilizes both a body-scale walking NL *and* a small desk-scale NL as intended means for students to ground the targeted mathematical procedures: students are to blend the egocentric and allocentric perceptual perspectives by shifting back and forth between full-body egocentric enactment and allocentric manipulatory re-enactment. If we want students to appropriate semiotic tools that afford an allocentric perspective (such as the NL), we must first provide students with dedicated opportunities to enact these tools' symbolic meaning (Radford, 2014). Therefore, in our design, students first walk along the body-scale, floor-based NL to solve integer arithmetic problems by enacting them (see Fig. 1).

A teacher then provides the student the three-phase instructions: (1) start by standing on the first number in the problem (not shown in the figure), (2) turn to the right (positive side of the NL) for addition problems (see "addition" sign on the classroom wall) or turn to the left (negative side of the NL) for subtraction (see

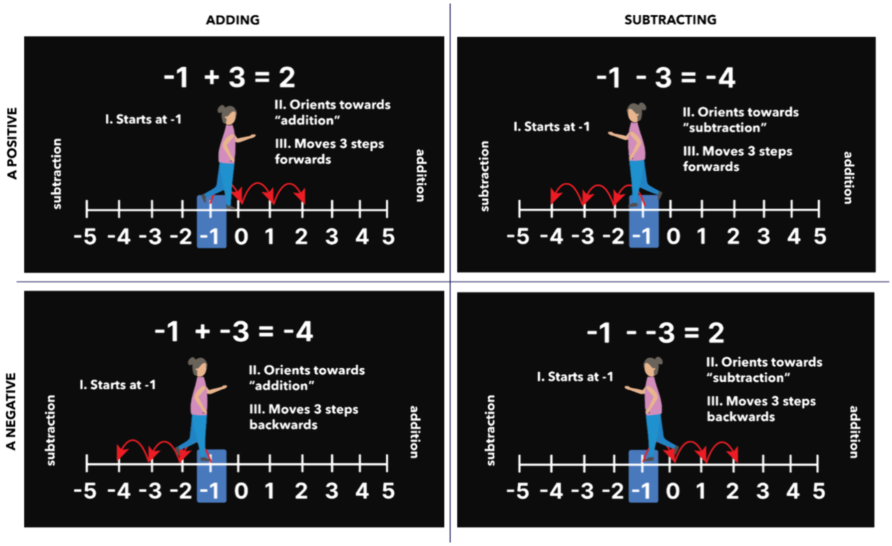

Fig. 1 Walking NL

“subtraction” sign on the classroom wall), and (3) walk the number of steps indicated by the second number in the problem (forward if the number is positive, backward if the number is negative). Figure 1 exemplifies enacted solution moves for the four possible combination schemes of adding or subtracting (columns) positive or negative integers (rows) on the walking NL as modeled by the teacher’s three-move instructions (see also Anton & Abrahamson, under review). Note here that students are experiencing the NL from an egocentric perspective (Tversky & Hard, 2009), whereby the NL is positioned on the sagittal (front–back) axis in respect to the body.

Next, students are invited to sit at their desks. They are offered a tablet-based NL, as well as a figurine. They are asked to use this action figure to re-enact their own body-scale arithmetic operation moves, now at desk-scale. Note here that, in this case, students are experiencing the NL from an allocentric (third-person) perspective (Herbst et al., 2017), even as the figurine “experiences” the NL from an egocentric perspective. Movement along a sagittal (front–back) axis has been shown to prioritize an egocentric perspective, while lateral (left–right) movement prioritizes an allocentric one (Marghetis et al., 2020).

We conjecture that having students experience the NL from both an egocentric and (by surrogate proxy) allocentric perspective will facilitate the form of perspectival co-ordination that students require, in order to make sense of the disciplinarily normative desk-scale NL in terms of their enactment on the body-scale NL; and that walking the action figure along *its* egocentric pathway, even while seeing it from an allocentric perspective, will create necessary cognitive circumstances for a phenomenological blending of the perspectives (e.g., that a student both adopts the perspective of the avatar and takes an allocentric, birds-eye-view perspective). We are intrigued by the cognitive mechanisms, challenges, and opportunities, of thus splitting and synergizing sensorimotor perspectives, where the eyes are *seeing* the NL while our operating hand is *being* the NL (cf. Gerofsky, 2011), as well as by the conceptual prospects of this *perspectival mutuality*, where an alternate perspective is used to complement one’s own perspective (Benally et al., 2022).

MOVES

This educational design utilizes MOVES to create three different NL-based interactions. The first interaction (Fig. 2) includes a NL ranging from − 5 to + 5 projected onto the floor along with either an addition or a subtraction problem projected onto the wall. The dual wall-and-floor projectors are co-ordinated through the *SENSEi* software (Gelsomini, 2023), which allows the motion sensor to track students’ position, orientation, and movement on the NL and, in response, mark the current position dynamically on the floor projection. In particular, when students stand on each hash mark along the NL, the number under their feet turns blue and a pleasant chime is sounded. This way, students can see and hear that the motion sensor is capturing their position. SENSEi’s interactive sonification affordances are particularly important for blind and visually impaired student accessibility, albeit the current article will not elaborate on the potential inclusivity parameters of future variants on MOVE–NL that will cater to students with sensorimotor diversity.

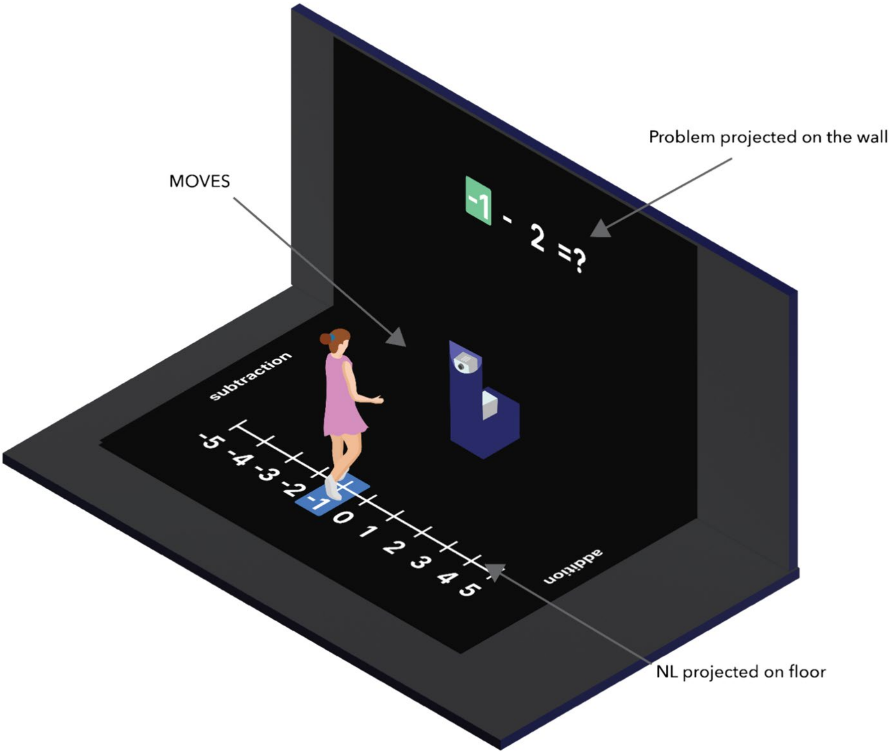

Fig. 2 Walking NL as projected by MOVES

In addition to recognizing student *position* on the NL, the projector registers student *orientation* and *movement*. As the student stands on each number on the floor, the corresponding number is highlighted in blue, and a chime is played, signifying to the student that the system recognizes their location. As the student performs the *correct* movements, each corresponding part of the problem on the wall lights up in green and a congratulatory sound is played, providing students with in-the-moment feedback on their whole-body movements. For example, given the problem "-1 – 2," the student would first stand on the NL's -1 hash mark. As they do so, the -1 on the floor-projected NL lights up in blue, and a chime is played. Concurrently, the -1 on the wall-projected NL in front of the student lights up in green. Next, the student needs to turn to the left, in order to orient themselves in the subtraction direction (still before moving). As soon as the student turns left, the subtraction sign on the wall turns green. Finally, the student needs to take 2 steps forward (i.e., in the direction they are facing, which is toward the lesser values on the NL). Once the student has taken the 2 steps, they raise their hands in the air to signal that they have reached the solution. If the solution is correct, the entire problem on the wall is highlighted in green, a congratulatory sound plays, and the solution is displayed.

The second interaction is the same as the first, only that a virtual avatar projected onto the wall mirrors the students' position and movements on the walking NL. See Fig. 3 for an illustration of the second interaction.

Here, the student receives the same feedback from the motion sensor as in the earlier floor-only activity. In addition, however, the avatar projected onto the wall mimics student movement. The design rationale of deploying a mirrored avatar in full view of the student is to support the student in bridging the egocentric experience of walking along the NL with the allocentric experience that is required in the final interaction, when the student is seated at a desk.

The third and final interaction involves only a tablet, which displays a smaller, desk-scale NL and, again, presents an addition or subtraction problem (see Fig. 4). During this interaction, the student re-enacts their previous whole-body movements by moving a tangible figurine (of identical appearance as the virtual avatar) along the small NL, just as they had moved their whole bodies on the walking NL.

Similarly to the previous levels of interaction, the tablet recognizes *where* the student places the figurine, in what *direction* the figurine is facing, and what *steps* the figurine is taking. When the student places the figurine in the correct location and facing the correct direction, the various corresponding screen elements of the displayed problem are highlighted in green and a congratulatory sound plays. In Fig. 4, the student has correctly completed the first phases of solving "-1 – 2 = ?" (begin by

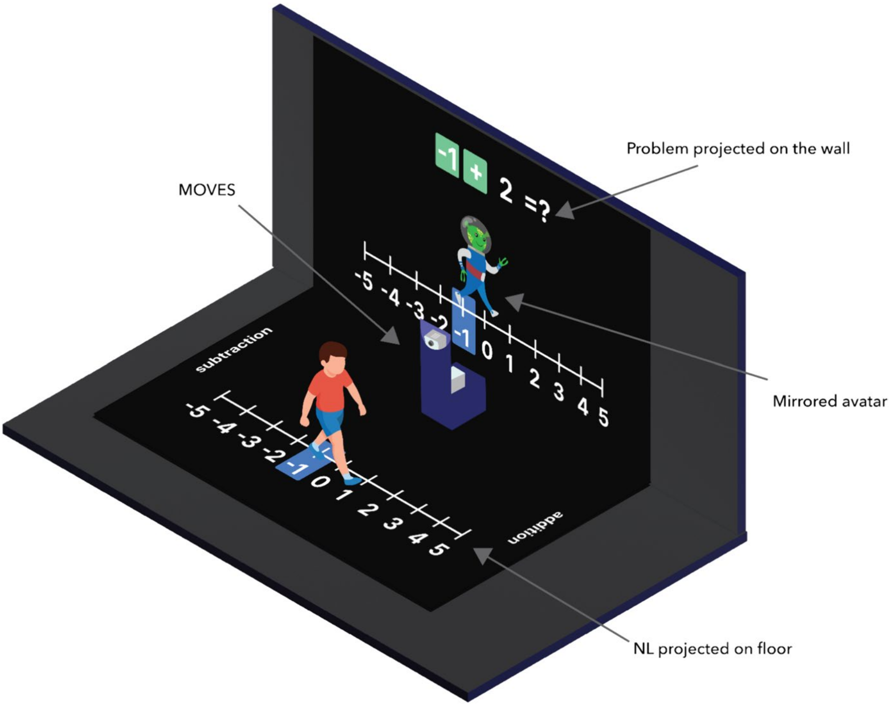

Fig. 3 Walking NL as projected by MOVES with the mirrored avatar functionality activated

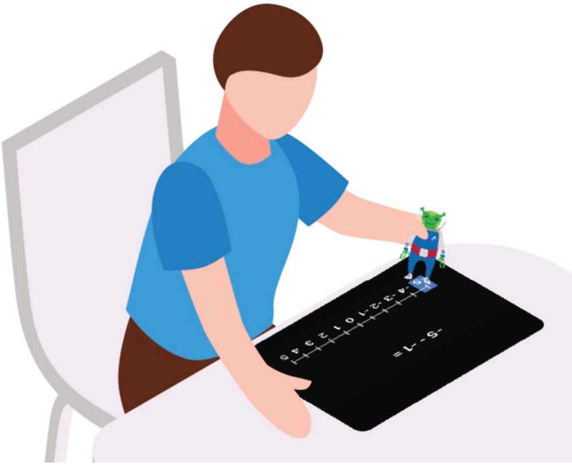

Fig. 4 Student moving a tangible figurine on the tablet NL

standing on -1; note that he has not yet performed the second phase of facing the avatar toward the lesser NL values per the item's subtraction operation symbol).

The MOVES Technological System

The purpose of this sub-section is to outline the technological system used in this design. While a paper version of the pedagogical design is feasible (see Anton & Abrahamson, under review), the MOVES technology allowed us to better support students in blending perceptual perspectives. The hardware structure of *MOVES* (see Fig. 5; Cosentino et al., 2023a) has wheels on its base (C) that can be locked for the duration of the activity, yet allow for easy repositioning and transport per diverse environments. The platform holds a mini-PC (E) that reads motion-sensing (RGB and depth) video and audio streams (Orbecc Astra Pro) (F) and outputs to two Ultra Short Throw LED projectors (G) using two independent video outputs (projecting the NL interactive image on the floor as well as the arithmetic problem on the wall). A router (I) provides wired connectivity to the PC, creating a local Wi-Fi network to which a controlling device (smartphone, tablet, or remote controller; L, M) is connected to control the experience. A coat made of a PVC layer covers the structure's front. The MOVES platform is equipped with SENSEi software (Gelsomini, 2023). *SENSEi* is a suite of software modules that enables the PC to manage several input and output devices. At a lower level, these devices are recognized and communicate with the PC through traditional drivers. At a higher level, they are accessible in the form of simple, homogeneous, and intuitive APIs, with which novice-to-skilled programmers can interface. SENSEi enables this simplification by accessing device providers' Software Development Kits (SDK) and translating them into a standardized documented form. The software is installed as a set of modules that interface with the sensing and actuation devices and a viewable layer to which contents are displayed.

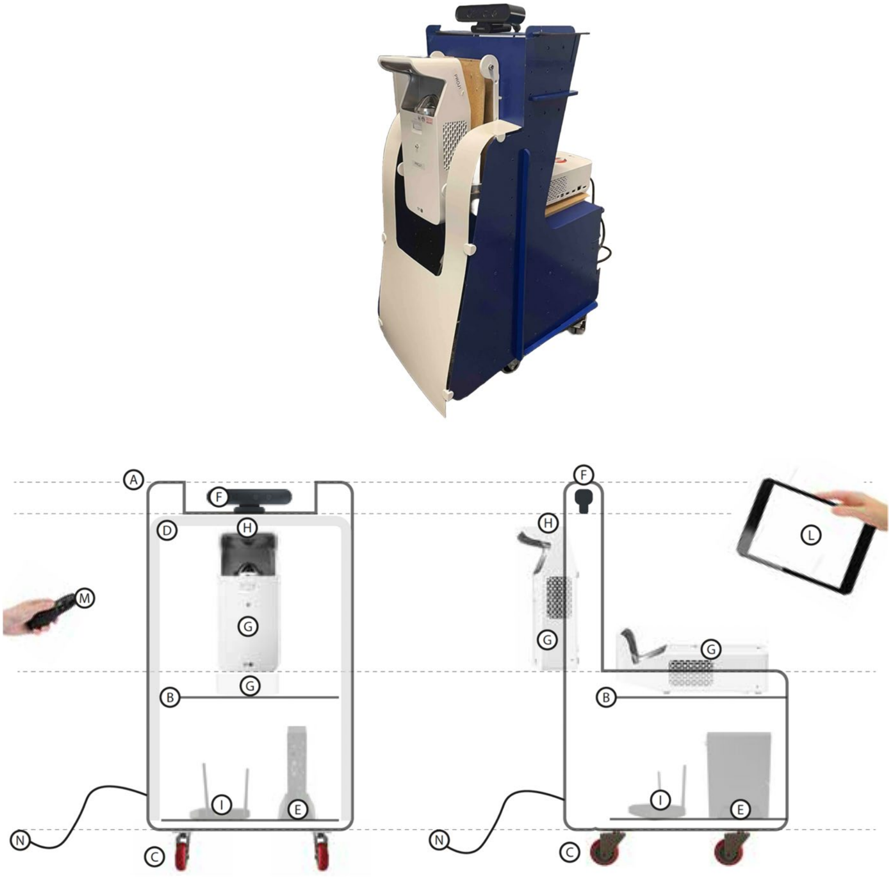

Fig. 5 The MOVES structure

Pilot Study

This section outlines preliminary data collected using MOVES-NL to teach integer arithmetic to twenty students in Grades 4–6 (9–11 years old) in Milan, Italy. (Note that, by this age, students will have studied basic arithmetic with negative numbers, albeit their understandings, per the literature, would not be robust at best.) Students began with the walking NL interaction (Fig. 2) and then operated the walking NL with a mirrored avatar (Fig. 3). Finally, students solved problems using the tablet (Fig. 4). Throughout the interactions, the researcher collected various observations and engaged students in a semi-structured interview (Ginsburg, 1997), seeking to gain deeper understanding of students' conceptual processes. All interactions were video recorded.

Analyzing this data, we observed that students' implicit confusions around negative integer arithmetic emerged as they were asked to enact procedures and solutions through whole-body movement. Specifically, we hypothesize that full-body movement disrupts students’ typical mathematics strategies and surfaces any misconceptions they harbor. Students typically completed the first two steps of the walking NL correctly (stand on the first number; face either addition or subtraction). However, students hesitated when it came time to take a step, often wondering in which direction they should walk. This hesitation is a case of misinterpreting the actionable meaning of the polarity sign of the second number (i.e., whether it is positive or negative). This finding serves as evidence supporting a claim that, whereas children have enacted arithmetic operations throughout their childhood, they have not had the opportunity to enact positive and negative polarity (Mock et al., 2019).

Furthermore, students’ hesitation to step either backwards or forwards could reflect the polysemy of the “- “ sign, as students struggled with determining the correct direction of movement both in items which added or subtracted a negative integer and in items which subtracted positive integers. This “- “ sign can either be operational in nature (i.e., subtraction) or it can denote polarity (i.e., negative). The walking NL asks students to address this symbolic ambiguity (Foster, 2011) by stipulating an action-based semiotic differentiation between the operation of subtraction and the polarity of a negative number. However, symbol polysemy is prevalent in mathematics, and students experience tension between the “obvious” well-known symbolic meaning and alternate meanings (Mamolo, 2010, p. 249). In this case, the “obvious” meaning of the “-” sign is subtraction, or “to take away.” The tension arises when students encounter this sign *after* another operational sign (i.e., $1+-3$, or $1--3$). Notwithstanding, this ambiguity *as evidenced in publicly displayed bodily enactment* provides an opportunity for students and teachers to engage in productive discourse around mathematical concepts (Abrahamson et al., 2009; Foster, 2011). In this instance, we believe that it is imperative for the teacher/researcher to step in, literally, and provide some context, usually by asking guiding questions (Ginsburg, 1997) that will elicit a productive negotiation toward a common understanding of why students should walk backwards or forwards.

After their first interaction, students watched an alien avatar mimic their whole-body movements (see Fig. 3). The purpose of this interaction was to facilitate students’ bi-perspectival co-ordination between their egocentric experience on the walking NL and, prospectively, their allocentric experience with the tablet (Fig. 4). In general, students seamlessly transitioned from the first two walking interactions to the tablet, where they mimicked their whole-body movement by operating the avatar “mini-me” action figure.

These students were able to co-ordinate their perspectives to achieve either perspectival mutuality (using a different perspective along with your own) or synergy (the emergence of a new composite perspective; see Benally et al., 2022). However, this perspectival co-ordination was, at times, brief; when using the tablet to solve problems that were similar to those they had already solved on the walking NL, students occasionally reverted to past “school” strategies and thus became “stuck.” It appears that when students are working on blending perspectives in the service of mathematical learning, they need to be given the opportunity to move back and forth between the enactive (walking NL), iconic (mirrored avatar), and symbolic (tablet) interactions (Dutton, 2018; cf., Bruner, 1966).

Conclusions

The pilot study led to three general conclusions. First, enacting arithmetic procedures as expansive embodied actions can productively disrupt students' solution-oriented routines, including their mathematically inappropriate heuristics. Mathematics instruction that prioritizes procedural rules over conceptual underpinnings can undermine students' understanding, agency, and development of content knowledge and epistemic practices (Erlwanger, 1973; Freudenthal, 1971; Kamii & Dominick, 1998; Nathan, 2012). Instead, activities that step aside from rote procedures, for example, whole-body arithmetic enactment, appear to refresh and reground students' mathematical perceptions in their common-sense situated know-how (Ma, 2016).

Second, if first-person immersive activities, such as those provided by virtual-reality technologies, are to ground students' conceptual understanding of mathematical concepts, then students should be given opportunities to co-ordinate these situated egocentric perspectives with allocentric perspectives on the analog symbolic procedures, which are more typically prioritized in classroom learning. More generally, utilizing multisensory technology that enables students to *co-ordinate* mathematics skills across multi-perspectival media may promote deeper conceptual understanding.

Third, observing students interact with mathematics content in many different ways may provide insight for teachers and practitioners struggling to convey concepts to their students. In our pilot study, the full-body movement initiated by the MOVES–NL design highlighted student misconceptions that might not have been apparent if we had engaged students in typical paper-and-pencil-based mathematics tasks. By collecting action logs and analytics from various biological sensors (e.g., physiological data from wristbands and skeletal data from motion sensors), researchers may be able to identify furthermore specific moments where students are struggling (e.g., experiencing difficulties in determining the correct orientation or moving slowly due to uncertainty).

Future Work

Future iterations of the MOVES-NL could be enhanced if the system provided further encouragement, guidance, and support for students struggling to blend their egocentric and allocentric perceptions of body-scale and desk-scale NL enactments. For example, allowing students to shift back and forth between these interactions, when they are stuck on a problem, could help to facilitate this co-ordination. In a sense, we are "*yes-and*ing" the received gospel from Jerome Bruner, often articulated as "enactive, iconic, symbolic" (Bruner, 1966) by way of supplementing "… and back again" (Abrahamson et al., 2012; Dutton, 2018). It appears students may sometimes require greater agency and latitude in organizing and pacing their own bilateral, situated co-ordination between perspectives (Kaput & West, 1994; Thompson, 2013).

One activity form that may occasion students opportunities to blend enactments across scale and perspective is for them to instruct another agent across the registers. For example, a student who is performing subtraction on the desk-scale NL may explain to a peer or an avatar—"walk them through," so to speak—how to enact the same arithmetic operation on the walking NL, and vice versa. These bilateral bridging activities could potentially help students sustain an enactive grounding of integer arithmetic, as they adopt symbolic forms of expression in their mathematics classrooms.

Author contributions The first and second authors (Jacqueline Anton and Giulia Cosentino) prepared the original draft of the manuscript, and all other authors (Mirko Gelsomini, Kshitij Sharma, Michail N. Giannakos, and Dor Abrahamson) reviewed the manuscript.

Funding The collaboration between the University of Berkeley, California, and the Norwegian University of Science and Technology is funded by the Peder Sather Grant program.

Data Availability Video and audio data from our study are not publicly available to preserve individual participants' privacy according to the Committee for the Protection of Human Subjects.

Declarations

Ethics Approval All authors confirm that any aspect of the work covered in this manuscript that has involved study participants has been conducted with the ethical approval of all relevant bodies and that such approvals are acknowledged within the manuscript. IRB approval was obtained before the collection of data (protocol ID 2022–10-15703). Written consent to publish potentially identifying information, such as details or the case and photographs, was obtained from the participants and/or their legal guardians.

Competing interests The authors declare no competing interests.

References

Abrahamson, D., Bryant, M., Gutiérrez, J., Mookerjee, A., Souchkova, D., & Thacker, I. (2009). Figuring it out: Mathematical learning as guided semiotic disambiguation of useful yet initially entangled intuitions, In S. Swars, D. Stinson & S. Lemons-Smith (Eds.), *Proceedings of the 31st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (vol. 5, pp. 662–670). PME-NA.

Abrahamson, D., Gutiérrez, J., Charoenying, T., Negrete, A., & Bumbacher, E. (2012). Fostering hooks and shifts: Tutorial tactics for guided mathematical discovery. *Technology, Knowledge and Learning, 17*(1–2), 61–86.

Anton, J., & Abrahamson, D. (under review). Walking the number line: Towards an enactive understanding of integer arithmetic. [Manuscript submitted for publication]

Benally, J., Palatnik, A., Ryokai, K., & Abrahamson, D. (2022). Learning through negotiating conceptually generative perspectival complementarities: The case of geometry. *For the Learning of Mathematics, 42*(3), 34–41.

Bossé, M. (2016). Using integer manipulatives: Representational determinism. *International Journal for Mathematics Teaching and Learning, 17*(3), 20.

Bruner, J. (1966). *Toward a theory of instruction*. Harvard University Press.

Cosentino, G., Gelsomini, M., & Giannakos, M. (2023). MOVES: Going beyond hardwired multisensory environments for children. In M. Horn (Ed.), *Proceedings of the 22nd annual meeting of interaction design and children (IDC '23)* (pp. 716–720). Association for Computing Machinery.

Cosentino, G., Gelsomini, M., Sharma, K., & Giannakos, M. (2023). Interaction modalities and children's learning in multisensory environments: Challenges and trade-offs. In M. Horn (Ed.), *Proceedings of the 22nd annual ACM interaction design and children conference (IDC '23)* (pp. 397–410*)*. Association for Computing Machinery.

Dutton, E. (2018). *Mathematics learning as perceptual reconstruction: The role of semiotic breakdown in collaborative problem solving*. Unpublished Masters thesis, University of California Berkeley. https://edrl.berkeley.edu/wp-content/uploads/2021/03/DuttonLizzy.2018.MACSME.thesis.PerceptualReconstruction.pdf. Accessed 10 Sept 2023.

Erlwanger, S. (1973). Benny's conception of rules and answers in IPI mathematics. *Journal of Children's Mathematical Behavior, 1*(2), 7–26.

Foster, C. (2011). Productive ambiguity in the learning of mathematics. *For the Learning of Mathematics, 31*(2), 3–7.

Freudenthal, H. (1971). Geometry between the devil and the deep sea. *Educational Studies in Mathematics, 3*(3–4), 413–435.

Gelsomini, M. (2023). SENSEi – Multisensory and multimodal research framework. SENSEi. (https://sensei.space/). Accessed 15 Sept 2023.

Gerofsky, S. (2011). Seeing the graph vs being the graph: Gesture, engagement and awareness in school mathematics. In G. Stam & M. Ishino (Eds.), *Integrating gestures* (pp. 245–256). John Benjamins.

Giannakos, M., Horn, M., Read, J., & Markopoulos, P. (2020). Movement forward: The continued growth of child–computer interaction research. *International Journal of Child-Computer Interaction, 26*, 100204.

Ginsburg, H. (1997). *Entering the child's mind* (ch. 4, pp. 115–158). Cambridge University Press.

Glenberg, A. (2010). Embodiment as a unifying perspective for psychology. *Wires Cognitive Science, 1*(4), 586–596.

Hawthorne, C., Philipp, R., Lamb, L., Bishop, J., Whitacre, I., & Schapelle, B. (2022). Reconceptualizing a mathematical domain on the basis of student reasoning: Considering teachers' perspectives about integers. *The Journal of Mathematical Behavior, 65*, 100931.

Herbst, P., Fujita, T., Halverscheid, S., & Weiss, M. (2017). *The learning and teaching of geometry in secondary schools: A modeling perspective*. Routledge.

Hourcade, J. (2015). *Child-computer interaction*. Self publication. http://homepage.cs.uiowa.edu/~hourcade/book/hourcade_cci_2nd_edition.pdf. Accessed 23 Sept 2023.

Kamii, C., & Dominick, A. (1998). The harmful effects of algorithms in grades 1–4. In L. Morrow & M. Kenney (Eds.), *The teaching and learning of algorithms in school mathematics, 1998 yearbook* (pp. 130–140). NCTM.

Kaput, J., & West, M. (1994). Missing-value proportional reasoning problems: Factors affecting informal reasoning patterns. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 237–287). SUNY Press.

Kosmas, P., Ioannou, A., & Zaphiris, P. (2019). Implementing embodied learning in the classroom: Effects on children's memory and language skills. *Educational Media International, 56*(1), 59–74.

Lee-Cultura, S., Sharma, K., Papavlasopoulou, S., Retalis, S., & Giannakos, M. (2020). Using sensing technologies to explain children's self-representation in motion-based educational games. In E. Rubegni & A. Vasalou (Chairs), *Proceedings of the annual meeting of Interaction Design and Children (IDC '20)* (pp. 541–555). Association for Computing Machinery.

Ma, J. (2016). Designing disruptions for productive hybridity: The case of walking scale geometry. *The Journal of the Learning Sciences, 25*(3), 335–371.

Malinverni, L., Schaper, M., & Pares, N. (2019). Multimodal methodological approach for participatory design of full-body interaction learning environments. *Qualitative Research, 19*(1), 71–89.

Mamolo, A. (2010). Polysemy of symbols: Signs of ambiguity. *The Montana Mathematics Enthusiast, 7*(2–3), 247–261.

Marghetis, T., McComsey, M., & Cooperrider, K. (2020). Space in hand and mind: Gesture and spatial frames of reference in bilingual Mexico. *Cognitive Science, 44*(12), 12920.

Mock, J., Huber, S., Cress, U., Nuerk, H., & Moeller, K. (2019). Negative numbers are not yet automatically associated with space in 6th graders. *Journal of Cognition and Development, 20*(4), 611–633.

Nathan, M. (2012). Rethinking formalisms in formal education. *Educational Psychologist, 47*(2), 125–148.

Nurnberger-Haag, J. (2018). Take it away or walk the other way? Finding positive solutions for integer subtraction. In L. Bofferding & N. Wessman-Enzinger (Eds.), *Exploring the integer addition and subtraction landscape* (pp. 109–141). Springer.

Papert, S. (2004). Keynote speech: i3 1 to 1 Notebook Conference. http://vimeo.com/9092144. Accessed 10 Nov 2022.

Radford, L. (2014). Towards an embodied, cultural, and material conception of mathematics cognition. *ZDM: The International Journal on Mathematics Education, 46*(3), 349–361.

Sharma, K., & Giannakos, M. (2021). Sensing technologies and child–computer interaction: Opportunities, challenges and ethical considerations. *International Journal of Child-Computer Interaction*, *30*, 100331.

Thompson, P. (2013). In the absence of meaning …. . In K. Leatham (Ed.), *Vital directions for mathematics education research* (pp. 57–94). Springer.

Tversky, B., & Hard, B. (2009). Embodied and disembodied cognition: Spatial perspective-taking. *Cognition, 110*(1), 124–129.

Varela, F., Thompson, E., & Rosch, E. (1992/2000). Cognition as embodied action. In *The embodied mind: Cognitive science and human experience* (pp. 172–180). MIT Press.

Varma, S., & Schwartz, D. (2011). The mental representation of integers: An abstract-to-concrete shift in the understanding of mathematical concepts. *Cognition, 121*(3), 363–385.

Zou, L., Tal, I., Covaci, A., Ibarrola, E., Ghinea, G., & Muntean, G. (2017). Can multisensorial media improve learner experience? In K.-T. Chen, P. Cesar & C.-H. Hsu (Eds.), *Proceedings of the 8th ACM on Multimedia Systems Conference* (pp. 315–320). Association for Computing Machinery.

This manuscript has not been presented publicly.