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ABSTRACT

A feature of the Levy-Klein solution of the Bethe-Salpeter equation for the deuteron is the elimination of ϕ_{+-}, ϕ_{-+}, and ϕ_{--} in terms of the ϕ_{++} component of the wave function via a perturbation expansion. To investigate the validity of this procedure, the coupled equations for the various components in the first non-adiabatic approximation to the Δ_+ interaction were examined. A set of first order radial equations with multiplicative potentials (in the region $r > 1/m$) was obtained, involving $\phi_{\pm\pm}(r) = \frac{1}{4}(1 \pm \beta_1)(1 \pm \beta_2)\phi(r)$. A rigorous elimination of ϕ_{+-} and ϕ_{-+} led to equations containing ϕ_{++} and ϕ_{--}. Neglecting velocity dependent terms, potentials of the type

$$\frac{\mu^2}{2m}\frac{g^2}{4\pi}\,\tau_1\cdot\tau_2\,\frac{Y(r)}{P_0+\frac{1}{2}\frac{g^2}{4\pi}\,\tau_1\cdot\tau_2\,Y(r)}$$

appear. Expanding the denominator yields the usual Yukawa second order potential plus a term proportional to $g^4(\tau_1\cdot\tau_2)^2$. Such an expansion however is poor for $r \gtrsim 1/\mu$, a pole actually existing near $r \sim .7/\mu$ for the charge singlet state. For the $J = 1$ the structure of the lowest order tensor interaction is greatly altered. Thus, the perturbation expansion appears to radically alter the structure of the equations.

EFFECT OF NEGATIVE ENERGY COMPONENTS IN THE TWO-NUCLEON SYSTEM

R. Arnowitt and S. Gasiorowicz

I. Introduction

The difficulties in obtaining solutions to the four-dimensional two-nucleon equation[1] have resulted in calculations being confined to approximations involving instantaneous interactions, and their non-adiabatic corrections. This approach, carried out in a number of papers by Levy[2] and Klein[3], led to the construction of a series of static potentials appearing in a non-relativistic Schrodinger type equation. A feature of the method used (from the point of view of the Bethe-Salpeter formalism) involves an elimination of the negative energy components of the wave function by a perturbation procedure. The nature of the pseudoscalar interaction and the large size of the coupling constant enhance the importance of the negative energy states, a fact brought out in a calculation for the meson-nucleon system[4]. It was therefore felt worthwhile to examine the validity of such an expansion.

[1] J. Schwinger, Proc. Natl. Acad. Sci. U.S. 37, 452, 455 (1951); E. Salpeter and H. A. Bethe, Phys. Rev. 84, 1232 (1951).

[2] M. M. Levy, Phys. Rev. 88, 72, 725 (1952), denoted hereafter by L1 and L2, respectively.

[3] A. Klein, Phys. Rev. 90, 1101 (1953).

[4] R. Arnowitt and S. Deser, Phys. Rev. 92, 1061 (1953).

For reasons of simplicity, and because comparison with experiment was not an object of this work, the calculations were limited to consideration of the lowest order interaction corresponding to the exchange of one meson. In particular, the coupled equations between the positive and negative energy components of the wave function for the first non-adiabatic approximation were examined. The "potential energy" parts of these equations are of the form of integral operators. The assumption of a short range hard core allows one to limit the investigation to the asymptotic region ($r \sim \hbar/\mu c$), where these integral operators may approximately be replaced by multiplicative potentials, correct to order μ/m of the leading term[5]. The velocity dependent terms were neglected.

[5] The inclusion of $(\mu/m)^2$ terms would, for consistency, require a treatment of the second non-adiabatic approximation.

Beyond this, no further approximations were made. Although the negative energy components were found to be $\mu/2m$ times smaller than the positive energy components, the perturbation expansion of Levy and Klein completely alters the structure of the potentials and thus appears to be incorrect.

While this conclusion could have been arrived at in a more direct fashion, the development of sections III and IV was included because of its more general applicability to equations of the type considered.

II. Wave Equation for the First Non-Adiabatic Approximation.

We briefly review the derivation of the first non-adiabatic approximation to the Δ_+ interaction given by Levy and Klein. The wave equation in momentum space, after the separation of the center of mass motion, is[6]

[6] The natural system of units $\hbar = c = 1$ is being used. Also $\gamma_\kappa = \beta\alpha_\kappa$, $\gamma_0 = \beta$ and $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_0$

$$\left[\left(\tfrac{1}{2}P_0 + p_0\right) - \mathcal{H}_1\right]\left[\left(\tfrac{1}{2}P_0 - p_0\right) - \mathcal{H}_2\right]\psi(p) = \int I(p,p')\psi(p')\,dp' \qquad (2.1)$$

$$\mathcal{H}_1 = \underset{\sim}{\alpha}^{(1)}\cdot\underset{\sim}{p} + \beta^{(1)}m \quad ; \quad \mathcal{H}_2 = -\underset{\sim}{\alpha}^{(2)}\cdot p + \beta^{(2)}m$$

P_0 = total energy

where

$$I(p,p') = -\frac{ig^2}{(2\pi)^4}\,\tau_1\cdot\tau_2\,\beta^{(1)}\gamma_5^{(1)}\beta^{(2)}\gamma_5^{(2)}\,\Delta_+(p-p') \qquad (2.2)$$

To obtain the first non-adiabatic approximation the following ansatz for $\psi(p)$ is inserted on the r.h.s. of equation (2.1):

$$\psi(p) = (2\pi)^{-1/2}\left[-2\pi i\left\{\left(\tfrac{1}{2}P_0 + p_0\right) - \mathcal{H}_1\right\}\left\{\left(\tfrac{1}{2}P_0 - p_0\right) - \mathcal{H}_2\right\}\right]^{-1}(P_0 - \mathcal{H}_1 - \mathcal{H}_2)\,\chi(\underset{\sim}{p}) \qquad (2.3)$$

where[7]

$$\chi_{\pm\pm}(\underset{\sim}{p}) = \pm\, C_{\pm}^{(1)}(\underset{\sim}{p})\, C_{\pm}^{(2)}(\underset{\sim}{p})\, \phi(\underset{\sim}{p})$$

$$\chi_{\pm\mp}(\underset{\sim}{p}) = C_{\pm}^{(1)}(\underset{\sim}{p})\, C_{\mp}^{(2)}(\underset{\sim}{p})\, \phi(\underset{\sim}{p})$$

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[7] As pointed out by Klein, such a definition of $\chi(\underset{\sim}{p})$ gives results that are equivalent to those obtained from the Tamm-Dancoff method, which allows one to overlook the fact that it is somewhat inconsistent.

$\phi(p)$ is the equal times wave function defined by

$$\phi(\underset{\sim}{p}) = (2\pi)^{-1/2} \int \psi(p)\, dp_o \tag{2.4}$$

and $C_{\pm}\,(\equiv(C_{1,2}))$ are the usual Casimir projection operators. Carrying out the indicated integration over p_o' and integrating both sides with respect to p_o, one obtains the desired three dimensional equations

$$(P_o - \mathcal{H}_1 - \mathcal{H}_2)\,\phi(\underset{\sim}{p})$$

$$= (2\pi)^{-3} \int d^3p'\, C_i^{(1)}(\underset{\sim}{p})\, C_j^{(2)}(\underset{\sim}{p})\, \lambda\, C_k^{(1)}(\underset{\sim}{p}')\, C_\ell^{(2)}(\underset{\sim}{p}')\, K_{ij}^{k\ell}(\underset{\sim}{p},\underset{\sim}{p}')\, \phi(\underset{\sim}{p}') \tag{2.5}$$

where

$$\lambda = g^2 \, \tau_1 \cdot \tau_2 \; \beta^{(1)} \gamma_5^{(1)} \beta^{(2)} \gamma_5^{(2)}$$

and

$$K_{ij}^{kl}\left(\underset{\sim}{p}, \underset{\sim}{p+k}\right) = \tfrac{1}{2}\left(1 + \epsilon_k + \epsilon_l - \epsilon_K \epsilon_l\right) \frac{1 - \tfrac{1}{4}(\epsilon_i - \epsilon_j)(\epsilon_k - \epsilon_l)}{4\,\omega_{\underset{\sim}{K}}}$$

$$\times \sum_{\substack{\mu = i,j \\ \nu = k,l}} \frac{\epsilon_\mu \epsilon_\nu}{\omega_{\underset{\sim}{K}} - \epsilon_\mu \Lambda_\mu\left(\underset{\sim}{p}\right) - \epsilon_\nu \Lambda_\nu\left(\underset{\sim}{p} + \underset{\sim}{K}\right)} \tag{2.6}$$

in the notation of Levy[8].

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[8] Cf. L1, equation (42). The factor $\frac{1}{2}(1 + \epsilon_K + \epsilon_l - \epsilon_K \epsilon_l)$ has been inserted to take account of an error in sign appearing in that equation.

It is convenient at this stage to return to coordinate space. Eq. (2.5) then becomes

$$\left(P_0 - \mathcal{H}_1 - \mathcal{H}_2\right) \Phi\left(\underset{\sim}{r}\right) = \int d^3 r' \; V\left(\underset{\sim}{r}, \underset{\sim}{r}'\right) \Phi\left(\underset{\sim}{r}'\right) \tag{2.7}$$

where

$$V(\underset{\sim}{r},\underset{\sim}{r}') = -(2\pi)^{-6}\int d^3p\, d^3k\; e^{i\underset{\sim}{p}\cdot(\underset{\sim}{r}-\underset{\sim}{r}')}\; C_i^{(1)}(\underset{\sim}{p})\, C_j^{(2)}(\underset{\sim}{p}) \times \lambda\, C_k^{(1)}(\underset{\sim}{p}-\underset{\sim}{k})\, C_\ell^{(2)}(\underset{\sim}{p}-\underset{\sim}{k})\, K_{ij}^{k\ell}(\underset{\sim}{p},\underset{\sim}{p}-\underset{\sim}{k})\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}'} \tag{2.8}$$

The assumption of a short range repulsive core limits the region of interest to values of $r \sim 1/\mu$. Since $V(r, r')$ is a function sharply peaked at $\underset{\sim}{r} \simeq \underset{\sim}{r}'$, $\int V(\underset{\sim}{r}, \underset{\sim}{r}')\, \phi(\underset{\sim}{r}')d^3r'$ may be replaced by

$$\int d^3r'\; V(\underset{\sim}{r},\underset{\sim}{r}')\, \Phi(\underset{\sim}{r}') = V(\underset{\sim}{r})\, \Phi(\underset{\sim}{r})$$

where

$$V(\underset{\sim}{r}) = -(2\pi)^{-3}\int d^3k\; e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}\; C_i^{(1)}(\underset{\sim}{k})\, C_j^{(2)}(\underset{\sim}{k})\, \lambda\, C_k^{(1)}(0)\, C_\ell^{(2)}(0) \times K_{ij}^{k\ell}(\underset{\sim}{k}, 0) \tag{2.9}$$

Furthermore the potential $V(r)$ can formally be expanded in the usual power series in μ/m . To order μ/m of the leading term, the functions $K_{ij}^{k\ell}$ become[9]

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[9] In the expression for $K_{ij}^{k\ell}$ in (2.10) the velocity dependent terms were neglected, since to investigate them consistently would require consideration of terms in the fourth order potential. It is not totally clear that they will cancel, as they do in the work of Klein.

$$K_{11}^{11} = \frac{1}{\omega_{\underset{\sim}{k}}^2} \; ; \; K_{11}^{12} = K_{11}^{21} = K_{12}^{11} = K_{21}^{11} = \frac{1}{2\omega_{\underset{\sim}{k}}^2} - \frac{1}{4E_{\underset{\sim}{k}}\omega_{\underset{\sim}{k}}}$$

$$K_{11}^{22} = -K_{22}^{11} = \frac{1}{2\omega_{\underset{\sim}{k}}E_{\underset{\sim}{k}}} - \frac{1}{4E_{\underset{\sim}{k}}^2} \; ; \; K_{12}^{12} = K_{21}^{21} = 0$$

(2.10)

$$K_{12}^{21} = K_{21}^{12} = \frac{1}{2\omega_{\underset{\sim}{k}}^2} - \frac{3}{8\omega_{\underset{\sim}{k}}E_{\underset{\sim}{k}}} \; ; \; K_{12}^{22} = K_{21}^{22} = -K_{22}^{12} = -K_{22}^{21}$$

$$= \frac{1}{8\omega_{\underset{\sim}{k}}E_{\underset{\sim}{k}}} - \frac{3}{32E_{\underset{\sim}{k}}^2} \; ; \; K_{22}^{22} = -\frac{1}{4E_{\underset{\sim}{k}}\omega_{\underset{\sim}{k}}} + \frac{1}{16E_{\underset{\sim}{k}}^2} \; ; \; E_{\underset{\sim}{k}} = (\underset{\sim}{k}^2 + m^2)^{1/2}$$

$$\omega_{\underset{\sim}{k}} = (\underset{\sim}{k}^2 + \mu^2)^{1/2}$$

By a straightforward but tedious calculation the potential may be recast in the form

$$V(\underset{\sim}{r}) = g^2 \tau_1 \cdot \tau_2 \Big[V_1(\underset{\sim}{r}) - (\beta^{(1)} + \beta^{(2)}) V_2(\underset{\sim}{r})$$

$$- (\underset{\sim}{\alpha}^{(1)} \cdot \underset{\sim}{p} - \underset{\sim}{\alpha}^{(2)} \cdot \underset{\sim}{p}) \frac{1}{m} V_3(\underset{\sim}{r}) - (\underset{\sim}{\gamma}^{(1)} \cdot \underset{\sim}{p} - \underset{\sim}{\gamma}^{(2)} \cdot \underset{\sim}{p}) \frac{1}{m} V_4(\underset{\sim}{r})$$

$$- (\beta^{(1)} \underset{\sim}{\gamma}^{(2)} \cdot \underset{\sim}{p} - \beta^{(2)} \underset{\sim}{\gamma}^{(1)} \cdot \underset{\sim}{p}) \frac{1}{m} V_5(\underset{\sim}{r}) - (\beta^{(1)} \underset{\sim}{\alpha}^{(2)} \cdot \underset{\sim}{p} - \beta^{(2)} \underset{\sim}{\alpha}^{(1)} \cdot \underset{\sim}{p}) \frac{1}{m} V_6(\underset{\sim}{r})$$

$$- \beta^{(1)} \beta^{(2)} V_7(\underset{\sim}{r}) \Big] \qquad (2.11)$$

where

$$V_i(\underset{\sim}{r}) = (2\pi)^{-3} \int d^3k \; e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} \; V_i(\underset{\sim}{k})$$

$$V_1(\underset{\sim}{k}) = \frac{m}{4\omega_{\underset{\sim}{k}}^2 E_{\underset{\sim}{k}}} - \frac{E_{\underset{\sim}{k}} + 2m}{16\omega_{\underset{\sim}{k}} E_{\underset{\sim}{k}}^2}\ ;\quad V_2(\underset{\sim}{k}) = \frac{5m + 3E_{\underset{\sim}{k}}}{32\omega_{\underset{\sim}{k}} E_{\underset{\sim}{k}}^2} - \frac{E_{\underset{\sim}{k}} - m}{8\omega_{\underset{\sim}{k}}^2 E_{\underset{\sim}{k}}}$$

$$V_3(\underset{\sim}{k}) = \frac{3m}{32\omega_{\underset{\sim}{k}} E_{\underset{\sim}{k}}^2}\ ;\quad V_4(\underset{\sim}{k}) = \frac{m}{8\omega_{\underset{\sim}{k}}^2 E_{\underset{\sim}{k}}} - \frac{m}{16\omega_{\underset{\sim}{k}} E_{\underset{\sim}{k}}^2} \qquad (2.12)$$

$$V_5(\underset{\sim}{k}) = \frac{m}{8\omega_{\underset{\sim}{k}}^2 E_{\underset{\sim}{k}}} + \frac{m}{16\omega_{\underset{\sim}{k}} E_{\underset{\sim}{k}}^2}\ ;\quad V_6(\underset{\sim}{k}) = \frac{m}{32\omega_{\underset{\sim}{k}} E_{\underset{\sim}{k}}^2}$$

$$V_7(\underset{\sim}{k}) = \frac{1}{4\omega_{\underset{\sim}{k}}^2} - \frac{m + 2E_{\underset{\sim}{k}}}{16\omega_{\underset{\sim}{k}} E_{\underset{\sim}{k}}^2}$$

The equation (2.7) involves the sixteen component wave function $\phi(r)$. The next section is devoted to a natural rearrangement of the component equations.

III. Reduction of the 16-Component Equation.

The 16-component wave function may conveniently be labeled by two spinor indices, (each running from 1 to 4) $\phi_{\alpha a}(\underset{\sim}{r})$, where the Greek index is chosen to refer to particle 1, and the Latin index to particle 2. In this scheme $\phi(\underset{\sim}{r})$ may be viewed as a 4 x 4 matrix. The left hand side of (2.11), when in matrix form gives:

$$\left(P_0 - \mathcal{H}_1 - \mathcal{H}_2\right)_{\alpha a,\beta b}\ \phi_{\beta b} = \left(P_0\,\delta_{\alpha\beta}\,\delta_{ab} - (\mathcal{H}_1)_{\alpha\beta}\,\delta_{ab} - (\mathcal{H}_2)_{ab}\,\delta_{\alpha\beta}\right)\phi_{\beta b}$$

$$= P_0\,\phi_{\alpha a} - \left(\mathcal{H}(\underset{\sim}{p})\,\phi\right)_{\alpha a} - \left(\phi\,\widetilde{\mathcal{H}}(-\underset{\sim}{p})\right)_{\alpha a} \tag{3.1}$$

where "$\sim$" denotes "transpose", and the derivatives in the last term act to the left. Defining a new function

$$\Phi_{\alpha a}(\underset{\sim}{r}) = \left(\phi(\underset{\sim}{r})\,C^{-1}\right)_{\alpha a} \tag{3.2}$$

where C is the usual unitary charge conjugation matrix[10], the r.h.s. of (3.1)

[10] $C\,\widetilde{\gamma}_\mu\,C^{-1} = -\gamma_\mu\ ;\quad \widetilde{C} = -C$

becomes

$$\left[\left(P_0\,\Phi - \mathcal{H}(\underset{\sim}{p})\,\Phi + \Phi\,\mathcal{H}(-\underset{\sim}{p})\right)C\right]_{\alpha a} \tag{3.3}$$

The matrix Φ may be expanded in a complete set of 4 x 4 matrices[11]

[11] Cf., for example, R. Finkelstein, Phys. Rev. 88, 555 (1952).

$$\Phi_{\alpha a}(\underset{\sim}{r}) = \sum_{\sigma=0}^{4} \sum_{\rho} \lambda_{\rho}^{\sigma}(\underset{\sim}{r}) \left(\Gamma_{\rho}^{\sigma}\right)_{\alpha a} \tag{3.4}$$

where the Γ_{ρ}^{σ} are the sixteen Dirac matrices:

$$\Gamma_{o}^{o} = 1 \; ; \quad \Gamma_{\mu}^{1} = (\gamma_1, \ldots \gamma_o) \; ; \quad \Gamma_{\rho}^{2} = (\gamma_2\gamma_3, \ldots \gamma_o\gamma_1, \ldots)$$

$$\Gamma_{\rho}^{3} = (\gamma_o\gamma_2\gamma_3, \ldots \gamma_1\gamma_2\gamma_3) \; ; \quad \Gamma_{o}^{4} = \gamma_5$$

The ρ index may alternately be specified by the set of numbers (s_1, s_2, s_3, s_0) which can take on values 0 or 1. They are defined by

$$\Gamma_{\rho}^{\sigma} = \pm (\gamma_1)^{s_1} \ldots (\gamma_o)^{s_o} \tag{3.5}$$

where $\sum_{\mu=0}^{3} s_\mu = \sigma$

Manipulations similar to the ones leading to (3.3) can be carried out for the right hand side of our equation. Inserting (3.4) in equation (2.7)

and cancelling C on both sides one obtains after some rearrangement

$$P_0 \lambda_\rho^\sigma \Gamma_\rho^\sigma + p_i \lambda_\rho^\sigma \left(1 + (-1)^{s_0+s_i}\right)\gamma_i \beta \Gamma_\rho^\sigma \tag{3.6}$$

$$- m \lambda_\rho^\sigma \left(1 - (-1)^{\sigma - s_0}\right)\beta \Gamma_\rho^\sigma = g^2 \tau_1 \cdot \tau_2 \left[\lambda_\rho^\sigma \left((-1)^{1+\sigma} V_1(r) + (-1)^{s_0+1} V_7(r)\right)\Gamma_\rho^\sigma\right.$$

$$+ \frac{1}{m} p_i \left(1 + (-1)^{s_0+s_i}\right)\left((-1)^{s_0} V_5(r) + 3(-1)^{1+\sigma} V_3(r)\right)\lambda_\rho^\sigma \gamma_i \beta \Gamma_\rho^\sigma$$

$$+ \lambda_\rho^\sigma (-1)^\sigma V_2(r)\left(1 - (-1)^{\sigma - s_0}\right)\beta \Gamma_\rho^\sigma$$

$$\left. + \frac{1}{m} p_i \left((-1)^\sigma V_4(r) + (-)^{1+s_0} V_6(r)\right)\left(1 + (-1)^{\sigma + s_i}\right)\gamma_i \Gamma_\rho^\sigma\right]$$

On the r.h.s., the p_i operate both on the potentials and λ_ρ^σ.

The equations for the $\lambda_\rho^\sigma(\underset{\sim}{r})$ may now be obtained by multiplying (3.6) by $\Gamma_{\rho'}^{\sigma'}$ and taking the spurs of both sides. Introducing the notation

$$\Lambda = \lambda_0^0 \,, \quad \underset{\sim}{a} = (\lambda_1^1, \ldots \lambda_3^1) \,, \quad a_0 = \lambda_0^1 \tag{3.7}$$

$$\underset{\sim}{H} = (\lambda_{23}^2, \ldots \lambda_{12}^2) \,, \quad \underset{\sim}{E} = (\lambda_{01}^2, \ldots \lambda_{03}^2)$$

$$\underset{\sim}{\lambda} = (\lambda_{023}^3, \ldots \lambda_{012}^3) \,, \quad \lambda_0 = \lambda_{123}^3 \,, \quad \Gamma = \lambda_0^4$$

one obtains the set of eight equations

$$P_0 \Lambda - 2\underset{\sim}{p}\cdot\underset{\sim}{E} = q^2 T_1\cdot T_2\left[-\Lambda(V_1+V_7)+\frac{2}{m}\underset{\sim}{p}\cdot\underset{\sim}{E}(V_3+V_5)+\frac{2}{m}\underset{\sim}{p}\cdot\underset{\sim}{a}(V_4+V_6)\right] \quad (3.8a)$$

$$-P_0\underset{\sim}{a} - 2\underset{\sim}{p}\times\underset{\sim}{\lambda} + 2m\underset{\sim}{E} = q^2 T_1\cdot T_2\left[-\underset{\sim}{a}(V_1-V_7) - 2\underset{\sim}{E}V_2 + \frac{2}{m}\underset{\sim}{p}\times\underset{\sim}{\lambda}(V_5-V_3)+\frac{2}{m}\underset{\sim}{p}\Lambda(V_6-V_4)\right]$$

$$a_0 = 0$$

$$P_0\underset{\sim}{E} - 2\underset{\sim}{p}\Lambda - 2m\underset{\sim}{a} = q^2 T_1\cdot T_2\left[-\underset{\sim}{E}(V_1-V_7) - \frac{2}{m}\underset{\sim}{p}\Lambda(V_5-V_3) - \frac{2}{m}\underset{\sim}{p}\times\underset{\sim}{\lambda}(V_6-V_4) - 2\underset{\sim}{a}V_2\right]$$

$$-P_0\underset{\sim}{\lambda} + 2\underset{\sim}{p}\times\underset{\sim}{a} = q^2 T_1\cdot T_2\left[-\underset{\sim}{\lambda}(V_1+V_7)+\frac{2}{m}\underset{\sim}{p}\times\underset{\sim}{a}(V_5+V_3)+\frac{2}{m}\underset{\sim}{p}\times\underset{\sim}{E}(V_6+V_4)\right]$$

$$-P_0\underset{\sim}{H} - 2\underset{\sim}{p}\Gamma = q^2 T_1\cdot T_2\left[\underset{\sim}{H}(V_1+V_7)+\frac{2}{m}\underset{\sim}{p}\Gamma(V_5+V_3)-\frac{2}{m}\underset{\sim}{p}\lambda_0(V_6+V_4)\right]$$

$$P_0\lambda_0 + 2m\Gamma = q^2 T_1\cdot T_2\left[\lambda_0(V_1-V_7) - 2\Gamma V_2 - \frac{2}{m}\underset{\sim}{p}\cdot\underset{\sim}{H}(V_6-V_4)\right]$$

$$P_0\Gamma + 2m\lambda_0 + 2\underset{\sim}{p}\cdot\underset{\sim}{H} = q^2 T_1\cdot T_2\left[-\Gamma(V_1-V_7)+2\lambda_0 V_2 + \frac{2}{m}\underset{\sim}{p}\cdot\underset{\sim}{H}(V_5-V_3)\right] \quad (3.8b)$$

As can be seen, the equations split into two uncoupled sets (3.8a and 3.8b) involving $\Lambda, \underset{\sim}{a}, \underset{\sim}{E}, \underset{\sim}{\lambda}$ and $\Gamma, \lambda_0, \underset{\sim}{H}$ respectively. In the non-relativistic limit, the first set corresponds to the charge singlet state and the second to the charge triplet state. Each set may further be rearranged in terms of positive and negative energy components by making use of the fact that

$$4\,\Lambda_{\pm\pm}\,\phi = \sum \lambda_\rho^\sigma \left\{ \left(1-(-)^{\sigma-s_0}\right)\Gamma_\rho^\sigma \pm \left(1-(-1)^{\sigma-s_0}\right)\beta\,\Gamma_\rho^\sigma \right\} C \qquad (3.10)$$

$$= 4\,\phi_{\pm\pm}$$

$$4\,\Lambda_{\pm\mp}\,\phi = \sum \lambda_\rho^\sigma \left\{ \left(1+(-1)^{\sigma-s_0}\right)\Gamma_\rho^\sigma \pm \left(1+(-1)^{\sigma-s_0}\beta\,\Gamma_\rho^\sigma \right\} C \qquad (3.11)$$

$$= 4\,\phi_{\pm\mp}$$

where[12]

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[12] These definitions differ from those of Levy and Klein by terms of order v/c. In their work Casimir operators replace our $\Lambda_\pm$.

$$4\,\Lambda_{++} = (1+\beta_1)(1+\beta_2)\,; \quad 4\,\Lambda_{+-} = (1+\beta_1)(1-\beta_2) \qquad (3.12)$$

etc.

Equations (3.10) and (3.11) imply that for the triplet state

$$\phi_{++} = \tfrac{1}{2}(1+\beta)\gamma_i C\,(a_i + E_i)\,; \quad \phi_{--} = \tfrac{1}{2}(1-\beta)\gamma_i C\,(a_i - E_i) \qquad (3.13)$$

$$\phi_{+-} = \tfrac{1}{2}(1+\beta)(\Lambda - i\,\underset{\sim}{\sigma}\cdot\underset{\sim}{\lambda})C\,; \quad \phi_{-+} = \tfrac{1}{2}(1-\beta)(\Lambda + i\,\underset{\sim}{\sigma}\cdot\underset{\sim}{\lambda})C$$

and for the singlet state

$$\Phi_{++} = \tfrac{1}{2}(1+\beta)\gamma_1\gamma_2\gamma_3 C(\lambda_0 - \Gamma);\ \Phi_{--} = \tfrac{1}{2}(1-\beta)\gamma_1\gamma_2\gamma_3 C(\lambda_0 + \Gamma) \tag{3.14}$$

$$\Phi_{+-} = \tfrac{1}{2}(1+\beta)(-i\underset{\sim}{\sigma}\cdot\underset{\sim}{H})C\ ;\ \Phi_{-+} = \tfrac{1}{2}(1-\beta)(-i\underset{\sim}{\sigma}\cdot\underset{\sim}{H})C$$

IV. Angular Separation.

The angular momentum integral allows one to separate the angular dependence of the wave function. For a system in a state J, M, one finds

$$\left[J^2 - L^2 - \frac{3}{2} + \frac{1}{2}(-1)^{\sigma+S_0}\left((-1)^{S_1} + (-1)^{S_2} + (-1)^{S_3}\right)\right] \lambda_\rho^\sigma \Gamma_\rho^\sigma \qquad (4.1)$$

$$= L_K \lambda_\rho^\sigma \sigma_K \Gamma_\rho^\sigma \left(1 - (-1)^{\sigma+S_0+S_K}\right)$$

$$\left[M - L_z\right] \lambda_\rho^\sigma \Gamma_\rho^\sigma = \frac{1}{2} \lambda_\rho^\sigma \sigma_z \Gamma_\rho^\sigma \left(1 - (-1)^{\sigma+S_0+S_3}\right) \qquad (4.2)$$

where $\underset{\sim}{L}$ is the orbital angular momentum operator. Hence the quantities Λ, Γ, λ_0 satisfy

$$\begin{aligned} (J^2 - L^2) S &= 0 \\ (M - L_z) S &= 0 \end{aligned} \qquad (4.3)$$

and the quantities $\underset{\sim}{a}$, $\underset{\sim}{E}$, $\underset{\sim}{\lambda}$, $\underset{\sim}{H}$ satisfy

$$(J^2 - L^2 - 2)\,\underset{\sim}{V} = 2i\,\underset{\sim}{L} \times \underset{\sim}{V} \qquad (4.4)$$

$$\begin{aligned} (M - L_z) V_x &= -i V_y \\ (M - L_z) V_y &= i V_x \\ (M - L_z) V_z &= 0 \end{aligned}$$

The latter are the equations for the vector spherical harmonics[13] $\underset{\sim}{\eta}_l^m$, $\underset{\sim}{\chi}_l^m$, $\underset{\sim}{\zeta}_l^m$

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13 Cf. Kemmer, Helv. Phys. Acta. 10, 47 (1937), H. Corben and J. Schwinger, Phys. Rev. 58, 953 (1940). The notation used is that of Kemmer. Note the three components of a $\underset{\sim}{\eta}_l^m$ etc. listed by him correspond to η_-, η_z and $-\eta_+$ etc.

To decide upon the correct combination of these functions we make use of the generalized parity operator

$$\Pi = P \beta^{(1)} \beta^{(2)} \tag{4.5}$$

(which reduces to the ordinary parity operator P in the non-relativistic limit). Since

$$\Pi \Phi = \sum (-1)^{\sigma - s_0 + 1} P \lambda_\rho^\sigma \Gamma_\rho^\sigma C \tag{4.6}$$

and since the triplet and singlet wave functions are separately eigenfunctions of Π (with eigenvalues ± 1), one finds that the most general form for the functions is

$$\Lambda = u_1(r)\, Y_J^M$$

$$\underset{\sim}{a} = u_2(r)\, \underset{\sim}{\chi}_{J-1}^M + u_3(r)\, \underset{\sim}{\zeta}_{J+1}^M$$

$$\underset{\sim}{E} = u_4(r)\,\underset{\sim}{\chi}^{M}_{J-1} + u_5(r)\,\underset{\sim}{\zeta}^{M}_{J+1} \qquad (4.7)$$

$$\underset{\sim}{\lambda} = u_6(r)\,\underset{\sim}{\eta}^{M}_{J}$$

$$\Gamma = v_1(r)\,Y^{M}_{J}$$

$$\lambda_0 = v_2(r)\,Y^{M}_{J}$$

$$\underset{\sim}{H} = v_3(r)\,\underset{\sim}{\chi}^{M}_{J-1} + v_4(r)\,\underset{\sim}{\zeta}^{M}_{J+1}$$

where $u_1, \ldots u_6$; $v_1, \ldots v_4$ are radial functions.

After substitution of these functions into (3.8a,b) the angular parts may be divided out and general radial equations obtained. We state the results only for the $J = 0$ (triplet charge state) and $J = 1$ (singlet charge state) cases:

$J = 0$

$$(P_0 - 2m) f_{++} + i\,(d/dr + 2/r)\,v_4 = g^2\,\tau_1\cdot\tau_2\left[(V_1 - V_7 - 2V_2) f_{--} + i/m\,(d/dr + 2/r)\left(v_4\,(V_6 + V_5 - V_4 - V_3)\right)\right] \qquad (4.8a)$$

$$(P_0 + 2m) f_{--} - i\,(d/dr + 2/r)\,v_4 = g^2\,\tau_1\cdot\tau_2\left[(V_1 - V_7 + 2V_2) f_{++} + i/m\,(d/dr + 2/r)\left(v_4\,(V_6 - V_5 - V_4 + V_3)\right)\right] \qquad (4.8b)$$

$$\left[P_0 + g^2 T_1 \cdot T_2 \,(V_1 + V_7)\right] v_4 \qquad (4.8c)$$

$$= -2i\, d/dr \left[f_{++} \left\{ 1 + (g^2 T_1 \cdot T_2/m)(V_3 + V_4 + V_5 + V_6) \right\} \right.$$

$$\left. - f_{--} \left\{ 1 + (g^2 T_1 \cdot T_2/m)(V_3 - V_4 + V_5 - V_6) \right\} \right]$$

where $f_{++} = \tfrac{1}{2}(v_2 - v_1)$ and $f_{--} = \tfrac{1}{2}(v_2 + v_1)$. These linear combinations, as may be seen from equations (3.14) and (4.7), are the radial parts of ϕ_{++} and ϕ_{--} respectively.

For the J = 1 case, we similarly define the quantities $f^S_{++} = \tfrac{1}{2}(u_2 + u_4)$ $f^S_{--} = \tfrac{1}{2}(u_2 - u_4)$, $f^D_{++} = \tfrac{1}{2}(u_3 + u_5)$, $f^D_{--} = \tfrac{1}{2}(u_3 - u_5)$ which are the appropriate radial functions for the $\ell = 0$ and $\ell = 2$ states. Writing $u_1' = -i\sqrt{3}\,u_1$, $u_6' = \sqrt{3}\,u_6$ we obtain

$$(P_0 - 2m) f^S_{++} + \tfrac{1}{3}(d/dr + 2/r)(2u_6' + u_1') = g^2 T_1 \cdot T_2 \left[f^S_{--} (V_1 - V_7 - 2V_2) \right. \qquad (4.9a)$$

$$\left. - \tfrac{1}{3} m (d/dr + 2/r) \left\{ (V_5 - V_4 + V_6 - V_3)(2u_6' - u_1') \right\} \right]$$

$$(P_0 + 2m) f^S_{--} + \tfrac{1}{3}(d/dr + 2/r)(2u_6' - u_1') = g^2 T_1 \cdot T_2 \left[f^S_{++} (V_1 - V_7 + 2V_2) \right. \qquad (4.9b)$$

$$\left. + \tfrac{1}{3} m (d/dr + 2/r) \left\{ (V_6 - V_5 - V_4 + V_3)(2u_6' + u_1') \right\} \right]$$

$$(P_o-2m)f^D_{++} + \tfrac{1}{3}(d/dr - 1/r)(u_6'-u_1') = g^2 T_1\cdot T_2\left[f^D_{--}(V_1-V_7-2V_2)\right. \quad (4.9c)$$
$$\left. - \tfrac{1}{3}m(d/dr - 1/r)\{(V_6+V_5-V_4-V_3)(u_6'+u_1')\}\right]$$

$$(P_o+2m)f^D_{--} + \tfrac{1}{3}(d/dr - 1/r)(u_6'+u_1') = g^2 T_1\cdot T_2\left[f^D_{++}(V_1-V_7+2V_2)\right. \quad (4.9d)$$
$$\left. + \tfrac{1}{3}m(d/dr - 1/r)\{(V_6+V_5-V_4-V_3)(u_6'-u_1')\}\right]$$

$$[P_o + g^2 T_1\cdot T_2(V_1+V_7)]u_1' = 2\,d/dr\,(f^S_{++} - f^S_{--})$$
$$- 4(d/dr + 3/r)(f^D_{++} - f^D_{--}) + 2g^2 T_1\cdot T_2/m\left[\frac{d}{dr}\{(V_3+V_5)(f^S_{++} - f^S_{--})\right. \quad (4.9e)$$
$$\left. + (V_6+V_4)(f^S_{++} + f^S_{--})\right\} - 2(d/dr + 3/r)\{(V_3+V_5)(f^D_{++} - f^D_{--})$$
$$\left. + (V_6+V_4)(f^D_{++} + f^D_{--})\right\}\Big]$$

$$[P_o - g^2 T_1\cdot T_2(V_1+V_7)]u_6' = 2\,d/dr\,(f^S_{++} + f^S_{--})$$
$$+ 2(d/dr + 3/r)(f^D_{++} + f^D_{--}) - 2g^2 T_1\cdot T_2/m\left[d/dr\{(V_3+V_5)(f^S_{++} + f^S_{--})\right. \quad (4.9f)$$
$$\left. + (V_6+V_4)(f^S_{++} - f^S_{--}) + (d/dr + 3/r)\{(V_3+V_5)(f^D_{++} + f^D_{--})\right.$$
$$\left. + (V_6+V_4)(f^D_{++} - f^D_{--})\right\}\Big]$$

V. Discussion of the J = 0 and J = 1 States.

The quantity v_4, which is related to the ϕ_{+-} and ϕ_{-+} components in the J = 0 case, may easily be eliminated from equations (4.8 a,b) with the aid of (4.8 c). One thus obtains a pair of coupled differential equations for f_{++} and f_{--}. Separating out, in the first equation, the quantity $-1/m\,(d^2/dr^2+(2/r)\,d/dr)f_{++}$ which is the kinetic energy structure with reduced mass, one obtains

$$(P_o-2m)f_{++} + 1/m\,(d^2/dr^2+(2/r)\,d/dr)\,f_{++} \tag{5.1a}$$

$$= \frac{2f_{++}}{P_o+g^2T_1\cdot T_2(V_1+V_7)}\left(-\frac{d^2}{dr^2}-\frac{2}{r}\frac{d}{dr}\right)\left(1+\frac{g^2T_1\cdot T_2}{m}(V_3+V_4+V_5+V_6)\right)$$

$$+\frac{2f_{++}}{\left[P_o+g^2T_1\cdot T_2(V_1+V_7)\right]^2}\frac{d}{dr}\left[P_o+g^2T_1\cdot T_2(V_1+V_7)\right]\frac{d}{dr}\left[1+\frac{g^2T_1\cdot T_2}{m}(V_3+V_4$$

$$+V_5+V_6)\right] + g^2T_1\cdot T_2(V_1-V_7-2V_2)f_{--} + \text{other terms}^{14}$$

$$(P_o+2m)f_{--} = g^2T_1\cdot T_2(V_1-V_7+2V_2)f_{++} + \text{other terms}^{14} \tag{5.1b}$$

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14 The remaining terms were not explicitly written down as their structure is not germane to the subsequent discussion. They consist of velocity dependent potentials generated by the elimination of the ϕ_{+-} and ϕ_{-+} components, and of μ/m corrections to the leading potentials.

Setting $P_o = 2m$ and neglecting μ/m corrections to the leading potentials, the first two potentials on the r.h.s. of (5.1a) become (see 2.12):

$$-(g^2\mu/4\pi)(\mu/2m)\,\tau_1\cdot\tau_2\;\frac{e^{-\mu r}/r}{2m+(g^2/8\pi)\,\tau_1\cdot\tau_2\;e^{-\mu r}/r} \tag{5.2a}$$

and

$$\mu/2\,(g^2\tau_1\cdot\tau_2/4\pi)^2\,(\mu/2m)\,\frac{\mu^2}{\left(2m+(g^2/8\pi)\tau_1\cdot\tau_2\;e^{-\mu r}/r\right)^2} \tag{5.2b}$$

$$\times\;(1+1/\mu r)^2\,e^{-2\mu r}/(\mu r)^2$$

(In obtaining (5.2a) the contact term has been dropped.) The first potential clearly corresponds to the usual Yukawa potential modified by the factor in the denominator. Upon expanding this denominator one obtains

$$-(g^2/4\pi)\,\tau_1\cdot\tau_2\,\mu\,(\mu/2m)^2\,e^{-\mu r}/r\;\left[1-\tfrac{1}{2}(g^2/4\pi)\,\tau_1\cdot\tau_2\,(\mu/2m)e^{-\mu r}/\mu r\right. \tag{5.3}$$

$$\left.+\;\cdots\cdot\right]$$

While the second term in (5.3) is of order $\mu/2m$ of the leading fourth order potential obtained by Levy and Klein, it appears from (5.2a) that an expansion of the denominator is poor in the region $r \lesssim 1/\mu$ (for $g^2/4\pi \simeq 2m/\mu$). The second potential, (5.2b), corresponds to part of the one-pair terms obtained by Klein, modified by a similar denominator factor. If one eliminates f_{--} by perturbation theory, which to first order would correspond to neglecting the "other terms" of (5.1b), one obtains a two-pair term which is however of order $\mu/2m$ of the dominant two-pair potential of Klein. This potential corresponds to the two pair structure when both are in the field at a given time. The latter

case arises in the second non-adiabatic approximation to the Δ_+ interaction (and in the crossed meson diagram (cf. L2, Fig. 1, a_1, a_2)), the first non-adiabatic approximation apparently not allowing "sufficient" retardation for such a situation to occur[15].

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15 We may remark parenthetically that the $\mu/2m$ two-pair terms mentioned above will cancel with the $\mu/2m$ correction to the dominant two-pair structure in a perturbation scheme, as shown by Klein. To see whether this will happen in the present approach, would require detailed calculation.

The rest of the one-pair terms come from a similar source.

Turning to the $J = 1$ case, it is clear that the elimination of U_1' and U_6' in (4.9 a-d) via (4.9 e, f) will yield denominators of the structure $P_0 \pm g^2 \tau_1 \cdot \tau_2 \, (V_1 + V_7)$. Since $\tau_1 \cdot \tau_2 = -3$ the upper sign produces a singularity which occurs approximately at the zero of $2m - 3/2\,(g^2/4\pi) e^{-\mu r}/r$ i.e., at $r \simeq 0.7\,\hbar/\mu c$ for $g^2/4\pi = 2m/\mu$.[16]

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16 A similar calculation using the lowest order kernel recently derived by Dyson, Phys. Rev. 91, 1543 (1953) in the new Tamm-Dancoff formalism does not alter this conclusion.

Although it is not necessarily true that the above singular behavior persists in the more rigorous equations containing integral operator potentials (because there the explicit elimination of the ϕ_{+-} and ϕ_{-+} components is not possible), it is certainly clear that the perturbation expansion is invalid for $g^2/4\pi \gtrsim 2m/\mu$.

If one now proceeds to eliminate all negative energy components in a perturbation fashion, one obtains the following equations, to lowest order

$$(P_o - 2m)\, u = -(1/m)(d^2/dr^2)\, u + V_c\, u + 2^{3/2}\, V_c\, w \qquad (5.4a)$$

$$(P_o - 2m)\, w = -(1/m)(d^2/dr^2 - 6/r^2)\, w + V_t\, w + 2^{3/2}\, V_t\, w \qquad (5.4b)$$

where $u = f^S_{++}/r$, $w = \sqrt{2}\, f^D_{++}/r$

$$V_c = -(g^2/4\pi)(\mu/2m)^2 e^{-\mu r}/r$$

$$V_t = -(g^2/4\pi)(\mu/2m)^2 \left(1 + 3/\mu r + 3/(\mu r)^2\right) e^{-\mu r}/r$$

It is interesting to observe that the tensor and central forces do not appear as in the usual structure (e.g., L2 equation (71)). The reason for this may be seen by returning to equation (2.9). Two of the terms appearing in the integrand of the potential are

$$\left(1/\omega^2_{\underset{\sim}{k}}\right) C^{(1)}_+(\underset{\sim}{p})\, C^{(2)}_+(\underset{\sim}{p})\, \beta^{(1)} \gamma_5^{(1)} \beta^{(2)} \gamma_5^{(2)}\, C^{(1)}_+(0)\, C^{(2)}_+(0) \qquad (5.5)$$

and

$$\left(1/\omega^2_{\underset{\sim}{k}}\right) C^{(1)}_-(\underset{\sim}{p})\, C^{(2)}_+(\underset{\sim}{p})\, \beta^{(1)} \gamma_5^{(1)} \beta^{(2)} \gamma_5^{(2)}\, C^{(1)}_+(0)\, C^{(2)}_+(0)$$

Taking the non-relativistic form of these two structures one obtains

$$(1/4m)^2 (\underset{\sim}{\sigma}^{(1)} \cdot \underset{\sim}{p})(1+\beta^{(1)})(\underset{\sim}{\sigma}^{(2)} \cdot \underset{\sim}{p})(1+\beta^{(2)})$$

and

(5.6)

$$-\tfrac{1}{2}(1/4m)\gamma_5^{(1)}(1+\beta^{(1)})\underset{\sim}{\sigma}^{(2)} \cdot \underset{\sim}{p}(1+\beta^{(2)}) - (1/4m)^2(\underset{\sim}{\sigma}^{(1)} \cdot \underset{\sim}{p})(1+\beta^{(1)}) \times (\underset{\sim}{\sigma}^{(2)} \cdot \underset{\sim}{p})(1+\beta^{(2)})$$

respectively. The first term is the one considered by Levy and clearly yields the usual tensor force in the ϕ_{++} state. However, adding the two expressions, the tensor part disappears, leaving only

$$-\tfrac{1}{2}(1/4m)\gamma_5^{(1)}(1+\beta^{(1)})\underset{\sim}{\sigma}^{(2)} \cdot \underset{\sim}{p}(1+\beta^{(2)}) \tag{5.7}$$

Had one made a perturbation expansion on the second of these terms, its effect would appear as a fourth and higher order potential. On the other hand, from the cancellation observed here and elsewhere in (2.9) one sees that these higher order potentials sum so as to replace $V_t w$ by $V_c w$ in (5.4a) and $(V_c - 2V_t)w$ by $V_t w$ in (5.4b).

VI. Conclusions.

The procedure utilized in the preceding sections allows one to examine the validity of the perturbation treatment of the negative energy components. It was hoped at the outset that the avoidance of one of the perturbation approximations inherent in the Levy-Klein program might lead to a convergent series of potentials. This now appears to be doubtful. The pole that appears in the potential for the deuteron seems to invalidate the μ/m expansions which are necessary to replace the integral operators by multiplicative potentials. Aside from this, the large number of μ/m corrections to the leading potentials tends to cast doubt on this method.

Perhaps a more disturbing feature is the change in the structure of the usual tensor interation. Here one has an example of a series of supposedly "small" potentials, adding up so as to modify the lowest order potential. This phenomenon is independent of the size of the coupling constant, and hence is characteristic of the pseudoscalar nature of the interaction only. The possibility is thus raised that other "leading" potentials may similarly be modified by series of terms not even considered in a perturbation scheme to a given order.

To see whether these difficulties occur in higher order approximations, would of course require detailed calculations.